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A New Best Proximity Point Results in Partial Metric Spaces Endowed with a Graph

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Abstract: For a given mapping f in the framework of different spaces, the fixed-point equations of the form $fx = x$ can model several problems in different areas, such as differential equations, optimization, and computer science. In this work, the aim is to find the best proximity point and to prove its uniqueness on partial metric spaces where the symmetry condition is preserved for several types of contractive non-self mapping endowed with a graph. Our theorems generalize different results in the literature. In addition, we will illustrate the usability of our outcomes with some examples. The proposed model can be considered as a theoretical foundation for applications to real cases.

Keywords: proximity point; partial metric spaces; G-contraction; connected graph

MSC: 54H25; 47H10

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Citation: Aloqaily, A.; Souayah, N.; Matawie, K.; Mlaiki, N.; Shatanawi, W. A New Best Proximity Point Results in Partial Metric Spaces Endowed with a Graph. *Symmetry* **2023**, *15*, 611. https://doi.org/10.3390/sym15030611

Academic Editors: Alexander Zaslavski and Sergei D. Odintsov

Received: 1 February 2023
Revised: 17 February 2023
Accepted: 21 February 2023
Published: 28 February 2023

1. Introduction

The increasing interest of fixed-point theory emerges due to its importance as a tool to solve nonlinear equations. Many problems can be formulated as nonlinear equations of the form $fx = x$, where f is a self-mapping. As shown by Banach [1], for every contractive self-mapping $f : X \longrightarrow X$, the fixed-point equation $fx = x$ has a unique solution in X. For more details, we refer to [2–11]. Nevertheless, if f is a non-self mapping, this type of equation does not necessarily have a solution. In this case, researchers tried different approaches, where they established an approximate solution that was the nearest possible point x to fx in the sense of the metric. This approximate point was said to be *best proximity point*, which is more general than the fixed point. We note that this solution x is optimal in the sense that the distance between fx and x is minimum. Recently, the best proximity point and fixed-point theory have been combined with graph theory. The first initiative was carried out by Jachymski [12]. He considered metric spaces with the structure of a graph as a part where the symmetry condition is preserved in relation to the fixed-point theory of contractive-type mappings. The principle of his work is that the fixed point needs only to be satisfied on certain pairs of points joined with the edges of the graph. Fixed-point and best-proximity-point theory on metric spaces with graphs have an application in diverse sciences, such as computer science and engineering. In fact, fixed-point theory is used to examine the stability analysis of complex neural networks. A new process of contraction mapping principle is employed to explore the stability of impulsive cellular neural networks with time-varying delays [13,14]. Chena et al. [15] introduced a suitable complete metric space and a contraction mapping of which the fixed point is a solution of the system given by a class of impulsive stochastic delayed neural networks and thus established the exponential stability of this system. Fixed-point theory can be applied also in a communication network, which can be considered as a space formed by the node iterative sequences of the path prediction algorithms [16,17]. The theory may be used to describe the relation of network nodes and reflect the physical relation characteristic presented by the network in general. The mapping f can be looked at as an operator used for multiple aspects, such as cost and energy.

Motivated by the importance of the fixed-point theory and its application, especially when it is coupled with graph theory, we focus in this paper on the best proximity point theorems on a partial metric space endowed with a graph that is more general than fixed point. Additionally, the partial metric is very useful in real work since the measure between two nodes, x and y, such that $x = y$ is not zero. This work can be considered a theoretical framework for applications to real cases.

In the following section, we will present some preliminary definitions.

2. Preliminaries

First, we start by reminding the reader of the definition of a best proximity point.

Definition 1. *Let (X, d) be a metric space, A_1 and A_2, two subsets of X, and a mapping $f : A_1 \longrightarrow A_2$. We denote by $d(A_1, A_2)$ the distance between A_1 and A_2 as follows:*

$$d(A_1, A_2) = \inf\{d(u_1, u_2) : u_1 \in A_1, \ \ u_2 \in A_2\}.$$

An element $u \in A_1$ is called a best proximity point of the mapping f if

$$d(u, fu) = d(A_1, A_2). \tag{1}$$

The best-proximity-point theory was introduced by Ky Fan [18]. He considered a continuous mapping $F : C \longrightarrow E$ where E, is a normed linear space and C is a compact convex subset of E. Ky Fan gave an approximate solution of $Fx = x$; unfortunately, his solution was not optimal. Later, many authors established existence and uniqueness theorems on best proximity point for contractive mapping [19–27].

Definition 2 ([28])**.** *Let X be a nonempty set, and if the function $p : X^2 \longrightarrow [0, \infty)$ satisfies the following assumptions for all $t, s, w \in X$*

(p_1) $t = s \Longleftrightarrow p(t,t) = p(t,s) = p(s,s)$,
(p_2) $p(t,t) \leq p(t,s)$,
(p_3) $p(t,s) = p(s,t)$,
(p_4) $p(t,s) \leq p(t,w) + p(w,s) - p(w,w)$.

Then, the pair (X, p) is said to be a partial metric space.

Definition 3 ([28])**.** *Consider a partial metric space (X, p). Then,*

1. *A sequence $\{a_n\}$ in X converges to a point a if and only if $\lim\limits_{n\to\infty} p(a_n, a) = p(a, a)$.*
2. *A sequence $\{a_n\}$ in X is called to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} p(a_n, a_m)$ exists and is finite.*
3. *(X, p) is called to be complete if each Cauchy sequence $\{a_n\}$ in X converges to a point $a \in X$ and $\lim\limits_{n,m\to\infty} p(a_n, a_m) = p(a, a)$.*
4. *Let $B_p(t_0, \delta)$ be an open ball in (X, p). A mapping $g : X \longrightarrow X$ is called to be continuous at $t_0 \in X$ if for each $\epsilon > 0$ there exists $\delta > 0$, so that $g(B_p(t_0, \delta)) \subset B_p(gt_0, \epsilon)$.*

Next, we present the definition of the best proximity point in the partial metric spaces (X, p).

Definition 4 ([28])**.** *Let A_1, A_2 be nonempty subsets of a partial metric space (X, p) and $f : A_1 \longrightarrow A_2$ be a given mapping.*

1. *We denote by $p(A_1, A_2) = \inf\{p(a_1, a_2) : a \in A_1, a_2 \in A_2\}$.*
2. *An element $u \in A$ is called a best proximity point for the mapping f if $p(u, fu) = p(A_1, A_2)$.*

Remark 1. *For a given map f, a best proximity point of f is a generalization of its fixed point.*

Consider the partial metric space (X, p). Let A_1 and A_2 be two nonempty subsets of (X, p); we denote by A_0 and B_0 the following sets:

$$A_0 = \{a_1 \in A_1 : p(a_1, a_2) = p(A_1, A_2) \text{ for some } a_2 \in A_2\} \tag{2}$$

$$B_0 = \{a_2 \in A_2 : p(a_1, a_2) = p(A_1, A_2) \text{ for some } a_1 \in A_1\}. \tag{3}$$

Note that A_0 and B_0 are nonempty sets [29].

Definition 5 ([27])**.** *Let (A_1, A_2) be a pair of nonempty subsets of (X, p) such that $A_0 \neq \emptyset$. The pair (A_1, A_2) is called to have the P-property if and only if for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$*

$$\left.\begin{array}{ll} p(x_1, y_1) & = p(A_1, A_2) \\ p(x_2, y_2) & = p(A_1, A_2) \end{array}\right\} \Longrightarrow p(x_1, x_2) = p(y_1, y_2).$$

For the convenience of the reader, we recall some basic concepts of graph theory which we will use later.

Now, let us recall some preliminaries from graph theory. Let (X, p) be a partial metric space and $\Delta = X \times X$. A graph G is determined by the given of a pair (V, E), where $V = V(G)$ is a set of vertices coinciding with X and $E = E(G)$ the set of its edges such that $\Delta \subset E(G)$. Additionally, we presume that graph G does not contain parallel edges. Graph G can be seen as a weighted graph by allocating to each edge the distance obtained by the p-metric between its vertices. Let G^{-1} be the graph defined as follows:

$$\left\{\begin{array}{ll} E(G^{-1}) & = \{(a, b) \in X^2 : (b, a) \in E(G)\} \\ V(G^{-1}) & = V(G). \end{array}\right.$$

It is clear that G^{-1} derives from graph G by reversing the direction of its edges. We denote by G^{-1}, the graph obtained from G by reversing the direction of its edges that can be defined as follows. Thereby, we denote by $\tilde{G}$ the undirected graph obtained by ignoring the direction of edges of G.

Definition 6. *Let u and u be two vertices in a graph G. A path in G from u to v of length s ($s \in \mathbb{N} \cup \{0\}$) is a sequence $(a_k)_{k=0}^{s}$ of $s + 1$ distinct vertices such that $a_0 = u$, $a_s = v$ and $(a_j, a_{j+1}) \in E(G)$ for $i = j, 2, \ldots, s$. We denote*

$$[u]_G^N = \{v \in X : \text{ there is a path in } G \text{ of length } N \text{ from } u \text{ to } v\}. \tag{4}$$

Definition 7.
- *If there is a path between any two vertices of a graph G, we say that G is connected.*
- *G is said to be weakly connected if $\tilde{G}$ is connected.*

Inspired by the work of Jachymski in [12] and the platform of graph theory that he introduced, in this paper, we generalize his results to a partial metric space for non-self mappings. Therefore, the fixed points do not necessarily exist; for this reason, we focus on the concept of the best proximity point on partial metric spaces endowed with a graph. Nevertheless, the distance between two vertices of the graph is given by the partial metric. Then, we can have nonzero self-distance for each vertex. Thereby, the theorems obtained represent a generalization of some results, and the essential feature of this work is that it is a further extension of partial metric spaces with a graph structure on them. In the following section, we will present our main results.

3. Main Results

Throughout the rest of the document, we consider (X, p) to be a partial metric space, and G is a directed graph without parallel edges such that $X = V(G)$.

Definition 8. *Let A_1 and A_2 be two nonempty subsets of (X, p). A mapping $f : A_1 \longrightarrow A_2$ is said to be G-contraction if for all $x, y \in A_1$ with $(x, y) \in E(G)$:*

(i) $p(fx, fy) \leq \alpha p(x, y)$ *for some* $\alpha \in [0, 1)$,

(ii) $\left.\begin{array}{ll} p(x_1, fx) & = p(A_1, A_2) \\ p(y_1, fy) & = p(A_1, A_2) \end{array}\right\} \Longrightarrow (x_1, y_1) \in E(G), \ \forall x_1, y_1 \in A_1.$

Theorem 1. *Let (X, p) be a complete partial metric space, A and B be two nonempty closed subsets of (X, p) such that (A, B) has the P-property. Let $f : A \longrightarrow B$ be a continuous G-contraction such that $f(A_0) \subseteq B_0$. Assume the following condition (C): x_0 and x_1 exist in A_0 such that there is a path in A_0 between them and $p(x_1, fx_0) = p(A, B)$. Then, there the sequence $\{x_n\}_{n\in\mathbb{N}}$ exists with $p(x_{n+1}, fx_n) = p(A, B)$ $\forall$ $n \in \mathbb{N}$, and f has a unique best proximity point.*

Proof. From condition (C), two points x_0 and x_1 in A_0 exist such that $p(x_1, fx_0) = p(A, B)$, and a path $(z_0^i)_{i=0}^N$ in G between them exists such that the sequence $(z_0^i)_{i=0}^N$ contains points of A_0. Subsequently, $z_0^0 = x_0$, $z_0^N = x_1$ and $(z_0^i, y_0^{i+1}) \in E(G)$ $\forall 0 \leq i \leq N$. Given that $z_0^1 \in A_0$, $f(A_0) \subseteq B_0$ and from the definition of A_0, $z_1^1 \in A_0$ exists such that $p(z_1^1, fz_0^1) = p(A, B)$. By proceeding this way, for $i = 2, \ldots, N$, $z_1^i \in A_0$ exists such that $p(z_1^i, fz_0^i) = p(A, B)$. Since $(z_0^i)_{i=0}^N$ is a path in G, then $(z_0^0, z_0^1) = (x_0, z_0^1) \in E(G)$. From the above, we have $p(x_1, fx_0) = p(A, B)$ and $p(z_1^1, fz_0^1) = p(A, B)$. f is a G-contraction; consequently, $(x_1, z_1^1) \in E(G)$. In the same way, we obtain

$$(z_1^{i-1}, z_1^i) \in E(G) \text{ for } i = 2, \ldots, N. \tag{5}$$

Consider $x_2 = z_1^N$. Therefore, $(z_1^i)_{i=0}^N$ is a path from $x_1 = z_1^0$ to $x_2 = z_1^N$. For each $i = 2, \ldots, N$, as $z_1^i \in A_0$ and $fz_1^i \in f(A_0) \subseteq B_0$, then by the definition of B_0 there exists $z_2^i \in A_0$ such that $p(z_2^i, fz_1^i) = p(A, B)$. Additionally, we have $p(x_2, fx_1) = p(A, B)$. Similar to the above, we obtain

$$(x_2, z_2^1) \in E(G) \text{ and } (z_2^{i-1}, z_2^i) \in E(G) \forall\, i = 1, 2, \ldots, N. \tag{6}$$

Let $x_3 = z_2^N$. Then, $(z_2^i)_{i=0}^N$ is a path from $z_2^0 = x_2$ and $z_2^N = x_3$. By repeating this process, for all $n \in \mathbb{N}$, we create a path $(z_n^i)_{i=0}^N$ from $x_n = z_n^0$ and $x_{n+1} = z_n^N$, which gives us a sequence $\{x_n\}_{n\in\mathbb{N}}$ where $x_{n+1} \in [x_n]_G^N$ and $p(x_{n+1}, fx_n) = p(A, B)$ such that

$$p(z_{n+1}^i, fz_n^i) = p(A, B) \ \ \forall \ \ i = 0, \ldots, N. \tag{7}$$

From (7) and the P-property, we obtain

$$p(z_n^{i-1}, z_n^i) = p(fz_{n-1}^{i-1}, fz_{n-1}^i) \ \forall\, i = 1, \ldots, N. \tag{8}$$

By the triangular inequality, we obtain for all $n \geq 0$,

$$\begin{aligned} p(x_n, x_{n+1}) &= p(z_n^0, z_n^N) \quad (9) \\ &\leq p(z_n^0, z_n^1) + p(z_n^1, z_n^2) + \ldots + p(z_n^{N-1}, z_n^N) - \sum_{i=1}^{N-1} p(z_n^i, z_n^i) \\ &\leq \sum_{i=1}^{N} p(z_n^{i-1}, z_n^i) \\ &= \sum_{i=1}^{N} p(fz_{n-1}^{i-1}, fz_{n-1}^i). \quad (10) \end{aligned}$$

Given that f is a G-contraction, for all $n \in \mathbb{N}$, $(z_{n-1}^{i-1}, z_{n-1}^i) \in E(G)$ and according to (10), we obtain

$$p(x_n, x_{n+1}) \leq \alpha \sum_{i=1}^{N} p(z_{n-1}^{i-1}, z_{n-1}^i) \;\; \forall n \in \mathbb{N}. \quad (11)$$

By induction, it results that $\forall n \in \mathbb{N}$

$$p(x_n, x_{n+1}) \leq \alpha^n \sum_{i=1}^{N} p(z_0^{i-1}, z_0^i) = \lambda\alpha^n. \quad (12)$$

where $\lambda = \sum_{i=1}^{N} p(z_0^{i-1}, z_0^i)$.

Now, we claim that the sequence $\{x_n\}$ is Cauchy. For $n, m \in \mathbb{N}$, $m \geq n$ and from property (p_4) of the partial metric, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m) - \sum_{i=n+1}^{m-1} p(x_i, x_i) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m) \\ &\leq \lambda\alpha^n + \lambda\alpha^{n+1} + \ldots + \lambda\alpha^{m-1} \\ &= \lambda\alpha^n(1 + \alpha + \ldots + \alpha^{m-n-1}) \\ &\leq \lambda\frac{\alpha^n}{1-\alpha}. \end{aligned}$$

Since $\alpha < 1$, then $\lim_{n,m \longrightarrow \infty} p(x_n, x_m) = 0$. Therefore, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and $u_0 \in A$ exists such that $\lim_{n \to \infty} x_n = u_0$. From the continuity of f, we obtain $fx_n \longrightarrow fu$ as $n \longrightarrow \infty$. Now, from the continuity of the partial metric function we obtain $p(x_n, fx_n)$ converges to $p(u, fu)$ as $n \longrightarrow \infty$. Since from the all beforementioned, we have $p(x_{n+1}, fx_n) = p(A, B)$ then $\{p(x_{n+1}, fx_n)\}_n$ is a constant sequence equal to $p(A, B)$. Finally, we obtain $p(u, fu) = p(A, B)$. Then, u_0 is a best proximity point of f.

Suppose that there exist s_1 and s_2 such that

$$\begin{aligned} p(s_1, fs_1) &= p(A, B) \quad (13) \\ p(s_2, fs_2) &= p(A, B). \quad (14) \end{aligned}$$

In order to obtain $s_1 = s_2$ in (X, p), we must prove that $p(s_1, s_2) = p(s_1, s_1) = p(s_2, s_2)$. Knowing that the pair (A, B) has the P-property, using (13) and (14), we obtain $p(s_1, s_2) = p(fs_1, fs_2)$. Since f is a G-contraction, we obtain $p(s_1, s_2) = p(fs_1, fs_2) \leq \alpha p(s_1, s_2)$ where $\alpha < 1$. Therefore,

$$p(s_1, s_2) = 0. \quad (15)$$

By the triangular inequality, we have

$$\begin{aligned} p(s_1,s_1) &\leq p(s_1,s_2)+p(s_2,s_1)-p(s_2,s_2) \\ &\leq \underbrace{2p(s_1,s_2)}_{=0}-p(s_2,s_2). \end{aligned}$$

Thus, $p(s_1,s_1)+p(s_2,s_2)=0$, which implies that

$$p(s_1,s_1)=p(s_2,s_2)=0. \tag{16}$$

Finally, (15) and (16) give that $s_1=s_2$. □

Example 1. *Consider $X=[0,\infty)$ and define $p:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ such that $p(a,b)=\max\{a,b\}$. Clearly, p satisfies the properties $(p_1)-(p_4)$ in Definition 2 and then it is a partial metric. Let $A=[3,4]$ and $B=[1,4]$, two closed subsets of X. It is easy to obtain $p(A,B)=\inf\{p(x_1,x_2):x_1\in A,x_2\in B\}=4$. Let us show that the pair (A,B) has the P-property. Let $x_1,x_2\in A$ and $y_1,y_2\in B$ such that $\left.\begin{array}{ll}\max\{x_1,y_1\}=4 & =p(A,B)=4\\ \max\{x_2,y_2\}=4 & =p(A,B)=4\end{array}\right\}\Longrightarrow$ $\max\{x_1,y_1\}=\max\{x_2,y_2\}$, which gives that $\max\{x_1,x_2\}=\max\{y_1,y_2\}=4\Longrightarrow p(x_1,x_2)=p(y_1,y_2)$. Then, the pair (A,B) has the P-property. Suppose that the map $f:A\longrightarrow B$ is defined as follows:*

$$f(a)=\frac{a+1}{2},\forall a\in A. \tag{17}$$

Consider a graph G with $V(G)=X$ and $E(G)=\{(a,b)\in X\times X:p(a,b)<5\}$. Let us prove that f is a G-contraction. Consider $x,y\in A=[3,4]$, $p(fx,fy)=\max\{fx,fy\}=\max\{\frac{x+1}{2},\frac{y+1}{2}\}=\dfrac{x+1}{2}$ w.l.o.g. Since $3\leq x\leq 4$ and $2\leq fx\leq 2.5$, then $\frac{x+1}{2}\leq\frac{2.5}{2}\max\{x,y\}$. Therefore, $p(fx,fy)\leq\alpha p(x,y)$ with $\alpha=\frac{2.5}{2}<1$. Now, let $x_1,y_1\in A$ and $(x,y)\in E(G)$ such that

$$\begin{aligned} p(x_1,fx) &= p(A,B)=4 && (18)\\ p(y_1,fy) &= p(A,B)=4. && (19) \end{aligned}$$

From (18), (19) and the P-property we obtain $p(x_1,y_1)=p(fx,fy)\leq\alpha p(x,y)<p(x,y)$. Since $(x,y)\in E(G)$, then $p(x,y)<5$, which gives $p(x_1,y_1)<5$; therefore, $(x_1,y_1)\in E(G)$. Hence, the map f is a G-contraction. Additionally, let $x_0\in A_0$, from (2), $x_0\in A=[3,4]$, and $y\in[1,4]$ exists such that $\max\{x_0,y\}=p(A,B)=4$. $fx_0=\dfrac{x_0+1}{2}$ such that $2\leq fx_0\leq\dfrac{5}{2}$ $\Longrightarrow fx_0\in B$, and $x\in A=[3,4]$ exists satisfying $\max\{x,fx_0\}=p(A,B)=4$; then $fx_0\in B_0$. Hence, $f(A_0)\subseteq B_0$. Consider $x_0,x_1\in A_0$, and let us check the condition (C). Let $x_0=3.5$, $x_1=4$ and $N=1$. Since $\max\{x_0,x_1\}=4<5$, then the pair $(x_0,x_1)\in E(G)$. From (17), $fx_0=2.25$, and we obtain $\max\{x_1,fx_0\}=\max\{4,2.25\}=4=p(A,B)$. Thus, condition (C) holds. Finally, all of the assumptions of Theorem 1 are satisfied. Hence, a unique best proximity point $u=4\in A=[3,4]$ exists that is $p(u,fu)=p(A,B)=4$. Indeed, $p(4,f(4))=\max\{4,2.25\}=4=p(A,B)$.

As a consequence of Theorem 1, if $X=A=B$, then we obtain a fixed point instead of the best proximity point, which generalizes many results in the literature.

Corollary 1. *Consider a complete partial metric space (X,p) and a continuous self-mapping $f:X\longrightarrow X$ such that for all $t,s\in X$, if $(t,s)\in E(G)$ then $(ft,fs)\in E(G)$ and $p(ft,fs)\leq\alpha p(t,s)$ where $\alpha\in[0,1)$. Then, the following statements hold:*

(i) if $(t,fs)\in E(G)$ then $\{f^n(t)\}_{n\in\mathbb{N}}$ converges to a fixed point of f,

(ii) if there is $x_0\in X$ such that $(x_0,fx_0)\in E(G)$ and G is weakly connected, then for all $t\in X$, $\{f^n(t)\}_{n\in\mathbb{N}}$ converges to a unique fixed point of f.

Definition 9 ([30]). *Let (X, p) be a partial metric space and $A, B \subseteq X$ such that $A \neq \emptyset$ and $B \neq \emptyset$. A mapping $f : A \longrightarrow B$ is said to be proximal-contraction if there exists $\alpha \in [0, 1)$ such that*

$$p(u, fx) + p(fx, fy) + p(fy, v) \leq \alpha p(x, y) \ \ \forall x, y \in A \tag{20}$$

satisfying $p(u, fx) = dist(A, B)$ and $p(v, fy) = dist(A, B)$ for some $u, v \in A$.

Definition 10. *Let A_1 and A_2 be two nonempty subsets of (X, p). A mapping $f : A_1 \longrightarrow A_2$ is called to be G_{prox}-contraction if for all $x_1, x_2 \in A_1$, with $(x_1, x_2) \in E(G)$:*

(i) $p(u, fx_1) + p(fx_1, fx_2) + p(fx_2, v) \leq \alpha p(x_1, x_2)$ *for some $u, v \in A_1$ and $\alpha \in [0, 1)$,*

(ii) $$\left.\begin{array}{ll} p(u, fx_1) & = p(A_1, A_2) \\ p(v, fx_2) & = p(A_1, A_2) \end{array}\right\} \Longrightarrow (u, v) \in E(G), \ \ \forall u, v \in A_1.$$

Theorem 2. *Let (X, p) be a complete partial metric space, and let A and B be two nonempty closed subsets of (X, p) such that (A, B) has the P-property. Let $f : A \longrightarrow B$ be a continuous G_{prox}-contraction such that $f(A_0) \subseteq B_0$. Assume that x_0 and x_1 exist in A_0 such that there is a path in A_0 between them and $p(x_1, fx_0) = p(A, B)$. Then, f has a unique best proximity point.*

Proof. It is enough to prove that the restriction $f_{|A_0}$ satisfies the conditions of Theorem 1. Let us start by proving that the map $f : A_0 \longrightarrow B_0$ is a (A_0, B_0) G-contraction mapping into B_0. Consider $t, s \in A_0$. From the definition of A_0 and B_0 and knowing that $f(A_0) \subseteq B_0$, $t_0, s_0 \in A_0$ exists such that $p(t_0, ft) = p(A, B)$ and $p(s_0, fs) = p(A, B)$. Then, using the P-property of (A, B), we obtain $p(t_0, s_0) = p(ft, fs)$. Hence, by the triangular inequality we obtain

$$\begin{array}{rcl} p(ft, fs) & = & p(t_0, s_0) \\ & \leq & p(t_0, ft) + p(ft, fs) + p(fs, s_0) \\ & - & p(ft, ft) - p(fs, fs) \\ & \leq & p(t_0, ft) + p(ft, fs) + p(fs, s_0) \\ & \leq & \alpha p(t, s). \end{array}$$

Therefore, $f : A_0 \longrightarrow B_0$ is a G-contraction mapping. On the other hand, as $p(A_0, B_0) = p(A, B)$, then the pair (A_0, B_0) has the P-property. Hence, by Theorem 1, we obtain the uniqueness of the best proximity point of f. □

Next, we propose a new concept of contractive mappings in a partial metric space.

Definition 11. *Let (X, p) be a partial metric space; A_1, A_2 are two subsets of X and G a directed graph. A non-self mapping $f : A_1 \longrightarrow A_2$ is said to be proximally G-edge conserving if for each $t_1, t_2, u_1, u_2 \in A_1$,*

$$\left.\begin{array}{ll} (t_1, t_2) & \in E(G) \\ p(u_1, ft_1) & = p(A_1, A_2) \\ p(u_2, ft_2) & = p(A_1, A_2) \end{array}\right\} \Longrightarrow (u_1, u_2) \in E(G).$$

Definition 12. *Let (X, p) be a partial metric space; A_1, A_2 are two subsets of X and G a directed graph. A non-self mapping $f : A_1 \longrightarrow A_2$ is said to be the G_p-proximal C-contraction if there $\alpha \in [0, 1)$ exists and for each $t_1, t_2, u_1, u_2 \in A$,*

$$\left.\begin{array}{ll} (t_1, t_2) & \in E(G) \\ p(u_1, ft_1) & = p(A_1, A_2) \\ p(u_2, ft_2) & = p(A_1, A_2) \end{array}\right\} \Longrightarrow p(u_1, u_2) \leq \alpha\Big[\frac{p(t_1, u_2) + p(t_2, u_1)}{2} - 2p(A_1, A_2)\Big].$$

Theorem 3. *Let (X, p) be a complete partial metric space. Let A and B be two nonempty closed subsets of X such that $A_0 \neq \emptyset$ and let $f : A \longrightarrow B$ a mapping satisfying the following properties:*

(i) *f is proximally G-edge conserving continuous and G_p-proximal C-contraction such that $f(A_0) \subseteq B_0$,*

(ii) *$w_0, w_1 \in A_0$, $p(w_1, fw_0) = p(A, B)$ exists and $(w_0, w_1) \in E(G)$.*

Then, f has a best proximity point in A, and there exists $u \in A$ such that $p(u, fu) = p(A, B)$. Moreover, the sequence $\{w_n\}$ defined by $p(w_n, fw_{n-1}) = p(A, B)$ $\forall n \in \mathbb{N}$ converges to u.

Proof. From the property (ii), $w_0, w_1 \in A_0$ exist such that

$$p(w_1, fw_0) = p(A, B) \text{ and } (w_0, w_1) \in E(G). \tag{21}$$

Since $f(A_0) \subset B_0$, we have $fw_1 \in B_0$; then, by definition of B_0, $w_2 \in A_0$ exist such that

$$p(w_2, fw_1) = p(A, B). \tag{22}$$

By the proximal G-edge preserving of f and from (21) and (22), we obtain $(w_1, w_2) \in E(G)$. Similarly, we create the sequence $\{w_n\}$ in A_0 such that

$$p(w_n, fw_{n-1}) = p(A, B) \text{ and } (w_{n-1}, w_n) \in E(G) \;\; \forall n \in \mathbb{N}. \tag{23}$$

Let us establish that the sequence $\{w_n\}$ is Cauchy. We notice that if $n_0 \in \mathbb{N}$, exists such that $w_{n_0} = w_{n_0+1}$, then from (23) w_{n_0} is a best proximity point of f. Let us suppose that $w_{n-1} \neq w_n$ $\forall n \in \mathbb{N}$. Since f is G-proximal C-contraction, we have for each $n \in \mathbb{N}$

$$\left.\begin{array}{ll} (w_{n-1}, w_n) & \in E(G) \\ p(w_n, fw_{n-1}) & = p(A, B) \\ p(w_{n+1}, fw_n) & = p(A, B) \end{array}\right\} \Longrightarrow$$

$$\begin{aligned} p(w_n, w_{n+1}) &\leq \alpha[\frac{p(w_{n-1}, w_{n+1}) + p(w_n, w_n)}{2} - 2p(A, B)] \\ &\leq \alpha[\frac{p(w_{n-1}, w_{n+1}) + p(w_n, w_n)}{2}]. \end{aligned}$$

Using the property (p_4) of the partial metric, we obtain $p(w_{n-1}, w_{n+1}) \leq p(w_{n-1}, w_n) + p(w_n, w_{n+1}) - p(w_n, w_n)$. Therefore,

$$p(w_n, w_{n+1}) \leq \frac{\alpha}{2}[p(w_{n-1}, w_n) + p(w_n, w_{n+1})]$$

which gives

$$(1 - \frac{\alpha}{2})p(w_n, w_{n+1}) \leq \frac{\alpha}{2}p(w_{n-1}, w_n).$$

Then,

$$p(w_n, w_{n+1}) \leq \frac{\alpha/2}{(1 - \frac{\alpha}{2})}p(w_{n-1}, w_n) = kp(w_{n-1}, w_n), \tag{24}$$

where $k = \dfrac{\alpha/2}{(1 - \frac{\alpha}{2})} < 1$.

By induction, we obtain

$$p(w_n, w_{n+1}) \leq k^n p(w_0, w_1) \;\; \forall n \in \mathbb{N}. \tag{25}$$

From (25), for each $n, m \in \mathbb{N}$ with $m > n$ and by the triangular inequality we have

$$\begin{aligned} p(w_n, w_m) &\leq p(w_n, w_{n+1}) + p(w_{n+1}, w_{n+2}) + \ldots + \\ &+ p(w_{m-1}, w_m) - \sum_{i=n+1}^{m-1} p(w_i, w_i) \\ &\leq \sum_{i=n}^{m-1} p(w_i, w_{i+1}) \\ &\leq \sum_{i=n}^{m-1} k^i p(w_0, w_1) \\ &\leq \frac{k^n}{1-k} p(w_0, w_1) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Then, the sequence $\{w_n\}$ is Cauchy. Since A is closed, there exists $u_0 \in A$ such that $\lim_{n \longrightarrow \infty} w_n = u_0$. Then, by the continuity of f, fw_n converges to fu_0. Since the partial metric is continuous, we obtain

$$p(w_{n+1}, fw_n) \longrightarrow p(u, fu) \text{ as } n \longrightarrow \infty. \quad (26)$$

Using (23), we obtain that $p(u, fu) = p(A, B)$. Hence, the point u is a best proximity point of f in A.

To prove the uniqueness, consider $t, s \in A$ two of the best proximity points of the mapping f. Then, $(t, s) \in E(G)$ and $p(t, ft) = p(s, fs) = p(A, B)$. Since f is G_p-proximal C-contraction,

$$\begin{aligned} p(t, s) &\leq \frac{\alpha}{2}[p(t, s) + p(s, t)] - 2p(A, B) \\ &\leq \alpha p(t, s). \end{aligned}$$

Therefore, $p(t, s) = 0$. On the other hand, the triangular inequality of the partial metric we obtain $p(t, t) \leq p(t, s) + p(s, t) - p(s, s)$, which implies that $p(t, t) + p(s, s) = 0 \Longrightarrow p(t, t) = p(s, s) = 0$. Given that $p(t, s) = 0$, we obtain $t = s$. □

4. Conclusions

In conclusion, we want to introduce some open questions.

Question 1: Let (X, m) be a complete M-metric space, and A and B be two nonempty closed subsets of (X, m). Let $f : A \longrightarrow B$ be a nonself continuous G-contraction satisfying the assumptions of Theorem 1. Does f have a unique best proximity point?

Question 2: Let (X, p) be a partial metric space. A mapping $f : X \longrightarrow X$ is said to be an expanding if $p(fx_1, fx_2) \geq \lambda(x_1, x_2)\ \forall x_1, x_2 \in X$ where $\lambda > 1$. Does f have a unique best proximity point ?

Author Contributions: Conceptualization, A.A., N.S., K.M. and N.M.; methodology, A.A.; investigation, W.S.; writing—original draft preparation, A.A., N.S., K.M. and N.M.; supervision, K.M. and N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors A. Aloqaily, N. Mlaiki, and W. Shatanawi would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Radenović, S.; Chandok, S.; Shatanawi, W. Some cyclic fixed point results for contractive mappings. *Bull. Nat. Sci. Res.* **2016**, *6*, 2. [CrossRef]
3. Shatanawi, W.; Pitea, A. Best Proximity Point and Best Proximity Coupled Point in a Complete Metric Space with (P)-Property. *Filomat* **2015**, *29*, 63–74. [CrossRef]
4. Shatanawi, W.; Pitea, A. Some coupled fixed point theorems in quasi-partial metric spaces. *Fixed-Point Theory Appl.* **2013**, *1*, 157. [CrossRef]
5. Souayah, N.; Mrad, M. Some Fixed Point Results on Rectangular Metric-Like Spaces Endowed with a Graph. *Symmetry* **2019**, *11*, 18. [CrossRef]
6. Acar, Ö.; Aydi, H.; De la Sen, M. New Fixed Point Results via a Graph Structure. *Mathematics* **2021**, *9*, 1013. [CrossRef]
7. Acar, O.; Altun, I. Multivalued F-contractive mappings with a graph and some fixed point results. *Publ. Math.* **2016**, *88*, 305–317. [CrossRef]
8. Nazam, M.; Ameer, E.; Mursaleen, M.; Acar, O. Nonlinear Inequalities and Related Fixed Point Problems. *J. Math. Inequal.* **2021**, *15*, 941–967. [CrossRef]
9. Asadi, M.; Karapinar, E.; Salimi, P. New extension of p-metric spaces with some fixed point results on M-metric spaces. *J. Inequal. Appl.* **2014**, *2014*, 18. [CrossRef]
10. Ullah, N.; Khan, M.B.; Aloraini, N.; Treanta, S. Some New Estimates of Fixed Point Results under Multi-Valued Mappings in G-Metric Spaces with Application. *Symmetry* **2023**, *15*, 517. [CrossRef]
11. Aloqaily, A.; Sagheer, D.-E.-S.; Urooj, I.; Batul, S.; Mlaiki, N. Solving Integral Equations via Hybrid Interpolative RJ-Type Contractions in b-Metric Spaces. *Symmetry* **2023**, *15*, 465. [CrossRef]
12. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **2008**, *136*, 1359–1373. [CrossRef]
13. Zhang, Y.; Luo, Q. Global exponential stability of impulsive cellular neural networks with time-varying delays via fixed-point theory. *Adv. Differ. Equ.* **2013**, *2013*, 23. [CrossRef]
14. Zhoun, L.; Zhang, Y. Global exponential stability of a class of impulsive recurrent neural networks with proportional delays via fixed-point theory. *J. Frankl. Inst.* **2016**, *353*, 561–575.
15. Chena, G.; Gaansa, O.N.; Lunel, S.V. Fixed points and pth moment exponential stability of stochastic delayed recurrent neural networks with impulses. *Appl. Math. Lett.* **2014**, *27*, 36–42. [CrossRef]
16. Hui, L.; Jun, Z.; Cheng, L.L. Application Examples of the Network fixed-point theory for Space-air-ground Integrated Communication Network. In Proceedings of the International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT), Moscow, Russia, 18–20 October 2010.
17. Souayah, N.; Mlaiki, M.; Mrad, M. The G_m-contraction principle for mappings on M-metric spaces endowed with a graph and fixed point theorems. *IEEE Access* **2018**, *6*, 25178–25184. [CrossRef]
18. Fan, K. Extensions of two fixed point theorems of F. E. Browder. *Math. Z.* **1969**, *122*, 234–240. [CrossRef]
19. Abdeljawad, T.; Alzabut, J.O.; Mukheimer, A.; Zaidan, Y. Best Proximity Points For Cyclical Contraction Mappings with 0-Boundedly Compact Decompositions. *J. Comput. Anal. Appl.* **2013**, *15*, 678–685.
20. Abdeljawad, T.; Alzabut, J.O.; Mukheimer, A.; Zaidan, Y. Banach contraction principle for cyclical mappings on partial metric spaces. *Fixed-Point Theory Appl.* **2012**, *2012*, 154. [CrossRef]
21. Alghamdi, M.A.; Shahzad, N.; Vetro, F. Best proximity points for some classes of proximal contractions. *Abstr. Appl. Anal.* **2013**, *2013*, 713252. [CrossRef]
22. Aydi, H.; Felhi, A. On best proximity points for various α-proximal contractions on metric-like spaces. *J. Nonlinear Sci. Appl.* **2016**, *9*, 5202–5218. [CrossRef]
23. Kumam, P.; Salimi, P.; Vetro, C. Best proximity point results for modified α-proximal C-contraction mappings. *Fixed-Point Theory Appl.* **2014**, *2014*, 99. [CrossRef]
24. Shatanawi, W. Best proximity point on nonlinear contractive condition. *J. Phys. Conf. Ser.* **2013**, *435*, 012006. [CrossRef]
25. Souayah, N.; Aydi, H.; Abdeljawad, T.; Mlaiki, N. Best Proximity Point Theorems on Rectangular Metric Spaces Endowed with a Graph. *Axioms* **2019**, *8*, 17. [CrossRef]
26. Sultana, A.; Vetrivel, V. Best proximity points of contractive mappings on a metric space with a graph and applications. *Appl. Gen. Topol.* **2017**, *18*, 13–21. [CrossRef]
27. Sankar, R.V. Best proximity point theorems for non-self mappings. *Fixed-Point Theory* **2013**, *14*, 447–454.
28. Pant, R.; Shukla, R.; Nashine, H.K.; Panicker, R. Some New Fixed Point Theorems in Partial Metric Spaces with Applications. *J. Funct. Spaces* **2017**, *2017*, 1072750. [CrossRef]

29. Kirk, W.A.; Reich, S.; Veeramani, P. Proximinal retracts and best proximity pair theorems. *Numer. Funct. Anal. Optim.* **2003**, *24*, 851–862. [CrossRef]
30. Sadiq, B.S. Extensions of Banach's contraction principle. *Numer. Funct. Anal. Optim.* **2010**, *31*, 5. [CrossRef]