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Research article

Iterative schemes for numerical reckoning of fixed points of new nonexpansive mappings with an application

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Abstract: The goal of this manuscript is to introduce a new class of generalized nonexpansive operators, called (α,β,γ)-nonexpansive mappings. Furthermore, some related properties of these mappings are investigated in a general Banach space. Moreover, the proposed operators utilized in the K-iterative technique estimate the fixed point and examine its behavior. Also, two examples are provided to support our main results. The numerical results clearly show that the K-iterative approach converges more quickly when used with this new class of operators. Ultimately, we used the K-type iterative method to solve a variational inequality problem on a Hilbert space.

Keywords: generalized operator; demiclosed principle; fixed point; convergence result; Banach space
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1. Introduction

In many areas of applied sciences, a given problem is often either very difficult or impossible to solve using the ordinary analytical techniques introduced in the present literature. In such a situation, the approximate value of the desired solution is always needed. Among other things, fixed point theory suggests very useful techniques for finding the approximate values of such solutions. The desired approximate solution for such types of problems can be written as the fixed point of an appropriate operator, i.e., as the solution of an equivalent fixed point equation

$$\mu = \mathcal{F}\mu,$$

where the self-map $\mathcal{F}$ is any suitable operator defined on a subset of a certain space. Some of these types of operators are already known in the literature, here, we present some of them. Suppose a self-map $\mathcal{F}$ of the given subset $\mathcal{W}$ of a Banach space (BS) is given. Then $\mathcal{F}$ is known as a Banach contraction if, for all two points μ, ν in the set $\mathcal{W}$, we have

$$\|\mathcal{F}\mu - \mathcal{F}\nu\| \leq c\|\mu - \nu\|, \tag{1.1}$$

where c is any fixed scalar in $[0, 1)$. Notice that when (1.1) true for the value c exactly equal to 1 then $\mathcal{F}$ is called nonexpansive. As almost always, we will write $F_{\mathcal{F}}$ for the fixed point set of $\mathcal{F}$, that is, $F_{\mathcal{F}} = \{\mu_0 \in \mathcal{W} : \mathcal{F}\mu_0 = \mu_0\}$. If $\mathcal{W}$ is closed subset of a BS (even of a complete metric space) then according to the Banach Contraction Principle (BCP, for short) (see, e.g., [1–5] and others) that each contraction $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ admits a unique fixed point, namely, μ_0 in $\mathcal{W}$ and the sequence of Picard iterates, $\mu_{\lambda+1} = \mathcal{F}\mu_\lambda$ converges strongly to μ_0 for all the initial values. The class of nonexpansive mappings shares important application in different fields of applied sciences [6]. The first important result for these mappings proved by Browder [7] (see, also [8, 9]) in a BS setting, that establishes an existence of fixed point for them. Precisely, Browder's result states that if $\mathcal{W}$ is any bounded closed convex subset of a uniformly convex BS (UCBS, for short), then each nonexpansive mapping $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ always admits a fixed point (however not necessary unique like in BCP). A natural question arises that whether the Picard iterates converge to a fixed of a nonexpansive mapping in general like in BCP? The following simple example shows that it is not the case.

Example 1.1. *Suppose* $\mathcal{W} = [0, 1]$ *and assume that* $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ *is read as:*

$$\mathcal{F}\mu = -\mu + 1 \; \forall \mu \in \mathcal{W}.$$

One can easily notices that $\mathcal{F}$ *is nonexpansive but not contraction and admits a unique fixed point* $\mu_0 = 0.5$. *For all* $\mu_1 = \mu \in \mathcal{W} - \{0.5\}$*, the Picard iterates produce the following sequence,*

$$\mu, (1 - \mu), \mu, (1 - \mu), \mu, (1 - \mu), ...,$$

this sequence is not convergent to the desired fixed point of $\mathcal{F}$.

Hence the Picard iterates are not necessarily convergent in the case of nonexpansive mappings. In this research, we are interested in providing a new class of nonlinear mappings that includes all nonexpansive mappings. The basic properties of these mappings will be established in the setting of Banach spaces. Using these properties, we prove the related convergence theorems using an appropriate, faster iterative method. The superiority of this method is compared with the other known methods by way of an example. We provide an application to our main results.

2. Preliminaries

We need some of the known results. Suppose a Banach space $\mathcal{L}$ is equipped with $\|.\|$. The space $\mathcal{L}$ will be called a UCBS [10] provided that for each choice of $0 \leq \varepsilon < 1$, a real number $0 < \delta < \infty$ can be found satisfying $\frac{\|\mu+\nu\|}{2} \leq (1-\delta)$, for all two elements $\mu, \nu \in \mathcal{L}$ with $\|\mu\| \leq 1$, $\|\nu\| \leq 1$ and $\|\mu+\nu\| \geq \varepsilon$. On the other side, if $\mathcal{L}$ satisfies the property that $\|\mu+\nu\| < 2$ for all two different $\mu, \nu \in \mathcal{L}$ with $\|\mu\| = \|\nu\| = 1$, then $\mathcal{L}$ is called strictly convex.

The space $\mathcal{L}$ is said to be equipped with the Opial's property [11], if and only if for any given weakly convergent sequence, namely, $\{\mu_\lambda\}$ in $\mathcal{L}$ having limit $\mu_0 \in \mathcal{L}$, then for all $\nu_0 \in \mathcal{L} - \{\mu_0\}$, one has

$$\limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| < \limsup_{\lambda\to\infty} \|\mu_\lambda - \nu_0\|.$$

Definition 2.1. *[12] A function $\mathcal{F}$ whose domain of definition is some subset $\mathcal{W}$ possibly of a BS is referred to as a equipped with a condition (I) if and only if there is a self-map $S : [0,\infty) \to [0,\infty)$ with the restrictions $S(0) = 0$ and $S(\nu) > 0$ for each scalar $\nu \in (0,\infty)$ and $\|\mathcal{F}\mu - \mu\| \geq S(d_s(\mu, F_{\mathcal{F}}))$ for each choice of $\mu \in \mathcal{W}$, where $d_s(\mu, F_{\mathcal{F}})$ is the norm distance between the element μ and the set $F_{\mathcal{F}}$.*

Definition 2.2. *Consider a bounded sequence and denote it by $\{\mu_\lambda\}$ in a BS $\mathcal{L}$ and $\emptyset \neq \mathcal{W} \subseteq \mathcal{L}$. The asymptotic radius of $\{\mu_\lambda\}$ corresponding to $\mathcal{W}$ we shall denote and define as $r(\mathcal{W}, \{\mu_\lambda\}) = \inf\{\limsup_{\lambda\to\infty} \|\mu_\lambda - \mu\| : \mu \in \mathcal{W}\}$. The asymptotic center of $\{\mu_\lambda\}$ corresponding to $\mathcal{W}$ we shall denote and define as $A(\mathcal{W}, \mu_\lambda\}) = \{\mu \in \mathcal{W} : \limsup_{\lambda\to\infty} \|\mu_\lambda - \mu\| = r(\mathcal{W}, \mu_\lambda)\}$.*

Remark 2.1. *The set $A(\mathcal{W}, \{\mu_\lambda\})$ contains only one point provided that $\mathcal{L}$ is a UCBS. The property that $A(\mathcal{W}, \{\mu_\lambda\})$ is convex is also known in the setting of weakly compact convex sets (see, e.g., [2, 3] and others).*

Every UCBS has the following important property [13].

Lemma 2.1. *Consider two sequences $\{\mu_\lambda\}$ and $\{\nu_\lambda\}$ in a UCBS $\mathcal{L}$ with $\limsup_{\lambda\to\infty} \|\mu_\lambda\| \leq \tau$, $\limsup_{\lambda\to\infty} \|\nu_\lambda\| \leq \tau$. In addition, if $0 < a \leq a_\lambda \leq b < 1$ and $\lim_{\lambda\to\infty} \|a_\lambda\mu_\lambda + (1-a_\lambda)\nu_\lambda\| = \tau$ for some $\tau \geq 0$, then $\lim_{\lambda\to\infty} \|\mu_\lambda - \nu_\lambda\| = 0$.*

3. The class of (α, β, γ)-nonexpansive mappings

We have precisely described some operators from the literature in the first section and provided some basic knowledge associated with them. Our strategy here is to first create a new class of operators and investigate their relationship to nonexpansive operators. Some elementary and basic results associated with these new operators (including convergence results) will also be established.

Definition 3.1. *A self-map $\mathcal{F}$ whose domain of definition is possibly a subset $\mathcal{W}$ of a BS is called (α, β, γ)-nonexpansive if for all $\mu, \nu \in \mathcal{W}$, one gets the following estimate:*

$$\|\mathcal{F}\mu - \mathcal{F}\nu\| \leq \alpha\|\mu - \nu\| + \beta\|\mu - \mathcal{F}\mu\| + \gamma\|\mu - \mathcal{F}\nu\|,$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$ are fixed scalars such that $\gamma \in [0, 1)$ and $\alpha + \gamma \leq 1$.

From the above definition, we have the statement of the following obvious proposition.

Proposition 3.1. *Suppose $\mathcal{F}$ is nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{W}$ of a BS, then $\mathcal{F}$ is (α, β, γ)-nonexpansive.*

Now one can thinks that whether the converse of the Proposition 3.1 holds in general? The following numerical example answers this question in negative.

Example 3.1. *We now suggest a self-map $\mathcal{F} : [0, 2] \to [0, 2]$ by the formula*

$$\mathcal{F}\mu = \begin{cases} 0 & \text{if } \mu \neq 2 \\ 1 & \text{if } \mu = 2. \end{cases}$$

Here $\mathcal{F}$ is discontinuous and so not nonexpansive. The aim is to prove that $\mathcal{F}$ is (α, β, γ)-nonexpansive. Put $\alpha = \beta = \gamma = \frac{1}{2}$. Now

Case (1). If $\mu, \nu \in [0, 2)$ or $\mu = \nu = 2$, then

$$\|\mathcal{F}\mu - \mathcal{F}\nu\| = 0 \leq \alpha\|\mu - \nu\| + \beta\|\mu - \mathcal{F}\mu\| + \gamma\|\mu - \mathcal{F}\nu\|.$$

Case (2). If $\mu \in [0, 2)$ and $\nu = 2$. Then

$$\begin{aligned} \|\mathcal{F}\mu - \mathcal{F}\nu\| &= 1 = \left\|\frac{\nu}{2}\right\| \\ &= \left\|\frac{\mu - \nu - \mu}{2}\right\| \\ &\leq \left\|\frac{\mu - \nu}{2}\right\| + \left\|\frac{\mu - 0}{2}\right\| \\ &\leq \left\|\frac{\mu - \nu}{2}\right\| + \left\|\frac{\mu - 0}{2}\right\| + \left\|\frac{\mu - 1}{2}\right\| \\ &= \alpha\|\mu - \nu\| + \beta\|\mu - \mathcal{F}\mu\| + \gamma\|\mu - \mathcal{F}\nu\|. \end{aligned}$$

Case (3). If $\nu \in [0, 2)$ and $\mu = 2$. Then

$$\begin{aligned} \|\mathcal{F}\mu - \mathcal{F}\nu\| &= 1 \leq \frac{3}{2} = \left\|\mu - \frac{1}{2}\right\| \\ &= \left\|\frac{2\mu - 1}{2}\right\| \\ &\leq \left\|\frac{(\mu - 1) + (\mu - 0)}{2}\right\| \\ &\leq \left\|\frac{\mu - 1}{2}\right\| + \left\|\frac{\mu - 0}{2}\right\| \\ &\leq \left\|\frac{\mu - \nu}{2}\right\| + \left\|\frac{\mu - 1}{2}\right\| + \left\|\frac{\mu - 0}{2}\right\| \\ &= \alpha\|\mu - \nu\| + \beta\|\mu - \mathcal{F}\mu\| + \gamma\|\mu - \mathcal{F}\nu\|. \end{aligned}$$

Since each of the case, we get the desired result. It immediately follows that in this example, $\mathcal{F}$ is (α, β, γ)-nonexpansive self-map on its domain of definition $[0, 2]$. Accordingly, the class of (α, β, γ)-nonexpansive self-maps properly contains as a subset the class of all nonexpansive self-maps.

Lemma 3.1. *Suppose $\mathcal{F}$ is (α,β,γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{W}$ of a BS with a fixed point, namely, μ_0. In such a case, the estimate $\|\mathcal{F}\mu - \mathcal{F}\mu_0\| \leq \|\mu - \mu_0\|$ holds for all $\mu \in \mathcal{W}$ and $\mu_0 \in F_{\mathcal{F}}$.*

Proof. Since μ_0 is fixed point of $\mathcal{F}$, we have $\mathcal{F}\mu_0 = \mu_0$. Hence

$$\begin{aligned}
\|\mathcal{F}\mu - \mathcal{F}\mu_0\| &\leq \alpha\|\mu - \mu_0\| + \beta\|\mu_0 - \mathcal{F}mu_0\| + \gamma\|\mu - \mathcal{F}\mu_0\| \\
&= \alpha\|\mu - \mu_0\| + \beta\|\mu_0 - \mathcal{F}\mu_0\| + \gamma\|\mu - \mu_0\| \\
&= \alpha\|\mu - \mu_0\| + \beta\|\mu_0 - \mu_0\| + \gamma\|\mu - \mu_0\| \\
&\leq \alpha\|\mu - \mu_0\| + \gamma\|\mu - \mu_0\| \\
&= (\alpha + \gamma)\|\mu - \mu_0\| \\
&\leq \|\mu - \mu_0\|.
\end{aligned}$$

Consequently $\|\mathcal{F}\mu - \mathcal{F}\mu_0\| \leq \|\mu - \mu_0\|$. This completes the required proof.

Now Lemma 3.1 suggests the following result .

Lemma 3.2. *Suppose $\mathcal{F}$ is (α,β,γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{W}$ of a BS $\mathcal{L}$. Consequently, the set $F_{\mathcal{F}}$ is closed. Also, the set $F_{\mathcal{F}}$ is convex provided that $\mathcal{W}$ is convex and the space $\mathcal{L}$ is strictly convex.*

The next lemma shows a very basic property of the (α,β,γ)-nonexpansive mappings.

Lemma 3.3. *Suppose $\mathcal{F}$ is (α,β,γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{W}$ of a BS. Then for all $\mu,\nu \in \mathcal{W}$, it follows that*

$$\|\mu - \mathcal{F}\nu\| \leq \frac{(1+\beta)}{(1-\gamma)}\|\mu - \mathcal{F}\mu\| + \frac{\alpha}{(1-\gamma)}\|\mu - \nu\|.$$

Proof. For any $\mu,\nu \in E$, we have

$$\begin{aligned}
\|\mu - \mathcal{F}\nu\| &\leq \|\mu - \mathcal{F}\mu\| + \|\mathcal{F}\mu - \mathcal{F}\nu\| \\
&\leq \|\mu - \mathcal{F}\mu\| + \alpha\|\mu - \nu\| + \beta\|\mu - \mathcal{F}\mu\| + \gamma\|\mu - \mathcal{F}\nu\| \\
&= (1+\beta)\|\mu - \mathcal{F}\mu\| + \alpha\|\mu - \nu\| + \gamma\|\mu - \mathcal{F}\nu\|.
\end{aligned}$$

Accordingly, we obtained

$$\|\mu - \mathcal{F}\nu\| \leq (1+\beta)\|\mu - \mathcal{F}\mu\| + \alpha\|\mu - \nu\| + \gamma\|\mu - \mathcal{F}\nu\|.$$

It follows that

$$\|\mu - \mathcal{F}\nu\| \leq \frac{(1+\beta)}{(1-\gamma)}\|\mu - \mathcal{F}\mu\| + \frac{\alpha}{(1-\gamma)}\|\mu - \nu\|.$$

This is what we need.

Now we prove a demiclosedness principle.

Lemma 3.4. *Suppose $\mathcal{F}$ is (α,β,γ)-nonexpansive self-map whose domain of definition is possibly a subset $\mathcal{W}$ of a BS. If the given BS satisfies the Opial's property, then the following implication holds.*

$$\mu_\lambda \in \mathcal{W},\ \|\mathcal{F}\mu_\lambda - \mu_\lambda\| \to 0 \text{ and } \mu_\lambda \rightharpoonup \mu_0 \Rightarrow\ \mu_0 \in F_{\mathcal{F}}.$$

Proof. From Lemma 3.3, we have

$$\|\mu_k - \mathcal{F}\mu_0\| \le \frac{(1+\beta)}{(1-\gamma)}\|\mu_k - \mathcal{F}\mu_\lambda\| + \frac{\alpha}{(1-\gamma)}\|\mu_\lambda - \mu_0\|.$$

Since $\alpha + \gamma \le 1$, so $\alpha \le 1 - \gamma$. It follows that

$$\limsup_{\lambda\to\infty} \|\mu_\lambda - \mathcal{F}\mu_0\| \le \limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\|.$$

Since the underlying space has the Opial's property, one get $\mathcal{F}\mu_0 = \mu_0$. This finishes the proof.

4. Results of fixed point convergence

The study of iterative scheme is an important area of research on its own [14, 15]. As we have seen in Example 1.1, Picard iteration is not necessarily convergent in the case of nonexpansive operators. This example suggests that we use other iterative methods. In the literature of fixed-point iterations, one can search for many iterative methods that converge in the case of nonexpansive operators and also suggest better accuracy as compared to the Picard iteration method. If $\mathcal{W}$ is a closed and convex subset of a Banach space, $\lambda \in \mathbb{N}$ and $a_\lambda, b_\lambda, c_\lambda \in (0,1)$. Then for $\mu_1 = \mu \in \mathcal{W}$, Mann [16], Ishikawa [17], Noor [18], Agarwal [19], Abbas [20], and Thakur [21] iterative methods respectively read as follows:

$$\left.\mu_{\lambda+1} = a_\lambda\mathcal{F}\mu_\lambda + (1-a_\lambda)\mu_\lambda, \right\} \tag{4.1}$$

$$\left.\begin{aligned} \mu_{\lambda+1} &= a_\lambda\mathcal{F}\nu_\lambda + (1-a_\lambda)\mu_\lambda, \\ \nu_\lambda &= b_\lambda\mathcal{F}\mu_\lambda + (1-b_\lambda)\mu_\lambda, \end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned} \mu_{\lambda+1} &= a_\lambda\mathcal{F}\nu_\lambda + (1-a_\lambda)\mu_\lambda, \\ \nu_\lambda &= b_\lambda\mathcal{F}\omega_\lambda + (1-b_\lambda)\mu_\lambda, \\ \omega_\lambda &= c_\lambda\mathcal{F}\mu_\lambda + (1-c_\lambda)\mu_\lambda, \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} \mu_{\lambda+1} &= a_\lambda\mathcal{F}\nu_\lambda + (1-a_\lambda)\mu_\lambda, \\ \nu_\lambda &= b_\lambda\mathcal{F}\mu_\lambda + (1-b_\lambda)\mathcal{F}\mu_\lambda, \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{aligned} \mu_{\lambda+1} &= a_\lambda\mathcal{F}\omega_\lambda + (1-a_\lambda)\mathcal{F}\nu_\lambda, \\ \nu_\lambda &= b_\lambda\mathcal{F}\omega_\lambda + (1-b_\lambda)\mathcal{F}\mu_\lambda, \\ \omega_\lambda &= c_\lambda\mathcal{F}\mu_\lambda + (1-c_\lambda)\mu_\lambda, \end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned} \mu_{\lambda+1} &= \mathcal{F}\nu_\lambda, \\ \nu_\lambda &= \mathcal{F}(a_\lambda\omega_\lambda + (1-a_\lambda)\mu_\lambda), \\ \omega_\lambda &= b_\lambda\mathcal{F}\mu_\lambda + (1-b_\lambda)\mu_\lambda. \end{aligned}\right\} \tag{4.6}$$

Remark 4.1. *It is known from [21] that the Thakur iterative method (4.6) converges faster than the iterative methods (4.1)–(4.5) under certain assumptions.*

A natural question arises: does there exist an iterative method that is essentially better than all of the above iterative methods, including the Thakur iterative method (4.6)? To answer this question, Hussain et al. [22] introduced and studied the following $K-$iterative method:

$$\left.\begin{array}{l} \mu_{\lambda+1} = \mathcal{F}\nu_\lambda, \\ \nu_\lambda = \mathcal{F}(a_\lambda \mathcal{F}\omega_\lambda + (1-a_\lambda)\mathcal{F}\mu_\lambda), \\ \omega_\lambda = b_\lambda \mathcal{F}\mu_\lambda + (1-b_\lambda)\mu_\lambda. \end{array}\right\} \tag{4.7}$$

Now, we apply the previously established properties of (α,β,γ)-nonexpansive mappings in this paper and prove the convergence of the K-iterative method (4.7) to the fixed point of these mappings. After this, we then provide another example of these mappings, which essentially exceed nonexpansive mappings, to compare the high accuracy of the K iterative method in this new setting. Using these convergence results, we suggest the K-type iterative method to solve variational inequality problems on Hilbert spaces. This will complete the paper's goals.

Lemma 4.1. *Suppose that $\mathcal{W}$ is a self-map whose domain is possibly a closed convex subset of a UCBS $\mathcal{L}$ and $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is any (α,β,γ)-nonexpansive mappings with $F_{\mathcal{F}} \neq \emptyset$. If $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7), then* $\lim_{\lambda\to\infty} \|\mu_\lambda - \mu_0\|$ *exists for all μ_0 in the set $F_{\mathcal{F}}$.*

Proof. To prove the required result, we take any $\mu_0 \in F_{\mathcal{F}}$. According to the (4.7) and Lemma 3.1,

$$\begin{aligned} \|\omega_\lambda - \mu_0\| &= \|b_\lambda \mathcal{F}\mu_\lambda + (1-b_\lambda)\mu_\lambda - \mu_0\| \\ &= \|b_\lambda(\mathcal{F}\mu_\lambda - \mu_0) + (1-b_\lambda)(\mu_\lambda - \mu_0)\| \\ &\leq b_\lambda\|\mathcal{F}\mu_\lambda - \mu_0\| + (1-b_\lambda)\|\mu_\lambda - \mu_0\| \\ &\leq b_\lambda\|\mu_\lambda - \mu_0\| + (1-b_\lambda)\|\mu_\lambda - \mu_0\| \\ &= \|u_\lambda - \mu_0\|. \end{aligned}$$

Hence

$$\begin{aligned} \|\mu_{\lambda+1} - \mu_0\| &= \|\mathcal{F}\nu_\lambda - \mu_0\| \leq \|\nu_\lambda - \mu_0\| \\ &= \|\mathcal{F}(a_\lambda \mathcal{F}\omega_\lambda + (1-a_\lambda)\mathcal{F}\mu_\lambda) - \mu_0\| \\ &\leq \|a_\lambda \mathcal{F}\omega_\lambda + (1-a_\lambda)\mathcal{F}\mu_\lambda - \mu_0\| \\ &= \|a_\lambda(\mathcal{F}\omega_\lambda - \mu_0) + (1-a_\lambda)(\mathcal{F}\mu_\lambda - \mu_0)\| \\ &\leq a_\lambda\|\mathcal{F}\omega_\lambda - \mu_0\| + (1-a_\lambda)\|\mathcal{F}\mu_\lambda - \mu_0\| \\ &\leq a_\lambda\|\omega_\lambda - \mu_0\| + (1-a_\lambda)\|\mu_k - \mu_0\| \\ &\leq a_\lambda\|\mu_\lambda - \mu_0\| + (1-a_\lambda)\|\mu_\lambda - \mu_0\| \\ &= \|\mu_\lambda - \mu_0\|. \end{aligned}$$

Accordingly, it is seen that $\{\|\mu_\lambda - \mu_0\|\}$ is bounded and non-increasing sequence of reals. It follows that $\lim_{\lambda\to\infty} \|\mu_\lambda - \mu_0\|$ exists for each μ_0 in the set $F_{\mathcal{F}}$.

The next theorem suggests the necessary and sufficient requirements for the existence of a fixed point for (α,β,γ)-nonexpansive self-maps.

Theorem 4.1. *Suppose that $\mathcal{F}$ is a self-map whose domain is possibly closed convex subset of a UCBS $\mathcal{L}$ such that $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is any (α, β, γ)-nonexpansive self-map with the fixed point set $F_{\mathcal{F}}$. Assume that $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7). Then the set $F_{\mathcal{F}} \neq \emptyset$ iff the sequence $\{\mu_\lambda\}$ is bounded as well as* $\lim_{\lambda\to\infty} \|\mathcal{F}\mu_\lambda - \mu_\lambda\| = 0$*.*

Proof. First prove the existence of a fixed point in the case when $\{\mu_\lambda\}$ is bounded and $\lim_{\lambda\to\infty} \|\mathcal{F}\mu_\lambda - \mu_\lambda\| = 0$.

To do this, we choose any $\mu_0 \in A(E, \{\mu_\lambda\})$. By Lemma 3.3, we have

$$\begin{aligned} r(\mathcal{F}\mu_0, \{\mu_\lambda\}) &= \limsup_{k\to\infty} \|\mu_\lambda - \mathcal{F}\mu_0\| \\ &\leq \limsup_{\lambda\to\infty} \left(\frac{(1+\beta)}{(1-\gamma)} \|\mu_\lambda - \mathcal{F}\mu_\lambda\| + \frac{\alpha}{(1-\gamma)} \|\mu_\lambda - \mu_0\| \right) \\ &= \limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| \\ &= r(\mu_0, \{\mu_\lambda\}). \end{aligned}$$

It is seen that $\mu_0, \mathcal{F}\mu_0$ are both the elements of $A(\mathcal{W}, \{\mu_\lambda\}$. Since $A(\mathcal{W}, \{\mu_\lambda\})$ is a singleton set, we have $\mu_0 = \mathcal{F}\mu_0$. Hence $\mathcal{F}$ has a fixed point, that is, $F_{\mathcal{F}} \neq \emptyset$.

In the converse, the aim is to prove that $\{\mu_\lambda\}$ is bounded and $\lim_{\lambda\to\infty} \|\mathcal{F}\mu_\lambda - \mu_\lambda\| = 0$ whenever $F_{\mathcal{F}} \neq \emptyset$. As by the assumption $F_{\mathcal{F}} \neq \emptyset$, we may choose any $\mu_0 \in \mathcal{W}$. So that by Lemma 4.1, $\lim_{\lambda\to\infty} \|\mu_\lambda - \mu_0\|$ exists as well as the sequence $\{\mu_\lambda\}$ is bounded. Thus we can set

$$\lim_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| = \tau. \tag{4.8}$$

It is proved in the proof of Lemma 4.1 that $\|\omega_\lambda - \mu_0\| \leq \|\mu_\lambda - \mu_0\|$. Accordingly, one has

$$\limsup_{\lambda\to\infty} \|\omega_\lambda - \omega_0\| \leq \limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| = \tau. \tag{4.9}$$

Now μ_0 is the point of $\mathcal{F}$, by Lemma 3.1, $\|\mathcal{F}\mu_\lambda - \mu_0\| \leq \|\mu_\lambda - \mu_0\|$. It follows that

$$\limsup_{\lambda\to\infty} \|\mathcal{F}\mu_\lambda - \mu_0\| \leq \limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| = \tau. \tag{4.10}$$

The proof of Lemma 4.1 that $\|\mu_{\lambda+1} - \mu_0\| \leq a_\lambda \|\omega_\lambda - \mu_0\| + (1 - a_\lambda)\|\mu_\lambda - \mu_0\|$. It follows that

$$\|\mu_{\lambda+1} - \mu_0\| - \|\mu_\lambda - \mu_0\| \leq \frac{\|\mu_{\lambda+1} - \mu_0\| - \|\mu_\lambda - \mu_0\|}{a_\lambda} \leq \|\omega_\lambda - \mu_0\| - \|\mu_\lambda - \mu_0\|.$$

Hence

$$\tau = \liminf_{\lambda\to\infty} \|\mu_{\lambda+1} - \mu_0\| \leq \liminf_{\lambda\to\infty} \|\omega_k - \mu_0\|. \tag{4.11}$$

From (4.9) and (4.11), we have

$$\tau = \lim_{\lambda\to\infty} \|\omega_\lambda - \mu_0\|. \tag{4.12}$$

From (4.12), we have

$$\tau = \lim_{\lambda\to\infty} \|\omega_k - \mu_0\| = \lim_{\lambda\to\infty} \|b_\lambda(\mathcal{F}\mu_\lambda - \mu_0) + (1 - b_\lambda)(\mu_\lambda - \mu_0)\|.$$

Hence,

$$\tau = \lim_{\lambda\to\infty} \|b_\lambda(\mathcal{F}\mu_\lambda - \mu_0) + (1 - b_\lambda)(\mu_\lambda - \mu_0)\|. \tag{4.13}$$

By Lemma 2.1, we have

$$\lim_{\lambda\to\infty} \|\mathcal{F}\mu_\lambda - \mu_\lambda\| = 0.$$

This finishes the proof.

Sometimes the strong convergence for a certain operator is not possible in general; therefore, we need the weak convergence in such a case. Under the following conditions, we establish the weak convergence result for (α, β, γ)-nonexpansive self-maps.

Theorem 4.2. *Suppose that $\mathcal{F}$ is a self-map whose domain is possibly closed convex subset of a UCBS $\mathcal{L}$ such that $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is any (α, β, γ)-nonexpansive self-map with $F_{\mathcal{F}} \neq \emptyset$. Assume that $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7) and $\mathcal{L}$ satisfies the Opial's property. Then the sequence $\{\mu_\lambda\}$ converges weakly to some fixed point of $\mathcal{F}$.*

Proof. The sequence $\{\mu_\lambda\}$ is bounded as shown in the Theorem 4.1. The space $\mathcal{L}$ is reflexive because of the convexity of $\mathcal{L}$. Accordingly, we can find a weakly convergent subsequence $\{\mu_{\lambda_m}\}$ of the sequence $\{\mu_\lambda\}$ with a some weak limit $\mu_0 \in \mathcal{W}$. Theorem 4.1 suggests that $\lim_{m\to\infty} \|\mathcal{F}\mu_{\lambda_m} - \mu_{\lambda_m}\| = 0$. All the requirements for Lemma 3.4 are proved and hence $\mu_0 \in F_{\mathcal{F}}$. If μ_0 is the weak limit of μ_λ then we have done. Suppose that μ_0 is not the weak limit of $\{\mu_\lambda\}$, that is, there exists ν_0 different form μ_0 and a subsequence $\{\mu_{\lambda_r}\}$ of $\{\mu_\lambda\}$ such that $\{\mu_{\lambda_r}\}$ converges weakly to ν_0. Using the previous technique, it follows that $\nu_0 \in F_{\mathcal{F}}$. By keeping Lemma 4.1 and Opial's property of $\mathcal{L}$, we have

$$\begin{aligned}
\limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| &= \limsup_{m\to\infty} \|\mu_{\lambda_m} - \mu_0\| < \limsup_{m\to\infty} \|\mu_{\lambda_m} - \mu_0\| \\
&= \limsup_{\lambda\to\infty} \|\mu_\lambda - \nu_0\| = \limsup_{r\to\infty} \|\mu_{\lambda_r} - \nu_0\| \\
&< \limsup_{r\to\infty} \|\mu_{\lambda_r} - \mu_0\| = \limsup \lim_{\lambda\to\infty} \|\mu_\lambda - \mu_0\|.
\end{aligned}$$

Consequently, we proved, $\limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\| < \limsup_{\lambda\to\infty} \|\mu_\lambda - \mu_0\|$. This is a contradiction and accordingly, we proved that μ_0 is the only weak limit of the sequence $\{\mu_\lambda\}$.

The next result is related to the strong convergence, which is based on the assumption that the domain of $\mathcal{F}$ is a compact set.

Theorem 4.3. *Suppose that $\mathcal{F}$ is a self-map whose domain is possibly compact convex subset of a UCBS $\mathcal{L}$ such that $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is any (α, β, γ)-nonexpansive self-map with $F_{\mathcal{F}} \neq \emptyset$. Assume that $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7). Then the sequence $\{\mu_\lambda\}$ converges strongly to some fixed point of $\mathcal{F}$.*

Proof. Since the set $\mathcal{W}$ is convex and compact, $\{\mu_\lambda\}$ contained in E and has a convergent subsequence. We denote this subsequence by $\{\mu_{\lambda_m}\}$ with a strong limit $q_0 \in E$, that is, $\lim_{\lambda_m\to\infty} \|\mu_{\lambda_m} - \mu_0\| = 0$. Suppose $\mu = \mu_{\lambda_m}$ and $\nu = \mu_0$, then applying Lemma 3.3, one has

$$\|\mu_{\lambda_m} - \mathcal{F}\mu_0\| \leq \frac{(1+\beta)}{(1-\gamma)}\|\mu_{\lambda_m} - \mathcal{F}\mu_{\lambda_m}\| + \frac{\alpha}{(1-\gamma)}\|\mu_{\lambda_m} - \mu_0\|. \tag{4.14}$$

By Theorem 4.1, $\lim_{\lambda_k\to\infty}\|\mu_{\lambda_k}-\mathcal{F}\mu_{\lambda_k}\| = 0$ and also from the above $\lim_{\lambda_m\to\infty}\|\mu_{\lambda_m}-q_0\| = 0$. Accordingly, (4.14) provides that $\mu_{\lambda_m}\to\mathcal{F}\mu_0$. It follows that $\mathcal{F}\mu_0=\mu_0$. By Lemma 4.1, $\lim_{k\to\infty}\|\mu_\lambda-\mu_0\|$ exists. Consequently, we have proved that $\mu_0\in F_{\mathcal{F}}$ and $\mu_\lambda\to\mu_0$. This finishes proof.

In the following result, we drop the assumption that the domain of $\mathcal{F}$ is a compact set.

Theorem 4.4. *Suppose that $\mathcal{F}$ is a self-map whose domain is possibly closed convex subset of a UCBS $\mathcal{L}$ such that $\mathcal{F}:\mathcal{W}\to\mathcal{W}$ is any (α,β,γ)-nonexpansive self-map with the fixed point set $F_{\mathcal{F}}$. Assume that $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7). If $\liminf_{\lambda\to\infty} d_s(\mu_\lambda,F_{\mathcal{F}})=0$ holds, then the sequence $\{\mu_\lambda\}$ converges strongly to some fixed point of $\mathcal{F}$.*

Proof. For any $\mu_0\in\mathcal{W}$, we have from Lemma 4.1 that $\lim_{\lambda\to\infty}\|\mu_\lambda-\mu_0\|$ exists. If follows that $\lim_{\mu\to\infty} d_s(\mu_\lambda,F_{\mathcal{F}})$ also exists. Accordingly, $\lim_{\lambda\to\infty} d_s(\mu_\lambda,F_{\mathcal{F}})=0$. Hence, two subsequences, namely, $\{\mu_{\lambda_m}\}$ of $\{\mu_\lambda\}$ and $\{p_m\}$ in $F_{\mathcal{F}}$ exists with property $\|\mu_{\lambda_m}-\mu_m\|\le\frac{1}{2^m}$. It is aim to prove that $\{p_m\}$ is Cauchy in $F_{\mathcal{F}}$. To do this, we can use Lemma 4.1 to write that $\{\mu_\lambda\}$ is non-increasing. Thus, we have

$$\|p_{m+1}-p_m\| \le \|p_{m+1}-\mu_{\lambda_{m+1}}\|+\|\mu_{\lambda_{m+1}}-p_m\|\le\frac{1}{2^{m+1}}+\frac{1}{2^m}.$$

It follows that $\lim_{m\to\infty}\|p_{m+1}-p_m\|=0$. This proves the required. It follows from Lemma 3.2 that $F_{\mathcal{F}}$ is closed hence $\{p_m\}$ converges to some $\mu_0\in F_{\mathcal{F}}$. By Lemma 4.1, $\lim_{\lambda\to\infty}\|\mu_\lambda-\mu_0\|$ exists and hence μ_0 is the strong limit of $\{\mu_\lambda\}$.

Now, we establish final result of this section, which is concerned with Senter and Dotson [12].

Theorem 4.5. *Suppose that $\mathcal{F}$ is a self-map whose domain is possibly closed convex subset of a UCBS $\mathcal{L}$ such that $\mathcal{F}:\mathcal{W}\to\mathcal{W}$ is any (α,β,γ)-nonexpansive self-map with the fixed point set $F_{\mathcal{F}}$. Assume that $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7). If $\mathcal{F}$ possess condition (I), then the sequence $\{\mu_\lambda\}$ converges strongly to some fixed point of $\mathcal{F}$.*

Proof. We prove this result using the Theorem 4.4. To do this, we can write from the Theorem 4.1, $\liminf_{\lambda\to\infty}\|\mathcal{F}\mu_\lambda-\mu_\lambda\|=0$. By applying condition (I) of $\mathcal{F}$, we have $\liminf_{\lambda\to\infty} d_s(\mu_\lambda,F_{\mathcal{F}})=0$. Now, we can apply Theorem 4.4 to obtain the required result. This finishes the proof.

5. Rate of convergence

In this section, we build a new example of an (α,β,γ)-nonexpansive mapping and demonstrate that it is not nonexpansive. Using this example, we compare our studied method with some other iterative methods (see tables and graphs below). According to the observations, the K-iterative method is this new class of mappings that converges faster than the corresponding iterative methods.

Example 5.1. *Suppose $\mathcal{W}=[0,1]$ and set $\mathcal{F}$ on $\mathcal{W}$ as:*

$$\mathcal{F}\mu=\begin{cases}\frac{\mu}{3} & on\ \ \mu\in[0,\frac{1}{2})\\ \\ \frac{\mu}{4} & on\ \ \mu\in[\frac{1}{2},1].\end{cases}$$

Case (1). If $\mu,\nu\in[0,\frac{1}{2})$. Then

$$\|\mathcal{F}\mu-\mathcal{F}\nu\| = \left\|\frac{\mu-\nu}{3}\right\|\le\left\|\frac{\mu-\nu}{2}\right\|$$

$$
\begin{aligned}
&\leq \left\|\frac{\mu-\nu}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{3})}{3}\right\| \\
&\leq \left\|\frac{\mu-\nu}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{3})}{3}\right\| + \left\|\frac{(\mu-\frac{\nu}{3})}{2}\right\| \\
&= \alpha\|\mu-\nu\| + \beta\|\mu-\mathcal{F}\mu\| + \gamma\|\mu-\mathcal{F}\nu\|.
\end{aligned}
$$

Case (2). If $\mu, \nu \in [\frac{1}{2}, 1]$. *Then*

$$
\begin{aligned}
\|\mathcal{F}\mu-\mathcal{F}\nu\| &= \left\|\frac{\mu-\nu}{4}\right\| \leq \left\|\frac{\mu-\nu}{2}\right\| \\
&\leq \left\|\frac{\mu-\nu}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{4})}{3}\right\| \\
&\leq \left\|\frac{\mu-\nu}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{4})}{3}\right\| + \left\|\frac{(\mu-\frac{\nu}{4})}{2}\right\| \\
&= \alpha\|\mu-\nu\| + \beta\|\mu-\mathcal{F}\mu\| + \gamma\|\mu-\mathcal{F}\nu\|.
\end{aligned}
$$

Case (3). If $\mu \in [0, \frac{1}{2})$ *and* $\nu \in [\frac{1}{2}, 1]$. *Then*

$$
\begin{aligned}
\|\mathcal{F}-\mathcal{F}\nu\| &= \left\|\frac{\mu}{3}-\frac{\nu}{4}\right\| \leq \left\|\frac{\nu}{4}\right\| + \left\|\frac{\mu}{3}\right\| \\
&\leq \left\|\frac{3\nu}{8}\right\| + \left\|\frac{4\mu}{9}\right\| \\
&= \left\|\frac{(\nu-\frac{\nu}{4})}{2}\right\| + \left\|\frac{4\mu}{9}\right\| \\
&= \left\|\frac{\left((\mu-\nu)-(\mu-\frac{\nu}{4})\right)}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{3})}{3}\right\| \\
&\leq \left\|\frac{\mu-\nu}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{3})}{3}\right\| + \left\|\frac{(\mu-\frac{\nu}{4})}{2}\right\| \\
&= \alpha\|\mu-\nu\| + \beta\|\mu-\mathcal{F}\mu\| + \gamma\|\mu-\mathcal{F}\nu\|.
\end{aligned}
$$

Case (4). If $\nu \in [0, \frac{1}{2})$ *and* $\mu \in [\frac{1}{2}, 1]$. *Then*

$$
\begin{aligned}
\|\mathcal{F}\mu-\mathcal{F}\nu\| &= \left\|\frac{\mu}{4}-\frac{\nu}{3}\right\| \leq \left\|\frac{\nu}{3}\right\| + \left\|\frac{\mu}{4}\right\| \\
&\leq \left\|\frac{2\nu}{6}\right\| + \left\|\frac{\mu}{2}\right\| \\
&= \left\|\frac{(\nu-\frac{\nu}{3})}{2}\right\| + \left\|\frac{6\mu}{12}\right\| \\
&= \left\|\frac{\left((\mu-\nu)-(\mu-\frac{\nu}{3})\right)}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{4})}{3}\right\| \\
&\leq \left\|\frac{\mu-\nu}{2}\right\| + \left\|\frac{2(\mu-\frac{\mu}{4})}{3}\right\| + \left\|\frac{(\mu-\frac{\nu}{3})}{2}\right\|
\end{aligned}
$$

$$= \alpha\|\mu - \nu\| + \beta\|\mu - \mathcal{F}\mu\| + \gamma\|\mu - \mathcal{F}\nu\|.$$

As a result, all of the preceding cases indicate that the operator $\mathcal{F}$ is (α,β,γ)-nonexpansive.

Now we use Example 5.1 to compare the K iteration and other iterations numerically and graphically. First we choose $a_\lambda = 0.7000, b_\lambda = 0.6500, c_\lambda = 0.9000$ and list the numerical data in the Table 1. Next for $u^* \in F_{\mathcal{F}}$, we keep $\|\mu_\lambda - \mu^*\| < 10^{-15}$ as our criterion for stopping point, and obtaining the graphs shown in Figure 1, where $a_\lambda = \frac{29\lambda+1}{31\lambda+3}, b_\lambda = \frac{4\lambda+1}{22\lambda+3}, c_\lambda = \frac{9\lambda+1}{20\lambda+4}$. Tables 1–4 and Figure 1 suggest the high accuracy of the K iterates to the fixed point of $\mathcal{F}$.

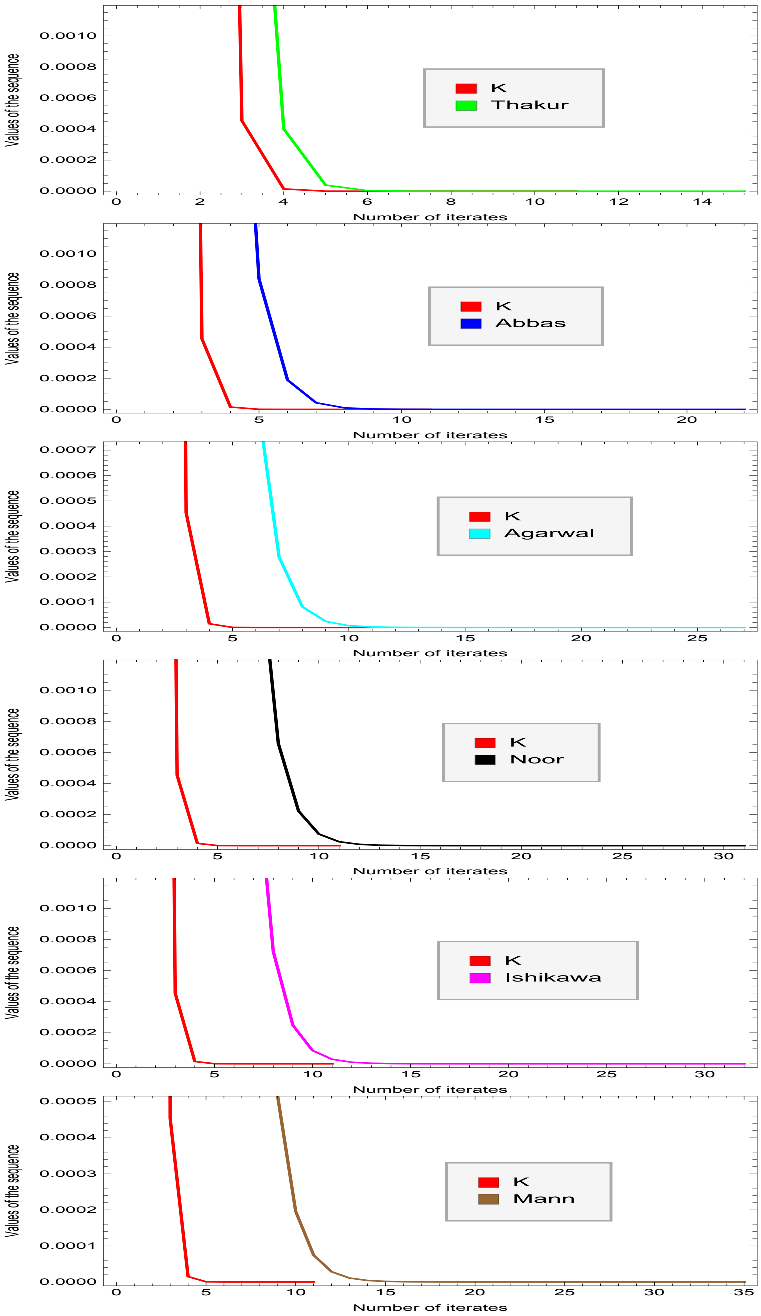

Figure 1. Comparison between K and other iterations with the help of graphs by using $\mathcal{F}$ of Example 5.1, where $\mu_1 = 0.5000$.

Table 1. Numerical date of the different iterates for $\mathcal{F}$ of Example 5.1.

Steps	K	Thakur	Abbas	Agarwal	Noor	Ishikawa	Mann
1	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
2	0.0108	0.0324	0.0458	0.0972	0.1990	0.2097	0.2375
3	0.0002	0.0025	0.0052	0.0225	0.0799	0.0906	0.1266
4	0.0000	0.0001	0.0005	0.0052	0.0321	0.0391	0.0675
5	0.0000	0.0000	0.0000	0.0012	0.0129	0.0169	0.0360
6	0.0000	0.0000	0.0000	0.0002	0.0051	0.0073	0.0192
7	0.0000	0.0000	0.0000	0.0000	0.0020	0.0031	0.0102
8	0.0000	0.0000	0.0000	0.0000	0.0008	0.0013	0.0054

Table 2. $a_\lambda = \frac{2\lambda}{5\lambda+4}$, $b_k = \frac{\lambda}{\sqrt{(3k+9)}}$ and $c_\lambda = \frac{4\lambda}{10\lambda+2}$.

Number of steps required to get the value of the fixed point			
Initial value	Abbas (4.5)	Thakur (4.6)	K(4.7)
0.13	16	12	09
0.31	16	13	09
0.53	16	13	09
0.74	16	13	09
0.98	16	13	09

Table 3. $a_\lambda = \frac{\lambda}{6\lambda+1}$, $b_k = \frac{\lambda}{9\lambda+9}$ and $c_\lambda = 1 - \frac{2\lambda}{\sqrt{9\lambda+1}}$.

Number of steps required to get the value of the fixed point			
Intial value	Abbas (4.5)	Thakur (4.6)	K (4.7)
0.13	18	14	10
0.31	19	15	10
0.53	19	15	10
0.74	19	15	10
0.98	20	15	10

Table 4. $a_\lambda = \frac{\lambda}{\sqrt[7]{12\lambda+27}}$, $b_\lambda = 1 - \left(\frac{1}{\lambda+4}\right)$ and $c_\lambda = \sqrt{\frac{2\lambda}{\lambda+5}}$.

Number of steps required to get the value of the fixed point			
Intial value	Abbas (4.5)	Thakur (4.6)	K (4.7)
0.13	22	13	08
0.31	23	14	08
0.53	23	14	08
0.74	24	15	08
0.98	24	15	08

6. Application

This section suggests an application of our main outcome. Suppose $\mathcal{H}$ denotes a Hilbert space, $\emptyset \neq \mathcal{W} \subset \mathcal{H}$ is convex as well closed. Then $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is known as a monotone mapping if and only if for all μ, ν in the domain, we have

$$\langle \mathcal{M}\mu - \mathcal{M}\nu, \mu - \nu \rangle \geq 0.$$

Notice that, we shall denote by $V(\mathcal{M}, \mathcal{W})$ a variational inequality problem endowed with $\mathcal{M}$ and $\mathcal{W}$ and define as

$$\text{find } \mu^* \in \mathcal{W} \ : \ \langle \mathcal{M}\mu^*, \mu - \mu^* \rangle \geq 0 \text{ for each } \mu \in \mathcal{H}.$$

Suppose that $I : \mathcal{H} \to \mathcal{H}$ and $P_{\mathcal{W}}$, respectively denote the identity self-map and the nearest point projection onto $\mathcal{W}$. Then according to the Bryne [23], if $\eta > 0$ then the point μ^* solves the $V(\mathcal{M}, \mathcal{W})$ if and only if μ^* solves the equation $P_{\mathcal{W}}(I - \eta\mathcal{M})u$. From now on, we denote by $S_{V(\mathcal{M},\mathcal{W})}$, the solution set of the $V(\mathcal{M}, \mathcal{W})$.

Under suitable assumptions, Byrne [23] shown that if $S_{V(\mathcal{M},\mathcal{W})}$ is nonempty and $I - \eta\mathcal{M}$, $P_{\mathcal{W}}(I - \eta\mathcal{M})$ are averaged nonexpansive, the sequence $\{\mu_\lambda\}$ generated by the iterative method $\mu_{\lambda+1} = P_{\mathcal{W}}(I - \eta\mathcal{M})\mu_\lambda$, converges weakly to a solution of a point of $S_{V(\mathcal{M},\mathcal{W})}$.

Now, we study a $V(\mathcal{M}, \mathcal{W})$ in the setting of (α, β, γ)-nonexpansive mappings that are discontinuous in general (as shown by two examples of this paper), instead of nonexpansive operators, which are already well-known to be uniformly continuous. We suggest K-type iterative method, which is better than the many other iterative methods as shown in this paper.

Since it is well-known that every Hilbert space satisfies the Opial's condition. Hence we have the following weak convergence result.

Theorem 6.1. *Suppose that $S_{V(\mathcal{M},\mathcal{W})}$ is non-empty and $\mathcal{F} := P_{\mathcal{W}}(I - \eta\mathcal{M})$ with $\eta > 0$ is (α, β, γ)-nonexpansive and $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7). Consequently, $\{\mu_\lambda\}$ converges weakly to a point of $S_{V(\mathcal{M},\mathcal{W})}$.*

Proof. According to the supposition, $\mathcal{F}$ is (α, β, γ)-nonexpansive. The conclusions now follows from the Theorem 4.2.

Theorem 6.2. *Suppose that $S_{V(\mathcal{M},\mathcal{W})}$ is non-empty and $\mathcal{F} := P_{\mathcal{W}}(I - \eta\mathcal{M})$ with $\eta > 0$ is (α, β, γ)-nonexpansive and $\{\mu_\lambda\}$ is a sequence of K iterative method (4.7). Consequently, $\{\mu_\lambda\}$ converges weakly to a point of $S_{V(\mathcal{M},\mathcal{W})}$ provided that* $\liminf_{\lambda\to\infty} d_s(\mu_\lambda, S_{V(\mathcal{M},\mathcal{W})}) = 0$.

Proof. According to the supposition, $\mathcal{F}$ is (α, β, γ)-nonexpansive. The conclusions now follows from the Theorem 4.4.

7. Conclusions and future works

The concept of (α, β, γ)-nonexpansive operators is introduced, and it has been shown that these operators are more general than the concept of nonexpansive operators. We studied the basic properties of these operators in a general Banach-space setting. The iterative method K is used to compute the fixed points of these operatos. The main result is used to solve variational inequality problems on Hilbert spaces. Next, we will try to use the concept of (α,β,γ)-nonexpansive mappings for solving some other problems involving differential and integral operators. As future works, the authors also have a plan to study multi-valued versions of these operators in order to solve Nash equilibrium, optimization, and inclusion problems in a more general setting of operators. Finally, we appointed the following:

(1) If we define a mapping $\mathcal{F}$ in a Hilbert space $\mathcal{H}$ endowed with inner product space, we can find a common solution to the variational inequality problem by using our iteration (4.7). This problem can be stated as follows: find $\wp^* \in \Delta$ such that

$$\langle \mathcal{F}\wp^*, \wp - \wp^* \rangle \geq 0 \text{ for all } \wp \in \mathcal{H},$$

where $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ is a nonlinear mapping. Variational inequalities are an important and essential modeling tool in many fields such as engineering mechanics, transportation, economics, and mathematical programming, see [24, 25].

(2) We can generalize our algorithm to gradient and extra-gradient projection methods, these methods are very important for finding saddle points and solving many problems in optimization, see [26].

(3) We can accelerate the convergence of the proposed algorithm by adding shrinking projection and CQ terms. These methods stimulate algorithms and improve their performance to obtain strong convergence, for more details, see [27–30].

(4) If we consider the mapping $\mathcal{F}$ as an α-inverse strongly monotone and the inertial term is added to our algorithm, then we have the inertial proximal point algorithm. This algorithm is used in many applications such as monotone variational inequalities, image restoration problems, convex optimization problems, and split convex feasibility problems, see [31–34]. For more accuracy, these problems can be expressed as mathematical models such as machine learning and the linear inverse problem.

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Conflict of interest

The authors declare that they have no conflict of interest.

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