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Research article

A novel iterative approach for resolving generalized variational inequalities

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Abstract: For figuring out general variational inequalities, we propose a novel and innovative iterative method. First, we demonstrate that the fixed point formulation and general vaiational inequality are equivalent. The fixed point formulation is used to formulate the explicit and implicit schemes. The general variational inequalities are the basis for the new algorithms. The newly developed algorithm is demonstrated numerically. For figuring out general variational inequalities, these new methods are innovative. Additionally, the convergence analysis is provided under certain favorable conditions.

Keywords: general variational inequalities; iterative methods; fixed point problem; convergence criteria; projection iterative process
Mathematics Subject Classification: 26A33, 26A51, 26D10

1. Introduction

Since its inception in the 1960s, variational inequality theory has inspired numerous mathematicians. It has been observed that the theory of variational inequalities(VI) now plays a significant role in both pure and applied mathematics, particularly in the field of scientific advancement. This theory is making a big difference in the main field of engineering's problem-solving and mathematical advancement. It has also seen significant expansion in its social, pure, and applied sciences, finance and economics, and industry fields. Variational inequalities have spawned a plethora of numerical approaches that have been developed over time [2–8, 10, 12, 15, 16, 18, 24, 26–30, 34, 35]. In addition, a variety of generalizations and refinements have been made to these methods for variational inequalities. [9, 11, 13, 14, 17, 19, 20, 22, 27, 28, 32, 33, 36] discuss the results of its applications in a variety of fields; however, this theory presented itself as the least artificial, clearest, most integrated, and most effective framework for resolving linear non-linear problems. It also suggests the general treatment they will receive, which is explicitly mentioned in [1, 9, 20, 21, 23, 25, 37, 38]. In addition, in 1988, Noor [26] proposed a diverse class of (VI) using two different operators. which were subsequently documented as general variational inequality(GVI). GVI are one-of-a-kind, brand-new, integrated, and simple methods used to investigate a wide range of that phenomenon in a variety of scientific fields. Noor [26] explored and created different inertial sort projection strategies and iterative plan for general variational imbalances. Under gentle conditions, assembly investigation pertinent to these strategies have been delineated too. The references therein [4, 9, 26, 31].

The exceptional implicit iterative approaches based on modified projection techniques were the subject of the current study. The new method is an extension of previously established variational inequalities. This is useful in applied science applications. This same formulation is frequently used in a number of numerical methods. It is highlighted that (GVI) is helpful to investigate a number of applied and pure sciences, including free and also moving boundary value related problems, odd-order classes, unilateral and non-symmetric obstacles, and so on. The proposed implicit method's convergence criteria are also specified for some mild cases, which would be helpful to students interested in mathematics research. The new findings are primarily motivated by the convergence analysis. The numerical example is provided for implementation.

2. Formulations and basic facts

Assume that convex set λ is in Hilbert space H. The notation of inner product and norm are $\langle \cdot, \cdot \rangle$ and $||\cdot||$ respectively. We assume that the mapping $T, \ \phi : H \longrightarrow H$ are continuous, the problem of getting the value of $\mathbb{C} \in H$, and $\phi(\mathbb{C}) \in \lambda$, we have

$$\langle T\mathbb{C}, \ \phi(t) - \phi(\mathbb{C}) \geq 0, \quad \forall \ \phi(t) \in \lambda, \ t \in H. \tag{2.1}$$

As a result of Noor [29], this class is called non-linear general variational inequality.

Special cases

(i) If we assume $\phi = I$, then (2.1) is considered to getting $\mathbb{C} \in \lambda$, we have

$$\langle T\mathbb{C}, \ t - \mathbb{C} \rangle \geq 0, \ \forall \ t \in \lambda. \tag{2.2}$$

This problem was originally introduced by Stampacchia [24] and is called variational inequality .

(ii) If $\mathrm{K}^* = \{\mathbb{C} \in H : \ \langle \mathbb{C}, t \rangle \geq 0, \ \forall \ t \in \lambda, \}$ is defined a polar cone (dual) of K in H, where λ is also defines as convex set in H, then (2.1) is modified to find $\mathbb{C} \in H$, satisfying the:

$$\phi(\mathbb{C}) \in H \ , \quad T(\mathbb{C}) \in \lambda^*, \qquad \langle \phi(\mathbb{C}), \ T\mathbb{C} \rangle = 0, \tag{2.3}$$

the equality (2.3) is defined as complementarity problem for nonlinear general variational inequality.

(iii) If $\lambda = H$, then (2.1) reduces to find $\mathbb{C}$,that is

$$\langle T\mathbb{C}, \ \phi(\mathbb{C}) \rangle = 0.$$

This is recognized as weak formulation in boundary value problem.

Definition 1. *The non-linear operator denoted by T and mapped from H to H is:*
(i) Strongly(monotone), for $\alpha > 0$, such that

$$\langle T\mathbb{C} - Tt,\ \mathbb{C} - t\rangle \geq \alpha \|\mathbb{C} - t\|^2,\ \forall\ \mathbb{C},\ t \in H.$$

(ii) Lipschitz continuous, for $\beta > 0$, such that

$$\|T\mathbb{C} - Tt\| \leq \beta \|\mathbb{C} - t\|,\ \forall\ \mathbb{C},\ t \in H.$$

(iii) Only Monotone, then

$$\langle T\mathbb{C} - Tt,\ \mathbb{C} - t\rangle \geq 0,\ \forall\ \mathbb{C},\ t \in H.$$

(iv) Called pseudo(monotone), we have

$$\langle T\mathbb{C},\ t - \mathbb{C}\rangle \geq 0 \Rightarrow \langle Tt,\ t - \mathbb{C}\rangle \geq 0,\ \forall\ \mathbb{C},\ t \in H.$$

Remark 1. *The conclusion is that strongly(monotonicity) mapping is a monotonicity and also monotonicity mapping implies a pseudo(monotonicity); however, the inverse does not exist.*

The role is to establish equivalence between fixed point problems and variational inequalities using known results relevant to projection lemma, also known as best projection lemma. Using these findings, we examine the convergence of newly considered approaches to solving optimization and variational inequalities-related problems.

Lemma 1. *[14, 30]: If $\lambda \in H$ be a convex and closed set, then, for $z \in H$, $\mathbb{C} \in \lambda$, satisfying the*

$$\langle \mathbb{C} - z,\ t - \mathbb{C}\rangle \geq 0, \quad \forall\ t \in \lambda, \tag{2.4}$$

if, $\mathbb{C} = P_\lambda \mathbb{C}$, where P_λ(is called $projection\ operator$) of H onto λ and is also called as non expansive operator.

$$\|P_\lambda(\mathbb{C}) - P_\lambda(t)\| \leq \|\mathbb{C} - t\|,\ \forall\ \mathbb{C}, t \in H.$$

3. Projection method and results

The new iterative schemes have been established by using the fixed point formulation for solving the GVI (2.1). The convergence analysis is also provided In this section. This is our main motivation and result.

Lemma 2. *[26, 30]: If λ($Convex\ set$) is in H($Helbert\ space$) and $\mathbb{C} \in H$ solution of the GVI (2.1) if and only if u satisfies the*

$$\phi(\mathbb{C}) = P_\lambda \left[\phi(\mathbb{C}) - \rho T\mathbb{C}\right], \tag{3.1}$$

the ρ is cited as constant and greater than zero and P_λ is defined as the projection from H onto λ.

We apply that the GVI (2.1) is regarded as equivalent to (3.1) from the projection lemma, and then we define the fixed point lemma and the problem. With the help of this formulation, we are able to establish a number of novel implicit schemes, algorithm (Algo) and approaches for figuring out how to solve general variational inequalities. The following new iterative approaches to figuring out the inequalities are denoted by (2.1).

Algo 3.1: For $\mathbb{C}_0 \in H$, approximate $\mathbb{C}_{n+1}$ by the formulation:

$$\phi(\mathbb{C}_{n+1}) \ = P_\lambda \left[\phi(\mathbb{C}_n) - \rho T \mathbb{C}_n\right], \qquad n = 0,\ 1,\ 2... \tag{3.2}$$

the formulation (3.2) has been established by using projection iterative scheme. This scheme has already been discussed many times [26].
Algo 3.2: For $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\phi(\mathbb{C}_{n+1)} = P_\lambda \left[\phi(\mathbb{C}_n) - \rho T \mathbb{C}_{n+1}\right], \qquad n = 0,\ 1,\ 2... \tag{3.3}$$

that is called extragradient technique and considers a new iterative scheme.

For $\phi = I$, we get

$$\mathbb{C}_{n+1} = P_\lambda \left[\mathbb{C}_n - \ \rho T \mathbb{C}_{n+1}\right], \qquad n = 0, 1, 2...$$

see Noot et al. [29].
Algo 3.3: For $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\phi\left(\mathbb{C}_{n+1}\right) = P_\lambda \left[\phi(\mathbb{C}_{n+1}) - \rho T \mathbb{C}_{n+1}\right], \qquad n = 0, 1, 2... \tag{3.4}$$

that is defined as modified projection technique and implicit scheme. We apply predictor- corrector scheme to make them explicit for working out general variational inequalities and can be modified and rewritten as:
Algo 3.4: For a taken $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\begin{aligned} y_n &= P_\lambda \left[\mathbb{C}_n - \rho T \mathbb{C}_n\right], \\ \phi(\mathbb{C}_{n+1}) \ &= \ P_\lambda \left[\phi(y_n) - \rho T y_n\right], \qquad n = 0, 1, 2... \end{aligned} \tag{3.5}$$

that is called double projection method(two step-method).

If $\phi = I$, then,

$$\begin{aligned} y_n &= P_\lambda \left[\mathbb{C}_n - \rho T \mathbb{C}_n\right], \\ \mathbb{C}_{n+1} &= P_\lambda \left[y_n - \rho T y_n\right], \qquad n = 0, 1, 2... \end{aligned}$$

see Noor et al. [30].

The Eq (3.1) can be written as:

$$\phi(\mathbb{C}) = P_\lambda \left[\frac{\phi(\mathbb{C}) + \phi(\mathbb{C})}{2} - \rho T \mathbb{C}\right]. \tag{3.6}$$

This is modified fixed point implicit formulation and is new one to consider the following scheme (implicit method) in Algo 3.5.
Algo 3.5: For a taken $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by iterative formulation:

$$\phi(\mathbb{C}_{n+1}) \ = \ P_\lambda \left[\frac{\phi(\mathbb{C}_n) \ + \phi(\mathbb{C}_{n+1})}{2} - \rho T \mathbb{C}_{n+1}\right]. \qquad n = 0, 1, 2... \tag{3.7}$$

For numerical output of Algo 3.5, we apply the technique of predictor-corrector for the following two steps method of iteration for solution of the GVI.

Algo 3.6: For $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\begin{aligned} y_n &= P_\lambda\left[\mathbb{C}_n - \rho T\mathbb{C}_n\right], \\ \phi(\mathbb{C}_{n+1}) &= P_\lambda\left[\frac{\phi(y_n)+\phi(\mathbb{C}_n)}{2} - \rho T(y_n)\right], \qquad n = 0,1,2... \end{aligned} \tag{3.8}$$

that is an explicit scheme for working out general variational inequalities.

Form Eq (3.1), we have

$$\phi(\mathbb{C}) = P_\lambda\left[\phi(\mathbb{C}) - \rho T(\frac{\mathbb{C}+\mathbb{C}}{2})\right]. \tag{3.9}$$

This scheme can be used to implement the iterative scheme for solving GVI of the following as:

Algo 3.7: For $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\phi(\mathbb{C}_{n+1}) = P_\lambda\left[\phi(\mathbb{C}_n) - \rho T(\frac{\mathbb{C}_n+\mathbb{C}_{n+1}}{2})\right]. \qquad n = 0,1,2... \tag{3.10}$$

For $\phi = I$, we obtain

$$\mathbb{C}_{n+1} = P_\lambda\left[\mathbb{C}_n - \rho T(\frac{\mathbb{C}_n+\mathbb{C}_{n+1}}{2})\right], \qquad \text{n= 0,1, 2...}$$

see Noor et al. [30].

For (3.10), we use the technique of predictor-corrector to convert the above implicit method into explicit method for working out general variational inequalities.

Algo 3.8: For a taken $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\begin{aligned} y_n &= P_\lambda\left[\mathbb{C}_n - \rho T\mathbb{C}_n\right], \\ \phi(\mathbb{C}_{n+1}) &= P_\lambda\left[\phi(\mathbb{C}_n) - \rho T(\frac{\mathbb{C}_n+\ y_n}{2})\right]. \qquad \text{n= 0,1, 2...} \end{aligned} \tag{3.11}$$

We see that (3.11) is the new iterative scheme(implicit midpoint) for solving the GVI. It is evident that different variants of the Eq (3.1) fixed point formulation have been suggested for Algos 3.7 and 3.8. This is the main reason for the paper: it can be combined with fixed point formulations to recommend an implicit scheme for GVI and other optimization problems.

The Eq (3.1) can be modified as:

$$\phi(\mathbb{C}) = P_\lambda\left[\frac{\phi(\mathbb{C})+\phi(\mathbb{C})}{2} - \rho T(\frac{\mathbb{C}+\mathbb{C}}{2})\right]. \tag{3.12}$$

We want to say that from (3.12), we develop the new algorithm called implicit scheme.For implementation of this scheme, we consider the predictor-corrector rule. For this, we take Algo 3.1 as predictor and Algo 3.9 as a corrector step. This procedure is called two steps method for the solution of the GVI.

This new equivalent formulation by using fixed point allows us to motivate the following scheme for the GVI.

Algo 3.9: For $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\phi(\mathbb{C}_{n+1}) = P_\lambda\left[\frac{\phi(\mathbb{C}_n) + \phi(\mathbb{C}_{n+1})}{2} - \rho T(\frac{\mathbb{C}_n + \mathbb{C}_{n+1}}{2})\right], \quad n = 0, 1, 2... \tag{3.13}$$

that is an implicit scheme.

It is again highlighted that the formulation made and constructed in the (3.13) is an implicit schem. For implementation of the modified implicit scheme, we apply predictor-corrector rule. Here, predictor step is consider as Algo 3.1 and corrector step as Algo 3.9 for solving the GVI. This process is also called two steps method and scheme is new for GVI.

Algo 3.10: For $\mathbb{C}_0 \in H$, calculate $\mathbb{C}_{n+1}$ by the formulation:

$$\begin{aligned} y_n &= P_\lambda\left[\mathbb{C}_n - \rho T\mathbb{C}_n\right], \\ \phi(\mathbb{C}_{n+1}) &= P_\lambda\left[\frac{\phi(\mathbb{C}_n) + \phi(y_n)}{2} - \rho T(\frac{\mathbb{C}_n + y_n}{2})\right], \quad \text{n=0,1, 2...} \end{aligned}$$

which is known as two-step method and considers to be new scheme. It is important to provide and prove the convergence analysis of the Algo 3.10 which is our main target and motivation of the new created scheme.

Theorem 1. *Let the mappings T, ϕ are strongly monotone with fixed $\alpha > 0$ and $\delta > 0$ are lipschitz contious with fixed $\beta > 0$ and $\sigma > 0$, respectively. Let $\mathbb{C} \in H$ be the solution of Eq (2.1) and $\mathbb{C}_{n+1}$ be the approximate solution obtained from algo 3.10. If there exists a constant $\rho > 0$, such that*

$$0 < \left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - 4\beta^2 k(1-k)}}{\beta^2}, \tag{3.14}$$

then the approximate solution $\mathbb{C}_{n+1}$ coverges to the exact solution $\mathbb{C} \in H$.

Proof. Let $\mathbb{C} \in H$ be the solution of Eq (1) and $\mathbb{C}_{n+1}$ be the approximate solution from Algo 3.10, then

$$\mathbb{C}_{n+1} = \mathbb{C}_{n+1} - \phi(\mathbb{C}_{n+1}) + P_\lambda\left[\frac{\phi(\mathbb{C}_n) + \phi(\mathbb{C}_{n+1})}{2} - \rho T(\frac{\mathbb{C}_n + \mathbb{C}_{n+1}}{2})\right] \tag{3.15}$$

$$\mathbb{C} = \mathbb{C} - \phi(\mathbb{C}) + P_\lambda\left[\frac{\phi(\mathbb{C}) + \phi(\mathbb{C})}{2} - \rho T(\frac{\mathbb{C} + \mathbb{C}}{2})\right]. \tag{3.16}$$

From Eqs (3.15) and (3.16) we can write

$$\|\mathbb{C}_{n+1} - \mathbb{C}\| = \begin{array}{r} \| \ \mathbb{C}_{n+1} - \phi(\mathbb{C}_{n+1}) + P_\lambda\left[\frac{\phi(\mathbb{C}_n) + \phi(\mathbb{C}_{n+1})}{2} - \rho T(\frac{\mathbb{C}_n + \mathbb{C}_{n+1}}{2})\right] - \mathbb{C} \\ +\phi(\mathbb{C}) - P_\lambda\left[\frac{\phi(\mathbb{C}) + \phi(\mathbb{C})}{2} - \rho T(\frac{\mathbb{C} + \mathbb{C}}{2})\right] \| \end{array}$$

as $\mathbb{C}_\lambda$ is non-expensiveu, the above equation can be written as:

$$\|\mathbb{C}_{n+1} - \mathbb{C}\| \quad \leq \quad \|\mathbb{C}_{n+1} - \mathbb{C} - \phi(\mathbb{C}_{n+1}) + \phi(\mathbb{C})\|$$

$$+\left\|\frac{\phi(\mathbb{C}_{n+1})+\phi(\mathbb{C}_n)}{2}-\frac{\phi(\mathbb{C})+\phi(\mathbb{C})}{2}-\rho T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})+\rho T(\frac{\mathbb{C}+\mathbb{C}}{2})\right\|.$$

Adding and subtracting $(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})$

$$\begin{aligned}||\mathbb{C}_{n+1}-\mathbb{C}|| &\leq ||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))|| \\ &\quad -+ \| (\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})+\frac{\phi(\mathbb{C}_{n+1})+\phi(\mathbb{C}_n)}{2}-\frac{\phi(\mathbb{C})+\phi(\mathbb{C})}{2} \\ &\quad +(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right) \|\end{aligned}$$

$$\begin{aligned}||\mathbb{C}_{n+1}-\mathbb{C}|| &\leq ||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))|| \\ &\quad +\left\|-(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})+\frac{\phi(\mathbb{C}_{n+1})+\phi(\mathbb{C}_n)}{2}-\frac{\phi(\mathbb{C})+\phi(\mathbb{C})}{2}\right\| \\ &\quad +\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\end{aligned}$$

$$\begin{aligned}||\mathbb{C}_{n+1}-\mathbb{C}|| &\leq ||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))|| \\ &\quad +\left\|-\left\{(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\frac{\phi(\mathbb{C}_{n+1})+\phi(\mathbb{C}_n)}{2}+\frac{\phi(\mathbb{C})+\phi(\mathbb{C})}{2}\right\}\right\| \\ &\quad +\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\end{aligned}$$

$$\begin{aligned}||\mathbb{C}_{n+1}-\mathbb{C}|| &\leq ||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))|| \\ &\quad +\frac{1}{2}\,||\mathbb{C}_{n+1}+\mathbb{C}_n-\mathbb{C}-\mathbb{C}-\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}_n)+\phi(\mathbb{C})+\phi(\mathbb{C})|| \\ &\quad +\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\end{aligned}$$

$$\begin{aligned}||\mathbb{C}_{n+1}-\mathbb{C}|| &\leq ||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))|| \\ &\quad +\frac{1}{2}\,||\mathbb{C}_{n+1}-\mathbb{C}-\phi(\mathbb{C}_{n+1})+\phi(\mathbb{C})+\mathbb{C}_n-\mathbb{C}-\phi(\mathbb{C}_n)+\phi(\mathbb{C})|| \\ &\quad +\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\end{aligned}$$

$$\begin{aligned}||\mathbb{C}_{n+1}-\mathbb{C}|| &\leq ||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))|| \\ &\quad +\frac{1}{2}\,||\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))+\mathbb{C}_n-\mathbb{C}-(\phi(\mathbb{C}_n)-\phi(\mathbb{C}))|| \\ &\quad +\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\end{aligned}$$

$$\begin{aligned}\|\mathbb{C}_{n+1}-\mathbb{C}\| &\leq \|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\| \\ &+\frac{1}{2}\|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\|+\frac{1}{2}\|\mathbb{C}_{n}-\mathbb{C}-(\phi(\mathbb{C}_{n})-\phi(\mathbb{C}))\| \\ &+\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\end{aligned}$$

$$\begin{aligned}\|\mathbb{C}_{n+1}-\mathbb{C}\| &\leq \frac{3}{2}\|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\|+\|\mathbb{C}_{n}-\mathbb{C}-(\phi(\mathbb{C}_{n})-\phi(\mathbb{C}))\| \\ &+\left\|(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2})-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|.\end{aligned}$$

Here we consider,

$$\|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\|^2=\|\mathbb{C}_{n+1}-\mathbb{C}\|^2-2\langle\mathbb{C}_{n+1}-\mathbb{C},\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C})\rangle+\|\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C})\|^2$$

$$\|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\|^2\leq\|\mathbb{C}_{n+1}-\mathbb{C}\|^2-2\delta\|\mathbb{C}_{n+1}-\mathbb{C}\|^2+\sigma^2\|\mathbb{C}_{n+1}-\mathbb{C}\|^2$$

$$\|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\|^2\leq\left(1-2\delta+\sigma^2\right)\|\mathbb{C}_{n+1}-\mathbb{C}\|^2$$

$$\|\mathbb{C}_{n+1}-\mathbb{C}-(\phi(\mathbb{C}_{n+1})-\phi(\mathbb{C}))\|\leq\sqrt{1-2\delta+\sigma^2}\,\|\mathbb{C}_{n+1}-\mathbb{C}\|. \tag{3.17}$$

Similarily,

$$\|\mathbb{C}_{n}-\mathbb{C}-(\phi(\mathbb{C}_{n})-\phi(\mathbb{C}))\|\leq\sqrt{1-2\delta+\sigma^2}\,\|\mathbb{C}_{n}-\mathbb{C}\|. \tag{3.18}$$

Also we can have,

$$\begin{aligned}&\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|^2 \\ =\ &\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}\right\|^2-2\rho\left\langle\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2},T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right\rangle \\ &+\rho^2\left\|T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right\|^2\end{aligned}$$

$$\begin{aligned}&\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|^2 \\ \leq\ &\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}\right\|^2-2\alpha\rho\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}\right\|^2 \\ &+\rho^2\beta^2\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}\right\|^2\end{aligned}$$

$$\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|^2\leq\left(1-2\alpha\rho+\rho^2\beta^2\right)\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}\right\|^2$$

$$\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|\leq\sqrt{1-2\alpha\rho+\rho^2\beta^2}\left\|\frac{\mathbb{C}_{n+1}-\mathbb{C}}{2}+\frac{\mathbb{C}_n-\mathbb{C}}{2}\right\|$$

$$\left\|\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2}-\frac{\mathbb{C}+\mathbb{C}}{2}-\rho\left(T(\frac{\mathbb{C}_{n+1}+\mathbb{C}_n}{2})-T(\frac{\mathbb{C}+\mathbb{C}}{2})\right)\right\|$$
$$\leq \frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\mathbb{C}_{n+1}-\mathbb{C}+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\,\|\mathbb{C}_n-\mathbb{C}\|. \quad (3.19)$$

Now,

$$\begin{aligned}\|\mathbb{C}_{n+1}-\mathbb{C}\| \leq\ & \frac{3}{2}\sqrt{1-2\delta+\sigma^2}\,\|\mathbb{C}_{n+1}-\mathbb{C}\|+\frac{1}{2}\sqrt{1-2\delta+\sigma^2}\,\|\mathbb{C}_n-\mathbb{C}\|\\ &+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\,\|\mathbb{C}_{n+1}-\mathbb{C}\|+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\,\|\mathbb{C}_n-\mathbb{C}\|\end{aligned}$$

$$\begin{aligned}&\|\mathbb{C}_{n+1}-\mathbb{C}\|-\frac{3}{2}\sqrt{1-2\delta+\sigma^2}\,\|\mathbb{C}_{n+1}-\mathbb{C}\|-\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\,\|\mathbb{C}_{n+1}-\mathbb{C}\|\\ \leq\ &\frac{1}{2}\sqrt{1-2\delta+\sigma^2}\,\|\mathbb{C}_n-\mathbb{C}\|+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\,\|\mathbb{C}_n-\mathbb{C}\|\end{aligned}$$

$$\begin{aligned}&\left(1-\frac{3}{2}\sqrt{1-2\delta+\sigma^2}-\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\right)\|\mathbb{C}_{n+1}-\mathbb{C}\|\\ \leq\ &\left(\frac{1}{2}\sqrt{1-2\delta+\sigma^2}+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}\right)\|\mathbb{C}_n-\mathbb{C}\|\end{aligned}$$

$$\|\mathbb{C}_{n+1}-\mathbb{C}\| \leq \frac{\frac{1}{2}\sqrt{1-2\delta+\sigma^2}+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}}{1-\frac{3}{2}\sqrt{1-2\delta+\sigma^2}-\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}}\,\|\mathbb{C}_n-\mathbb{C}\|$$
$$\|\mathbb{C}_{n+1}-\mathbb{C}\| \leq \theta\,\|\mathbb{C}_n-\mathbb{C}\|. \quad (3.20)$$

Where,

$$\theta=\frac{\frac{1}{2}\sqrt{1-2\delta+\sigma^2}+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}}{1-\frac{3}{2}\sqrt{1-2\delta+\sigma^2}-\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}}.$$

For contract solution, $\theta<1$, then,

$$\frac{\frac{1}{2}\sqrt{1-2\delta+\sigma^2}+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}}{1-\frac{3}{2}\sqrt{1-2\delta+\sigma^2}-\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}}<1$$

$$\frac{1}{2}\sqrt{1-2\delta+\sigma^2}+\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}<1-\frac{3}{2}\sqrt{1-2\delta+\sigma^2}-\frac{1}{2}\sqrt{1-2\alpha\rho+\rho^2\beta^2}$$
$$\sqrt{1-2\alpha\rho+\rho^2\beta^2}<1-2\sqrt{1-2\delta+\sigma^2}.$$

Let $k=\sqrt{1-2\delta+\sigma^2}$, then,

$$\sqrt{1-2\alpha\rho+\rho^2\beta^2}<1-2k$$
$$1-2\alpha\rho+\rho^2\beta^2<1+4k^2-4k$$
$$\rho^2\beta^2-2\alpha\rho+4k-4k^2<0$$
$$\rho^2\beta^2-2\alpha\rho+4k(1-k)<0.$$

Apply quadratic formula,

$$\rho < \frac{2\alpha \pm \sqrt{4\alpha^2 - 16\beta^2 k(1-k)}}{2\beta^2}$$

$$\rho < \frac{2\alpha \pm 2\sqrt{\alpha^2 - 4\beta^2 k(1-k)}}{2\beta^2}$$

$$\rho < \frac{\alpha}{\beta^2} \pm \frac{\sqrt{\alpha^2 - 4\beta^2 k(1-k)}}{\beta^2}$$

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - 4\beta^2 k(1-k)}}{\beta^2}$$

where, $k > 1$.

Hence,

$$0 < \left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - 4\beta^2 k(1-k)}}{\beta^2}$$

where,

$$\alpha > 2\beta\sqrt{k(1-k)}$$

and $0 < k < 1$.

From Eq (3.22), we have

$$\|\mathbb{C}_{n+1} - \mathbb{C}\| \leq \Pi_{i=0}^{\infty} \theta_i \|\mathbb{C}_0 - \mathbb{C}\|$$

$$\Pi_{i=0}^{\infty} \theta_i = 0.$$

Cosequently, $\lim_{n\to\infty} \|\mathbb{C}_{n+1} - \mathbb{C}\| \to 0$,

$$\lim_{n\to\infty} \|\mathbb{C}_{n+1} - \mathbb{C}\| \to 0$$

$$\lim_{n\to\infty} \mathbb{C}_{n+1} = \mathbb{C}.$$

Which satisfies the general variational inequalities. From (3.14), it follows that $\theta < 1$. This shows that the $\mathbb{C}_{n+1}$ created from the the new Algo (3.10) called approximate solution and has converged to exact solution $\mathbb{C} \in \lambda$ satisfy the inequality (2.1).

4. Numerical example and discussion

Problem 1. *We consider the problem related to general variational inequality (2.1), with* $\phi(\mathbb{C}) = B\mathbb{C} + q$ *and* $T\mathbb{C} = \mathbb{C}$, *where*

$$B = \begin{bmatrix} 4 & -2 & 0 & \cdots & 0 & 0 \\ 1 & 4 & -2 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & -2 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

For out put of the result the following domains and parameters are considered.

$\mathbf{M} = \{\alpha \in R^n / \, 0 \leq \alpha_i \leq 1, \text{ for } i = 1, 2, 3, ...n\}$. Tables 1 and 2 mention the output for the Algo 3.10 with starting initial point $\mathbb{C}^0 = -B^{-1}q$ for the matrix of order $n = 100$. For all output, we set, $\mu, \delta \in (0, 1), \gamma \in [1, 2]$ and $\rho > 0$. The process of iteration will stop when $\|R(\mathbb{C}_n, \rho_n)\| \leq 10^{-7}$. Tables 1 and 2 provide the output of and results of the new establised algorithm (Algo 3.10). From these values, we have seen and observed that by varying of the parameters δ, ρ, and μ, the number of iterations also vary. If we set the parameters accordingly, the number of iterations reduce significantly.

Table 1. Numberical results for Algo 3.10.

Parameters	$\rho = 5, \delta = 0.2,$ $\mu = 0.6$	$\rho = 5, \delta = 0.1,$ $\mu = 0.7$	$\rho = 4, \delta = 0.04,$ $\mu = 0.6$
Iterations	6	10	14

Table 2. Numberical results for Algo 3.10.

Parameters	$\rho = 5, \delta = 0.3,$ $\mu = 0.5$	$\rho = 7, \delta = 0.2,$ $\mu = 0.6$	$\rho = 8, \delta = 0.05,$ $\mu = 0.6$
Iterations	3	4	12

5. Conclusions

In order to establish equivalence and make recommendations for new iterative approaches for solving general variational inequilities, this research paper makes use of the fixed point formulation and general variational inequalities. Under certain favorable condition for the established method's, the convergence analysis is examined. As special cases, the extragradient method and modified double projection methods are among these novel implicit methods. Several novel implicit methods for resolving GVI and related problems can be recommended using the methods and procedures described in this paper. For use, a numerical example is provided.

Conflict of interest

There are no conflicts interest by all authors.

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