



Research article

Synergic ranking of fuzzy Z-numbers based on vectorial distance and spread for application in decision-making

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Abstract: Decision science has a wide range of applications in daily life. Decision information is usually incomplete and partially reliable. In the fuzzy set theory, Z-numbers are introduced to handle this situation because they contain the restriction and reliability components, which complement the impaired information. The ranking of Z-numbers is a challenging task since they are composed of pairs of fuzzy numbers. In this research, the vectorial distance and spread of Z-numbers were proposed synergically, in which the vectorial distance measures how much the fuzzy numbers are apart from the origin, which was set as a relative point, and their spreads over a horizontal axis. Furthermore, a ranking method based on the convex compound was proposed to combine the restriction and reliability components of Z-numbers. The proposed ranking method was validated using several empirical examples and a comparative analysis was conducted. The application of the proposed ranking method in decision-making was illustrated via the development of the Analytic Hierarchy Process-Weighted Aggregated Sum Product Assessment (AHP-WASPAS) model to solve the prioritization of public services for the implementation of Industry 4.0 tools. Sensitivity analysis was also conducted to evaluate the performance of the proposed model and the results showed that the proposed model has improved its consistency from 66.67% of the existing model to 83.33%. This research leads to a future direction of the application of ranking based on the vectorial distance and spread in multi-criteria decision-making methods, which use Z-numbers as linguistic values.

Keywords: Z-numbers; vectorial distance; spread; ranking; decision-making; AHP-WASPAS
Mathematics Subject Classification: 91B06

1. Introduction

Decision-making is a crucial task in almost all daily activities since it is highly related to cognitive thinking. Decisions are made based on available knowledge and understandable information. Most of the time, the information is indefinite and imperfect to some decision makers, leading to imprecise conclusions. The decisions are also highly influenced by the decision makers' intuitions and personal biases. According to [1], as normal human beings, decision makers are always restricted to the level of competency, psychological biases, and insufficient information.

Zadeh [2] proposed the concept of Z-numbers as a means of overcoming this issue. Generally, Z-numbers are composed of two components, namely the restriction and reliability parts. The restriction part restricts the value that a variable can take, while the reliability part describes the degree of certainty of the restriction. Z-numbers are also described as a pair of two fuzzy numbers, which are expressed in a natural language. According to [3], Z-numbers have a higher level of generality than fuzzy numbers, intervals, and crisp values. Besides, Z-numbers have a high-level computational complexity since the arithmetic operations must deal with two fuzzy numbers in pairs [3]. The decision-making information is usually expressed in the natural language and is always imperfect. Hence, the application of Z-numbers in decision-making is essential as they better depict imperfect information, which is as close as possible to the natural language [4].

Correspondingly, many decision-making methods have been developed based on Z-numbers such as the Analytic Hierarchy Process (AHP) [5], Technique for Order of Preferences by Similarity to Ideal Solution (TOPSIS) [6,7], Decision-Making Trial and Evaluation Laboratory (DEMATEL) [8], Weighted Aggregated Sum Product Assessment (WASPAS) [9], including the full consistency method and multi-attributive border approximation area comparison [10], Viekriterijumsko Kompromisno Rangiranje (VIKOR) [11], Approximate calculation method (ACM) [12], Elimination and Choice Translating Reality (ELECTRE) [13], Stepwise Weight Assessment Ratio Analysis (SWARA) [14], and Combinative Distance-based Assessment (CODAS) [15].

Kang et al. [16], for instance, converted Z-numbers into regular fuzzy numbers using the fuzzy expectation of fuzzy numbers. In their method, the reliability part was converted into a weight, which was then added to the restriction part. However, it should be noted that converting Z-numbers into regular fuzzy numbers does not preserve the initial decision information in form of pairs of fuzzy numbers, which finally leads to a significant loss of information [3].

Furthermore, Aliev et al. [17,18] proposed the arithmetic operations of Z-numbers, which allow for direct calculation without the need to convert them into regular fuzzy numbers. Essentially, the advantage of this approach is that it preserves the Z-numbers and avoids the loss of initial information. However, working directly with Z-numbers, which extensively include linear programming, leads to high computational complexity [3]. Hence, the bandwidth method was suggested in [19] to perform the operations between Z-numbers, especially on a larger scale.

Many methods were also developed to rank fuzzy numbers. The ranking based on the centroid and spread of fuzzy numbers was used in [20], in which Z-numbers were first converted into regular fuzzy numbers. According to Abdullahi et al. [3], this conversion has caused information loss. Alternatively, Jiang et al. [21] defined a score function that considers the centroid point, the Minkowski degree of fuzziness and spread. The score function was calculated for both the restriction

and reliability components and combined altogether to obtain the final ranking of Z-numbers.

Meanwhile, using the conversion of Z-numbers into regular fuzzy numbers, Mohamad et al. [22] calculated the score value of the standardized defuzzified fuzzy number; however, the weakness of this method lies in its impurity of Z-numbers, which has undergone the transformation process. Recently, Farzam et al. [23] defined the magnitude of Z-numbers to determine their ranking. In their method, the magnitudes of restriction and reliability components were combined using the convex compound to avoid the conversion of Z-numbers into regular fuzzy numbers.

Nevertheless, in the existing ranking methods of Z-numbers, it was proposed that the Z-numbers were either converted into regular fuzzy numbers before the final ranking was calculated or the ranking functions of the restriction and reliability components of Z-numbers were first calculated before being combined to obtain the final ranking. Since transforming Z-numbers into regular fuzzy numbers has caused a great loss of information [3], ranking methods that use this transformation process are, therefore, not really reliable.

Fuzzy numbers are usually used to represent the linguistic terms in the decision-making information. In this regard, when ranking such fuzzy numbers, there is a reference point to which the fuzzy numbers are compared. Furthermore, the distance between the fuzzy numbers and the reference point measures which fuzzy numbers should also be ranked higher. Therefore, a new ranking method that works directly on both the components of Z-numbers is proposed in this research, in which the spread and distances of fuzzy numbers from a reference point, namely origin, are considered. The following entails some novelties of this research:

- (a) The proposed ranking method is based on the vectorial distances of fuzzy numbers from the origin. The defined distances measure how much the fuzzy numbers are distant from the origin, which will then determine their ranking. This concept is highly suitable when comparing fuzzy numbers that represent the linguistic terms with ordinal scales, which determine the decision maker's level of importance of criteria and alternatives. Moreover, the spread of the fuzzy numbers is also considered in the ranking process to make a difference in the size of the fuzzy numbers, which is closely related to the level of fuzziness such that it works synergically with the defined vectorial distance.
- (b) The employment of the convex compound in defining the final ranking of Z-numbers could avoid the loss of information when processing the decision information. Using the proposed method, the Z-numbers are not converted into regular fuzzy numbers but are processed separately on their restriction and reliability components, which are further combined using the convex compound. A parameter is defined in the convex compound to ensure that the weightage of the restriction component is at least equal to that of the reliability component.
- (c) The proposed ranking method is further integrated into the development of a novel Z-number-based AHP-WASPAS. The Z-AHP is used to quantify the criteria weights considering the decision maker's reliability, while the alternatives are ranked using the Z-WASPAS. As a validation, a numerical example from a real case study was employed to illustrate the proposed Z-AHP-WASPAS model.

In this paper, a novel method of ranking Z-numbers is proposed based on the vectorial distances and spread of fuzzy numbers. The ranking of both the restriction and reliability components was combined using a convex compound, which prioritizes the restriction part for ranking the Z-numbers. The proposed method was validated using some empirical examples and a comparative analysis was then performed to observe the comparison of the proposed ranking method with the existing methods. A novel AHP-WASPAS model was also developed based on the proposed ranking method of Z-numbers and the real case example was adopted from [9] to illustrate the proposed model.

2. A review on the ranking of Z-numbers

In this section, the existing methods of ranking Z-numbers are reviewed to give an overview of the strengths and weaknesses of the methods. Hence, the novelty of the newly proposed ranking method can be highlighted in the next section. Assume that $Z = (A, R) = \langle (a_1, a_2, a_3, a_4; h_A), (r_1, r_2, r_3, r_4; h_R) \rangle$.

2.1. Centroid and spread

In [20], Bakar and Gegov ranked Z-numbers based on the horizontal and vertical centroid, including spread. Three steps were developed to construct the final ranking index, as follows:

Step 1: The centroid index for A is calculated using (1) and (2) for the horizontal and vertical axes, respectively,

$$x_A^* = \frac{\int xf(x)dx}{\int f(x)dx} \quad (1)$$

$$y_A^* = \frac{\int_0^{h_A} \alpha |a_i^\alpha| d\alpha}{\int_0^{h_A} |a_i^\alpha| d\alpha} \quad (2)$$

where $|a_i^\alpha|$ is the length of α -cuts of A.

Step 2: The spread of A is calculated using (3).

$$s_A = |a_4 - a_1| \times y_A^*. \quad (3)$$

Step 3: The final ranking function then combines the horizontal and vertical centroid with the spread, as shown in (4).

$$s_A = x_A^* \times y_A^* \times (1 - s_A). \quad (4)$$

2.2. Fuzzy score of standardized defuzzified fuzzy numbers

Mohamad et al. [22] proposed the ordering of Z-numbers via conversion into regular fuzzy numbers. The fuzzy numbers are standardized and defuzzified before calculating the fuzzy score, considering the spread. Suppose the Z-numbers are represented by $Z = (A, R) = \langle (a_1, a_2, a_3), (r_1, r_2, r_3) \rangle$, the conversion into crisp numbers is shown in (5).

$$\alpha = \frac{\int x\mu_R dx}{\int \mu_R dx}. \quad (5)$$

The weight from (5) is then added to A to obtain the weighted Z-number as shown in (6).

$$A^\alpha = \{x, \mu_{A^\alpha}(x) \mid \mu_{A^\alpha} = \alpha\mu_A(x), x \in [0, 1]\}. \quad (6)$$

Hence, a generalized fuzzy number is obtained as $A^\alpha = (a_1, a_2, a_3; h_A)$ where $\alpha = h_A$. This generalized fuzzy number is then standardized as shown in (7),

$$\mathcal{A}^\alpha = \left(\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}; h_A \right) = (\mathcal{A}_1^\alpha, \mathcal{A}_2^\alpha, \mathcal{A}_3^\alpha; h_A) \quad (7)$$

where $k = \max_{ij}(|a_{ij}|, 1)$. Subsequently, the standardized generalized fuzzy number is defuzzified, as shown in (8), and its spread is calculated as shown in (9).

$$x_A^* = \frac{\mathcal{A}_1^\alpha + \mathcal{A}_2^\alpha + \mathcal{A}_3^\alpha}{3} \quad (8)$$

$$s_A = \left(\frac{(\mathcal{A}_1^\alpha - x_A^*)^2 + (\mathcal{A}_2^\alpha - x_A^*)^2 + (\mathcal{A}_3^\alpha - x_A^*)^2}{3} \right)^2. \quad (9)$$

Hence, the final ranking score can be calculated as

$$\text{Rank}(A) = \frac{x_A^* \times h_A}{1 + s_A}. \quad (10)$$

2.3. Value and ambiguity

In [24], Chutia proposed a new method of ranking Z-numbers based on the value and ambiguity at decision-making levels. Firstly, the values $V_\alpha(A_i), V_\alpha(R_i)$ and ambiguities $A_\alpha(A_i), A_\alpha(R_i)$ are calculated. Next, their distances from the reference point (0,0) are also calculated as shown in (11).

$$(d_V)_\alpha = \sqrt{V_\alpha(A_i)^2 + V_\alpha(R_i)^2}, \quad (d_A)_\alpha = \sqrt{A_\alpha(A_i)^2 + A_\alpha(R_i)^2}. \quad (11)$$

Subsequently, the value index and ambiguity index are constructed at the decision level α as shown in (12) and (13), respectively.

$$(I_V)_\alpha = \frac{V_\alpha(A_i) + 2(d_V)_\alpha}{3} \quad (12)$$

$$(I_A)_\alpha = \frac{A_\alpha(A_i) + 2(d_A)_\alpha}{3}. \quad (13)$$

Finally, the following ranking rules are used to rank Z-numbers based on their value and ambiguity indices such that

- (i) $(I_V)_\alpha(Z_1) > (I_V)_\alpha(Z_2) \Rightarrow Z_1 \text{ f } Z_2$;
- (ii) $(I_V)_\alpha(Z_1) < (I_V)_\alpha(Z_2) \Rightarrow Z_1 \text{ p } Z_2$; and
- (iii) if $(I_V)_\alpha(Z_1) = (I_V)_\alpha(Z_2)$, then
 - $(I_A)_\alpha(Z_1) > (I_A)_\alpha(Z_2) \Rightarrow Z_1 < Z_2$
 - $(I_A)_\alpha(Z_1) < (I_A)_\alpha(Z_2) \Rightarrow Z_1 > Z_2$
 - $(I_A)_\alpha(Z_1) = (I_A)_\alpha(Z_2) \Rightarrow Z_1 \sim Z_2$.

2.4. Magnitude of Z-numbers

Farzam et al. [23] used the magnitude of fuzzy numbers to rank Z-numbers. The convex compound is used to combine the restriction and reliability components, in which a parameter λ is used to control the weightage of each component. However, the parameter λ was set such that the restriction component majorly determines the ranking of Z-numbers. The magnitude of A is given by

$$\text{Mag}(A) = \frac{(3h_A^2 + 2)(a_2 + a_3)}{12h_A} + \frac{(3h_A - 2)(a_1 + a_4)}{12h_A} \quad (14)$$

while the magnitude of R is given by

$$\text{Mag}(R) = \frac{(3h_R^2 + 2)(r_2 + r_3)}{12h_R} + \frac{(3h_R - 2)(r_1 + r_4)}{12h_R}. \quad (15)$$

Hence, the final ranking of Z-numbers is determined by a ranking function that combines $\text{Mag}(A)$ and $\text{Mag}(R)$, as shown in (16).

$$\text{Rank}(Z_i) = \lambda \text{Mag}(A_i) + (1 - \lambda) \text{Mag}(R_i). \quad (16)$$

3. Synergic ranking based on vectorial distance and spread

In this section, a novel synergic ranking method of fuzzy Z-numbers based on the vectorial distance and spread of fuzzy numbers representing the restriction and reliability components of Z-numbers is proposed. Some numerical examples are also explicated, followed by the comparative analysis with the previous ranking methods of Z-numbers.

3.1. Vectorial distance and spread of fuzzy numbers

A fuzzy number can be regarded as a shape on the vectorial plane in which the distance of all vertices of such a shape can be calculated as vectorial distances with respect to the origin. Suppose a point is located on a vectorial xy -plane as shown in Figure 1, then the vectorial distance of the point can be calculated as

$$\text{distance} = \sqrt{b^2 + h^2}. \quad (17)$$

If point b lies on the x -axis in which $b > 0$, then its vectorial distance is simply given by

$$\text{distance} = \sqrt{b^2 + 0^2} = \sqrt{b^2} = |b| = b. \quad (18)$$

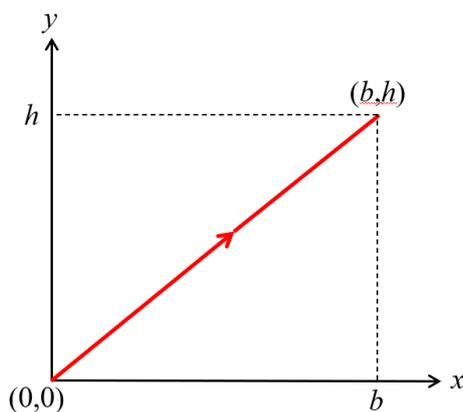


Figure 1. A vectorial distance on the xy -plane.

Hence, if the fuzzy number entails a triangular fuzzy number as shown in Figure 2, then its average of vectorial distances is given by the mean of vectorial distances of three vertices a , b , and c from the origin, as shown in (19). Generally, positive fuzzy numbers are often used in the application for solving decision-making problems. The farther the values deviate from the origin, the larger their values, which subsequently determine the larger fuzzy numbers. Hence, the distance of the fuzzy numbers from the origin is a strong reason for defining the ranking of fuzzy numbers. Likewise, the same concept is applied to the heights of the fuzzy numbers, in which the origin is set as a reference point to determine the ranking of fuzzy numbers since fuzzy numbers with longer heights are ranked higher, which are mainly characterized by their maximum membership values.

$$\text{distance} = \frac{1}{3} \left(a + \sqrt{b^2 + h^2} + c \right). \quad (19)$$

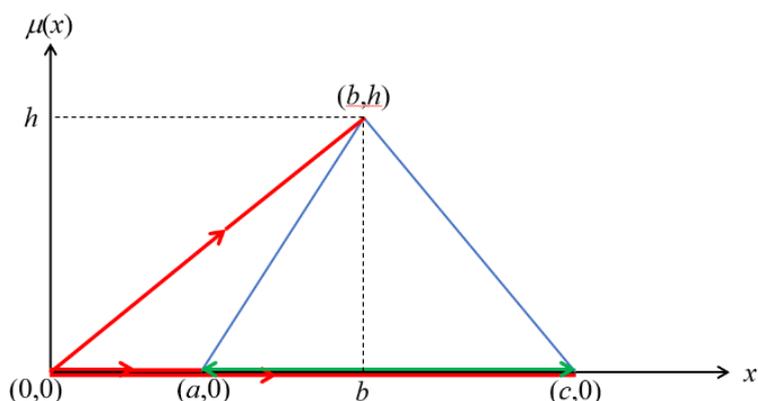


Figure 2. Vectorial distance and spread of a triangular fuzzy number.

The spread of such triangular fuzzy numbers can be assumed as the difference between points c and a . Hence, combining the vectorial distances and spread of the triangular fuzzy number $A = (a, b, c; h)$ synergically, its ranking can be defined as follows:

$$\text{Rank}(A) = \frac{1}{3} \left(\frac{a + \sqrt{b^2 + h^2} + c}{1 + c - a} \right). \quad (20)$$

In most cases, trapezoidal fuzzy numbers are used in decision-making instead of triangular fuzzy numbers due to their ability to process ill-defined quantities [25]. Hence, the triangular fuzzy number A can be generalized into a trapezoidal fuzzy number $A = (a, b, c, d; h)$ as shown in Figure 3, in which its rank is defined by

$$\text{Rank}(A) = \frac{1}{6} \left(\frac{2a + \sqrt{b^2 + h^2} + \sqrt{c^2 + h^2} + 2d}{1 + d - a} \right) \times (1 + c - b). \quad (21)$$

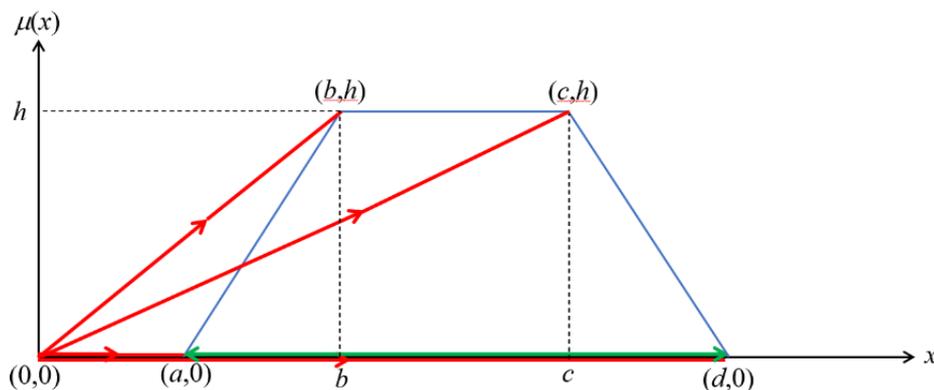


Figure 3. Vectorial distance and spread of a trapezoidal fuzzy number.

Remark: The ranking of the trapezoidal fuzzy number denotes a generalization of the triangular fuzzy numbers.

Proof. The trapezoidal fuzzy number is reduced to the triangular fuzzy number when $b = c$. Substituting this equality in (5) yields

$$\begin{aligned} \text{Rank}(A) &= \frac{1}{6} \left(\frac{2a + \sqrt{b^2 + h^2} + \sqrt{b^2 + h^2} + 2d}{1 + d - a} \right) \times (1 + b - b) \\ &= \frac{1}{6} \left(\frac{2a + 2\sqrt{b^2 + h^2} + 2d}{1 + d - a} \right) \times 1 \\ &= \frac{1}{3} \left(\frac{a + \sqrt{b^2 + h^2} + d}{1 + d - a} \right) \end{aligned}$$

in which the final result is equivalent to (20). Hence, the rank of the trapezoidal fuzzy number is reduced to the rank of the triangular fuzzy number.

3.2. Novel ranking of Z-numbers

A Z-number is a pair of fuzzy numbers, $Z = (A, R)$, in which A represents the restriction part while R describes how reliable the first part is. Both components of Z-numbers can be assumed as triangular fuzzy numbers, as shown in Figure 4.

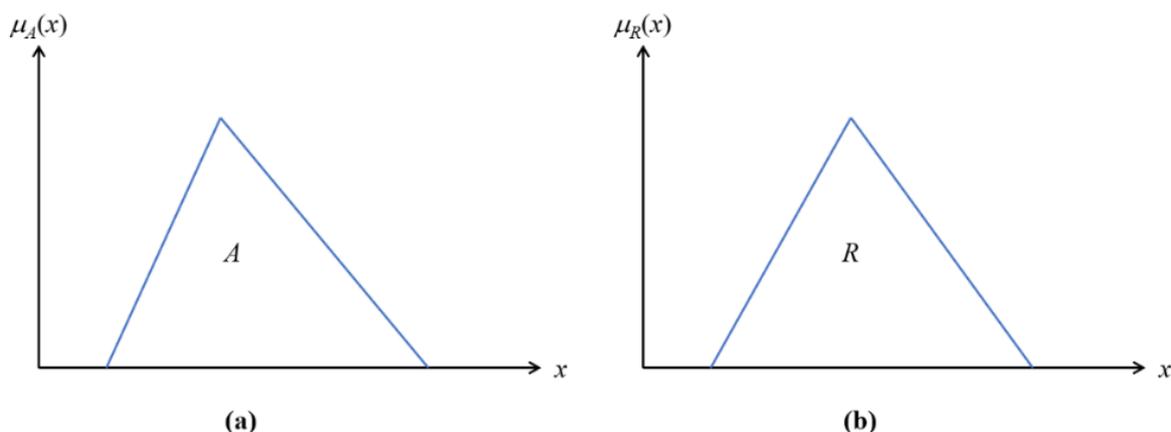


Figure 4. A Z-number: (a) restriction part; (b) reliability part.

In this paper, a novel method of ranking Z-numbers, $Z = (A, R) = \langle (a_1, a_2, a_3; h_A), (r_1, r_2, r_3; h_R) \rangle$ was proposed by combining the restriction and reliability parts using a convex compound as shown in (22),

$$\text{Rank}(Z) = \lambda \text{Rank}(A) + (1 - \lambda) \text{Rank}(R) \quad (22)$$

where λ takes any value in the interval $[0.5, 1]$. This condition governs the compound such that the restriction part, A , majorly represents the Z-number rather than the reliability part, R [23]. $\text{Rank}(A)$ and $\text{Rank}(R)$ in (22) constitute the ranking of triangular fuzzy numbers A and R based on the vectorial distance and spread as shown in (23) and (24), respectively.

$$\text{Rank}(A) = \frac{1}{3} \left(\frac{a_1 + \sqrt{a_2^2 + h_A^2} + a_3}{1 + a_3 - a_1} \right) \quad (23)$$

$$\text{Rank}(R) = \frac{1}{3} \left(\frac{r_1 + \sqrt{r_2^2 + h_R^2} + r_3}{1 + r_3 - r_1} \right). \quad (24)$$

If both components of Z-numbers are represented by trapezoidal fuzzy numbers, then the ranking function in (21) will replace both (23) and (24). Next, the following rules are used to rank Z-numbers based on (22):

- (i) If $\text{Rank}(Z_1) > \text{Rank}(Z_2)$, then Z_1 f Z_2 (Z_1 is ranked higher than Z_2)
- (ii) If $\text{Rank}(Z_1) < \text{Rank}(Z_2)$, then Z_1 p Z_2 (Z_1 is ranked lower than Z_2)
- (iii) If $\text{Rank}(Z_1) = \text{Rank}(Z_2)$, then $Z_1 : Z_2$ (Z_1 is ranked the same as Z_2).

4. Empirical validation and comparative analysis

The proposed ranking of Z-numbers was subsequently validated using some numerical examples, which entail sets of Z-numbers taken from [24,26]. Table 1 displays the sets of Z-numbers used for the validation of the proposed ranking method.

Table 1. Sets of examples of Z-numbers [24,26].

Set	Restriction Part	Reliability Part
1.1	$A_1=(0.1,0.3,0.5;1.0)$	$R_1=(0.1,0.3,0.5;1.0), R_2=(0.2,0.3,0.4;1.0)$
1.2	$A_1=(0.1,0.3,0.5;1.0)$	$R_3=(0.1,0.2,0.4,0.5;1.0), R_4=(0.1,0.3,0.5;1.0)$
1.3	$A_1=(0.1,0.3,0.5;1.0)$	$R_5=(0.1,0.3,0.5;0.8), R_6=(0.1,0.3,0.5;1.0)$
1.4	$A_1=(0.1,0.3,0.5;1.0)$	$R_7=(0.1,0.2,0.4,0.5;1.0), R_8=(0.1,0.3,0.5;0.8)$
1.5	$A_1=(0.1,0.3,0.5;1.0)$	$R_9=(0.1,0.2,0.4,0.5;1.0), R_{10}=(0.3,0.3,0.3;1.0)$
1.6	$A_1=(0.1,0.3,0.5;1.0)$	$R_{11}=(0.1,0.3,0.5;1.0), R_{12}=(0.3,0.3,0.3;1.0)$
2.1	$A_2=(0.1,0.4,0.6;1.0)$	$R_{13}=(0.1,0.4,0.5;1.0), R_{14}=(0.2,0.3,0.6;1.0)$
2.2	$A_2=(0.1,0.4,0.6;1.0)$	$R_{15}=(0.1,0.4,0.7;1.0), R_{16}=(0.2,0.3,0.5,0.6;1.0)$
2.3	$A_2=(0.1,0.4,0.6;1.0)$	$R_{17}=(0.2,0.3,0.4,0.5;1.0), R_{18}=(0.2,0.3,0.5,0.6;1.0)$
3.1	$A_3=(0.1,0.4,0.6;1.0)$	$R_{19}=(0.1,0.3,0.5;1.0), R_{20}=(0.3,0.5,0.7;1.0)$
3.2	$A_3=(0.1,0.4,0.6;1.0)$	$R_{21}=(0.1,0.2,0.4,0.5;1.0), R_{22}=(1.0,1.0,1.0;1.0)$
3.3	$A_3=(0.1,0.4,0.6;1.0)$	$R_{23}=(0.4,0.5,1.0;1.0), R_{24}=(0.4,0.7,1.0;1.0),$ $R_{25}=(0.4,0.9,1.0;1.0)$

The weightages for λ used for the empirical validation are 0.5, 0.7, and 0.9 to show that the proposed ranking method is reasonable for several situations: $\lambda = 0.5$ means that the rank of Z-numbers is balanced by both components; $\lambda = 0.7, 0.9$ means that the rank of the first component is higher than the second one; $\lambda = 1.0$ is not considered in this case because it reduces the Z-numbers to regular fuzzy numbers since the reliability part has completely vanished. Table 2 displays the ranking results obtained using the proposed method and their comparisons with existing methods.

Table 2. Sets of examples of Z-numbers [24,26].

Set	Proposed	[20]	[21]	[22]	[27]	[28]	[24]	[29]	[23]
1.1	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$
1.2	$Z_1 > Z_2$	$Z_1 \sim Z_2$	$Z_1 > Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$
1.3	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$
1.4	$Z_1 > Z_2$	$Z_1 \sim Z_2$	$Z_1 > Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 > Z_2$	$Z_1 \sim Z_2$	$Z_1 > Z_2$
1.5	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$
1.6	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$
2.1	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 > Z_2$
2.2	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 > Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$
2.3	$Z_1 > Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 \sim Z_2$	$Z_1 > Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$
3.1	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$
3.2	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 > Z_2$	$Z_1 < Z_2$
3.3	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 < Z_2$	$Z_1 \sim Z_2$	$Z_1 < Z_2$
	$< Z_3$	$< Z_3$	$< Z_3$	$< Z_3$	$< Z_3$	$< Z_3$	$< Z_3$	$\sim Z_3$	$< Z_3$

For Example 1.1, both R_1 and R_2 are triangular fuzzy numbers having the same central value with different spreads, as shown in Figure 5(a). According to [30], fuzzy sets with higher means and smaller spreads are preferred. Hence, theoretically through $Z_2 > Z_1$ and by using the proposed method, the same result was obtained. The ranking methods from [21,24,29] also ranked Z_2 higher than Z_1 ,

whereas Z_1 is of the same rank as Z_2 using the methods from [20,22,23,27,28].

Example 1.2 compares the triangular and trapezoidal fuzzy numbers with the same spread. The proposed method ranked $Z_1 > Z_2$, similar to [21], which indicates that the trapezoidal fuzzy number is ranked higher than the triangular fuzzy number since the triangular fuzzy number only contains a single point with the highest membership value. This is also supported by the fact that the trapezoidal fuzzy number has a better ability to process ill-defined quantities [25].

In Example 1.3, two triangular fuzzy numbers with different heights are compared. In reference to Figure 5(c), $Z_2 > Z_1$ since Z_2 clearly has a higher maximum membership value than Z_1 and the same result was obtained using the proposed ranking method. On the other hand, using the ranking of Z-numbers proposed in [20,22,27–29], the ranking result $Z_1 \sim Z_2$ was obtained.

Example 1.4 ranked Z_1 greater than Z_2 , in which Z_1 is a trapezoidal fuzzy number with higher maximum membership values. Using the magnitude of Z-numbers [23], value and ambiguity [24], and centroid, fuzziness degree, and spreads of fuzzy numbers [21], $Z_1 > Z_2$ was also obtained. In Example 1.5, the trapezoidal fuzzy number (Z_1) was compared to a singleton (Z_2) and the proposed ranking based on vectorial distance and spread ranked Z_2 higher than Z_1 . The same result was also obtained in [21,24] due to the fact that the singleton is a crisp value, which does not have a degree of fuzziness. Moreover, the same result was also obtained when the singleton was compared to the triangular fuzzy numbers, as illustrated in Example 1.6.

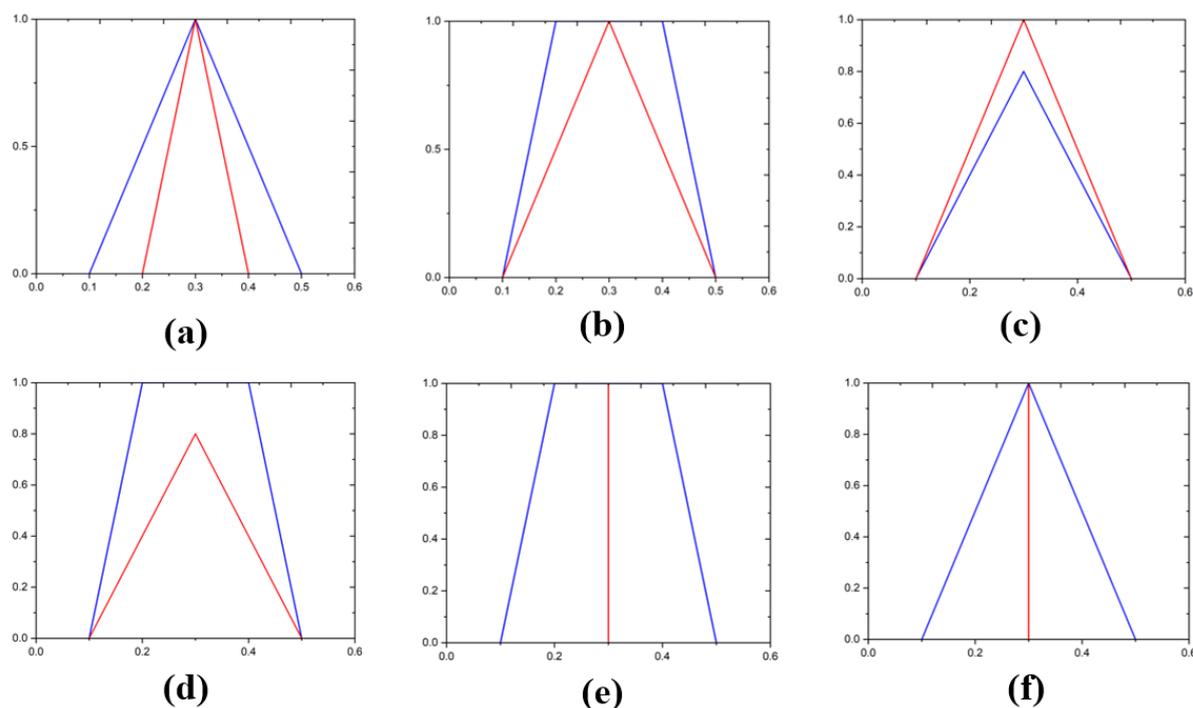


Figure 5. Example 1: (a) set 1.1; (b) set 1.2; (c) set 1.3; (d) set 1.4; (e) set 1.5; (f) set 1.6.

Example 2.1 compares two triangular fuzzy numbers with different spreads; one is farther from the origin, while the other is nearer to the origin. Since they share the same length of spread, vectorial distances of vertices were used to determine the rank of fuzzy numbers. Hence, $Z_2 > Z_1$ was obtained using the proposed method. The same result was also obtained using the existing ranking

methods of Z-numbers, except for the magnitude proposed in [23], in which Z_1 was ranked higher.

In Example 2.2 as illustrated in Figure 6(b), the trapezoidal fuzzy number (Z_2) was ranked higher than the triangular fuzzy number (Z_1). This is supported by the fact that the trapezoidal fuzzy number has a smaller spread and many points with maximum membership values. The same ranking result was also obtained using the methods from [21,29]. Example 2.3 illustrates two trapezoidal fuzzy numbers with different spreads, in which Z_1 was ranked higher since it has a smaller spread than Z_2 . This is supported by the fact that fuzzy numbers with smaller spreads are preferred [30].

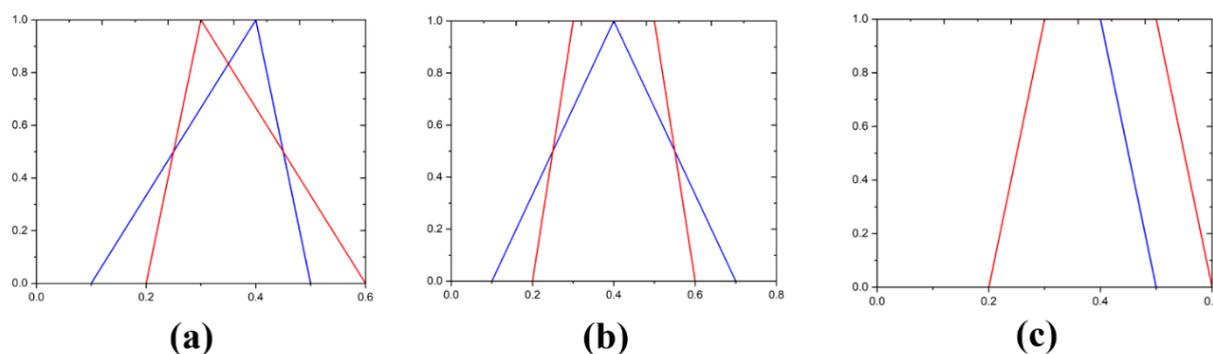


Figure 6. Example 2: (a) set 2.1; (b) set 2.2; (c) set 2.3.

In reference to Figure 7, Example 3.1 ranked $Z_2 > Z_1$ since the position of Z_2 is farther from the origin compared to Z_1 although they have the same spread length. In this case, the vectorial distances determine the ranking of the triangular fuzzy numbers. The same result was obtained in [20–24,27–29]. Meanwhile, in Example 3.2, the singleton was compared to the trapezoidal fuzzy number, in which the singleton is extensively farther from the origin compared to the trapezoidal fuzzy number. Hence, the proposed method ranked the singleton (Z_2) higher than the trapezoidal fuzzy number (Z_1). A similar result was also obtained from the existing methods of ranking Z-numbers, except for the method from [29].

In Example 3.3, three triangular fuzzy numbers that share the same spread were compared. Since Z_1 is the nearest to the origin, it is ranked the lowest, followed by Z_2 and Z_3 . In this situation, the vectorial distance of the peak of the triangular fuzzy numbers majorly determines the ranking order. The same ranking result was also obtained using existing methods of ranking Z-numbers, except [29]. According to [29], the three triangular fuzzy numbers have the same ranking.

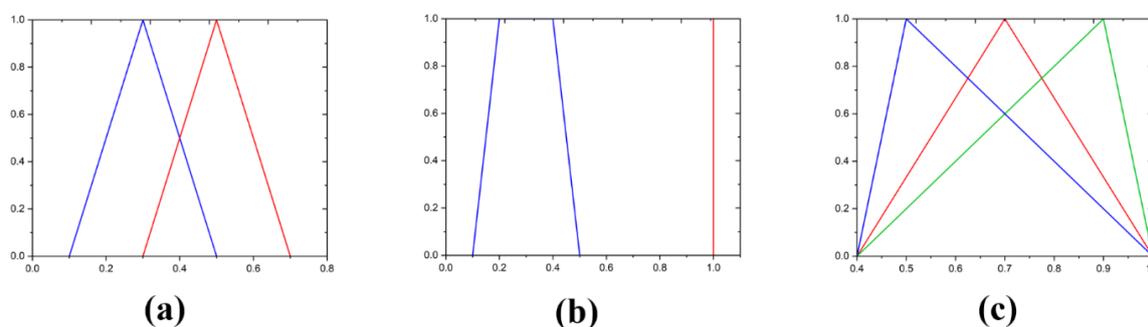


Figure 7. Example 3: (a) set 3.1; (b) set 3.2; (c) set 3.3.

5. Application in decision-making

5.1. Proposed Z-AHP-WASPAS

AHP-WASPAS is extended to Z-number's application in obtaining criteria weights and ranking desired alternatives. This proposed model will utilize the ranking approach based on the vectorial distances and spread of Z-numbers to defuzzify the decision information into crisp values. The AHP model was used to obtain the criteria weights, while the WASPAS model was developed to rank the alternatives. The following entails the detailed steps of the proposed AHP-WASPAS model:

Step 1: Construct the hierarchical structure consisting of three layers: the top layer is the aim of the decision-making problem, the middle layer consists of the criteria or attributes, and the bottom layer lists all the alternatives.

Step 2: Obtain the decision information from the decision maker using pairwise comparison matrices as shown in (25). Note that (a_{ij}, r_{ij}) is a Z-number in which a_{ij} and r_{ij} represent the restriction and reliability, respectively, on the degree of importance that the i -th criterion is preferred to the j -th criterion. From this point onwards, a_{ij} and r_{ij} are both treated as triangular fuzzy numbers, denoted by $(a_{ij1}, a_{ij2}, a_{ij3})$ and $(r_{ij1}, r_{ij2}, r_{ij3})$, respectively. Hence, all arithmetic operations involving these elements are based on the arithmetic operations of triangular fuzzy numbers.

$$\begin{pmatrix} (a_{11}, r_{11}) & (a_{12}, r_{12}) & L & (a_{1n}, r_{1n}) \\ (a_{21}, r_{21}) & (a_{22}, r_{22}) & L & (a_{2n}, r_{2n}) \\ M & M & O & M \\ (a_{n1}, r_{n1}) & (a_{n2}, r_{n2}) & L & (a_{nn}, r_{nn}) \end{pmatrix}. \quad (25)$$

Step 3: The fuzzy numbers representing restriction and reliability are separately aggregated using the defined operator in (26),

$$\Phi((a_{i1}, r_{i1}), (a_{i2}, r_{i2}), \dots, (a_{in}, r_{in})) = ((a_{i1} + \dots + a_{in}), (r_{i1} + \dots + r_{in})) = (\mathcal{A}_i^{\phi}, \mathcal{R}_i^{\phi}) \quad (26)$$

where Φ is the arithmetic aggregation operators and $a_{i1}, \dots, a_{in}, r_{i1}, \dots, r_{in}$ are all triangular fuzzy numbers. Meanwhile, $(\mathcal{A}_i^{\phi}, \mathcal{R}_i^{\phi})$ is the aggregated triangular fuzzy number.

Step 4: The fuzzy weights are then evaluated using the formula below,

$$(A_i, R_i) = \left(\mathcal{A}_i^{\phi} \cdot \left(\sum_{i=1}^n \mathcal{A}_i^{\phi} \right)^{-1}, \mathcal{R}_i^{\phi} \cdot \left(\sum_{i=1}^n \mathcal{R}_i^{\phi} \right)^{-1} \right) \quad (27)$$

where $\left(\sum_{i=1}^n \mathcal{A}_i^{\phi} \right)^{-1}$ and $\left(\sum_{i=1}^n \mathcal{R}_i^{\phi} \right)^{-1}$ are the inverses of the triangular fuzzy numbers.

Step 5: The fuzzy weights (A_i, R_i) are then defuzzified into $\text{Rank}(Z_i)$ using (22) for all $i = 1, 2, \dots, n$. Hence, the final weights, $W(Z_i)$, are obtained by normalizing $\text{Rank}(Z_i)$ using the formula

$$W(Z_i) = \frac{\text{Rank}(Z_i)}{\sum_{i=1}^n \text{Rank}(Z_i)}, \quad (28)$$

such that $W(Z_1) + W(Z_2) + \dots + W(Z_n) = 1$.

Step 6: Next, the decision maker's evaluation of each alternative with respect to all criteria is obtained as shown in (29),

$$\begin{pmatrix} (b_{11}, r_{11}) & (b_{12}, r_{12}) & L & (b_{1n}, r_{1n}) \\ (b_{21}, r_{21}) & (b_{22}, r_{22}) & L & (b_{2n}, r_{2n}) \\ M & M & O & M \\ (b_{m1}, r_{m1}) & (b_{m2}, r_{m2}) & L & (b_{mn}, r_{mn}) \end{pmatrix}, \quad (29)$$

where (b_{ij}, r_{ij}) is a Z-number in which b_{ij} and r_{ij} represent restriction and reliability, respectively, on the level of importance of the j -th criterion for the i -th alternative. Note that b_{ij} and r_{ij} are both triangular fuzzy numbers, denoted by $(b_{ij1}, b_{ij2}, b_{ij3})$ and $(r_{ij1}, r_{ij2}, r_{ij3})$, respectively.

Step 7: The decision matrix (29) is normalized using

$$\left(\beta_{ij}^{\%}, \rho_{ij}^{\%} \right) = \left(\frac{b_{ij}}{\max b_{ij3}}, \frac{r_{ij}}{\max r_{ij3}} \right). \quad (30)$$

The j -th criterion is a benefit criterion and $\max b_{ij3}$ is the maximum value of the rightmost element in the triangular fuzzy numbers b_{ij} from each column and row. $\max r_{ij3}$ is defined analogously, but for the reliability component. Otherwise, the following formula is used:

$$\left(\beta_{ij}^{\%}, \rho_{ij}^{\%} \right) = \left(\frac{\min b_{ij3}}{b_{ij}}, \frac{\min r_{ij3}}{r_{ij}} \right) \quad (31)$$

$\min b_{ij3}$ is the minimum value of the rightmost element in the triangular fuzzy numbers b_{ij} from each column and row. $\min r_{ij3}$ is defined similarly for the triangular fuzzy numbers representing the reliability components.

Step 8: The weighted sum matrix is then constructed by evaluating the weighted sum of the normalized decision matrix using the following formula:

$$\left(WS(\beta_{ij}^{\%}), WS(\rho_{ij}^{\%}) \right) = \left(\beta_{ij}^{\%} \times W(Z_j), \rho_{ij}^{\%} \times W(Z_j) \right) \quad (32)$$

$W(Z_j)$ is the weight of the j -th criterion and $\beta_{ij}^{\%}$ and $\rho_{ij}^{\%}$ are the normalized triangular fuzzy numbers representing the restriction and reliability components, respectively, for the j -th criterion and i -th alternative. The weighted product matrix is also constructed by evaluating the weighted product of the normalized decision matrix using the following formula:

$$\left(WP(\beta_{ij}^o), WP(\rho_{ij}^o) \right) = \left(\beta_{ij}^{oW(z_j)}, \rho_{ij}^{oW(z_j)} \right). \tag{33}$$

Step 9: The utility function combining the weighted sum and weighted product for the i -th alternative is calculated using the following formula:

$$(B_i, R_i) = \left(\sigma \sum_{j=1}^m WS(\beta_{ij}^o) + (1 - \sigma) \prod_{j=1}^m WP(\beta_{ij}^o), \sigma \sum_{j=1}^m WS(\rho_{ij}^o) + (1 - \sigma) \prod_{j=1}^m WP(\rho_{ij}^o) \right) \tag{34}$$

$\sum_{j=1}^m WS(\beta_{ij}^o)$ and $\sum_{j=1}^m WS(\rho_{ij}^o)$ are the summation of the weighted sum of fuzzy numbers for all criteria, while $\prod_{j=1}^m WP(\beta_{ij}^o)$ and $\prod_{j=1}^m WP(\rho_{ij}^o)$ are the product of the weighted product of fuzzy numbers for all criteria, and $\sigma \in [0,1]$.

Step 10: Finally, the Z-numbers (B_i, R_i) representing the utility functions are defuzzified into $\text{Rank}(Z_i)$ using (22). The alternatives are then ranked based on the calculated defuzzified values. The greater the defuzzified value, the higher the alternative is ranked and vice versa. The developed steps for the Z-AHP-WASPAS above are summarized and illustrated in Figure 8.

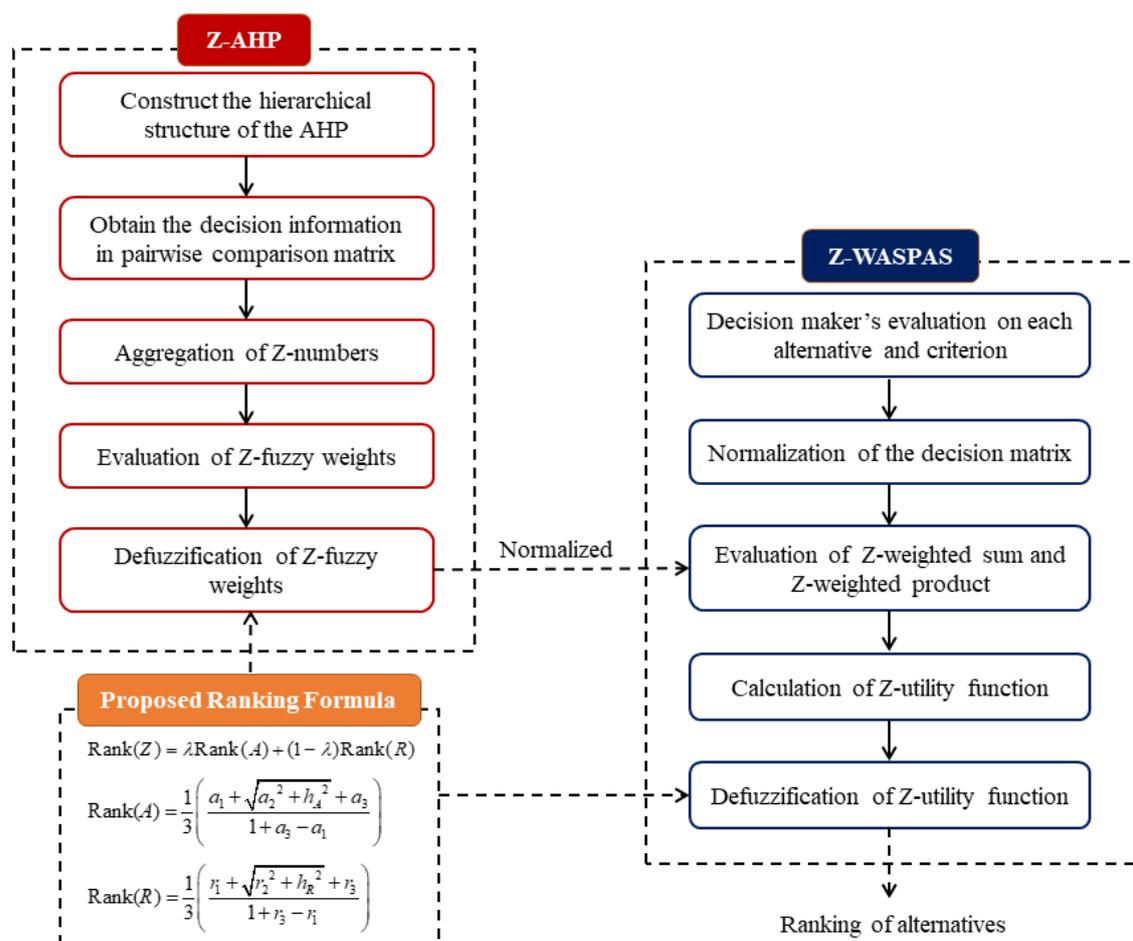


Figure 8. The proposed Z-AHP-WASPAS.

5.2. Real case example

The proposed Z-AHP-WASPAS model was subsequently implemented in the prioritization of public services for the implementation of Industry 4.0 tools adopted from [9].

Step 1: The hierarchical structure is first constructed as shown in Figure 9.

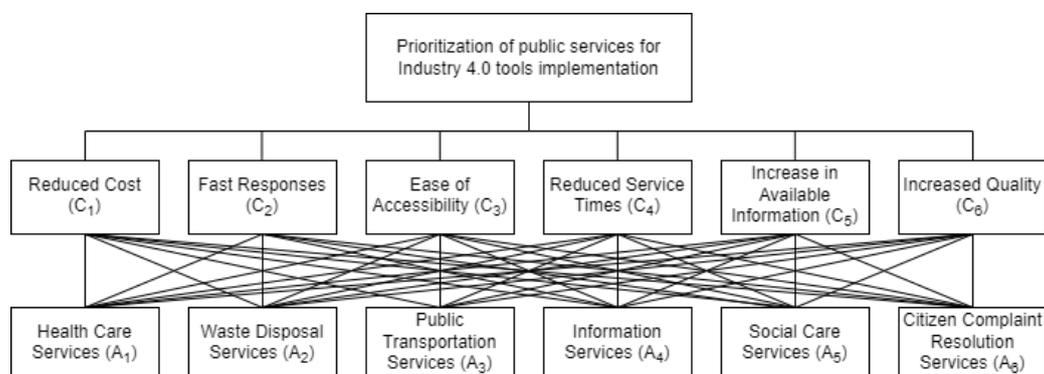


Figure 9. AHP hierarchical structure.

Step 2: The decision information on the pairwise comparison matrix is obtained as displayed in Table 3.

Table 3. Pairwise comparison matrix of AHP.

Criterion	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
C ₁	(1, 9)	(1/2, 4)	(1/3, 5)	(1/3, 5)	(4, 7)	(2, 7)
C ₂	(2, 7)	(1, 9)	(1/2, 4)	(1/2, 4)	(4, 7)	(3, 5)
C ₃	(3, 5)	(2, 7)	(1, 9)	(2, 7)	(5, 8)	(3, 5)
C ₄	(3, 5)	(2, 7)	(1/2, 4)	(1, 9)	(4, 7)	(3, 5)
C ₅	(1/4, 4)	(1/4, 4)	(1/5, 2)	(1/4, 4)	(1, 9)	(1/2, 4)
C ₆	(1/2, 4)	(1/3, 5)	(1/3, 5)	(1/3, 5)	(2, 7)	(1, 9)

The decision information in Table 3 is subsequently converted into Z-numbers, in which the restriction and reliability components will be represented by triangular fuzzy numbers that correspond to the crisp values, as shown in Table 4.

Table 4. Z-Numbers representation for the restriction and reliability components.

Crisp	Fuzzy Numbers (Restriction)	Crisp	Fuzzy Numbers (Reliability)
1	(1, 1, 1)	1	(0.0, 0.1, 0.2)
2	(1, 3, 5)	2	(0.1, 0.2, 0.3)
3	(3, 5, 7)	3	(0.2, 0.3, 0.4)
4	(5, 7, 9)	4	(0.3, 0.4, 0.5)
5	(7, 9, 9)	5	(0.4, 0.5, 0.6)
1/2	(1/5, 1/3, 1)	6	(0.5, 0.6, 0.7)
1/3	(1/7, 1/5, 1/3)	7	(0.6, 0.7, 0.8)
1/4	(1/9, 1/7, 1/5)	8	(0.7, 0.8, 0.9)
1/5	(1/9, 1/9, 1/7)	9	(1.0, 1.0, 1.0)

Step 3: The Z-numbers for all criteria are aggregated horizontally using the operator in (26). For example, when the Z-numbers representing C_1 are aggregated, the following calculation is performed:

$$\begin{aligned} & (1,1,1) + (1/5,1/3,1) + (1/7,1/5,1/3) + (1/7,1/5,1/3) + (5,7,9) + (1,3,5) \\ &= \left(1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{7} + 5 + 1, 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + 7 + 3, 1 + 1 + \frac{1}{3} + \frac{1}{3} + 9 + 5 \right) \\ &= (7.486, 11.733, 16.667) \end{aligned}$$

$$\begin{aligned} & (1.0, 1.0, 1.0) + (0.3, 0.4, 0.5) + (0.4, 0.5, 0.6) + (0.4, 0.5, 0.6) + (0.6, 0.7, 0.8) + (0.6, 0.7, 0.8) \\ &= (1.0 + 0.3 + 0.4 + 0.4 + 0.6 + 0.6, 1.0 + 0.4 + 0.5 + 0.5 + 0.7 + 0.7, \\ & \quad 1.0 + 0.5 + 0.6 + 0.6 + 0.8 + 0.8) \\ &= (3.300, 3.800, 4.300) \end{aligned}$$

Hence, the aggregated Z-number for C_1 is obtained as $((7.486, 11.733, 16.667), (3.3, 3.8, 4.3))$. Additionally, the Z-numbers for the rest of the criteria are aggregated analogously and the results are presented in Table 5.

Table 5. Aggregated Z-numbers.

Criterion	Aggregated Z-Numbers
C_1	$((7.486, 11.733, 16.667), (3.300, 3.800, 4.300))$
C_2	$((10.400, 16.667, 24.000), (3.200, 3.700, 4.200))$
C_3	$((16.000, 26.000, 34.000), (3.700, 4.200, 4.700))$
C_4	$((13.200, 21.333, 30.000), (3.300, 3.800, 4.300))$
C_5	$((1.644, 1.873, 2.743), (2.300, 2.800, 3.300))$
C_6	$((2.629, 4.933, 8.000), (3.100, 3.600, 4.100))$

Step 4: The fuzzy weight for each criterion is then calculated using (27) and the results are presented in Table 6. For example, the calculation for obtaining the fuzzy weight for C_1 will be shown. The sum of the aggregated Z-numbers is first obtained by summing up all the aggregated Z-numbers from Table 5 and the total is $((51.359, 82.540, 115.410), (18.9, 21.9, 24.9))$. Subsequently, the inverse of such numbers is calculated as follows:

$$(51.359, 82.540, 115.410)^{-1} = \left(\frac{1}{115.410}, \frac{1}{82.540}, \frac{1}{51.359} \right) = (0.0087, 0.0121, 0.0195)$$

$$(18.9, 21.9, 24.9)^{-1} = \left(\frac{1}{24.9}, \frac{1}{21.9}, \frac{1}{18.9} \right) = (0.0402, 0.0457, 0.0529)$$

Next, the fuzzy weight for C_1 can be obtained by multiplying the aggregated Z-number for C_1 with the obtained inverse triangular fuzzy number, as shown below.

$$\begin{aligned}
& (7.486, 11.733, 16.667) \times (0.0087, 0.0121, 0.0195) \\
& = (7.486 \times 0.0087, 11.733 \times 0.0121, 16.667 \times 0.0195) \\
& = (0.065, 0.142, 0.325)
\end{aligned}$$

$$\begin{aligned}
& (3.30, 3.80, 4.30) \times (0.0402, 0.0457, 0.0529) \\
& = (3.30 \times 0.0402, 3.80 \times 0.0457, 4.30 \times 0.0529) \\
& = (0.133, 0.174, 0.228)
\end{aligned}$$

Step 5: The restriction and reliability components of the obtained fuzzy weights from the previous step are then defuzzified. As an illustration, the calculation for defuzzifying the restriction and reliability components representing the fuzzy weight of C_1 is shown below.

$$\begin{aligned}
\text{Rank}(A) &= \frac{1}{3} \left(\frac{0.065 + \sqrt{0.142^2 + 1^2 + 0.325}}{1 + 0.325 - 0.065} \right) = 0.370 \\
\text{Rank}(R) &= \frac{1}{3} \left(\frac{0.133 + \sqrt{0.174^2 + 1^2 + 0.228}}{1 + 0.228 - 0.133} \right) = 0.419
\end{aligned}$$

The fuzzy weights of the rest of the criteria are then defuzzified similarly as shown in Table 6.

Table 6. Fuzzy weights.

Criterion	Fuzzy Weights	Rank(A)	Rank(R)
C_1	((0.065, 0.142, 0.325), (0.133, 0.174, 0.228))	0.370	0.419
C_2	((0.090, 0.202, 0.467), (0.129, 0.169, 0.222))	0.382	0.416
C_3	((0.139, 0.315, 0.662), (0.149, 0.192, 0.249))	0.405	0.429
C_4	((0.114, 0.258, 0.584), (0.133, 0.174, 0.228))	0.393	0.419
C_5	((0.014, 0.023, 0.053), (0.092, 0.128, 0.175))	0.343	0.393
C_6	((0.023, 0.060, 0.156), (0.124, 0.164, 0.217))	0.347	0.413

The defuzzified weights for the restriction and reliability components are subsequently combined using (22) by choosing $\lambda = 0.75$ to highlight that the role of the restriction component is more important than its reliability [23]. The obtained weights are then normalized using (28) and the results are presented in Table 7.

Table 7. Normalized weights.

Criterion	Final Weight	Normalized Weight	Priority Rank
C_1	0.382	0.166	4
C_2	0.390	0.170	3
C_3	0.411	0.178	1
C_4	0.399	0.173	2
C_5	0.355	0.154	6
C_6	0.364	0.158	5

Step 6: The decision maker's evaluation for all alternatives with respect to each criterion is obtained as shown in Table 8.

Table 8. Decision information matrix of WASPAS.

Alternative	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
A ₁	(5, 8)	(4, 7)	(5, 8)	(5, 8)	(4, 7)	(4, 7)
A ₂	(2, 5)	(4, 7)	(3, 6)	(4, 7)	(2, 5)	(3, 6)
A ₃	(3, 6)	(3, 6)	(4, 7)	(4, 7)	(3, 6)	(4, 7)
A ₄	(3, 6)	(4, 7)	(5, 8)	(3, 6)	(3, 6)	(3, 6)
A ₅	(4, 7)	(5, 8)	(4, 7)	(4, 7)	(3, 6)	(5, 8)
A ₆	(5, 8)	(4, 7)	(5, 8)	(3, 6)	(2, 5)	(4, 7)

The above decision matrix is further transformed into Z-numbers using the Z-numbers representation, as shown in Table 4.

Step 7: Since all the criteria constitute benefit criteria, the decision matrix in the form of Z-numbers is, therefore, normalized using (30). Note that $\max b_{ij3} = 9$, all elements b_{ij} in the decision matrix is consequently divided by 9. For example, $b_{11} = (7,9,9)$ is normalized as follows:

$$\frac{1}{9}(7,9,9) = (0.778, 1.000, 1.000).$$

Step 8: The weighted sum and weighted product matrices are constructed using (32) and (33), respectively. Considering the normalized $(\beta_{11}^0, \rho_{11}^0) = ((0.778, 1.000, 1.000), (0.778, 0.889, 1.000))$ and the criteria weight for C₁, which is 0.166, the weighted sum and weighted product for $(\beta_{11}^0, \rho_{11}^0)$ is, hence, calculated as follows:

$$\begin{aligned} (WS(\beta_{11}^0), WS(\rho_{11}^0)) &= ((0.778 \times 0.166, 1.000 \times 0.166, 1.000 \times 0.166), \\ &\quad (0.778 \times 0.166, 0.889 \times 0.166, 1.000 \times 0.166)) \\ &= ((0.129, 0.166, 0.166), (0.129, 0.148, 0.166)) \end{aligned}$$

$$\begin{aligned} (WP(\beta_{11}^0), WP(\rho_{11}^0)) &= ((0.778^{0.166}, 1.000^{0.166}, 1.000^{0.166}), (0.778^{0.166}, 0.889^{0.166}, 1.000^{0.166})) \\ &= ((0.959, 1.000, 1.000), (0.959, 0.981, 1.000)) \end{aligned}$$

Step 9: The utility function for each alternative is calculated using (34) by taking $\sigma = 0.5$, which represents the equal weightage for the weighted sum and weighted product. For example, the utility function for A₁ will be illustrated. The total sum of the weighted sum, $\sum_{j=1}^m WS(\beta_{ij}^0)$

and the total product of the weighted product, $\prod_{j=1}^m WP(\rho_{ij}^0)$ for the restriction component are first calculated and obtained as (0.665, 0.886, 0.997) and (0.658, 0.882, 1.000), respectively. As for the reliability component, the total sum of the weighted sum,

$\sum_{j=1}^m WS(\frac{\#}{j})$ and the total product of the weighted product, $\prod_{j=1}^m WP(\frac{\#}{j})$ are obtained as (0.720, 0.831, 0.941) and (0.721, 0.832, 0.943), respectively. Hence, the Z-numbered utility function for A_1 is calculated as follows:

$$\begin{aligned}(B_1, R_1) &= (0.5 \times (0.665, 0.886, 0.997) + 0.5 \times (0.658, 0.882, 1.000), \\ &\quad 0.5 \times (0.720, 0.831, 0.941) + 0.5 \times (0.721, 0.832, 0.943)) \\ &= ((0.661, 0.884, 0.998), (0.720, 0.831, 0.942))\end{aligned}$$

Meanwhile, the defuzzification of the restriction and reliability components will be performed separately as follows:

$$\begin{aligned}\text{Rank}(B_1) &= \frac{1}{3} \left(\frac{0.661 + \sqrt{0.884^2 + 1^2} + 0.998}{1 + 0.998 - 0.884} \right) = 0.747 \\ \text{Rank}(R_1) &= \frac{1}{3} \left(\frac{0.720 + \sqrt{0.831^2 + 1^2} + 0.942}{1 + 0.942 - 0.720} \right) = 0.808\end{aligned}$$

The utility functions of the rest of the alternatives are evaluated analogously as presented in Table 9 with their corresponding defuzzified values.

Table 9. Utility function values in terms of Z-numbers.

Alternative	Utility Function in Terms of Z-Numbers	Rank(B)	Rank(R)
A ₁	((0.661, 0.884, 0.998), (0.720, 0.831, 0.942))	0.747	0.808
A ₂	((0.304, 0.542, 0.772), (0.554, 0.667, 0.780))	0.502	0.689
A ₃	((0.441, 0.676, 0.913), (0.621, 0.738, 0.856))	0.580	0.734
A ₄	((0.433, 0.665, 0.863), (0.615, 0.730, 0.845))	0.582	0.731
A ₅	((0.572, 0.784, 0.927), (0.667, 0.772, 0.875))	0.681	0.774
A ₆	((0.474, 0.705, 0.859), (0.633, 0.740, 0.847))	0.615	0.748

Step 10: Finally, the defuzzified restriction and reliability components of the utility functions are combined using (22), taking $\lambda = 0.75$, similar to the AHP model. Table 10 shows the final priority weights for all alternatives and their ranking.

Table 10. Final priority weights and ranking of alternatives.

Alternative	Final Priority Weight	Priority Rank
A ₁	0.762	1
A ₂	0.549	6
A ₃	0.618	5
A ₄	0.619	4
A ₅	0.704	2
A ₆	0.648	3

5.3. Sensitivity analysis and discussion

The priority ranking of alternatives obtained using the proposed Z-AHP-WASPAS model sensitivity follows similar order as [9], which is $A_1 > A_5 > A_6 > A_4 > A_3 > A_2$. The sensitivity analysis was further conducted to highlight the advantage of the proposed model compared to the previous one. Each of the criteria weights were increased by 50% and the effects on the ranking order of alternatives were observed. Table 11 shows the results for the final priority weights obtained using the proposed Z-AHP-WASPAS model when the weight of each criterion is increased by 50%.

Table 11. Sensitivity analysis for the proposed Z-AHP-WASPAS model.

Alternative	Criterion with Increased Weights					
	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
A ₁	0.804	0.754	0.754	0.754	0.755	0.755
A ₂	0.551	0.558	0.551	0.551	0.550	0.550
A ₃	0.618	0.618	0.634	0.618	0.618	0.618
A ₄	0.619	0.619	0.619	0.636	0.619	0.619
A ₅	0.701	0.701	0.700	0.701	0.730	0.701
A ₆	0.647	0.647	0.647	0.647	0.647	0.668

Next, the sensitivity analysis was performed on the Z-AHP-WASPAS model proposed in [9] to be compared with the results obtained using the proposed model in this paper. The results in Table 12 show that the ranking of alternatives changes when the weights of C₃ and C₄ are increased by 50%.

Table 12. Sensitivity analysis for Z-AHP-WASPAS [9].

Alternative	Criterion with Increased Weights					
	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
A ₁	0.891	0.871	0.907	0.898	0.887	0.884
A ₂	0.532	0.567	0.527	0.580	0.547	0.548
A ₃	0.642	0.633	0.676	0.665	0.650	0.654
A ₄	0.679	0.694	0.769	0.650	0.689	0.684
A ₅	0.766	0.785	0.754	0.760	0.765	0.772
A ₆	0.752	0.741	0.806	0.691	0.736	0.742

The ranking orders $A_1 > A_6 > A_4 > A_5 > A_3 > A_2$ and $A_1 > A_5 > A_6 > A_3 > A_4 > A_2$ were observed when the weights of C₃ and C₄ were changed, respectively. The ranking order remained unchanged when the other criteria weights were changed, resulting in a 66.67% consistency. Comparing this result to the proposed Z-AHP-WASPAS model, the ranking order maintained its stability when all criteria weights were changed, except for C₃, resulting in an 83.33% consistency. In fact, only the interchange of ranking between A₃ and A₄ was observed when the weight of C₃ was increased by 50%. The comparison of the sensitivity analysis is illustrated in Figure 10.

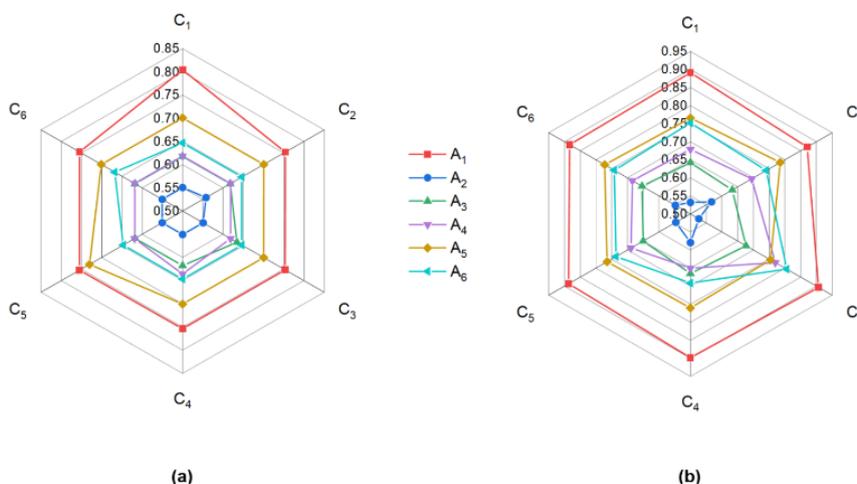


Figure 10. The radar diagram for the sensitivity analysis: (a) Proposed Z-AHP-WASPAS; (b) Z-AHP-WASPAS [9].

The consistency of the proposed Z-AHP-WASPAS model is influenced by many factors. First, this includes the preservation of decision information in the form of Z-numbers until the final steps of the AHP and WASPAS. The proposed model does not transform the Z-numbers into regular fuzzy numbers since this process is believed to lead to a great loss of information [3]. Second, the defuzzification of Z-numbers into crisp values for the valuation of the criteria weights and priority weights of alternatives makes use of the proposed ranking approach of Z-numbers based on the vectorial distances and spread. In addition, the comparative analysis has also shown that the proposed method could rank Z-numbers better than the existing approaches. In fact, when applying the proposed method for ranking Z-numbers in the decision-making model, the sensitivity analysis has shown an improvement in consistency. Third, the use of the convex compound in combining the restriction and reliability components of Z-numbers allows the level of importance of each component to depict the Z-numbers. In this example, the variable λ was set as 0.75 to highlight that the role of the restriction component is more important than its reliability [23].

Furthermore, the proposed Z-AHP-WASPAS model was compared with the classical AHP method in obtaining the criteria weights. For this purpose, the classical AHP based on crisp scales [31] was used. In the classical AHP, the decision maker's evaluation is kept as crisp values and the degree of reliability of the decision maker's preferences is completely ignored, as shown in Table 13.

Table 13. Criteria weights obtained using classical AHP.

Criterion	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆	Weight
C ₁	1	1/2	1/3	1/3	4	2	0.123
C ₂	2	1	1/2	1/2	4	3	0.183
C ₃	3	2	1	2	5	3	0.321
C ₄	3	2	1/2	1	4	3	0.246
C ₅	1/4	1/4	1/5	1/4	1	1/2	0.047
C ₆	1/2	1/3	1/3	1/3	2	1	0.080

Using the obtained criteria weights, the weighted sum and weighted product of the fuzzy WASPAS model were evaluated. The ranking of alternatives was obtained as $A_1 > A_5 > A_6 > A_3 > A_4 > A_2$. The

sensitivity analysis was further performed to see the effects of increasing the weight of each criterion by 50%. Table 14 presents the results of the sensitivity analysis when the weights obtained using the classical AHP are integrated into the WASPAS model.

Table 14. Sensitivity analysis for the WASPAS model with criteria weights obtained using classical AHP [31].

Alternative	Criterion with Increased Weights					
	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
A ₁	0.754	0.726	0.721	0.724	0.729	0.728
A ₂	0.502	0.508	0.509	0.506	0.501	0.502
A ₃	0.594	0.591	0.653	0.587	0.598	0.596
A ₄	0.585	0.584	0.582	0.622	0.587	0.586
A ₅	0.685	0.686	0.688	0.687	0.679	0.685
A ₆	0.633	0.634	0.640	0.637	0.631	0.618

Similar to the Z-AHP-WASPAS model from [9], the ranking order changed when the weights of C₃ and C₄ were increased by 50%. The increase in weights for C₃ and C₄ resulted in the ranking orders A₁ > A₅ > A₃ > A₆ > A₄ > A₂ and A₁ > A₅ > A₆ > A₄ > A₃ > A₂, respectively. Hence, the consistency performance was only 66.67% when the criteria weights were quantified using the classical AHP and further applied in the fuzzy WASPAS.

In general, the results are affected by the ignorance of the reliability parts, which represent the degree of sureness when the decision maker is making the evaluation. This shows that the application of Z-numbers in decision-making is essential as it can depict imperfect information. Besides, the use of crisp numbers in the pairwise comparison matrix of the classical AHP could not present the decision information in the best way. This is due to imprecision and uncertainties when the decision maker is hesitant on giving discrete values on the relative importance of the criteria. Hence, the use of fuzzy numbers can better describe the criteria preferences [32].

However, it is still important to note that this research is limited to the application of the proposed ranking method in the Z-AHP-WASPAS model. Since the proposed ranking of Z-numbers is based on the vectorial distance and spread of fuzzy numbers, other decision-making models are based on distance such as fuzzy TOPSIS, CODAS, and evaluation based on distance from average solution (EDAS). Therefore, in future works, the proposed method should be extended to other methods, which are regressively discussed in the current decision-making methods such as preference relation [33–35] and the consensus model for group decision-making [36,37].

6. Conclusions

Ranking Z-numbers is a complex task since they are fashioned by a pair of fuzzy numbers, namely the restriction and reliability components. When ranking Z-numbers, it is essential to keep the information so that it can be processed by making use of the meaning of the Z-numbers. In addition, the issue of information loss when converting Z-numbers into regular fuzzy numbers should be overcome. Hence, a novel ranking method of Z-numbers was proposed in this paper based on the vectorial distances and spread. The advantages of the study can be summarized as follows:

- (a) The vectorial distances in the proposed ranking method measure the distances of all vertices,

which form the shape of fuzzy numbers from a reference point, namely the origin. The spread is also considered in ranking the Z-numbers to cater to various situations in which Z-numbers have different spreads. Besides, this ranking approach is very suitable to be adopted in any decision-making model that uses linguistic terms to describe the level of importance of criteria and alternatives. The Z-numbers, which represent the linguistic terms, are appropriately ranked based on their distances and spread from the origin.

- (b) The proposed ranking of Z-numbers does not convert them into regular fuzzy numbers but instead keeps the restriction and reliability components of the Z-numbers until a final ranking is obtained. The ranking method makes use of the convex compound to combine the restriction and reliability components in which a variable is introduced to allow for at least equal weightage for both components; however, in most situations, it is assumed that the restriction component is given more priority compared to its reliability. Hence, a comparative analysis was conducted to validate the proposed ranking method and compare some examples of Z-numbers with the existing ranking methods. Overall, it was observed that the proposed method has a better capability of ranking Z-numbers, which could be further used in solving decision-making problems.
- (c) The ranking method of Z-numbers proposed in this study was further extended to develop a novel AHP-WASPAS model based on the Z-numbers, in which the decision information was preserved as Z-numbers and the proposed ranking method was used to defuzzify the processed Z-numbers information into crisp values to obtain the criteria weights and priority weights of alternatives. Additionally, a numerical example was also adopted to illustrate the proposed Z-AHP-WASPAS model and the sensitivity analysis was further conducted to evaluate the consistency of the model. In essence, the study observed that the proposed model has improved its consistency compared to the existing model.

The improvement of the Z-AHP-WASPAS model using the proposed ranking approach could be influenced by several factors such as the preservation of Z-numbers information, the use of the improved ranking method of Z-numbers, and the choice of variable value in controlling the weightages of the restriction and reliability components. However, it is important to note that this research is limited to the application of the Z-number-based ranking approach in the AHP-WASPAS model. The ranking approach could be applied when developing other fuzzy decision-making models that require linguistic evaluation from decision makers. Besides, the proposed model does not consider group decision-making that involves many decision makers and consequently requires further exploration in aggregating all the decision makers' preferences, such as the consensus model. In future works, the proposed ranking approach of Z-numbers could also be applied when developing other decision-making models such as TOPSIS, DEMATEL, and Best Worst Method (BWM).

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Conflict of interest

All the authors declare that there is no conflict of interest.

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