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Research article

A result about the atomic decomposition of Bloch-type space in the polydisk

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Abstract: The aim of the paper is to obtain a interesting result about the atomic decomposition of Bloch-type space in the polydisk. The existing similar results have been applied many times to the atomic decompositions of Bloch-type and weighted Bergman spaces in the unit ball.

Keywords: normal function; Bloch-type space; weighted-type space; atomic decomposition; unit polydisk
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1. Introduction

Let $\mathbb{C}$ be the complex plane. Denote by $D(a, r)$ the open disk in $\mathbb{C}$ centered at a with radius r, by $\mathbb{C}^N$ the N-dimensional complex Euclidean space with the inner product $\langle z, w\rangle = \sum_{j=1}^{N} z_j\overline{w}_j$, by $\mathbb{D}$ the open unit disk in $\mathbb{C}$, by $\mathbb{D}^N$ the open unit polydisk in $\mathbb{C}^N$, and by $\mathbb{B}^N$ the open unit ball in $\mathbb{C}^N$. For given $z \in \mathbb{C}^N$, write $|z|_\infty = \max_{1\le j\le N} |z_j|$. Let $H(\mathbb{D}^N)$ be the space of all holomorphic functions on $\mathbb{D}^N$ and $H^\infty(\mathbb{D}^N)$ the space of all bounded holomorphic functions on $\mathbb{D}^N$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}^N} |f(z)|$.

A positive continuous radial function μ on the interval $[0, 1)$ is called normal (see, for example, [9]), if there are $\lambda \in [0, 1)$, a and b $(0 < a < b)$ such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\lambda, 1), \quad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\lambda, 1), \quad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = +\infty.$$

For such function, the following examples were given in [9]:

$$\mu(r) = (1 - r^2)^\alpha, \quad \alpha \in (0, +\infty),$$

$$\mu(r) = (1 - r^2)^\alpha \{\log 2(1 - r^2)^{-1}\}^\beta, \ \ \alpha \in (0, 1), \ \ \beta \in [\frac{\alpha - 1}{2} \log 2, 0],$$

and

$$\mu(r) = (1 - r^2)^\alpha \{\log \log e^2(1 - r^2)^{-1}\}^\gamma, \ \ \alpha \in (0, 1), \ \ \gamma \in [\frac{\alpha - 1}{2} \log 2, 0].$$

The following fact can be used to prove that there exist lots of non-normal functions. It follows from [14] that if μ is normal, then for each $s \in (0, 1)$ there exists a positive constant $C = C(s)$ such that

$$C^{-1}\mu(t) \le \mu(r) \le C\mu(t) \tag{1.1}$$

for $0 \le r \le t \le r + s(1 - r)$. From (1.1), it is easy to check the following functions are non-normal

$$\mu(r) = \Big|\sin\big(\log \frac{1}{1 - r}\big)\Big| v_\alpha(r) + 1$$

and

$$\mu(r) = \Big|\sin\big(\log \frac{1}{1 - r}\big)\Big| v_\alpha(r) + \frac{1}{e^{e^{\frac{1}{1-r}}}},$$

where

$$v_\alpha(r) = [(1 - r)\big(\log \frac{e}{1 - r}\big)^\alpha]^{-1}.$$

The functions $\{\mu, \nu\}$ will be called a normal pair if μ is normal and for b in above definition of normal function, there exists $\beta > b$ such that

$$\mu(r)\nu(r) = (1 - r^2)^\beta.$$

If μ is normal, then there exists ν such that $\{\mu, \nu\}$ is normal pair (see [17]). Note that if $\{\mu, \nu\}$ is a normal pair, then ν is also normal. One of the purposes of introducing normal pair is to characterize the duality of spaces defined by the normal function (see [4, 25]).

Given a normal function μ, the Bloch-type space $\mathcal{B}_\mu(\mathbb{D}^N)$ consists of all $f \in H(\mathbb{D}^N)$ such that

$$\beta_\mu(f) = \sup_{z \in \mathbb{D}^N} \sum_{k=1}^{N} \mu(z_k) \Big|\frac{\partial f}{\partial z_k}(z, \ldots, z_N)\Big| < +\infty.$$

Endowed with the norm $\|f\|_{\mathcal{B}_\mu(\mathbb{D}^N)} = |f(0)| + \beta_\mu(f)$, it is a Banach space. There were a handful of studies on the space $\mathcal{B}_\mu(\mathbb{D}^N)$ (see, for example, [5, 6]). But the spaces with some special normal functions defined on the unit ball or unit disk and some operators have been extensively studied (see, for example, [3, 12, 19, 20, 28], where [28] contains the elementary knowledge of such space).

Given a normal function μ, the weighted-type space $H^\infty_\mu(\mathbb{D}^N)$ consists of all $f \in H(\mathbb{D}^N)$ such that

$$\|f\|_{H^\infty_\mu(\mathbb{D}^N)} = \sup_{z \in \mathbb{D}^N} \prod_{k=1}^{N} \mu(z_k) \big|f(z, \ldots, z_N)\big| < +\infty.$$

$H^\infty_\mu(\mathbb{D}^N)$ is a Banach space with the norm $\|\cdot\|_{H^\infty_\mu(\mathbb{D}^N)}$. There are lots of studies about such spaces on the unit ball or unit disk and some operators (see, for example, [7, 8, 10, 11, 21, 22, 24]).

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$ and $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ the weighted Lebesgue measure on $\mathbb{D}$, where $-1 < \alpha < +\infty$. For given $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$, $-1 < \alpha_j < +\infty$, $j = 1, \ldots, N$, and $0 < p < +\infty$, the weighted Bergman space $A^p_{\vec{\alpha}}(\mathbb{D}^N)$ consists of all $f \in H(\mathbb{D}^N)$ such that

$$\|f\|^p_{A^p_{\vec{\alpha}}(\mathbb{D}^N)} = \int_{\mathbb{D}^N} |f(z)|^p dA_{\vec{\alpha}}(z) < +\infty,$$

where $dA_{\vec{\alpha}}(z) = dA_{\alpha_1}(z_1) \ldots dA_{\alpha_N}(z_N)$. For some information about this space, see, for example, [23]. When $p \geq 1$, the weighted Bergman space with the norm $\|\cdot\|_{A^p_{\vec{\alpha}}(\mathbb{D}^N)}$ becomes a Banach space. While $p \in (0, 1)$, it is a Fréchet space with the translation invariant metric

$$d(f, g) = \|f - g\|^p_{A^p_{\vec{\alpha}}(\mathbb{D}^N)}.$$

The reason why people study the atomic decompositions for holomorphic function spaces is that it is very useful in operator theory. In particular, it can be used to describe dual spaces or to study basic questions such as the boundedness, the compactness or the Schatten class membership of concrete operators (see, for example, [1, 13–15, 18]). The atomic decompositions for holomorphic function spaces have been studied. For example, Coifman and Rochberg in [2] studied this problem on the weighted Bergman space. Zhu in [28] modified the proof of [2] and gave the atomic decomposition for the weighted Bloch space on the unit ball. Motivated by the previous studies, Zhang et al. in [26] characterized the atomic decomposition for the μ-Bergman space on the unit ball. Later, Zhang et al. in [27] also considered the atomic decomposition for μ-Bloch space on the unit ball. The works of [26, 27] extended the corresponding results in [28].

We find that in above-mentioned works, the following result have been used many times in the atomic decompositions (see [28] for some details):

For any $p > 0$ and $\alpha > -1$, there exists a positive constant C independent of the separation constants r and η such that

$$|f(z) - Sf(z)| \leq C\sigma \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{(pb-N-1-\alpha)/p}}{|1 - \langle z, a_k\rangle|^b} \Big(\int_{D(a_k,2r)} |f(w)|^p dv_\alpha(w) \Big)^{\frac{1}{p}}$$

for all $r \in (0, 1)$, $z \in \mathbb{B}^N$ and $f \in H(\mathbb{B}^N)$, where σ is some constant related to r and η, and the operator S on $H(\mathbb{B}^N)$ is defined by

$$Sf(z) = \sum_{k=1}^{N} \sum_{j=1}^{J} \frac{v_\beta(D_{kj}) f(a_{kj})}{(1 - \langle z, a_{kj}\rangle)^b}. \tag{1.2}$$

A very natural problem is to extend this useful result to some other domains in $\mathbb{C}^N$. For example, here we will consider the result on $\mathbb{D}^N$. Another reason of our extension is that $\mathbb{D}^N$ and $\mathbb{B}^N$ are completely different domains in $\mathbb{C}^N$ (see, for example, [16]), which may produce some differences in methods and techniques.

In this paper, positive numbers are denoted by C, and they may vary in different situations. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive number C such that $a \leq Cb$ (resp. $a \geq Cb$). When $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

2. Preliminary results

In this section, we will obtain several elementary results which are used to prove the main results. We first have the following basic one (see, for example, [28]).

Lemma 2.1. *There exists a positive integer* $\widehat{N}$ *such that for any* $0 < r < 1$, *we can find a sequence* $\{a_k\}$ *in* $\mathbb{D}$ *and a set* $\{D_k\}$ *consisting of Lebesgue measurable sets satisfying the following conditions:*

(1) $\mathbb{D} = \cup_{k=1}^{\infty} D(a_k, r)$;

(2) $D_k \cap D_j = \emptyset$ *for* $k \neq j$;

(3) $D(a_k, \frac{r}{4}) \subset D_k \subset D(a_k, r)$ *for every* $k \in \mathbb{N}$;

(4) Each point $z \in \mathbb{D}$ *belongs to at most* $\widehat{N}$ *of the sets* $D(a_k, 4r)$.

Let $\vec{r} = (r_1, \dots, r_N)$, $0 < r_j < 1$, $j = 1, \dots, N$. Then for each fixed r_j there exist a sequence $\{a_{jk}\}$ and a set $\{D_{jk}\}$ satisfy Lemma 2.1. For convenience, we denote by $a_k = (a_{1k}, \dots, a_{Nk})$, by $\widehat{D}(a_k, \vec{r}) = D(a_{1k}, r_1) \times \cdots \times D(a_{Nk}, r_N)$, and by $\widehat{D}_k = D_{1k} \times \cdots \times D_{Nk}$.

By using Lemma 2.1, we obtain the following similar disjoint decomposition of $\mathbb{D}^N$.

Lemma 2.2. *There exists a positive integer* $\widehat{N}$ *such that for any* $\vec{r} = (r_1, \dots, r_N)$, $0 < r_j < 1$, $j = 1, \dots, N$, *we can find a sequence* $\{a_k\}$ *in* $\mathbb{D}^N$ *and a set* $\{\widehat{D}_k\}$ *consisting of Lebesgue measurable sets satisfying the following conditions:*

(1) $\mathbb{D}^N = \cup_{k=1}^{\infty} \widehat{D}(a_k, \vec{r})$;

(2) $\widehat{D}_k \cap \widehat{D}_j = \emptyset$ *for* $k \neq j$;

(3) $\widehat{D}(a_k, \frac{1}{4}\vec{r}) \subset \widehat{D}_k \subset \widehat{D}(a_k, \vec{r})$ *for every* $k \in \mathbb{N}$;

(4) Each point $z \in \mathbb{D}^N$ *belongs to at most* $\widehat{N}^N$ *sets* $\widehat{D}(a_k, 4\vec{r})$.

The following integral expression for the functions in $A^1_\alpha(\mathbb{B}^N)$ is well known (see [28]).

Lemma 2.3. *If* $\alpha > -1$ *and* $f \in A^1_\alpha(\mathbb{B}^N)$, *then*

$$f(z) = \int_{\mathbb{B}^N} \frac{f(w) dv_\alpha(w)}{(1 - \langle z, w \rangle)^{\alpha+N+1}}.$$

Let $x^* = (x_1, \dots, x_{j-1})$ and $x_* = (x_{j+1}, \dots, x_N)$, $j = 1, \dots, N$. As an application of Lemma 2.3 for $N = 1$, if we regard the function $f(z_1, \dots, z_N) \in A^1_{\vec{\alpha}}(\mathbb{D}^N)$ as a one-variable function $f(z^*, z_j, z_*) \in A^1_{\alpha_j}(\mathbb{D})$, $j = 1, \dots, N$, then we have the following integral expression on $A^1_{\vec{\alpha}}(\mathbb{D}^N)$.

Lemma 2.4. *If* $\alpha_j > -1$, $j = 1, \dots, N$, *and* $f \in A^1_{\vec{\alpha}}(\mathbb{D}^N)$, *then*

$$f(z) = \int_{\mathbb{D}^N} \frac{f(w) dA_{\vec{\alpha}}(w)}{\prod_{j=1}^N (1 - \overline{w}_j z_j)^{\alpha_j+2}}.$$

Remark 2.1. In order to obtain Theorem 3.1, we need give a key decomposition for $\mathbb{D}^N$. We first further partition the sets $\{\widehat{D}_k\}$ in Lemma 2.2. Actually, we partition the set $\widehat{D}_1$ and use automorphisms to carry the partition to other $\widehat{D}_k$'s. To this end, we let $\vec{\eta} = (\eta_1, \dots, \eta_N)$, where each η_j denotes a positive number that is much smaller than the separation constant r_j in $\vec{r} = (r_1, \dots, r_N)$, in the sense that the quotient η_j / r_j is small. We fix a finite sequence $\{z_1, \dots, z_J\}$ in $\widehat{D}(0, \vec{r})$, depending on $\vec{\eta}$, such that $\{\widehat{D}(z_j, \vec{\eta})\}$ covers

$\widehat{D}(0,\vec{r})$ and that $\{\widehat{D}(z_j,\frac{1}{4}\vec{\eta})\}$ are disjoint. We then can enlarge each set $\widehat{D}(z_j,\frac{1}{4}\vec{\eta})\cap\widehat{D}(0,\vec{r})$ to a Borel set E_j in the way that $E_j\subset\widehat{D}(z_j,\vec{\eta})$ and that

$$\widehat{D}(0,\vec{r})=\bigcup_{j=1}^{J}E_j$$

is a disjoint union. Here we just give a instruction. If you would like, you can see the proof of Lemma 2.28 in [28] for how to achieve this.

Let $w=(w_1,\dots,w_N)\in\mathbb{D}^N$ and $\varphi_w(z)$ be the involutive automorphism of $\mathbb{D}^N$. Then

$$\varphi_w(z)=(\varphi_{w_1}(z_1),\dots,\varphi_{w_N}(z_N)),$$

where

$$\varphi_{w_j}(z_j)=\frac{w_j-z_j}{1-\overline{w}_jz_j}$$

is the involutive automorphism of $\mathbb{D}$.

For fixed $j\in\{1,\dots,J\}$ and $a_k=(a_{1k},\dots,a_{Nk})$, we define $b_{jk}=\varphi_{a_k}(z_j)$. For $l\in\{1,\dots,N\}$, we define $E_{lj}=\{z_l\in\mathbb{D}:(z^*,z_l,z_*)\in E_j\}$. Let $B_{ljk}=D_{lk}\cap\varphi_{a_{lk}}(E_{lj})$, where sets $\{D_{lk}\}$ are in Lemma 2.1 for $\{a_{lk}\}$ and $l\in\{1,\dots,N\}$. Since

$$D_{lk}=\bigcup_{j=1}^{J}B_{ljk}$$

is a disjoint union for every l and k, we obtain a disjoint decomposition

$$\mathbb{D}^N=\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{J}\widehat{B}_{jk}$$

of $\mathbb{D}^N$, where $\widehat{B}_{jk}$ is the Cartesian product of B_{ljk}, $l=1,\dots,N$.

By using the decomposition of $\mathbb{D}^N$, we give the following definition, which is similar to those of (1.2).

Definition 2.1. Let $b_l>1$ and $\beta_l=b_l-2$, $l=1,\dots,N$. Then the operator S on $H(\mathbb{D}^N)$ is defined by

$$Sf(z)=\sum_{k=1}^{\infty}\sum_{j=1}^{J}\frac{A_{\vec{\beta}}(\widehat{B}_{jk})f(b_{jk})}{\prod_{l=1}^{N}(1-\overline{b}_{ljk}z_l)^{b_l}}.$$

Remark 2.2. From the definition of $dA_{\vec{\beta}}$, it follows that

$$A_{\vec{\beta}}(\widehat{B}_{jk})=\prod_{l=1}^{N}A_{\beta_l}(B_{ljk}).$$

3. Main results and proofs

After discussions in Part 2, we now return to the following result.

Theorem 3.1. *For $p > 0$, $\alpha_l > -1$, $l = 1, \dots, N$, there exists a positive constant C independent of the separation constants $\vec{r}$, $\vec{\eta}$, $f \in H(\mathbb{D}^N)$ and $z \in \mathbb{D}^N$, such that*

$$\left|f(z) - Sf(z)\right| \leq C\sigma \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{b_l - \frac{\alpha_l+2}{p}}}{\prod_{l=1}^{N}|1-\overline{a}_{lk}z_l|^{b_l}} \Big(\int_{\widehat{D}(a_k, 2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w) \Big)^{\frac{1}{p}}$$

for all $\vec{r} = (r_1, \dots, r_N)$, $0 < r_l < 1$, $l = 1, \dots, N$, where

$$\sigma = \frac{|\vec{\eta}'|}{(r_1' r_2' \cdots r_N')^{\frac{2}{p}-1}} + \eta_1\eta_2\cdots\eta_N, \quad r_l' = \tanh(r_l), \quad \vec{\eta}' = (\tanh(\eta_1), \dots, \tanh(\eta_N)),$$

and $\tanh(\cdot)$ *is the hyperbolic tangent function.*

Proof. Without loss of generality, we may assume that $f \in A^1_{\vec{\beta}}(\mathbb{D}^N)$. By the integral expression of the functions in $A^1_{\vec{\beta}}(\mathbb{D}^N)$, we have

$$f(z) = \int_{\mathbb{D}^N} \frac{f(w)dA_{\vec{\beta}}(z)}{\prod_{l=1}^{N}(1-\overline{w}_l z_l)^{b_l}}.$$

Since $\{\widehat{B}_{jk}\}$ is a partition of $\mathbb{D}^N$, we can write

$$f(z) - Sf(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \int_{\widehat{B}_{jk}} \Big[\frac{f(w)}{\prod_{l=1}^{N}(1-\overline{w}_l z_l)^{b_l}} - \frac{f(b_{jk})}{\prod_{l=1}^{N}(1-\overline{b}_{ljk} z_l)^{b_l}}\Big] dA_{\vec{\beta}}(w). \tag{3.1}$$

Since $\widehat{B}_{jk}$ is the Cartesian product of B_{ljk}, $l = 1, \dots, N$, it follows that

$$\int_{\widehat{B}_{jk}} dA_{\vec{\beta}}(w) = \int_{B_{1jk}} \int_{B_{2jk}} \cdots \int_{B_{Njk}} dA_{\vec{\beta}}(w). \tag{3.2}$$

By using the triangle inequality in (3.1), we have

$$\left|f(z) - Sf(z)\right| \leq I(z) + H(z),$$

where by (3.2) we get

$$I(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \frac{1}{\prod_{l=1}^{N}|1-\overline{b}_{ljk}z_l|^{b_l}} \int_{B_{1jk}} \int_{B_{2jk}} \cdots \int_{B_{Njk}} \left|f(w) - f(b_{jk})\right| dA_{\vec{\beta}}(w)$$

and

$$H(z) = \sum_{k=1}^{\infty}\sum_{j=1}^{J} \frac{1}{\prod_{l=1}^{N}|1-\overline{b}_{ljk}z_l|^{b_l}} \int_{B_{1jk}} \int_{B_{2jk}} \cdots \int_{B_{Njk}} \Big|\frac{\prod_{l=1}^{N}(1-\overline{b}_{ljk}z_l)^{b_l}}{\prod_{l=1}^{N}(1-\overline{w}_l z_l)^{b_l}} - 1\Big| \left|f(w)\right| dA_{\vec{\beta}}(w).$$

We first estimate $I(z)$. Let

$$I_{jk} = \int_{B_{1jk}} \int_{B_{2jk}} \cdots \int_{B_{Njk}} \left| f(w) - f(b_{jk}) \right| dA_{\vec{\beta}}(w). \tag{3.3}$$

By a change of variables in (3.3), we have

$$I_{jk} = \prod_{l=1}^{N} (1 - |b_{ljk}|^2)^{b_l} \int_{E_{1jk}} \int_{E_{2jk}} \cdots \int_{E_{Njk}} \frac{|f \circ \varphi_{b_{jk}}(w) - f \circ \varphi_{b_{jk}}(0)|}{\prod_{l=1}^{N} |1 - \overline{b}_{ljk} w_l|^{2b_l}} dA_{\vec{\beta}}(w), \tag{3.4}$$

where

$$\begin{aligned} E_{ljk} &= \varphi_{b_{ljk}}(B_{ljk}) \subset \varphi_{b_{ljk}} \circ \varphi_{a_{lk}}(E_{lj}) \subset \varphi_{b_{ljk}} \circ \varphi_{a_{lk}}(D(z_{lj}, \eta_l)) \\ &= \varphi_{b_{ljk}}(D(b_{ljk}, \eta_l)) = D(0, \eta_l). \end{aligned}$$

For $w_l \in E_{ljk}$, the quantities $(1 - |w_l|^2)^{\beta_j + 2}$ and $|1 - \overline{b}_{ljk} w_l|$ are both bounded from below and from above. Also, since each $b_{ljk} \in D(a_{lk}, r_l)$, the quantities $1 - |b_{ljk}|^2$ and $1 - |a_{lk}|^2$ are comparable. Therefore, there exists a positive constant C independent of $\vec{r}$ and $\vec{\eta}$, such that

$$I_{jk} \le C \prod_{l=1}^{N} (1 - |a_{lk}|^2)^{b_l} \int_{E_{1jk}} \int_{E_{2jk}} \cdots \int_{E_{Njk}} |f \circ \varphi_{b_{jk}}(w) - f \circ \varphi_{b_{jk}}(0)| dA(w). \tag{3.5}$$

Write $r_l' = \tanh(r_l)$, $\eta_l' = \tanh(\eta_l)$. Since each η_l is much smaller than r_l for $l \in \{1, \ldots, N\}$, we may as well assume that $R = \max_{1 \le l \le N} \eta_l'/r_l' \le \frac{1}{2}$. By Lemma 2.4, there exists a positive constant C such that

$$|\nabla g(z)| \le C \Big(\int_{\mathbb{D}^N} |g(w)|^p dA(w) \Big)^{\frac{1}{p}}, \quad |z| \le R,$$

for $g \in H(\mathbb{D}^N)$. Let $\vec{r'} = (r_1', \ldots, r_N')$ and $\vec{\eta'} = (\eta_1', \ldots, \eta_N')$. Consider $g(z) = h(\vec{r'} z)$, where $h(z) = f \circ \varphi_{b_{jk}}(z)$, $\vec{r'} z = (r_1' z_1, \ldots, r_N' z_N)$, $z \in \mathbb{D}^N$. After a change of variables, we obtain

$$r_1' r_2' \cdots r_N' |\nabla h(\vec{r'} z)| \le C \Big(\frac{1}{(r_l' r_2' \cdots r_N')^2} \int_{\widehat{D}(0, \vec{r})} |h(w)|^p dA(w) \Big)^{\frac{1}{p}}$$

for all $|z| \le R$. That is,

$$|\nabla h(\vec{r'} z)| \le \frac{C}{(r_1' r_2' \cdots r_N')^{1+2/p}} \Big(\int_{\widehat{D}(0, \vec{r})} |h(w)|^p dA(w) \Big)^{\frac{1}{p}} \tag{3.6}$$

for all $z \in \widehat{D}(0, \vec{\eta})$. For any $w \in \widehat{E}_{jk} \subset \widehat{D}(0, \vec{\eta})$, the identity

$$h(w) - h(0) = \int_0^1 \Big(\sum_{l=1}^{N} w_l \frac{\partial h}{\partial w_l}(tw) \Big) dt$$

directly leads to

$$|h(w) - h(0)| \le |\vec{\eta'}| \sup \{ |\nabla h(u)| : u \in \widehat{D}(0, \vec{\eta}) \}. \tag{3.7}$$

From (3.5) and (3.7), we therefore have

$$I_{jk} \le C|\vec{\eta}'| \prod_{l=1}^{N} (1-|a_{lk}|^2)^{b_l} A(\widehat{E}_{jk}) \sup\{|\nabla h(u)| : u \in \widehat{D}(0,\vec{\eta})\}, \tag{3.8}$$

where $\widehat{E}_{jk}$ is the Cartesian product of E_{ljk}, $l = 1,\dots,N$. Combining (3.6) with (3.8), we obtain

$$I_{jk} \le \frac{C|\vec{\eta}'|}{(r_1' r_2' \cdots r_N')^{1+2/p}} \prod_{l=1}^{N} (1-|a_{lk}|^2)^{b_l} A(\widehat{E}_{jk}) \Big(\int_{\widehat{D}(0,\vec{r})} |h(w)|^p dA(w) \Big)^{\frac{1}{p}}. \tag{3.9}$$

By a change of variables again,

$$\int_{\widehat{D}(0,\vec{r})} |h(w)|^p dA(w) = \int_{D(b_{1jk},r_1)} \int_{D(b_{2jk},r_2)} \cdots \int_{D(b_{Njk},r_N)} |f(w)|^p \prod_{l=1}^{N} \frac{(1-|b_{ljk}|^2)^2}{|1-\overline{b}_{ljk} w_l|^4} dA(w). \tag{3.10}$$

It is easy to see that the quantities $1-|b_{ljk}|^2$ and $|1-\overline{b}_{ljk} w_l|$ are both comparable to $1-|a_{lk}|^2$ for $w_l \in D(b_{ljk}, r_l)$. This along with the fact that for each $l \in \{1,\dots,N\}$ it follows that $D(b_{ljk}, r_l) \subset D(a_{lk}, 2r_l)$ shows that

$$\begin{aligned} &\int_{D(b_{1jk},r_1)} \int_{D(b_{2jk},r_2)} \cdots \int_{D(b_{Njk},r_N)} |f(w)|^p \prod_{l=1}^{N} \frac{(1-|b_{ljk}|^2)^2}{|1-\overline{b}_{ljk} w_l|^4} dA(w) \\ &\le C \prod_{l=1}^{N} \frac{1}{(1-|a_{lk}|^2)^2} \int_{D(a_{1k},2r_1)} \int_{D(a_{2k},2r_2)} \cdots \int_{D(a_{Nk},2r_N)} |f(w)|^p dA(w). \end{aligned}$$

Since $1-|a_{lk}|^2$ is comparable to $1-|w_l|^2$ for $w_l \in D(a_{lk}, 2r_l)$, we have

$$\begin{aligned} &\int_{D(b_{1jk},r_1)} \int_{D(b_{2jk},r_2)} \cdots \int_{D(b_{Njk},r_N)} |h(w)|^p \prod_{l=1}^{N} \frac{(1-|b_{ljk}|^2)^2}{|1-\overline{b}_{ljk} w_l|^4} dA(w) \\ &\le C \prod_{l=1}^{N} \frac{1}{(1-|a_{lk}|^2)^{\alpha_l+2}} \int_{D(a_{1k},2r_1)} \int_{D(a_{2k},2r_2)} \cdots \int_{D(a_{Nk},2r_N)} |f(w)|^p dA_{\vec{\alpha}}(w). \end{aligned} \tag{3.11}$$

From (3.9)–(3.11), we have

$$I_{jk} \le \frac{C|\vec{\eta}'|}{(r_1' r_2' \cdots r_N')^{1+2/p}} \prod_{l=1}^{N} (1-|a_{lk}|^2)^{b_l - \frac{\alpha_l+2}{p}} A(\widehat{E}_{jk}) \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w) \Big)^{\frac{1}{p}}. \tag{3.12}$$

Since

$$\sum_{j=1}^{J} A(\widehat{E}_{jk}) = \sum_{j=1}^{J} \prod_{l=1}^{N} A_l(E_{ljk}) \le \sum_{j=1}^{J} \prod_{l=1}^{N} A_l(D(0,\eta_l) = J\eta_1^2 \eta_2^2 \cdots \eta_N^2$$

and

$$A(\widehat{D}(0,\vec{r})) = \sum_{j=1}^{J} A(E_j) \ge \sum_{j=1}^{J} A(\widehat{D}(z_j,\vec{\eta})) = \sum_{j=1}^{J} \prod_{l=1}^{N} A_l(D(z_{lj},\eta_l)) \ge CJ\eta_1^2 \eta_2^2 \cdots \eta_N^2,$$

where the last inequality follows from Lemma 1.23 in [28], we have

$$\sum_{j=1}^{J} A(\widehat{E}_{jk}) \leq CA(\widehat{D}(0, \vec{r'})) = C(r_1')^2(r_2')^2 \cdots (r_N')^2. \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$\sum_{j=1}^{J} I_{jk} \leq \frac{C|\vec{\eta'}|}{(r_1'r_2' \cdots r_N')^{\frac{2}{p}-1}} \prod_{l=1}^{N}(1-|a_{lk}|^2)^{b_l-\frac{\alpha_l+2}{p}} \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w)\Big)^{\frac{1}{p}}.$$

For each $k \in \mathbb{N}$ and $1 \leq j \leq J$, it follows from Lemma 2.27 in [28] for $N = 1$ that $|1-\overline{b}_{ljk}z_l|^{b_l}$ is comparable to $|1-\overline{a}_{lk}z_l|^{b_l}$. Therefore,

$$I(z) \leq \frac{C|\vec{\eta'}|}{(r_1'r_2' \cdots r_N')^{\frac{2}{p}-1}} \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{b_l-\frac{\alpha_l+2}{p}}}{\prod_{l=1}^{N}|1-\overline{a}_{lk}z_l|^{b_l}} \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w)\Big)^{\frac{1}{p}}.$$

Now we estimate $H(z)$. Let

$$H_{jk} = \int_{B_{1jk}} \int_{B_{2jk}} \cdots \int_{B_{Njk}} \Big| \frac{\prod_{l=1}^{N}(1-\overline{b}_{ljk}z_l)^{b_l}}{\prod_{l=1}^{N}(1-\overline{w}_l z_l)^{b_l}} - 1 \Big| \big|f(w)\big| dA_{\vec{\beta}}(w).$$

By Lemma 2.27 for $N = 1$ in [28], we have

$$H_{jk} \leq C\eta_1\eta_2 \ldots \eta_N \prod_{l=1}^{N}(1-|b_{ljk}|^2)^{\beta_l} \int_{B_{1jk}} \int_{B_{2jk}} \cdots \int_{B_{Njk}} \big|f(w)\big| dA(w).$$

From [23], there is a positive constant C independent of $f \in H(\mathbb{D}^N)$ and $z \in \mathbb{D}^N$ such that

$$|f(z)|^p \leq \frac{C}{\prod_{l=1}^{N}(1-|z_l|^2)^{\alpha_l+2}} \int_{\widehat{D}(z,\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w). \tag{3.14}$$

By the definition of B_{ljk} and since $\widehat{B}_{jk}$ is the Cartesian product of B_{ljk}, $l = 1, \ldots, N$, we deduce that $\widehat{B}_{jk} \subseteq \widehat{D}_k$, and then by Lemma 2.2 we further obtain that $\widehat{B}_{jk} \subseteq \widehat{D}_k \subseteq \widehat{D}(a_k, 2\vec{r})$. From this and replacing z by a_k in (3.14), for every $w \in \widehat{B}_{jk}$ we have

$$|f(w)| \leq \frac{C}{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{\frac{\alpha_l+2}{p}}} \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w)\Big)^{\frac{1}{p}}.$$

Then

$$H_{jk} \leq C\eta_1\eta_2 \ldots \eta_N \frac{\prod_{l=1}^{N}(1-|b_{ljk}|^2)^{\beta_l}}{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{\frac{\alpha_l+2}{p}}} A(\widehat{B}_{jk}) \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w)\Big)^{\frac{1}{p}}.$$

Since

$$\sum_{j=1}^{J} A(\widehat{B}_{jk}) = A(\widehat{D}_k) \leq A(\widehat{D}(a_k, \vec{r})) \leq \prod_{l=1}^{N}(1-|a_{lk}|^2)^2,$$

we deduce that

$$\sum_{j=1}^{J} H_{jk} \leq C\eta_1\eta_2 \ldots \eta_N \frac{\prod_{l=1}^{N}(1-|b_{ljk}|^2)^{\beta_l}}{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{\frac{\alpha_l+2}{p}-2}} \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w)\Big)^{\frac{1}{p}}.$$

From above same reasons, we have seen that $1-|b_{ljk}|^2$ and $1-|a_{lk}|^2$ are comparable, and $|1-\overline{b}_{ljk}z_l|$ and $|1-\overline{a}_{lk}z_l|$ are also comparable. It follows that

$$H(z) \leq C\eta_1\eta_2 \ldots \eta_N \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{b_l-\frac{\alpha_l+2}{p}}}{\prod_{l=1}^{N}|1-\overline{a}_{lk}z_l|^{b_l}} \Big(\int_{\widehat{D}(a_k,2\vec{r})} |f(w)|^p dA_{\vec{\alpha}}(w)\Big)^{\frac{1}{p}}.$$

This completes the proof. □

As an important application of Theorem 3.1, we obtain the next result, which shows that the operator $I-S$ is bounded on $H_\mu^\infty(\mathbb{D}^N)$.

Theorem 3.2. *Let μ be normal on $[0,1)$. Then there exist a positive constant C independent of $f \in H_\mu^\infty(\mathbb{D}^N)$, $\vec{r}$ and $\vec{\eta}$, such that*

$$\|(I-S)f\|_{H_\mu^\infty(\mathbb{D}^N)} \leq C\|f\|_{H_\mu^\infty(\mathbb{D}^N)}.$$

Proof. We first choose $\vec{\alpha}=0$ and $p=1$ in Theorem 3.1. Then we obtain

$$|f(z)-Sf(z)| \leq C\sigma \sum_{k=1}^{\infty} \frac{\prod_{l=1}^{N}(1-|a_{lk}|^2)^{b_l-2}}{\prod_{l=1}^{N}|1-\overline{a}_{lk}z_l|^{b_l}} \int_{\widehat{D}(a_k,2\vec{r})} |f(w)|dA(w) \tag{3.15}$$

for $\vec{r}=(r_1,\ldots,r_N)$, $0<r_l<1$, $l=1,\ldots,N$, $z\in\mathbb{D}^N$. From Lemma 2.3 for $N=1$ in [26], there exists a positive constant A_l for each $0 \leq l \leq N$ such that

$$A_l^{-1} \leq \frac{|1-\bar{a}_{lk}z_l|}{|1-\overline{w}_l z_l|} \leq A_l. \tag{3.16}$$

Then, from (3.15) and (3.16), it follows that there exists a positive constant C independent of $\vec{\eta}$ and $\vec{r}$ such that

$$\begin{aligned} |f(z)-Sf(z)| &\leq C\sigma \sum_{k=1}^{\infty} \int_{\widehat{D}(a_k,2\vec{r})} \frac{\prod_{l=1}^{N}(1-|w_l|^2)^{b_l-2}}{\prod_{l=1}^{N}|1-\overline{w}_l z_l|^{b_l}} |f(w)|dA(w) \\ &\leq C\widehat{N}^N\sigma \int_{\mathbb{D}^N} \frac{\prod_{l=1}^{N}(1-|w_l|^2)^{b_l-2}}{\prod_{l=1}^{N}|1-\overline{w}_l z_l|^{b_l}} |f(w)|dA(w). \end{aligned}$$

Which shows that

$$\prod_{l=1}^{N}\mu(z_l)\big|f(z)-Sf(z)\big| \leq C\widehat{N}^N\sigma \int_{\mathbb{D}^N} \frac{\prod_{l=1}^{N}(1-|w_l|^2)^{b_l-2}\prod_{l=1}^{N}\mu(z_l)}{\prod_{l=1}^{N}|1-\overline{w}_l z_l|^{b_l}} |f(w)|dA(w)$$

$$= C\widehat{N}^N\sigma \int_{\mathbb{D}^N} \frac{\prod_{l=1}^N (1-|w_l|^2)^{b_l-2} \prod_{l=1}^N \mu(z_l) \prod_{l=1}^N \mu(w_l)}{\prod_{l=1}^N |1-\overline{w}_l z_l|^{b_l} \prod_{l=1}^N \mu(w_l)} |f(w)| dA(w)$$
$$\leq C\widehat{N}^N\sigma \|f\|_{H_\mu^\infty(\mathbb{D}^N)} \int_{\mathbb{D}^N} \frac{\prod_{l=1}^N (1-|w_l|^2)^{b_l-2} \prod_{l=1}^N \mu(z_l)}{\prod_{l=1}^N |1-\overline{w}_l z_l|^{b_l} \prod_{l=1}^N \mu(w_l)} dA(w). \tag{3.17}$$

By Lemma 2.2 in [26], for each $l \in \{1, \ldots, N\}$ we have

$$\frac{\mu(z_l)}{\mu(w_l)} \leq \Big(\frac{1-|z_l|^2}{1-|w_l|^2}\Big)^a + \Big(\frac{1-|z_l|^2}{1-|w_l|^2}\Big)^b.$$

We therefore obtain

$$\prod_{l=1}^N \frac{\mu(z_l)}{\mu(w_l)} \leq \prod_{l=1}^N \Big[\Big(\frac{1-|z_l|^2}{1-|w_l|^2}\Big)^a + \Big(\frac{1-|z_l|^2}{1-|w_l|^2}\Big)^b\Big]. \tag{3.18}$$

For convenience, write

$$x_l = \frac{1-|z_l|^2}{1-|w_l|^2},$$

then from (3.17) and (3.18), we have that

$$\prod_{l=1}^N \mu(z_l)\big|f(z) - Sf(z)\big| \leq C\widehat{N}^N\sigma \|f\|_{H_\mu^\infty(\mathbb{D}^N)} \int_{\mathbb{D}^N} \frac{\prod_{l=1}^N (1-|w_l|^2)^{b_l-2}}{\prod_{l=1}^N |1-\overline{w}_l z_l|^{b_l}} \prod_{l=1}^N (x_l^a + x_l^b) dA(w). \tag{3.19}$$

It is easy to see that there are 2^N terms in $\prod_{l=1}^N (x_l^a + x_l^b)$. For arbitrary term $h(x_1, \ldots, x_N)$, without lose of generality, we may assume that

$$h(x_1, \ldots, x_N) = x_1^a \cdots x_l^a x_{l+1}^b \cdots x_j^b x_{j+1}^a \cdots x_k^a x_{k+1}^b \cdots x_m^b x_{m+1}^a \cdots x_N^a. \tag{3.20}$$

Substituting x_l in (3.20), we obtain

$$h(x_1, \ldots, x_N) = \frac{\prod_{l_j=1}^l (1-|z_{l_j}|^2)^a \prod_{l_j=l+1}^j (1-|z_{l_j}|^2)^b \prod_{l_j=j+1}^k (1-|z_{l_j}|^2)^a}{\prod_{l_j=1}^l (1-|w_{l_j}|^2)^a \prod_{l_j=l+1}^j (1-|w_{l_j}|^2)^b \prod_{l_j=j+1}^k (1-|w_{l_j}|^2)^a}$$
$$\times \frac{\prod_{l_j=k+1}^m (1-|z_{l_j}|^2)^b \prod_{l_j=m+1}^N (1-|z_{l_j}|^2)^a}{\prod_{l_j=k+1}^m (1-|w_{l_j}|^2)^b \prod_{l_j=m+1}^N (1-|w_{l_j}|^2)^a}.$$

Then by Theorem 2.12 in [28] we obtain

$$\int_{\mathbb{D}^N} \frac{\prod_{l=1}^N (1-|w_l|^2)^{b_l-2}}{\prod_{l=1}^N |1-\overline{w}_l z_l|^{b_l}} h(x_1, \ldots, x_N) dA(w) \asymp 1.$$

From this and (3.19), the desired result follows. This completes the proof. □

4. Conclusions

In this paper, a interesting result in the polydisk about the atomic decomposition of Bloch-type space has been obtained. It is well known that the existing similar results in the unit ball have been applied many times to the atomic decompositions of Bloch-type and weighted Bergman spaces. Hope that this study can attract people's more attention for the atomic decomposition of Bloch-type space.

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Conflict of interest

The authors declare that they have no competing interests.

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