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ITERATIVE METHODS FOR EIGENSENSITIVITY ANALYSIS - A REVIEW

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Abstract

The dynamic behavior of a structural system is characterized by its eigendata. The partial derivatives of eigenvalues and eigenvectors of mechanical system with respect to the design parameters have attracted extensive attention for the last four decades because of their various applications, such as optimal dynamic design, machinery failure diagnostic, parameter identification, model modification and automative control. A more general problem of structural dynamic analysis has three important aspects. Firstly, the observed physical structure is represented by initial finite element model. Modeling is based on numerous idealizing approximations within an exaggerated elaboration of details, which in essence does not significantly improve the accuracy of output data, especially having available powerful computers and appropriate software packages. Optimal alternative is to have the possibility of verifying outputted data that were measured on a prototype or real structure. Secondly, the dynamic characteristics of construction under reanalysis are analyzed. What is basically observed are eigenvalues and main forms of oscillations as characteristic variables that can invoke inadequate actual dynamic behavior. Thirdly, on the basis of the analysis of actual dynamic behavior, modification steps are proposed after which a modified model is obtained. Having in mind that mechanical structures are most often very complex, the most convenient modification steps are not easily obtained. The most straightforward approach for calculating the derivatives is the finite difference method. There mainly exist three categories in the literature: the modal method, the direct method and the iterative method. Several methods for the computation of eigenvector derivatives is analyzed with emphasis on the iterative methods.

Key words: eigensensitivity, structural optimization, repeated frequencies

1. INTRODUCTION

Fox and Kapoor [1] derived the direct and modal methods. Nelson [2] simplified the calculation of the direct method. The eigenvector derivatives with repeated eigenvalues are derived by Ojalvo [3], Mills-Curran [4] and Dailey [5]. Reference [6]-[17] presented reviews for the early work in this area. Reference [18] compared the operation counts of the modal method and the modified Rudisill and Chu's iterative algorithm [19]-[20]. The relative efficiencies are surveyed in Ref [21] for the finite difference method, modal method, Wang's modified modal method and Nelson's direct method on the basis of central processor second.

Although numerical methods for computing eigenvalues and matrix exponentials have been well studied in the literature, there is a lack of analysis in inexact iterative methods for eigenvalue computation and certain variants of the Krylov subspace methods for approximating the matrix exponentials. Ping Zhang [22] has proposed an inexact inverse subspace iteration method that generalizes the inexact inverse iteration for computing multiple and clustered eigenvalues of a generalized eigenvalue problem. The well-known iterative methods for solving eigenvalue problems are the power method (the inverse iteration), the subspace iteration, the Krylov subspace methods and the Jacobi-Davidson algorithm. Traditionally, if the extreme eigenvalues are not well separated or the eigenvalues sought are in the interior of the spectrum, a shift-and-invert transformation (a preconditioning technique) has to be used in combination with these eigenvalue problem solvers. The shiftand-invert transformation requires the solution of shifted linear systems at each iteration. Owing to the size of the matrices, direct solution of the shifted matrix (i.e. factorization) may not be practical. Alternatively, iterative method (inner iterations) can be used to solve the shift-and-invert equation, which leads to two levels of iterations, called inner-outer iterations. The use of inner-outer iterations (or inexact i terations) has been studied for several methods, such as the Davidson and the Lanczos algorithm, the inverse iteration, the rational Arnoldi algorithm and truncated RQ (Rayleigh Quotient) iterations and the Jacobi-Davidson method.

2. ITERATIVE METHODS FOR DESIGN SENSITIVITY ANALYSIS (IEM)

When the measured coordinates are incomplete, measured modes must be expanded for any direct method to be applied, which may be an erroneous procedure which jeopardizes updating. The use of mode expansion can be avoided by using an IEM (or similar method) where only the coordinates which have been measured in the test are used for updating. Collins et. al. [23] first introduced the IEM for model updating. Later, Chen et. al. [24] modified Collins' method by proposing a matrix perturbation method to calculate the sensitivity matrix and to compute the new modal parameters for the parameter estimation procedure. The inverse eigensensitivity method uses modal parameters of an analytical model as initial values and the parameters are updated iteratively based on the differences between the analytical and measured values. Consider mathematically well-behaved functions $f_i(i=1,2,...,m)$ of L variables $b_j,(j=1,2,...,L)$. If we denote b as the entire vector of values b_j, then in the neighbourhood of b_0, the functions can be expanded in the Taylor series:

$$f_i(b) = f_i(b_0) + \sum_{j=1}^{L} \frac{\partial f_i}{\partial b_j}\Delta b_j + \sum_{j=1}^{L}\sum_{k=1}^{L} \frac{\partial^2 f_i}{\partial b_j \partial b_k}\Delta b_j \Delta b_k + ... \tag{1}$$

By neglecting terms of second and higher order, equation (1) can be approximated as:

$$f_i(b) - f_i(b_0) = \sum_{j=1}^{L} \frac{\partial f_i}{\partial b_j}\Delta b_j \tag{2}$$

or in matrix form

$$\begin{Bmatrix} f_1(b) - f_1(b_0) \\ . \\ . \\ f_m(b) - f_m(b_0) \end{Bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial b_1} & \frac{\partial f_1}{\partial b_2} & \cdots & \frac{\partial f_1}{\partial b_L} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial f_m}{\partial b_1} & \frac{\partial f_m}{\partial b_2} & \cdots & \frac{\partial f_m}{\partial b_L} \end{bmatrix} \begin{Bmatrix} \Delta b_1 \\ . \\ . \\ \Delta b_L \end{Bmatrix} \tag{3}$$

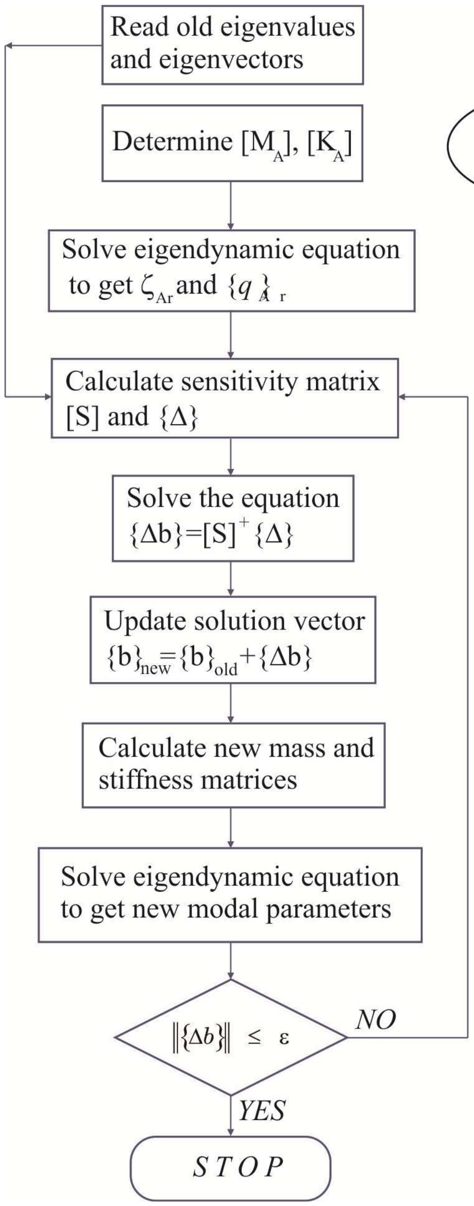

Fig. 1
The flowchart of the inverse iteration method

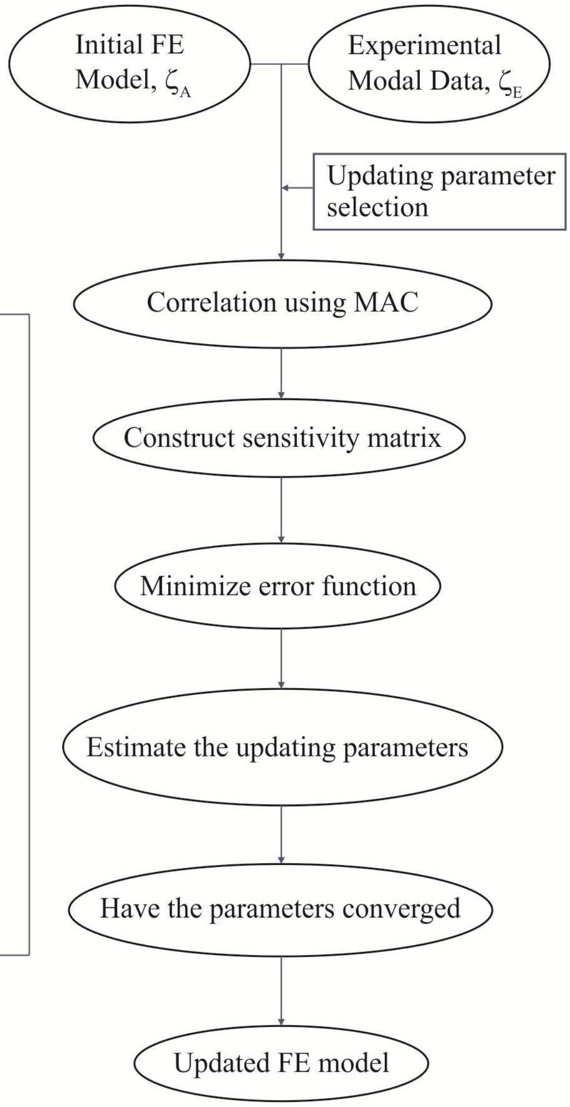

Fig. 2
The flowchart for the model updating method

For a structure under study, the parameters b are to be identified and b_0 are the corresponding values used in the initial analysis. If the updated mass and stiffness matrices are written as in equation

$$\left[M_U\right]=\sum_{j=1}^{L_1} a_j\left[M\right]_j \quad \text{and} \quad \left[K_U\right]=\sum_{j=1}^{L_2} d_j\left[K\right]_j \tag{5}$$

where a_j and d_j are correction factors to be determined and $\left[M\right]_j$ and $\left[K\right]_j$ are submatrices of system matrices such as: sub element matrices, finite element matrices and macro element matrices. The number of unknowns becomes L_1 and L_2. Functions $f_i(b)$ represent the measured modal parameters and $f_i(b_0)$ are the corresponding modal parameters obtained from the initial model. Equation (3) can be written as:

$$\begin{Bmatrix} \Delta\zeta_1 \\ (\Delta q)_1 \\ \cdot \\ \cdot \\ \Delta\zeta_m \\ (\Delta q)_m \end{Bmatrix} = \begin{bmatrix} \frac{\partial\zeta_{A1}}{\partial a_1} & \cdots & \frac{\partial\zeta_{A1}}{\partial a_{L_1}} & \frac{\partial\zeta_{A1}}{\partial d_1} & \cdots & \frac{\partial\zeta_{A1}}{\partial d_{L_2}} \\ \frac{\partial(q_A)_1}{\partial a_1} & \cdots & \frac{\partial(q_A)_1}{\partial a_{L_1}} & \frac{\partial(q_A)_1}{\partial d_1} & \cdots & \frac{\partial(q_A)_1}{\partial d_{L_2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial\zeta_{Am}}{\partial a_1} & \cdots & \frac{\partial\zeta_{Am}}{\partial a_{L_1}} & \frac{\partial\zeta_{Am}}{\partial d_1} & \cdots & \frac{\partial\zeta_{Am}}{\partial d_{L_2}} \\ \frac{\partial(q_A)_m}{\partial a_1} & \cdots & \frac{\partial(q_A)_m}{\partial a_{L_1}} & \frac{\partial(q_A)_m}{\partial d_1} & \cdots & \frac{\partial(q_A)_m}{\partial d_{L_2}} \end{bmatrix} \begin{Bmatrix} \Delta a_1 \\ \cdot \\ \Delta a_{L_1} \\ \Delta d_1 \\ \cdot \\ \Delta d_{L2} \end{Bmatrix} \tag{6}$$

or

$$\{\Delta\}_{m(n+1)\times 1} = [S]_{m(n+1)\times(L_1+L_2)}\{\Delta b\}_{(L_1+L_2)\times 1} \tag{7}$$

The elements of the sensitivity matrix can be expressed as

$$\frac{\partial\zeta_r}{\partial b_i} = \{q\}_r^T \frac{\partial[K]}{\partial b_i}\{q\}_r - \zeta_r\{q\}_r^T \frac{\partial[M]}{\partial b_i}\{q\}_r \text{ and } \frac{\partial\{q\}_r}{\partial b_i} = \sum_{j=1}^{N} c_{rj}^i \{q\}_j , \tag{8}$$

where ζ_r and $\{q\}_r$ are eigenvalues and eigenvectors, where

$$c_{rk}^i = \begin{cases} \dfrac{\{q\}_k^T \left(\dfrac{\partial[K]}{\partial b_i} - \lambda_r \dfrac{\partial[M]}{\partial b_i} \right) \{q\}_r}{\zeta_r - \zeta_k}, & (k \neq i) \\ -\dfrac{1}{2}\{q\}_k^T \dfrac{\partial[M]}{\partial b_i}\{q\}_r . & (k = i) \end{cases} \tag{9}$$

If the number of measured modes m is greater than $(L_1 + L_2)/(n+1)$, equation (7) becomes overdetermined and the unknown vector $\{\Delta b\}$ can be calculated by premultiplying equation (7) by $[S]^+$

$$\{\Delta b\} = [S]^+\{\Delta\} \tag{10}$$

where „+“ is the Moore-Penrose generalized inverse. The corrections ate than added to the solution vector

$$\{b\}_{new} = [b]_{old} + \{\Delta b\} \tag{11}$$

and the process is repeated iteratively to the convergence because equation (6) is formulated based on the first-order approximation. The flowchart of the whole procedure could be seen in Fig. 1. Iterative methods or the sensitivity methods, which concern of reducing an objective function that is generally a non-linear function of selected updating parameters, are carried out by either using eigendata or frequency response function (FRF) data. Therefore it provides wider choice of parameters for updating. These methods considered as capable of overcoming the limitations of the direct methods. It also has been applied successfully to large-scale industrial problems. A brief explanation and tutorial on this sensitivity method in finite element model updating is provided by Mottershead, Link and Friswell [25]. Example of model updating of a helicopter airframe is also showed in the paper. The sensitivity method is based upon linearization of the generally non-linear relationship between the measurable outputs such as natural frequencies, mode shapes or displacement responses, of the model's parameters in need of emendation. The most important quality is to define an error function of modal data obtained

from computer simulation and experimental. The estimated parameters are attained by minimizing the error function with respect to the updating parameters.
The simplified flow diagram for the model updating method is shown in figure 2.

3. ITERATIVE METHOD BASED ON THE RAYLEIGH QUOTIENT APPROXIMATION

The relationship of a natural frequency or corresponding eigenvalue to its associated eigenvector and the system's stiffness and mass is expressed by Rayleigh's quotient

$$\zeta = \omega^2 = \frac{\{q\}_r^T [K] \{q\}_r}{\{q\}_r^T [M] \{q\}_r} = \frac{E_p}{E_k} \quad \text{for all} \quad \{q\} \in W \;,\; \{q\} \neq \{0\} \qquad (12)$$

where the modal strain energy E_p and the modal kinetic energy E_k are the sum of the strain and kinetic energy, respectively, from each of the elements [26]-[30].

4. EXAMPLE

Iterative method based on using Rayleigh's quotient is presented on the following example.
Consider a cantilever beam with one joint of length 1 m, rectangular cross-section, 100 mm and 50 mm, divided into 5 finite elements (Fig. 3). In designations, in the tables and diagrams, the beam is referred to as the original or initial beam.

Table 1. Few initial eigenvalues for the original cantilever beam and the modified one, where the height, as a construction variable, is increased by 10%

Initial beam		Height increased by 10% across the entire length		Modified shape, I, II,III,IV,V Δh[%], using Mat Lab: 15.17 -0.47 6.34 8.55 2.03	
Frequencies, f_{0i}[Hz]	Eigenvalues, ζ_i	Frequencies f_{0i}[Hz]	Eigenvalues, ζ_i	Frequencies f_{0i}[Hz]	Eigenvalues, ζ_i
1243.79	61073360.50	1368.17	73898766.21	1318.61	68642050.36
591.37	13806531.28	650.51	16705902.85	633.17	15826955.55
182.05	1308456.41	**200.26**	1583232.26	**202.34**	1616377.13
		$\Delta\zeta_1$ = 21 %, Δf_{01} = 10 %			

For the analysis of sensitivity to changes, the original beam is modified across the entire length, with small modifications[1]. That beam is called a modified cantilever beam (Fig. 4). In this case, the chosen construction variable b is the height of the rectangular cross-section h. Calculations are performed with the software package MatLab that possesses the function for calculating eigenvalues and eigenvectors. The lowest frequencies are always of the utmost interest for analysis. The Table 1 shows a few initial eigenvalues for the original beam and the modified one, where the height, as a construction variable, is increased by 10%.
Fig. 5 displays the diagram of potential, E_p, and kinetic, E_k, energy distributions, and their mutual difference, $E_p - E_k$, initial cantilever beam for all elements in a row, for the first

[1] In the literature dealing with dynamic reanalysis it is stressed that modifications should be small, so that the chosen modification process converges to the desired eigenvalues of the pairs, however it is not easy to determine what is 'small';

oscillation mode, where the first eigenfrequency is f_{01} = 182.05Hz, and the first eigenvalue is ζ_1=1308456.41s^{-2}.

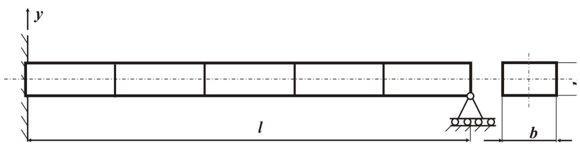

Fig. 3 Initial beam, ρ=7833kg/m^3, E=206840000000 N/m^2;
b=0.1m, h=0.05m, l=1m

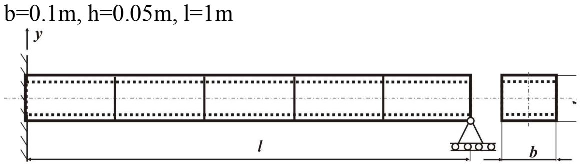

Fig. 4 Modified beam, b_1=b, h_1=1.1h

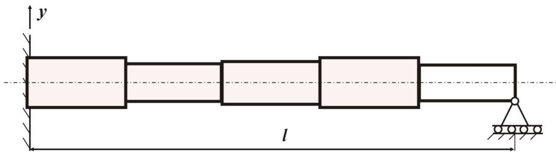

Fig. 7 Modified beam after the first iterative step

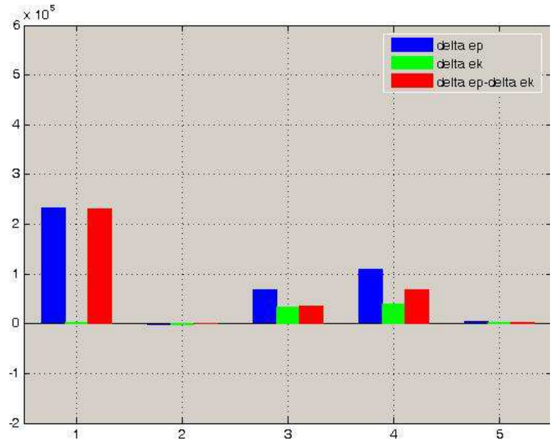

Fig 8 Diagram of potential and kinetic energy growth rate distributions and their mutual difference for the modified cantilever beam after the first iterative step, and the original cantilever beam [J].

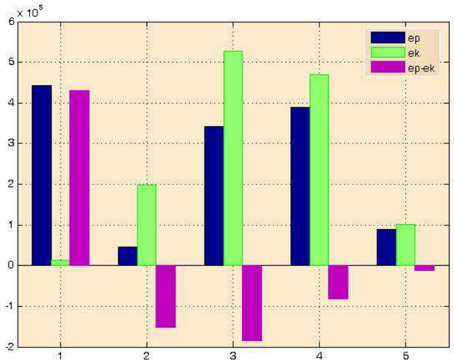

Fig. 5 Diagram of potential and kinetic energy distributions and their mutual difference for the original cantilever beam [J].

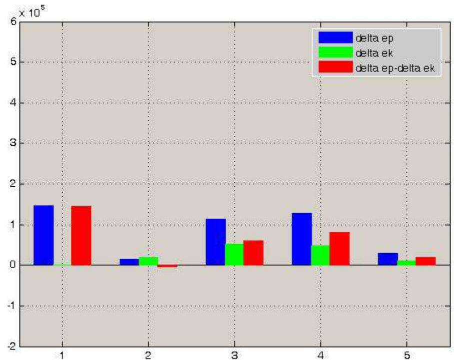

Fig. 6 Diagram of potential and kinetic energy growth rate distributions and their mutual difference for modified and original cantilever beam [J].

Fig. 6 shows a diagram of potential, ΔE_p, and kinetic, ΔE_k, energy growth rates and their difference $\Delta E_p - \Delta E_k$ for the increased height across the entire beam length, by 10 %, for the first oscillation mode. The first frequency of a modified cantilever beam is f'_{01} = 200.26Hz, while the first eigenvalue is ζ_1'=1583232.26s^{-2}. It is noticeable that the first eigenvalue growth rate is $\Delta\lambda_1$ = +21 %, and the corresponding eigenfrequency growth rate is $\Delta f_{01} = f'_{01} - f_{01}$ = +10 %
Fig. 8 displays a diagram of potential, ΔE_p, and kinetic, ΔE_k, energy growth rates and their difference $\Delta E_p - \Delta E_k$ for the modified cantilever beam after the first iterative step (Fig. 7) for the first oscillation mode. The aim of modification is to increase the frequency by 10 %. Note the

convergence compared to the previous diagram, which is evidenced by reduced 'columns' characterizing the change in potential and kinetic energy growth rates. Also, a significant conclusion related to the cantilever beam cross-section modification is that stiffness, i.e. cross-section height, should be increased in the fixed-point zone.

Tab. 2 Differences in potential and kinetic energy growth rates as well cantilever cross-section heights after the iteration procedure

Element 1	Element 2	Element 3	Element 4	Element 5
ΔE_{pk1}	ΔE_{pk2}	ΔE_{pk3}	ΔE_{pk4}	ΔE_{pk5}
145033.92	-4498.74	60599.49	81700.29	19418.48
$h_1 = 1.1517h$	$h_2 = 0.9953h$	$h_3 = 1.0634h$	$h_4 = 1.0855h$	$h_5 = 1.0203h$

Differences in potential and kinetic energy growth rates as well cantilever cross-section heights after the iteration procedure are shown in the table 2.

5. CONCLUDING REMARKS

Studying the dynamic behavior of a construction can predict its response to change in shape, changes in size of its elements or change in materials used. Generally, the aim of system modification with respect to improvements in dynamic behavior is to increase eigenfrequencies and widen the distance between two neighboring frequencies. The specific importance lies in lowest frequencies and those close to the system exciting frequencies. The resulting Rayleigh Quotient Approximation has the important and unique characteristic. Although the numerical examples were simple, the theory itself does not share their limitations. The example presented here and similar examples in the literature did demonstrate that Rayleigh Quotient Approximation based on the distribution kinetic and potential energy should be especially important for complex structures. When groups suitable for reanalysis are located, a detailed (fine) analysis of a separated subgroup is undertaken. Most often it is necessary to make a modified model which is used for comparison to the original one, and on the basis of Rayleigh Quotient, new guidelines are reached. There are clear, mathematically expressed, unambiguous guidelines for further conducting the modification procedure.

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