Nonlinear Analysis: Modelling and Control, Online First, 1–25
https://doi.org/10.15388/namc.2023.28.31728

Existence of positive solutions for singular p-Laplacian Hadamard fractional differential equations with the derivative term contained in the nonlinear term*

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Received: May 5, 2022 / **Revised:** February 8, 2023 / **Published online:** March 11, 2023

Abstract. In this paper, based on the properties of Green function and the eigenvalue of a corresponding linear operator, the existence of positive solutions is investigated by spectral analysis for a infinite-points singular p-Laplacian Hadamard fractional differential equation boundary value problem, and an example is given to demonstrate the validity of our main results.

Keywords: Hadamard fractional differential equation, spectral analysis, positive solution, infinite points.

1 Introduction

We consider the following singular Hadamard fractional differential equation:

$$ {}^H D^{\alpha}_{1^+}\big(\varphi_p\big({}^H D^{\gamma}_{1^+} u\big)\big)(t) + \hbar\big(t, u(t), {}^H D^{\mu}_{1^+} u(t)\big) = 0, \quad 1 < t < \mathrm{e}, \tag{1} $$

with infinite-point boundary condition

$$ u^{(j+\mu)}(1) = 0, \quad j = 0, 1, 2, \dots, n-2; \qquad {}^H D^{r_1}_{1^+} u(\mathrm{e}) = \sum_{j=1}^{\infty} \eta_j {}^H D^{r_2}_{1^+} u(\xi_j), \tag{2} $$

*This research was supported by the National Natural Science Foundation of China (12101086, 12272064, 62273061), Changzhou Science and Technology Planning Project (CJ20220238, CE20215057, CZ20220030), Changzhou Leading Innovative Talents Cultivation Project (CQ20220092), and Major Project of Basic Science (Natural Science) Research in colleges and universities of Jiangsu Province (22KJA580001).

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$$ {}^HD^{\gamma}_{1^+}u(1) = 0; \qquad \varphi_p\big({}^HD^{\gamma}_{1^+}u(\mathrm{e})\big) = \sum_{j=1}^{\infty} \zeta_j \varphi_p\big({}^HD^{\gamma}_{1^+}u(\xi_j)\big), \tag{3} $$

where $\alpha, \gamma, \mu \in \mathbb{R}^+ = [0, +\infty)$, $1 < \alpha \leqslant 2$, $n < \gamma \leqslant n+1 (n \geqslant 3)$, $r_1, r_2 \in [2, n-2]$, $r_2 \leqslant r_1$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p, q > 1$, $1/p + 1/q = 1$, $0 < \mu \leqslant n-2$ and $0 < \eta_i, \zeta_i < 1$, $1 < \xi_i < \mathrm{e}$ $(i = 1, 2, \dots, \infty)$, $\hbar \in C((1, \mathrm{e}) \times \mathbb{R}_+ \times \mathbb{R}_+))(\mathbb{R}_+ = [0, +\infty)$ and may be singular at $t = 1, \mathrm{e}$, and ${}^HD^{\alpha}_{1^+}u$, ${}^HD^{\gamma}_{1^+}u$, ${}^HD^{\mu}_{1^+}u$, ${}^HD^{r_i}_{1^+}u$ $(i = 1, 2)$ are the standard Hadamard fractional derivatives.

Compared with classical integer-order differential equations, fractional-order differential model has the advantages of simple modeling, accurate description, and clear physical meaning of parameters for complex problems, and it is one of the important tools for mathematical modeling of complex system such as physics [21], chemistry [23], astronomy [22], artificial intelligence [20], population dynamics [4], and financial modeling [25]. It has been noticed that most of the work on the topic is based on Riemann–Liouville and Caputo derivatives; for more details, readers can refer to [3, 6, 9, 11–16, 19, 27, 31–33] and the references therein. There is another kind of fractional derivatives in the literature due to Hadamard [18], which is named as Hadamard derivative and differs from the preceding ones in the sense that its definition involves logarithmic function of arbitrary exponent. Although many researchers are paying more and more attention to Hadamard fractional differential equation, but the solutions of Hadamard fractional differential equations are still very few, the study of the topic is still in its primary stage. About the details and recent developments on Hadamard fractional differential equations, we refer the reader to [1, 2, 5, 26, 28]. In [29], Zhang et al. considered the following Hadamard fractional integral boundary value problem:

$$ \begin{aligned} -{}^HD^{\alpha}_{1^+}u(t) &= f_1\big(t, u(t), v(t)\big), \quad 1 \leqslant t \leqslant \mathrm{e}, \\ -{}^HD^{\alpha}_{1^+}v(t) &= f_2\big(t, u(t), v(t)\big), \quad 1 \leqslant t \leqslant \mathrm{e}, \end{aligned} $$

with boundary conditions

$$ u^j(1) = v^j(1) = 0, \qquad u(\mathrm{e}) = \int_1^{\mathrm{e}} h(t)u(t)\frac{\mathrm{d}t}{t}, \qquad v(\mathrm{e}) = \int_1^{\mathrm{e}} h(t)v(t)\frac{\mathrm{d}t}{t}, $$

where $\alpha \in (n-1, n]$ is a real number with $n \geqslant 3$, $j = 0, 1, 2, \dots, n-2$, and ${}^HD^{\alpha}_{1^+}$ denotes Hadamard fractional derivative of order α. The nonlinearities $f_i \in C([1, \mathrm{e}] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, and $\int_1^{\mathrm{e}} h(t)(\log t)^{\alpha-1}\mathrm{d}t/t \in [0, 1)$. In [24], Zhang and Wang considered the following Hadamard fractional differential equation:

$$ {}^HD^{\beta}_{1^+}\phi_p\big({}^HD^{\alpha}_{1^+}u(t)\big) = f\big(t, u(t)\big), \quad t \in (1, \mathrm{e}), $$

with boundary conditions

$$ \begin{gathered} u(1) = u'(1) = u'(\mathrm{e}) = 0, \qquad {}^HD^{\alpha}_{1^+}u(1) = 0, \\ \varphi_p\big({}^HD^{\alpha}u(\mathrm{e})\big) = \mu \int_1^{\mathrm{e}} \varphi_p\big({}^HD^{\alpha}u(t)\big)\frac{\mathrm{d}t}{t}, \end{gathered} $$

where ${}^{H}D_{1+}^{\beta}$ and ${}^{H}I_{1+}^{\sigma}$ denote Hadamard fractional derivative of order $\alpha \in (2,3)$, $\beta \in (1,2]$, and $\mu \in [0,\beta)$, $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, where $1/p + 1/q = 1$. In [28], Yukunthorn and Ahmad considered the following fractional differential equation:

$$-{}^{H}D_{1+}^{\beta}\phi_p\big(-{}^{H}D_{1+}^{\alpha}u(t)\big) = f\big(t, u(t), u\big(\theta(t)\big)\big),$$

subject to integral boundary condition

$${}^{H}D_{1+}^{\alpha}u(1) = {}^{H}D_{1+}^{\alpha}u(\mathrm{e}) = 0, \qquad u(1) = 0, \qquad {}^{H}D_{1+}^{\alpha-1}u(1) = \eta\, {}^{H}D_{1+}^{\alpha-1}u(\mathrm{e}),$$

where $1 < \alpha, \beta \leqslant 2$, $\eta \in \mathbb{R}$, $J = [1,\mathrm{e}]$, $\theta \in C[J,J]$, $t \in J$, $f \in C(\mathbb{R}^2,\mathbb{R})$ ${}^{H}D_{1+}^{\alpha}$ and ${}^{H}I_{1+}^{\beta}$ denote Hadamard fractional derivative of order α and Hadamard fractional integral of order β, and $f, g : [1,\mathrm{e}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In [30], the author considered the following fractional differential equation:

$$D_{0+}^{\alpha}u(t) + g(t)f\big(t, u(t)\big) = 0, \quad 0 < t < 1,$$

with boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j),$$

where $\alpha \in \mathbb{R}^+$, $n < \alpha \leqslant n+1$, $n > 3$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2, \dots)$, f is allowed to have singularities with respect to both time and space variables. Various theorems were established for the existence and multiplicity of positive solutions. The existence of positive solutions are established under some sufficient conditions by u_0-positive linear operator and the fixed point theorem.

Motivated by the excellent results above, in this paper, we investigate the existence of positive solutions by spectral analysis for singular Hadamard fractional differential equation with infinite-point boundary value conditions (1)–(3). Compared with [24, 30], the fractional derivative is involved in the nonlinear term in this paper, and the method used in this paper is spectral analysis. Compared with [30], the derivatives used in this paper are Hadamard fractional derivatives.

2 Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas, which are useful for the following research and which can be found in the recent literature such as [18].

Definition 1. (See [18].) The Hadamard fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}_+$ is given by

$${}^{H}I_{1+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s}\,\mathrm{d}s.$$

Definition 2. (See [18].) The Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}_+$ is given by

$$ {}^H D^{\alpha}_{1+} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{\mathrm{d}}{\mathrm{d}t} \right)^n \int_1^t \frac{y(s)}{s(\ln \frac{t}{s})^{\alpha-n+1}} \, \mathrm{d}s, $$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 1. (*See* [18]). *If* $\alpha, \beta > 0$*, then*

$$ {}^H I^{\alpha}_a \left(\ln \left(\frac{t}{a} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\ln \frac{x}{a} \right)^{\beta+\alpha-1}, $$

$$ {}^H D^{\alpha}_a \left(\ln \left(\frac{t}{a} \right)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{x}{a} \right)^{\beta-\alpha-1}. $$

Lemma 2. (*See* [18]). *Suppose that* $\alpha > 0$ *and* $u \in C[1, \infty) \cap L[1, \infty)$*, then the solution of Hadamard fractional differential equation* ${}^H D^{\alpha}_{1+} u(t) = 0$ *is*

$$ u(t) = c_1 (\ln t)^{\alpha-1} + c_2 (\ln t)^{\alpha-2} + \cdots + c_n (\ln t)^{\alpha-n}, $$

$c_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n)$, $n = [\alpha] + 1$.

Lemma 3. (*See* [18]). *Suppose that* $\alpha > 0$*,* α *is not natural number. If* $u \in C[1, \infty) \cap L[1, \infty)$*, then*

$$ u(t) = {}^H I^{\alpha}_{1+} {}^H D^{\alpha}_{1+} u(t) + \sum_{k=1}^{n} c_k (\ln t)^{\alpha-k} $$

for $t \in (1, \mathrm{e}]$*, where* $c_k \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$ *and* $n = [\alpha] + 1$.

Let $u(t) = {}^H I^{\mu}_{1+} v(t)$, $v(t) \in C[1, \mathrm{e}]$, then the BVP (1)–(3) reduced to the following modified boundary value problem:

$$ {}^H D^{\alpha}_{1+} \big(\varphi_p \big({}^H D^{\gamma-\mu}_{1+} v \big) \big)(t) + \hbar \big(t, {}^H I^{\mu}_{1+} v(t), v(t) \big) = 0, \quad 1 < t < \mathrm{e}, \tag{4} $$

with nonlocal boundary conditions

$$ \begin{aligned} & v^{(j)}(1) = 0, \quad j = 0, 1, 2, \ldots, n-2; \qquad {}^H D^{r_1-\mu}_{1+} v(\mathrm{e}) = \sum_{j=1}^{\infty} \eta_j {}^H D^{r_2-\mu}_{1+} v(\xi_j), \\ & {}^H D^{\gamma-\mu}_{1+} v(1) = 0; \qquad \varphi_p \big({}^H D^{\gamma-\mu}_{1+} v(\mathrm{e}) \big) = \sum_{j=1}^{\infty} \zeta_j \varphi_p \big({}^H D^{\gamma-\mu}_{1+} v(\xi_j) \big). \end{aligned} \tag{5} $$

Lemma 4. (See [10].) *Let* $g \in L(1, \mathrm{e}) \cap C(1, \mathrm{e})$*, then the equation of the BVPs*

$$ -{}^H D^{\gamma-\mu}_{1+} v(t) = g(t), \quad 1 < t < \mathrm{e}, $$

$$ v^{(j)}(1) = 0, \quad j = 0, 1, 2, \ldots, n-2; \qquad {}^H D^{r_1-\mu}_{1+} v(\mathrm{e}) = \sum_{j=1}^{\infty} \eta_j {}^H D^{r_2-\mu}_{1+} v(\xi_j) $$

has integral representation

$$v(t) = \int_1^{\mathrm{e}} \mathfrak{I}(t,s)\frac{g(s)}{s}\,\mathrm{d}s, \tag{6}$$

where

$$\mathfrak{I}(t,s) = \frac{1}{\Delta\Gamma(\gamma-\mu)}\begin{cases}\Gamma(\gamma-\mu)(\ln t)^{\gamma-\mu-1}Q(s)(\ln\frac{\mathrm{e}}{s})^{\gamma-r_1-1} \\ \quad -\Delta(\ln\frac{t}{s})^{\gamma-\mu-1}, \quad 1 \leqslant s \leqslant t \leqslant \mathrm{e}, \\ \Gamma(\gamma-\mu)(\ln t)^{\gamma-\mu-1}Q(s)(\ln\frac{\mathrm{e}}{s})^{\gamma-r_1-1}, \\ \quad 1 \leqslant t \leqslant s \leqslant \mathrm{e}, \end{cases}$$

in which

$$Q(s) = \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)}\sum_{s\leqslant\xi_j}\eta_j\left(\frac{\ln\frac{\xi_j}{s}}{\ln\frac{\mathrm{e}}{s}}\right)^{\gamma-r_2-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{r_1-r_2},$$

$$\Delta = \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j\ln\xi_j^{\gamma-r_2-1} \neq 0.$$

Proof. The method used in this paper is similar with [10], we omit it here. □

Lemma 5. *The functions $\mathfrak{I}(t,s)$ given by* (6) *has the following properties*:

- (i) $\mathfrak{I}: [1,\mathrm{e}]\times[1,\mathrm{e}] \to \mathbb{R}_+$ *is continuous, and* $\mathfrak{I}(t,s) > 0$ *for all* $t, s \in (1,\mathrm{e})$:
- (ii) $(\ln t)^{\gamma-\mu-1}\mathfrak{I}(\mathrm{e},s) \leqslant \mathfrak{I}(t,s) \leqslant \mathfrak{I}(\mathrm{e},s)$.

Proof. By direct calculation we get $Q'(s) \geqslant 0$, $s \in [1,\mathrm{e}]$, and so $Q(s)$ is nondecreasing with respect to s. For $r_2 \leqslant r_1$, $s \in [1,\mathrm{e}]$, we get

$$\begin{aligned}\Gamma(\gamma-\mu)Q(s) &= \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_2)}\sum_{s\leqslant\xi_j}\eta_j\left(\frac{\ln\frac{\xi_j}{s}}{\ln\frac{\mathrm{e}}{s}}\right)^{\gamma-r_2-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{r_1-r_2} \\ &\geqslant \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j\ln\xi_j^{\gamma-r_2-1} = \Delta,\end{aligned}$$

hence, for $1 < s \leqslant t < \mathrm{e}$, we have

$$\begin{aligned}\mathfrak{I}(t,s) &= \frac{1}{\Delta\Gamma(\gamma-\mu)}\left[\Gamma(\gamma-\mu)(\ln t)^{\gamma-\mu-1}Q(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-r_1-1} - \Delta\left(\ln\frac{t}{s}\right)^{\gamma-\mu-1}\right] \\ &= \frac{1}{\Delta\Gamma(\gamma-\mu)}\ln t^{\gamma-\mu-1}\left[\Gamma(\gamma-\mu)Q(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-r_1-1} - \Delta\left(1-\frac{\ln s}{\ln t}\right)^{\gamma-\mu-1}\right] \\ &\geqslant \frac{1}{\Delta\Gamma(\gamma-\mu)}\ln t^{\gamma-\mu-1}\left[\Gamma(\gamma-\mu)Q(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-r_1-1} - \Delta(1-\ln s)^{\gamma-\mu-1}\right] \\ &\geqslant \ln t^{\gamma-\mu-1}\mathfrak{I}(\mathrm{e},s).\end{aligned} \tag{7}$$

For $1 < t \leqslant s < \mathrm{e}$, we have

$$\mathfrak{I}(t,s) = \frac{1}{\Delta}(\ln t)^{\gamma-\mu-1}Q(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma-r_1-1} \geqslant (\ln t)^{\gamma-\mu-1}\mathfrak{I}(\mathrm{e},s). \tag{8}$$

By calculation we have

$$ {}^{H}D^{\kappa}_{1+}\mathfrak{I}(t,s) = \frac{1}{\Delta\Gamma(\gamma-\mu-\kappa)}\begin{cases} \ln t^{\gamma-\mu-1-\kappa}\Gamma(\gamma-\mu)Q(s)(1-s)^{\gamma-p_1-1} \\ \quad -\Delta(\ln\frac{t}{s})^{\gamma-\mu-1-\kappa}, \quad 1\leqslant s\leqslant t\leqslant \mathrm{e}, \\ (\ln t)^{\gamma-\mu-1-\kappa}\Gamma(\gamma-\mu)Q(s)(\ln\frac{\mathrm{e}}{s})^{\gamma-p_1-1}, \\ \quad 1\leqslant t\leqslant s\leqslant \mathrm{e}. \end{cases} \tag{9}$$

By simple computation we get ${}^{H}D^{\kappa}_{1+}\mathfrak{I}(t,s) \geqslant 0$, then we have $\partial\mathfrak{I}(t,s)/\partial t \geqslant 0$, hence, $\mathfrak{I}(t,s)$ is increasing with respect to t, thus, we have

$$\max_{t\in[1,\mathrm{e}]}\mathfrak{I}(t,s) = \mathfrak{I}(\mathrm{e},s).$$

By (7) and (8) we have that (i) holds. By (9) we get (ii) holds. □

Lemma 6. *Let* $\hbar \in C((1,\mathrm{e})\times(0,+\infty)^2,[0,+\infty))$*, then the BVP* (4), (5) *has a unique solution*

$$v(t) = \int_1^{\mathrm{e}}\mathfrak{I}(t,s)\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau, {}^{H}I^{\mu}_{1+}v(\tau), v(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s},$$

where

$$\mathfrak{R}(t,s) = \mathfrak{R}_1(t,s) + \mathfrak{R}_2(t,s), \tag{10}$$

in which

$$\begin{aligned} \mathfrak{R}_1(t,s) &= \frac{1}{\Gamma(\alpha)}\begin{cases} (\ln t)^{\alpha-1}(\ln\frac{\mathrm{e}}{s})^{\alpha-1} - (\ln\frac{t}{s})^{\alpha-1}, & 1\leqslant s\leqslant t\leqslant \mathrm{e}, \\ (\ln t)^{\alpha-1}(\ln\frac{\mathrm{e}}{s})^{\alpha-1}, & 1\leqslant t\leqslant s\leqslant \mathrm{e}, \end{cases} \\ \mathfrak{R}_2(t,s) &= \frac{(\ln t)^{\alpha-1}}{\overline{\Delta}\Gamma(\alpha)}\left[\sum_{\xi_j>s}\zeta_j\left[\xi_j^{\alpha-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\alpha-1} - \left(\ln\frac{\xi_j}{s}\right)^{\alpha-1}\right]\right. \\ &\quad \left. + \sum_{s\geqslant\xi_j}\zeta_j\ln\xi_j^{\alpha-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{\alpha-1}\right], \quad t,s\in[1,\mathrm{e}]. \end{aligned} \tag{11}$$

Here $\overline{\Delta} = 1 - \sum_{i=1}^{\infty}\zeta_i\ln\xi_i^{\alpha-1}$. *Easily, we have*

$$\mathfrak{R}_2(t,s) = \frac{1}{\overline{\Delta}}\sum_{i=1}^{\infty}\zeta_i\mathfrak{R}_1(\xi_i,s)\cdot\ln t^{\alpha-1}. \tag{12}$$

Proof. The proof is similar to the proof of Lemma 2.2 in [17], we omit it here. □

Lemma 7. *Let $\overline{\Delta} > 0$, then the Green functions $\mathfrak{R}(t,s)$ defined by* (10) *satisfies*:

(i) $\mathfrak{R}: [1,\mathrm{e}] \times [1,\mathrm{e}] \to \mathbb{R}_+$ *is continuous, and* $\mathfrak{R}(t,s) > 0$ *for all* $t, s \in (1,\mathrm{e})$;

(ii)
$$\frac{1}{\overline{\Delta}}\bar{j}(s)(\ln t)^{\alpha-1} \leqslant \mathfrak{R}(t,s) \leqslant b^*(\ln t)^{\alpha-1}, \quad t,s \in [1,\mathrm{e}], \tag{13}$$
$$\bar{j}(s) = \sum_{i=1}^{\infty} \zeta_i \mathfrak{R}_1(\eta_i, s), \qquad b^* = \frac{1}{\overline{\Delta}\Gamma(\alpha)}\left(\overline{\Delta} + \sum_{i=1}^{\infty} \zeta_i\left(\ln \xi_i^{\alpha-1}\right)\right),$$

$\overline{\Delta}$ *is defined as in* (11).

Proof. Since $\mathfrak{R}_1(t,s) \geqslant 0$ for all $(t,s) \in [1,\mathrm{e}] \times [1,\mathrm{e}]$, we have
$$\mathfrak{R}(t,s) \geqslant \mathfrak{R}_2(t,s) = \frac{\bar{j}(s)}{\overline{\Delta}}(\ln t)^{\alpha-1}.$$
On the other hand, we get
$$\begin{aligned}
\mathfrak{R}(t,s) &= \mathfrak{R}_1(t,s) + \frac{1}{\overline{\Delta}}\sum_{i=1}^{\infty} \zeta_i \mathfrak{R}_1(\xi_i, s) \cdot (\ln t)^{\alpha-1} \\
&\leqslant \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\overline{\Delta}\Gamma(\alpha)}\sum_{i=1}^{\infty} \zeta_i (\ln \xi_i)^{\alpha-1}(\ln t)^{\alpha-1} \\
&\leqslant \frac{1}{\overline{\Delta}\Gamma(\alpha)}\left(\overline{\Delta} + \sum_{i=1}^{\infty} \zeta_i\left(\ln \xi_i^{\alpha-1}\right)\right) \cdot (\ln t)^{\alpha-1} = b^*(\ln t)^{\alpha-1}.
\end{aligned}$$
□

Let $E = C[1,\mathrm{e}]$, $\|v\| = \max_{1\leqslant t\leqslant \mathrm{e}} |v(t)|$, then $(E, \|\cdot\|)$ is a Banach space. In this paper,
$$\begin{aligned}
P &= \{v \in E\colon\ v(t) \geqslant 0,\ t \in [1,\mathrm{e}]\}, \\
K &= \{v \in P\colon\ v(t) \geqslant (\ln t)^{\gamma-\mu-1}\|v\|,\ t \in [1,\mathrm{e}]\}.
\end{aligned}$$
Obviously, K is a subcone of P in Banach space E, and (E,K) is an ordering Banach space. Let $K_r = \{v \in K\colon\ \|v\| < r\}$, $\partial K_r = \{v \in K\colon\ \|v\| = r\}$, and $\overline{K}_r = \{v \in K\colon\ \|v\| \leqslant r\}$.

We list the following conditions used in this paper for convenience.

(H1) $f \in C((1,\mathrm{e}) \times (0,+\infty)^2, \mathbb{R}_+)$ is continuous, and for any $0 < r < R < +\infty$,
$$\limsup_{n\to+\infty}\left\{\sup \int_{e(n)} \mathfrak{I}(\mathrm{e},s)\varphi_q\left(\int_1^{\mathrm{e}} \hbar\bigl(\tau, x(\tau), y(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\colon\ x \in K_{\overline{R}},\ y \in \overline{K}_R \setminus K_r\right\} = 0,$$
where $e(n) = [1,\ 1+1/n] \cup [\mathrm{e}-1/n,\ \mathrm{e}]$, $\overline{R} = R/\Gamma(\mu+1)$;

(H2) $0 < \mu \leqslant n-2$ and $r_1, r_2 \in [2, n-2]$, $r_2 \leqslant r_1$;
(H3) For $0 < \eta_i, \zeta_i < 1$, $1 < \xi_i < \mathrm{e}$ $(i = 1, 2, \dots, \infty)$, the following formulas hold:

$$\frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j \ln \xi_j^{\gamma-r_2-1} < \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_1)}, \qquad \sum_{i=1}^{\infty}\zeta_i \ln \xi_i^{\alpha-1} < 1.$$

Define two operators $A : K \setminus \{0\} \to P$ and $T : E \to E$ as follows:

$$(Av)(t) = \int_1^{\mathrm{e}} \Im(t,s)\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^{H}I_{1+}^{\mu}v(\tau), v(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, \quad t \in [1,\mathrm{e}], \tag{14}$$

$$(Tv)(t) = \int_1^{\mathrm{e}} \Im(t,s)v(s)\frac{\mathrm{d}s}{s}, \quad t \in [1,\mathrm{e}]. \tag{15}$$

Lemma 8 [Krein–Rutmann theorem]. (*See* [8].) *Let $T : E \to E$ be a continuous linear operator, P be a total cone, and let $T(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a positive constant c such that $cT(\psi) \geqslant \psi$, then the spectral radius $r(T) \neq 0$ and has a positive eigenfunction corresponding to its first eigenvalue $\lambda = r(T)^{-1}$.*

Lemma 9 [Gelfand's formula]. (*See* [8].) *For a bounded linear operator T and the operator norm $\|\cdot\|$, the spectral radius of T satisfies*

$$r(T) = \lim_{n\to+\infty}\left\|T^n\right\|^{1/n}.$$

Lemma 10. *Assume that* (H2)–(H3) *hold, then $T : K \to K$ defined by* (15) *is a completely continuous linear operator, and the spectral radius $r(T) \neq 0$, moreover, T has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof. For any $v \in K$, by Lemma 5 we have

$$\|Tv\| = \max_{t\in[1,\mathrm{e}]}\int_1^{\mathrm{e}} \Im(t,s)v(s)\frac{\mathrm{d}s}{s} \leqslant \int_1^{\mathrm{e}} \Im(\mathrm{e},s)v(s)\frac{\mathrm{d}s}{s}. \tag{16}$$

On the other hand, from Lemma 5 we also have

$$Tv(t) \geqslant (\ln t)^{\gamma-\mu-1}\int_1^{\mathrm{e}} \Im(\mathrm{e},s)v(s)\frac{\mathrm{d}s}{s}, \quad t \in [1,\mathrm{e}]. \tag{17}$$

Then (16) and (17) lead to $T : K \to K$. From the uniform continuity of $\Im(t,s)$ on $[1,\mathrm{e}] \times [1,\mathrm{e}]$ and (H2)–(H3) we have that $T : K \to K$ is a completely continuous linear operator.

In the following, we show that T has the first eigenvalue $\lambda_1 > 0$ by using Krein–Rutmann's theorem. In fact, by Lemma 5 there is $t_0 \in (1,\mathrm{e})$ such that $\Im(t_0,t_0) > 0$.

Hence, there exists $[a, \tilde{a}] \subset (1, \mathrm{e})$ such that $t_0 \in (a, \tilde{a})$ and $\mathfrak{I}(t, s) > 0$ for all $t, s \in [a, \tilde{a}]$. Choose $v \in K$ such that $v(t_0) > 0$ and $v(t) = 0$ for all $t \notin [a, \tilde{a}]$. Then for $t \in [a, \tilde{a}]$, we have

$$(Tv)(t) = \int_1^{\mathrm{e}} \mathfrak{I}(t, s)v(s)\frac{\mathrm{d}s}{s} \geqslant \int_a^{\tilde{a}} \mathfrak{I}(t, s)v(s)\frac{\mathrm{d}s}{s} > 0.$$

So there exists $\mu > 0$ such that $\mu(Tv)(t) \geqslant v(t)$ for $t \in [1, \mathrm{e}]$. It follows from Lemma 8 that the spectral radius $r(T) \neq 0$, and moreover, A has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$ such that

$$\lambda_1 T\varphi^* = \varphi^*.$$

The proof is completed. □

Lemma 11. *If* (H1)–(H3) *holds, then* $A : \overline{K}_R \setminus K_r \to K$ *is completely continuous.*

Proof. First, we prove $A(\overline{K}_R \setminus K_r) \subset K$. In fact, for any $v \in \overline{K}_R \setminus K_r$, $t \in [1, \mathrm{e}]$, by Lemma 5 we have

$$\begin{aligned}(Av)(t) &= \int_1^{\mathrm{e}} \mathfrak{I}(t, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s, \tau)\hbar\left(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &\leqslant \int_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s, \tau)\hbar\left(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s},\end{aligned}$$

then

$$\left\|(Av)\right\| \leqslant \int_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s, \tau)\hbar\left(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}.$$

On the other hand, by Lemma 5 we also have

$$\begin{aligned}(Av)(t) &= \int_1^{\mathrm{e}} \mathfrak{I}(t, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s, \tau)\hbar\left(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &\geqslant (\ln t)^{\gamma-\mu-1} \int_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s, \tau)\hbar\left(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\ &\geqslant (\ln t)^{\gamma-\mu-1}\|Av\|, \quad t \in [1, \mathrm{e}].\end{aligned}$$

Hence, $A(\overline{K}_R \setminus K_r) \subset K$.

Next, for any $r > 0$, we show that

$$\sup_{v \in \overline{K}_R \setminus K_r} \int_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s, \tau)\hbar\left(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} < +\infty, \tag{18}$$

which implies that $A : \overline{K}_R \setminus K_r \to K$ is well defined. In fact, it follows from (H1) that there exists a natural number $n_0 > 1$ such that

$$\sup_{v \in \overline{K}_R \setminus K_r} \int_{e(n_0)} \mathfrak{I}(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\big(\tau, {}^H I_{1^+}^{\mu} v(\tau), v(\tau)\big) \frac{\mathrm{d}\tau}{\tau}\right) \frac{\mathrm{d}s}{s} < 1. \tag{19}$$

Thus, for any $v \in \overline{K}_R \setminus K_r$, we have

$$(\ln t)^{\gamma-\mu-1}\|v\| \leqslant v(t) \leqslant \|v\| \leqslant R, \quad t \in [1, \mathrm{e}], \tag{20}$$

$$\begin{aligned} {}^H I_{1^+}^{\mu} v(t) &= \frac{1}{\Gamma(\mu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\mu-1} \frac{v(s)}{s}\,\mathrm{d}s \leqslant \frac{\|v\|}{\Gamma(\mu)} \int_1^t \left(\ln\frac{t}{s}\right)^{\mu-1} \mathrm{d}(\ln s) \\ &= -\frac{\|v\|}{\Gamma(\mu)} \int_1^t (\ln t - \ln s)^{\mu-1}\,\mathrm{d}(\ln t - \ln s) \\ &\leqslant \frac{1}{\Gamma(\mu+1)}\|v\|, \quad t \in [1, \mathrm{e}], \end{aligned} \tag{21}$$

$$\begin{aligned} {}^H I_{1^+}^{\mu} v(t) &= \frac{1}{\Gamma(\mu)} \int_1^t \left(\ln\frac{t}{s}\right)^{\mu-1} \frac{v(s)}{s}\,\mathrm{d}s \\ &\geqslant \frac{\|v\|}{\Gamma(\mu)} \int_1^t (\ln t - \ln s)^{\mu-1}(\ln s)^{\gamma-\mu-1}\,\mathrm{d}(\ln s). \end{aligned} \tag{22}$$

Let $\ln s = \tau \ln t$, then we have

$$\begin{aligned} &\frac{\|v\|}{\Gamma(\mu)} \int_1^t (\ln t - \tau\ln t)^{\mu-1}(\tau\ln t)^{\alpha-\mu-1}\ln t\,\mathrm{d}\tau \\ &\quad = \frac{\|v\|}{\Gamma(\mu)}(\ln t)^{\mu-1} \int_0 \tau^{\alpha-\mu-1}(1-\tau)^{\mu-1}\,\mathrm{d}\tau \\ &\quad = \frac{\|v\|}{\Gamma(\mu)}(\ln t)^{\mu-1} B(\alpha-\mu, \mu). \end{aligned} \tag{23}$$

So for any $1 + 1/n_0 \leqslant t \leqslant \mathrm{e} - 1/n_0$, by (20)–(23) we get

$$\begin{aligned} &r\left(\ln\left(1 + \frac{1}{n_0}\right)\right)^{\alpha-\mu-1} \leqslant v(t) \leqslant R, \\ &\left(\ln\left(1+\frac{1}{n_0}\right)\right)^{\mu-1} \frac{B(\alpha-\mu,\mu)}{\Gamma(\mu)} r \leqslant^H I_{1^+}^{\mu} v(t) \leqslant \frac{R}{\Gamma(\mu+1)}. \end{aligned} \tag{24}$$

From (19) and (24), by Lemma 7, we have

$$\begin{aligned}
&\sup_{v\in\overline{K}_R\setminus K_r}\int_1^{\mathrm{e}}\mathfrak{I}(\mathrm{e},s)\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v(\tau),v(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\
&\quad\leqslant\sup_{v\in\overline{K}_R\setminus K_r}\int_{e(n_0)}\mathfrak{I}(\mathrm{e},s)\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v(\tau),v(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\
&\qquad+\sup_{v\in\overline{K}_R\setminus K_r}\int_{1+1/n_0}^{\mathrm{e}-1/n_0}\mathfrak{I}(\mathrm{e},s)\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v(\tau),v(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\
&\quad\leqslant 1+(D_1)^{q-1}\big(b^*\big)^{q-1}\int_1^{\mathrm{e}}\mathfrak{I}(\mathrm{e},s)\frac{\mathrm{d}s}{s}<+\infty,
\end{aligned}$$

where

$$D_1=\max\left\{\hbar(t,x_1,x_2)\colon (t,x_1,x_2)\in\left[1+\frac{1}{n_0},\ \mathrm{e}-\frac{1}{n_0}\right]\times[a,R]\times\left[b,\frac{R}{\Gamma(\mu+1)}\right]\right\},$$

$a=r(\ln(1+1/n_0))^{\gamma-\mu-1}$, $b=(\ln(1+1/n_0))^{\mu-1}(B(\alpha-\mu,\,\mu)/\Gamma(\mu))r$. Hence, (18) is true, and this implies that A is uniformly bounded on any bounded set.

Next, we prove that $A:\overline{K}_R\setminus K_r\to K$ is continuous. Let $v_k,v_0\in\overline{K}_R\setminus K_r$ and $\|v_k-v_0\|\to 0$ $(k\to\infty)$. For any $\epsilon>0$, by (H1) there exists a natural number $m_0>1$ such that

$$\sup_{v\in\overline{K}_R\setminus K_r}\int_{e(m_0)}\mathfrak{I}(\mathrm{e},s)\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v(\tau),v(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}<\frac{\epsilon}{4}. \tag{25}$$

Since $f(t,x_1,x_2)$ is uniformly continuous on $[1{+}1/n_0,\ \mathrm{e}{-}1/n_0]{\times}[a,R]{\times}[b,R/\Gamma(\mu{+}1)]$, so

$$\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v(\tau),v(\tau)\big)\frac{\mathrm{d}\tau}{\tau}$$

is uniformly continuous on $[1+1/n_0,\ \mathrm{e}-1/n_0t]\times[a,R]\times[b,R/\Gamma(\mu+1)]$. Hence, we get that

$$\begin{aligned}
\lim_{k\to+\infty}\Bigg|&\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v_k(\tau),v_k(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\\
&-\varphi_q\left(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\big(\tau,{}^HI_{1+}^{\mu}v_0(\tau),v_0(\tau)\big)\frac{\mathrm{d}\tau}{\tau}\right)\Bigg|=0
\end{aligned}$$

holds uniformly for $t \in [1 + 1/m_0, \mathrm{e} - 1/m_0]$. It follows from the Lebesgue control convergence theorem and (H1) that

$$\int_{1+1/m_0}^{\mathrm{e}-1/m_0} \Im(\mathrm{e}, s)\left|\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_k(\tau), v_k(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right) - \varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_0(\tau), v_0(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\right| \frac{\mathrm{d}s}{s} \to 0 \quad \text{as } k \to \infty.$$

Hence, for the above $\epsilon > 0$, there exists a natural number N such that for $k > N$, we get

$$\int_{1+1/m_0}^{\mathrm{e}-1/m_0} \Im(\mathrm{e}, s)\left|\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_k(\tau), v_k(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right) - \varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_0(\tau), v_0(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\right| \frac{\mathrm{d}s}{s} < \frac{\epsilon}{2}. \tag{26}$$

According to (25) and (26), when $k > N$, we have

$$\begin{aligned}
&\|Av_k - Av_0\| \\
&\leqslant \sup_{v_k \in \overline{K}_R \setminus K_r} \int_{e(m_0)} \Im(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_k(\tau), v_k(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right) \frac{\mathrm{d}s}{s} \\
&\quad + \sup_{v_0 \in \overline{K}_R \setminus K_r} \int_{e(m_0)} \Im(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_0(\tau), v_0(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right) \frac{\mathrm{d}s}{s} \\
&\quad + \int_{1+1/m_0}^{\mathrm{e}-1/m_0} \Im(\mathrm{e}, s)\left|\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_k(\tau), v_k(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\right. \\
&\quad \left. - \varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v_0(\tau), v_0(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right)\right| \frac{\mathrm{d}s}{s} \\
&< 2 \times \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, $A : \overline{K}_R \setminus K_r \to K$ is continuous.

For any bounded set $B \subset \overline{K}_R \setminus K_r$, we will prove that $A(B)$ is equicontinuous. In fact, by (H1), for any $\epsilon > 0$, there exists a natural number $k_0 > 1$ such that

$$\sup_{v \in \overline{K}_R \setminus K_r} \int_{e(k_0)} \Im(\mathrm{e}, s)\varphi_q\left(\int_1^{\mathrm{e}} \Re(s,\tau)\hbar\left(\tau, {}^H I_{1^+}^{\mu} v(\tau), v(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\right) \frac{\mathrm{d}s}{s} < \frac{\epsilon}{4}.$$

Let

$$D_2 = \max\left\{\hbar(t,x_1,x_2)\colon (t,x_1,x_2)\in\left[1+\frac{1}{k_0},\, \mathrm{e}-\frac{1}{k_0}\right]\times[a,R]\times\left[b,\frac{R}{\Gamma(\mu+1)}\right]\right\}.$$

Since $G(t,s)$ is uniformly continuous on $[1,\mathrm{e}]\times[1,\mathrm{e}]$, for the above $\epsilon>0$, there exists $\delta>0$ such that for all $s\in[1+1/k_0,\, \mathrm{e}-1/k_0]$,

$$\left|\Im(t,s)-\Im(t',s)\right| \leqslant \frac{\epsilon}{2}\left(D_2\int\limits_{1+1/k_0}^{\mathrm{e}-1/k_0}\Im(\mathrm{e},s)\,\frac{\mathrm{d}s}{s}\right)^{-1}$$

for $|t-t'|<\delta, t,t'\in[1,\mathrm{e}]$. Hence, for $|t-t'|<\delta$, $t,t'\in[1,\mathrm{e}]$ and $v\in B$, we get

$$\begin{aligned}
&\left\|Av(t)-Av(t')\right\| \\
&\quad\leqslant 2\sup_{v\in\overline{K}_R\setminus K_r}\int\limits_{e(k_0)}\Im(\mathrm{e},s)\varphi_q\left(\int\limits_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\bigl(\tau,{}^HI^{\mu}_{1+}v(\tau),v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\
&\qquad+\sup_{v\in\overline{K}_R\setminus K_r}\int\limits_{1+1/k_0}^{\mathrm{e}-1/k_0}\left|\Im(t,s)-\Im(t',s)\right|\varphi_q\left(\int\limits_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar\bigl(\tau,{}^HI^{\mu}_{1+}v(\tau),v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s} \\
&\quad< 2\times\frac{\epsilon}{4}+\frac{\epsilon}{2}=\epsilon,
\end{aligned}$$

which implies that $A(B)$ is equicontinuous. From the Arzelà–Ascoli theorem $A:\overline{K}_R\setminus K_r\to K$ is completely continuous, then the proof is completed. □

Lemma 12. (*See* [7,8].) *Let K is a cone in Banach space E. Suppose that $A:\overline{K}_r\to K$ is a completely continuous operator. If there exists $u_0\in K\setminus\{\theta\}$ such that $u-Au\neq\mu u_0$ for any $u\in\partial K_r$ and $\mu\geqslant 0$, then $i(A,K_r,K)=0$.*

Lemma 13. (*See* [7,8].) *Let K is a cone in Banach space E. Suppose that $A:\overline{K}_r\to K$ is a completely continuous operator. If $Au\neq\mu u$ for any $u\in\partial K_r$ and $\mu\geqslant 1$, then $i(A,K_r,K)=1$.*

3 Main results

Theorem 1. *Suppose conditions* (H1)–(H3) *are satisfied, and*

$$\liminf_{\substack{x_i\to 0^+\\ i=1,2}}\frac{\varphi_q\bigl(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\frac{\mathrm{d}\tau}{\tau}\bigr)}{x_1+x_2}>\lambda_1, \tag{27}$$

$$\limsup_{\substack{x_1+x_2\to+\infty\\ x_2\to+\infty}}\frac{\varphi_q\bigl(\int_1^{\mathrm{e}}\mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\frac{\mathrm{d}\tau}{\tau}\bigr)}{x_2}<\lambda_1 \tag{28}$$

uniformly hold for $s\in[1,\mathrm{e}]$, $x_i\in[0,+\infty)$ $(i=1,2)$, where λ_1 is the first eigenvalue of T defined by (15). *Then the BVP* (1)–(3) *has at least one positive solution.*

Proof. It follows from (27) that there exists $r > 0$ such that

$$\begin{gathered}\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\,\frac{\mathrm{d}\tau}{\tau}\Biggr) \geqslant \lambda_1(x_1+x_2),\\ |x_1| \leqslant \frac{r}{\Gamma(\mu+1)},\quad |x_2| \leqslant r,\quad t\in[1,\mathrm{e}],\end{gathered} \tag{29}$$

and thus, for every $v \in \partial K_r$, we have

$$\bigl|{}^H I_{1+}^{\mu} v(s)\bigr| \leqslant \frac{r}{\Gamma(\mu+1)},\quad \bigl|v(s)\bigr| \leqslant r. \tag{30}$$

From (29) and (30) we have that

$$\begin{aligned}(Av)(t) &= \int_1^{\mathrm{e}} \mathfrak{I}(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau,{}^H I_{1+}^{\mu} v(\tau),v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s}\\ &\geqslant \lambda_1\int_1^{\mathrm{e}} \mathfrak{I}(t,s)\bigl({}^H I_{1+}^{\mu} v(s)+v(s)\bigr)\,\frac{\mathrm{d}s}{s}\lambda_1(Tv)(t),\quad t\in[1,\mathrm{e}].\end{aligned} \tag{31}$$

By Lemma 8 T has a positive eigenfunction φ^* corresponding to λ_1, that is, $\varphi^* = \lambda_1 T\varphi^*$. In the following, we shall prove that

$$v - Av \neq \mu\varphi^*,\quad v\in\partial K_{r_0},\ \mu\geqslant 0. \tag{32}$$

If not, then there exist $v_0 \in \partial K_{r_0}$ and $\mu_0 \geqslant 0$ such that $v_0 - Av_0 = \mu_0\varphi^*$, and then $\mu_0 > 0$ and $v_0 = Av_0 + \mu_0\varphi^* \geqslant \mu_0\varphi^*$. Let $\overline{\mu} = \sup\{\mu\colon v_0 \geqslant \mu\varphi^*\}$, then $\overline{\mu} \geqslant \mu_0$, $v_0 \geqslant \overline{\mu}\varphi^*$, $\lambda_1 Tv_0 \geqslant \lambda_1\overline{\mu}T\varphi^* = \overline{\mu}\varphi^*$. Therefore, by (31) we have

$$v_0 = Av_0 + \mu_0\varphi^* \geqslant \lambda_1 Tv_0 + \mu_0\varphi^* \geqslant \overline{\mu}\varphi^* + \mu_0\varphi^* = (\overline{\mu}+\mu_0)\varphi^*,$$

which contradicts the definition of $\overline{\mu}$. So (32) holds, and from Lemma 7 it follows that

$$i(A, K_{r_0}, K) = 0.$$

Now we choose a constant $0 < \sigma < 1$ such that

$$\limsup_{x_2\to+\infty}\max_{s\in[1,\mathrm{e}]}\frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\,\frac{\mathrm{d}\tau}{\tau})}{x_2} < \sigma\lambda_1,$$

and we define a linear operator $\widetilde{T}y = \sigma\lambda_1 Ty$. Then $\widetilde{T}: E \to E$ is a bounded linear operator, and $\widetilde{T}(K) \subset K$. Moreover, $\widetilde{T}\varphi^* = \sigma\lambda_1 T\varphi^* = \sigma\varphi^*$, and so the spectral radius of $\widetilde{T}$ is $r(\widetilde{T}) = \sigma$, and $\widetilde{T}$ also has the first eigenvalue $r^{-1}(\widetilde{T}) = \sigma^{-1} > 1$. By Gelfand's formula we know

$$\sigma = \lim_{n\to+\infty}\bigl\|\widetilde{T}^n\bigr\|^{1/n}. \tag{33}$$

Let $\varepsilon_0 = (1-\sigma)/2$, and by (33) there exists a sufficiently large natural number N such that $n \geqslant N$ implies that $\|\widetilde{T}^n\| \leqslant [\sigma+\epsilon_0]^n$. For any $v \in E$, define

$$\|v\|^* = \sum_{i=1}^{N} [\sigma+\epsilon_0]^{N-i} \big\|\widetilde{T}^{i-1} v\big\|, \tag{34}$$

where $\widetilde{T}^0 = I$ is the identity operator. Clearly, $\|\cdot\|^*$ is also the norm of E.

On the other hand, it follows from (28) that there exists $R_1 > r$ such that

$$\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\,\frac{\mathrm{d}\tau}{\tau}\Biggr) \leqslant \sigma\lambda_1 x_2 \quad \text{for } x_2 \geqslant R_1,\ x_1 \geqslant 0,\ t \in [1,\mathrm{e}]. \tag{35}$$

Taking

$$R > \max\biggl\{R_1,\ \frac{2(\sigma+\epsilon_0^{N-1})^{-1}}{\epsilon_0}\,C^*\biggr\}, \quad C^* = \|C\|^*,$$

and

$$C = \sup_{v\in\partial K_{R_1}} \int_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e},s)\varphi_q\biggl(\mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\biggr)\frac{\mathrm{d}s}{s} < +\infty \quad \bigl(\text{by (18)}\bigr).$$

Next, we show that

$$Av \neq \mu v, \quad v \in \partial K_R,\ \mu \geqslant 1. \tag{36}$$

Otherwise, if there exist $v_1 \in \partial K_R$ and $\mu_1 \geqslant 1$ such that $Av_1 = \mu_1 v_1$, let $\widetilde{v}(t) = \min\{v_1(t), R_1\}$ and $D(v_1) = \{t \in [1,\mathrm{e}] : v_1(t) > R_1\}$, and let

$$\xi(t) = (\ln t)^{\gamma-\mu-1}.$$

From $\widetilde{v} \in C([1,\mathrm{e}],[0,+\infty))$ it follows $\xi(t)R \leqslant v_1(t) \leqslant \|v_1\| = R$, and by $Av_1 = \mu_1 v_1$ we get that v_1 satisfies boundary conditions, so we have $v_1(0) = 0$. Then there exists $1 < t_0 \leqslant \mathrm{e}$ such that $v_1(t_0) = R$. Thus, $\widetilde{v}(t) = \min\{v_1(t), R_1\} \leqslant \min\{R, R_1\} = R_1$ for $t \in [1,\mathrm{e}]$, and $\widetilde{v}(t_0) = \min\{v_1(t_0), R_1\} = \min\{R, R_1\} = R_1$. Then we have $\|\widetilde{v}\| = R_1$. Since $\widetilde{v}(t) = \min\{v_1(t), R_1\} \geqslant \min\{\xi(t)R, R_1\} \geqslant R_1\xi(t)$, $t \in [1,\mathrm{e}]$, so $\widetilde{y} \in \partial K_{R_1}$. From (35) and Lemma 5, for $t \in D(v_1)$, $v_1(t) \geqslant R_1$, ${}^H I_{1+}^{\mu} x_1(t) \geqslant 0$, we get

$$\begin{aligned}
&(Av_1)(t) \\
&\quad= \int_1^{\mathrm{e}} \mathfrak{I}(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\quad\leqslant \int_{D(v_1)} \mathfrak{I}(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\qquad+ \int_{[1,\mathrm{e}]\setminus D(v_1)} \mathfrak{I}(\mathrm{e},s)\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$\leqslant \sigma\lambda_1 \int_1^{\mathrm{e}} \mathfrak{I}(t,s))v_1(s)\,\mathrm{d}s + \int_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e},s)\varphi_q\Bigg(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\big(\tau, {}^H I_{1^+}^{\mu}\widetilde{v}(\tau), \widetilde{v}(\tau)\big)\,\frac{\mathrm{d}\tau}{\tau}\Bigg)$$
$$\leqslant (\widetilde{T}v_1)(t) + C, \quad t \in [1,\mathrm{e}]. \tag{37}$$

Noticing that $\widetilde{T}: K \to K$ is a bounded linear operator, by (37) we have

$$0 \leqslant \big(\widetilde{T}^j(Av_1)\big)(t) \leqslant \big(\widetilde{T}^j(\widetilde{T}v_1 + C)\big)(t), \quad j = 0,1,2,\dots,N-1,\ t \in [1,\mathrm{e}]. \tag{38}$$

Then (38) yields

$$\big\|\big(\widetilde{T}^j(Av_1)\big)\big\| \leqslant \big\|\big(\widetilde{T}^j(\widetilde{T}v_1 + C)\big)\big\|, \quad j = 0,1,2,\dots,N-1,$$

which leads to

$$\|Av_1\|^* = \sum_{i=1}^{N}[\sigma+\epsilon_0]^{N-i}\big\|\widetilde{T}^{i-1}(Av_1)\big\|$$
$$\leqslant \sum_{i=1}^{N}[\sigma+\epsilon_0]^{N-i}\big\|\widetilde{T}^{i-1}(\widetilde{T}v_1 + C)\big\| = \|\widetilde{T}v_1 + C\|^*. \tag{39}$$

Since $v_1 \in \partial K_R$ and $\|v_1\| = R$, from (34) we get

$$\|v_1\|^* > [\sigma+\epsilon_0]^{N-1}\|v_1\| = [\sigma+\epsilon_0]^{N-1}R > \frac{2}{\epsilon_0}C^*,$$

which implies that

$$C^* < \frac{\epsilon_0}{2}\|v_1\|^*. \tag{40}$$

By using (34), (39) and (40) we have

$$\mu_1\|v_1\|^* = \|Av_1\|^* \leqslant \|\widetilde{T}v_1\|^* + C^* = \sum_{i=1}^{N}[\sigma+\epsilon_0]^{N-i}\big\|\widetilde{T}^i v_1\big\| + C^*$$
$$= [\sigma+\epsilon_0]\sum_{i=1}^{N-1}[\sigma+\epsilon_0]^{N-i-1}\big\|\widetilde{T}^i v_1\big\| + \big\|\widetilde{T}^N v_1\big\| + C^*$$
$$\leqslant [\sigma+\epsilon_0]\sum_{i=1}^{N-1}[\sigma+\epsilon_0]^{N-i-1}\big\|\widetilde{T}^i v_1\big\| + [\sigma+\epsilon_0]^N\|v_1\| + C^*$$
$$= [\sigma+\epsilon_0]\sum_{i=1}^{N}[\sigma+\epsilon_0]^{N-i}\big\|\widetilde{T}^{i-1} v_1\big\| + C^*$$
$$= [\sigma+\epsilon_0]\|v_1\|^* + C^* \leqslant [\sigma+\epsilon_0]\|v_1\|^* + \frac{\epsilon_0}{2}\|v_1\|^*$$
$$= \left[\frac{1}{4}\sigma + \frac{3}{4}\right]\|v_1\|^*.$$

Notice that $\mu_1 \geqslant 1$, we obtain $\sigma/4 + 3/4 \geqslant 1$, and then $\sigma \geqslant 1$, which is a contradiction with $0 < \sigma < 1$. Thus, (36) is indeed true, and by Lemma 13 we get

$$i(A, K_R, K) = 1. \tag{41}$$

It follows from (36) and (41) that

$$i(A, K_R \setminus \overline{K}_{r_0}, R) = i(A, K_R, K) - i(A, K_{r_0}, K) = 1.$$

Hence, A has at least one fixed point in $K_R \setminus \overline{K}_{r_0}$. Consequently, the BVP (4), (5) has at least one positive solution, which implies that BVP (1)–(3) also has at least one positive solution. □

Now we consider another case of problem (1)–(3). For this, we define a linearoperator T_ϵ for any sufficiently small $0 < \epsilon < 1$ as follows:

$$(T_\epsilon v)(t) = \int_{1+\epsilon}^{\mathrm{e}-\epsilon} \Im(t, s) v(s) \frac{\mathrm{d}s}{s}, \quad t \in [1, \mathrm{e}].$$

From Lemma 5 we know that $T_\epsilon : K \to K$ is also a completely continuous linear operator, the spentral radius $r(T_\epsilon) \neq 0$, and moreover, T_ϵ has a positive eigenfunction φ_ϵ corresponding to its first eigenvalue $\lambda_\epsilon = (r(T_\epsilon))^{-1}$.

Lemma 14. *Suppose that* (H2)–(H3) *hold, then there exists an eigenvalue $\widetilde{\lambda}_1$ of T such that*

$$\lim_{\epsilon \to 0^+} \lambda_\epsilon = \widetilde{\lambda}_1.$$

Proof. Take $\epsilon_1 \geqslant \epsilon_2 \geqslant \cdots \geqslant \epsilon_n \geqslant \cdots$ and $\epsilon_n \to 0$ $(n \to +\infty)$. So for any $m > n$ and $\varphi \in E$, we have

$$(T_{\epsilon_n} \varphi)(t) \leqslant (T_{\epsilon_m} \varphi)(t) \leqslant (T\varphi)(t), \quad t \in [1, \mathrm{e}],$$

and

$$(T_{\epsilon_n}^k \varphi)(t) \leqslant \bigl(T_{\epsilon_m}^k \varphi\bigr)(t) \leqslant \bigl(T^k \varphi\bigr)(t), \quad t \in [1, \mathrm{e}],\ k = 2, 3, \dots,$$

where $T_{\epsilon_n}^k = T(T_{\epsilon_n}^{k-1})$ $(k = 2, 3, \dots)$. Consequently, $\|T_{\epsilon_n}^k\| \leqslant \|T_{\epsilon_m}^k\| \leqslant \|T^k\|$ $(k = 1, 2, \dots)$. From Gelfand's formula we get $\lambda_{\epsilon_n} \geqslant \lambda_{\epsilon_m} \geqslant \lambda_1$, where λ_1 is the first eigenvalue of T. Since $\{\lambda_{\epsilon_n}\}$ is monotonous with lower boundedness λ_1, let

$$\lim_{n \to +\infty} \lambda_{\epsilon_n} = \widetilde{\lambda}_1.$$

Now we will show that $\widetilde{\lambda}_1$ is an eigenvalue of T. Suppose φ_{ϵ_n} is a positive eigenfunction of T_{ϵ_n} corresponding to λ_{ϵ_n} with $\|\varphi_{\epsilon_n}\| = 1$ $(n = 1, 2, \dots)$, i.e.,

$$\varphi_{\epsilon_n}(t) = \lambda_{\epsilon_n} \int_{1+\epsilon_n}^{\mathrm{e}-\epsilon_n} \Im(t, s) \varphi_{\epsilon_n}(s) \frac{\mathrm{d}s}{s} = \lambda_{\epsilon_n} T_{\epsilon_n} \varphi_{\epsilon_n}(t), \quad t \in [1, \mathrm{e}].$$

Notice that

$$\|T_{\epsilon_n}\varphi_{\epsilon_n}\| = \max_{1\leqslant t\leqslant \mathrm{e}} \int\limits_{1+\epsilon_n}^{\mathrm{e}-\epsilon_n} \mathfrak{I}(t,s)\varphi_{\epsilon_n}(s)\,\frac{\mathrm{d}s}{s} \leqslant \int\limits_1^{\mathrm{e}} \mathfrak{I}(\mathrm{e},s)\,\frac{\mathrm{d}s}{s}, \quad n = 1,2,\dots,$$

and thus, $\{T_{\epsilon_n}\varphi_{\epsilon_n}\} \subset E$ is uniformly bounded. On the other hand, for any $n \in \mathbb{N}$ and $t_1, t_2 \in [1,\mathrm{e}]$, we have

$$\bigl|T_{\epsilon_n}\varphi_{\epsilon_n}(t_1) - T_{\epsilon_n}\varphi_{\epsilon_n}(t_2)\bigr| \leqslant \int\limits_{1+\epsilon_n}^{\mathrm{e}-\epsilon_n} \bigl|\mathfrak{I}(t_1,s) - \mathfrak{I}(t_2,s)\bigr|\varphi_{\epsilon_n}(s)\,\frac{\mathrm{d}s}{s}.$$

So as $G(t,s)$ is uniformly continuous on $[1,\mathrm{e}] \times [1,\mathrm{e}]$, it follows that $\{T_{\epsilon_n}\varphi_{\epsilon_n}\} \subset E$ is equicontinuous. By the Arzelà–Ascoli theorem and $\lim_{n\to+\infty}\lambda_{\epsilon_n} = \widetilde{\lambda}_1$ we get that $\varphi_{\epsilon_n} \to \varphi_0$ as $n \to +\infty$. This leads to $\|\varphi_0\| = 1$, and then by (41) we have

$$\varphi_0(t) = \widetilde{\lambda}_1 \int\limits_1^{\mathrm{e}} \mathfrak{I}(t,s)\varphi_0(s)\,\frac{\mathrm{d}s}{s}, \quad t \in [1,\mathrm{e}],$$

that is, $\varphi_0 = \widetilde{\lambda}_1 T\varphi_0$. □

Theorem 2. *Suppose* (H1)–(H3) *hold, and*

$$\limsup_{\substack{x_i\to0^+\\ i=1,2}} \frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\,\frac{\mathrm{d}\tau}{\tau})}{x_2} < \lambda_1, \tag{42}$$

$$\liminf_{x_1+x_2\to+\infty} \frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\,\frac{\mathrm{d}\tau}{\tau})}{x_1+x_2} > \widetilde{\lambda}_1, \tag{43}$$

uniformly on $t \in [1,\mathrm{e}]$, where λ_1, $\widetilde{\lambda}_1$ are the eigenvalues of T, and λ_1 is the first eigenvalue of T. Then the BVP (4), (5) *has at least one positive solution, that is, the BVP* (1)–(3) *has at least one positive solution.*

Proof. Firstly, by (42), for any $s \in [1,\mathrm{e}]$, there exists $r > 0$ such that

$$\varphi_q\Biggl(\int\limits_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau,x_1,x_2)\,\frac{\mathrm{d}\tau}{\tau}\Biggr) \leqslant \lambda_1 x_2, |x_1| \leqslant \frac{r}{\Gamma(\mu+1)}, \quad |x_2| \leqslant r. \tag{44}$$

Thus, for every $v \in \partial K_r$, by $|{}^H I_{1^+}^{\mu} v(s)| \leqslant r/\Gamma(\mu+1)$, $|v(s)| \leqslant r$, we have

$$\begin{aligned}(Av)(t) &= \int\limits_1^{\mathrm{e}} \mathfrak{I}(t,s)\varphi_q\Biggl(\int\limits_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1^+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\,\frac{\mathrm{d}s}{s} \\ &\leqslant \lambda_1 \int\limits_1^{\mathrm{e}} \mathfrak{I}(t,s)v(s)\,\frac{\mathrm{d}s}{s} = \lambda_1(Tv)(t), \quad t \in [1,\mathrm{e}]. \end{aligned} \tag{45}$$

In fact, we suppose that A has no fixed point on ∂K_r. Now we prove that

$$Av \neq \mu v \quad \text{for any } v \in \partial K_r,\ \mu \geqslant 1.$$

Otherwise, there exist $v_0 \in \partial K_r$ and $\mu_0 \geqslant 1$ satisfying $Av_0 = \mu_0 v_0$. We know $\mu_0 > 1$ and from (45) we have

$$\mu_0 v_0 = Av_0 \leqslant \lambda_1 T v_0. \tag{46}$$

By induction for (46), we get

$$\mu_0^n v_0 \leqslant \lambda_1^n T^n v_0, \quad n = 1, 2, \dots,$$

which implies that

$$\|T^n\| \geqslant \frac{\|T^n v_0\|}{\|v_0\|} \geqslant \frac{\mu_0^n \|v_0\|}{\lambda_1^n \|v_0\|} = \frac{\mu_0^n}{\lambda_1^n}.$$

On the other hand, by the Gelfand formula we have

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} \geqslant \frac{\mu_0}{\lambda_1} > \frac{1}{\lambda_1},$$

which is a contradiction with $r(T) = \lambda_1^{-1}$. So (45) holds. By Lemma 13 we have

$$i(A, K_r, K) = 1. \tag{47}$$

It follows from (43) and $\lim_{\epsilon\to 0^+} \lambda_\epsilon = \widetilde{\lambda}_1$ that there exist a sufficiently small $\epsilon \in (1,\ 1 + 1/2)$ and $R > r$ such that

$$\varphi_q\left(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar(\tau, x_1, x_2)\,\frac{\mathrm{d}\tau}{\tau}\right) \geqslant \lambda_\epsilon(x_1 + x_2) \geqslant \rho_\epsilon R, \quad s \in [1, \mathrm{e}], \tag{48}$$

where λ_ϵ is the first eigenvalue of T_ϵ, $\rho_\epsilon = (\ln(1+\epsilon)^{\mu-1}/\Gamma(\mu))\varrho$. Let φ_ϵ be the positive eigenfunction of T_ϵ corresponding to λ_ϵ, then $\varphi_\epsilon = \lambda_\epsilon T_\epsilon \varphi_\epsilon$.

For any $v \in \partial K_R$, $s \in [1+\epsilon,\ \mathrm{e}-\epsilon]$, by (12)–(14) we have

$$\begin{aligned}
{}^H I_{1^+}^{\mu} v(s) + v(s) &= \frac{1}{\Gamma(\mu)} \int_1^t \left(\ln\frac{t}{s}\right)^{\mu-1} \frac{v(s)}{s}\,\mathrm{d}s + v(s) \\
&\geqslant \|v\|(\ln t)^{\mu-1} B(\gamma-\mu,\ \mu) + (\ln t)^{\gamma-\mu-1}\|v\| \\
&\geqslant \frac{\ln(1+\epsilon)^{\mu-1}}{\Gamma(\mu)}\|v\|\varrho = \rho_\epsilon R.
\end{aligned} \tag{49}$$

By (48) and (49) we have

$$
\begin{aligned}
(Av)(t) &= \int_1^{\mathrm{e}} \mathfrak{I}(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1^+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\geqslant \int_{1+\epsilon}^{\mathrm{e}-\epsilon} \mathfrak{I}(t,s)\varphi_q\Biggl(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)\hbar\bigl(\tau, {}^H I_{1^+}^{\mu} v(\tau), v(\tau)\bigr)\,\frac{\mathrm{d}\tau}{\tau}\Biggr)\frac{\mathrm{d}s}{s} \\
&\geqslant \lambda_\epsilon \int_{1+\epsilon}^{\mathrm{e}-\epsilon} \mathfrak{I}(t,s)\bigl({}^H I_{1^+}^{\mu} v(s) + v(s)\bigr)\,\frac{\mathrm{d}s}{s} \geqslant \lambda_\epsilon \int_{1+\epsilon}^{\mathrm{e}-\epsilon} \mathfrak{I}(t,s)v(s)\,\frac{\mathrm{d}s}{s} \\
&= \lambda_\epsilon (T_\epsilon v)(t), \quad t \in [1,\mathrm{e}].
\end{aligned}
$$

Proceeding as in the proof of Theorem 1, we have

$$v - Av \neq \mu\varphi_\epsilon, \quad v \in \partial K_R,\ \mu \geqslant 0,$$

then by Lemma 7 we get

$$i(A, K_R, K) = 0. \tag{50}$$

By (47) and (50) we have

$$i(A, K_R \setminus \overline{K}_r, K) = i(A, K_R, K) - i(A, K_r, K) = -1.$$

Hence, A has at least one fixed point in $K_R \setminus \overline{K}_r$, which is the positive solution of the BVP (4), (5). Certainly, it is also the positive solution of the BVP (1)–(3). The proof is completed. □

4 An example

Consider the following infinite-point p-Laplacian fractional differential equations:

$$
\begin{aligned}
&{}^H D_{1^+}^{3/2}\bigl(\varphi_3\bigl({}^H D_{1^+}^{11/2} u\bigr)\bigr)(t) + f\bigl(t, u(t), {}^H D_{1^+}^{1/2} u(t)\bigr) = 0, \quad 1 < t < \mathrm{e}, \\
&u(1) = u'(1) = 0, \qquad {}^H D_{1^+}^{7/2} u(\mathrm{e}) = \sum_{j=1}^{\infty} \eta_j D_{0^+}^{5/2} u(\xi_j), \\
&{}^H D_{1^+}^{11/2} u(1) = 0, \qquad \varphi_p\bigl({}^H D_{1^+}^{11/2} u(\mathrm{e})\bigr) = \sum_{j=1}^{\infty} \zeta_i \varphi_p\bigl({}^H D_{1^+}^{11/2} u(\eta_i)\bigr),
\end{aligned}
\tag{51}
$$

where $\gamma = 11/2$, $\alpha = 3/2$, $\mu = 1/2$, $r_1 = 7/2$, $r_2 = 5/2$, $p = 3$, $q = 3/2$, $\eta_j = j^4/2$, $\xi_j = \mathrm{e}^{1/j^4}$, $\zeta_j = 1/(4j^2)$, $f(t,x,y) = (x+y)^{-1/2} + |\ln(\ln y)|$, f may be singular at $t = \mathrm{e}$ and $x = y = 0$. $u(t) ={}^H I_{1^+}^{1/2} v(t)$, $v(t) \in C[1,\mathrm{e}]$, then the BVP (51) reduced to the following modified boundary value problem:

$$\begin{aligned}
&{}^{H}D_{1^+}^{3/2}\big(\varphi_3\big({}^{H}D_{1^+}^{5}v\big)\big)(t) + f\big(t, {}^{H}I_{1^+}^{1/2}(t), v(t)\big) = 0, \quad 1 < t < \mathrm{e},\\
&v(1) = v'(1) = 0, \qquad {}^{H}D_{1^+}^{3}v(\mathrm{e}) = \sum_{j=1}^{\infty}\eta_j^{H}D_{1^+}^{2}v(\xi_j),\\
&{}^{H}D_{1^+}^{5}u(1) = 0, \qquad \varphi_p\big({}^{H}D_{1^+}^{5}v(\mathrm{e})\big) = \sum_{j=1}^{\infty}\zeta_i\varphi_p\big({}^{H}D_{1^+}^{5}v(\eta_i)\big).
\end{aligned} \tag{52}$$

By simple calculation we have

$$\begin{aligned}
\varDelta &= \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma-\mu)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j \ln \xi_j^{\gamma-r_2-1}\\
&= \frac{\Gamma(5)}{\Gamma(\frac{11}{2}-\frac{7}{2})} - \frac{\Gamma(5)}{\Gamma(\frac{11}{2}-\frac{5}{2})}\sum_{j=1}^{\infty}\frac{j^4}{2}\left(\frac{1}{j^4}\right)^2 \approx 17.51,\\
\overline{\varDelta} &= 1 - \sum_{i=1}^{\infty}\zeta_i \ln \xi_i^{\alpha-1} = 1 - \sum_{i=1}^{\infty}\zeta_i(\ln\xi_i)^{1/2} = 1 - \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{j^4}\\
&\approx 1 - 0.5411 = 0.4589,\\
b^* &= \frac{1}{\overline{\varDelta}\Gamma(\alpha)}\left(1 + \sum_{i=1}^{\infty}\zeta_i\big(\ln\xi_i\big)^{\alpha-1}\right) = \frac{1}{0.4589\frac{\sqrt{\pi}}{2}}\left(1 + \frac{1}{4}\sum_{i=1}^{\infty}\frac{1}{j^4}\right)\\
&\approx 3.1242.
\end{aligned}$$

Clearly, we have

$$\mathfrak{I}(t,s) = \frac{1}{\varDelta\Gamma(5)}\begin{cases}\Gamma(5)(\ln t)^4 P(s)(\ln\frac{\mathrm{e}}{s}) - \varDelta(\ln\frac{t}{s})^4, & 1\leqslant s\leqslant t\leqslant \mathrm{e},\\ \Gamma(5)(\ln t)^4 P(s)(\ln\frac{\mathrm{e}}{s}), & 1\leqslant t\leqslant s\leqslant \mathrm{e},\end{cases}$$

$$\mathfrak{R}_1(t,s) = \frac{1}{\Gamma(\frac{3}{2})}\begin{cases}(\ln t)^{1/2}P(s)(\ln\frac{\mathrm{e}}{s})^{1/2} - (\ln\frac{t}{s})^{1/2}, & 1\leqslant s\leqslant t\leqslant \mathrm{e},\\ (\ln t)^{1/2}P(s)(\ln\frac{\mathrm{e}}{s})^{1/2}, & 1\leqslant t\leqslant s\leqslant \mathrm{e},\end{cases}$$

$$\begin{aligned}
\mathfrak{R}_2(t,s) = \frac{(\ln t)^{1/2}}{\overline{\varDelta}\Gamma(\frac{3}{2})}\Bigg[&\sum_{\xi_j>s}\zeta_j\left[\xi_j^{\alpha-1}\left(\ln\frac{\mathrm{e}}{s}\right)^{1/2} - \left(\ln\frac{\xi_j}{s}\right)^{1/2}\right]\\
&+ \sum_{s\geqslant\xi_j}\zeta_j \ln\xi_j^{1/2}\left(\ln\frac{\mathrm{e}}{s}\right)^{1/2}\Bigg], \quad t,s\in[1,\mathrm{e}].
\end{aligned}$$

For any $0 < r < R < +\infty$ and $v \in \overline{K}_R \setminus K_r$, we have

$$(\ln t)^4 r \leqslant v(t) \leqslant R,$$

$$\frac{B(5,\frac{1}{2})}{\sqrt{\pi}}r(\ln t)^{-1/2} \leqslant u(t) = {}^{H}I_{1^+}^{1/2}v(t) \leqslant \frac{2R}{\sqrt{\pi}}, \quad t\in[1,\mathrm{e}].$$

Since $|\ln(\ln y)|$ is decreasing on $(1, \mathrm{e})$ and increasing on $(\mathrm{e}, +\infty)$, we get

$$\left|\ln\bigl(\ln y(t)\bigr)\right| \leqslant 2\left|\ln(\ln R)\right| + \left|(\ln t)^4\right|, \quad t \in [1, \mathrm{e}],$$

$$\bigl[x(t) + y(t)\bigr]^{-1/3} \leqslant \left[(\ln t)^4 r + \frac{B(5, \frac{1}{2})}{\Gamma(\mu)} r (\ln t)^{-1/2}\right]^{-1/3}, \quad t \in [1, \mathrm{e}].$$

The absolute continuity of the integral yields that

$$\lim_{m\to\infty} \int\limits_{e(m)} \mathfrak{I}(\mathrm{e}, s)\left((\ln t)^4 r + \left[(\ln t)^4 r + B\left(5, \frac{1}{2}\right) r (\ln t)^{-1/2}\right]^{-1/3}\right) \mathrm{d}s = 0.$$

Hence,

$$\begin{aligned}
&\limsup_{m\to+\infty} \sup_{\substack{x\in K_{\overline{\Lambda}}\setminus K_r\\ y\in \overline{K}_R\setminus K_r}} \int\limits_{e(m)} \mathfrak{I}(\mathrm{e}, s) f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s \\
&\leqslant \limsup_{m\to+\infty} \sup_{\substack{x\in K_{\overline{\Lambda}}\setminus K_r\\ y\in \overline{K}_R\setminus K_r}} \frac{1}{\Lambda} \int\limits_{e(m)} P(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma - r_1 - 1} \bigl[(x+y)^{-1/3} + |\ln y|\bigr]\,\mathrm{d}s \\
&\leqslant \limsup_{m\to+\infty} \frac{1}{\Lambda} \int\limits_{e(m)} Q(s)\left(\ln\frac{\mathrm{e}}{s}\right)^{\gamma - r_1 - 1} \\
&\qquad \times \left((\ln t)^4 r + \left[(\ln t)^4 r + B\left(5, \frac{1}{2}\right) r (\ln t)^{-1/2}\right]^{-1/3}\right) \mathrm{d}s = 0,
\end{aligned}$$

where $\Lambda = R/\Gamma(3/2)$, so (H1) holds. On the other hand, it is obvious that

$$\begin{aligned}
&\liminf_{\substack{x_1\to 0^+\\ x_2\to 0^+}} \inf_{t\in[1,\mathrm{e}]} \frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau) f(\tau, x_1, x_2)\,\frac{\mathrm{d}\tau}{\tau})}{x_1 + x_2} \\
&\quad = \liminf_{\substack{x_1\to 0^+\\ x_2\to 0^+}} \inf_{t\in[1,\mathrm{e}]} \frac{((\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)(x_1+x_2)^{-1/3} + |\ln x_2|)\frac{\mathrm{d}\tau}{\tau})^2}{x_1 + x_2} = +\infty, \\
&\limsup_{\substack{x_1+x_2\to+\infty\\ x_2\to+\infty}} \sup_{t\in[1,\mathrm{e}]} \frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau) f(\tau, x_1, x_2)\frac{\mathrm{d}\tau}{\tau})}{x_2} \\
&\quad = \limsup_{\substack{x_1+x_2\to+\infty\\ x_2\to+\infty}} \sup_{t\in[1,\mathrm{e}]} \frac{((\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau)(x_1+x_2)^{-1/3} + |\ln x_2|)\frac{\mathrm{d}\tau}{\tau})^2}{x_2} = 0,
\end{aligned}$$

which imply that

$$\liminf_{\substack{x_1\to 0^+\\ x_2\to 0^+}} \inf_{t\in[1,\mathrm{e}]} \frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau) f(\tau,x_1,x_2)\frac{\mathrm{d}\tau}{\tau})}{x_1+x_2} > \lambda_1$$

$$> \limsup_{\substack{x_1+x_2\to +\infty\\ x_2\to +\infty}} \sup_{t\in[1,\mathrm{e}]} \frac{\varphi_q(\int_1^{\mathrm{e}} \mathfrak{R}(s,\tau) f(\tau,x_1,x_2)\frac{\mathrm{d}\tau}{\tau})}{x_2}.$$

Therefore, all assumptions of Theorem 1 are satisfied. Thus, Theorem 1 ensures that problem (52) has at least one positive solution, that is to say, (51) has at least one positive solution.

Acknowledgment. The authors would like to thank the referee for his/her valuable comments and suggestions.

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