Financial market dynamics: essay in agent-based exploration

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Introduction

The present PhD thesis in Mathematical Finance is the result of almost four years of research, ranging from the study of individual expectations and strategies implemented in strictly theoretical contexts, such as pure exchange economies and asset pricing models, to those ones of aggregate dynamics of content spread over a social network structure in combination with behavioral data, passing through robustness and sensitivity analysis of the models themselves. The modeling framework employed for approaching social, economic and financial contexts laid out in this thesis is the one called: agent-based model. Withing the scope of my thesis, agent-based modeling is the computational investigation of economies modelled as evolving systems of interacting autonomous agents. It is thus a specialisation in economics of the basic paradigm of complex adaptive systems (Anderson, 1972; Gallegati and Kirman, 2012; Grazzini, 2012; Holland, 1992; Tesfatsion and Judd, 2006).

The novelty in agent-based modeling of economies lies on harnessing very powerful computational means, in particular object-oriented programming. Four key features let the extension of self-organization and economic evolution topics in literature by ABM investigators. Firstly, ABMs allow to computationally set up socio-economic environments populated by heterogeneous agents, interacting one another and with their environment, basing on internal behaviors, social norms and data learnt from the past observations. In this way, agents observe a comprehensive internal cognitive frame which makes economic agents much more *realistic* and takes into account actual ways of forming expectations and performing learning. This is in opposition with the conventional way agents have been meant to behave in economic models. In second place, a wide variety of agent behaviour and interactions can be allowed for in such economic settings, with predatory and collaborative relationships alongside price and quantity connections. Agents constantly adjust their actions in reaction to interactions both one another or with the environment in an effort to fulfill their needs and desires. In other words, the rules of behaviour are constrained by the state, and agents co-adjust their behaviour in an entangling interaction network. Economic systems can thus show up self-organisation. Thirdly, agents can be involved in open-ended testing with adaptive behavior rules changing over time; in other words, economic agents co-evolve. The latter is allowed by evolutionary process with can be thought as acting directly on agent behavioral features rather than figuring laws of motion governing the dynamic of the agent population. Finally, economic environments can be observed along a time evolution; such opportunity makes agent-based model pretty realistic. By setting boundary conditions and model parameters, the dynamic of an economic model evolves by itself without any later exogenous interventions, unless explicitly wanted (e.g. policy interventions, structural breaking, changing due to uncertainty, etc). In order to recap, agent-based modeling in socio-economic environments combines instruments and concepts from cognitive science, evolutionary economics, computer science, applied mathematics and dynamical system physics, in a way that may brings to very relevant developments. In first instance, the construct foundation of economic theories in the beliefs and interactions of independent agents; secondly, the verification, fine-tuning, and

extension of those theories by means of accurate computational exercises, statistical evaluation of results, and suitable pairwise comparisons with analytical research, econometric analysis, field trials, and laboratory investigations of human subjects; finally, the development and verification of socioeconomic conceptually embedded theories that are consistent with data and theory from many distinct social science fields thought as separated, in principle (Tesfatsion, 2001).

The current PhD thesis mostly relates to financial context. In the middle and late 20th century, finance experienced a major transformation. The emergence of the efficient market hypothesis efficient markets, the capital pricing model and the Black and Scholes option pricing formula brought to a new and "scientific" basis. This new realm of finance was founded on the assumption that asset markets were strong computational motors, capable of aggregating and processing the traders' beliefs and demands, allowing the full range of correctly processed information actually available to be captured in prices (i.e. representative agent endowed of rational expectation). It is worth noting that these bases came with a very significant computational scale (LeBaron, 2006). In agent-based models the computational resource is again at the centre of a shift in thinking about financial markets. This time it is supporting the pursuit of a way of looking at the world in where agents can differ in many directions, not only in their information, but in their capacity to elaborate information, in their approaches to risk, and in many other aspects. Agent-based models in finance consider financial markets as interacting communities of learning, boundedly rational agents. In such contexts, analytical derivations are most of the times impossible; computation simulations are then necessary. In ABM, dynamic heterogeneity is critical. This heterogeneity is a distribution of agents, or wealth, across a set of fixed or changing strategies. Financial markets are highly interesting new applications for agent-based methods for a number of reasons. First of all, the fundamental challenges in finance on market efficiency and rationality are still unanswered. Secondly, financial time series include many intriguing complexities which are not properly understood. In the third place, financial markets yield a richness of price and quantity data that can be investigated. And fourth, when looking at evolution, markets give a fair approximation to a rough fitness measure by means of return and wealth. Lastly, there are close linkages with significant experimental findings that in some instances work on the same time scales as time scales of real financial markets(LeBaron, 2006).

The academic finance has been discussing the issue of market efficiency for a while. The notion of market efficiency has both theoretical and empirical support that should not be disregarded (Fama, 1970; Fama and French, 1998). From a theoretical point of view, the contention is that investors with strategies that are less than rational will vanish, and if prices include any predictable elements in their historical series or linked to fundamentals, the rest of the rational traders will drop them to zero. Such a strong concept is still in force in much of the academic financial world, and can be found in articles like Rubinstein et al. (2001). As attractive as this notion is, it is worth pointing out that there has never been a properly agreed dynamic process depicting how the efficient market hypothesis is realized. The second cornerstone of efficient market theories, endorsed by most of the first empirical work on financial markets, is that markets are much more unpredictable than the financial practitioners' world indicates (Fama, 1970). Early conceptualization of efficient markets has led to a strong model representation for financial markets: the representative agent. This model gives a formal connection between asset prices and the

expectations of a single aggregate actor who can thus be related to several macroeconomic state variables.

Rationality generally address three pillars: consistency, maximisation of expected utility and, if learning is entailed, Bayesian probability updating. Savage (1972), the founder of modern Bayesian decision theory, identified two necessary conditions for these three pillars of rationality: the perfect foresight of future realizations of the world and the perfect foresight of their consequences. Savage called the latter a "small world". In a small world, all the possible future states of the world, their outcomes and the probability associated to them are know for certain. Savage himself clearly constrained his theory only to small worlds. For example, he stressed that his theory does not hold under those conditions in which the set of future realizations of the world and their outcomes is not known. In this case, Bayesian updating is unachievable since one cannot assign prior probabilities to future world states that sum up to one. In line with seminal contributions of Knight (1921) and Simon (1955), the latter conditions are referred to the term *uncertainly*; most of the situations faced in real words do not meet the assumptions of small world (i.e. risk) for expected utility maximization and Bayesian updating. Chapter 1 joins this discourse by exploring a standard security market economy where agents have to form beliefs on future realizations of states of the world relying on a set of predicting models that does not include the true one. The latter is the source of uncertainly I embed in the model. The main goal of this chapter is to assess accuracy of predictions of boundedly rational agents, rely on several learning protocols and beliefs, compared to the rational agent (i.e. Bayesian updating benchmark). The crucial and very relevant research question is the phenomenology of asset prices if no agent knows the actual distribution (i.e. *the truth*), knowing that, in case a Bayesian updating agent knows it, it would be unbeatable. My analysis is consistent with the literature on market selection and long-run asset prices(Blume and Easley, 1992, 2009; Bottazzi et al., 2018; Bottazzi and Giachini, 2019; Dindo and Massari, 2020; Sandroni, 2000) Given an economy characterized by independent and identically distributed (i.i.d.) states of the world, I find that a boundedly rational trader observing under-reacting, who gives a sufficiently high weight to the prior, drives out of the market any other trader who does not learn the truth asymptotically (i.e. the Bayesian one included). By performing numerical exercises, I provide a better characterization of the advantages of an under-reacting rule. In particular, I show that the accuracy of predictions (and, thus, the selection advantage) increases with the degree of under-reaction the agent exhibit. I generalize such an idea showing that, with i.i.d. states, an agent can make an opponent vanish by building its beliefs as a moving average of the opponent's predictions. I show that my results persist when the sequence of states of nature is driven by Markov or Polya urn processes. In this respect, extreme models, giving near full probability to different states, represent an optimal base on which an agent should root its updating when the true model is not available. The latter consideration is totally in line with the concept that behavioral protocols are properly modeled by heuristics. An heuristic is "a strategy that ignores part of the information, with the goal of making decisions more quickly, frugally and/or accurately than more complex methods" (Gigerenzer and Brighton (2009), 454).

The theoretical aspects of the notion of efficient markets have been under attack for a while. In general, the nature of an efficient market rests on two main concepts: available information is already reflected in stock prices and traders cannot earn risk-weighted excess returns (Degutis and Novickytė, 2014). Rationality of agents

is a key prerequisite for its efficiency. In this perspective, an important argument is derived by Grossman and Stiglitz (1980). In this paper, rational expectation agents are given with the option of buying an information signal about an asset. Under a perfectly efficient scenario with a cost on the signal (i.e. information), nobody would have an incentive to purchase it. Although it is accepted that not all market players act rationally, efficient markets supporters argue that the trades of irrational traders are considered *random*, they don't affect the market price and they are ruled out of the market on the long-run (Fama, 1970, 1965). As I pointed out above by mentining market selection literature, the latter is a controversial issue theoretically. Moreover, in the noise trading literature, De Long et al. (1990) present the crucial idea that rational risk-averse traders may not be able to drive the dynamics and rule out less rational approaches, since they trade in a less aggressive way as they are sensitive to risk generated by noisy agents. Chapter 2 explores these settings. To the best of my knowledge, my accurate dynamical reproduction of Grossman and Stiglitz (1980)' static settings through agent-based modeling is pretty original in literature, in particular from a theoretical perspective. Firstly, I set up a theoretical framework where efficient market hypothesis should definitely hold: all the agents know that the expected fundamental value of the risky asset evolves according to the expected dividends. Agents choose to adopt a *fundamentalist* trading strategy: they expect future asset price will converge to the fundamental one, so they trade accordingly to exploit possible imbalances. A crucial issue is to provide information a dynamical character. To this end, expected dividends, and accordingly fundamental price, play that key role; this way of modeling matches information characterization in the original model, including the idiosyncratic shock on the return. Moreover, I characterize the demand of an agent á la Grossman-Stiglitz and how informed and uninformed formally differ from each other. By performing both theoretical and numerical explorations, two different asymptotic scenarios arise: one in which the population distributes in half informed and half uninformed over time, and one in which the whole population joins uninformed strategy. This split crucially depends on the magnitude of information cost and it is controlled by a threshold which I derived analytically. In other words, an informed agent never performs better than an uninformed one and, if the cost of information is too high w.r.t. current earnings, it is driven out the market. In any case, market price follows the fundamental one. In this framework, I prove dynamically the paradox found by (Grossman and Stiglitz, 1980); i.e. if price would reflect all information available, no trader would pay the cost, as they could learn information from the price; on the contrary, if no one does, then prices would reflect no information, and it would be profitable to buy information. In line with noisy traders literature (De Long et al., 1990) and ecological rationality literature (Gigerenzer and Brighton, 2009; Gigerenzer and Gaissmaier, 2011), I bring into the model the heuristic rule which makes agents forming their expectations at each time on the previous observed return (i.e. *recency* heuristic). Their presence destabilizes the market, making market price deviating from fundamental value and price volatility dramatically increasing. Moreover, also in this case the informed agent never performs better that others and for a price of information higher than a certain price threshold (that can be analytically derived) it is ruled out of the market. Uniformed and heuristic agents survive in the market and the dynamic matches Fehr and Tyran (2005)' findings: strategies may be either complements or substitutes is the actual main driver in determining aggregate outcomes. I found that when the two strategies are complements, even a small amount of heuristic makes aggregate outcomes diverge from rational ones. On the contrary, when agents' strategies are substitutes, even a small amount of fundamental agents leads to a convergence to

rational outcomes.

As previously introduced, financial market is a good agent-based test bed. Financial data are generally copious, detailed and available on many various dimensions. Quality time series of at least sixty years are available on prices and books; historical data on lower frequencies and for particular stocks up to one hundred years are available as well. In the last forty years, very high frequency data have become accessible. Such series often track every transaction or order that enters a financial market, and sometimes incorporate identity information on traders. Thus, researchers have a very accurate overview of precisely how the market is performing and the market clearing dynamics. This level of detail on the behaviour of individuals is crucial in the interest of agent-based models building and validation. Empirical data are available that can be employed to align and calibrate agent behaviour. Since ABMs aim to explore dynamic features that cannot be studied analytically, cause-and-effect mechanisms that cannot always be properly identified, and emergent relationships that cannot be simply inferred by aggregating micro-level interactions, there is a urgent need for suitable and reliable tools to explore the model's emergent behaviour in terms of different parameter setting exploration and sensitivity analysis (Fagiolo et al., 2019; Lamperti et al., 2018; Lee et al., 2015; Roventini and Fagiolo, 2017). In particular, the main challenge in ABM parameter space exploration and calibration is the increasing number of parameters arising from the increasingly realistic dynamics of ABMs. In that respect, a particularly interesting approach has been presented by Lamperti et al. (2018); they propose a novel method to quickly learn a cheaper proxy (i.e. surrogate model) for replacing an ABM, using a restricted number of ABM valuations and approximating the non-linear relationship between inputs and outputs. Along this line, Chapter 3 originally investigates opportunities and limits of calibration, parameter range exploration and sensitivity analysis. I propose a novel, general and trustworthy approach for building a cheaper proxy for replacing an ABM, in order to quickly perform reliable sensitivity analysis. As far as I know, my methodology has no former proposals in financial ABM literature. Through an intelligent exploration of parameter ranges, I select parameter intervals over which surrogate models can be reliably trained. I show the dramatic gaining in performances that I obtain by using my proposed procedure. Then, I employ and compare those trained surrogate models with the true model for sensitivity analysis purposes and I show that my approach lets building up trustworthy proxies which can be employed for sensitivity analysis in place of the original ABM, with almost zero computational effort.

It sounds pretty paradigmatic that last century's academics believed that the more information to process was available, the more an agent (e.g. an individual, a firm, an institution, an organization, etc) would make a better decision, for instance in rational expectation's formalization, and nowadays, the enormous information availability on web platforms brings to opposite conclusions. Individuals have never been so cognitively impoverished to the extent that a "wealth of information creates a poverty of attention" (p. 41, Simon (1996)). Large web platforms like Google and Facebook act as hubs, distributors and curators (Kalogeropoulos et al., 2019; Lorenz-Spreen et al., 2020). Their algorithms are essential to navigate the immense digital landscape and enable bottom-up engagement in the production, dissemination and spreading of information. Data curation companies exploit this crucial role to secure the most valuable resource in the online marketplace: *human attention*. Using algorithms that learn people's behavioural patterns (Boerman et al., 2017; Kosinski et al.,

2013; Ruths and Pfeffer, 2014), these companies target their customers with advertising and shape the users' information and choice experiences (Lorenz-Spreen et al., 2020). Their usage is far from being transparent and undermines *user autonomy* (e.g. polarizing individual opinions, mis- and dis- information spreading). Moreover, it has been profoundly under-researched academically, in particular collective behavior effects. Chapter 4 explores this academic research gap by tackling a methodology successfully performed by private companies to increase user engagement and satisfaction about online features: A/B testing. My center of focus is to determine how and under which conditions A/B testing affects the distribution of content on the collective level, specifically on different social network structures. In order to achieve that, I leverage once again the so powerful framework of agent-based modeling, by reproducing social interaction and an individual decision-making model. The latter approach is completely new in literature and my results are particularly exiting. I find and show that A/B testing has a substantial influence on the qualitative dynamics of information dissemination on a social network. In particular, I observe that the A/B testing mechanism increases the homogeneity of information that is spread. In other words, I observe that A/B testing performed on a social network structure reduces the exploration and amplifies exploitation of successful features of early pieces of information, ignoring others. Moreover, my modeling framework promisingly embeds conjecturing policy (e.g. nudging, boosting) interventions.

Chapter 1

Market selection and learning under model misspecification

1.1 Introduction

The main goal of this Chapter[1] is to assess accuracy of predictions, compared to the rational agent (i.e. Bayesian benchmark), in a security market economy where agents have to form beliefs on future realizations of states of the world relying on a set of predicting models that does not include the true one. They rely on several learning protocols and beliefs which are, in a way, *bounded* versions of the rational agent benchmark. The assumption that agents behave are fully rational is the essential requirement of asset pricing standard models as the one of Lucas (1978). Fully rational agents are aware of all (relevant) available information about the economy and their behavior is either consistent with or comes from utility function maximization. Moreover, they are endowed with rational expectations and they can exactly compute every realization of future states of the economy. In order to model a decision maker, come by itself remarking how unrealistic the latter assumption is. As a matter of fact, a crucial and very relevant research question is the phenomenology of asset prices if no agent knows the actual distribution (i.e. *the truth*). A way to tackle this topic is the one of Blume and Easley (2009); they analyze an Arrow-security economy with complete markets and assume agents observe their "best" rational behavior conditioned to their expectations; both heterogeneous and incorrect (i.e. in the sense of not knowing the true distribution of future states). Their main finding relies on showing that asset prices on the long-run reflect beliefs of that agent whose belief was the closest to the actual distribution of asset prices. Furthermore, in the transition from *fully rationality* to *bounded rationality* (i.e. the assumption that agent behavior is not relying on rational beliefs but rather on heuristics), it is commonly thought to obtain a worsening in asset pricing, market dynamics, etc. In this regard, Gigerenzer and Brighton (2009) exactly assert the opposite, so that they proved that simple *fast and frugal* rules actually improve outcomes in complex environments.

This chapter analyzes market selection in a security market economy where agents' learning process is characterized by model misspecification. More specifically, agents have to form beliefs on future realizations of states of the world relying on a set of predicting models that does not include the true one. In this setting, a Bayesian learner would asymptotically predict as the best model in its set (Berk, 1966). According to Massari (2020), the Bayesian learner could be driven out of the market by a trader who under-reacts to information. Under-reaction amounts to

[1] Bottazzi G., Giachini D., M. Ottaviani (2021). "Market selection and learning under model misspecification", *forthcoming*.

update the weight a model has in the agent's prediction according to a convex combination of the Bayesian update with the prior and Massari (2020) points out that the dynamics of weights and predictions of an under-reacting rule matches those of relative consumption levels and risk neutral probabilities in an Arrow-Debreu security market as in Dindo and Massari (2020). In order to build up the connection between learning and market dynamics, I follow the literature on market selection (Blume and Easley, 1992; Bottazzi et al., 2018; Bottazzi and Giachini, 2019; Dindo and Massari, 2020; Sandroni, 2000). Relative entropy turns out as the correct measure of accuracy of a trader's beliefs and its infinite time average is deeply connected to asymptotic dynamics. I argue that, in an economy characterized by independent and identically distributed (i.i.d.) states of the world, an under-reacting trader who gives a sufficiently high weight to the prior drives out of the market any other trader who does not learn the truth asymptotically. By means of numerical exercises, I provide a better characterization of the advantages of an under-reacting rule. In particular, I show that the accuracy of predictions (and, thus, the selection advantage) increases with the degree of under-reaction the agent exhibit. Such a superior performance results from a combination of the good properties of the Bayesian update with the systematic smoothing of predictions operated by an under-reacting rule. I generalize such an idea showing that, with i.i.d. states, an agent can make an opponent vanish by building its beliefs as a moving average of the opponent's predictions. I show that my results persist when the sequence of states of nature is driven by Markov or Polya urn processes. In this respect, extreme models, giving near full probability to different states, represent an optimal base on which an agent should root its updating when the true model is not available. Indeed, learning over a set of extreme models coupled with smoothing beliefs by means of a combination of under-reaction and moving average increases the chances an agent has to survive the market selection struggle. The latter consideration is totally in line with the concept that behavioral protocols are properly modeled by heuristics. An heuristic is "a strategy that ignores part of the information, with the goal of making decisions more quickly, frugally and/or accurately than more complex methods" (Gigerenzer and Brighton (2009), 454).

1.1.1 Literature review

In this chapter I build up the connection between learning and market dynamics through analyzing market selection in a security market economy where agents' learning process is characterized by model misspecification. How the presence of different heterogeneous boundedly rational agents affects asset pricing and market dynamics in general has been a relevant threat in literature. In their seminal review, Fehr and Tyran (2005) collects and discuss how individual *irrationality* (in their words) affects the economic aggregate outcomes. They observe that many many economists are aware they live in a world where, obviously, individuals are not fully rational. Nonetheless, they often make use of models which analyze economic interactions of fully rational agents, in their professional life. In order to reduce the "cognitive dissonance" of this clear paradoxical situation, many economists believe that market interactions actually offset or even correct individually "anomalous behaviors". The authors report five common hypotheses which support the latter credence. Firstly, there is the assumption that *random* deviations from full rationality offset on average during aggregation of expectations. The second hypothesis states that agent learning protocols tend to the rational one in the long-run: "learning from their own mistakes" makes agents become rational over time. The third one argues

that market structure is the actual driver to rational aggregate outcomes; it relies on results obtained by Gode and Sunder (1993) on allocative efficiency of markets with zero-intelligence traders. The fourth assumption relates market selection results, in line with Alchian (1950); Friedman (1953): rational agents outperform irrational ones, make much higher profits and in the long-rung, they drive irrational agents out of the market. The last hypothesis argues that rational agents hold marginal buying or selling positions while irrational ones extreme positions, so in a very populated market, in equilibrium, only rational agents really matter. Since there exist many cases in which all these mechanisms don't work at all, Fehr and Tyran (2005) conclude that strategies may be either complements or substitutes is the actual main driver in determining aggregate outcomes. In case strategies are complements, even a small amount of individual rationality makes aggregate outcomes diverge from rational ones. On the contrary, when agents' strategies are substitutes, even a small amount of rational agents leads to a convergence to rational outcomes. The connection between individual rationality and aggregate outcomes that Fehr and Tyran (2005) illustrate is thus clear: a failure of the former is a potential damage to the latter.

The idea that agents' bounded rationality accounts for deviations from rational outcomes underlies the entire domain of behavioural economics (Camerer et al., 2011). This literature is based on the work of Tversky and Kahneman (1974), which deals with cognitive biases due to judgement heuristics. The basic claim is that widespread heuristics negatively affect decision-making by enabling agents to make non-optimal decisions. This view is partially challenged by the work of Gigerenzer and Brighton (2009); Gigerenzer and Gaissmaier (2011); Gigerenzer et al. (1999). Gigerenzer and his co-authors even argue that heuristics can actually enhance decision-making. The contrast between the contributions of Tversky and Kahneman and the work of Gigerenzer and other authors is that the latter examine individuals in a complex environment in which an optimal choice is inconceivable. Therefore, heuristics turn into simple tricks to get the best out of a setting that is uncertain in Knight (1921) terms. In addition, the authors illustrate that attempting to increase the "rationality" of a decision (e.g. by collecting more information) leads in many cases to a decrease in performance due to a variance-bias trade-off. In other words, sophisticated statistical models have poor predictive power in some situations due to overfitting. This idea is explored by Dosi et al. (2017) in the setting of a macroeconomic agent-based model in which aggregate economic performance is measured across different expectation formation rules. In a complex evolving environment, they demonstrate how agents' use of sophisticated expectation formation rules leads to a deterioration in terms of both individual performance and the economy's aggregate outcome compared to the case where they assume simple myopic expectations. The underlying idea is that any statistical model based on past observations will be biased if the data generating process is altered over time. Therefore, a simple prediction scheme, such as assuming that the next value of a variable is the same as the last observed one, can outperform sophisticated models based on long time series. They conclude that under Knightian uncertainty arising from technical change, interaction and imperfect information, recourse to simple heuristics removes the trade-off between variance and bias and enhances overall performance. A somewhat related perspective arises from the analysis of Kirman (2010a): the interaction of simple heterogeneous individuals, such as shoppers in fish markets (Gallegati et al., 2011; Härdle and Kirman, 1995; Kirman and Vriend, 2000), often leads to aggregate outcomes that behave well, while the interaction of more rational agents can lead to outcomes that are far from the rational benchmark.

The analysis presented in this chapter is consistent with the literature on market selection and long-run asset prices. Blume and Easley (2009) examine an Arrow security economy with complete markets, constant total endowments and agents maximising the expectation of their geometrically discounted utility of consumption over an infinite horizon. Given an agent-specific survival index that includes the discount factor and belief accuracy, they demonstrate that long-run prices generally reflect the beliefs of the agent with the highest survival index, i.e. the one for whom markets choose. They also find that there are cases where at least two agents survive (same maximal survival index), and this implies that prices vary between the evaluations of these traders. Dindo and Massari (2020) illustrate how such situations turn generic when agents adjust their beliefs dogmatically with equilibrium prices. Indeed, this kind of belief correction enhances agents' accuracy and equilibrium prices become more accurate than those of the most precise agent. They further provide evidence that this accuracy increases with the degree of adjustment. The authors therefore conclude that market efficiency is a self-fulfilling prophecy: when market participants believe that prices provide valuable information, they in fact become informative. Massari (2017), on the other hand, investigates selection and price formation in large economies (i.e. economies populated by a continuum of traders) and concludes that efficient prices emerge even when accurate traders disappear due to selection forces at work in the market.

I follow the literature on market selection (Blume and Easley, 1992; Bottazzi et al., 2018; Bottazzi and Giachini, 2019; Dindo and Massari, 2020; Sandroni, 2000). Relative entropy turns out as the correct measure of accuracy of a trader's beliefs and its infinite time average is deeply connected to asymptotic dynamics.

1.2 The Model

Consider an Arrow-Debreu economy with infinite horizon and discrete time (indexed by $t = 0, 1, \ldots$). There is a homogeneous consumption good and markets are complete. Call $s_t \in \{1, 2, \ldots, S\}$ the state realized at time $t > 0$. I indicate with $\sigma = (s_1, s_2, \ldots, s_t, \ldots)$ a path and with $\sigma^t = (s_1, s_2, \ldots, s_t)$ a partial history until time t. The set of all the possible paths is Σ while Σ^t indicates the set of all partial histories until time t. Let $\mathcal{C}(\sigma^t) = \{\sigma \in \Sigma | \sigma = (\sigma^t, \ldots)\}$ be the cylinder with base σ^t, $\mathcal{F}_t$ is the σ-algebra generated by the cylinders $\mathcal{C}(\sigma^t)$. Then, by construction, $(\mathcal{F}_t)_{t=0}^{\infty}$ is a filtration and I indicate with $\mathcal{F}$ the σ-algebra generated by the union of filtrations. I indicate with p the true probability measure on $(\Sigma, \mathcal{F})$, such that $(\Sigma, \mathcal{F}, p)$ is a well-defined probability space. I assume that p follows a discrete-time Markov chain with transition matrix P, that is $p(s_{t+1}|\sigma^t) = P_{s_t, s_{t+1}}$ $\forall t, \sigma$ and $p(s|\sigma^0) = p_{s,0}$ with $p_{s,0} > 0$ $\forall s \in \{1, \ldots, S\}$. Expectation is denoted with E and, when there is no subscript or superscript, it is computed with respect to p.

The economy is populated by N agents indexed by $i = 1, 2, \ldots, N$. Every agent i is endowed with a stream of consumption good for any path σ, $(e_{i,t}(\sigma))_{t=0}^{\infty}$, and has subjective beliefs on the realizations of the states of the world. Basically any agent has a subjective measure p_i on $(\Sigma, \mathcal{F})$.

I indicate with $c_{i,t}(\sigma)$ the consumption of agent i at time t along path σ and her consumption levels are chosen such as to solve

$$\begin{aligned} &\max_{\{c_{i,t}(\sigma),\,\forall t,\sigma\}} \sum_{t=0}^{\infty} \sum_{\sigma^t \in \Sigma^t} \beta_i^t p_i(\sigma^t) u_i\left(c_{i,t}(\sigma)\right) \\ &\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{\sigma^t \in \Sigma^t} q(\sigma^t)\left(e_{i,t}(\sigma) - c_{i,t}(\sigma)\right) \geq 0\,, \end{aligned} \tag{1.1}$$

where $\beta_i \in (0,1)$ is agent i's discount factor, u_i is the instantaneous utility of consumption of agent i, and $q(\sigma^t)$ is the price of the Arrow-Debreu security paying 1 if partial history σ^t is realized. I shall perform my analysis assuming the competitive equilibrium: consumption plans maximize (1.1) for any i and markets clear in every period. Thus, defining the aggregate endowment at time t along sequence σ as $e_t(\sigma) = \sum_{i=1}^{N} e_{i,t}(\sigma)$, I assume that $\forall t, \sigma$ it is

$$\sum_{i=1}^{N} c_{i,t}(\sigma) = e_t(\sigma)\,. \tag{1.2}$$

Following Peleg and Yaari (1970), to ensure that a competitive equilibrium exists I have to assume the following:

i) $\forall i \in \{1,2,\ldots,N\}$, u_i is continuously differentiable, increasing, strictly concave, and satisfies the Inada condition at 0;

ii) $\forall i \in \{1,2,\ldots,N\}$ it is $p(\sigma^t) > 0 \rightarrow p^i(\sigma^t)$ and $\exists \epsilon > 0$ such that $\forall i, t, \sigma$ it is $p^i(s_t|\sigma^t) > \epsilon$;.

iii) $\forall t, \sigma$, the aggregate endowment $e_t(\sigma)$ is uniformly bounded from above and away from zero.

1.2.1 Consumption asymptotic behavior

My main goal is to evaluate the selection dynamics taking place in competitive markets under different learning protocols. To do that I need to introduce some important definitions and results concerning the asymptotic behavior of the system in connection to evolutionary dynamics. Let us start clarify the meaning of *vanishing*, *surviving*, and *dominating*.

Definition 1.2.1. An agent i:

- *vanishes* on a path σ if and only if $\lim_{t\to\infty} c_{i,t}(\sigma) = 0$;
- *survives* on a path σ if and only if $\lim_{t\to\infty} \sup_t c_{i,t}(\sigma) > 0$;
- *dominates* on a path σ if and only if $\lim_{t\to\infty} c_{i,t}(\sigma) = \lim_{t\to\infty} e_t(\sigma)$.

I say that trader i *vanishes*, *survives*, *dominates*, if the previous limits hold respectively $p-$almost surely.

Next, I have to introduce relative entropy and its infinite time average. Indeed, following the literature on market selection (Blume and Easley, 1992; Bottazzi et al., 2018; Bottazzi and Giachini, 2019; Dindo and Massari, 2020; Sandroni, 2000), relative entropy turns out as the correct measure of accuracy of a trader's beliefs and its infinite time average is deeply connected to asymptotic dynamics. I also clarify what I mean when I compare agents in terms of accuracy.

Definition 1.2.2. I indicate the relative entropy of agent i's beliefs with respect to the truth p given partial history σ^t as

$$D_p(p_i, \sigma^t) = \sum_{s=1}^{S} p(s|\sigma^t) \log \frac{p(s|\sigma^t)}{p_i(s|\sigma^t)},$$

while I indicate its infinite time average on a path σ as

$$\overline{D}_p(p_i, \sigma) = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} D_p(p_i, \sigma^\tau).$$

I say that agent j is more accurate than agent i on a path σ if $\overline{D}_p(p_j, \sigma) < \overline{D}_p(p_i, \sigma)$ while I say that agent j is more accurate than agent i if the inequality holds p-almost surely. I say that agent j is as accurate as agent i on a path σ if $\overline{D}_p(p_j, \sigma) = \overline{D}_p(p_i, \sigma)$ while I say that agent j is as accurate as agent i if the equality holds p-almost surely.

I report in the following Proposition a fundamental result proved by Sandroni (2000) that shall guide my analysis.

Proposition 1.2.1 (Sandroni (2000))**.** *An agent i vanishes if there exists an agent j such that, p-almost surely, it is*

$$\log \beta_j - \overline{D}_p(p_j, \sigma) > \log \beta_i - \overline{D}_p(p_i, \sigma).$$

According to Proposition 1.2.1, asymptotic outcomes in terms of selection depend upon time preferences and accuracy. Indeed, an agent who is more patient and more accurate than another drives the latter out of the market. In what follows, I will often neglect time preferences (e.g. assuming discount factor homogeneity) and focus on devising the average relative entropy of agents' beliefs under some specific assumptions. In particular, I will focus on two different, whereas connected, definitions of learning with under-reaction.

1.3 Learning protocols

As mentioned above, each agent i has a subjective measure p_i on $(\Sigma, \mathcal{F})$. These subjective measures derive from a learning process and I assume that agents believe that the true probability measure is i.i.d. and they can rely on K i.i.d. models $\pi_1, \ldots, \pi_K$. An i.i.d. model k is simply a probability measures on $(\Sigma, \mathcal{F})$ with $\pi_k(s_t|\sigma^{t-1}) = \pi_k(s_t) > 0 \ \forall k, t, \sigma$. The spirit of these assumptions is that the model misspecification I want to address is not limited to not having the true model in the set of models agents over which agents are learning, but it also must be intended as the true model being more complex than those the agents can rely upon.

The baseline learning protocol I consider is *Bayesian learning*. The conditional beliefs of an agent i who learns in a bayesian way the true model is

$$p_i(s_t|\sigma^{t-1}) = \sum_{k=1}^{K} \pi_k(s_t) w_{i,k}(\sigma^{t-1}) \quad \forall t, \sigma, \tag{1.3}$$

with $w_{i,k}(\sigma^{t-1})$ denoting the weight agent i attaches to model k after having observed the partial history σ^{t-1}. Those weights evolve according to Bayes rule, that is

$$w_{i,k}(\sigma^t) = \frac{\pi_k(s_t)w_{i,k}(\sigma^{t-1})}{p_i(s_t|\sigma^{t-1})} \quad \forall k, t, \sigma\,. \tag{1.4}$$

Indeed, the weight $w_{i,k}(\sigma^t)$ can be considered the probability agent i attaches to the event "model k is the true one" conditional upon the observation of partial history σ^t. Notice that Eq. (1.3) coupled with $\pi_k(s) > 0\ \forall k, s$ ensures that assumption $ii)$ is satisfied.

The second learning protocol I consider consists in a modification of Eq. (1.4) according to the notion of under-reaction proposed in Epstein et al. (2010) and Massari (2020). Thus, an agent i that learns with under-reaction builds its conditional beliefs as in Eq. (1.3) while the weights evolve according to the rule

$$w_{i,k}(\sigma^t) = \lambda_i w_{i,k}(\sigma^{t-1}) + (1-\lambda_i)\frac{\pi_k(s_t|\sigma^{t-1})w_{i,k}(\sigma^{t-1})}{p_i(s_t|\sigma^{t-1})} \quad \forall k, t, \sigma\,, \tag{1.5}$$

with $\lambda_i \in [0, 1)$. Notice that setting $\lambda_i = 0$ I recover bayesian learning. Thus, such a learning protocol can be considered a form of "moderate" Baysian learning where the probability attached to the event "model k is the true one" in obtained taking a convex combination of Bayes rule with the prior probability. This *underreaction* can be proven to be a robust learning strategy against model misspecification since it may outperform bayesian learning (Massari, 2020). Moreover, equations (1.3) and (1.5) constitute the *Soft-Bayes* algorithm of Orseau et al. (2017), match the dynamics of prices and wealth in the prediction market model of Bottazzi and Giachini (2019), and describe the risk neutral probabilities and consumption shares in the pure exchange economy model analyzed by Dindo and Massari (2020). Finally, as in the case of Bayesian learning, Eq. (1.3) and $\pi_k(s) > 0\ \forall k, s$ let assumption $ii)$ be respected.

The idea of under-reacting in the sense of moderating a learning protocol can be also obtained by means of a moving average. That is, assume agent i can observe $\forall t, \sigma$ the conditional probabilities $p^*(s_t|\sigma^{t-1})$ generated by a given learning protocol respecting assumption $ii)$. Then, provided a finite natural number M_i, it builds its beliefs according to

$$p_i(s_t|\sigma^{t-1}) = \begin{cases} p^*(s_t|\sigma^{t-1}) & \text{if } t < M_i\,, \\ M_i^{-1}\sum_{m=1}^{M_i} p^*(s_t|\sigma^{t-m}) & \text{if } t \geq M_i\,. \end{cases} \tag{1.6}$$

This procedure can be considered a form of under-reaction since, for $t > M_i$, it moderates the probabilistic prediction the learning model p^* produces averaging it with its previous predictions. However, in what follows, in order to do not generate confusion, I shall refer to under-reacting by applying a moving average as "smoothing", while I will keep the term "under-reaction" to refer to the learning protocol defined by Eq. (1.3)-(1.5). Also in this case, the moving average procedure over probabilistic predictions respecting assumption $ii)$ generates beliefs that respect $ii)$.

Finally, I propose a limited memory version of Bayesian learning. That is, assume that an agent i build its beliefs as in Eq. (1.3) but now the weight it assigns to model

k after a partial history σ^t reads

$$w_{i,k}(\sigma^t) = \frac{\pi_k(s_t) w_{i,k}(\sigma^0)}{\sum_{k'=1}^{K} \pi_{k'}(s_t) w_{i,k'}(\sigma^0)} \tag{1.7}$$

such that

$$p_i(s_t|\sigma^{t-1}) = p_i(s_t|s_{t-1}) = \sum_{k=1}^{K} \pi_k(s_t) \frac{\pi_k(s_{t-1}) w_{i,k}(\sigma^0)}{\sum_{k'=1}^{K} \pi_{k'}(s_{t-1}) w_{i,k'}(\sigma^0)} \,. \tag{1.8}$$

Basically agent i in any period t is forgetting all the sequence of states occurred until $t-1$ (excluded) and restarts its Bayesian learning procedure simply considering the previous state and the initial distribution of weights, that is, the initial prior $w_{i,1}(\sigma^0), w_{i,2}(\sigma^0), \ldots, w_{i,K}(\sigma^0)$. This can be considered a simplified and extreme form of *over-reaction*. Indeed, the agent does not only give higher weight to the last event (as it would be in the definition provided by Epstein et al., 2010), but it arrives to completely forget realizations older than one period ago.

1.4 Learning accuracy and market selection outcomes

I start reporting two important results – one for Bayesian learning and the other for under-reaction – that shall be fundamental to develop my analysis. They are mainly adaptations to the framework at hand of the results of Berk (1966) and Massari (2020). The first one states that, provided the existence of unique "best" model, Bayesian learning lets an agent select the most accurate model in its set.

Proposition 1.4.1. *Assume that $\overline{D}_p(\pi_n, \sigma)$ exists finite $\forall n \in \{1, 2, \ldots, K\}$ p-almost surely and suppose that there exists a k such that $\overline{D}_p(\pi_k, \sigma) < \overline{D}_p(\pi_j, \sigma)$ p-almost surely $\forall j \neq k$. Then, for a Bayesian agent i it is p-almost surely $\lim_{t\to\infty} w_{i,k}(\sigma^t) = 1$, $\lim_{t\to\infty} p_i(s|\sigma^t) = \pi_k(s)$ $\forall s$, and $\overline{D}_p(p_i, \sigma) = \overline{D}_p(\pi_k, \sigma)$.*

Proof. Consider the quantity

$$\frac{1}{t} \log \frac{w_{i,k}(\sigma^t)}{w_{i,j}(\sigma^t)} = \frac{1}{t} \log \frac{\pi_k(\sigma^t)}{\pi_j(\sigma^t)} + \frac{1}{t} \log \frac{w_{i,k}(\sigma^0)}{w_{i,j}(\sigma^0)} \,,$$

taking the limit for $t \to \infty$ and invoking the Strong Law of Large Numbers for Martingale Differences, one obtains p-a.s.

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{w_{i,k}(\sigma^t)}{w_{i,j}(\sigma^t)} = \lim_{t\to\infty} \frac{1}{t} \log \frac{\pi_k(\sigma^t)}{\pi_j(\sigma^t)} = \overline{D}_p(\pi_j, \sigma) - \overline{D}_p(\pi_k, \sigma) > 0 \,.$$

This implies $\lim_{t\to\infty} w_{i,j}(\sigma^t) = 0$ p-a.s. $\forall j \neq k$. Hence, it is $\lim_{t\to\infty} w_{i,k}(\sigma^t) = 1$ p-a.s., which in turn implies, p-a.s., $\lim_{t\to\infty} p_i(s|\sigma^t) = \pi_k(s)$ $\forall s$, and $\overline{D}_p(p_i, \sigma) = \overline{D}_p(\pi_k, \sigma)$. □

Thus, a Bayesian agent is as accurate as the best i.i.d. model in its set. As showed by Massari (2020), an under-reacting agent is at least as accurate as a Bayesian almost surely with respect to the *empirical distribution of states*. The empirical distribution of states correspond to the true measure when the latter one is i.i.d.. In the other cases, the true probability measure cannot does not correspond with the one derived from the empirical distribution. Thus, stating that a result holds almost surely with respect to the empirical distribution is not generically equivalent

to having that the same result holds almost surely with respect to the true measure. However, the formal arguments Massari (2020) proposes can be easily generalized. Indeed, provided the existence of average relative entropies (i.e. the existence of the limits involved in their definition), under-reaction is never worse than Bayesian learning in terms of accuracy also for more general processes, like the Markovian one I shall consider.[2] To formally show that, I define the *mixed model* k $\rho_{i,k}(s_{t+1}|\sigma^t) = \lambda_i p_i(s_{t+1}|\sigma^t) + (1-\lambda_i)\pi_k(s_{t+1})$, such that Eq. (1.5) can be rewritten as $w_{i,k}(\sigma^t) = w_{i,k}(\sigma^{t-1})\rho_{i,k}(s_t|\sigma^{t-1})/p_i(s_t|\sigma^{t-1})$. The following Proposition shows that any mixed model is at least as accurate as the corresponding model and an under-reacting agent is at least as accurate as the the best mixed model. Thus, an under-reacting agent cannot be less accurate than a Bayesian.

Proposition 1.4.2. *Consider an under-reacting agent i and assume that $\overline{D}_p(p_i,\sigma)$, $\overline{D}_p(\rho_{i,k},\sigma)$, and $\overline{D}_p(\pi_k,\sigma)$ exist finite $\forall k \in \{1,2,\dots,K\}$ p-almost surely. Then, p-almost surely, it is*

$$\overline{D}_p(p_i,\sigma) \leq \overline{D}_p(\rho_{i,k},\sigma) \leq \overline{D}_p(\pi_k,\sigma) \quad \forall k \in \{1,2,\dots,K\}\,.$$

Proof. Let us start noticing that, $\forall t,\sigma$, it is

$$\sum_{k=1}^{K} \rho_{i,k}(s_{t+1}|\sigma^t) w_{i,k}(\sigma^t) = \lambda_i p_i(s_{t+1}|\sigma^t) + \sum_{k=1}^{K} \pi_k(s_{t+1}) w_{i,k}(\sigma^t) = p_i(s_{t+1}|\sigma^t)\,.$$

Thus, considering $p_i(\sigma^t)$ and iteratively substituting with (1.5), one obtains

$$p_i(\sigma^t) = p_i(\sigma^{t-1}) \sum_{k=1}^{K} \rho_{i,k}(s_t|\sigma^{t-1}) w_{i,k}(\sigma^{t-1}) = \cdots = \sum_{k=1}^{K} \rho_{i,k}(\sigma^t) w_{i,k}(\sigma^0)\,.$$

Hence, $\forall k,t,\sigma$, it is $p_i(\sigma^t) \geq \rho_{i,k}(\sigma^t) w_{i,k}(\sigma^0)$, which implies $\log p_i(\sigma^t) \geq \log \rho_{i,k}(\sigma^t) + \log w_{i,k}(\sigma^0)$. Multiplying both sides by $1/t$ and taking the limit for $t \to \infty$, it is

$$\lim_{t\to\infty} \frac{1}{t} \log p_i(\sigma^t) \geq \lim_{t\to\infty} \frac{1}{t} \log \rho_{i,k}(\sigma^t)$$

and the first inequality in the statement follows multiplying both sides by -1, adding to both sides $\lim_{t\to\infty} t^{-1} \log p(\sigma^t)$ and invoking the Strong Law of Large Numbers for Martingale Differences. Then, focus on $\rho_{i,k}(\sigma^t)$ and notice that, by Jensen's inequality, it is

$$\log \rho_{i,k}(\sigma^t) = \sum_{\tau=1}^{t} \log(\lambda_i p_i(s_\tau|\sigma^{\tau-1}) + (1-\lambda_i)\pi_k(s_\tau)) \geq \lambda_i \log p_i(\sigma^t) + (1-\lambda_i)\log \pi_k(\sigma^t)\,.$$

[2] When the true probability measure is defined by a Markov chain admitting a unique invariant distribution, such an invariant distribution is indeed the empirical distribution of states. Notice, however, that computing the average relative entropy with respect to the invariant is, in general, not equivalent to computing the average relative entropy with respect to the transition probabilities of the chain. Moreover, the implications derived computing the average relative entropy with respect to the invariant hold almost surely with respect to the i.i.d. measure defined by the invariant distribution. In the generic Markov case this amounts to selecting those sequences of states that seem i.i.d., which is a zero measure set with respect to the true (Markov) measure.

Thus, multiplying both sides by $1/t$, taking the limit for $t \to \infty$, and invoking the previous result, one obtains

$$\lim_{t\to\infty} \frac{1}{t} \log \rho_{i,k}(\sigma^t) \geq \lambda_i \lim_{t\to\infty} \frac{1}{t} \log \rho_{i,k}(\sigma^t) + (1-\lambda_i) \lim_{t\to\infty} \frac{1}{t} \log \pi_k(\sigma^t)\,.$$

Rearranging terms, dividing both sides by $1-\lambda_i$, multiplying both sides by -1, adding to both sides $\lim_{t\to\infty} t^{-1} \log p(\sigma^t)$, and invoking the Strong Law of Large Numbers for Martingale Differences, the second inequality follows. □

Notice that, as long as the limits involved exist, there are not particular requirements I have to impose on the true probability measure to let the statements hold. Hence, the Markovian assumption on the true probability is not needed for the previous two Propositions to hold. In what follows, instead, it shall be central for my analysis. I will first impose that all the rows of the transition matrix are equal, such that the i.i.d. special case is obtained. I will refer to this as the *parameter misspecification* case. Indeed, the models the agent use belong to the same class of the true one, but their parameters are, generically, not correct. Under this assumption I will prove that under-reaction is very effective when the truth belongs to the convex hull of i.i.d. models and the degree of under-reaction is high. Such an advantage can be further exploited by an agent that adopts a moving average approach using as reference the beliefs of an under-reacting agent. Then, I focus on the general Markovian case. I will refer to it as the proper *model misspecification* case, since the models agents use belong to a different (and less general) class than the truth. I show that the average accuracy of a model (and, thus, of a Bayesian) is the sum of two components: the relative entropy of the i.i.d. model with respect to the invariant distribution and the average accuracy of the invariant with respect to the transition probabilities. I proceed showing that the advantage under-reaction has when the truth is i.i.d. fades away when the true process is increasingly different from the i.i.d. one and in those cases a limited memory Bayesian can make more accurate predictions.

1.4.1 Parameter misspecification: i.i.d. states

I start assuming that states of nature follow an i.i.d. process, such that models agents use belong to the same family of the truth but have misspecificated parameters. This case can be easily recovered from my Markovian assumption for the true probability measure assuming that all the rows of the transition matrix are equal.

A1 The transition matrix P of the Markov chain defining the true probability measure p is such that $P_{s',s} = \pi(s)\ \forall s, s'$ with $\pi = (\pi(1), \dots, \pi(S)) \in \Delta_+^{S-1}$.[3]

I provide a novel result on under-reaction inspired from previous finding from the market selection literature[4].

Proposition 1.4.3. *Define* $\mathcal{H} = \{v \in \Delta^{S-1} \,|\, v = \sum_{k=1}^K \zeta_k \pi_k,\ \zeta_k > 0\ \forall k,\ \sum_{k=1}^K \zeta_k = 1\}$ *and assume* $\pi \in \mathcal{H}$. *Consider an under-reacting agent* i, *then, given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* $|\lambda_i - 1| < \delta$ *then* $\overline{D}_p(p_i, \sigma) < \varepsilon$ $\overline{p}$*-almost surely.*

[3] With Δ_+^{S-1} I refer to the interior of the $(S-1)$-simplex.

[4] See section 5 of Bottazzi and Giachini (2017) and Theorem 1 of Dindo and Massari (2020).

Proof. By assumption there exists a $\zeta \in \Delta^{K-1}$ such that $\pi = \sum_{k=1}^{K} \zeta_k \pi_k$. From Lemma A.2.1 in appendix A.2, $\forall \sigma$ it is

$$\lim_{t\to\infty} \frac{1-\lambda_i}{t} \sum_{\tau=1}^{t} \sum_{s=1}^{S} \pi(s) \left(\frac{\pi(s)}{p_i(s|\sigma^{\tau-1})} - 1 \right) - |O((1-\lambda_i)^2)| =$$
$$= \overline{D}_p(p_i, \sigma) - \sum_{k=1}^{K} \zeta_k \overline{D}_p(\rho_{i,k}, \sigma) = \sum_{k=1}^{K} \zeta_k (\overline{D}_p(p_i, \sigma) - \overline{D}_p(\rho_{i,k}, \sigma)) .$$

By Proposition 1.4.2 it is $\overline{D}_p(p_i, \sigma) \leq \overline{D}_p(\rho_{i,k}, \sigma)$ $\forall k$ p-almost surely, thus

$$\lim_{t\to\infty} \frac{1-\lambda_i}{t} \sum_{\tau=1}^{t} \sum_{s=1}^{S} \pi(s) \left(\frac{\pi(s)}{p_i(s|\sigma^{\tau-1})} - 1 \right) \leq |O((1-\lambda_i)^2)| \quad p\text{-almost surely}$$

and, since $\log(x) \leq x - 1$, one has p-almost surely

$$(1-\lambda_i)\overline{D}_p(p_i, \sigma) \leq \lim_{t\to\infty} \frac{1-\lambda_i}{t} \sum_{\tau=1}^{t} \sum_{s=1}^{S} \pi(s) \left(\frac{\pi(s)}{p_i(s|\sigma^{\tau-1})} - 1 \right) \leq |O((1-\lambda_i)^2)|$$

and the statement follows. □

Proposition 1.4.3 shows that when the truth belongs to the convex hull of the i.i.d. models, then an under-reacting agents becomes extremely accurate for a sufficiently high level of under-reaction.

Indeed, provided a market in which no one can asymptotically learn the truth – a common consequence of model misspecification – an under-reacting can survive increasing enough its degree of under-reaction. I formalize this point in the following Proposition.

Proposition 1.4.4. *Assume $\pi \in \mathcal{H}$ and suppose that there exists a number $\epsilon' > 0$ and a set of agents $I \subset \{1, 2, \ldots, N\}$ with $I = \{i \in \{1, 2, \ldots, N\} \,|\, \overline{D}_p(p_i) > \epsilon'\}$. Assume homogeneity in discount factors across agents and the presence in the economy of an under-reacting agent $j \in \{1, 2, \ldots, N\}$, with $j \notin I$, whose λ_j is sufficiently close to one, then every agent in I vanishes.*

Proof. The statement is a straightforward consequence of Proposition 1.2.1 and of Proposition 1.4.3. Indeed, consider a sequence of under-reacting agents characterized by $\lambda_{j_1}, \lambda_{j_2}, \ldots$ with $\lambda_{j_n} = 1 - 1/n$. Then, Proposition 1.4.3 implies that there exists a natural number $n_{\epsilon'}$ such that for all $n > n_{\epsilon'}$ it is $\overline{D}_p(p_{j_n}, \sigma) < \epsilon'$. Choosing $1 - 1/n_{\epsilon'} < \lambda_j < 1$ and imposing discount factor homogeneity, Proposition 1.2.1 directly delivers the statement. □

In the following, I leverage an illustrative example in order to lay out a graphic representation of the results coming from Propositions 1.4.3 and 1.4.4. Let us consider a particular case of the economy in which there exist only two possible states of the world, A and B. Without loss of generality, an equivalent description is given by an event that, at a certain time t, may either occur, then $s_t = A = 1$, or not, $s_t = B = 0$. N agents repeatedly bet on the latter binary event which every time may take place or not. The event occurrence is based on an independent Bernoulli trial s_t; at every time t, only two realizations of the world are possible: $s_t = 1$ with probability p or $s_t = 0$ with probability $1 - p$. Agents are unaware of the process success probability p. Nevertheless, they know that two models which estimate the

true probability are available. These are two misspecified models, a and b, and accordingly their estimates probabilities $\pi_a(s_t|\sigma^{t-1})$ and $\pi_b(s_t|\sigma^{t-1})$. Let us consider the case they are fixed over time,i.e. $\pi_a(1|\sigma^{t-1}) = \pi_a$ and $\pi_b(1|\sigma^{t-1}) = \pi_b$, $\forall t$ and, without loss of generality, let us set $\pi_a < \pi_b$. Since the number of models provided in the economy is reduced to $K = 2$, the weights assign to models by each agent i are now related: $w_{i,a}(\sigma^t) + w_{i,b}(\sigma^t) = 1$ $\forall t$. Then, I am allowed to set $w_{i,a}(\sigma^t) = w_i(\sigma^t)$ and $w_{i,b}(\sigma^t) = 1 - w_i(\sigma^t)$. The probability of agent i of observing state $s_t = 1$ after σ^{t-1} (Eq. 1.3) becomes:

$$p_i(1|\sigma^{t-1}) = \pi_a\, w_i(\sigma^{t-1}) + \pi_b\,(1 - w_i(\sigma^{t-1}))\,, \tag{1.9}$$

and it is, of course, always bounded by the two models: $\pi_i \in [\pi_a, \pi_b]$, $\forall t$. After the realization of s_t, the updating of $w_i(\sigma^t)$ is:

$$w_i(\sigma^t) = \lambda_i w_i(\sigma^{t-1}) + (1 - \lambda_i) \begin{cases} \dfrac{\pi_a\, w_i(\sigma^{t-1})}{p_i(1|\sigma^{t-1})}\,, & \text{if } s_t = 1 \\ \dfrac{(1-\pi_a)\, w_i(\sigma^{t-1})}{(1 - p_i(1|\sigma^{t-1}))}\,, & \text{if } s_t = 0\,. \end{cases} \tag{1.10}$$

Substituting $w_i(\sigma^t)$ in (1.9) the dynamics of $p_i(1|\sigma^t)$ can be derived (see also Bottazzi and Giachini, 2017, 2019), obtaining

$$p_i(1|\sigma^t) = p_i(1|\sigma^{t-1}) + (1-\lambda_i)\,\frac{(\pi_a - p_i(1|\sigma^{t-1}))\,(p_i(1|\sigma^{t-1}) - \pi_b)(s_t - p_i(1|\sigma^{t-1}))}{p_i(1|\sigma^{t-1})(1 - p_i(1|\sigma^{t-1}))}\,. \tag{1.11}$$

According to Prop.1.2.1 and in line with my illustrative example where two misspecified models π_a and π_b are given, I notice that asymptotically two cases are possible: either the agent prediction converges to the best model or a more accurate combination is adopted. Given the settings above and in line with both Propositions 1.4.3 and 1.4.4, I show in Fig.1.1 to what extend a sub-Bayesian learning protocol overperforms a Bayesian one in case of λ_i greater than zero, otherwise its best model identifies the Bayesian one. Furthermore, in a two-model sub-bayesian learning system, I can recover sufficient and necessary conditions adapting the results in Proposition 3.1 of Bottazzi and Giachini (2019). In particular, there exists a $\underline{\lambda}_i$ such that for $\lambda_i < \underline{\lambda}_i$ I have the convergence to the best model and for $\lambda_i > \underline{\lambda}_i$ I have the emergence of the more accurate mixture. The latter is easily shown by Fig.1.1; the plot suggests the existence of $\underline{\lambda}_i$ with respect to p, π_a and π_b values. The conditions, however, are implicit in λ_i.

Let us consider the setting $\pi_a < p < \pi_b$, and let us call $\tilde{\pi}_a < p$ the value which satisfies the equation: $\overline{D}_p(\tilde{\pi}_a) = \overline{D}_p(\pi_b)$; the inequality $\overline{D}_p(\pi_a) < \overline{D}_p(\pi_b)$ always holds for $\pi_a < \tilde{\pi}_a$. By following the analysis shown in appendix A.3, the functional form derived is then:

$$\underline{\lambda}_i(\pi_a) = \frac{\tilde{\pi}_a - \pi_a}{\pi_b - \pi_a}\,. \tag{1.12}$$

In the settings above, Eq.1.12 is represented by the dashed line in Fig.1.1; as I may notice, the theoretical model exactly matches the the transition point which separate the two regimes mentioned above: for $\lambda_i < \underline{\lambda}_i$ I have the convergence to the best Bayesian model and for $\lambda_i > \underline{\lambda}_i$ I have the emergence of the more accurate mixture where then a sub-Bayesian protocol overperforms the Bayesian one. In Fig.1.1 some illustrative trends of the sub-Bayesian learning protocol for different p's are given in

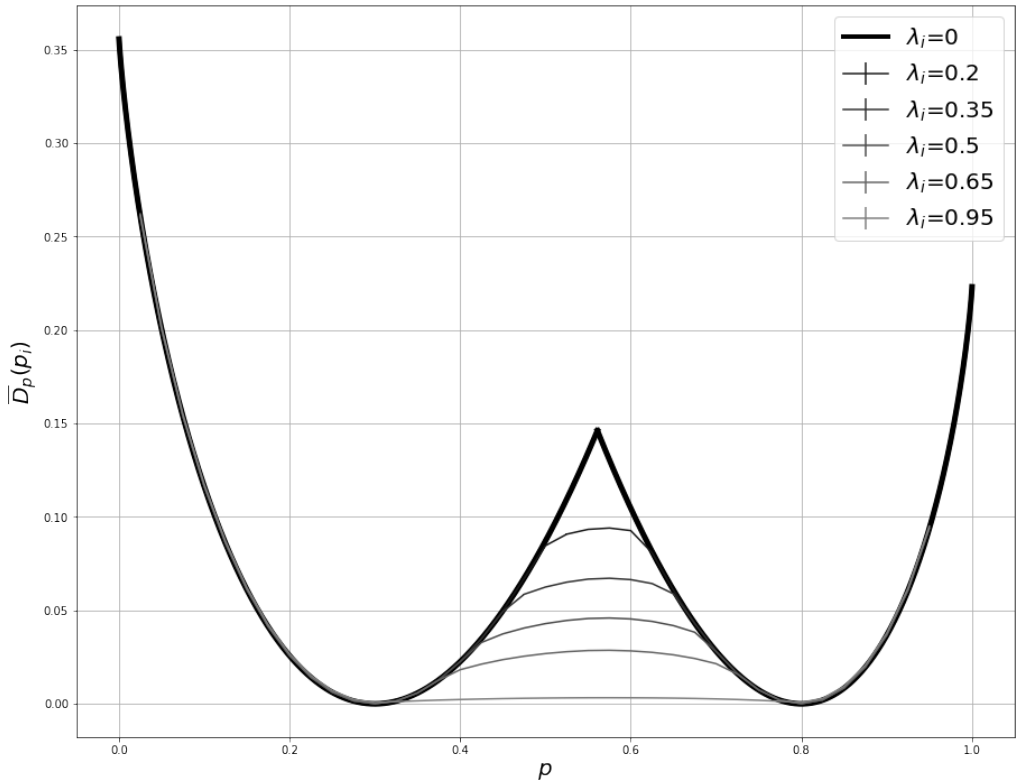


FIGURE 1.1: Relative entropy of a sub-bayesian agent probability distribution w.r.t. the true probability distribution p computed for different λ_i. The thicker black line indicates the relative entropy of a Bayesian agent; it switches between observing one misspecified model instead of the other according to which is the *closest* (i.e. in term of relative entropy distance) to the true probability distribution p. It can be exactly computed since the analytical form is available. Conversely, the thinner lines deviating from the thicker one indicate sub-Bayesian performances for different λ_i. The latter need to be simulated since analytical results don't hold in this case. Furthermore, as I may notice, the greater is λ_i, the lower is the relative entropy value and the wider is the p range where a sub-Bayesian learning performs better. The confidence bounds in the simulations are negligible since much smaller than 1% of $\overline{D}_p(p_i)$.

order to show the latter agreement.

To further explore the mechanism that makes a λ_i close to 1 cause an improvement in predicting accuracy, consider the conditional expected variation of agent i's belief. From (1.11), it reads

$$\mathrm{E}\left[p_i(1|\sigma^t) - p_i(1|\sigma^{t-1})\middle|p_i(1|\sigma^{t-1}) = p_i\right] = (1-\lambda_i)\,\frac{(\pi_a - p_i)\,(p_i - \pi_b)(p - p_i)}{p_i(1-p_i)}\,. \tag{1.13}$$

It follows that the conditional expected variation is positive if the previous belief was smaller than p and negative if it was bigger than p. Then, once $p_i(1|\sigma^t)$ arrives in the neighborhood of p, it starts to fluctuate around it, since, whenever the belief goes away from p, it tends to come back. From (1.13) one immediately notices that λ_i modulates the amplitude of fluctuations. Thus, the larger λ_i is, the closer to p the

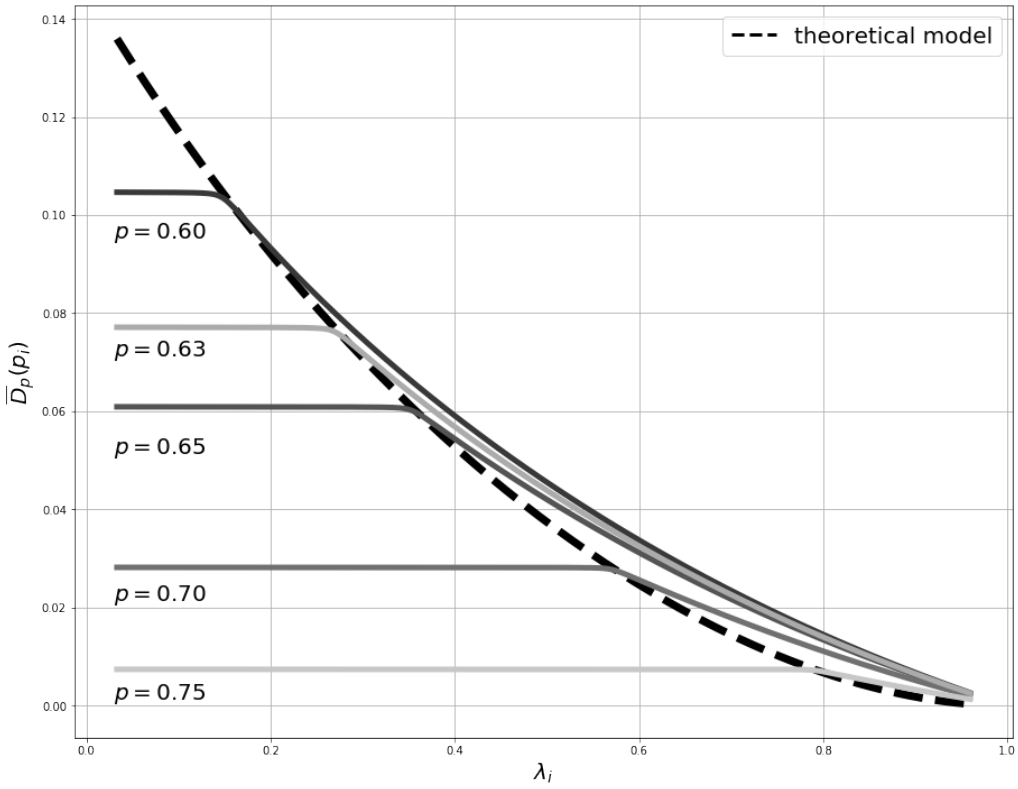


FIGURE 1.2: Relative entropy of an agent probability distribution w.r.t. its λ_i computed for different p. Each curve is characterized by two different regimes and a transition point. The thicker black line indicates the range in which observing the best misspecified model by a Bayesian agent overperforms a sub-Bayesian one. The latter takes place when satisfying the conditions to dominate in Proposition 3.1 of (Bottazzi and Giachini, 2019) and this regime can be then exactly computed. Conversely, the thinner line corresponds to the regime in which mixing the two misspecified models by a sub-Bayesian dominates and no analytical result holds; then, I may characterize it only by performing simulations. The presence of the star highlights that transition point $\underline{\lambda}_i$ analytically derivable according to (A.11); here the relative entropy of performing the best misspecified model by a Bayesian agent equates mixing the two models by a sub-Bayesian one. The confidence bounds are negligible since much smaller than 1% of $\overline{D}_p$.

belief of agent i tends to stay asymptotically.[5] Hence, it seems that damping fluctuations one may experience an improvement in accuracy. Thus, other smoothing operations on the learning updates may produce similar improvements.

Finally, let us explore this idea in the following: I show that smoothing may have an advantage in terms on selection when the learning protocol it leverages is adopted by another agent in the economy.

Indeed, the following Proposition shows that in such a case, provided the assumption $A1$, a smoothing agent cannot be less accurate than the agent whose probabilistic predictions it averages.

[5]Indeed, the side effect of a large λ_i (and the resulting small variations) is that, if the initial belief is far away from the truth, it takes many periods to arrive in a neighborhood of p. However, this is not particularly relevant for the analysis at hand, which concerns asymptotic market selection outcomes.

Proposition 1.4.5. *Consider an under-reacting agent i and assume that $\forall t, \sigma$ another agent, say j, builds its beliefs according to Eq. (1.6) with $p^* = p_i$. Then, under **A1**, p-almost surely it is*

$$\overline{D}_p(p_j, \sigma) \leq \overline{D}_p(p_i, \sigma)\,.$$

If, moreover, $\exists k' \in \{1, 2, \ldots, K\}$ such that $\overline{D}_p(\lambda_i \tilde{\pi} + (1-\lambda_i)\pi_{k'}) < \overline{D}_p(\tilde{\pi})$, with $\tilde{\pi} = \arg\min_{k\in\{1,2,\ldots,K\}} \overline{D}_p(\pi_k)$, then p-almost surely it is

$$\overline{D}_p(p_j, \sigma) < \overline{D}_p(p_i, \sigma)\,.$$

Proof. Consider the difference between the average relative entropy of agent j and of agent i, then one has p-almost surely

$$\begin{aligned}
&\overline{D}_p(p_j, \sigma) - \overline{D}_p(p_i, \sigma) = \\
&\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=M_j}^{t} \left(\sum_{s=1}^{S} \pi(s) \log \frac{\pi(s)}{M_j^{-1} \sum_{m=1}^{M_j} p_i(s|\sigma^{\tau-m})} - D_p(p_i, \sigma^\tau) \right) \leq \\
&\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=M_j}^{t} \left(M_j^{-1} \sum_{m=1}^{M_j} D_p\left(p_i, \sigma^{\tau-m}\right) - D_p(p_i, \sigma^\tau) \right) = \\
&\lim_{t\to\infty} \frac{1}{t} M_j^{-1} \sum_{m'=0}^{M_j-1} \sum_{\tau'=m'}^{M_j-1} D_p\left(p_i, \sigma^{\tau'}\right) + \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=M_j}^{t} \left(M_j^{-1} \sum_{m=1}^{M_j} D_p(p_i, \sigma^\tau) \right) - D_p(p_i, \sigma^\tau) \\
&= 0
\end{aligned} \tag{1.14}$$

and the first statement follows. For the second statement it is enough to notice that the condition implies that p_i never settles on a unique model (see Lemma 5 of Massari, 2020) and, from Eq. (1.5), one notice that settling on a unique model is the only case in which $p_i(s|\sigma^t) = p_i(s|\sigma^{t-1})\ \forall s$ and for some t, σ. Hence, the inequality in Eq. (1.14) is strict and the statement is proven. □

Proposition 1.4.5 shows that, when state of the worlds are i.i.d., if an agent j builds its beliefs taking the arithmetic average of the last M_j beliefs generated by agent i, agent j is never less accurate than agent i. This confirms the intuition I provided in advance: averaging (i.e.a simple form of smoothing) can be beneficial in terms of accuracy. The reason for that can be understood noticing how a trader j, averaging the beliefs of trader i, incorporates in its prediction for state s_{t+1} after a partial history σ^t the forecast of i for s_{t+1} after the (sub-)partial histories $\sigma^t, \sigma^{t-1}, \ldots, \sigma^{t-M_j}$. Fig.1.3 shows us a graphic demonstration of Proposition 1.4.5 through the same particular case of before; the dashed lines are those relative entropies by performing moving averages on sub-Bayesian learning protocols of Fig.1.1 with respect to themselves. Thus, averaging, and smoothing in general, allows to incorporate more information: it simultaneously evaluates the occurrence of a given state after different partial histories. Obviously, such improvement in forecasting accuracy translates in a selection advantage, especially if the beliefs it is averaging derive from under-reaction.

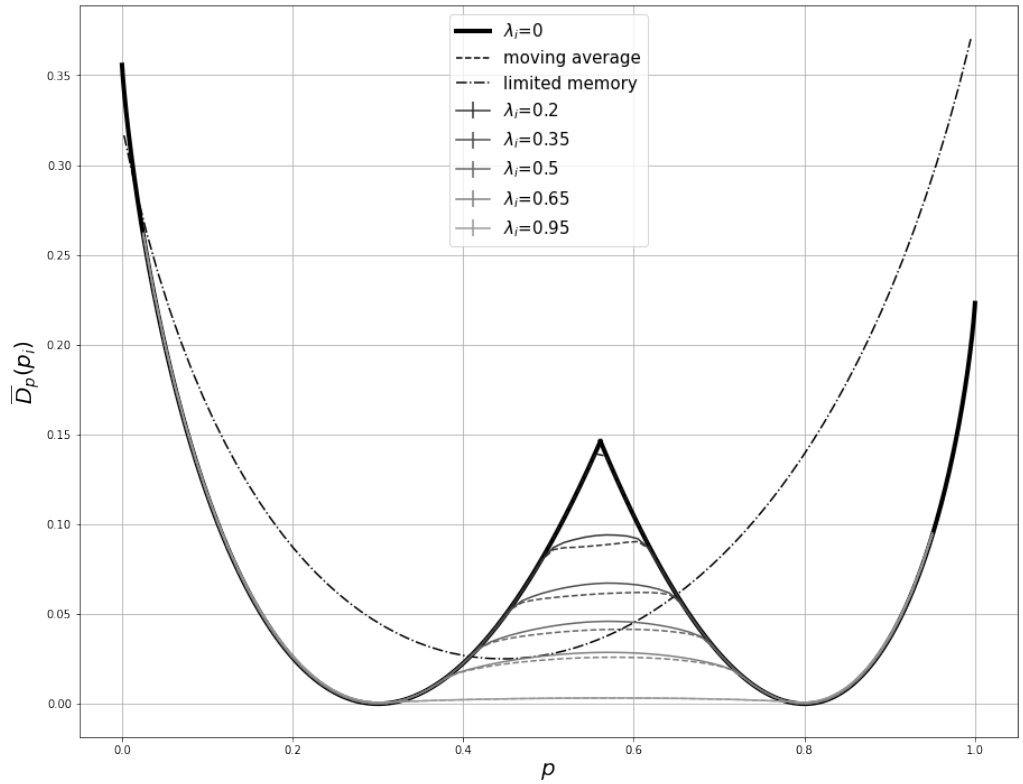


FIGURE 1.3: I report Fig.1.1 as a baseline, i.e. Relative entropy of a sub-bayesian agent probability distribution w.r.t. the true probability distribution p computed for different λ_i. Conversely, the dashed line deviating from the thinner one (i.e. sub-Bayesian) indicates the relative entropy of an agent implementing a moving average on its corresponding sub-Bayesian learning protocol, in line with Eq.(1.6). According to proposition 1.4.5, an agent computing a moving average on the sub-Bayesian learning protocol may perform better than the latter. Moreover, for average λ's, MA protocol's gains are maximized and the greater is λ (i.e. that tends to 1), the more this gain tends to vanish. Finally, the dashed dot-line shape represents the $\overline{D_p}(p_i)$ for a *limited memory* Bayesian learning protocol, according to Eq.(1.8). As shown on the plot, the performances of a Bayesian agent not embedding the available information about past realizations over time may even overperform all the other learning protocols (which in principle they would be thought "more accurate"). The latter results depend on the fixed weights assigned to the models available, in line with Eq.(1.7), the models themselves and the true probability distribution. The confidence bounds in the simulations are negligible since much smaller than 1% of $\overline{D}_p(p_i)$.

1.4.2 Model misspecification: Markov states

In this subsection I consider more general Markov processes assuming that the true transition matrix is composed by strictly positive entries. In this way assumption $ii)$ of section 1.2 is automatically satisfied for the learning protocols I consider and I can analyze the asymptotic outcomes emerging when the true process belongs to a more general class than the one the models agents use are part of.

A2 The transition matrix P of the Markov chain defining the true probability measure p is such that $P_{s',s} > 0 \; \forall s', s$.

An immediate consequence of assumption **A2** is that a unique invariant distribution $\pi = (\pi(1), \pi(2), \ldots, \pi(S))$ with $\pi(s) > 0 \ \forall s$ exists. Indeed, the strict positiveness of the entries of the transition matrix implies that the Markov chain defining the true probability measure p is irreducible with only ergodic states (i.e. aperiodic and persistent with finite mean recurrence time; see Feller, 1968, sections XV.6 and XV.7). This, in turn, delivers the existence of the unique invariant probability distribution π (see Feller, 1968, page 393). The invariant distribution emerges in the computation of models' accuracy.

Proposition 1.4.6. *Under A2, for any i.i.d. model k it is p-almost surely*

$$\overline{D}_p(\pi_k, \sigma) = \sum_{s=1}^{S} \pi(s) \log \frac{\pi(s)}{\pi_k(s)} + \sum_{s'=1}^{S} \pi(s') \sum_{s=1}^{S} P_{s',s} \log \frac{P_{s',s}}{\pi(s)} . \tag{1.15}$$

Proof. From the definition of $\overline{D}_p(\pi_k, \sigma)$ one has p-almost surely

$$\overline{D}_p(\pi_k, \sigma) = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \sum_{s=1}^{S} p(s|\sigma^\tau) \log \frac{p(s|\sigma^\tau)}{\pi_k(s)} = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \sum_{s=1}^{S} P_{s_\tau,s} \log \frac{P_{s_\tau,s}}{\pi_k(s)} =$$
$$= \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \sum_{s'=1}^{S} \mathbb{1}_{s',s_\tau} \sum_{s=1}^{S} P_{s',s} \log \frac{P_{s',s}}{\pi_k(s)} = \sum_{s'=1}^{S} \pi(s') \sum_{s=1}^{S} P_{s',s} \log \frac{P_{s',s}}{\pi_k(s)} ,$$

where $\mathbb{1}_{s',s}$ represents the indicator function ($\mathbb{1}_{s',s} = 1$ if and only if $s' = s$ and 0 otherwise) and the last equality is an application of the Strong Law of Large Numbers. The equation in the statement directly follows adding and subtracting $\sum_{s=1}^{S} \pi(s) \log \pi(s)$ and exploiting the properties of the invariant distribution, i.e. $\pi(s) = \sum_{s'=1}^{S} P_{s',s} \pi(s') \ \forall s$. □

Proposition 1.4.6 shows that the average relative entropy of an i.i.d. model in a Markov world results from the sum of two components. The fist can be considered the relative entropy of the model with respect to the "best" i.i.d. distribution (i.e. the invariant distribution of the chain), while the second represents the average relative entropy of the "best" i.i.d. model with respect to the transition probabilities. This has an immediate consequence for Bayesian learning. Indeed, coupling it with Proposition 1.4.1, one obtains that a Bayesian is increasingly inaccurate as the true Markov model is increasingly different from the i.i.d. special case. Such a loss of accuracy derived from learning over i.i.d. models when the truth is Markov can be quantified by the second term of Eq. (1.15).

Hence, when the true process driving the realizations of states of nature belongs to a more general class than the one of models over which the agents are learning, accuracy is obviously negatively affected. My analysis shows that in the Bayesian case one can be able to split the accuracy loss in two components, one identified by how inaccurate is the best model in the set with respect to the best model possible in the class of i.i.d. models and the other consisting in how inaccurate is the best possible i.i.d. model with respect to the true transition probabilities of the chain.

Next, I investigate the performance of an under-reacting agent i focusing on the simple example I have introduced in advance. Figure 1.4 shows the average relative entropy estimated over 100 independent realizations of 10000 steps each of the underlying Markov chain with for different values of $P_{1,1}$, $P_{2,1}$, and λ_i. As one can notice, increasing the degree of under-reaction provides a clear advantage in terms of accuracy when the transition probabilities can be written as a convex combination

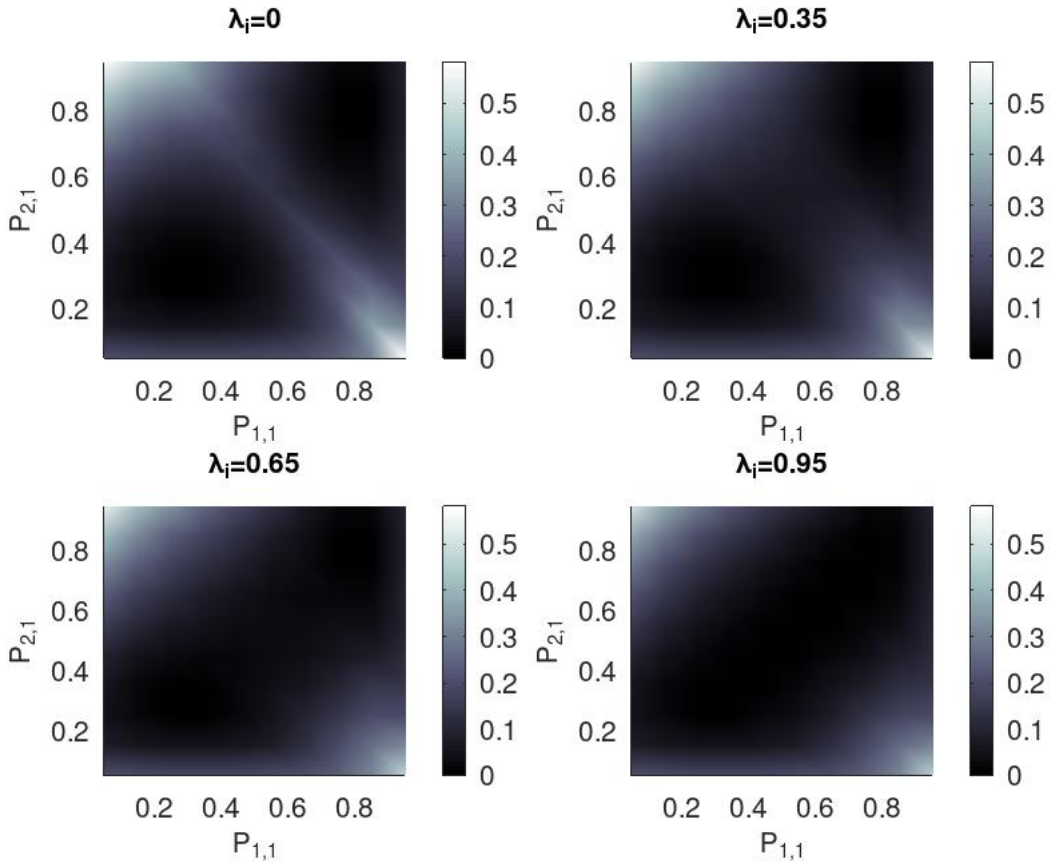


FIGURE 1.4: Average relative entropy of an under-reacting agent computed for different λ_i, $P_{1,1}$ and $P_{2,1}$.

of the two i.i.d. models the agent is using. This is especially true when $P_{1,1}$ and $P_{2,1}$ are equal: in such a case **A1** is recovered and Proposition 1.4.3 holds. Hence, by continuity of the average relative entropy, an high degree of under-reaction is effective also when $P_{1,1}$ and $P_{2,1}$ are not exactly equal but close. This advantage disappears as $P_{1,1}$ and $P_{2,1}$ become increasingly different. This clearly appears from Figure 1.5, where I plot the mean average relative entropy recorded for different λ_i as a function of $|P_{1,1} - P_{2,1}|$ with $P_{1,1}, P_{2,1} \in \{0.25, 0.35, 0.45, 0.55, 0.65, 0.75\}$. Indeed, one observes that high levels of under-reaction are still beneficial in terms of selection, but the extreme accuracy achieved when the truth belongs to the same class of models populating the set over which the agent is learning, fades away as transition probability become increasingly different.

The fact that increasing the level of under-reaction improves the accuracy and, thus, the selection advantage of an agent, suggests that also in the generic Markov case a moving average approach should be beneficial. In Figure 1.6 I repeat the exercise of Figure 1.5 for the case of a moving average agent i with $M_i = 500$ that uses as reference an under-reacting learning protocol with under-reaction degree λ_i. Comparing the two Figures one notices that applying the moving average procedure provides indeed an advantage over the reference under-reaction protocol and, as in the i.i.d. case, such advantage becomes weaker as the underlying degree of under-reaction increases. Again, this is due to the damping of fluctuations that increasing the degree of under-reaction causes: as the level of under-reaction grows the magnitude of jumps in the probabilistic prediction decreases. Thus, in the long-run an under-reacting agent with high degree of under-reaction narrowly fluctuates around

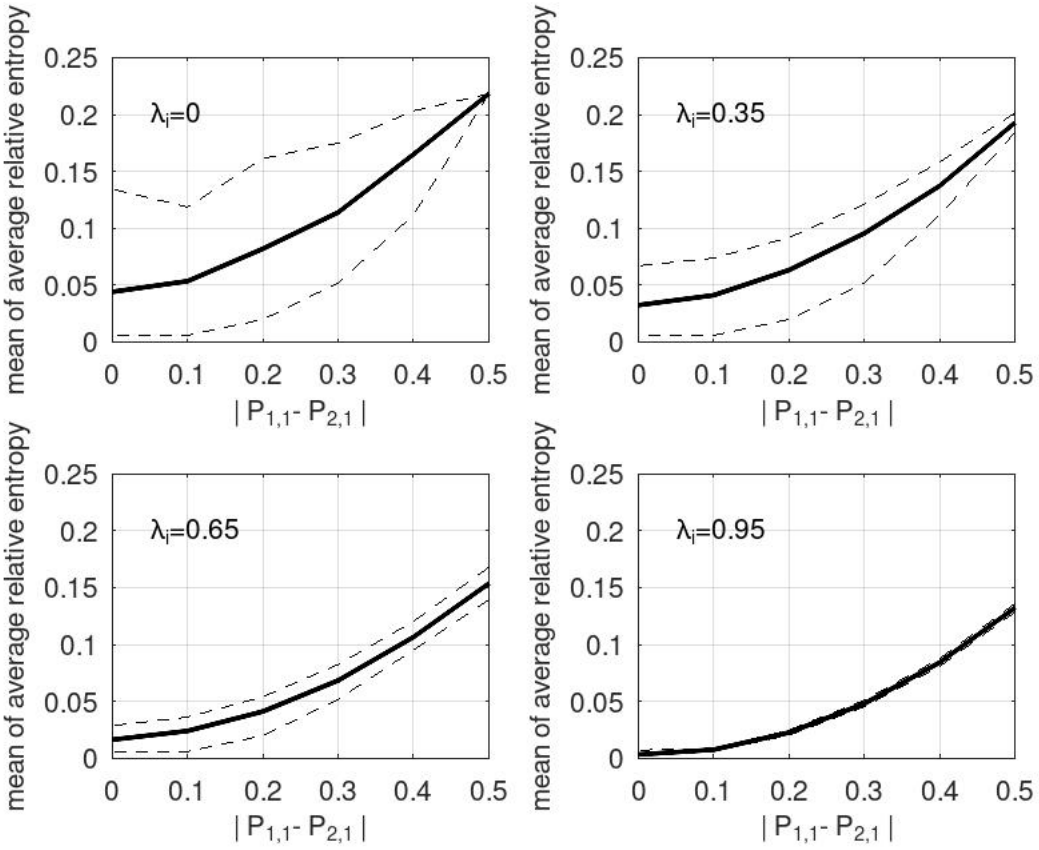


FIGURE 1.5: Mean average relative entropy of an under-reacting agent i computed for different λ_i as a function of $|P_{1,1} - P_{2,1}|$ with $P_{1,1}, P_{2,1} \in \{0.25, 0.35, 0.45, 0.55, 0.65, 0.75\}$. Dashed lines represent minimum and maximum recorded values of average relative entropy.

the best i.i.d. model, that is, the invariant distribution of the chain. Thus, the damping effect that is key to generate the advantage of the moving average approach with under-reaction reference (almost) disappears when the reference under-reaction protocol has a degree of under-reaction extremely close to 1.

Finally, notice that "fluctuations in beliefs" are, in principle, less detrimental when the truth is Markov than in the i.i.d. case. Indeed, when such fluctuations are somehow aligned with the true transition probabilities, accuracy increases. This situation clearly emerges when one considers the limited memory Bayes protocol defined by Eq. (1.8). Here the limitation in the number of observations the agent adopts let its prediction move in a Markovian way. Thus, in those cases in which the i.i.d. models it uses and the initial weights it attaches are such that the resulting probabilistic prediction are close to true transition probabilities, it shows an high level of accuracy.

Proposition 1.4.7. *Under* **A2**, *for an agent i that uses the limited memory Bayes protocol in Eq.* (1.8) *it is p-almost surely*

$$\overline{D}_p(p_i, \sigma) = \sum_{s'=1}^{S} \pi(s') \sum_{s=1}^{S} P_{s',s} \log \frac{P_{s',s}}{\sum_{k=1}^{K} \pi_k(s) \frac{\pi_k(s') w_{i,k}(\sigma^0)}{\sum_{k'=1}^{K} \pi_{k'}(s') w_{i,k'}(\sigma^0)}}. \tag{1.16}$$

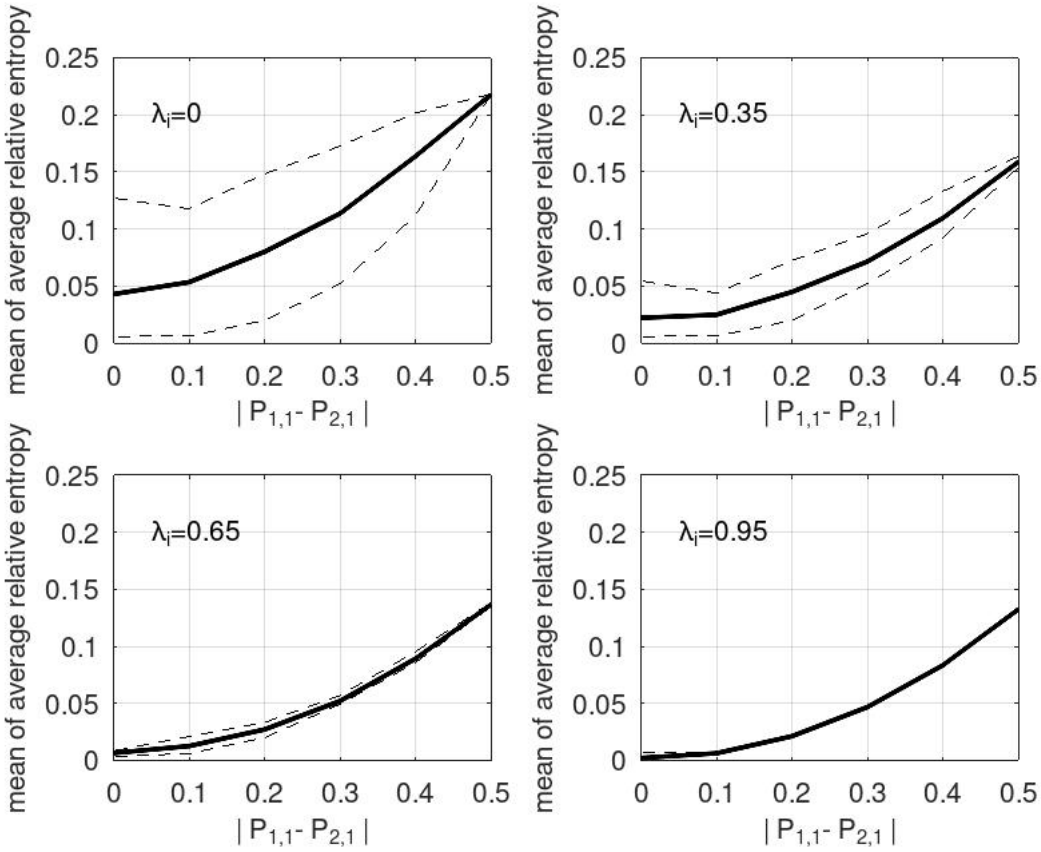


FIGURE 1.6: Mean average relative entropy of an agent i who computes its beliefs according the moving average protocol with $M_i = 500$. Means are computed for different λ_i as a function of $|P_{1,1} - P_{2,1}|$ with $P_{1,1}, P_{2,1} \in \{0.25, 0.35, 0.45, 0.55, 0.65, 0.75\}$. Dashed lines represent minimum and maximum recorded values of average relative entropy.

Proof. The statement follows from Eq. (1.8) and the application of the Strong Law of Large Numbers as in the proof of Proposition 1.4.6. □

A straightforward consequence of Proposition 1.4.7 is that the limited memory Bayesian, depending on the available i.i.d. models and the weights, can be maximally accurate. I show this point in Figure 1.7, where I consider again my example and I set $w_{i,1}(\sigma^0) = w_{i,2}(\sigma^0) = 0.5$. As one can notice, there generically exist combinations of $P_{!,1}$ and $P_{2,1}$ such that the limited memory Bayesian can be more accurate than an agent that strongly under-reacts. In the right panel of Figure 1.7 I also added the average accuracy of the best i.i.d. model, that is, the invariant distribution. As one can notice the average accuracy of the invariant looks like a lower bound for the under-reaction protocol, while the limited memory Bayesian can, in some cases, be more accurate.

This shows how in general cases of model misspecification – i.e. when the true model is more complicated than those the agents use to learn – there generically exist cases in which naive approaches outperform more sophisticated ones. Indeed, looking at Figure 1.7 one notices that strong under-reaction can produce beliefs approximately as accurate as the best i.i.d. model, but it cannot improve upon that. The

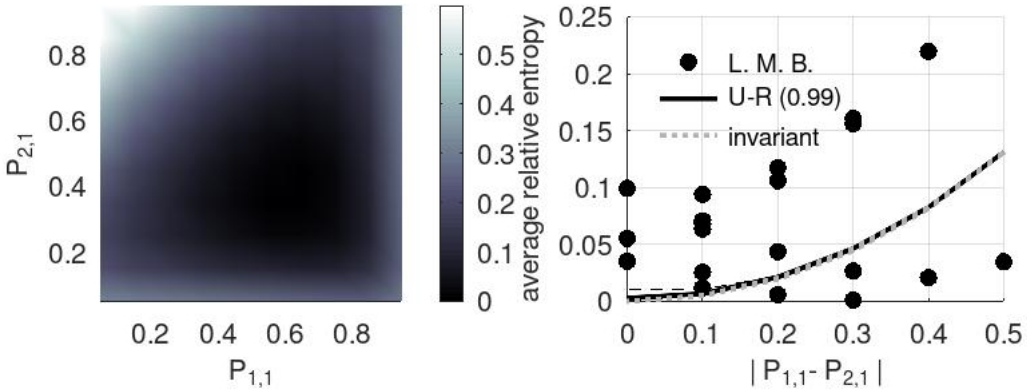


FIGURE 1.7: Left: average relative entropy of an agent i who computes its beliefs according the limited memory Bayes approach for different values of $P_{1,1}$ and $P_{2,1}$. Right: the dots represent the average relative entropy of an agent i who computes its beliefs according to the limited memory Bayes protocol as a function of $|P_{1,1} - P_{2,1}|$ with $P_{1,1}, P_{2,1} \in \{0.25, 0.35, 0.45, 0.55, 0.65, 0.75\}$; the black line represents the mean of average relative entropy of an under-reacting agent j with $\lambda_j = 0.99$ as a function of $|P_{1,1} - P_{2,1}|$. Dashed lines represent minimum and maximum recorded values of average relative entropy for such under-reacting agent. The dotted gray line represents the mean average relative entropy of the invariant distribution. Minimum and maximum values for the average accuracy of the invariant are not reported since differences in recorded values are smaller than 10^{-3}.

limited memory Bayes approach, instead, naturally generates beliefs that change depending on the previous state, that is, it naturally produces Markovian beliefs. This can more than compensate for its inability of exploiting information for selecting the best model in the set (the key feature of Bayesian learning) or getting close to the best convex combination of models (the key feature of strong under-reaction).

1.5 Conclusions

In this chapter I build up the connection between learning and market dynamics through analyzing market selection in a security market economy where agents' learning process is characterized by model misspecification. In this setting, a Bayesian learner would asymptotically predict as the best model in its set (Berk, 1966). According to Massari (2020), the Bayesian learner could be driven out of the market by a trader who under-reacts to information. I follow the literature on market selection (Blume and Easley, 1992; Bottazzi et al., 2018; Bottazzi and Giachini, 2019; Dindo and Massari, 2020; Sandroni, 2000). Relative entropy turns out as the correct measure of accuracy of a trader's beliefs and its infinite time average is deeply connected to asymptotic dynamics. In an economy characterized by independent and identically distributed (i.i.d.) states of the world, I find that an under-reacting trader who gives a sufficiently high weight to the prior drives out of the market any other trader, the Bayesian one included, who does not learn the truth asymptotically. By performing numerical exercises, I provide a better characterization of the advantages of an under-reacting rule. In particular, I show that the accuracy of

predictions (and, thus, the selection advantage) increases with the degree of under-reaction the agent exhibit. Such a superior performance results from a combination of the good properties of the Bayesian update with the systematic smoothing of predictions operated by an under-reacting rule. I generalize such an idea showing that, with i.i.d. states, an agent can make an opponent vanish by building its beliefs as a moving average of the opponent's predictions. I show that my results persist when the sequence of states of nature is driven by Markov or Polya urn processes. In this respect, extreme models, giving near full probability to different states, represent an optimal base on which an agent should root its updating when the true model is not available. Indeed, learning over a set of extreme models coupled with smoothing beliefs by means of a combination of under-reaction and moving average increases the chances an agent has to survive the market selection struggle. The latter consideration is totally in line with the concept that behavioral protocols are properly modeled by heuristics. An heuristic is "a strategy that ignores part of the information, with the goal of making decisions more quickly, frugally and/or accurately than more complex methods" (Gigerenzer and Brighton (2009), 454).

Chapter 2

The Grossman and Stiglitz model: an agent-based extension

2.1 Introduction

In broad terms, an efficient stock market is one in which share prices reflect the fundamental information about agents. In that case, the market price reflects changes in a way that is close to its intrinsic value. Such changes are not in line with the value and do not restrain from financial assets trading. The disparities in agent awareness and uneven costs of transaction avoid fundamental values variations to be totally and immediately reflected in market prices (Koller et al., 2010). Eakins and Mishkin (2012) claimed that an efficient market is one in which asset prices fully reflect all available information. In general, the nature of an efficient market rests on two main concepts: available information is already reflected in stock prices in efficient markets; traders cannot earn risk-weighted excess returns in efficient markets (Degutis and Novickytė, 2014). The efficient market hypothesis (EMH) is intimately linked to other financial models and assumptions. Firstly, the absolute or partial rationality of the market agents is a key prerequisite for its efficiency. It is frequently assumed that not all market players act rationally, which results in that some trading is not rational analysis based. Conversely, the trades of irrational traders are considered *random* and in this perspective it should not affect the market price. The EMH achieved its popularity peak in the 1980s (Shiller, 2003) and the greatest contribution was made by E. Fama, whose works have become classics in the area of market efficiency. Fama (1965) affirmed the randomness of stock prices and defined the concept of the "efficient market" for the first time. Fama defined an efficient market as the market in which information is "fully reflected" and suggested that market efficiency tests should be conducted along the lines of asset valuation tests (Fama, 1970). Grossman (1976) highlighted the paradox of market efficiency: The more traders believe in market efficiency, the less efficient the market gets. Grossman argued that if there exists a universal agreement that the market is efficient, agents start trading passively and stop gathering information, then leading to inefficiency. Later, Grossman and Stiglitz (1980) claimed that markets cannot be efficient because there are information costs. The return on investment must therefore be higher than the information costs, otherwise the inclination to invest would vanish. Grossman and Stiglitz (1980) developed a rational-expectation equilibrium model of financial markets, introduced by Lucas (1978). In this model, agents trade an asset with uncertain value; they submit a demand curve of the asset quantity they would buy for every price would be realized. wherein “fundamentalist” traders can choose to be informed or uninformed with respect to the future return of the asset. In the first case, they pay a cost to become informed, while in the latter they estimate the future return. Hence, the total population (which is fixed) splits into two fractions endogenously

determined. This concept comes form the necessity of including and recognizing the costly activity of those who arbitrage in a competitive economy, who would not pay for information if there were no profit. Within their model, Grossman and Stiglitz showed the existence of a unique equilibrium, as the benefit of being informed decreases in the number of informed agents. The latter occurs either because the more people are informed, the more informative the price is, or a higher fraction of informed agents means that pro capita gains decrease. More importantly, Grossman and Stiglitz showed that prices cannot be fully efficient: they cannot reflect all the available information. Indeed, if that were the case, no agent would pay the cost of becoming informed, as they could just learn information from the price. In contrast, if no trader produces information, then prices would have no information content inside, and it would be advantageous to produce it. In other words, a competitive equilibrium cannot exist if information is costly and prices are fully informative: prices partially reveal information (i.e. with some noise), making agents willing to acquire it.

The aim of this Chapter is to reproduce the settings in which EMH should definitely hold in a dynamical framework. In order to do the latter, I port the model in an agent-based framework explicitly considering the interactions of heterogeneous traders. Agent-based models represent the economy as a complex, evolving system populated by heterogeneous, interacting agents (Dosi et al., 2012; Farmer and Foley, 2009; Kirman, 2010b; Leal et al., 2016; LeBaron and Tesfatsion, 2008; Tesfatsion and Judd, 2006). In line with the settings EMH should definitely hold, all the agents know that the expected fundamental value of the risky asset evolves according to the expected dividends. Agents choose to adopt a *fundamentalists* trading strategy: they expect future asset price will converge to the fundamental one, so they trade accordingly to exploit possible imbalances. In this case, some agents will decide to pay the cost to acquire information about the value of the expected dividends (informed "fundamentalists"), while the others will infer it from past realizations of the dividend process (uninformed "fundamentalist"). My first research question lies on what is the effect on economic aggregate outcomes, market stability and if informed traders really earn more the uninformed ones by paying a cost for private information.

Later, a third category of agents is introduced. They are agnostic about the "laws of motion" of the market and trade according to heuristics. They can be considered as "noisy" traders, or simply, even if they think that the share price will revert to its fundamental value, the believe that world is too complex to follow "fundamentalist" strategies. In line with Grossman and Stiglitz (1980), each trader submits a demand curve for the asset-quantity for every price and an exogenous noisy supply for the asset occurs. Then, the price is such that market clears and earnings of being informed of not can be computed. Every types of agents can adaptively learn, i.e. they can compute the performance of all the available strategies and switch to that which guaranteed the highest return in the previous period. The resulting model is an adaptive interacting system: the interactions of heterogeneous population of agents with different trading strategies will determine the evolution of asset prices and strategy profits which in turn will affect the share of traders following specific rules. In such a framework one can study the informative content of the price level vis-á-vis the fundamental and, more generally, the efficiency of the financial market. Moreover, exploiting the information gathered at the microeconomic level, one can condition the previous results to the evolution of the different trading strategies. Are noisy traders destabilizing the markets or is it rational to follow heuristics? How noisy traders are affecting the economic aggregate outcomes? The entire perspective,

so that a dynamical adaptive framework in which a fundamental value is ensured and agents may trade w.r.t. it, paying for a private information or simply following heuristics, has never been explored in literature previously. In the next Section 2.3, I briefly describe the paper Grossman and Stiglitz (1980) from which I take inspiration for the model I set up in this chapter. In Section 2.4, I formally describe the model, including how market prices are set and fundamental value and dividends evolve. In Subsection 2.4.1, I focus on the possible strategies followed by the agents: I characterize the demand of an agent á la Grossman-Stiglitz and how informed and uninformed formally differ from each other, and I briefly describe three heuristic rules, *recency*, *adaptive* and *chartist*. In Subsection 2.4.2, the dynamics of the model is presented, showing how agents compare their performances after market price has been set, updating of their wealths, the way in which they choose to keep their previous strategy or move to another and what one. In Section 2.5, I analyze a market composed by agents provided of the Informed and Uninformed strategies and I describe my numerical and analytical findings. By exploring the parameter space, two different asymptotic scenarios arise: one in which the population distributes in half informed and half uninformed over time, and one in which the whole population joins uninformed strategy. This split crucially depends on the magnitude of C and it is controlled by a threshold (i.e. C^*). In Subsection 2.5.1 I show how I derived it analytically. In Subsection 2.5.3 I lay out an illustrative example in which I prove the paradox found by (Grossman and Stiglitz, 1980); i.e. if price would reflect all information available, no trader would pay C, as they could learn information from the price; on the contrary, if no one does, then prices would reflect no information, and it would be profitable to buy information. Finally, in Section 2.6, I go through a preliminary exploration of all the possible market combination, considering three different agent demands: Informed, Uninformed and the heuristic Recency. Moreover, I take into account also another dividend process setting: the *structural break* case; i.e. a periodic switching of the dividends between two different GBM processes. In particular, I analyze the case in which informed agents that are actually so rich that the cost of being informed is always negligible compared to their wealth (i.e. $C = 0$).

2.2 Literature review

An efficient stock market is one in which share prices reflect the fundamental information about agents. In that case, the market price reflects changes in a way that is close to its intrinsic value. Such changes are not in line with the value and do not restrain from financial assets trading. The disparities in agent awareness and uneven costs of transaction avoid fundamental values variations to be totally and immediately reflected in market prices (Koller et al., 2010). Eakins and Mishkin (2012) claimed that an efficient market is one in which asset prices fully reflect all available information. In general, the nature of an efficient market rests on two main concepts: available information is already reflected in stock prices in efficient markets; traders cannot earn risk-weighted excess returns in efficient markets (Degutis and Novickytė, 2014). The efficient market hypothesis (EMH) is intimately linked to other financial models and assumptions. Firstly, the absolute or partial rationality of the market agents is an key prerequisite for its efficiency. It is frequently assumed that not all market players act rationally, which results in that some trading is not rational analysis based. Conversely, the trades of irrational traders are considered *random* and in this perspective it should not affect the market price. The EMH

achieved its popularity peak in the 1980s (Shiller, 2003) and the greatest contribution was made by E. Fama, whose works have become classics in the area of market efficiency. Fama (1965) affirmed the randomness of stock prices and defined the concept of the "efficient market" for the first time. Fama defined an efficient market as the market in which information is "fully reflected" and suggested that market efficiency tests should be conducted along the lines of asset valuation tests (Fama, 1970). Grossman (1976) highlighted the paradox of market efficiency: The more traders believe in market efficiency, the less efficient the market gets. Grossman argued that if there exists a universal agreement that the market is efficient, agents start trading passively and stop gathering information, then leading to inefficiency. Leter, Grossman and Stiglitz (1980) claimed that markets cannot be efficient because there are information costs. The return on investment must therefore be higher than the information costs, otherwise the inclination to invest would vanish. Grossman and Stiglitz (1980) developed a rational-expectation equilibrium model of financial markets, introduced by Lucas (1978). In this model, agents trade an asset with uncertain value; they submit a demand curve of the asset quantity they would buy for every price would be realized.

The issues of information aggregation through prices and the information efficiency of markets in the face of heterogeneous information have been extensively studied in the literature. In a seminal paper, Grossman (1978, 1976) illustrates how prices can perfectly aggregate information and replace private information in markets. This result brings up the potential for an interesting paradox that Grossman and Stiglitz (1980) have highlighted: If prices aggregate costly private information in a perfect way, there would be no incentive for traders to purchase such information, since they could merely observe prices, but then no information would be available to aggregate prices. A standard method of overcoming this paradox is to incorporate into prices aggregate noise, following the approach in Hellwig (1980) and Diamond and Verrecchia (1981). The introduction of aggregate noise overcomes the paradox because prices in equilibrium are only partially informative, and hence agents have an interest in acquiring private information.

Another possibility to tackle the paradox is to examine markets with a finite amount of traders who hold market power, such as in Kyle (1989): market power guarantees that prices only partially disclose private information, so that agents have an incentive to buy that information. Another workaround to the paradox, suggested lately by Vives (2014), is to differentiate between common and private value components in traders' evaluations, where signals yield only bundled information on such components.

In all these papers, static situations are considered where the information is received only once. However, noisy equilibria with rational expectations have been analysed in dynamic scenarios as well. Vives (1993) investigates the rate at which dispersed information converges to the prices of risk-neutral firms. This convergence is slow with finite accuracy of private information, but a crowd of perfectly informed agents accelerates the convergence.

Kyle (1985) studies how quickly new private information about the underlying value of a speculative asset enters market prices and how it affects the market's liquidity. In the model, there are three types of traders: a single risk-neutral insider, random noise traders and competitive risk-neutral market makers. The insider makes profits positively by leveraging his power as a monopolist, where the noise trader hides their trading from the market makers. In the limiting case, when the time between auctions approaches zero, all private information is integrated into prices through trading.

Amador and Weill (2012) look at a dynamic framework in which agents learn from the trading actions of others through two channels: a public channel, like equilibrium market prices, and a private channel, like local interactions: In the case where agents learn only through the public channel, an initial disclosure of public information raises knowledge and welfare, while the result is reversed when there is also a private learning channel, and an increment of initial public information decreases agents' knowledge asymptotically. It is important to note that initial realisations of signals focus on the true state of the world, while following realisations focus on the endogenous actions of the agents: This means that subsequent to the initial period, all the incoming information on the state of the world stems from others and there is no extra incoming exogenous information.

The assumption that agents behave are fully rational is the essential requirement of asset pricing standard models as the one of Lucas (1978). Fully rational agents are aware of all (relevant) available information about the economy and their behavior is either consistent with or comes from utility function maximization. Moreover, they are endowed with rational expectations and they can exactly compute every realization of future states of the economy. In order to model a decision maker, come by itself remarking how unrealistic the latter assumption is.

Actually, it is even not that clear what "rational" expectations would mean, even in concept, in the presence of Knightian uncertainty; i.e. when there are radical changes in policies Dosi et al. (2017); Stiglitz (2015); **?** and/or *structural breaks* in the underlying distributions on which agents form their predictions (Hendry and Mizon, 2010).

The idea that agents' *bounded rationality* accounts for deviations from rational outcomes underlies the entire domain of behavioural economics (Camerer et al., 2011). This literature is based on the work of Tversky and Kahneman (1974), which deals with cognitive biases due to judgement heuristics. The basic claim is that widespread heuristics negatively affect decision-making by enabling agents to make non-optimal decisions. This view is partially challenged by the works: Gigerenzer and Brighton (2009); Gigerenzer and Gaissmaier (2011); Gigerenzer et al. (1999). Gigerenzer and his co-authors even argue that heuristics can actually enhance decision-making. The contrast between the contributions of Tversky and Kahneman and the work of Gigerenzer and other authors is that the latter examine individuals in a complex environment in which an optimal choice is inconceivable. Therefore, heuristics turn into simple tricks to get the best out of a setting that is uncertain in Knight (1921) terms. In addition, the authors illustrate that attempting to increase the "rationality" of a decision (e.g. by collecting more information) leads in many cases to a decrease in performance due to a variance-bias trade-off. In other words, sophisticated statistical models have poor predictive power in some situations due to overfitting. This idea is explored by Dosi et al. (2017) in the setting of a macroeconomic agent-based model in which aggregate economic performance is measured across different expectation formation rules. In a complex evolving environment, they demonstrate how agents' use of sophisticated expectation formation rules leads to a deterioration in terms of both individual performance and the economy's aggregate outcome compared to the case where they assume simple myopic expectations. The underlying idea is that any statistical model based on past observations will be biased if the data generating process is altered over time. Therefore, a simple prediction scheme, such as assuming that the next value of a variable is the same as the last observed one, can outperform sophisticated models based on long time series. They conclude that under Knightian uncertainty arising from technical change, interaction and imperfect information, recourse to simple

heuristics removes the trade-off between variance and bias and enhances overall performance. A somewhat related perspective arises from the analysis of Kirman (2010a): the interaction of simple heterogeneous individuals, such as shoppers in fish markets (Gallegati et al., 2011; Härdle and Kirman, 1995; Kirman and Vriend, 2000), often leads to aggregate outcomes that behave well, while the interaction of more rational agents can lead to outcomes that are far from the rational benchmark.

In line with the latter contributions, I go through an alternative path based on the seminal papers of Cyert et al. (1963); March and Simon (1993); Simon (1955). Firstly, expectations and behaviour cannot be neatly separated in a complex environment, and secondly, behavioural patterns are appropriately incorporated by heuristics that, under Knightian uncertainty and the non-stationarity of economic fundamentals, may not only be "reasonable" but also rational in a particular sense. If all individuals use these heuristics, both the accuracy of their predictions and their welfare may be higher than if they follow apparently more complicated rules. A heuristic is "a strategy that ignores some information with the aim of making decisions faster, more economically and/or more accurately than more complex methods" (Gigerenzer and Gaissmaier, 2011). The heuristics I examine below exhibit the so-called "less-is-more" effect, which arises in response to the "bias-variance dilemma" - basically the possible trade-off between accuracy in interpolating past observations and accuracy of prediction-also well known in machine-learning and statistical inference Dosi et al. (2017). However, heuristics are not "biases" based on "fast thinking" producing suboptimal behaviours (as one would understand from Kahneman (2003) and from the field of behavioural economics), but may well be resilient strategies that result in performance that is better than in purely "rational" choices in changing worlds characterised by substantive and procedural uncertainty (Dosi et al., 2017). The reason for this seemingly abnormal outcome is that how expectations are formed affects how the system performs; more "rational" expectations may lead to poorer system performance, where the difficulties of prediction increase, in some sense, quicker than the sophistication of the forecasting method.

I study Grossman and Stiglitz (1980) settings under different rules of expectation formation and behavior in an agent-based framework. ABMs represent the economy as a complex, evolving system populated by heterogeneous, interacting agents (Dosi et al., 2012; Farmer and Foley, 2009; Kirman, 2010b; Leal et al., 2016; LeBaron and Tesfatsion, 2008; Tesfatsion and Judd, 2006). Agent-bases modeling lets me obtain a dynamical counterpart of the static considerations by Grossman and Stiglitz (1980), poorly present in literature.

2.3 Grossman and Stiglitz (1980)

The model developed by Grossman and Stiglitz (1980) extends from the combinations of Lucas's model and the informational flows among traders due to Green (1973).

In that model, they consider just two financial assets: one bond B, a safe asset which guarantees a fixed return R at the end of the whole period T; one stock S, a risky asset that observes a random return u varying with time. The return of S is a variable composed of two terms:

$$u = \theta + \epsilon \tag{2.1}$$

where θ and ϵ are both random variables; the primer is knowable at a cost C while the latter is unobservable. In the model, the difference between informed and uninformed traders depends on whether they paid of not C to obtain the information, otherwise they are fully identical one another. Given P as the price of the risky asset S, The demand functions for the two different kind of agents shall own the form:

$$D_I(\theta, P): \quad \text{informed traders} \qquad\qquad D_U(P, F_u): \quad \text{uninformed traders}$$

where P is the risky asset price and F_u is the cumulative distribution of return. It is assumed the uninformed have *rational expectations*: "rational" means they rely on the relationship between F_u and P in order to derive the demand. Given a percentage of informed traders λ, an equilibrium can be defined through a price function of the risky asset $P_\lambda(\theta, x)$ such that demand equals x, that is the supply:

$$\sum_i D_i = D_I(\theta, P) + D_U(P, F_u) = x$$

In order to prevent uniformed traders from learning θ from observing $P_\lambda(\theta, x)$, it shall be assumed that they do not observe x. In this way, variations in price observed by uninformed traders may be due both to changes in informed choices and changes in aggregate supply. Agents' switching mechanism from informed to uninformed and viceversa depends on the expected utility $E[V]$: some individuals from the state (U or I) with the smaller expected utility shift to the opposite state (I or U). It is taken into account the cost C paid to observe the information for the informed traders.
That is, at a certain time t:

$$\text{if} \quad E_I[V_{t-\delta t}] > E_U[V_{t-\delta t}] \qquad \text{then} \qquad \lambda(t) > \lambda(t - \delta t) \qquad \text{(and conversely).}$$

The use of δt is just to remark the causal relationship: for a certain time t the two different expectations are computed and then someone gets to the other side for convenience.
An overall equilibrium needs $E_I[V_{t-\delta t}] = E_U[V_{t-\delta t}]$.
With an increasing in λ, i.e. the more uninformed ones become informed, the ratio $\frac{E_I[V]}{E_U[V]}$ decreases for two reasons:

- the aggregate demand (and thus the asset price) is very sensitive to changes in θ, since the more traders observe θ, the more the information of informed is available to the uninformed. Thus, the price system becomes more informative and the gain of informed trading with the uninformed is reduced.
- Even if the above effect would not happen, anyhow the increase in the number of informed traders leads to per capita gain reduction.

A list of conjectures is provided in order to summarize what above about equilibrium description when price carries information.

Proposition 2.3.1. *The more agents that are informed, the more informative the system is ("more informative" shall be properly defined below.)*

Proposition 2.3.2. *The more agents who are informed, the lower is the ratio of the expected utility of informed to the uninformed.*

Critical parameters of the economy, which the equilibrium number of informed depends on, are: C cost of information, "how much" the price system is informative (i.e. noise that interferes with the info carried by price system), "how informative" is the information bought from informed people.

Proposition 2.3.3. *The more expensive the information cost C, the smaller will be λ, the equilibrium percentage of informed people.*

Proposition 2.3.4. *If information quality increases, informed agents' demands will vary better according with information and, thus, the price will vary better according θ. The price system becomes more informative. In this case, ratio of informed to uniformed may be either decreased or increased, because both the value of being informed has increased, due to higher θ quality, and the value of being uninformed, due to the more informative price system.*

Proposition 2.3.5. *The higher the noise, the less informative will the price system be, hence the lower $E_U[V]$. At equilibrium, the higher the noice, the larger λ.*

Proposition 2.3.6. *In the no noise limit, price conveys all information, so there is no reason to buy information. Hence, the only feasible equilibrium is the one with no information. However, in a totally uninformed world, it may be convenient for someone to purchase information. Hence, there doesn't exist a competitive equilibrium (Grossman and Stiglitz, 1980).*

In other words, if anyone is informed, by buying information would be the only way to get informed, since the price system cannot reveals information. By paying C, a trader would be able to predict better than others, hence he/she would have an higher expected utility in comparison with uniformed, gross of information cost. So, for a sufficiently low price C, everyone would desire to get informed. Reasons of trading among people could be: tastes (risk aversion), endowments and beliefs. Focusing on the latter, it is worth to emphasize that, at equilibrium, there could exist two situations in which all individuals have identical beliefs: when all of them are informed and when not. That is:

Proposition 2.3.7. *Considering all the remaining conditions fixed, markets will be thinner under those conditions in which λ is either near zero ore one.*

Grossman and Stiglitz were not able to obtain a general proof for any of the propositions. They studied a particular example, the one in which the utility function employed is the Constant Absolute Risk-Aversion (CARA) and random variables are normally distributed. With these assumptions, the equilibrium price can be calculated and all the conjectures above can be verified.
The two kind of agent demands employed by Grossmn and Stiglitz could be easily linked to both the forms known in literature for rational expectation. For the informed agent, a stronger form of rational expectation (Fama, 1972): one has access to all relevant available information about the structure of their environment, makes optimal use of this information in forming his expectation and the latter will be correct up to unsystematic (unavoidable) errors, so $P = F + \epsilon$, where P is the actual price (my case u), F the fundamental price (my case θ), and ϵ is an error (in my case i.i.d. with zero mean and variance σ_ϵ).

2.4 The model

I develop a pure exchange economy model in which trading occurs in discrete time. Two goods are traded: a risky asset (stock), which pays a positive random dividend y_t anytime a period t starts, and a riskless asset (bond) that has a constant interest rate $r > 0$. The price of the riskless asset is set to 1, as it is the numéraire of the economy. At any time step t, the price of the risky asset is determined through market-clearing condition, taking into account the aggregate demand and supply emerging from the decentralized decisions of heterogeneous traders. I assume that

dividends grow at a positive rate. This implies that both the fundamental price and total wealth will increase over time.

In the market, there are N heterogeneous interacting traders, indexed by i. The wealth $W_{i,t}$ of the traders evolves over time according to their trading performance. At time t, each agent i fixes its (excess) demand function $D^s_{i,t}(P_t)$ for the risky asset adopting different trading strategies s. The different strategies are spelled out in Section 2.4.1 below.

Let us identify X_t the aggregate supply at time t.Under a Walrasian auctioneer scenario, the market clearing condition at time t implicitly defines the market price P_t:

$$\sum_{i=1}^{N} D^s_{i,t}(P_t) = X_t \qquad . \tag{2.2}$$

Furthermore, let us consider the following notion of fundamental price as benchmark:

$$F_t = \sum_{k=1}^{\infty} \frac{E_t[y_{t+k}]}{(1+r)^k} = \frac{(1+\mu)\, y_t}{r-\mu} \qquad , \tag{2.3}$$

where $0 < \mu < r$ is the positive expected dividend growth rate. As showed in Appendix B.1, it originates from the special case where there is only a rational-expectation representative agent and zero supply of output shares (i.e. $x_t = 0\ \forall t$). In that setting, the market converges to it.

The evolution of the fundamental price depends on the dividends $\{y_t\}$ which follows a geometric brownian motion:

$$y_{t+1} = (1 + \mu + \sigma\, \zeta_{t+1})\, y_t \qquad \mu > 0 \quad , \quad \sigma > 0 \quad , \tag{2.4}$$

where $\zeta_t \sim N(0,1)$ are i.i.d. random shocks, μ represents the expected growing rate of the dividends and σ the standard deviation of the dividend growth rate. Consistently with Eq.2.3, fundamental price process $\{F_t\}$ will follow a similar behavior, together with same value of expected growth rate and standard deviation. Note that the information about the last realization of the dividend process will allow the agents to compute the exact value of the fundamental price and consequently the possible difference with the market price. It is worth to mention that dividends play a crucial role in this model, since the knowledge of its process coincides with information one chooses to buy or not.

When it comes to simulating the model, I assume that dividend process $\{y_t\}$ follows a geometric Brownian motion, since Eq.B.3 is consistent with this assumption:

$$y_{t+1} = (1 + \mu + \sigma\, \zeta_{t+1})\, y_t \qquad \mu \geq 0 \quad , \quad \sigma > 0 \quad , \tag{2.5}$$

where $\zeta_t \sim N(0,1)$ are i.i.d. random shocks, μ represents the expected growing rate of dividends and σ the standard deviation of the dividend growth rate. By definition of fundamental price F_t in Eq.2.3, the latter will follow a similar process as well Chiarella et al. (2009), with same value of expected growth rate and standard deviation.

2.4.1 Heterogenous strategies

Given the general market set up spelled out in the last Section, how do agents compute their demand for the risky asset? Agents can choose among different trading strategies which involve different degree of rationality and learning capabilities. I start with a framework á la Grossman and Stiglitz where agents form expectations

about the fundamental price, and they can choose to buy or not the costly information concerning the future dividend. The cost of such information is $C > 0$. Agents that buy the information follow an Informed (I) "fundamentalist" strategy, while the other an Uninformed (U). Both I and U agents face the same demand function:

$$D^s_{i,t}(P_t) = K^s \cdot (E^s_t[F_{t+1}] + E^s_t[y_{t+1}] - RP_t) \qquad s \in \{I, U\} \quad ,$$

where $R = 1 + r$, is the compound interest rate of the risk-free asset.
However, informed agents are aware of the exact value of expected dividend growth rate μ, so they can correctly compute both expectations of next dividend $E^I_t[y_{t+1}] = (1+\mu)\, y_t$ and consequently next fundamental, $E^I_t[F_{t+1}] = (1+\mu)\, F_t$. Uninformed ones rely instead on the expected dividend growth rate $\hat{\mu}$ estimated according to past dividend observations (more details in Appendix B.2). In that respect, their expectations will be: $E^U_t[y_{t+1}] = (1+\hat{\mu})\, y_t$ and $E^U_t[F_{t+1}] = (1+\hat{\mu})\, \hat{F}_t$. Uninformed agents compute the fundamental at time t following Eq.2.3 correctly but $\hat{\mu}$, i.e. $\hat{F}_t = \frac{(1+\hat{\mu})\, y_t}{r-\hat{\mu}}$. Hence, the demands of informed and uninformed agents are equal to:

$$\begin{aligned} D^I_{i,t}(P_t) &= K^I \cdot ((1+\mu)\, F_t + (1+\mu)\, y_t - RP_t) \qquad ; \\ D^U_{i,t}(P_t) &= K^U \cdot ((1+\hat{\mu})\, \hat{F}_t + (1+\hat{\mu})\, y_t - RP_t) \qquad . \end{aligned} \tag{2.6}$$

In the first set of simulations, I will consider only the Informed and Uninformed fundamentalist agents in order to have an agent-based counterpart of the original model. Later, I will let agents to trade according to different heuristics. Why do agents should choose heuristics? One possible explanations is that the market is populated by noisy traders De Long et al. (1990). Or there are some forms of bounded rationality. Finally, agents could rationally choose to follow heuristics because the price dynamics is too much complex to trade according to the evolution of the fundamental price even if they perfectly know it. This is in line with the idea of ecological rationality **?** and with the results obtained in Dosi et al. (2017). In the first *recency demand* (N) heuristics, agents think that the past is the best guess for the future:

$$D^R_{i,t}(P_t) = (P_t + y_t - RP_{t-1}) = \rho_{t-1}(P_t) \qquad ; \tag{2.7}$$

in other words, recency agents fully rely on the previous realized excess return $\rho_{t-1}(P_t)$, that is over the period $[t-1, t]$. In the *adaptive demand* (A) heuristics, agents correct their past demand forecast mistakes:

$$D^A_{i,t}(P_t) = D^A_{t-1}(P_{t-1}) + \omega_A[\rho_{t-1}(P_t) - D^A_{t-1}(P_{t-1})] \qquad ; \tag{2.8}$$

where $0 < \omega_A < 1$ is the intensity of correction. Finally, agents can follow the *chartist demand* (C) heuristics taking into account the market trend:

$$D^C_{i,t}(P_t) = \rho_{t-1}(P_t) + \omega_C[\rho_{t-1}(P_t) - \rho_{t-2}(P_{t-1})] \qquad ; \tag{2.9}$$

where $0 < \omega_C < 1$ is the value of the parameter weighting past demand changes.

2.4.2 Market dynamics

Agents can choose among different strategies at any time step. Let S be the number of performable strategies and N^s_t the number of agents following the strategy s at time t, then: $N = \sum_{s=1}^{S} N^s_t$. Hereafter let us consider population fractions: $n^s_t = \frac{N^s_t}{N}$, which implies $\sum_{s=1}^{S} n^s_t = 1$. I observe this rephrasing in order to directly match how

the fraction of informed agents is expressed in Grossman and Stiglitz (1980). At the beginning of any time t, the population is divided in S fractions, n_t^s, where index s refers to all the strategies defined above, i.e. $s \in \{I, U, N, A, C\}$. Any agent (thereby any fraction) will compute the desired amount of risky asset $D_{i,t}^s(P_t)$. In order to observe same unit, let us define $x_t = \frac{X_t}{N}$ which denotes the average aggregate supply of shares per agent at time t. Formally, x_t are independent random variables with mean $\overline{x}$ and variance σ_x^2.[1]

Traders provide to the auctioneer their excess demand schedules and it aggregates them in order to obtain the market price Chiarella et al. (2009). Hence, the market clearing condition will fix the prevailing price P_t of the stock as follows:

$$\sum_{s=1}^{S} n_t^s \cdot D_{i,t}^s(P_t) = x_t \qquad . \tag{2.10}$$

Once price P_t is determined, the risky asset return referred to $[t-1, t]$ period is known: $\rho_{t-1}(P_t) = P_t + y_t - RP_{t-1}$. Then the realized profit of the previous period $\pi_{i,t}^s$ for any agent i who followed strategy s can be accordingly computed:

$$\begin{aligned} \pi_{i,t}^I &= D_{i,t-1}^I(P_{t-1}) \cdot \rho_{t-1}(P_t) - C \qquad s = I \\ \pi_{i,t}^s &= D_{i,t-1}^s(P_{t-1}) \cdot \rho_{t-1}(P_t) \qquad s \in \{U, N, A, C\} \ . \end{aligned}$$

Every agent wealth at time t can be updated:

$$\begin{aligned} W_{i,t} &= D_{i,t-1}^I(P_{t-1}) \cdot \rho_{t-1}(P_t) + (W_{i,t-1} - D_{i,t-1}^I(P_{t-1}) \cdot P_{t-1} - C)\, R \qquad s = I \\ W_{i,t} &= D_{i,t-1}^s(P_{t-1}) \cdot \rho_{t-1}(P_t) + (W_{i,t-1} - D_{i,t-1}^s(P_{t-1}) \cdot P_{t-1})\, R \qquad s \in \{U, N, A, C\} \ . \end{aligned}$$

At this point, agents can switch strategies according to their past performance. In line with Brock and Hommes (1998b), the performance $U_{i,t}^s$ of agent i who has followed strategy s is updated according to the resulting profits:

$$U_{i,t}^s = \pi_t^s + \eta\, \pi_{t-1}^s \qquad s \in \{I, U, N, A, C\} \ ; \tag{2.11}$$

where $0 \leq \eta \leq 1$ is a memory parameter measuring the relative weight attributed by agents to past profits of the same strategy they just performed. According to discrete choice probability approach Brock and Hommes (1998b), agents adopt a given trading strategy for the next period with a probability $n_{i,t+1}^s$, which is updated as follows:

$$n_{i,t+1}^s = \frac{\exp\left[\beta\, U_{i,t}^s\right]}{\sum_{s=1}^{S} \exp\left[\beta\, U_{i,t}^s\right]} \ , \tag{2.12}$$

where β measures intensity of choice, i.e. how fast agents switch to more successful strategies. In this way, there will be an ecology of strategies evolving over time and relates them to the performance of the market.

[1] The aggregate supply is given by the composition of all individual supplies at agent level. If the agent supplies are independent normally distributed variables, with mean $\overline{x}^i$ and variance ${\sigma^i}^2$, their average aggregate supply per agent will be $x_t = \frac{X_t}{N} = \frac{N\overline{x}^i}{N} + \frac{\sigma^i}{\sqrt{N}}\xi_t = \overline{x} + \sigma_x\, \xi_t$, where $\xi_t \sim N(0,1)$ is the idiosyncratic shock any agent is exposed to.

2.5 Informed vs Uninformed: numerical and analytical model explorations

I started my numerical analysis by considering the market only provided of the two *fundamentalist* strategies mentioned above, Informed and Uninformed. I carried out a comprehensive parameter space exploration and a first output surface response. In Table 2.1 I report all the parameters that make up the model, together with their explored ranges. I begin underlying the main features arising from the simulation.

TABLE 2.1: Parameters involved in the market model composed by Informed and Uninformed strategies, together with their corresponding numerically explored ranges.

	Parameter	Explored range
μ	expected dividend growth rate	$[1e-4, 9e-4]$
r	constant interest rate	$[1e-2, 1e-3]$
C	cost of information	$[1e-3, 1e10]$
σ_y^2	variance of the dividend growth rate	$[1e-15, 9e-6]$
σ_x^2	variance of aggregate supply	$[1e-13, 1e5]$
K^I,K^U	positive constants in demands	$[0, 1]$
K^I,K^U	positive constants in demands	$[0, 1]$
$n_{i,0}^s$	initial fraction of agent i with strategy s	$[0, 1]$
β	agent intensity of choice	$[1e-7, 9e-1]$

I fix K^I,K^U to the same value; i.e. same risk preferences and same beliefs on the variance of returns for both kind of agents. In this way I am not losing of generality. After some analytical considerations, tested by numerical simulations as well, I also keep μ and r fixed. The latter because either there are some constraints given by the model (e.g. the definition of the fundamental price imposes $\mu < r$), or their relationship of proportionality sets other parameter ranges' magnitude. The initial value of informed traders fraction ($n_{i,0}^s = \lambda(t = 0)$) has no impact on model dynamics; it only affects the first time steps before trading settles down. Hence, I am mainly considering the following parameter combination: $\{\sigma_y^2, \sigma_x^2, \beta, C\}$. As it arises from simulations, it is worth to mention that price always follows fundamental value except for large σ_x^2 and/or σ_y^2; in these cases, price keeps anyways the same trend of fundamental but fluctuating over/below or around it. By exploring the parameter space, two different asymptotic scenarios arise: one in which the population distributes in half informed and half uninformed over time (Scenario A), and one in which the whole population joins uninformed strategy (Scenario B). The intensity of choice β is responsible for slowing down (if smaller) or speeding up (if larger) the asymptotic convergence to a given Scenario.

2.5.1 Profit threshold for the cost of information

The split mentioned above crucially depends on C magnitude, both in short and long runs. The latter suggests the existence of a threshold C^* for the price, over which the dynamics converges to Scenario B and under which to Scenario A. This intuition is supported and confirmed by easy analytical derivations. The condition under which the benefit of being informed is exactly offset by its cost (i.e. C, the cost of information), so that the net profit is as much as the one of the Uninformed agent, is the equality of the probabilities of joining a given strategy for an agent i (or,

in other words, the equality of the two population fractions). Then, in line with Eq. 2.12:

$$n^I_{i,t+1} = \frac{\exp\left[\beta\, U^I_{i,t}\right]}{\sum_{s=1}^{S} \exp\left[\beta\, U^s_{i,t}\right]} = \frac{\exp\left[\beta\, U^U_{i,t}\right]}{\sum_{s=1}^{S} \exp\left[\beta\, U^s_{i,t}\right]} = n^U_{i,t+1}, \quad \text{where } s \in \{I, U\}\ . \tag{2.13}$$

The latter becomes an equality of the two exponents indicating agent performances, as in Eq. 2.11. By considering $\eta = 0$ for both strategies, for an agent i:

$$\pi^I_t = D^I_{i,t-1}(P_{t-1}) \cdot \rho_{t-1}(P_t) - C = D^U_{i,t-1}(P_{t-1}) \cdot \rho_{t-1}(P_t) = \pi^U_t\ . \tag{2.14}$$

The model consistency allows to elide the index i, since performances, profits and demands are the same for all the agents sharing the same strategy (i.e. in this case, Informed of Uninformed). By making the profits explicit and together with some computations, I derive an expression for a threshold cost of information:

$$C^*_t = (P_t + y_t - RP_{t-1})\left(D^I_{t-1}(P_{t-1}) - D^U_{t-1}(P_{t-1})\right) \qquad ; \tag{2.15}$$

it turns out to be the realized excess return times the difference in demands. If the cost of information is lower that C^*_t, then the Informed agent is gaining more than the Uniformed; on the contrary, if it is greater the C^*_t, the Uninformed is earning more than the Informed. Let us describe Scenarios in these terms in the following. It is worth to mention that, clearly, $C^*_t \xrightarrow[t\to\infty]{} 0$ by definition, but it crucial how long it takes.

2.5.2 Remarks on simulation constraints

The cost of information C actually is not fixed but increases over simulated time, proportionally to the dividend growth rate, in order to keep the same proportionality among relationships. This because the dividend growth rate is positive, so this implies that both the fundamental price and total wealth will increase over time. If C weren't so, I would asymptotically see only Scenario A; in other words, at a certain point of the simulation, any fixed C would always be too small, compared to earnings at stake in that moment, in order to be worth. Moreover, an higher or lower value of the interest rate r doesn't alter how the model behaves qualitatively. Only the proportion between earnings of risky asset and risk less one changes; at the same conditions then, a competitive price for information C is meant to be lower or higher to matter.

2.5.3 Scenario A

This is the case the cost of information C is comparable with C^*_t. Mainly two cases are distinguishable. In the first one, $\lambda(t) \cong 0.5$, $\forall t$. Here profits of U and I strategies are comparable over time; i.e. population is half informed indefinitely, no one switches. By observing $C \simeq C^*_t\ \forall t \in [0, T]$, it means that realized extra profit obtained being informed is totally offset by the cost of information. The fact that this relation keeps constant over time happens for two reasons: C^*_t roughly keeps the same value for a long period because the decrease in $(D^U - D^I)$ is counterbalanced by increasing in realized excess returns, as dividends grow over time; once that C^*_t reaches zero, since $(D^U - D^I)$ does asymptotically, C is so small that the relationship $\pi^I_{i,t} \simeq \pi^U_{i,t}$ preserves over time anyways. Anyhow, the most interesting case is:

$\frac{1}{T}\sum_{t=1}^{T}\lambda(t) \cong 0.5$. Here, for different time steps t, cases in which $C > C_t^*$ and others in which $C < C_t^*$ take turns. Larger σ_y^2 results in a $(D^U - D^I) \xrightarrow[t\to\infty]{} 0$ much slower convergence. Larger σ_x^2 and/or σ_y^2 produce (wider) fluctuations in price which cause (very) profitable earnings and/or (severe) losses (variations in realized profits). As a consequence, that causes fluctuations in performances and then fluctuations in populations. A particular case is the one where agents switch altogether. Since the latter is particularly illustrative of the model dynamics, I show a representative simulation of Scenario A in Fig.2.1.

Degenerate ecology and deviation of P_t from F_t

I recall the paradox found by (Grossman and Stiglitz, 1980): if price would reflect all information available, no trader would pay C, as they could learn information from the price; on the contrary, if no one does, then prices would reflect no information, and it would be profitable to buy information. An illustrative example (shown in Fig.2.3) takes place in case of large σ_y^2, moderate or large β, C comparable to earnings (Scenario A), σ_x^2 large or small. Agents periodically alternate according to how the paradox prescribes: when all uninformed, they realize that it would have been more profitable to be informed (by comparing the previous strategy performances), then all agents pay C and join I. Now, as response to N informed demands by the whole population, the price approaches the fundamental, revealing then all information available. Once there, anyone leaves I strategy because performing less profit than U one, so all agents return to be uninformed and price goes back to reflect only the estimations carried out so far without paying C.

2.5.4 Scenario B

This is the case in which the cost of information $C > C_0^* > \cdots > C_{t-1}^* > C_t^*$, $\forall t \in [0, T]$. The more time goes by, the greater C w.r.t. C_t^* is; in other words, C_t^* decreases as time passes by, so profits of Informed are more and more unprofitable and the whole population gradually joins Uninformed. Here the main reason of C_t^* decreasing is the progressive tendency of $(D^U - D^I) \to 0$, for $t \to \infty$, since σ_y^2 perturbation caused on uninformed learning is little relevant. β impacts driving this process quicker, since it acts making agents choose the best strategy even for a very small profit difference. An illustrative example of this settings in Fig. 2.4; I show the dynamics of the fraction of informed population, the price compared to its fundamental one and the profits of both strategies.

2.6 Numerical exploration when C is negligible w.r.t. wealth of Informed

Here I go through a preliminary exploration of all the possible market combination, considering three different agent demands: D^I for the informed expectation, D^U the uninformed one and D^R the recency (i.e. the heuristic expectation I take into account) one. Moreover, I take into account also another dividend process setting: the *structural break* case. The latter consists of a periodic switching of the dividends between two different GBM processes. However, a random switching instead of a periodic one doesn't impact on the models dynamics. The most mattering parameters to calibrate are still the ones mentioned in the subsection above. Anyhow, in

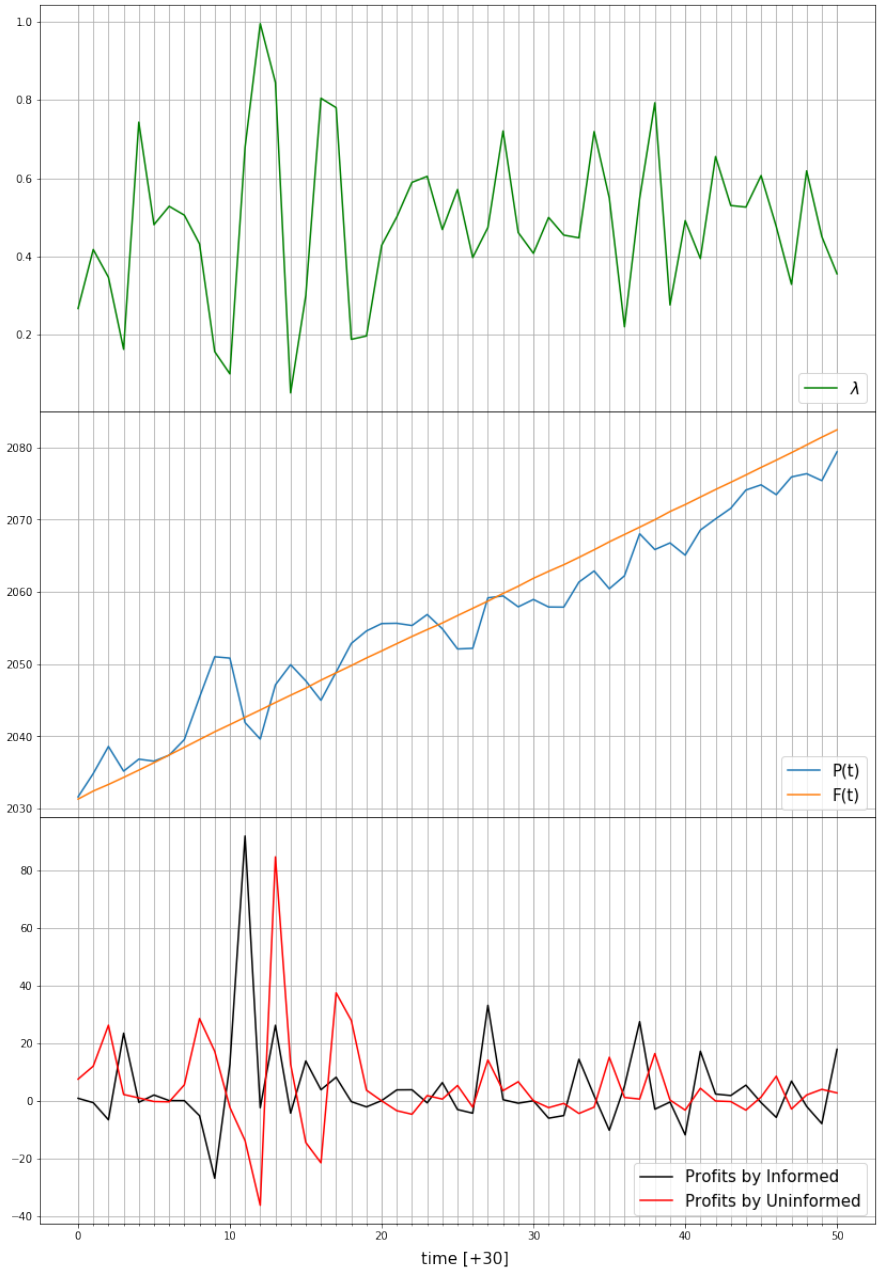


FIGURE 2.1: Here a representative simulation of Scenario A. In green the percentage of informed; in blue the price and in orange the fundamental; in black profits by informed and in red the ones by uninformed. Note that here, the maximum percentage deviation of price from fundamental value is about 0.6%. The plots should be read from top to bottom: at time t, fractions of I and U proportionally contribute to determine market price, alongside exogenous aggregate supply; once P_t is given, profits of the two strategies can be computed and, accordingly, new population fractions for next time step, $t + 1$.

this case I prefer an exploration of the parameters without considering the cost of information C (i.e. $C = 0$ for each time step). I have opted for this initial screening in order to calibrate all the models before including C impact; its later introduction in already calibrated models facilitates me for analytical assessing. Moreover, the case of $C = 0$ may be seen in the terms of informed agents that actually are so rich that

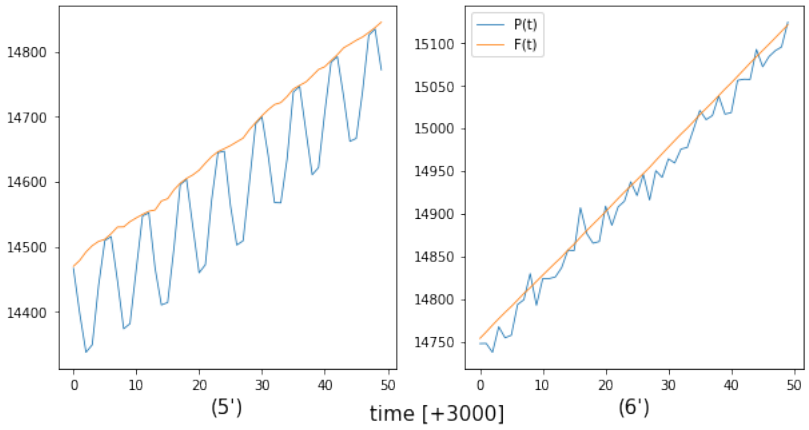


FIGURE 2.2: Here I show how price behaves w.r.t. fundamental in case of large σ_y^2 or large σ_x^2. Maximum percentage deviation of market price from fundamental value may reach about 20%. Both $5'$ and $6'$ below belong to a particular degenerate case of Scenario A, in which agents switch altogether every a few steps. Reading it from the C_t^* perspective, wide fluctuations in $5'$ are due to either the presence of y_t in the realised profits, or the difference in demands, since y_t and F_t are mattering. On the contrary, in $6'$ only the fuctuations in realised profits are mattering.

the cost of being informed is always negligible compared to their wealth. I discuss hereafter the numerical analysis of each possible market made of pairwise combinations of the different strategies together with a market in which all three strategies are available. I performed numerical analysis on four different market combinations. The cases I explored are three two-strategy markets, i.e. Informed vs Uninformed, Informed vs Recency, Uniformed vs Recency, and a three-strategy market, composed of Uninformed, Informed and Recency. Each of the latter is a different model and they may differ one another with respect to parameter settings. After wide parameter explorations employing several sampling methods (e.g. Sobol', NOLH, Random Sampling, etc) and sensitivity analysis, I identified the parameters are most impacting on model dynamics and outputs. Moreover, I calibrated each model taking into account model constraints that allow to set restrictions for the parameter ranges. As mentioned before, the parameters impacting the most are: $\sigma_y^2, \sigma_x^2, \beta$. Model dynamics and outputs are robust with respect to $K_I, K_U, K_R, n_{i,0}^S$ variation. Then, I am allow to keep them fixed without loss of generality. The couple μ, r, bound together by the relationship $\mu < r$, set the order of magnitude of the other parameters. After parameter screenings, I set their two values fixed for my numerical exercises. The latter are then numerically defining the order of magnitude of demands, performances, profits and, consequently, $\sigma_y^2, \sigma_x^2, \beta$. As I pointed out in Subsection 2.5.1, the cost of information is relevant with respect to a threshold which is directly depending on the different in demands and the realized returns; that means the order of magnitude of the model has to be made firstly without C, since C^* explicitly depends on $\sigma_y^2, \sigma_x^2, \beta$. Then, the first analysis it performed keeping $C = 0$. In particular, a crucial constraint for the parameter settings is the non-negativity of the equilibrium price generated at each time step by market clearing. This condition implies setting down further the bounds of the main drivers of model dynamics. Moreover, I tuned the agent intensity of choice β specifically for each model, in order to always keep agents at their optimal trading activity; the latter means that either they are not underreacting to

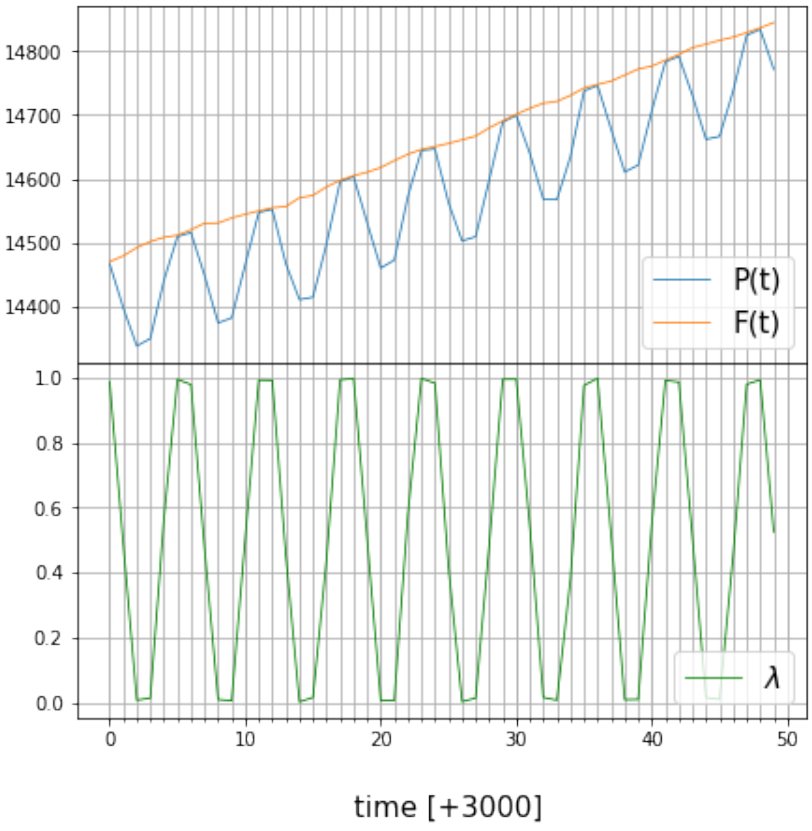


FIGURE 2.3: An illustrative example of *degenerate ecology*, which I argued it can be seen as the dynamical version of the Grossman and Stiglitz (1980) paradox. Prices cannot reflect all the information available to informed agents. Prices reveal information with some noise, making traders willing to but it.

the possibility of switching, nor overreacting, bringing the population moving from a strategy to the other always all at one. All these considerations led me to draw up the Table 2.2, where I indicated for each model its own calibrated parameter combination. In principle, it is not possible an absolute comparative analysis among all of them of strategy profitability, deviation of the equilibrium price from the fundamental one and the daily volatility of price; each different market has to be taken into account specifically. Anyhow, some models have very close parameter combinations and then it is possible to bring them into comparison. I carried out the same numerical exploration for a different setting of the dividend dynamics: dividends periodically switch between two different geometric Brownian processes, $\{y_{t,1}\}$ and $\{y_{t,2}\}$ (i.e. *structural break*); its results are shown in Table 2.3. I briefly discuss the results from the numerical analysis, either with a geometric Brownian process leading the dividends, or with dividends periodically switching between two different geometric Brownian processes.

In calibrating the eight different market cases where they exist, I notice that the models with a simple GBM leading dividends allow wider explorations of the parameter ranges with respect to the *structural break* case; the latter gives me a glimpse of analytical conditions for the models existence. An interesting perspective is the comparison of the different strategy combinations for the two settings. In the two-agent market composed by Informed and Uninformed, when the pure GBM leads dividends, no strategy overperforms the other; the market price does not deviate

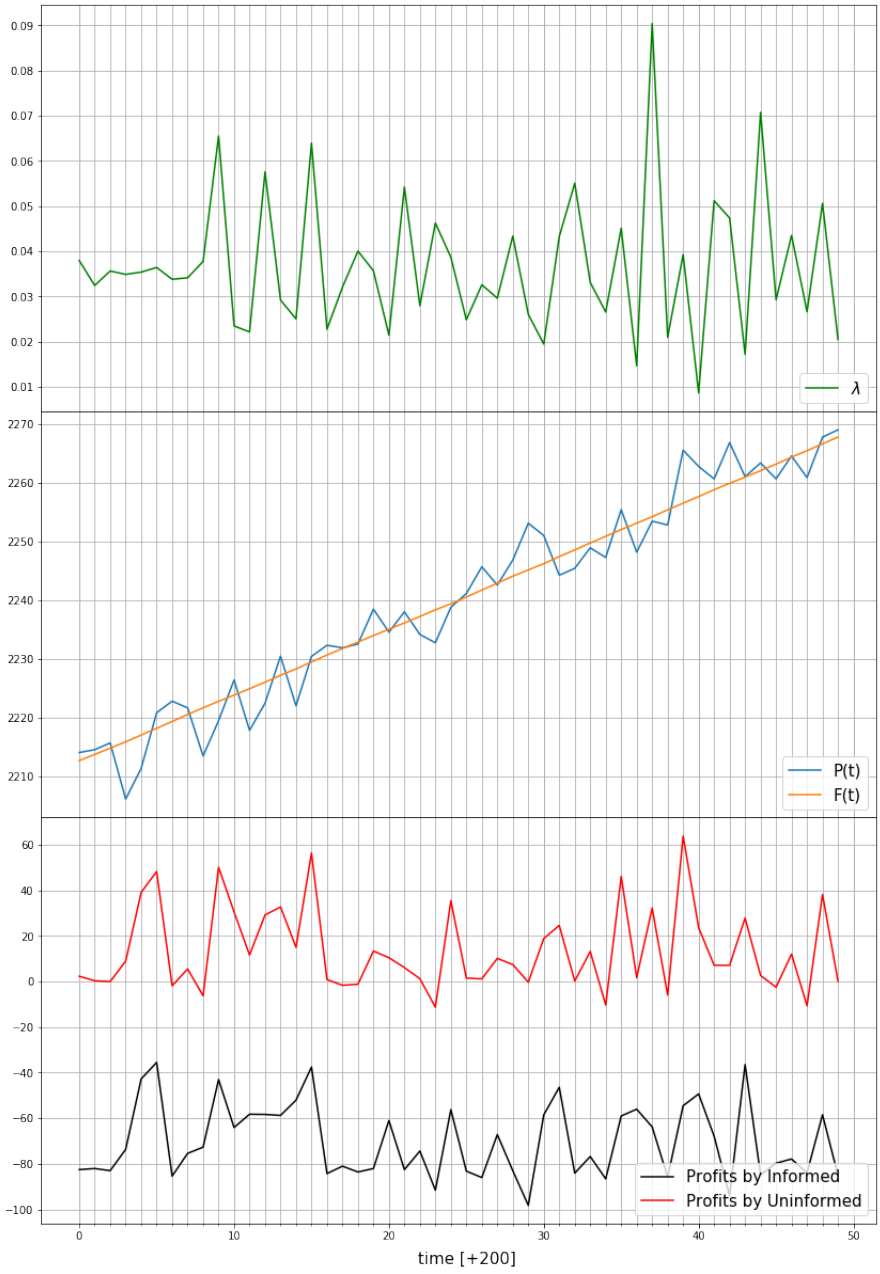


FIGURE 2.4: Here a representative simulation of Scenario *B*. In this range of simulation, the maximum percentage deviation of market price from fundamental value is about 0.5%. Since informed fraction is at most 9% in some spikes, their demand little contributes to price formation; price is then determined almost only by uninformed one and exogenously shifted by fluctuations of aggregate supply.

from the fundamental and price volatility increases as σ_y^2 increases. In the *structural break* setting, the Informed has a benefit in profits w.r.t. Uniformed. This advantage increases as the difference in the two dividends increases. Clearly, the more the two dividend processes get closer one another, the more *structural break* tends to pure GBM. In this case, market price deviates from fundamental of about -90% and price volatility reaches 40%. In the Informed-Recency market and pure GBM setting, no strategy overcomes the other, market price deviates from the fundamental of 50% and price volatility may reach 40%. In the *structural break*, the Informed

TABLE 2.2: Case of *pure* GBM dividend process: parameters involved in the simulation and their corresponding calibrated numerical ranges. Intensity of choice, β, has been the last one calibrated after μ, r, σ_y^2, σ_x^2. At the bottom, relative difference in profits between the strategies present in the market, price deviation w.r.t. the fundamental value and price volatility are reported.

Parameter	D^I **vs** D^U	D^I **vs** D^R	D^U **vs** D^R	D^I **vs** D^U **vs** D^R
μ	$5e-4$	$5e-4$	$5e-4$	$5e-4$
r	$1e-3$	$1e-3$	$1e-3$	$1e-3$
σ_y^2	$[0, 1e-7]$	$[0, 1e-7]$	$[0, 1e-8]$	$[0, 1e-7]$
σ_x^2	$[0, 1e3]$	$[0, 1e3]$	$[0, 1e3]$	$[0, 1e3]$
β	$[1.5e+1, 1.5e-4]$	$[4.01e-7, 3.7e-7]$	$[0, 3.52e-7]$	$[1e-7, 4.7e-7]$
Relative profit difference	$[0.5 \pm 0.0\%, 0.5 \pm 0.0\%]$	$[0.4 \pm 0.0\%, 0.7 \pm 0.0\%]$	$[0.4 \pm 0.0\%, 0.3 \pm 0.0\%]$	$[0.0 \pm 0.0\%, 0.0 \pm 0.0\%]$ $[-2.2 \pm 0.0\%, 0.0 \pm 0.0\%]$
P deviation w.r.t. F	$[0.0 \pm 0.0\%, 0 \pm 0.2\%]$	$[55.5 \pm 0.0\%, 54.6 \pm 0.0\%]$	$[55.5 \pm 0.0\%, 54.0 \pm 0.0\%]$	$[48.2 \pm 0.0\%, 38.3 \pm 0.1\%]$
volatility of P	$[0.0 \pm 0.0\%, 4 \pm 0.3\%]$	$[40.6 \pm 0.0\%, 31.4 \pm 0.7\%]$	$[42.4 \pm 0.0\%, 22.3 \pm 0.8\%]$	$[47.9 \pm 0.0\%, 8.7 \pm 0.7\%]$

TABLE 2.3: Case of *structural break* (i.e. dividends switch between two GBM processes periodically): parameters involved in the simulation and their corresponding calibrated numerical ranges. Intensity of choice, β, has been the last one calibrated after μ, r, $\sigma_{y,1}^2$, $\sigma_{y,2}^2$, σ_x^2. At the bottom, relative difference in profits between the strategies present in the market, price deviation w.r.t. the fundamental value and price volatility are reported.

Parameter	D^I **vs** D^U	D^I **vs** D^R	D^U **vs** D^R	D^I **vs** D^U **vs** D^R
μ_1	$3e-4$	$3e-4$	$6e-4$	$3e-4$
μ_2	$8e-4$	$8e-4$	$8e-4$	$8e-4$
r	$1e-3$	$1e-3$	$1e-3$	$1e-3$
$\sigma_{y,1}^2$	$[0, 1e-8]$	$[0, 1e-8]$	$[0, 1e-9]$	$[0, 1e-9]$
$\sigma_{y,2}^2$	$[0, 1e-8]$	$[0, 1e-8]$	$[0, 1e-9]$	$[0, 1e-9]$
σ_x^2	$[0, 1e3]$	$[0, 1e3]$	$[0, 1e3]$	$[0, 1e3]$
β	$[3.8e-6, 1.e-6]$	$[9.5e-7, 8.5e-7]$	$[4.2e-8, 3.6e-8]$	$[8.7e-8, 7e-8]$
Relative profit difference	$[25 \pm 0.0\%, 7.5 \pm 1.0\%]$	$[2.1 \pm 0.0\%, 1.0 \pm 0.0\%]$	$[-1.4 \pm 0.0\%, -1.2 \pm 0.0\%]$	$[2.0 \pm 0.0\%, 1.6 \pm 0.0\%]$ $[-0.3 \pm 0.0\%, -0.2 \pm 0.0\%]$
P deviation w.r.t. F	$[-60.0 \pm 0.0\%, -87.0 \pm 2.8\%]$	$[54.8 \pm 0.0\%, 53.0 \pm 0.0\%]$	$[23.5 \pm 0.0\%, 22.6 \pm 0.0\%]$	$[-25.0 \pm 0.0\%, -24 \pm 0.0\%]$
volatility of P	$[36.8 \pm 0.0\%, 43.4 \pm 0.5\%]$	$[71.0 \pm 0.0\%, 35.0 \pm 0.6\%]$	$[72.3 \pm 0.0\%, 59.2 \pm 0.5\%]$	$[75.3 \pm 0.0\%, 60.7 \pm 0.2\%]$

strategy overperforms the Recency one (anyway less than if it were Uninformed); as above, the more the two dividends are different, the wider this advantage is. Market price deviation from the fundamental and price volatility are the same. In the Uninformed-Recency market, the parameter ranges explorable are tighter than the two cases above: the market is more unstable and exists for small fewer parameter combinations. In the pure GBM case, no strategy overcomes the other, market price deviates of 50% from fundamental and price volatility of 30%; in the structural break case, Recency overcomes Uninformed, market price deviates from fundamental of 63% and price volatility is 63%. In a market composed by all three strategies I took into account, agents don't switch from a strategy to another as frequently as in the cases above; there are much less appealing profit opportunities over time. When a pure GBM leads dividends, no strategy overcomes the others, market price deviated form fundamental of 34% and price volatility is 6%. In the structural break case, the Informed has a small advantage on the Uninformed but Recency has also a small advantage in profit with respect to the Informed. Market price deviates from the fundamental as before and price volatility become 30%.

2.7 Conclusions

The nature of an efficient market rests on two main concepts: available information is already reflected in stock prices and traders cannot earn risk-weighted excess returns (Degutis and Novickytė, 2014). Rationality of agents is a key prerequisite for its

efficiency. In this perspective, an important argument is derived by Grossman and Stiglitz (1980). In this paper, rational expectation agents are given with the option of buying an information signal about an asset. Under a perfectly efficient scenario with a cost on the signal (i.e. information), nobody would have an incentive to purchase it.

This chapter explores these settings. To the best of my knowledge, my accurate dynamical reproduction of Grossman and Stiglitz (1980)' static settings through agent-based modeling is pretty original in literature, in particular from a theoretical perspective. Firstly, I set up a theoretical framework where efficient market hypothesis should definitely hold: all the agents know that the expected fundamental value of the risky asset evolves according to the expected dividends. Agents choose to adopt a *fundamentalist* trading strategy: they expect future asset price will converge to the fundamental one, so they trade accordingly to exploit possible imbalances. A crucial issue is to provide information a dynamical character. To this end, expected dividends, and accordingly fundamental price, play that key role; this way of modeling matches information characterization in the original model, including the idiosyncratic shock on the return. Moreover, I characterize the demand of an agent á la Grossman-Stiglitz and how informed and uninformed formally differ from each other. By performing both theoretical and numerical explorations, two different asymptotic scenarios arise: one in which the population distributes in half informed and half uninformed over time, and one in which the whole population joins uninformed strategy. This split crucially depends on the magnitude of information cost and it is controlled by a threshold which I derived analytically. In other words, an informed agent never performs better than an uninformed one and, if the cost of information is too high w.r.t. current earnings, it is driven out the market. In any case, market price follows the fundamental one. In this framework, I prove dynamically the paradox found by (Grossman and Stiglitz, 1980); i.e. if price would reflect all information available, no trader would pay the cost, as they could learn information from the price; on the contrary, if no one does, then prices would reflect no information, and it would be profitable to buy information. In line with noisy traders literature (De Long et al., 1990) and ecological rationality literature (Gigerenzer and Brighton, 2009; Gigerenzer and Gaissmaier, 2011), I bring into the model the heuristic rule which makes agents forming their expectations at each time on the previous observed return (i.e. *recency* heuristic). Their presence destabilizes the market, making market price deviating from fundamental value and price volatility dramatically increasing. Moreover, also in this case the informed agent never performs better that others and for a price of information higher than a certain price threshold (that can be analytically derived) it is ruled out of the market. Uniformed and heuristic agents survive in the market and the dynamic matches Fehr and Tyran (2005)' findings: strategies may be either complements or substitutes is the actual main driver in determining aggregate outcomes. I found that when the two strategies are complements, even a small amount of heuristic makes aggregate outcomes diverge from rational ones. On the contrary, when agents' strategies are substitutes, even a small amount of fundamental agents leads to a convergence to rational outcomes.

Chapter 3

Modern Tools for Agent-Based Model Sensitivity Analysis

3.1 Introduction

This chapter[1] suggests a new approach to model calibration, parameter space exploration and sensitivity analysis in agent-based models (ABMs). It couples supervised machine learning, ad-hoc sampling (because of ABMs constrains) together with traditional and machine-learning sensitivity analysis methods in the development of a meaningful parameter exploration in order to produce a reliable surrogate metamodel that is a computationally cheap approximation to the real model. The latter can be then employed for sensitivity analysis purposes with almost zero computational effort.

ABMs are concerned with the investigation of socio-ecological systems that can be accurately conceptualised through a set of micro- and macro-relations. An inherent problem of this approach is that the statistical properties of interest are unknown a priori, even to the designer of the model. Indeed, such features arise from repeated interactions involving ecologies of heterogeneous, boundedly rational and adaptive agents. This leads to dynamic features that cannot be studied analytically, cause-and-effect mechanisms that cannot always be properly identified, and emergent relationships that cannot be simply inferred by aggregating micro-level interactions (Anderson (1972); Gallegati and Kirman (2012); Grazzini (2012); Tesfatsion and Judd (2006)). Consequently, there is a need for suitable tools to explore the model's emergent behaviour in terms of different parameter settings, random seeds and initial conditions.

The main challenge in ABM parameter space exploration and calibration is the increasing number of parameters arising from the increasingly realistic dynamics of ABMs. For instance, more recent macroeconomic models employ dozens of parameters to account for the complexity of micro-founded, cross-sectoral and cross-country phenomena (see Roventini and Fagiolo (2017), for a recent review). Existing direct estimation and global sensitivity analysis tools (often advocated as a natural approach to explore ABM (Ten Broeke et al. (2016); Thiele et al. (2014)), are computationally unaffordable and require time and computational resources often not at the disposal of researchers or practitioners. This growth in the size of the parameter set leads to what has been called " the curse of dimensionality", i.e. the convergence of any estimator to the true value of a smooth function defined in a high dimensional

[1] Ottaviani M., Sani A., Lamperti F., A. Roventini (2021). "Modern Tools for Agent-Based Model Sensitivity Analysis", *forthcoming*.

parameter space is extremely slow (De Marchi and De Marchi (2005)). There is potentially an exponential number of local critical values in parameter space that can be mistaken for global maxima or minima.

Calibrating ABMs conventionally involves three computationally intensive steps: running the model, calibrating quality, and identifying the parameters of interest (for more on validating ABMs, see Fagiolo et al. (2019)). As Grazzini et al. (2017a) note, these steps account for more than half of the time needed to estimate ABMs, even for very simple models. Thus, suitable tools need to be developed to search quickly for "meaningful" parameters and initial conditions. An approach is to use a cheaper proxy to replace the computationally intensive ABM. This is the goal of metamodels or surrogates that approximate the relationship between the inputs and outputs of ABMs (Fagiolo et al. (2019); Lee et al. (2015)) to explore the parameter space quickly. Traditionally, surrogate models are used as fast approximations of complex phenomena that are expensive to evaluate in real world or simulation (Booker et al. (1999)), and are regularly used to find promising parameter combinations and avoid expensive computations. If the error of approximation is small, the surrogate can be considered a suitable substitute for the original ABM in parameter space exploration, calibration and sensitivity analysis.

Kriging and XGboost have recently been implemented as a surrogate modelling method to support the exploration of the parameter space and sensitivity analyses of ABMs (Bargigli et al. (2020); Dosi et al. (2018a,b); Lamperti et al. (2018); Salle and Yildizoglu (2014a)). If, however, the response surface of the model is unknown entirely and may contain non-smooth regions, as is typically the case for ABMs, kriging and XGboost require a high number of evaluations and large exploratory data analyses that scale up with the parameter space size. Such limitations also apply to state-of-the-art extensions (Herlands et al. (2016); Wilson et al. (2015)), requiring large-scale ABM modellers to arbitrarily specify a subset of parameters when the parameter space is large (Barde and Van Der Hoog (2017).

Lamperti et al. (2018) suggested a novel approach which "learns" a quick surrogate metamodel using a restricted number of ABM valuations and approximates the non-linear relationship between inputs and outputs. Performance is measured using the asset pricing model of Brock and Hommes (1998a) and the endogenous growth model "Islands" of Fagiolo and Dosi (2003). The findings show that the machine learning surrogates obtained with the proposed iterative learning procedure replace the actual model fairly precisely and drastically reduce the computational time required for the exploration and calibration of the parameter space on a large scale. Sensitivity Analysis (SA hereafter) are precious tools when one builds and uses numerical methods. They allow to study how the uncertainty in the output of a model can be apportioned to different sources of uncertainty in the model input A Saltelli and Scott (2000). Through the identification of the parameter or set of parameters that have the greatest influence on the model output, SA is very useful for the determination of the most contributing input variables to an output behavior as well as the non-relevant inputs and also the assessment of interaction effects within the model. SA application can be summarized as four main steps: (i) understanding the input–output relationship, (ii) determining to what extent uncertainty in structural model parameters contribute to the overall variability in the model output, (iii) identifying the important and influential parameters that drive model outputs and magnitudes, and (iv) guiding future experimental designs Iooss and Lemaître (2015); Kiparissides et al. (2009); Zhang, Trame, Lesko, and Schmidt (Zhang et al.). From model builders and users perspectives, it is also a useful tool to check the model structure, uncertainty around the input parameters and feedback

into the model refinement to gain additional confidence in the model. Especially in a very complex model, the results of sensitivity analysis will help the model builders to focus on the critical parameters that determine the model output Zhang, Trame, Lesko, and Schmidt (Zhang et al.). Since their usage directly on an ABM may result computationally prohibitive, an attempt is to employ meta-models in order to replace the original model with a cheaper proxy which whose computational effort is way more affordable. I look for an 'hand-off' approach to employing meaningful sensitivity analysis methods together with accounting for limited computational resources of the researcher/practitioner. As illustrative examples, I explore the same asset-pricing model of Brock and Hommes (1998a) and the limit-order book model of Franke and Westerhoff (2012a). Despite their apparent simplicity, both models might exhibit multiple equilibria, let several behavioural attitudes and allow a pretty broad range of dynamics, which essentially depends on their inputs. My approach is made up with the precise purpose of being general, without either knowledge of the model output surface or parametric assumptions.

Along the line of Lamperti et al. (2018), I treasure their results in order to set up a cheap way of performing sensitivity analysis without losing of generality of any kind of agent-based model. Since my two illustrative examples are two financial models, I choose as outputs four representative stylized facts of financial markets. In this way, either I obtain a very rich description of model outputs, or I can calibrate them at the same time. The first key limitations in the parameter space exploration lies on the impossibility of employing already set up gridded sampling methods (e.g. Sobol', LOAH, etc). The reason why they are not working has to be found in the nature of agent-based models. ABMs are stochastic systems; they are strongly non-linear and they present tipping points. Some parameter combinations automatically computed by gridded sampling methods may not be available. This limitation drives me to the design on an intelligent sampling method in order to obtain meaningful parameter combinations to explore. Still exploring the parameter space by performing the most common local sensitivity analysis method (i.e. One-Factor-A-Time, OFAT), I realize how dangerous and misleading can be even the best calibrated parameter combination. Parameter ranges may present areas in which, for the same inputs, the model can behave in very far away manners. This phenomenon has again its reason in the ABM nature. The existence of this ambiguity makes the training of any kind of surrogate model extremely hazardous and misleading. A robustness assessing of model outputs for the same parameter combinations is absolutely essential. I propose a novel self-consistent approach to analyze parameter ranges robustness and point out stable areas over which perform either sensitivity analysis or surrogate model training. I called it Monte Carlo-OFAT; it leverages the OFAT sensitivity analysis method combining it with a Monte Carlo replica robustness assessing and finally pointing out trustworthy areas of parameter ranges though a normality test. The meta-models I employ are three different versions of Kriging and XGboost. I show how their prediction power dramatically increases after the introduction in the analysis of my proposed approach MC-OFAT. Moreover, I propose a further design approach which increases much more surrogate performances. That is a further selection in the pool of parameter combinations, this time based on model outputs. I employ three different global analysis methods on the two illustrative ABMs: the standard and most common Sobol' first order index (Sobol (1993)), Borgonovo's Delta approach (Borgonovo (2007); Plischke et al. (2013)) and XGboost-based Shapley effects assessing Shapley (1953); Song et al. (2016). Also in this case, the application of my proposed approach MC-OFAT makes surrogate performances dramatically better than without its usage. In the end, I conclude that MC-OFAT

approach lets the design of solid, resilient and trustworthy surrogate models and I show they can reliably replace the original model for assessing the most impacting parameters in the model though global sensitivity analysis methods.

Letting replace the original ABM by its cheap meta-model version in a delicate task like performing sensitivity analysis gives enormous room of manoeuvre to a researcher or practitioner in a policy making perspective.

3.2 Literature Review

3.2.1 Agent Based Model Calibration

The relationships between ABMs and real-world data are fundamental in informing methodological research and theoretical analysis.Recently, many studies dealt with the issue of estimating and calibrating ABMs Fagiolo et al. (2017). As well-described by Chen et al. (2012), ABMs world needs to translate with proper tools the qualitative capability to grow stylized facts to a quantitative perspective in which parameters are chosen according to sound econometric methods. In those situations in which the model is quite simple and well behaved, it's allowed to derive a closed form solution for the properties of the distribution of a certain model output, and then estimating the parameters determining such distributions Alfarano et al. (2005, 2006); Boswijk et al. (2007). However, when model complexity prevents analytical solutions, more sophisticated techniques are required. Amilon (2008) estimates a model of financial markets with 15 parameters (with only 2 or 3 agents) using the method of simulated moments[2], reporting high model sensitivity to assumptions made on the noise term and stochastic component of the procedure. Gilli and Winker (2003) and Winker et al. (2007) introduce an algorithm and objective function to estimate exchange-rate models by indirect inference[3], pushing them closer to the properties of real data. Franke (2009) refines on this framework to estimate 6 parameters of an asset pricing model. Franke and Westerhoff (2012b) propose a model contest over structural stochastic volatility models, but the models are defined by only a few parameters.[4] Finally, Recchioni et al. (2015) use a simple gradient-based procedure for calibration, evaluating performance based on out-of-sample forecast errors.

A common limitation shared by all these methods is the cost of simulating ABMs. As well discussed in Grazzini et al. (2017b), simulating the ABM is the most expensive step in calibration, estimation and validation; lots of parameter vectors, initial conditions and seeds of the pseudo-random number generators may hinder proper model evaluations.In addiction, the understanding of model's response to possibly combined variations in some parameter values or initial conditions is essential to evaluate the robustness of models' output as well as to draw robust conclusions from policy simulations. It's the aim of Sensitivity Analysis; however, the latter in ABM often implies high computational costs deriving by simulating the model. Salle and Yıldızoglu Salle and Yildizoglu (2014b) were the firsts suggesting to combine "design of experiments" and surrogate modelling in order to tackle the problem in economics literature. The strategy they propose is straightforward. DoE allows to minimize the sample size of parameter configurations under the constraint on their

[2]The method of simulated moments was introduced as an approach to estimating moment functions when they can not be evaluated directly. See Franke and Westerhoff (2012b); Gilli and Winker (2003) for more information on its use in the macro literature.

[3]Note that the method of simulated moments is a form of indirect inference.

[4]See also Grazzini and Richiardi (2015) and Fabretti (2012) for other applications of the same approach.

representativeness. Based on the data collected through that sample, the original model is approximated with a meta-model, which is then employed to connect the parameters to the variables of interest at virtually zero computational costs. Building on such an approach, Dosi et al. (2016) provide a global sensitivity analysis for a relatively simple model of industry dynamics. Meta-modeling was achieved through kriging (or Gaussian process regression); it is a simple and efficient method for investigating the behavior of simulation models (Krige, 1951).

This spatial interpolation technique estimates the ABM response over the full parameter space from a finite sample of ABM evaluations to generate the best unbiased linear predictor through knowledge of the true variogram or true degree of spatial dependence in the data. In the case of spatially homogeneous data, kriging only requires 30 points to estimate the spatial structure. However, when the spatial distribution of the data is unknown, as is often the case with ABMs, kriging requires specialist knowledge of variography to empirically estimate the spatial dependence of the data. This generally requires a large number of ABM evaluations and extensive exploratory data analysis that increases with the size of the parameter space. Unfortunately, the performance of any kriging model depends on the accuracy of estimating this true variogram, as the empirical variogram asymptotically converges to the true one when the number of ABM evaluations reaches infinity.

Very recently, Lamperti et al. (2018) have shown that reducing computational time can be achieved in a meaningful way by efficiently training a surrogate model over multiple rounds to approximate the mapping between ABM inputs and the response of the ABM output to a user-defined calibration criterion. In the latter, kriging meta-modeling is replaced by XGBoost, a surrogate machine-learning approach, allowing to overcome above-mentioned kriging limitations at negligible computational costs. That procedure has some similarities to the one of Dawid et al. (2014), where penalized splines methods are employed to shortcut parameter exploration and unravel the dynamic effects of policies on the economic variables of interest. However, Lamperti et al. (2018) especially focuses on computational efficiency and therefore builds on two pillars: surrogate modelling and intelligent sampling. In particular, no parametric assumptions or knowledge of the topology governing the spatial distribution of the data is required. Just to give a flavour of the procedure presented by Lamperti et al. (2018), it can be summed up in the following manner: first, by drawing a relatively large pool of parameter combinations using any standard sampling routine, where each combination contains a value for each initial condition. This pool acts as a proxy for the full parameter space. Next, a small random subset of combinations are drawn without replacement from the pool to initialize the learning procedure. The ABM is then evaluated for each of these initial combinations and its outputs receive a label. Those outputs satisfying a user-defined calibration criterion are assigned to a positive category, otherwise to a negative one. A surrogate is then learned over the combinations using the selected surrogate algorithm. The first surrogate is used to predict the probability that unlabeled combinations in the pool belong to the positive category. This concludes the first round. In the second and subsequent rounds, a very small subset of the pool is drawn according to the predicted positive probability. These selections are evaluated in the ABM to learn their true labels and aggregated to the set of all other combinations that have been sampled during the previous rounds. This continues over multiple rounds a predefined level of performance is achieved.

Many works form different fields enhanced discussions on sensitivity analysis tools, coming to very powerful methods, also in case of input dependance; up to now, the latter is the most difficult case to interpret. For example, the well-known

Sobol' indices, which are based on the functional variance analysis, present a difficult interpretation in the presence of statistical dependence between inputs.

3.2.2 Sensitivity Analysis

Sensitivity Analysis (SA hereafter) are precious tools when one builds and uses numerical methods. They allow to study how the uncertainty in the output of a model can be apportioned to different sources of uncertainty in the model input (A Saltelli and Scott, 2000). Through the identification of the parameter or set of parameters that have the greatest influence on the model output, SA is very useful for the determination of the most contributing input variables to an output behavior as well as the non-relevant inputs and also the assessment of interaction effects within the model. The purposes of SA are numerous; few examples could be: model verification and understanding, model simplifying and validation of a computer code (Iooss and Lemaître, 2015). Sensitivity analysis has been widely used in many fields such as risk assessment, economics, engineering, biology and so on; its application can be summarized as four main steps: (i) understanding the input–output relationship, (ii) determining to what extent uncertainty in structural model parameters contribute to the overall variability in the model output, (iii) identifying the important and influential parameters that drive model outputs and magnitudes, and (iv) guiding future experimental designs (Iooss and Lemaître, 2015; Kiparissides et al., 2009; Zhang, Trame, Lesko, and Schmidt, Zhang et al.). From model builders and users perspectives, it is also a useful tool to check the model structure, uncertainty around the input parameters and feedback into the model refinement to gain additional confidence in the model. Especially in a very complex model, the results of sensitivity analysis will help the model builders to focus on the critical parameters that determine the model output (Zhang, Trame, Lesko, and Schmidt, Zhang et al.).

Moreover, these tools have obtained large interests by numerical model users and modelers for industrial and environmental applications, taking full advantages of the advent on computing materials and numerical methods (E de Rocquigny and Tarantola, 2008; Helton, 1993). According with A Saltelli and Ratto (2004) and F Pappenberger and Vandenberghe (2010), before making a SA, the following goals need to be plainly specified (Iooss and Lemaître, 2015): identify and prioritize the most influential inputs; identify non-influential inputs in order to fix them to nominal values; map the output behavior in function of the inputs by focusing on a specific domain of inputs if necessary; calibrate some model inputs using some available information (e.g. real output observations, constraints). Three kinds of SA methods can be distinguished:

- *screening*: rough sorting of the most relevant inputs among a large number of them;
- *local* : importance measures of input variables for small perturbations, basen on partial derivatives;
- *global*: deep exploration of the model behaviour by ranking input random parameters with respect to their effects in the output uncertainty.

Screening methods refer to a discretization of the inputs in levels allowing, in this manner, a fast exploration of the code; their aim is the "screening" of non-influential inputs with very few model calls while making proper hypotheses on the model complexity. The local approach, instead, is the first historical approach to Sensitivity Analysis. Local sensitivity analysis evaluates changes in the model outputs with

respect to variations in a single parameter input. The input parameters are typically changed one at a time in relatively small increments and the effect of this individual parameter perturbation on the model output is calculated using local sensitivity indices. It consists fundamentally in calculating or estimating the partial derivatives of the model at a specific point (Iooss and Lemaître, 2015) Local sensitivity analysis may only be used when the model output is linearly related to the parameters near a specific nominal value (the mean of a random variable for instance). The main limitation of a local sensitivity analysis is that it evaluates parameters one at a time, and it does not allow for the evaluation of simultaneous changes in all model parameters. In addition, the interaction between parameters cannot be evaluated using a local sensitivity analysis Zhang, Trame, Lesko, and Schmidt (Zhang et al.).

From the late eighties, to overcome the limitations of local methods, such as linearity and normality assumptions, a new class of methods has been developed in a statistical framework. It is referred to as global sensitivity analysis since it takes into account the whole variation range of the inputs, in contrast to local sensitivity analysis (A Saltelli and Scott, 2000). In this paper, I am particularly interested in the latter. In a global sensitivity analysis, by varying at the same time all parameters over the entire parameter space, simultaneous evaluations of the relative contributions of each individual parameter, as well as the interactions between parameters, can be carried out, in order to investigate the model output variance. However, for a wider overview about global sensitivity anaylsis I suggest Iooss and Lemaître (2015). Among global SA methods, variance-based ones are a class of probabilistic approaches that measure the share of variance of the model output which is a consequence of the variance of a particular input Benoumechiara and Elie-Dit-Cosaque (2018). The latter were popularized by Sobol (1993) who introduced the famous first-order *Sobol' indices*.

The first-order Sobol index of X_i, denoted S_i, represents the amount of the output variance solely due to X_i. The second-order Sobol index (S_{ij}) expresses the contribution of the interactions of the pairs of variables X_i and X_j, and so on for the higher orders. Thereafter, Homma and A.Saltelli (1996) introduced the total-order Sobol' index: it measures the contribution to the output variance of X_i taking into account any possible interaction, of any order, with any combination of other input variables. However, it is worth to underline that Sobol' sensitivity analysis is not aimed at identify reasons of input variability; it just shows impact and size on model output. As a matter of fact, it cannot be used to identify sources of variance. One of the main points to be well-established in any sensitivity analysis, either local or global, is the definition of the model output to be used. Sobol SA's major advantages are, for instance, the feature that no assumption is needed between model input and output and the opportunity of evaluating the full range of each input and their interactions. At the contrary, the main drawback is the high computation request (Zhang, Trame, Lesko, and Schmidt, Zhang et al.).

As it will be well-explained in **??**, these sensitivity indices are built on the functional ANalyse Of VAriance (ANOVA), which is unique only under the assumption of independence between the input variables. Anyhow, this hypothesis is sometimes not verified, letting their interpretation much harder to understand. Few works dealt with this issue and extended Sobol' indices to the case of a stochastic dependence among input variables, as Benoumechiara and Elie-Dit-Cosaque (2018); Kucherenko et al. (2012); Mara and Tarantola (2012); Mara et al. (2015). As a matter of fact, very commonly input variables related to one another through a statistical dependence structure; it could be imposed, for example, either by a probabilistic dependence function (Kurowicka and Cooke, 2006), like a copula function, or physical

constraints upon the input or the output space (A Lopez-Benito, A Lopez-Benito; S Kucherenko, 2017). The estimation and interpretation, in similar cases, are not easy at all; there were several suggestions in literature but, unfortunately, they are not always easy to interpret (A Saltelli, 2002; S Da Veiga, 2009).

3.2.3 Local Sensitivity Analysis

One-factor-at-time (OFAT)

One-factor-at-time sensitivity analysis refers to locate a benchmark parameter combination and varying one input at a time while keeping fixed all the other ones (see, for example, Campolongo et al. (2007)). The latter is the reason why is referred to a local procedure. The main and crucial use of this sensitivity analysis method lies in uncovering form and shape of the relationship between the varied input and the model output, keeping all the other parameters at their benchmark values. For example, the latter may show whether the model response is linear or not, or whether there exists tipping points where the model response changes dramatically even for small input variations. Moreover, a benchmark parameter combination may be the one matches the most with observed data in order to calibrate a model describing real world as in my case. For this reason, a local analysis in this direction is worth it and I perform it in the first place. In order to locate a benchmark parameter combination, I draw 10000 samples and I choose the one closest to observed data features.

3.2.4 Global Sensitivity Analysis

Sobol' indices

Let us consider the input vector of the model $\mathbf{X} = (X_1, \dots, X_d) \in \mathbb{R}^d$, where the variables are mutually independent; each input is considered to range over some finite interval which can be assumed, after rescaling, to be in $[0, 1]$. I restrict the study to a scalar output $Y \in \mathbb{R}$ of a deterministic model $f(\cdot)$:

$$Y = f(\mathbf{X}) \; ; \tag{3.1}$$

in my model, $f(.)$, is a square-integrable function and it is defined on the unit hypercube $[0, 1]^d$. In my context, $\mathbf{X}$ is a random vector defined by a probability distribution $p(\mathbf{X})$ which follows a d-dimensional uniform distribution; for simplicity, I have dropped the pdf's. However, these results can be extended to any marginal distributions.

The main idea of Sobol' method consists of variance decomposition into contributions from effects of single inputs, joint effects of input pairs etc. The Hoeffding decomposition, introduced in Hoeffding (1948) and known also as high dimensional model representation (HDMR), allows to decompose $f(\mathbf{X})$ as a sum of elementary functions:

$$f(\mathbf{X}) = f_0 + \sum_{i=1}^{d} f_i(X_i) + \sum_{i<j}^{d} f_{ij}(X_i, X_j) + \cdots + f_{12\dots d}(\mathbf{X}) \; , \tag{3.2}$$

for some $f_\emptyset, f_i, \dots, f_{1,\dots,d}$ set of functions. In this expression, f is decomposed in 2^d terms. Due to the infinite possible choice for the latter, the expansion (Eq. 3.2) is not unique. Thanks to the following orthogonality constraint, shown in Sobol (1993), the

unicity condition is ensured by:

$$\int_0^1 f_{i_1\dots i_s}(x_{i_1},\dots,x_{i_s})dx_{i_k} = 0\,, \quad 1 \le k \le s\,,\ \{i_1,\dots,i_s\} \subseteq \{1,\dots,d\}\,.$$

This implies that f_0 is a constant. Furthermore, the crucial consequence of this condition is that all the terms in (Eq. 3.2) are orthogonal each other. This property means mutually independence among random variables X_i and let us express the functions $f_{i_1\dots i_s}$ in Eq. 3.2 in the following way:

$$f_\varnothing = \mathbb{E}(Y), \tag{3.3}$$
$$f_i(X_i) = \mathbb{E}_{\mathbf{X}_{-i}}(Y|X_i) - \mathbb{E}(Y), \tag{3.4}$$
$$f_{i,j}(X_i,X_j) = \mathbb{E}_{\mathbf{X}_{-ij}}(Y|X_i,X_j) - f_i - f_j - \mathbb{E}(Y) \tag{3.5}$$

where $\mathbf{X}_{-i}$ is the vector $(X_1,\cdots,X_d)$ not containing X_i. As a result of the uniqueness condition above, by squaring and integrating both sides of Eq. 3.2, a functional decomposition of the variance (often called ANOVA) is available (Efron and Stein, 1981):

$$\mathrm{Var}(Y) = \sum_{i=1}^{d} D_i(Y) + \sum_{i<j}^{d} D_{ij}(Y) + \cdots + D_{12\dots d}(Y) \tag{3.6}$$

where $D_i(Y) = \mathrm{Var}[f_i(X_i)]$, $D_{ij}(Y) = \mathrm{Var}[f_{i,j}(X_i,X_j)] - D_i(Y) - D_j(Y)$ and so on for higher order interactions (Iooss and Prieur, 2018). The Sobol sensitivity indices (Sobol, 1993) are now deduced by dividing with $\mathrm{Var}[Y]$ both sides of Eq. 3.6 :

$$S_i = \frac{D_i(Y)}{\mathrm{Var}(Y)}, \quad S_{ij} = \frac{D_{ij}(Y)}{\mathrm{Var}(Y)}, \quad \cdots \tag{3.7}$$

For example, the first Sobol' index S_i means the first-order contribution from i_{th} input to the variance of Y; the second one, instead, expresses the second-order contribution from interaction between i_{th} and j_{th} inputs. Finally, total-order sensitivity indices, introduced by Homma and A.Saltelli (1996), are defined as the sum of all the indices as follows:

$$\begin{aligned} S_{T_i} &= S_i + \sum_{i<j} S_{ij} + \sum_{j\neq i, k\neq i, j<k} S_{ijk} + \dots + S_{1,\dots,d} \\ &= 1 - \frac{\mathrm{Var}_{\mathbf{X}_{-i}}[\mathbb{E}_{X_i}(Y|\mathbf{X}_{-i})]}{\mathrm{Var}(Y)} = \frac{\mathbb{E}_{\mathbf{X}_{-i}}[\mathrm{Var}_{X_i}(Y|\mathbf{X}_{-i})]}{\mathrm{Var}(Y)}\,. \end{aligned} \tag{3.8}$$

The last identity is due to a known result coming from the law of total variance (Mood et al., 1974). In practice, it measures the first and higher order effects, the interactions, of input X_i. One manner to visualize this is by taking into account that $\mathrm{Var}_{\mathbf{X}_{-i}}[\mathbb{E}_{X_i}(Y|\mathbf{X}_{-i})]$ is the first order effect of X_{-i}, so that $V(Y)$ minus it has to give the contribution of all terms in the variance decomposition which do include X_i. As a matter of fact, given that the number of variance shares due to ANOVA decomposition with d inputs grows as 2^d, one usually compute only the d first-order effects and the d total ones, in that way giving a good information on the model sensitivities (Iooss and Prieur, 2018). As mentioned befofe, Eq. (3.6) holds only if the X_is are independent; different approaches exist to deal with the case of dependent input. Just to give the idea about one of them, in T.Mara (2015), for example, the authors propose a strategy based on the estimation of four sensitivity indices per input, namely $S_{(i)}$, $S_{T_{(i)}}$, $S^{\mathrm{ind}}_{(i)}$ and $S^{\mathrm{ind}}_{T_{(i)}}$. $S_{(i)} = S_i$ and $S^{\mathrm{ind}}_{T_{(i)}} = S_{T_i}$ are the classical

Sobol' indices, while $S_{T_{(i)}}$ and $S^{\text{ind}}_{(i)}$ are new ones that can be expressed by means of Rosenblatt transform (M.Rosenblatt, 1952). In particular, two different types of indices are identified:

- the *full* Sobol' indices, $S_{(i)}$ and $S_{T_{(i)}}$, which include for each iput its dependencies with other inputs;
- the *independent* Sobol' indices, $S^{\text{ind}}_{(i)}$ and $S^{\text{ind}}_{T_{(i)}}$, which describe the impact of each input without its dependencies with other ones.

Note that $S^{\text{ind}}_{(i)} \leq S^{\text{ind}}_{T_{(i)}} = S_{T_i}$ and that $S_i = S_{(i)} \leq S_{T_{(i)}}$, but other inequalities are not known. In order to obtain a more detailed discussion about dependent Sobol' indices I send to thw works of Iooss and Prieur (2018) and Benoumechiara and Elie-Dit-Cosaque (2018).

In order to compute the above-mentioned two indices, from the practical point of view, it's due to Saltelli et al. (2010) the most comprehensive comparison among different Monte Carlo based estimators, experimental designs and sampling methods, testing all of them by numerical simulations. In particular, Saltelli et al. (2010) discusses existing estimators to compute in a single set of simulations, that is an individual output value of a sampled input set, compares them, show the best design and proves that Jansen's method Jansen (1999) is the best among the available ones. For exploring the input factor space in this case, moreover, the most suitable sampling method is an updated version of Sobol' quasi-random sequence (Saltelli et al., 2010; Sobol, 2001). The latter is specifically designed to generate samples of input variables as uniformly as possible in the unit hypercube. Unlike random numbers, successive quasi-random points take into account the position of previously sampled points and draw the new ones into the gaps between them (they are not random then). Sobol' sequences, characterized by what is called low 'discrepancy', outperform raw Monte Carlo sampling in my particular case (Sobol and Kucherenko, 2009).

The just described optimal operational way to compute $S_{(i)}$ and $S^{\text{ind}}_{T_{(i)}}$ is the one taken into account for my analysis, shown in the Section 3.3, within *SAlib*, a collection of sensitivity analysis libraries in Python (Herman and Usher, 2017).

Shapley effects

The aim of Sobol' indices is the decomposition of $\text{Var}(Y)$ allocating to a subset of inputs while the Shapley effects decompose the total variance and allocate it to each single input. This distinction allows to consider any inputs without taking account of their dependence with the other ones. In cooperative games theory, an important issue is to define a significant way to distribute the earnings for each player. Shapley (1953) proposed a fair allocation of earnings between d players. According to E Song (2016), a d-player game is defined as a real-valued function which maps a subset of D players ($D = \{1, 2, \ldots, d\}$) to its related cost; i.e., $c : 2^D \mapsto R$ with $c(\emptyset) = 0$. Hence, $c(J)$ represents the cost that results from the participation of J players to the game. The Shapley value of player i with respect to the cost function $c(\cdot)$ is :

$$Sh_i^q = \sum_{J \subseteq D \setminus \{i\}} \frac{(k - |J| - 1)!|J|!}{d!} \left(c\left(J \cup \{i\}\right) - c\left(J\right)\right) \ . \tag{3.9}$$

In other words, it is the incremental cost of including player i in set J averaged over all sets $J \subseteq D \ \{i\}$ for a given data instance q. In order to clarify the connection with

my aims of learning and prediction: the 'game' is the prediction goal for a single instance of the dataset (i.e. a certain parameter configuration, a sample); the 'gain' is the real prediction for the instance minus the average prediction of all samples; the 'players' are the feature values of the instance, which work together to obtaining the gain (i.e. predict a certain value). Considering the set of inputs of $f(\cdot)$ as the set of players D, Eq. 3.9 can be applied to global sensitivity analysis (Owen, 2014). In order to do that, I need a cost function that verifies $c(\emptyset) = 0$ and $c(D) = \text{Var}(Y)$ and, for $J \subseteq D$, $J \subseteq D$, $c(J)$ must coincide to the part of variance of Y caused by the uncertainty of the inputs in J.

E Song (2016) proved that it is equivalent to define $c(\cdot)$ as $\mathbb{E}[\text{Var}(Y|\boldsymbol{X}_{-D})]/\text{Var}(Y)$ or as $\text{Var}(\mathbb{E}[Y|\boldsymbol{X}_D])/\text{Var}(Y)$. The Shapley effects rely on a well-balanced allocation of a variance share of the output to each input. Then, it is not any linear combination of Sobol' indices; actually, the primer equitably shares interaction effects of a subset of inputs with each individual input within the subset. Results of this share allocation are that Shapley effects are non negative and sum up to one, allowing, in this manner, an easy interpretation for ranking inputs. The Shapley value is the only attribution method that meets the following properties, allowing to think about it as a fair distribution way:

- Efficiency: $\sum_{i=1}^{p} Sh_i^k = \hat{f}(x_i) - E_X(\hat{f}(X))$; the effects of a feature have to sum up to the difference of prediction for a certain instance and the average.
- Simmetry: the contribution for two features doesn't change if they contribute in the same way to all possible coalitions.
- Dummy: a feature which does not change the predicted value has got a null Shapley value.
- Additivity: for a game with combined cost functions, also the respective Shapley values have to behave in that manner.

The Shapley value works for both classification and regression. A common way to make the primer into action could be to use the Shapley value to analyze the predictions of a random forest model predicting a certain event like, for example, a particular economic scenario. For estimating a Shapley value, all possible sets have to be estimated, with and without the feature of interest in order to calculate the Shapley value for one certain feature value. Unfortunately, just few features involved and the purpose of obtaining an exact solution becomes computationally unfeasible, since the number of possible coalitions increases exponentially with the number of features. In that respect, the necessity of an "intelligent" way of sampling is crucial for calculating Shapley values: tentatives in that direction could be found in Štrumbelj and Kononenko (2014) where the authors propose an approximation with Monte-Carlo sampling, in Lundberg and Lee (2016) they suggest a computation method that includes weight kernels and regularised linear regression and, finally, in Lundberg et al. (2018), deriving an algorithm for tree ensambles like XGBoost and LightGBM. The latter is very recent and it allows to reduce the complexity of computing exact Shapley values from $O(TL2^k)$ to $O(TLd^2)$ where T is the number of ensamble trees, L is the maximum number of leaves for every tree, k is the number of features, and d is the maximum tree depth. This kind of reduction in complexity allows forecasts from models which were not possible to work on before. I take advantage of the version leveraging XGBoost for my computations in the following.

TABLE 3.1: Brock and Hommes (1998a): Theoretical support ranges for the parameters involved in the model.

Parameter	Description	Support
β	Agent 1 intensity of choice	$[0; +\infty)$
n_1	initial share of Agent 1	$[0.0; 1.0]$
b_1	Agent 1 bias	$(-\infty; +\infty)$
b_2	Agent 2 bias	$(-\infty; +\infty)$
g_1	Agent 1 trend component	$(-\infty; +\infty)$
g_2	Agent 2 trend component	$(-\infty; +\infty)$
C	cost of obtaining type 1 forecasts	$[0; +\infty)$
ω	weight to past profits type 1 traders	$[0.0, 1.0]$
σ	asset volatility	$(0; +\infty)$
ν	risk aversion	$[0; +\infty]$
r	risk-free return	$(1; +\infty)$
ϵ_μ	Random Normal Dividend μ	$(0; 1)$
ϵ_σ	Random Normal Dividend σ	$(0; 1)$
p^*	fundamental price	$\mathcal{R}$
x_0	Initial price deviation from the fundamental price	$\mathcal{R}$
y_0	Initial dividend	$\mathcal{R}$
T_{BH}	number of periods	$\mathcal{N}$

3.3 Results

3.3.1 Choice of ABMs output

The two illustrative examples of ABM are: the asset-pricing model of Brock and Hommes (1998a) and the limit-order book model Franke and Westerhoff (2012a). Very briefly, Brock and Hommes (1998a) is an asset pricing model where heterogeneous population of agents trade a generic asset according to different strategies (fundamentalists, chartists, etc). Franke and Westerhoff (2012a) is instead a limit-order book where there are two types of traders, chartists and fundamentalists. The fundamental is fixed and known. There is a market maker clearing mechanism which adjusts prices according to excess demand. Finally, there is an adaptive mechanism that moves agent densities to better performing strategies. This will be a key component which is adjusted across many of the different market mechanisms considered. Since both of them are financial markets, their simulations produce time series of price. In terms of model comparisons, one way could be to combine the latter with a kind of output which would have a consistence about what I am working for. Even better, the goal would be to make synthetic historical data match with the one I actually observe in real world through some features. In this regard, I choose to use as outputs for my illustrative ABM models some of the empirical properties of asset returns widely agreed in finance (Cont, 2001). This procedure is clearly inspired by Methods for Simulated Moments (MSM), in line with Lamperti (2018). MSM are usually employed in order to point out those parameter values that "calibrate" the model, within certain confidence bounds, matching some financial stylized facts considered particularly crucial. In the latter, a loss function which combines distances between simulated and empirical moment values is minimized. In my case, I suggest to use as outputs of ABMs four stylized facts. That is remarkably useful for some reasons: in first place, it is an easy way to get meaningful and expressive outputs for the meta-modeling process later on. Secondly, it is an automatic procedure to calibrate data coming from simulations. It lets me quickly skim meaningless parameter combinations automatically, ruling out those parameter combinations with

even only one output larger that an order of magnitude with respect to the empirical one. Therefore, I achieve two goals at the same time: I obtain solid and reliable outputs for the models for surrogate models training and I calibrate ABMs through stylized facts. The following are the stylized facts employed in my analysis:

Absence of autocorrelations :
Linear autocorrelations of asset returns are insignificant, in particular regarding my time scales. They would have them for very small intraday time scales only, around 20 minutes, for which peculiar effects, known as microstructure ones, take place.

Heavy tails :
Returns distribution appears to show a tail like power-law or Pareto-like. From data sets studied in Cont (2001), it seems tail index of returns is difficult to determine precisely, but however, it is finite, higher than two and less than five. As a matter of fact, this excludes stable laws with the normal distribution and infinite variance. This evidence translates in measuring kurtosis of returns distribution.

Volatility clustering :
It was pointed out a positive autocorrelation in several measures of volatility, letting us think that high-volatility events tend to cluster in time.

Long-memory returns :
It is referred also to slow decay of autocorrelation in absolute returns; it measures the decay of autocorrelation of absolute returns as a function of time lags. I employed Hurst exponent as measure to calculate it.

In particular, I rely on 502 observations totally, daily adjusted closing prices for the S&P 500 going from December 09, 2013 to December 07, 2015; this number of observations comes from consideration on relaxation time for obtaining asymptotic quantities of the four stylized facts mentioned above. This opportunity may be seen either as a measure of how far the simulated model is from its realistic counterpart or to calibrate the model on producing realistic price series and, consequently, obtaining the parameter set matching this scenario. In Table 3.2, I report the values obtained on the historical observed data (i.e. S&P 500).

TABLE 3.2: Return autocorrelation, kurtosis (i.e. a measure for fat tails), absolute return autocorrelation and Hurst exponent (i.e. a measure for memory) computed on historical data of S&P 500 index. These values are employed to calibrate the illustrative ABMs I employed in my analysis; 502 observations totally, daily adjusted closing prices for the S&P 500 going from December 09, 2013 to December 07, 2015. Such number of observations is enough for obtaining asymptotic quantities of the four stylized facts mentioned above.

	Return autocorr.	**Price series kurtosis**	**Abs. return autocorr.**	**Hurst exp.**
S&P 500	-0.001169	2.5067	0.004125	-0.0072579

3.3.2 Requirement of robust parameter space exploration

Sampling the parameter space

One of the main goals of this Chapter is to refine robust and at the same time computationally cheap methods for the sensitivity analysis of realistic ABMs that otherwise

would be prohibitive to explore, since the high number of inputs involved. In principle, surrogate models may offer a solution to this computational burden Lamperti et al. (2018); I explore the opportunity of their usage for sensitivity scopes. First of all, it may sound trivial but I believe it is worth to point out that meta-models (i.e. surrogate models) need a representative and reliable pool of parameter combinations (and, accordingly, their corresponding model responses) in order to be trained and working as a trustworthy 'proxy' for an ABM. Once trained on a large part of the data set (i.e. *training set*), the goodness of a meta-model is then quantified by prediction accuracy measures (e.g. RMSE, MSE, MAE, etc), by testing its results on a small data set not employed for model training (i.e. *test set*). Usually a 10/20% of the whole data set is reserved for serving this purpose and the remaining amount of data as training set. Since my illustrative examples are two financial ABMs, so that the main endogenous response is an ideal stock price series, I use as output the combination of the four stylized facts of financial markets mentioned above: zero autocorrelation, fat tails, cluster volatility and memory (Hurst exponent). A relevant advantage of their employing relies on serving as model response that may be calibrate with observed data values; I drew the latter from the historical series of S&P 500 index. In particular, I rely on 502 observations totally, daily adjusted closing prices for the S&P 500 going from December 09, 2013 to December 07, 2015; this number of observations comes from consideration on relaxation time for obtaining asymptotic quantities of the four stylized facts mentioned above. This opportunity may be seen either as a measure of how far the simulated model is from its realistic counterpart or to calibrate the model on producing realistic price series and, consequently, obtaining the parameter set matching this scenario. Just to go back, in order to train a surrogate model, a wide parameter space sampling is required. Here one faces the first ambiguity between the theoretical model and its numerical counterpart: how theoretical model supports translate into numerical ones? A wide sampling of the parameter space by combining all the possible values along all the whole parameter ranges is necessary. This is a first obstacle that may require computational effort. By the way, a first screening may be to set order of magnitudes for the parameter ranges because of inherent model proportionality coming from interaction relationships of parameters. I perform several sampling methods and the first consideration coming up from my analysis is the following: gridded sampling methods (e.g. LHS, NOLH, etc) may not work for ABMs. The reason why mainly lays on the fact that ABMs are stochastic and, then non-linear systems; for specific parameter combinations the model may not exist at all or dynamically tends to a tipping point. In practice, the simulation may not start or stop after a few time steps. Since a gridded sampling algorithm requires precise parameter combinations because of its planned parameter space partitioning, one or more parameter combinations which don't generate a meaningful model response would make the gridded sampling algorithm not working, completely meaningless and even misleading if working. The output surface of an agent-based model may be imagined as pitted; gridded sampling methods may choose those points (i.e. parameter combinations) for which the model is not defined. In order to overcome the latter and collect only meaningful parameter combinations, I propose the following *intelligent* sampling algorithm: I perform a random sampling on all the possible combinations of parameter ranges and, in order to admit them, I combine a double check of the existence of the model for that parameter combination, , through the four outputs (i.e. if the model dynamic is taking place and outputs are "meaningful"). I perform a deep analysis of parameter ranges, in line with One-Factor-A-Time sensitivity analysis. In particular, I set as benchmark the parameter combination which better match all the four

values computed from empirical price series (i.e. through observed stylized facts of S&P 500 index historical data, see Table 3.2). In order to locate it for each ABM employed in the analysis, I explored over 100000 samples generated according to random sampling and double check of model existence, as I mentioned above. It is worth to remark that although local sensitivity analysis is meant to be working with local minima/maxima, on the other hand it may be possible to assess the robustness of the benchmark parameter combination chosen. As a matter of fact, I studied the latter by drawing a ranking list of the best calibrated parameter combinations and I noticed that there exists a sort of smooth progressive tendencies towards the fist ranked one (i.e. my benchmark). The latter is pretty reassuring; even if I cannot state for certain that my benchmark is a global minimum, I can state that the local minima I find are coherent one another.

Non-stability areas of parameter ranges

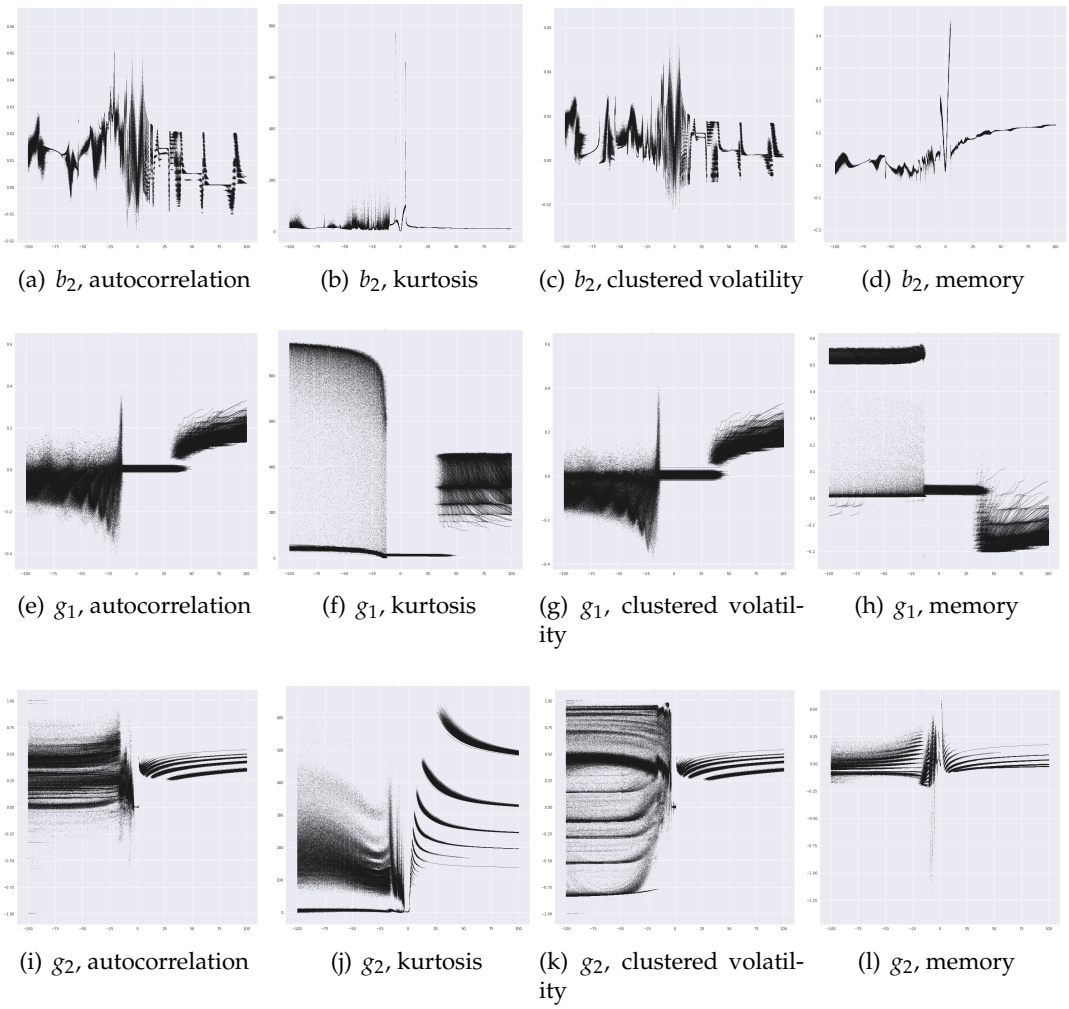
The procedure mentioned above (i.e. random sampling and double checking of model existence) is still not enough for the goodness of sampling selection, either in the perspective of building a reliable surrogate model or in the exploration of the model itself. As I ascertained by simulating the two illustrative ABMs employed in this chapter, even by choosing through random sampling those parameter combinations for which the model shows satisfying outputs (i.e. the simulation is matching observed data) may be misleading. I further elaborate this point, since it it crucial. I notice that, for a given parameter combination, depending on the Monte Carlo replica I account for (i.e. the random seed set for the model simulation), the ABM response could produce time series of price either pretty calibrated with observed data or very far from it. This cannot be admissible for many crucial reasons. In particular, in my case surrogate models usage means to let algorithms train input-output pairs in order to reproduce the functional form which would approximate the ABM. Agent-based models are stochastic systems; they may observe strong variability of their response, for particular parameter range areas; this is due to the presence of randomness sources. In light of that, they require robustness assessing of thier response by Monte Carlo raplicas next to their sampling: each experiment (i.e. the simulation of a certain parameter combination) needs to be repeated through different MOnte Carlo replicas (i.e. different random seeds) in order to obtain the statistical robustness whichi is necessary for testing hypotheses and discerning multiple scenarios (Lee et al., 2015). The "random seed" uniquelly characterizes the pseudo-random sequences employed for a given experiment. Monte Carlo dispersion over many random seeds naturally comes from the presence of stochastic variables into the agent-based model. Fixed a certain parameter combination, a Monte Carlo replica is a model simulation featured by a unique random series which feature model randomness. Each parameter combination identifies a precise model setting; I refer to such model setting as *stable* if different Monte Carlo replicas bring to the same outputs. For a given parameter combination, if the system dynamics of different Monte Carlo replicas bring to very different scenarios (i.e. very different outputs), I refer to the model as *non-stable*. What does it mean for me? Without a robust checking of Monte Carlo replicas for the same parameter combinations, calibration and training a meta-model would dramatically bring to misleading and meaningless results. Hence, performing sensitivity analysis on a surrogate model in this way would be totally meaningless.

3.3.3 My proposal of sensitivity analysis: Monte Carlo-One-Factor-At-a-Time (MC-OFAT)

Inspired by bifurcation diagrams (crucial in dynamical systems analysis for physical systems), I propose a procedure which either shows how choosing reliable parameter combinations to ensure robustness or allows operationally the employment of surrogate models to perform sensitivity analysis. What I propose may be seen in some ways as a modified version of One-factor-at-time (OFAT) local sensitivity analysis method: Monte Carlo-OFAT (MC-OFAT). OFAT is one of the simplest and most common local sensitivity analysis approaches and consists of moving One-Factor-At-a-Time fixing a parameter combination of baseline values; it is widely employed for macro and financial ABMs. My proposal is somehow enriching OFAT of Monte Carlo replicas: I extended OFAT sensitivity analysis by performing Monte Carlo replicas for each parameter combination computed by OFAT (i.e. for each OFAT computed point). I explore each parameter throughout its own support range (i.e. the thoeretical support within the mathematical model exists) in the same way I would do for OFAT, extending it by replacing each OFAT point (i.e. a parameter combination) with 1000 Monte Carlo replicas of the same parameter combination. In case the MC-OFAT curve (i.e. one of the four outputs and one parameter) collapses on the standard OFAT curve, the model dynamics is *stable* for that parameter setting, so it makes sense to account it for surrogate model training and model exploration. On the contrary, in case the MC-OFAT curve features dispersed output values for the same parameter combination, then the model dynamics is *non-stable* and a training of the surrogate model has no meaning.

I lay out one of the main results of this chapter: I find out that for some input ranges I observe a very large dispersion for the same parameter combination that makes the output response surface extremely ambiguous. In some cases, I even observe Monte Carlo replicas splitting into different "areas of attractions". The latter is a very tricky issue to solve, both for simulating the model and even worse for meta-modeling it. Moreover, even in a calibration setting, or making a skim of the most relevant parameter combinations (i.e. those ones which generate output values the closest to empirical ones), I may be deceived. In particular, in those areas it would be "by chance" to have picked a MC replica which match or not empirical values. Monte Carlo replicas "dispersion" results in a dramatic performance for any kind of surrogate model that can may be employed. Every meta-model trying to replicate the dynamics of an agent-based model would definitely fail for those areas in which different Monte Carlo replicas result in very far away output values. The main restriction comes up from the fact that surrogate models are trained on one or a few MC replicas. The possibility to run many MCs is directly proportional to how much the ABM into question is computationally expensive; moreover, that is the same trade-off which actually brings a researcher/ practitioner to look for a cheaper proxy (i.e. surrogate model) to simulate the ABM. Furthermore, another limitation comes from the fact that the benchmark I choose is a local minimum; it may not be the global one. As I mentioned before, this problem may be partly overcome by looking at the input values distribution of a ranking list of local minima. That said, my procedure aims to define a a general approach to ensure meaningful training of surrogate models, in order to employ them for global sensitivity analysis purposes. In Fig. 3.1 I report four stylized facts as outputs for the three main parameters of Brock-Hommes agent-based model which present strong instability with different random-seeds. In the first line I report the outputs for b_2, i.e. the bias of the second agent: for values less then zero, kurtosis can either match calibrated values (i.e.

toward 10) or quite far form it, even more than 200. Moreover, the two spikes that are present distribute from the bottom till 800; this are two points to be avoided for sure. It can be due to the particular combination I chose as benchmark. About the other two inputs, g_1 and g_2, which are the trend components of both type of agents, I have a much more representative example of the problem I explained before. For example, in g_1 I can easily notice that for all 4 plots there is an area (form about -13 to about 30) which conserves stable with a admissible variance around the mean; in the remaining part of the range values can be very far one another, in some cases there are even discrete steps on which random seeds distribute. The last parameter g_2 looks like the more dangerous, having real independent patterns followed by different random seeds; here I had to dramatically reduce the range. As I can notice, it is strongly unstable and it could take to considerations strongly depending from the random seed on which I are relying.



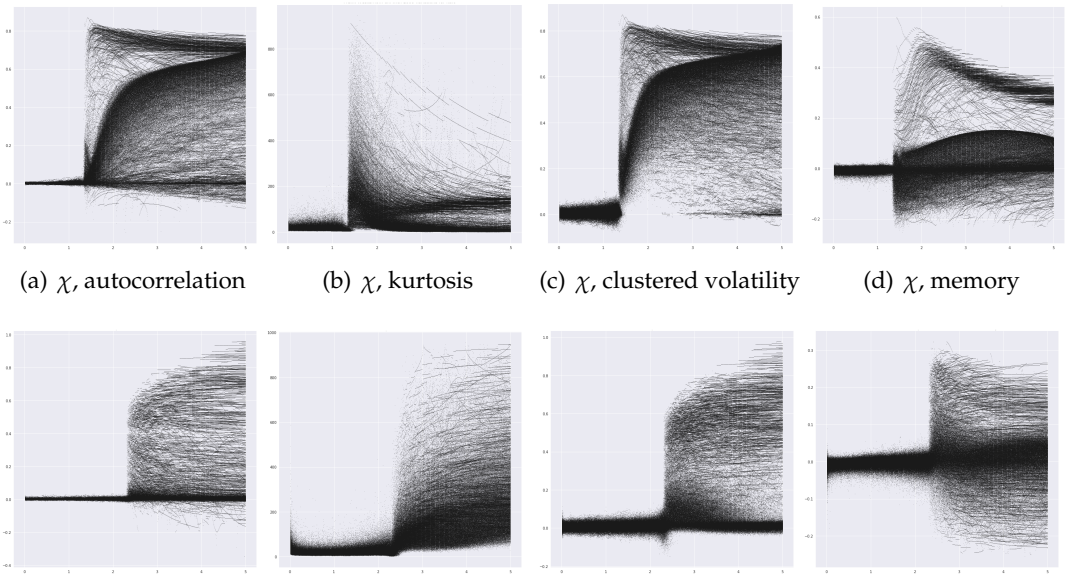
(a) b_2, autocorrelation (b) b_2, kurtosis (c) b_2, clustered volatility (d) b_2, memory

(e) g_1, autocorrelation (f) g_1, kurtosis (g) g_1, clustered volatility (h) g_1, memory

(i) g_2, autocorrelation (j) g_2, kurtosis (k) g_2, clustered volatility (l) g_2, memory

FIGURE 3.1: **MC-OFAT sensitivity analysis for Brock & Hommes ABM**. Examples of three parameters in which montecarlo realizations crucially matter: bias b_2 of one trader and trend components g_1, g_2 of both two traders.

Regarding Franke-Westerhoff ABM, the two main parameters showing similar behaviors are χ and ϕ, i.e. the strength at which the two type of agents (chartists and fundamentalists) respectively react. In Fig. 3.2 I show the latter, according to



(a) χ, autocorrelation (b) χ, kurtosis (c) χ, clustered volatility (d) χ, memory

(e) ϕ, autocorrelation (f) ϕ, kurtosis (g) ϕ, clustered volatility (h) ϕ, memory

FIGURE 3.2: **MC-OFAT sensitivity analysis for Franke & Westerhoff ABM**. Two parameters, χ and ϕ in which the exploration of MC different replicas crucially matters. One thousand MC replicas have been performed on each parameter combination

MC-OFAT.

The detection of a regime in which there is Monte Carlo dispersion (i.e. non-stability) for a given parameter quantifies the uncertainty of the parameter and its impact (then in a sensitivity analysis sense) on the model output. It is very relevant for the model understanding and that crucially requires an ex-ante economic question. Furthermore, in light of all the considerations above, it would seem pretty clear that a non-stable area of the parameter range doesn't give any room to work on it, from a surrogate modeling perspective. As mentioned above, in order to work properly and keep being computationally cheap, a surrogate model should train on robust parameter combinations (and then ranges) which produces unambiguous model response. In light of the latter considerations, the need which arises is to locate those parameter areas where the stability regime holds, for the possible parameter combinations; in other words, I need a robust procedure in order to locate the meaningful parameter areas and to rule out the non-stable ones. I have tried several methods for assessing the stability of a given parameter window of the explored range and I found out that a meaningful observable for this task is the normality distribution of Monte Carlo replicas. Since the two illustrative ABMs I am employing reach after few time steps the steady-state, the only non-stability source come from random sources sequences. In a *stable* world, the average of an observable (i.e. a model output) over the phase space must identify its time average. In my case, the phase space is determined by all the possible Monte Carlo replicas. A Monte Carlo replica is identified by a unique set of random variables which participates the ABM in the form of random sources. That said, what I expect from a stable regime is that the model responses of MC replicas behave as i.i.d. variables and hence, for a number of MCs which tends to infinity and in line with the Central Limit Theorem, they distribute normally. This intuition has been confirmed numerically. Normality tests have been performed on MC-OFAT, as in Fig.3.1 and Fig.3.2; I noticed that the more

MC replicas are, the better normality tests perform. Moreover, Jacque-Bera normality test turns out to be the best choice for my needs, matching those areas I can notice they are stable graphically.

3.3.4 MC-OFAT as a tool to screen parameter ranges before surrogate model training

A meta-model needs a pool of parameter combinations and model response pairs to train on, in order to produce a functional approximation of the model. This means that if a parameter combination of a non-stable regime is used to train the meta-model, the latter will build up its function (i.e. the functional input-output form to replace the ABM) on that MC replica, regardless the fact that in such area the model response be very different for another MC replica. In principle, I could also train the meta-model with different replicas of the same parameter combination but it would require much more computational effort and no gain is guaranteed. First of all, the goal of surrogate modeling is to save computational expense and, secondly, even feeding the meta-model with some MC replicas, the latter will consider mean and variance of the model output proportional to which particular MC replicas came up by chance; this doesn't assure any improvement with respect to the surrogate trained on a single MC. The strength of meta-modeling lies on extracting a general cheaper proxy for the model, valid wherever, trained "on a few" simulations of the model right because the ABM is computationally expensive and I choose to surrogate it. But in case I would need so many more simulations of the ABM to train the meta-model, the sense of using it would be missing and I experience only the drawbacks of meta-modeling. By absurd, in case I would be able to draw a huge number of MC replicas, I'd rather perform my analysis on the real model than on its surrogate. As a matter of fact, the drawbacks of meta-modeling may easily overcome the benefits:

- meta-models will never reproduce in a satisfying way the output surface of an ABM (since it is strongly non-linear and present tipping points); in general, the more random sources there are, the less it will be working;
- singularity points and other non-linear effects cannot be reproduced fairly; the interpolation of points in the training set may result very misleading in the model response surface reconstruction;
- it is almost unfeasible to reproduce the uncertainty of the model by meta-modeling;
- using a parameter combination of a non-stable area to train the meta-model may result in very misleading surface reconstruction by the latter.

A way to take advantage of meta-modeling is to directly bound the parameter ranges for its training to the only ranges in which the model is stable, assessing them through Jacque-Bera normality test. In this way, I expect that meta-model performances increase. In order to measure an eventual improving in performances, I use Mean Absolute Error assessment before and after applying my procedure MC-OFAT.

Surrogate model performances of my procedure MC-OFAT

The surrogate models I employ and compare each other are: the decision tree-based XGBoost and three different implementations of Kriging. For the latter, I employed

different covariance functions: Matérn model with $\nu = 5/2$, Matérn model with $\nu = 3/2$ and exponential model, all together with a constant covariance basis (with dimension equals to number of inputs). Very briefly, in Kriging meta-model (Gaussian process regression), the original model response for the unknown points can be predicted by a linear combination of the responses at the closest known points, similarly to an ordinary multivariate linear regression, but taking the spatial information into consideration, in a Bayesian framework. XGBoost (Gradient boosting decision tree) instead consists in a machine learning technique for regression and classification problems. Such a model is an ensemble of simpler decision trees, which are aggregated to improve the overall prediction performance. In order to calibrate XGBoost surrogate model, I optimized the so called *pipeline*; i.e. I tune the surrogate model arguments (e.g number of estimators, learning rate, maximum depth of the decision trees etc) by minimizing an objective cost function depending on them. The results I present onwards are given for those optimized conditions in which Kriging and XGBoost methods perform the best. Here I go through a comprehensive comparison of XGBoost and Kriging meta-models predictions with respect to the agent-based model prediction. In particular, I compute how the prediction quality of the four surrogate models in question varies increasing the number of samples (form 200 to 1000 samples) on which I train them. I perform the latter in both cases: either before and after using my proposed procedure MC-OFAT (i.e. narrowing parameter ranges where to train the model according to a stable regime), for the four stylized facts used as outputs. I compute the Mean Absolute Error for each stylized fact, for each set of N samples and for 50 Monte Carlo iterations (i.e. initializing a different random seed anytime). I show the average MAE of the 50 MC replicas with two times their standard error of the mean. Hence, for each meta-model and each stylized fact s, its Mean Absolute Error (MAE) is computed as follows:

$$\text{MAE}_i = \frac{1}{N} \sum_{j=1}^{N} y_j^i - y_{rd}^i \tag{3.10}$$

where N is the number of samples, y_j^i is the surrogate prediction of the stylized fact j for the sample i and y_{rd}^i is the real data value for that stylized fact (the one computed on the real price series). I report in Fig. 3.3 and in Fig. 3.4 the surrogate models performances evaluated through MAE on, respectively, Brock and Hommes (1998a) and Franke and Westerhoff (2012a) models. For each ABM, I show a comparison of surrogate models performances before and after applying my proposed procedure (i.e. MC-OFAT sensitivity analysis and normality test assessing of stable parameter ranges), for each of the four outputs (i.e. autocorrelation, fat tails, clustered volatility, memory). For each set of MC replicas, I compute the mean of their MAE values, together with twice their Standard Error of Mean (i.e. 95% confidence intervals) and I show graphically each value of assessed performance. As I notice analyzing the plots, there is always a relevant gain in adopting my procedure, with Brock-Hommes ABM in particular (e.g. an order of magnitude for all the four outputs). Anyhow, I notice each output reacts differently and in some cases surrogate models find the training very difficult. As I show in Fig. 3.3 and in Fig. 3.2, in some cases Kriging happens to work better and in some others Xgboost, but still none of them is exhaustively working properly. In general, I obtain the best MAE values for Franke-Westerhoff model but it may be due to its lower number of inputs, 8 model parameters, compared to the 15 inputs for Brock-Hommes. Even for Franke-Westerhoff, the results are not satisfying (i.e. comparing with the order of

magnitude of the benchmark stylized facts reported in Table 3.2), in particular for the output which takes into account the stylized fact of fat tails (i.e. kurtosis). Moreover, Xgboost keeps pretty wide standard errors of the mean for all cases, compared to Kriging.

I refine a further limitation on the parameter combinations to take into account for surrogate training. In a few words, the latter is a skim on the pool of parameter combinations used for training a surrogate model, selecting as suitable for training the only parameter combinations whose outputs don't diverge beyond a certain distance from real data stylized facts (i.e. S&P 500 historical data benchmark). Along this way, all the parameter combinations I take into account are implicitly relevant from a calibration point of view, since they are very close to real data. In this direction, I could say to have set quantities that are either consistent measures for an ABM output or measures of closeness to real world at the same time. In Fig. 3.5 and Fig. 3.6, I report for both agent-based models and for each output a box plot of surrogate models performances through Mean Absolute Error. The results obtained are pretty satisfying; looking at MAEs w.r.t. each output, I was able to make meta-models reproduce model responses with relative small errors, compared to benchmark values (see Table 3.2). Unfortunately, I don't obtain one surrogate model overperforming the others for all output choices but their results are roughly comparable.

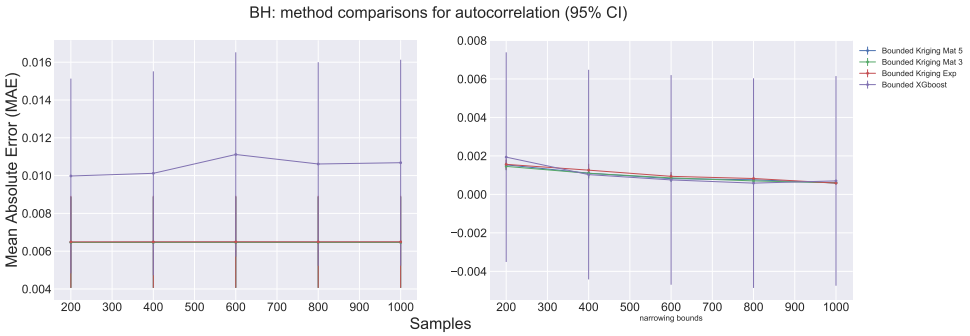
3.3.5 Global sensitivity analysis

The sampling method which one uses in order to do any kind of other measure, like surrogate modeling or sensitivity analysis, is quite crucial. There exist lots of way of sampling (e.g. Latin Hypercube Sampling, Random Sampling etc) and each one has its relevance for precise purposes. In particular, regarding what concerns to me, several methods for estimate SA indices have been implemented for a given sampling method each; they are well-built only on a way of sampling. The latter because covering in a certain manner some distances in every parameter range would lead to do a weighted assessing of their influence on the model response.

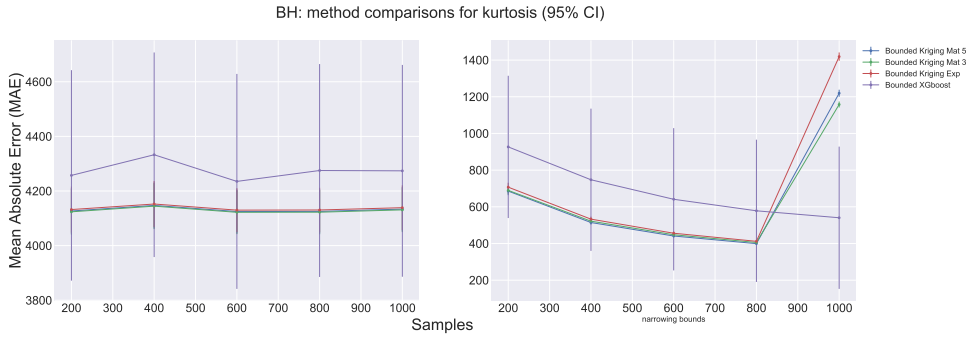
As laid out in Subsection 3.3.2, unfortunately I cannot take advantage of ordinary gridded sampling methods in my case. As a matter of fact, this limitation rules me out from usage of most of global sensitivity analysis method, either already implemented (e.g. SA libraries, such as in python libraries like *Openturns* and *SALib*) or to implement, since the regularity in parameter ranges slicing lets analytical and numerical computations easier and more versatile. In light of that, I am prevented to use the most employed global sensitivity analysis methods except the ones perform sensitivity analysis of given data. In the following, I present global sensitivity analysis measures obtained through methods which allow to assess parameter impacts from data I obtain with my ad-hoc sampling methods, before and after applying my MC-OFAT procedure. These are: Sobol' first indices (Sobol, 1993), Borgonovo's Delta values (Borgonovo, 2007) and Shapley values (Shapley, 1953; Song et al., 2016).

Borgonovo's Delta and Sobol' First Indices

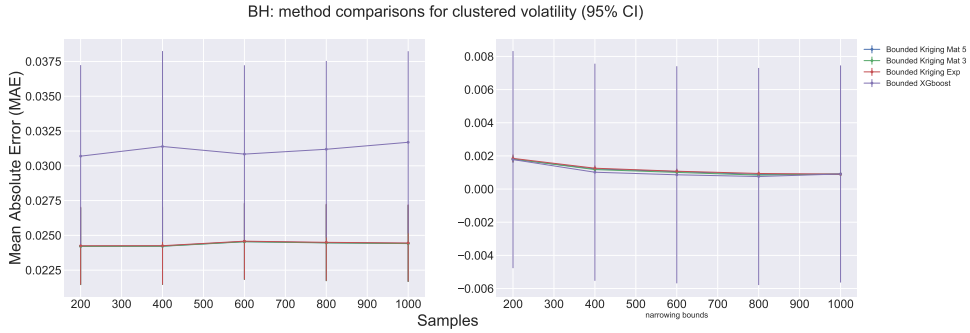
In line with the work of Borgonovo (2007) and the one of Plischke et al. (2013), I make usage of their derivations and compute the Sobol' indices for each parameter. In particular, what I leverage are methods of estimating density-based indicators from given samples. The main problem I deal with when using 'standard' and 'most common' sensitivity analysis measures is that they quite always require a given sampling gridded partition. Such methods are totally calibrated on given distance for



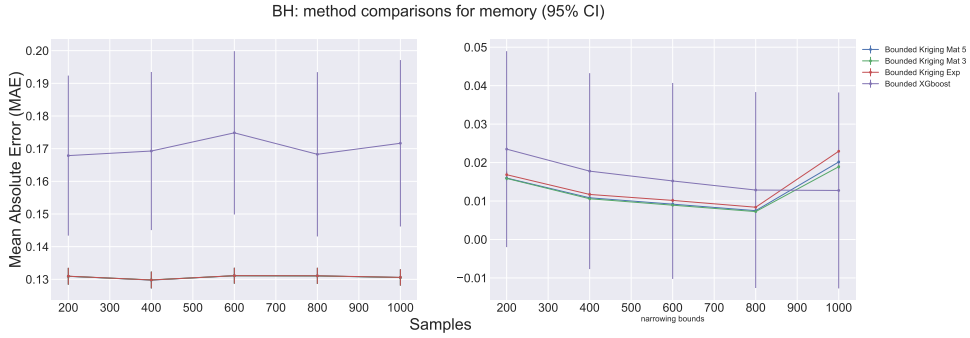
(a) autocorrelation



(b) kurtosis

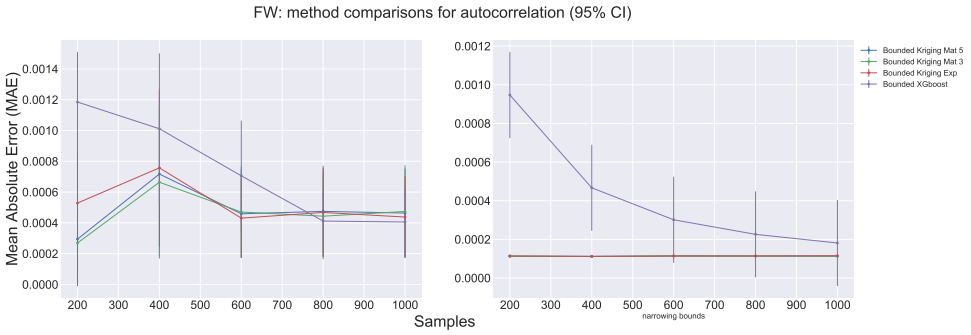


(c) clustered volatility

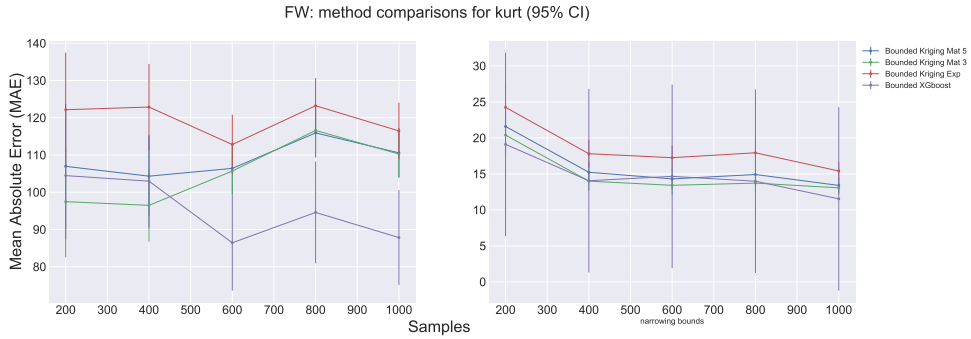


(d) memory

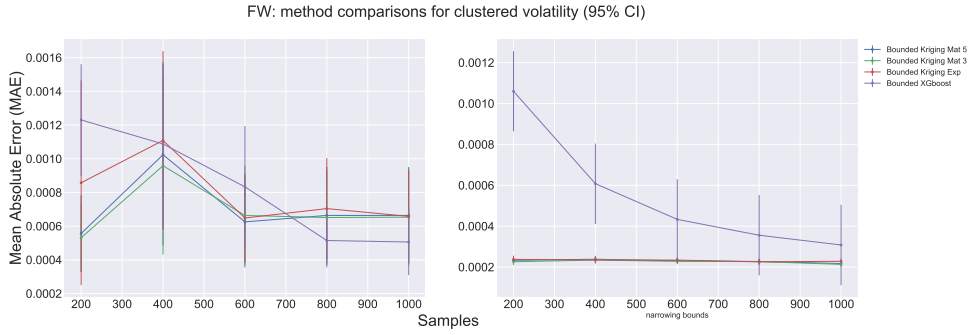
FIGURE 3.3: **Brock & Hommes.** Performances of Kriging and XGboost surrogate models for the four outputs. I compute different sample sizes, each of them repeated for 50 MC replicas, in the two cases: on the left-hand side, before to apply the my procedure (i.e. parameter ranges are the ones of the theoretical support) and, on the right-hand side, after its application (narrowing each parameter range to the only stable regime areas.



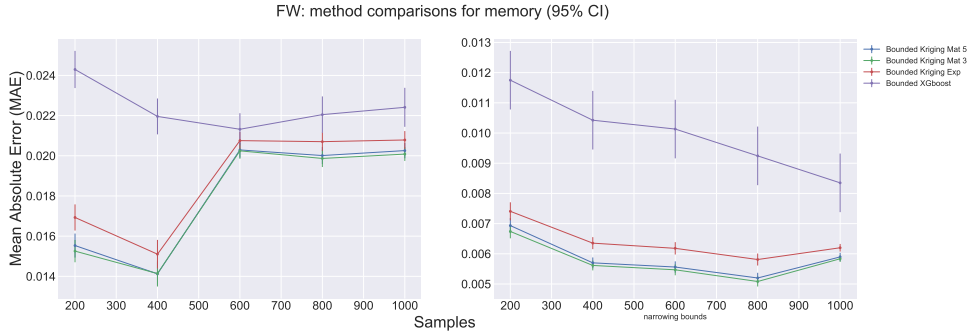
(a) autocorrelation



(b) kurtosis

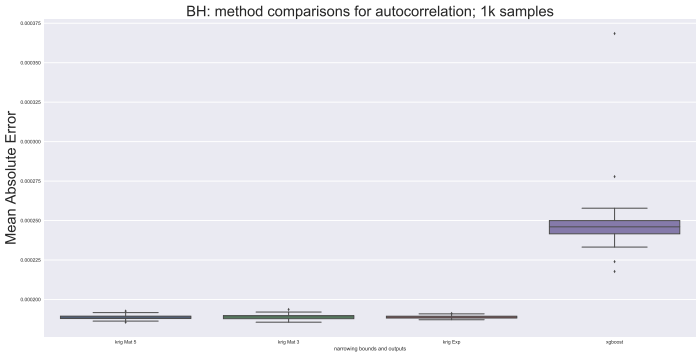


(c) clustered volatility

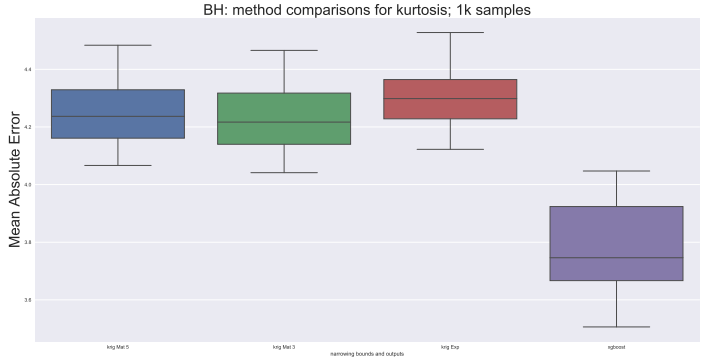


(d) memory

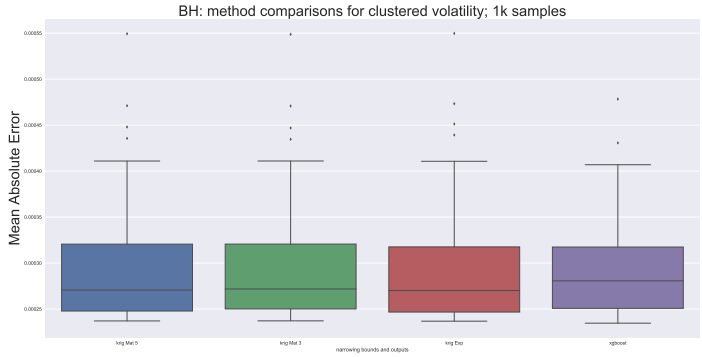
FIGURE 3.4: **Franke & Westerhoff.** Performances of Kriging and XGboost surrogate models for the four outputs. I compute different sample sizes, each of them repeated for 50 MC replicas, in the two cases: on the left-hand side, before to apply the my procedure (i.e. parameter ranges are the ones of the theoretical support) and, on the right-hand side, after its application (narrowing each parameter range to the only stable regime areas.



(a) autocorrelation



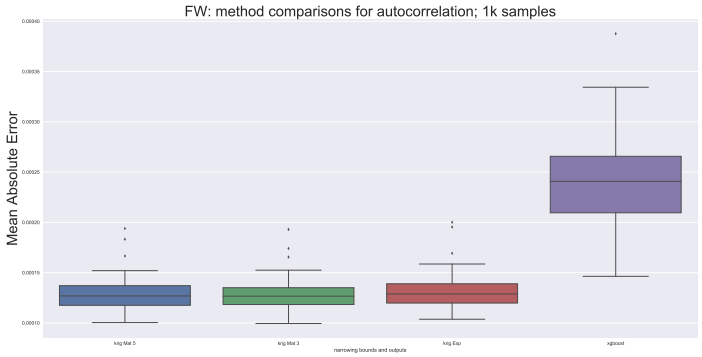
(b) kurtosis



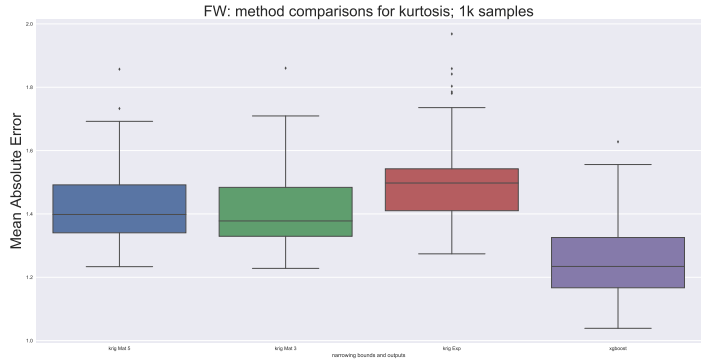
(c) clustered volatility

(d) memory

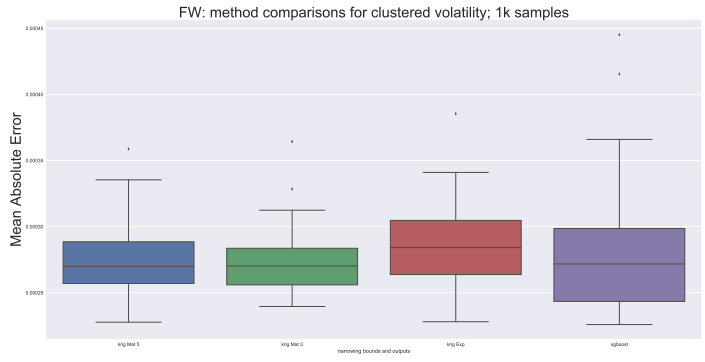
FIGURE 3.5: **Brock & Hommes.** Box plots of the four meta-models performances.



(a) autocorrelation



(b) kurtosis



(c) clustered volatility

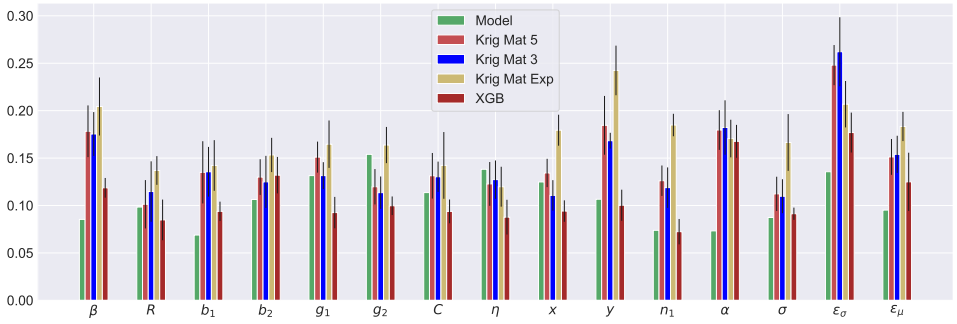
(d) memory

FIGURE 3.6: **Franke & Westerhoff.** Box plots of the four meta-models performances.

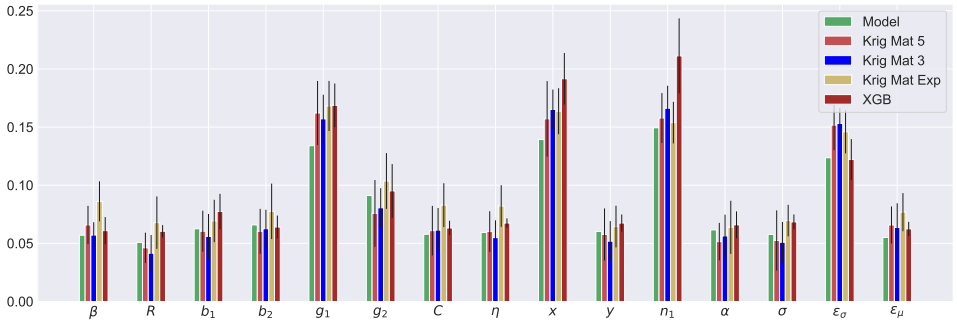
any parameter. As mentioned before, since I am dealing with stochastic systems, there may be some parameter combinations not available and they may match with those ones required by gridded sampling approaches. For this reason, I have opted for methods which are based on given data. Borgonovo (2007) and Plischke et al. (2013) take advantage of a wide literature on distribution separation measurements and they measure the separation between the distribution of the model output and the same distribution conditioned to the decision maker's degree-of-belief about the model output Y. In the end, they come up defining the importance of a given input X_i on the model output Y through a function called $\delta(Y, X_i)$. Moreover, Plischke et al. (2013) defines a consistent estimator for $\delta(Y, X_i)$ in order to provide an estimation strategy. The kind of measure differs form the classical ones from the fact that are denisty-based rather than variance-based. It has been shown in Plischke et al. (2013) how this difference could bring to measures more inclusive towards dependent interactions between inputs. In Fig. 3.7 I show the computation of Borgonovo's Delta indicators for the output kurtosis. I employ four surrogate models trained on a set of 50000 samples and tested on 5000 and I compare their sensitivity analysis results with the real model ones (i.e. my "ground truth"), before and after MC-OFAT approach application. As can be seen from the plots in comparison, before MC-OFAT application no surrogate model is able to reproduce the same input impacts produced by the model, neither some proportionality among them. After MC-OFAT application, all surrogate models are able to reproduce the impact of parameters on the model response. I obtain the same satisfying result in applying MC-OFAT approach for Sobol' first order indices, as shown in Fig. 3.8.

Shapley effects

As mentioned in Subsection 3.2.4, Sobol' sensitivity indices and are based on a functional, the ANalysis Of VAriance (ANOVA); its decomposition is unique only if the input variables (that are random) are assumed independent. This also applies to Borgonovo's Delta indicators. The latter is a huge limitation since the nature and scope of ABM, so that to model real world frameworks as complex evolving interacting environments; it is much harder to imagine independent inputs in an ABM rather than dependent one another. Anyhow, several works have been carried out to deal with this issue. Some works as Kucherenko et al. (2012); Mara and Tarantola (2012) propose an extension to the case of a stochastic dependence between inputs. Up to that, it is still hard their estimation and interpretation as well. Owen (2014) threaded the connection between Sobol' indices and Shapley values straight out game theory. Recently, E Song (2016) establish a relation between the Shapley values, coming from the field of game theory, and Sobol' indices. Furthermore, A Owen (2017) show relevant Shapley features regarding dependent inputs. In that case, Shapley values can be a good option to Sobol' dipendent input extensions. The study of Shapley effects would require ideally the computation of all the possible combinations of the inputs; this requirement would rule out a feasible computation and surrogate models come to its assistance. I will take advantage of a meta-model I employed before, XGBoost, that greatly takes over the model together with the application of MC-OFAT approach, dramatically reducing the computational cost of the evaluation, as shown in Fig. 3.5 and in Fig. 3.6. In this direction, my strong assumption is that XGBoost is replacing the true model in a satisfactory way and thanks to the results shown in Subsection 3.3.5, that could be roughlt stated. Since the computational constrains, I cannot be given of a the true model sensitivity analysis impacts as in Subsection



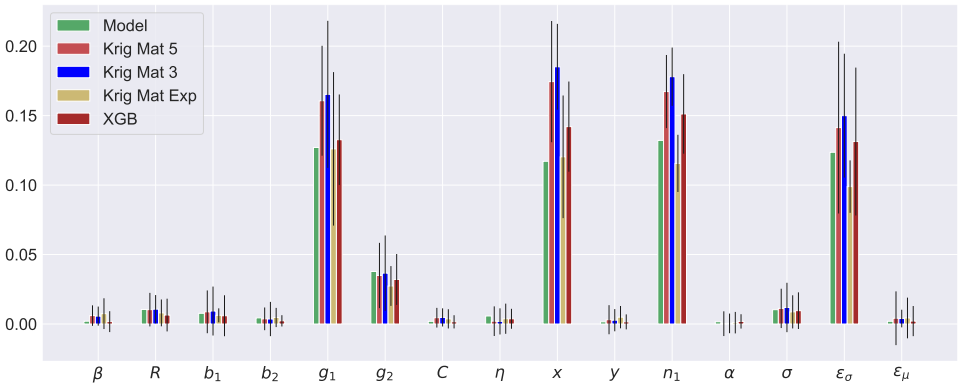
(a) before MC-OFAT



(b) after MC-OFAT

FIGURE 3.7: **Delta measure on Brock & Hommes**. In line with Borgonovo (2007) and Plischke et al. (2013) Borgonovo's Density-based sensitivity analysis measures computed on three Kriging and XGboost surrogate models and the true model (in green), for the output kurtosis, before and after applying my proposed procedure MC-OFAT.

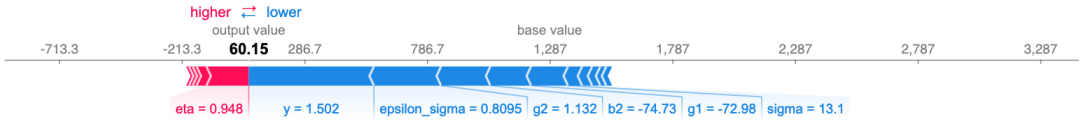
(a) before MC-OFAT



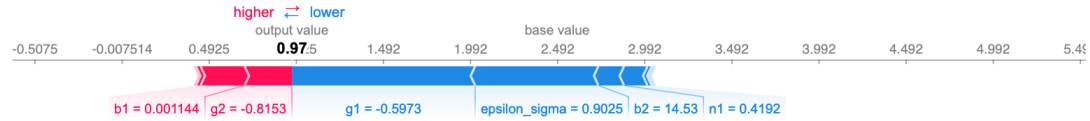
(b) after MC-OFAT

FIGURE 3.8: **Sobol' first indices on Brock & Hommes**. First Order Sobol' sensitivity analysis measures computed on three Kriging and XGboost surrogate models and the true model (in green), for the output kurtosis, before and after applying my proposed procedure MC-OFAT.

3.3.5. In order to test the validity of computing Shapley values on XGBoost surrogate model, I may compare Shapley effects before and after MC-OFAT application. I compute Shapley effects taking advantage of the estimation method proposed by Lundberg et al. (2018) and their interpretation: the SHapley Additive exPlanations (SHAP); it is an approach that aims to explain the output of any machine learning model. In Fig. 3.9 and Fig. 3.10, I show *force plots* for Shapley affects (Lundberg et al., 2018), for both illustrative ABMs. Force plots aim to represent which and how much relevant parameters push the model response down or up from the base value. Inputs which drive the model response down are drawn in blue and, conversely, those ones which drive the model response up are drawn in red. This way of representation remind like 'forces' which act on the whole model Lundberg et al. (2018).



(a) kurtosis; before MC-OFAT



(b) kurtosis; after MC-OFAT

FIGURE 3.9: **Force Plots of Shapley effects for Brock and Hommes (1998a).**

(a) kurtosis; before MC-OFAT

(b) kurtosis; after MC-OFAT

FIGURE 3.10: **Force Plots of Shapley effects for Franke and Westerhoff (2012a).**

3.4 Conclusions

In this chapter, I propose a novel self-consistent approach to analyze parameter ranges robustness of agent-based models in order to build up cheap but reliable surrogate models to replace the true model for global sensitivity analysis purposes.

To the best of my knowledge, such one is the first attempt in literature to propose a general approach for constructing surrogate models with the purpose of performing trustworthy sensitivity analysis on, in a financial and economic agent-based framework.

The results yielded with two illustrative financial agent-based models, i.e. the Brock and Hommes (1998a) asset pricing model and the limit-order book model of Franke and Westerhoff (2012a), show that my proposed procedure (i.e. MC-OFAT) to analyze parameter ranges robustness before the surrogate training provides a reliable dramatically cheaper replacement for the agent-based models which lets performing sensitivity analysis with an high level of confidence. The key advantage of my approach lies on its practical usefulness. As a matter of fact, the meta-model can be trained at very low computational cost and it needs a little amount of time to predict the impact of parameters on model response the researcher/ practitioner should rely on with satisfying good results. As I show in Section 3.3, MC-OFAT approach lets the design of solid, resilient and trustworthy surrogate models. I show they can reliably replace the original model for assessing the most impacting parameters in the model though global sensitivity analysis methods (see Subsection 3.3.5). Letting replace the original ABM by its cheap meta-model version in a delicate task like performing sensitivity analysis gives enormous room of manoeuvre to a researcher or practitioner in a policy making perspective.

Chapter 4

How A/B testing changes the dynamics of information spreading

4.1 Introduction

The online ecosystem has been an exceptional ground for private companies to build choice environments that draw people's attention in order to make and increase profits. Although their goal may vary from influencing political opinions to selling a pair of shoes, the methods which are involved often share the same algorithmic principles. One example in that respect is A/B testing, a method that optimizes pieces of the online environment to some goal function via a randomized experiment with two variants, the currently used version A (control) and its modification in some respect B (treatment). By testing a subject's response to A against B, this methodology aims to determine which one is more effective in optimizing to the set goal function. A/B tests are generally performed by private companies to increase user engagement and satisfaction about online features. Large platforms like Facebook, Instagram, Google, Groupon, LinkedIn, Microsoft, Netflix, Yahoo and Amazon use A/B testing to make user experiences more profitable and as a way to streamline interface of their services (Kohavi et al., 2014; Xu et al., 2015). Surfing the web, users undergo A/B tests without awareness; for example different users may see different versions of the same web page at the same moment.

Through A/B testing, private companies, online newspapers etc, may shape users' online experience, in an adaptive way. bias and polarise individual opinions with commercial or political contents etc. In general, the online environments they set up can undermine user autonomy and even spread mis- and dis- information. These concerns coming up are pretty natural given a bunch of remarks. Thereby, the A/B testing machinery is far from being transparent; private companies don't share their data, their methods, their direct purposes and, particularly, the different versions they are testing. Moreover, the nature of A/B testing of allocating different variants to different people adds an even more complex layer to the issue: it makes it impossible to keep an overview of who is seeing what version of a website or a story on the internet. A/B testing implications on decision-making and effects on collective dynamics academically underresearched.

The motivation of my work presented in this chapter[1] may identify two levels of impact: the way A/B testing influence individual decision making and the way how it scales up to collective behavior. Changing environments with respect to users' profiling actually let A/B test's author to pursue same goals by adapting to individual decision-making. The first research question that I address here in this

[1] Ottaviani M., Nickl P. L., Herzog S., P. Lorenz-Spreen (2021). "How A/B testing changes the dynamics of information spreading", *forthcoming*.

Chapter aims to find out which are the most relevant features that A/B testing is typically amplifying. In order to do that, I leverage real world data from a particular field of application of A/B testing. In the second research question I ask how and under which conditions A/B testing affects the distribution of content on the collective level, specifically on different social network structures. A third research question extends the latter in finding out measures which aim to dampen potentially unwanted outcomes of A/B testing and may be implemented in a policy perspective. Understanding the impact A/B testing has on the online ecosystem and the mechanisms at work may help to develop intervention and tools that dampen potentially unwanted outcomes of A/B testing and help to promote "user autonomy" (Kozyreva et al., 2019; Lorenz-Spreen et al., 2020). For my analysis, I leverage a crucial case study regarding the effectiveness of A/B testing: the Upworthy archive Matias and Munger (2019). In November 2019 Good & Upworthy and a team of researchers announced a dataset of 32,487 A/B tests carried out by Upworthy from January 2013 to April 2015 (Matias and Munger, 2019). The company conducted its A/B tests on its own website, randomly varying different presentations for a single story. Presentations were *packages* of news/story headlines that were randomly assigned to people on the website as part of a test. A/B test machinery compared package "success" to pick winners employing the click-through rate (CTR; i.e. the ratio of clicks to impressions). The present data set is by definition a clickbait dataset: The different headlines where designed and tested in an effort to maximise clicks (Blom and Hansen, 2015; Chen et al., 2015). This data is particularly suitable for systematically analyzing which linguistic features successfully attract clicks and are then amplified through A/B testing.

Linguistic cues occur on different levels. Particular attention in each package should be paid on the lexical or semantic level, the structural or grammatical one and the formal or formatting one. For example, different words may recall different emotional reactions, some topics could have been more successful than others, a specific text and word lengths may impact in a certain way. Particular attention should be drawn on the characteristics of *clickbait style*: it is a specific way of writing messages with the only purpose of maximizing the number of their views and the Upworthy dataset belongs to it. It is worth to point them out because of what kind of linguistic features either to expect or to follow up on.

I combine rule-based and machine learning methods to extract linguistic features. For the semantic analysis I use a topic model and a text analysis tool that incorporates a number of freely available sentiment dictionaries. I use topic modeling to detect the presence of semantic topics (e.g. feminism, racial equality, LGBT+ issues, etc). For the topic modelling I use Latent Dirichlet Allocation (LDA). LDA is an unsupervised learning algorithm that detects the co-occurence of words across documents (e.g. a headline) in a corpus (the collection of all headlines). Words that cluster together form a "topic," which may or may not be an interpretable semantic topic. In order to detect the presence of emotion, arousal etc. I use Sentiment Analysis and Cognition Engine (Crossley et al., 2017), a freely available text analysis tool that embeds a number of sentiment dictionaries. Mainly I am using SEANCE to analyze headlines in terms of valence, arousal and specific emotions. Once I obtained linguistic features characterizing packages, I associate to each package an input vector made of its extracted features. The decision-making model DM I assume is a simple linear relationship that binds the linguistic features present in a generic package with its probability to be clicked, i.e., the click-through rate of that package. I assume that the decision-making model holds globally for each package of the Upworthy data set. The continuous outcome for each package (i.e. the click-through

rate), together with the latter assumption, allows for a linear regression method. Therefore, I perform a LASSO regression (Least Absolute Shrinkage and Selection Operator). The regression coefficients coming from its training quantify the overall impact of each linguistic feature on click-through rate; they serve the decision-making function for simulating an individual interacting in front of an hypothetical (e.g. clickbait) headline. Therefore, I put forward a data-driven human decision-making model for clicking headlines online, based on linguistic/semantic features as cues. An agent is modeled as a click rate made up by linearly compounding the impact of linguistic features present in the package. To asses the global impact of A/B testing on the distribution of content, I compare two situations: (1) clicks on headline refer to a social sharing process, thereby no global actor has an agency and just the attractiveness of linguistic features plays a role in the distribution and (2) a global A/B tester is introduced who is aiming to additionally maximize the clicks by systematically varying the headlines following an A/B testing scheme. Let us consider the case of simulating an ideal social environment in which individuals interact sharing contents to their friends and acquaintances. The two main assumptions of my agent-based model lay on:

- all agents are equal: each of them reacts to a given package following the same decision-making model (mentioned above);
- the click for an agent means "sharing" a package (partially of entirely) to the nearest neighbors (with a friction parameter: the infection rate).

In order to mimic an online discourse I couple agents via a stylized social network structure, along which they can share information (headlines in this case). I employ several network topologies (e.g. Albert-Barabasi, Erdős-Rényi, Stochastic Block Model) and tune their densities (i.e. sparsity of the network). I consider synthetic randomly generated headline, which consists of a random combination of a fixed number of linguistic features. I have tested the procedure mentioned above. It has been achievable thanks to the availability of an exploratory data set from the Upworthy Research Archive Matias and Munger (2019). In particular, the final distributions of feature distribution from my agent-based model, either in a pure social spreading scenario or in the A/B testing condition. I could observe that the A/B testing mechanism increases the homogeneity of information that is spread. In other words, I observe that A/B testing performed on a social network structure reduces the exploration and amplifies exploitation of successful features of early pieces of information, ignoring others.

4.1.1 Brief literature review on *attention economy*

People have never been so cognitively impoverished to the extent that a "wealth of information creates a poverty of attention" (p. 41, Simon (1996)). Large web platforms like Google and Facebook act as hubs, distributors and curators (Kalogeropoulos et al., 2019; Lorenz-Spreen et al., 2020). Their algorithms are essential to navigate the immense digital landscape and enable bottom-up engagement in the production, dissemination and spreading of information. Data curation companies exploit this crucial role to secure the most valuable resource in the online marketplace: *human attention*. Using algorithms that learn people's behavioural patterns (Boerman et al., 2017; Kosinski et al., 2013; Ruths and Pfeffer, 2014), these companies target their customers with advertising and shape the users' information and choice experiences (Lorenz-Spreen et al., 2020). There is a deep asymmetry in the relationship between

platforms and people: platforms have extensive knowledge about the behaviour of users, while users are unaware of how their data is collected, how it is used for political or commercial purposes, and how their data and the data of other people are managed to design their online experience.

Such asymmetries in the business model of big tech have generated an opaque information ecology that undermines *user autonomy*. The web is permeated by multiple troublesome social phenomena, such as the spread of misinformation (Lazer et al., 2018; Mocanu et al., 2015; Vargo et al., 2018; Vosoughi et al., 2018). The latter may include disinformation, i.e. deliberately created untruths, and misinformation, i.e. unintentional spread of untruths, such as under-checked content or already partisan/biased reporting, and attitudinal and emotional polarization (Abramowitz and Saunders, 2008; Baldassarri and Gelman, 2008), political sorting (Fiorina and Abrams, 2008) and polarisation on controversial debates (Cota et al., 2019; McCright and Dunlap, 2011).

The role of research (and behavioural sciences in general) should be not just to drive active scientific discourse about the sources and extent of misinformation (Barberá et al., 2015; DiMaggio et al., 1996; Guess et al., 2020; Lorenz-Spreen et al., 2020) or increasing mass polarisation (Del Vicario et al., 2016; Evans, 2003; Lelkes, 2016). It would be desirable to explore new approaches to foster the potential of the internet to support, rather than undermine, democratic societies (Watts, 2017). Solving global challenges, from pandemics to climate change, requires a strong highly coordinated collective response in contrast with private companies which would just exploit any kind of new technology channels in order to make their own profits.

4.2 A/B testing

Private companies which sell their products online, including Amazon, eBay, Etsy, Facebook, Google, Groupon, LinkedIn, Microsoft, Netflix or Yahoo (Kohavi et al., 2014; Xu et al., 2015), have been using *online controlled experiments* (i.e. A/B testing) for at least two decades, in order to shape their platforms. However, in their words, the aim that drives them is the continuous improvement of users' online experiences. Companies run and analyse thousands concurrent experiments per day in order to validate new ideas, which materially translate in changes in the online environment shown to customers. The latter range from entire redesigns and infrastructure changes (e.g. a post on a social network, mobile app interface, etc) to bug fixes. Finally, the whole collection of successes and failures is then employed in learnings on customer behaviour (Kaufman et al., 2017; Xu et al., 2015). Private companies are used to build in-house experiment infrastructures to maximize their products' success, strongly relying on A/B testing (Kaufman et al., 2017; Xu et al., 2015).

As briefly mentioned before, A/B testing is a randomised controlled trial for assessing the causal effect of introducing a new idea (i.e. treatment) on some wished output. Traditionally, the effect of the treatment is estimated on the whole intended population, since the outcome for a given individual cannot be computed for both versions A and B. It is referred to as the Average Treatment Effect (ATE; Rubin, 1974): it measures the difference in the randomized treatment and control groups as following: $ATE = E[Y = 1] - E[Y = 0]$. The ATE is then the difference in the expected values of the treatment and control group's output. For a given causal effect of interest, the ATE value would suggest the effectiveness of some treatment applied. Statistical inference is usually employed to determine whether an ATE estimate is

statistically consistent (either positively or negatively; Kaufman et al., 2017). However, since ATE is a measure of the average effect, a positive or negative ATE does not assess whether and how a particular individual would react to a given treatment. In the last years, much more attention has been focused on the latter requirement. In particular, the rising of huge data sets containing personalized treatments, along with their individual profiles, has allowed to explore how treatment effects vary across individuals. Accordingly, a new measure has been put forward: Conditional Average Treatment Effect (CATE, $\tau(x) = E[Y(1) - Y(0)|X = x]$) which is the treatment effect conditional on observed covariates. Meta-algorithms are employed in order to compute CATE. The most common of them takes two steps: it employs the so called "base learners" to measure the conditional expectations of the outcomes both for control and treatment groups, separately; afterwards, it takes the difference between these estimates. To date, most used meta-algorithms include:*T-learner*, *S-learner* and *X-learner* (Künzel et al., 2019).

4.3 A case of study: Upworthy.com

I took into account a crucial case study regarding the effectiveness of A/B testing: Upworthy Archive. Upworthy.com is a company founded in 2012 by the author of *The Filter Bubble* (Pariser, 2011) and cofounder Peter Koechley; their aim was to reach large audiences with pieces of news (in their words "stuff that matters") which would have drawn most of people's attention on the web. A key component of their strategy was to setting up attractive "packages," each of them consisting of a headline, a subheading and an image, similar to the one individuals were shown when an article was posted on a social media. In order to establish which package would influence the most, Upworthy adopted A/B testing practices. They were already popular among technology industry and political campaigns. The company optimized content to deliver packages, measure responses, and compute probability of a viewer clicking on different versions of the same story. In the history of American media, Upworthy was a leading actor from 2013–2015; people associated its success to the idea of "clickbait." Speaking at The Guardian's Changing Media Summit in March of 2015, the cofounder Koechley apologized for the just three-years-old online platform saying: 'sorry we kind of broke the internet last year' (O'Reilly, 2015). Upworthy carried out its experiments on its website Upworthy.com, both on the homepage and the article pages, randomly allocating and/or recommending different (usually four) versions (i.e. packages) of the same story to different readers. Editors and website engineers reported that they set only one experiment per page in order to reduce correlations and/or dependencies among experiments. The system recorded the number of individuals that were shown a given package (i.e. *impressions*) and the number that clicked on the package (i.e. *clicks*). After a while, editors decided either to choose the best performing package for finalizing or to keep modifying packages for increasing performance (Matias and Munger, 2019). In November 2019 Good & Upworthy (Upworthy merged with Good Worldwide in 2017) and a team of researchers composed by J. Nathan Matias (Cornell University), Kevin Munger (Penn State University), and Marianne Aubin Le Quere (Cornell University) announced *The Upworthy Research Archive*, a dataset of 32,487 A/B tests carried out by Upworthy from January 2013 to April 2015. In their words, the release of the dataset aimed to enhance knowledge in many fields, including: communication, political science, psychology, statistics, and computer science (Matias and Munger, 2019).

4.3.1 The Upworthy Research Archive

The Upworthy Research Archive dataset consists of 32,488 tests carried out from January 24, 2013 through April 14, 2015, just after an editorial shifts was announced by the company. For each test, the dataset includes viewer reactions to each package in an experiment; there is a median of 4 packages per test. The dataset contains over 150 thousand packages. These packages altogether received over 538 million impressions and over 8 million clicks. Each test includes a median of 14,342 impressions and a median of 201 clicks per test. Each package consists of: the experiment ID; the headline; the subhead, the social media summary (where Upworthy used them); the preview image; the number of impressions and clicks received by the package during the test (see Fig. 4.1; Matias and Munger, 2019).

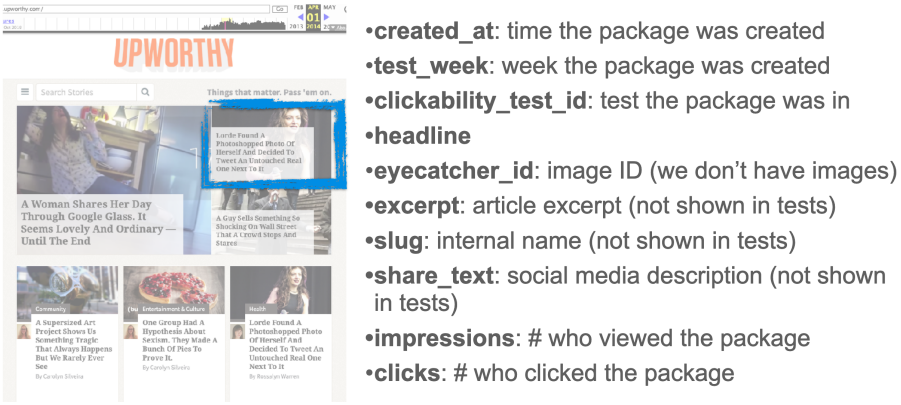


FIGURE 4.1: An example of package features from the Upworthy Research Archive, as described in Matias (2020).

4.3.2 Clickbait: definition and characteristics, linguistic features

Potthast et al. (2018) distinguish between four definitions of clickbait by: emergence, intention, effect and perception. I may define clickbait by its effect (as whatever attracts the most clicks) or as a certain style (e.g. anything being perceived as sensationalist). A New York Times headline might be very effective without employing clickbait style. It is worth distinguishing between clickbait in the effective definition and "clickbait style." It could be that clickbait style actually is not the most effective way of grabbing a reader's attention, or that it even annoys some people—for example the makers and users of downworthy (ZOT), a browser extension that changes clickbait style elements, replacing words like "Awesome" and "WHOA." While I expect my present dataset to fulfil all four definitions of clickbait, it is important to keep in mind the distinction between clickbait as a certain style and clickbait as a click-optimised piece of text.

4.3.3 Clickbait style

The present data set is by definition a clickbait dataset: It is the result of an effort to optimize for number of clicks. Though the effectiveness of any headline may not be due to clickbait style, it is worth examining this style in the context of the Upworthy dataset, as it actually coined much of what I think of as "clickbait style." In fact, studies on clickbait actually take material from upworthy (Blom and Hansen,

2015; Chen et al., 2015). Clickbait style includes formal features (the two-sentence headline, the listicle, unusual punctuation) as well as semantic features (colloquial, overly emotional language). I cast a wide net: I try to include as many clickbait characteristics from the literature as possible.

4.4 Methods

In Fig. 4.2 I show a broad outline of the main steps of the procedure I follow. After employing linguistic methods on the data set in order to infer linguistic features, I convert my data set in input-output relationships for each package, where the features presented in each bundle are linked to its click-though rate. By performing LASSO regression over the latter, I infer feature coefficients which serve us their overall impacts (negatively or positively) on the click-though rate. My first research question about what are the linguistic features most mattering in an A/B testing can be addressed with this procedure. For the second research question I assume a simple linear model for decision making that binds the linguistic features present in a generic package with its probability to be clicked, directly linking to the click-through rate of that package. LASSO regression coefficients are part of this individual decision-making model to quantify the feature impacts. In order to pursue my second research question, i.e. under which conditions a message enhanced by AB testing spread in a social network structure, I need to set up a general framework in which I can ideally 'switch on' the A/B testing machinery and make a comparison afterwards. Therefore, I design a simplistic agent-based model, where I can distinguish between two scenarios. In the first one, messages shared by individuals undergo a pure social spreading; in the second one, instead, A/B tests influences the message selection dynamics. In the following I go through a full description of all methods that I employ for achieving this.

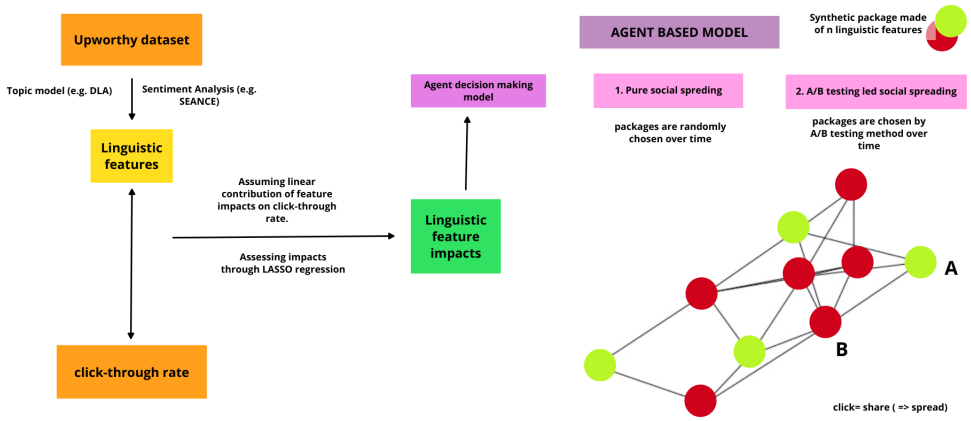


FIGURE 4.2: Broad round-up of the temporal order of my procedure.

4.4.1 Linguistic features extraction

I combine rule-based and machine learning methods to extract formal and semantic linguistic features. Most linguistic features are extracted by the application of an explicit rule, e.g. the presence of a certain character or word. For the topic modelling,

I employ a data-driven approach, by capturing the underlying semantic structure of the corpus. All code is implemented in python. For preprocessing I employ a separate pipelines based on the python libraries nltk, and spaCy separately, as they come with different functions for tokenization, lemmatization and different word lists e.g. for stop word recognition. The output of the feature extraction process is a pandas dataframe. It is a headline-feature matrix with the headlines as rows and the extracted linguistic features, as well as the click-rate as columns. I use this matrix both to describe the upworthy corpus linguistically as well as to see how features are associated with the click rate.

Formal Features

Clickbait style is characterised by formal features, such as a preference for fully formed sentences as titles. This entails overall more words, a higher stop-word to content-word ratio and longer syntactic dependencies in comparison to the succinct headlines of established newspapers, while average word length might be low. Such features are easy to operationalise. For my feature extraction, I take inspiration from Kuiken et al. (2017) and Zheng et al. (2017) who list these and more features (see table 1 for their respective lists). As formal features Safran (2013) lists five headlines types: normal, Question (headline forms a question), How to (headline starts with "how to"), Number (headline introduces a listicle), and reader-addressing (contains a form of "you"). I would like to include the feature of forward referencing (Blom and Hansen, 2015), another characteristic of clickbait style: In headlines like "She Did Not Expect THIS," or "What Happened Next Will Blow Your Mind." Here, one has to click the article to resolve what these headlines are referring to. I am still working on an adequate operationalization of this feature, perhaps simplifying it to the presence of a demonstrative and/or pronoun.

Zheng et al. (2017)	Kuiken et al. (2017)
Presence/number of exclamation mark	Number of characters
Presence/number of question mark	Number of words
Presence of pronoun	Average word length
Presence of interrogative	Number of sentences
Number of words	Number of sentimental words
Number of dots	Readability score
Stop words ratio	Containing question
tf-idf weight of words	Containing quote
tf-idf weight of bigrams	Containing signal words
	Containing pronouns
	Containing number
	First word type

TABLE 4.1: Clickbait feature lists to compute a clickbait score.

Semantic features

Emotional valence (positive / negative) and arousal (strong / weak) are important factors for reading behavior (Chen et al., 2015). Clickbait is further associated with overly positive sentiment. For sentiment analysis, I use the Sentiment Analysis

and Cognition Engine (Crossley et al., 2017). SEANCE is a large knowledge base including many word lists for nuanced sentiment analysis (251 core indeces). Since SEANCE includes word lists that are interpretable as a topic (e.g. "politics, economics, and religion," "Dominance, respect, money, and power," "social relations," "Arts and Academics"), it may also serve as a knowledge base to detect these topics.

In addiction, I employ topic modeling to detect the presence of semantic topics (e.g. feminism, racial equality, LGBT+ issues). For the topic modelling I use Latent Dirichlet Allocation (LDA). LDA is an unsupervised learning algorithm that detects the co-occurence of words across documents (e.g. a headline) in a corpus (the collection of all headlines). Words that cluster together form a "topic," which may or may not be an interpretable semantic topic. For the LDA, I define a document as each unique story plus all its different headlines, i.e. I combine the unique "lede" with all its unique headlines. This yields larger, more semantically coherent documents, which give LDA a better chance of picking up semantically related words. This means that I perform LDA on the story-level, not the headline level. Since I aim to detect semantic topics, preprocessing for the LDA include lowercasing, lemmatization, stopword removal, punctuation removal. While I may lose potentially informative formal features by removing formal features as "noise," I amplify the semantic signal. I retrieve information about stopwords, uppercasing and punctuation in other feature extraction steps.

4.4.2 Assessing linguistic features importance in the data set

Through the linguistic methods mentioned above, I obtain in total M linguistic features extracted along with their relative weights. In other words, for each package k and for each feature i included in it, a relative weight F_i^k is measured. Each package k identifies an online environment and it can be formally described as follows: $\{F_1^k, ..., F_M^k\}$, where F_i^k is the weight assigned by linguistic methods to the linguistic feature i for the package k; $F_i^k = 0$ if the linguistic feature i was not found in the package k.

Individual decision-making model

My main purpose is to build up a decision-making model able to surrogate the interaction of an individual with a news headline presented online. The latter may be thought of consisting of a set of linguistic features: $OE_k = \{j, n, ..., s\}$, where $j, n, ..., s$ are the indices belonging to the linguistic features present in the headline k; $k = 1, ..., K$, K is the total number of packages available. Let us call Y_k the click-through rate value for package k.

The decision-making model DM I assume is a simple linear relationship that binds the linguistic features present in a generic package k with the click-through rate of that package. Formally:

$$DM_k = \sum_{i \in OE_k} w_i = Y_k \qquad \propto p_k \tag{4.1}$$

where w_i are the coefficients quantifying the overall impact of feature i and $p_k \in [0, 1]$.

In order to assess features importance w_i's, let us assume that my decision-making model (Eq. 4.1) holds for each package of the Upworthy data set, that is a simple linear relationship which binds the linguistic features present in the package

with its click-through rate. Since I am provided of a continuous outcome for each package (i.e. the click-through rate), together with the latter assumption, then a linear regression method fits these requirements.

LASSO

In order to measure parameters for all features, I perform a Lasso regression (Least Absolute Shrinkage and Selection Operator). It is a modification of linear regression. In Lasso, the loss function is modified to minimize the complexity of the model by limiting the sum of the absolute values of the model coefficients (also called the $l1-$norm).

Formally, the estimates of LASSO coefficient $(w_1, ..., w_M)$ are the quantities that minimize:

$$\sum_{k=1}^{K} Y_k - \left(w_0 + \sum_{i=1}^{M} w_i F_i^k \right)^2 + \lambda \sum_{i=1}^{M} |w_i| \tag{4.2}$$

where w_0 is the regression intercept and λ is a tuning parameter which to be determined apart. The LASSO regression searches for the best coefficient estimates that fit the data set.

LASSO differs form other regulation methods in this context (as Ridge, OLS, etc) thanks to its second term, which is called "shrinkage penalty;" it has the effect of shrinking the coefficient estimates towards zero. While several regression methods tend to generate a model involving all features, the LASSO not only shrinks parameter estimates but it also pushes some of them to zero. I could say that LASSO performs variable selection, in a certain sense (Georges and Pereira, 2019).

LASSO tuning parameter selection

LASSO implementation needs a way to selecting the tuning parameter λ. Cross-validation (CV) provides a simple method to achieve this task and it is consistent with the goal of minimizing overfitting. In particular, I employ k-fold CV that consists in randomly splitting the set of observed data into k folds of equal size. The first fold works as a validation set, and the LASSO regression is performed on the remaining $k-1$ folds. The fold kept out (i.e. the validation set) is used to compute the mean squared error, MSE_1. By iterating this procedure k times with a different fold of data set used as the validation set, the process generates k estimates of the test error, $MSE_1, MSE_2, ..., MSE_k$. The average of the latter yields the k-fold CV error:

$$CV_k = \frac{1}{k} \sum_{j=1}^{k} MSE_j \ . \tag{4.3}$$

The choices of λ and k are affected by a bias-variance trade-off. It has been shown empirically that $k = 5$ and $k = 10$ values produce test error rate estimates with relatively small bias and variance and it is then common practise to compute k-fold CV employing them (Georges and Pereira, 2019). In light of this, I perform LASSO with respect to a grid of λ tuning parameters, I compute the cross-validation error for each of them, and then I select the one for which the k-fold CV error is the lowest value. Once the tuning parameter is chosen, I finally obtain the best set of LASSO regression coefficients; they represent the overall impact of features on the click-through rate. An illustrative example of the outcome of this procedure is Fig.4.3

where they are shown the top ten impacting linguistic features on the click-though rate.

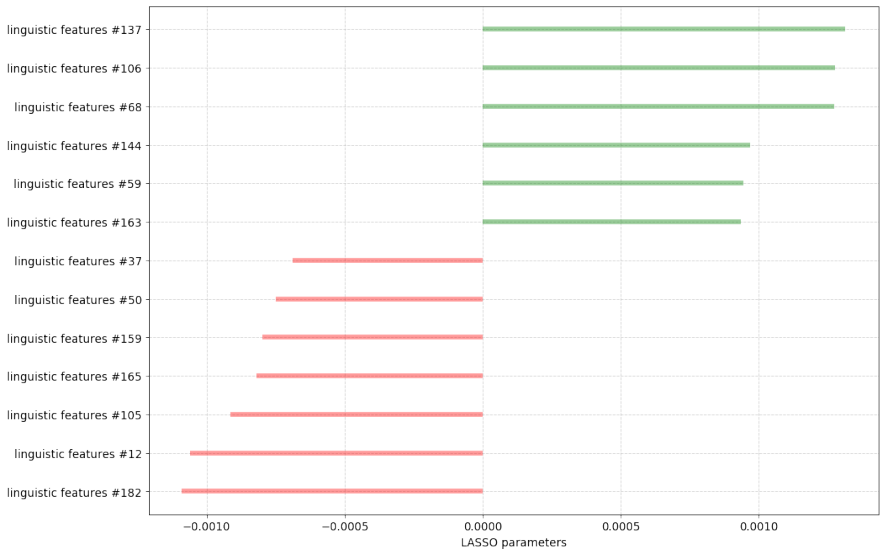


FIGURE 4.3: By having performed 10-fold CV for a bunch of λ values, I select the LASSO coefficients for the best performance and show the first ten ranked in order of decreasing impact on the click-though rate.

Since the focus of my work is on the collective behaviour which emerges from the A/B testing methodology employment in a social spreading framework, the results coming from the most impacting linguistic are not shownin order not to focus on hypothetical linguistic features and their meanings. All the procedure I described so far remains unaltered. This choice is motivated by the fact that linguistic methods settings depend on the data set. The latter exploration would make more accurate the answer to the research question: what features of a written piece of news (e.g. clickbait) make them gain more clicks or, in other words, what features an AB testing mechanism may leverage.

4.4.3 Agent based model of messages social spreading

Once obtained from LASSO performance the impact of each feature on the click-though rate, the decision-making function shown in Eq. 4.1 is fully working for simulating an individual interaction in front of an hypothetical (e.g. clickbait) online environment. Let us figure individuals in a social network which are given of different stimuli (in the spirit of Upworthy); their social interactions may promote sharing of popular messages (i.e. linguistic features). A synthetic reproduction of that dynamics may be thought as following. In my simulation frameworks, N agents are given. My first assumption lies on the fact that they are all equal: they behave through the same decision-making function (Eq. 4.1) in front of a package, provided of the same feature coefficients. The agents are linked one another with respect of a network structure which reproduce social interconnections among them. In order to mimic a social network structure, the best modeling tool which fits my requirements is borrowed from graph theory; I am endowed with a stylized complex network whose nodes are agents presented in the environment and whose edges are the possible interactions among different individuals. I employ several network topologies (e.g. Albert-Barabasi, Erdős-Rényi, Stochastic Block Model) and tune their densities (i.e. sparsity of the network). Let us consider synthetic packages generated by

randomly picking n linguistic features. When an agent is given of a package, its individual decision-making generates an output with respect to the features presented in the package. This response is then converted in a probability to click and a random threshold determines if the agent clicks or not on the content. Once an agent clicks, it *shares* the successful package to its directly connected nearest neighbors, according to an infection rate $\eta \in (0, 1]$. The latter is my second main assumption: the click for an agent means "sharing" a package to the nearest neighbors (with a friction parameter η). In order to pursue my second research question, i.e. under which conditions a piece of news enhanced by A/B testing spread in a social network structure, I need to set up a general framework in which I can ideally 'switch on' the A/B testing machinery and make a comparison afterwards. Therefore, I design the agent-based model and I can distinguish between two scenarios. In the first scenario, messages shared by individuals undergo a pure social spreading; in the second one, instead, A/B tests lead the message selection dynamics.

Pure social spreading scenario

The case of a pure social spreading is meant to be my benchmark scenario. The simulation starts by drawing two randomly generated packages (i.e. packages are done of n linguistic features randomly picked) and are randomly allocated to agents all over the network, package A to half population and package B to the other half. Agents click or not according to my decision-making model and the ones who clicked may spread (according to the infection rate η) to those nearest neighbors of theirs who were shown the other package. This spreading dynamics stops in a few steps, once the opportunities of sharing a new content to neighbors saturate (i.e. each agent may at most see both packages and share them around if it clicked). At this point, a new two packages drawing takes place: one between the old A and B is kept and the other one is partially reassembled according to a mutation rate μ; i.e. μ is the percentage of package features which is randomly varied. Another test administration round starts again by randomly splitting the population in two and randomly assigning the two packages to agents and so on.

A/B testing led social spreading scenario

The case of social spreading in which A/B testing is performed by a third party (e.g. a private company) is a slight but pretty crucial modification of the benchmark framework mentioned above. The A/B testing machinery takes place from the second round package selection on. By comparing click results between A and B of the previous round, it chooses for the new test administration the B package if its performance overcomes A click-though rate and builds a new package by varying B of a mutation rate. In the opposite case, A is kept for the new round and a new package B is built by varying A package. Exploring A/B testing literature, several statistical methods have been employed over time for assessing if B click-through rate greater than A one is statistically consistent in an A/B test. Nevertheless, the main difference is between *frequentist* and *Bayesian* approaches.

A/B testing methodologies In general, one performing an A/B test is basically collecting click-through rates from the control A and the variation (B) and then uses a statistical method to determine which of the two performed better. In a frequentist world, one would use p-values and make a choice between the null hypothesis (i.e. there is no relevant difference between A and B , then I keep A) and the alternative

hypothesis that the variation B is better than the control A. As soon as the p-value reaches statistical significance or large amount of data, the experiment is considered done. The frequentist approach requires a wider difference in performances in order to prefer B.

On the contrary, following the Bayesian methodology, the click-though rate of each variant is modeled as a random variable with a probability distribution. Even in case the improvement of a new variant is small, Bayesian statistics is more willing to choose the variation B against the control A. On one hand, this means that this methodology results more prone to false positive rate; on the other hand, it lets either to control the magnitude of wrong decisions or to perform a quick series of experiments so that one may accumulate small gains from incremental improvements once making a lot of variations is effortless and free of charge (Frasco, 2018).

Since my aim is to simulate how a private company would perform A/B testing, my choice of which methodology to reproduce solely depends on which choice they would make. They actually depend on their purposes and their contexts. For example, one may prefer the latter in situations in which the treatment requires relevant expenses (e.g. engineering maintenance, disruption to the user experience, the costs of implementing it, etc) which would only be offset by large benefits (i.e. wide difference between A and B click-through rates). On the contrary, for situations similar to mine, a Bayesian approach is rather more preferred. Each small improvement is very welcome since no additional expenses one incurs in choosing the variation B against A, even if it is a false positive. It doesn't worth putting effort against the false positive rate; it is much more relevant administrate more tests (i.e. explore wider all the possible combinations) and quickly.

Bayesian A/B testing

The Bayesian methodology for performing A/B testing lies in a nicety of the Bayes theorem in combination with the nature of processes underlying A/B tests. It is usually very hard or impossible in most of cases to obtain closed form solutions to the Bayes theorem; for this reason approximation methods (e.g. Markov Chain Monte Carlo) have been developed. An A/B test can be described by Bernoulli trial since it is a random experiment with only two possible outcomes: "click" and "not to click." The Beta distribution is the conjugate prior of such process; that means the posterior function of the Bayes Theorem lies in the same family of the prior one and then a final function can be built with an iterative process. Therefore, an exact solution of the Bayes formula (i.e. closed-form solution) exists. As just mentioned, a convenient prior distribution form modeling a binomial parameter q is the beta distribution. Starting from a flat, uninformative prior, defined by $Beta(1,1)$, the distribution of p after C clicks and F failures (i.e. impressions minus clicks) is given by $Beta(C+1, F+1)$. $C+1$ and $F+1$ are the two parameters of the beta distribution for the belief. Therefore, in my case I have two beta functions and accordingly their two Bayesian beliefs; they are one for the experimental branch A and the other one for B:$p_A \sim Beta(C_A+1, F_A+1)$ and $p_B \sim Beta(C_B+1, F_A+1)$.

In line with Miller (2015) and without explicitly report the calculation, the probability that B performs better than A in the long run is given by:

$$Pr(p_B > p_A) = \sum_{i=0}^{C_B-1} \frac{Beta(C_A+i, F_B+F_A)}{(F_B+i)Beta(1+i, F_B)Beta(C_A, F_A)} \,. \tag{4.4}$$

Let us define the *uplift* as how much B variation increases the click-through rate with respect to the control A:

$$uplift = \frac{c_B - c_A}{c_A} , \tag{4.5}$$

where c_A and c_B are click-through rates. Let us suppose the latter is positive. In order to assess how trustful is the result, Eq. 4.4 estimates the probability that B option to perform better than A. Note that in the frequentist approach there is no way to calculate this probability. In that case, one computes a p-value and in case it falls under an arbitrary threshold, one would state that "within a certain percentage of confidence level (i.e. the α value, usually 95%), the null hypothesis can be rejected." This is a way different from the Bayesian case here where one would state "this hypothesis is better then the other with a certain probability percentage."

4.5 Results

All the procedure explained in details above has been tested exploratory data set from the Upworthy Research Archive (Matias and Munger, 2019). All the machinery described above may be considered pretty general and, in principle, it could employ successfully other similar data sets.

In the following I present some results obtained from my procedure's performance. In particular, I set up a social spreading framework according to the settings previously described. I simulate $N = 500$ agents allocated according to the Albert-Barabasi network structure for total $T = 500$ time steps. Each 5 time steps a new A-B couple drawing takes place. I perform this simulation either the pure social spreading framework or in the A/B led one. The first A-B couple drawing is common to both of the settings: n_F features (in the shown example $n_F = 7$) are randomly picked for generating the control package and the variation B is assembled varying the control. From the second drawing on, the two setting differ in the A-B couples way of selection. In the pure social spreading case, one of the two previous packages is chosen for the new round (new control A) and challenged against its varied version (new variation B). In the A/B led social spreading setting, the Bayesian A/B testing methodology is employed on the gathered click-through rates at the end of every round; in case the variation B overperforms the control A with a probability greater than a fixed threshold (in the shown simulation the threshold is set at 95%), then the successful B becomes the new control A for the next round and a new variation B is generated varying the control. Al the contrary, if successful conditions for the A/B are not satisfied, the control A remains the same and a new variation is generated from it. In every case, generating the variation depends on a mutation rate μ(in the example shown, $\mu = 3/7$). The simulation of every scenario was performed for $R = 100$ Monte Carlo replicas. Every replica was built according to a unique set of random seeds for pseudo-random generators; this guarantees either that replicas differ one another in random drawings employed or simulation reproducibility.

A crucial measure has been assessing the dynamics of the most successful linguistic features over time. I show a comparison of the two scenarios in Fig. 4.4. In a pure social spreading, the 'winning' linguistic feature behavior over time (averages over 100 replicas) shows an evolution roughly fitted by a third degree polynomial function. At the contrary, in an A/B testing led scenario, the successful linguistic feature behavior follows a trend which can be fitted by a linear function.

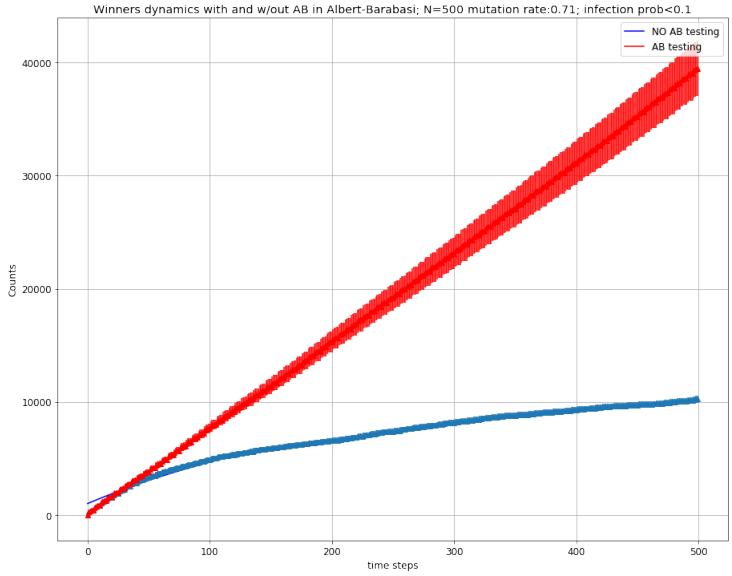


FIGURE 4.4: Comparison of the fist-ranked linguistic feature dynamics in both explored scenarios, a pure social spreading setting and an A/B testing led one. The network structure employed in the simulation is the Albert-Barabasi one.

The results coming up from this exploration are indicating that A/B testing has a substantial influence on the qualitative dynamics of information dissemination on a social network. Through measuring homogeneity and heterogeneity of final distribution of successful messages in a social network framework, either in a pure social spreading or in a AB testing-leaded one, I clearly observe (Fig. 4.5) that the A/B testing-led framework changes the dynamics of the pure social spreading qualitatively, by drawing crowd attention on leading linguistic features. In other words, I observe that A/B testing performed on synthetic social networks structure kills message heterogeneity by promoting the most successful linguistic features identified during the dynamics. In the subsequent data analysis, leveraging the full, confirmatory upworthy data set, I am able to quantify specifically which of the linguistic features are the ones particularly favoured by A/B testing strategies and allow inferences about its potential impact on the online discourse. Due to the dramatic shifts of the public discourse to algorithmically driven platforms, this assessment is of interest for the future design of platforms and regulation that preserve an exchange of arguments while content delivery stays relevant.

4.6 Conclusions

The following Chapter tackles a methodology generally performed by private companies to increase user engagement and satisfaction about online features: A/B testing. Their usage is far from being transparent and may undermine user autonomy (e.g. polarizing individual opinions, mis- and dis- information spreading). Moreover, it has been profoundly under-researched academically, in particular collective behavior effects. For my analysis I leverage a crucial case study dataset (i.e. Upworthy) where news headlines were allocated to users and reshuffled for optimizing clicks. My center of focus is to determine how and under which conditions A/B

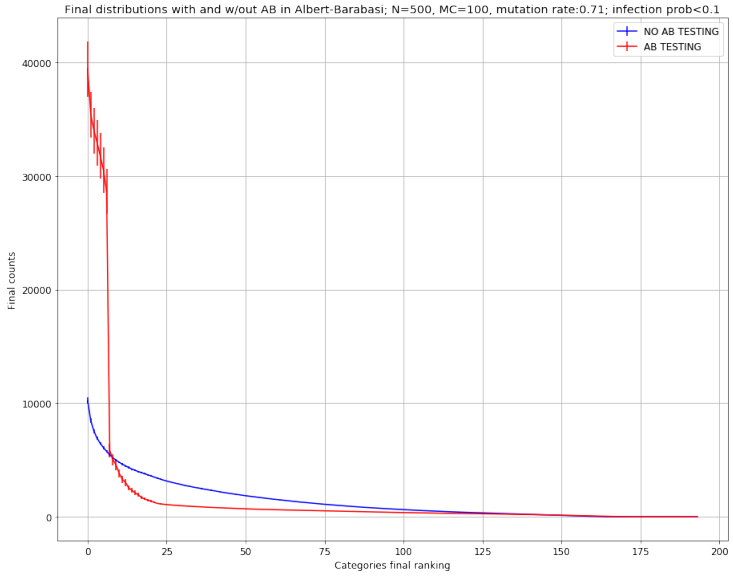


FIGURE 4.5: Comparison of the final ranking distributions of messages in both explored scenarios, a pure social spreading setting and an A/B testing led one. The network structure employed in the simulation is the Albert-Barabasi one.

testing affects the distribution of content on the collective level, specifically on different social network structures. In order to achieve that, I set up an agent-based model reproducing social interaction and an individual decision-making model. I employ several network topologies (e.g. Albert-Barabasi, Erdős-Rényi, Stochastic Block Model) and tune their densities (i.e. sparsity of the network). I consider synthetic randomly generated headline, which consists of a random combination of a fixed number of linguistic features. My results indicate that A/B testing has a substantial influence on the qualitative dynamics of information dissemination on a social network. In particular, the final distributions of feature distribution from my agent-based model, either in a pure social spreading scenario or in the A/B testing condition. I could observe that the A/B testing mechanism increases the homogeneity of information that is spread. In other words, I observe that A/B testing performed on a social network structure reduces the exploration and amplifies exploitation of successful features of early pieces of information, ignoring others. Moreover, my modeling framework promisingly embeds conjecturing policy (e.g. nudging, boosting) interventions.

Appendix A

Market selection and learning under model misspecification

A.1 Proof of Proposition 1.2.1

Solving the problem in (1.1) by Lagrangian method, the system of agent i First Order Conditions is:

$$\begin{cases} u_i'(c_{i,0}) = \mu_i \\ u_i'(c_{i,t}(\sigma)) = \dfrac{\mu_i\, q(\sigma^t)}{\beta_i^t\, p_i(\sigma^t)} \\ \sum\limits_{t=0}^{\infty} \sum\limits_{\sigma^t \in \Sigma^t} q(\sigma^t)\,(e_{i,t}(\sigma) - c_{i,t}(\sigma)) = 0 \quad , \end{cases} \tag{A.1}$$

where μ_i is the Lagrange multiplier associated with agent i's budget constraint and $q(\sigma^0) = 1$. Defining

$$x_{i,t}(\sigma) = \frac{1}{u_i'(c_{i,t}(\sigma))}\,, \quad q(s_t|\sigma^{t-1}) = \frac{q(\sigma^{t-1})}{q(\sigma^t)}\,, \quad p_i(s_t|\sigma^{t-1}) = \frac{p_i(\sigma^t)}{p_i(\sigma^{t-1})}\,,$$

the marginal utility dynamics for agent i can be derived from Eq. A.1 and reads:

$$x_{i,t}(\sigma) = \beta_i\, \frac{p_i(s_t|\sigma^{t-1})}{q(s_t|\sigma^{t-1})}\, x_{i,t-1}(\sigma)\ . \tag{A.2}$$

From the Inada condition at 0 for u_i stated above, the following limit must hold for each agent i and time t, along a given path σ:

$$\lim_{c_{i,t}(\sigma)\to 0^+} x_{i,t}(\sigma) = 0\ . \tag{A.3}$$

By iterating Eq.A.2, I can express the ratio of the marginal utility dynamics of two agents j and i for any path σ at time t, that is

$$\frac{x_{j,t}(\sigma)}{x_{i,t}(\sigma)} = \frac{x_{i,0}}{x_{j,0}} \left(\frac{\beta_j}{\beta_i}\right)^t \frac{p_j(\sigma^t)}{p_i(\sigma^t)}\ . \tag{A.4}$$

Taking the logarithm and dividing by t on both sides of eq. A.4, one obtains

$$\frac{1}{t}\log\frac{x_{j,t}(\sigma)}{x_{i,t}(\sigma)} = \frac{1}{t}\log\frac{x_{i,0}}{x_{j,0}} + \log\frac{\beta_j}{\beta_i} + \frac{1}{t}\sum_{\tau=1}^{t}\left(\log\, p_j(s_\tau|\sigma^{\tau-1}) - \log\, p_i(s_\tau|\sigma^{\tau-1})\right)\ . \tag{A.5}$$

Adding and subtracting $t^{-1}\sum_{\tau=1}^{t}\log p(s_\tau|\sigma^{\tau-1})$ at the right hand side of eq. A.5, taking the limit for $t\to\infty$, and invoking the strong law of large numbers for martingale

differences, one has

$$\lim_{t\to\infty}\frac{1}{t}\log\frac{x_{j,t}(\sigma)}{x_{i,t}(\sigma)}=\left(\log\beta_j-\overline{D}_p(p_j)\right)-\left(\log\beta_i-\overline{D}_p(p_i)\right)\quad\text{p-a.s.}\,. \tag{A.6}$$

Eq. A.6, together with eq. A.3 and the boundedness of aggregate endowments, delivers the statement.

A.2 Auxiliary results

Lemma A.2.1. *Assume* $\exists\zeta\in\Delta^{K-1}$ *such that* $\pi=\sum_{k=1}^{K}\zeta_k\pi_k$. *Then, considering an under-reacting agent* i, *it is*

$$\overline{D}_p(p_i)-\sum_{k=1}^{K}\zeta_k\overline{D}_p(\rho_{i,k},\sigma)=\lim_{t\to\infty}\frac{1-\lambda_i}{t}\sum_{\tau=T+1}^{t}\sum_{s=1}^{S}\pi(s)\left(\frac{\pi(s)}{p_i(s|\sigma^{\tau-1})}-1\right)-\left|O((1-\lambda_i)^2)\right|.$$

Proof. Applying the Taylor expansion argument of Dindo and Massari (2020) Lemma 8 to $\log((\lambda_i p_i(s|\sigma^{t-1})+(1-\lambda_i)\pi_k(s))/p_i(s|\sigma^{t-1}))$, one obtains $\forall\sigma,t$

$$\sum_{s=1}^{S}\pi(s)\log\frac{\lambda_i p_i(s|\sigma^{t-1})+(1-\lambda_i)\pi_k(s)}{p_i(s|\sigma^{t-1})}=(1-\lambda_i)\sum_{s=1}^{S}\pi(s)\left(\frac{\pi_k(s)}{p_i(s|\sigma^{t-1})}-1\right)-\left|O((1-\lambda_i)^2)\right|,$$

with $O(f(x))$ such that $\limsup_x|O(f(x))/f(x)|<\infty$. Since, by the definition of relative entropy, one has $\forall\tau\in\mathbb{N}$

$$\sum_{s=1}^{S}\pi(s)\log\frac{\lambda_i p_i(s|\sigma^{\tau-1})+(1-\lambda_i)\pi_k(s)}{p_i(s|\sigma^{\tau-1})}=D_p(p_i,\sigma^{t-1})-D_p(\rho_{i,k},\sigma^{t-1})\,.$$

Multiplying on both sides of each equation k by the respective ζ_k and summing up, one obtains

$$D_p(p_i,\sigma^{t-1})-\sum_{k=1}^{K}\zeta_k D_p(\rho_{i,k},\sigma^{t-1})=(1-\lambda_i)\sum_{s=1}^{S}\pi(s)\left(\frac{\pi(s)}{p_i(s|\sigma^{\tau-1})}-1\right)-\left|O((1-\lambda_i)^2)\right|.$$

Multiplying both sides by $1/t$ and taking the limit for $t\to\infty$, the statement follows. □

A.3 Minimum λ dynamics for a sub-Bayesian agent to over-perform Bayesian one

According to sufficient and necessary conditions coming from the results in Proposition 3.1 of Bottazzi and Giachini (2019) and in line with my illustrative example where two misspecified models π_a and π_b are given, I notice that asymptotically

two cases are possible: either the agent prediction converges to the best model or a more accurate combination is adopted. Proposition 3.1 of Bottazzi and Giachini (2019) provides a sufficient condition for such improvement in accuracy. In a two-model sub-bayesian learning system, I can recover sufficient and necessary conditions adapting the results in Proposition 3.1 of Bottazzi and Giachini (2019). In particular, there exists a $\underline{\lambda}_i$ such that for $\lambda_i < \underline{\lambda}_i$ I have the convergence to the best model and for $\lambda_i > \underline{\lambda}_i$ I have the emergence of the more accurate mixture. The conditions, however, are implicit in λ_i. there exists a $\underline{\lambda}_i \in [0,1)$ such that, if $\lambda_i > \underline{\lambda}_i$, a sub-Bayesian agent observing λ_i overperforms a Bayesian one. In K-L distance terms: $\overline{D}_p(\lambda_i \pi_{a\wedge b} + (1-\lambda_i)\pi_{b\wedge a}) < \min\{\overline{D}_p(\pi_a), \overline{D}_p(\pi_b)\}$.

Let us consider the setting $\pi_a < p < \pi_b$, and let us call $\tilde{\pi}_a < p$ the value which satisfies the equation: $\overline{D}_p(\tilde{\pi}_a) = \overline{D}_p(\pi_b)$; the inequality $\overline{D}_p(\pi_a) < \overline{D}_p(\pi_b)$ always holds for $\pi_a < \tilde{\pi}_a$. However, a more accurate mixture may perform better that being Bayesian and following the misspecified model π_b. In other words, there exists a $\underline{\lambda}_i \in [0,1)$ such that, if $\lambda_i > \underline{\lambda}_i$, a sub-Bayesian agent observing λ_i overperforms the Bayesian on π_b; in K-L distance terms: $\overline{D}_p(\lambda_i \pi_b + (1-\lambda_i)\pi_a) < \overline{D}_p(\pi_b)$. Then, the value $\underline{\lambda}_i$ solves the equality:

$$p \log \frac{\underline{\lambda}_i \pi_b + (1-\underline{\lambda}_i)\pi_a}{\pi_b} + (1-p) \log \frac{1-\underline{\lambda}_i \pi_b + (1-\underline{\lambda}_i)\pi_a}{1-\pi_b} = 0 \quad . \tag{A.7}$$

Let us set $\pi_{\lambda_i} = \lambda_i \pi_b + (1-\lambda_i)\pi_a$; by investigating $\underline{\lambda}_i$ as a function of π_a, for the implicit function theorem:

$$\left(\frac{p}{\pi_{\underline{\lambda}_i}} - \frac{1-p}{1-\pi_{\underline{\lambda}_i}} \right) \left((\pi_b - \pi_a) \frac{d\,\underline{\lambda}_i}{d\,\pi_a} + (1-\underline{\lambda}_i) \right) = 0 \quad ; \tag{A.8}$$

since $\underline{\lambda}_i \neq p$, the first term in the parenthesis is never zero; this implies:

$$\frac{d\,\underline{\lambda}_i}{d\,\pi_a} = -\frac{1-\underline{\lambda}_i}{\pi_b - \pi_a} \ , \text{ which can be written as: } \frac{d}{d\,\pi_a} \log(1-\underline{\lambda}_i) = -\frac{d}{d\,\pi_a} \log(\pi_b - \pi_a) \ . \tag{A.9}$$

By writing explicitly the dependency from π_a (omitted above for simplicity) :

$$\log(1-\underline{\lambda}_i(\pi_a)) = \frac{c}{\pi_b - \pi_a} \ , \text{ then: } \underline{\lambda}_i(\pi_a) = 1 - \frac{c}{\pi_b - \pi_a} \ ; \tag{A.10}$$

since I know from above that $\underline{\lambda}_i(\tilde{\pi}_a) = 0$, I easily obtain $c = \pi_b - \tilde{\pi}_a$. The functional form is then:

$$\underline{\lambda}_i(\pi_a) = \frac{\tilde{\pi}_a - \pi_a}{\pi_b - \pi_a} \ . \tag{A.11}$$

In the end, it is worth to notice that the quantity in (A.7) is monotone in λ_i; i.e. its partial derivative w.r.t. λ_i is:

$$\left(\frac{p}{\pi_{\underline{\lambda}_i}} - \frac{1-p}{\pi_{1-\underline{\lambda}_i}} \right) (\pi_b - \pi_a) > 0 \ . \tag{A.12}$$

If is mixing for some λ_i, there is also mixing for $\lambda_i + \delta$, where $\delta > 0$.

Appendix B

The Grossman and Stiglitz model: an agent-based extension

B.1 Rational expectations model derivation

I restrict the model to the case:

- agents are homogeneous and provided of rational expectations;
- zero supply of output shares (i.e. $x_t = 0 \ \forall t$).

Since agents are homogeneous, the strategy j is the only one performed, i.e. $K^i = K$. Relying on Muth's concept of rational expectations, rational agents use publicly available information efficiently. That means no systematic errors affect their expectations. Moreover, they fully understand the structure of the economy (i.e. model) and are able to rely on this knowledge when computing the expectations of variables. In this setting, Eq. 2.2 reduces to:

$$P_t = \frac{E_t[P_{t+1} + y_{t+1}]}{R} \qquad ; \tag{B.1}$$

$\{y_t\}$ is the dividend process and the agents are assumed to form correct expectations about, then for any agent i:

$$E_t^i[y_{t+k}] = E_t[y_{t+k}] \qquad k = 1, 2, \dots . \tag{B.2}$$

In addiction, dividend payments are expected to grow at a constant rate s.t. :

$$E_t[y_{t+k}] = (1+\mu)^k y_t \qquad k = 1, 2, \dots , \tag{B.3}$$

where $0 < \mu < r$ is the positive expected dividend growth rate. Since expectation is a linear operator, the pricing equation could be written:

$$P_t = \frac{1}{1+r} \cdot (E_t[P_{t+1}] + E_t[y_{t+1}]) \qquad , \tag{B.4}$$

which holds in all periods; under the assumption of rational expectation, agents fully understand Eq. B.4 and then formulate their expectations consistently with it. Since it holds in all periods, they devide the solution by repeated substitution. In order to make it clear, let us write Eq. B.4:

$$P_{t+k} = \frac{1}{1+r} \cdot (E_{t+k}[P_{t+k+1}] + E_{t+k}[y_{t+k+2}]) \qquad ; \tag{B.5}$$

iteratively substituting the latter in Eq. B.4 for $0 \leq k \leq N-1$ further steps, together with the law of total expectations (which in my case implies: $E_t[E_{t+k}[P_{t+k+1}]] = E_t[P_{t+k+1}]$, $E_t[E_{t+k}[y_{t+k+1}]] = E_t[y_{t+k+1}]$), Eq. B.4 becomes:

$$P_t = \sum_{k=1}^{N} \left(\frac{1}{1+r}\right)^k E_t[y_{t+k}] + \left(\frac{1}{1+r}\right)^N E_t[P_{t+N}] \qquad . \tag{B.6}$$

An assumption usually made in this context is that the last term tends to zero as N tends to ∞; this is also called "no-bubble condition", since it rules out the possibility a bubble could occur for several reasons (e.g. negative bubbles, asset price with a maximum, the present of risk averse investors etc). Then:

$$\lim_{N\to\infty} \left(\frac{1}{1+r}\right)^N E_t[P_{t+N}] = 0 \qquad ; \tag{B.7}$$

another explanation is that if I would set all future values of dividends to zero, the asset price would be still positive; but an asset that never pays out should be inherently worthless, so Eq. B.7 condition excludes this possibility.

What remains of Eq. B.6 states that equilibrium price should equal a discounted present-value sum of expected future dividends; moreover, as I assumed dividend payments are expected to grow (Eq. B.3), Eq. B.6 becomes:

$$F_t = \sum_{k=1}^{\infty} \left(\frac{1}{1+r}\right)^k E_t[y_{t+k}] = \frac{1}{1+r} \sum_{k=1}^{\infty} \left(\frac{1+\mu}{1+r}\right)^{k-1} E_t[y_{t+1}]. \tag{B.8}$$

If the condition $\mu < r$ is satisfied, then the term $\left(\frac{1+\mu}{1+r}\right) < 1$ and the summation of infinite geometric series will converge. A fixed point for price has then been reached:

$$F_t = \frac{(1+\mu)\, y_t}{r-\mu} \qquad . \tag{B.9}$$

As a matter of fact, taking over Eq. B.9 into Eq. B.4 I would obtain an identity. In order to sum up, the latter is the price value the market would converge itself in case I would have homogeneous rational agents trading a risk free asset and a risky asset, with constant expected dividend growth. Actually, what I derived is in the spirit of *Gordon dividend growth model*. Eq. B.9 also implies that:

$$E_t[F_{t+k}] = (1+\mu)^k F_t \qquad k = 1, 2, \ldots \, . \tag{B.10}$$

B.2 Uninformed agent strategy

In line with Eq. 2.4, dividends $\{y_t\}$ and then fundamental prices $\{F_t\}$ are Geometric Brownian motion (GBM) processes. Since uniformed agents fully understand the structure of the economy, they are aware of these processes' nature as well. That means they expect that percentage changes over equal non-overlapping intervals are i.i.d., and if they observe a number k of y_t realizations, $y_0, \ldots, y_k$, the ratios $\frac{y_{t+1}}{y_t}$, $0 \leq t \leq k-1$, have to be independent and identically distributed Marathe and Ryan (2005). Let us define $u_t = \frac{y_{t+1}}{y_t}$ and take logs, one gets:

$$\ln(y_{t+1}) = \ln(y_t) + \ln(u_t). \tag{B.11}$$

Let $w(t) = \ln(y_{t+1}) - \ln(y_t)$; if $\{y_t\}$ is GBM then $\{w(t)\}$ are i.i.n. variables with mean $\hat{\mu}_k$ and variance $\hat{\sigma}_k$; k index states the sample size they are relying on when they compute. Hence, at $t = k$, i.e. after having collected k dividend realizations, uninformed agents will infer $\hat{\mu}_k$ by simply computing:

$$\hat{\mu}_k = \frac{1}{k-1} \sum_{t=1}^{k-1} w(t) \qquad . \tag{B.12}$$

For $t = k$, once provided of $\hat{\mu}_k$, they are able to form expectations about future dividend and fundamental price in their demand, Eq. 2.6. Their precision on mean computation increases the more time goes on. Another way accounted for uninformed learning is leveraging Ordinary Least Square inference. Uninformed agents suppose that dividend process is a linear function of time plus some random noise:

$$y_t = \beta_0 + \beta_1 \, \delta_t + e_t \ . \tag{B.13}$$

The parameters β_0 and β_1 can be estimated consistently by OLS:

$$\{\hat{\beta_0}, \hat{\beta_1}\} = \underset{\{\beta_0, \beta_1\}}{\arg\min} \sum_{t=1}^{n} (y_t - \beta_0 - \beta_1 \, \delta_t)^2 \ . \tag{B.14}$$

Then, $\hat{\mu} = y_t$. Although the first method is pretty more accurate that the second one, model dynamics doesn't change qualitatively.

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