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Optimal control results for impulsive fractional delay integrodifferential equations of order $1 < r < 2$ via sectorial operator

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Abstract. This research investigates the existence of nonlocal impulsive fractional integrodifferential equations of order $1 < r < 2$ with infinite delay. To begin with, we discuss the existence of a mild solution for the fractional derivatives by using the sectorial operators, the nonlinear alternative of the Leray–Schauder fixed point theorem, mixed Volterra–Fredholm integrodifferential types, and impulsive systems. Furthermore, we develop the optimal control results for the given system. The application of our findings is demonstrated with the help of an example.

Keywords: fractional derivative, infinite delay, impulsive systems, integrodifferential systems, sectorial operators, nonlocal conditions.

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1 Introduction

Fractional-order calculus has been a 300-year-old mathematical notion as a generalization of integer-order differentiation and integration to arbitrary noninteger order. In the past decades, it found that there existed fractional-order calculus in both theoretical and applied aspects of numerous branches of science and engineering such as viscoelastic systems, dielectric polarization, electromagnetic waves, colored noise, heat conduction, robotics, biological systems, finance, and so on. Fractional differential equations have recently emerged as a very significant area of research due to their steadily increasing number of applications in several branches of applied science and engineering. For more information, see the books [9, 17, 26, 27]. In comparison to differential equations represented by standard integer-order derivatives, fractional-order differential equations yield a better and more realistic scenario for explaining a wide range of physical processes. There are several methods for interpolating integer-order definitions into noninteger order. Riemann–Liouville and Caputo derivatives are two of the most known. As a consequence, a large number of scholars have lately played an important role in [7, 18, 20, 29]. The authors recently developed the well-posedness of semilinear Rayleigh–Stokes problem with fractional derivative on $\mathbb{R}^N$ in [8, 28, 30]. Additionally, the authors discussed the Hilfer fractional derivatives on infinite interval. In [1,24], the researchers investigated the optimal control results for fuzzy fractional differential systems by using stability analysis, impulsive systems, nonlocal conditions, and state variables.

In engineering and many other scientific disciplines, particularly for fractional differential evolution systems, fractional calculus has transformed the optimal control issue into a fractional-order problem. The authors of [10, 11] investigated the general formulation and solution scheme for fractional optimal control problems and optimal control of systems governed by partial differential equations. Moreover, the authors of [2] have described a formulation for fractional optimal control problems defined in multiple dimensions. In [5, 6], the researchers discussed the fractional derivatives with optimal control problems by referring to the impulsive systems, integrodifferential equations, stochastic systems, Clarke subdifferential-type, and Hilfer derivative. Currently, the researchers in [14] have established the optimal control for fractional derivatives of order $(1, 2)$ by applying the cosine families, sectorial operators, continuous dependence, fixed point theorems, and integrodifferential systems.

Physical issues gave rise to the concept of nonlocal conditions. Nonlocal functional differential systems were introduced by Byszewski in [4]. Subsequently, utilizing nondense domains, semigroups, cosine functions, multivalued maps, and noncompactness measures, the authors tested fractional differential systems with nonlocal conditions. See the articles [7, 13, 23] for more information. The topic of impulsive functional differential systems has recently provided a natural structure for mathematical modeling of several real-world situations, particularly, in the control, biology, and medicinal areas. The explanation for this applicability is that impulsive differential issues are a suitable model for explaining changes that occur in their state rapidly at some points and cannot be represented using traditional differential equations. For additional information on this theory and its applications, we suggest the papers and books [3, 5, 6, 25]. By using the upper and lower solution approaches, sectorial operators, and nonlocal conditions the investigators of [20, 21] constructed the existence and uniqueness outcomes for fractional differential systems of $1 < \alpha < 2$. The authors [7, 29] established fractional differential systems of order $(1, 2)$ with control problems utilizing nonlocal conditions, mild solutions, cosine families, the measure of noncompactness, the Laplace transform as well as other fixed point theorems.

In [20], the authors looked into the existence and uniqueness of fractional differential equations of order $(1, 2)$. The authors of [21] utilized the upper and lower solution method to investigate the existence of the extremal solutions for a class of fractional partial differential equations of order $(1, 2)$. The existence of mild solutions for Caputo fractional derivatives of $(1, 2)$ was also addressed by the authors in [19]. In [5], researchers used stochastic systems, MNC, control problems, the fixed point theorem of Mönch, nonlocal circumstances, and sectorial operators to derive Riemann–Liouville fractional derivatives of order $1 < q < 2$. Furthermore, in [13–15], the researchers developed the existence, optimal controls, and approximate controllability results for fractional derivatives of order $(1, 2)$ by utilizing the sectorial operators, integrodifferential systems, multivalued functions, and various fixed point techniques.

Inspired by the above work, in this paper, we deduce the existence of mild solution for fractional mixed Volterra–Fredholm integrodifferential equations with infinite delay of order $r \in (1, 2)$ via the sectorial operator of type (P, κ, r, γ). Furthermore, we discuss the optimal control results for the given nonlocal fractional delay integrodifferential systems of order $r \in (1, 2)$. The main contribution of this paper: A new set of sufficient conditions are formulated and used to prove the existence and optimal controls for the Caputo fractional derivative of order $r \in (1, 2)$ with infinite delay by using impulsive systems, mixed Volterra–Fredholm integrodifferential systems, the nonlinear alternative of the Leray–Schauder fixed point theorem, and sectorial operator of type (P, κ, r, γ). Finally, the application of our findings is demonstrated with the help of an example.

The following form, which is inspired by the aforementioned facts, is appropriate for impulsive nonlocal fractional delay mixed Volterra–Fredholm integrodifferential systems of order $r \in (1, 2)$ with control problems

$$
\begin{aligned}
{}^{C}D_t^r z(t) &= Az(t) + \int_0^t h(t, s, z_s)\,\mathrm{d}s \\
&\quad + f\bigl(t, z_t, (Kz_t), (Wz_t)\bigr) + B(t)u(t), \quad t \in V,\ t \neq t_j, \\
\Delta z(t_j) &= m_j, \quad \Delta z'(t_j) = \widetilde{m}_j, \quad j = 1, 2, \dots, n, \\
z(0) &= \phi(0) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i}) \in \mathcal{Q}_g, \quad t \in (-\infty, 0],\ z'(0) = z_1.
\end{aligned} \tag{1}
$$

In the above,

$$
(Kz)(t) = \int_0^t k(t, s, z_s)\,\mathrm{d}s, \qquad (Wz)(t) = \int_0^J w(t, s, z_s)\,\mathrm{d}s,
$$

where ${}^{C}D_t^r$ is the Caputo fractional derivative of order $r \in (1,2)$; $A : D(A) \subset \mathbb{Y} \to \mathbb{Y}$ is sectorial operator of type (P, κ, r, γ) on the Banach space $\mathbb{Y}$; the continuous function $f : [0, \jmath] \times \mathcal{Q}_g \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$, and $k, w : \mathcal{S} \times \mathcal{Q}_g \to \mathbb{Y}$ are appropriate functions, and $h : \mathcal{S} \times \mathcal{Q}_g \to \mathbb{Y}$, where $\mathcal{S} = \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant \jmath\}$. The functions $m_j, \widetilde{m}_j : \mathbb{Y} \to \mathbb{Y}$, and $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = \jmath$; $\Delta z(t_j) = z(t_j^+) - z(t_j^-)$, where $z(t_j^+) = \lim_{\epsilon^+ \to 0} z(t_j + \epsilon)$ and $z(t_j^-) = \lim_{\epsilon^- \to 0} z(t_j + \epsilon)$ represent the right and left limits of $z(t)$ at $t = t_j$, respectively. $\Delta z'(t_j)$ has also a same explanation. Consider the expression $z_t(\theta) = z(t + \theta)$, $\theta \leqslant 0$, and associate it to the abstract phase space $\mathcal{Q}_g$. In the above system, $0 < t_1 < t_2 < t_3 < \cdots < t_i < \jmath$, $i \in \mathbb{N}$, and the appropriate function $\mathcal{E} : \mathcal{Q}_g^i \to \mathcal{Q}_g$.

The following sections represent the remaining portions of this article: Section 2 starts with a description of some basic concepts and the results of the preparation. In Section 3, we use the fixed point theorem to look at the existence results for the Caputo fractional delay of Volterra–Fredholm type with impulses (1). Moreover, we investigate the optimal control results for equations (1). Finally, an application for establishing the theory of the main results is shown.

2 Preliminaries

We will provide some definitions, lemmas, fractional derivatives and integrals definitions, sectorial operator assumptions, and preliminaries in this section, which will be used throughout the article.

Let $C(V, \mathbb{Y}) : V \to \mathbb{Y}$ be the space of all continuous functions with the norm $\|z\|_C = \sup_{t \in V} \|z(t)\|$, and let $L(\mathbb{Y})$ be the Banach space of every linear and bounded operators on $\mathbb{Y}$.

Suppose that $g : (-\infty, 0] \to (0, +\infty)$ is continuous function with $l = \int_{-\infty}^0 g(t)\,\mathrm{d}t < +\infty$. For any $b > 0$, we introduce

$$\mathcal{Q} = \bigl\{\mathfrak{I} : [-b, 0] \to \mathbb{Y} \bigm| \mathfrak{I}(t) \text{ is bounded and measurable}\bigr\}$$

and equip the space $\mathcal{Q}$ with the norm

$$\|\mathfrak{I}\|_{[-b,0]} = \sup_{-b \leqslant s \leqslant 0} \bigl|\mathfrak{I}(s)\bigr| \quad \forall\, \mathfrak{I} \in \mathcal{Q}.$$

Let us define

$$\mathcal{Q}_g = \left\{\mathfrak{I} : (-\infty, 0] \to \mathbb{Y} \;\middle|\; \forall d > 0,\ \mathfrak{I}|_{[-d,0]} \in \mathcal{Q},\ \int_{-\infty}^{0} g(s)\|\mathfrak{I}\|_{[s,0]}\,\mathrm{d}s < +\infty\right\}.$$

If $\mathcal{Q}_g$ is supplied with the norm

$$\|\mathfrak{I}\|_{\mathcal{Q}_g} = \int_{-\infty}^{0} g(s)\|\mathfrak{I}\|_{[s,0]}\,\mathrm{d}s \quad \forall\, \mathfrak{I} \in \mathcal{Q}_g,$$

then $(\mathcal{Q}_g, \|\cdot\|_{\mathcal{Q}_g})$ is a Banach space.

Now, we consider the space

$$\mathcal{Q}'_g = \bigl\{z : (-\infty, \jmath] \to \mathbb{Y} \bigm| z_j \in C(V_j, \mathbb{Y}),\ \text{and there exist } z\bigl(t_j^+\bigr), z\bigl(t_j^-\bigr) \colon z\bigl(t_j^-\bigr) = z\bigl(t_j\bigr), \\ z_0 - \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i}) = \phi \in \mathcal{Q}_g,\ j = 0, \dots, n\bigr\},$$

where z_j is the restriction of z to $V_j = (t_j, t_{j+1}]$. Let $\|\cdot\|_\jmath$ be a seminorm in $\mathcal{Q}'_g$ defined by

$$\|z\|_\jmath = \|\phi\|_{\mathcal{Q}_g} + \sup_{0 \leqslant s \leqslant \jmath} \bigl|z(s)\bigr|, \quad z \in \mathcal{Q}'_g.$$

Lemma 1. (*See* [25].) *Assume that $z \in \mathcal{Q}'_g$, then for $t \in V$, $z_t \in \mathcal{Q}_g$. Moreover,*

$$l\bigl|z(t)\bigr| \leqslant \|z_t\|_{\mathcal{Q}_g} \leqslant \|\phi\|_{\mathcal{Q}_g} + l \sup_{0 \leqslant s \leqslant t} \bigl|z(s)\bigr|,$$

where $l = \int_{-\infty}^0 g(t)\,\mathrm{d}t < +\infty$.

Definition 1. (See [17].) The Riemann–Liouville fractional integral of order $r \in \mathbb{R}^+$ with the lower limit zero for $f : [0, \infty) \to \mathbb{R}^+$ is defined by

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}}\,\mathrm{d}s, \quad t > 0,$$

if the right side is point-wise defined on $[0, \infty)$.

Definition 2. (See [17].) The Riemann–Liouville fractional derivative of order $r \in \mathbb{R}^+$ with the lower limit zero for f is given by

$${}^L D^r f(t) = \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t f(s)(t-s)^{n-r-1}\,\mathrm{d}s, \quad t > 0,\ n-1 < r < n.$$

Definition 3. *(*See *[17].)* The Caputo fractional derivative of order $r \in \mathbb{R}^+$ with the lower limit zero for f is defined by

$${}^C D^r f(t) = {}^L D^r \left(f(t) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} t^m \right), \quad t > 0,\ n-1 < \beta < n.$$

Definition 4. (See [20].) Let $A : D \subseteq \mathbb{Y} \to \mathbb{Y}$ be a closed and linear operator. A is said to be sectorial operator of type (P, η, r, γ) if there exist $\gamma \in \mathbb{R}$, $0 < \eta < \pi/2$, and $P > 0$ such that the r-resolvent of A exists outside the sector,

$$\gamma + \mathcal{S}_\eta = \bigl\{\gamma + \mu^r \colon \mu \in \mathbb{C},\ \bigl|\mathrm{Arg}(-\mu^r)\bigr| < \eta\bigr\},$$

and

$$\bigl\|\bigl(\mu^r I - A\bigr)^{-1}\bigr\| \leqslant \frac{P}{|\mu^r - \gamma|}, \quad \mu^r \notin \gamma + \mathcal{S}_\eta.$$

Further, if A is a sectorial operator of type (P, η, r, γ), then it is not difficult to see that A is the infinitesimal generator of a r-resolvent family $\{K_r(t)\}_{t \geqslant 0}$ in a Banach space, where

$$K_r(t) = \frac{1}{2\pi \mathrm{i}} \int_c \mathrm{e}^{\mu r} R\big(\mu^r, A\big)\,\mathrm{d}\mu.$$

Definition 5. (See [20].) A continuous function $z : (-\infty, \jmath] \to \mathbb{Y}$ is said to be a mild solution of system (1) if $z_0 = (\phi(0) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(0)) \in \mathcal{Q}_g$ on $(-\infty, 0]$, $z'(0) = z_1$ such that

$$\begin{aligned} z(t) &= S_r(t)\big(\phi(0) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(0)\big) + Q_r(t) z_1 \\ &\quad + \int_0^t K_r(t-s) \int_0^s h(s, \nu, z_\nu)\,\mathrm{d}\nu\,\mathrm{d}s \\ &\quad + \int_0^t K_r(t-s) f\big(s, z_s, (K z_s), (W z_s)\big)\,\mathrm{d}s \\ &\quad + \int_0^t K_r(t-s) B(s) u(s)\,\mathrm{d}s + \sum_{j=1}^n S_r(t-t_j) m_j \\ &\quad + \sum_{j=1}^n Q_r(t-t_j) \widetilde{m}_j, \quad t \in V, \end{aligned} \tag{2}$$

where

$$S_r(t) = \frac{1}{2\pi \mathrm{i}} \int_c \mathrm{e}^{\mu t} \mu^{r-1} R\big(\mu^r, A\big)\,\mathrm{d}\mu, \qquad Q_r(t) = \frac{1}{2\pi \mathrm{i}} \int_c \mathrm{e}^{\mu t} \mu^{r-2} R\big(\mu^r, A\big)\,\mathrm{d}\mu,$$
$$K_r(t) = \frac{1}{2\pi \mathrm{i}} \int_c \mathrm{e}^{\mu t} R\big(\mu^r, A\big)\,\mathrm{d}\mu$$

with c being a sufficient path such that $\mu^r \notin \gamma + \mathcal{S}_\eta$ for μ in c.

Theorem 1. (*See* [20, 21].) *Let A be a sectorial operator of type (P, η, r, γ). Then the following estimates on $\|S_r(t)\|$ hold*:

(i) *Let $\gamma \geqslant 0$. For $\psi \in (0, \pi)$, we get*

$$\begin{aligned} \big\|S_r(t)\big\| &\leqslant \frac{M_1(\eta, \psi) P \exp\{[M_1(\eta, \psi)(1 + \gamma t^r)]^{1/r}[(1 + \frac{\sin\psi}{\sin(\psi - \eta)})^{1/r} - 1]\}}{\pi \sin^{1+1/r} \eta} \\ &\quad \times \big(1 + \gamma t^r\big) + \frac{\Gamma(r) P}{\pi(1 + \gamma t^r)|\cos\frac{\pi - \psi}{r}|^r \sin\eta \sin\psi} \end{aligned}$$

for $t > 0$, where $M_1(\eta, \psi) = \max\{1, \sin\eta / \sin(\psi - \eta)\}$.

(ii) *Let* $\gamma < 0$. *For* $0 < \psi < \pi$, *we get*

$$\|S_r(t)\| \leqslant \left(\frac{\mathrm{e}P[(1+\sin\psi)^{1/r}-1]}{\pi|\cos\psi|^{1+1/r}} + \frac{\Gamma(r)P}{\pi|\cos\psi||\cos\frac{\pi-\psi}{r}|^r}\right)\frac{1}{1+|\gamma|t^r}, \quad t > 0.$$

Theorem 2. (*See* [20, 21].) *Let* A *be a sectorial operator of type* (P, η, r, γ). *Then the following estimates on* $\|K_r(t)\|$, $\|Q_r(t)\|$ *hold*:

(i) *Let* $\gamma \geqslant 0$. *For* $\psi \in (0, \pi)$, *we get*

$$\begin{aligned}\|K_r(t)\| \leqslant{}& \frac{P[(1+\frac{\sin\psi}{\sin(\psi-\eta)})^{1/r}-1]}{\pi\sin\eta}(1+\gamma t^r)^{1/r}t^{r-1}\mathrm{e}^{[M_1(\eta,\psi)(1+\gamma t^r)]^{1/r}} \\ &+ \frac{Pt^{r-1}}{\pi(1+\gamma t^r)|\cos\frac{\pi-\psi}{r}|^r\sin\eta\sin\psi}, \\ \|Q_r(t)\| \leqslant{}& \frac{P[(1+\frac{\sin\psi}{\sin(\psi-\eta)})^{1/r}-1]M_1(\eta,\psi)}{\pi\sin\eta^{(r+2)/r}}\big(1+\gamma t^r\big)^{(r-1)/r}t^{r-1} \\ &\times \mathrm{e}^{[M_1(\eta,\psi)(1+\gamma t^r)]^{1/r}} \\ &+ \frac{Pr\Gamma(r)}{\pi(1+\gamma t^r)|\cos\frac{\pi-\psi}{r}|^r\sin\eta\sin\psi}\end{aligned}$$

for $t > 0$, *where* $M_1(\eta, \psi) = \max\{1, \sin\eta/\sin(\psi-\eta)\}$.

(ii) *Let* $\gamma < 0$. *For* $\psi \in (0, \pi)$, *we get*

$$\begin{aligned}\|K_r(t)\| &\leqslant \left(\frac{\mathrm{e}P[(1+\sin\psi)^{1/r}-1]}{\pi|\cos\psi|} + \frac{P}{\pi|\cos\psi||\cos\frac{\pi-\psi}{r}|}\right)\frac{t^{r-1}}{1+|\gamma|t^r}, \\ \|Q_r(t)\| &\leqslant \left(\frac{\mathrm{e}P[(1+\sin\psi)^{1/r}-1]t}{\pi|\cos\psi|^{1+2/r}} + \frac{r\Gamma(r)P}{\pi|\cos\psi||\cos\frac{\pi-\psi}{r}|}\right)\frac{1}{1+|\gamma|t^r}\end{aligned}$$

for $t > 0$.

Theorem 3 [Nonlinear alternative of the Leray–Schauder principle]. (*See* [16].) *Let* $\varPhi$ *be a convex subset of a normed linear space* $\mathbb{Y}$, *and assume that* $0 \in \varPhi$. *Let* $\varPsi : \varPhi \to \varPhi$ *be a completely continuous operator, and let*

$$\mathbb{U}(\varPsi) = \big\{z \in \varPhi\colon\ z = \varLambda\varPsi z \text{ for some } \varLambda \in (0,1)\big\}.$$

Then either $\varPsi$ *has a fixed point or* $\mathbb{U}(\varPsi)$ *is bounded.*

3 Main results

In this section, we investigate the existence of mild solutions for impulsive nonlocal fractional delay mixed Volterra–Fredholm integrodifferential systems of order $r \in (1, 2)$. Based on Section 2, we consider that the operators $S_r(t)$, $Q_r(t)$ and $K_r(t)$ are bounded.

(H1) The linear operator A, which is a sectorial accretive operator of type (P, κ, r, γ), generates the compact r-resolvent families $S_r(t)$, $Q_r(t)$, and $K_r(t)$ for every $t \in V$, and there exists $\widehat{P} > 0$ such that

$$\sup_{t\in[0,\jmath]} \|S_r(t)\| \leqslant \widehat{P}, \qquad \sup_{t\in[0,\jmath]} \|Q_r(t)\| \leqslant \widehat{P}, \qquad \sup_{t\in[0,\jmath]} \|K_r(t)\| \leqslant \widehat{P}.$$

(H2) (i) There exist positive constants E_f and $\widehat{E}_f$ such that for every $t \in V$ and $z_1, y_1, u_1, kz_2, y_2, u_2 \in \mathcal{Q}_g$,

$$\begin{aligned} &\|f(t, z_1, y_1, u_1) - f(t, z_2, y_2, u_2)\|_{\mathcal{Q}_g} \\ &\quad \leqslant E_f\bigl(\|z_1 - z_2\|_{\mathcal{Q}_g} + \|y_1 - y_2\|_{\mathcal{Q}_g} + \|u_1 - u_2\|_{\mathcal{Q}_g}\bigr) \end{aligned}$$

and $\widehat{E}_f = \sup_{t\in V} \|f(t, 0, 0, 0)\|_{\mathcal{Q}_g}$.

(ii) There exist constants $\eta_1, \eta_2 > 0$ such that

$$\left\| \int_0^t \bigl[k(t, s, z) - k(t, s, y)\bigr]\,\mathrm{d}s \right\|_{\mathcal{Q}_g} \leqslant \eta_1 \|z - y\|_{\mathcal{Q}_g},$$

$$\left\| \int_0^\jmath \bigl[w(t, s, z) - w(t, s, y)\bigr]\,\mathrm{d}s \right\|_{\mathcal{Q}_g} \leqslant \eta_2 \|z - y\|_{\mathcal{Q}_g}$$

for any $\widehat{\eta}_1 = \sup_{t\in V} \|\int_0^t k(t, s, 0)\,\mathrm{d}s\|_{\mathcal{Q}_g}$, $\widehat{\eta}_2 = \sup_{t\in V} \|\int_0^t w(t, s, 0)\,\mathrm{d}s\|_{\mathcal{Q}_g}$ and $z, y \in \mathcal{Q}_g$.

(H3) (i) For any $(t, s) \in \mathcal{S}$, the continuous function $h(t, s, \cdot) : \mathcal{Q}_g \to \mathbb{Y}$, and for any $z \in \mathcal{Q}_g$, the strongly measurable function $h(\cdot, \cdot, z) : \mathcal{S} \to \mathbb{Y}$ such that

$$\left\| \int_0^t h(t, s, z)\,\mathrm{d}s \right\| \leqslant c w(t) \varPi_1\bigl(\|z\|_{\mathcal{Q}_g}\bigr),$$

where $w : [0, \jmath] \to [0, \infty)$ is an integrable function, and c is a positive constant, the continuous and increasing function $\varPi_1 : [0, \infty) \to (0, \infty)$.

(ii) For $z, y \in \mathcal{Q}_g$, there is $\zeta_h > 0$ such that

$$\left\| \int_0^t \bigl[h(t, s, z) - h(t, s, y)\bigr]\,\mathrm{d}s \right\|_{\mathcal{Q}_g} \leqslant \zeta_h \|z - y\|_{\mathcal{Q}_g},$$

and $\widehat{\zeta}_h = \sup_{t\in V} \|\int_0^t h(t, s, 0)\,\mathrm{d}s\|_{\mathcal{Q}_g}$.

(H4) Let $\mathcal{X}$ be a separable reflexive Banach space from which the control u take the values. Operator $B \in L_\infty(V, L(\mathcal{X}, \mathbb{Y}))$, $\|B\|_\infty$ denotes the norm of operator B on the Banach space $L_\infty(V, L(\mathcal{X}, \mathbb{Y}))$.

(H5) The closed, convex, and bounded valued function $\lambda(\cdot) : V \rightrightarrows 2^{\mathcal{X}} \setminus \{\emptyset\}$, which is graph measurable and $\lambda(\cdot) \subseteq \Theta$, where Θ is a bounded set of $\mathcal{X}$.

(H6) $\mathcal{E} : \mathcal{Q}_g^i \to \mathcal{Q}_g$ is continuous function such that

$$\left\|\mathcal{E}(x_1, x_2, x_3, \dots, x_i) - \mathcal{E}(v_1, v_2, v_3, \dots, v_i)\right\| \leqslant \sum_{\mathfrak{z}=1}^{i} M_{\mathfrak{z}}(\mathcal{E}) \|x - v\|_{\mathcal{Q}_g},$$

where $M_{\mathfrak{z}}(\mathcal{E}) > 0$ for any $x_i, v_i \in \mathcal{Q}_g$, and assume that

$$\mathcal{P}_{\mathcal{E}} = \sup\left\{\left\|\mathcal{E}(x_1, x_2, x_3, \dots, x_i)\right\|,\ x_i \in \mathcal{Q}_g\right\}.$$

(H7) There exists a constant $\mathcal{I} > 0$ such that

$$\frac{[1 - l\widehat{P}\jmath(\zeta_h + E_f(1 + \eta_1 + \eta_2))]\mathcal{I}}{l\widehat{P}[\zeta_h\jmath + \jmath E_f(\widehat{\eta}_1 + \widehat{\eta}_2) + \widehat{E}_f\jmath + \mathcal{O}]} > 1,$$

where $l\widehat{P}\jmath(\zeta_h + E_f(1 + \eta_1 + \eta_2)) < 1$.

Fix the admissible set

$$\lambda_{\mathrm{ad}} = \left\{y(\cdot) \mid V \to \mathcal{X} \text{ strongly measurable},\ y(t) \in \lambda(t) \text{ a.e. } t \in V\right\}.$$

Additionally, $\lambda_{\mathrm{ad}} \neq \emptyset$ from [22], and $\lambda_{\mathrm{ad}} \subset L^q(V, \mathcal{X})$ $(1 < q < +\infty)$. $Bu \in L^q(V, \mathbb{Y})$ for any $u \in \lambda_{\mathrm{ad}}$.

Theorem 4. *If hypotheses* (H1)–(H7) *are satisfied, then system* (1) *has a mild solution on* $[0, \jmath]$.

Proof. Let $\Theta : \mathcal{Q}_g' \to \mathcal{Q}_g'$ given by

$$\Theta z(t) = \begin{cases} \phi(t) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(t), \quad t \in (-\infty, 0], \\ S_r(t)(\phi(0) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(0)) + Q_r(t)z_1 \\ \quad + \int_0^t K_r(t-s) \int_0^s h(s, \nu, z_\nu)\,\mathrm{d}\nu\,\mathrm{d}s \\ \quad + \int_0^t K_r(t-s) f(s, z_s, (Kz_s), (Wz_s))\,\mathrm{d}s + \int_0^t K_r(t-s)B(s)u(s)\,\mathrm{d}s \\ \quad + \sum_{j=1}^n S_r(t-t_j)m_j + \sum_{j=1}^n Q_r(t-t_j)\widetilde{m}_j, \quad t \in V. \end{cases}$$

For verify that Θ has a fixed point, we deduce that it should be the solution to system (1).

Since ϕ in $\mathcal{Q}_g$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(t), & t \in (-\infty, 0], \\ S_r(t)(\phi(0) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(0)) + Q_r(t)z_1, & t \in V, \end{cases}$$

then $\widehat{\phi}$ in $\mathcal{Q}_g'$.

Consider $z(t) = y(t) + \widehat{\phi}(t)$, $t \in (-\infty, \jmath)$, we deduce that y fulfills (2) iff y fulfills $y_0 = 0$ and

$$y(t) = \int_0^t K_r(t-s) \int_0^s h(s,\, \nu,\, y_\nu + \widehat{\phi}_\nu)\, \mathrm{d}\nu\, \mathrm{d}s + \int_0^t K_r(t-s) f\bigl(s,\, y_s + \widehat{\phi}_s,\, \bigl(K(y_s + \widehat{\phi}_s)\bigr),\, \bigl(W(y_s + \widehat{\phi}_s)\bigr)\bigr)\, \mathrm{d}s + \int_0^t K_r(t-s) B(s) u(s)\, \mathrm{d}s + \sum_{j=1}^n S_r(t-t_j) m_j + \sum_{j=1}^n Q_r(t-t_j) \widetilde{m}_j, \quad t \in V.$$

Consider $\mathcal{Q}_g'' = \{y \in \mathcal{Q}_g' \colon y_0 = 0 \in \mathcal{Q}_g\}$. For every $y \in \mathcal{Q}_g''$,

$$\|y\|_\jmath = \|y_0\|_{\mathcal{Q}_g} + \sup\bigl\{\bigl|y(s)\bigr| \colon 0 \leqslant s \leqslant \jmath\bigr\} = \sup\bigl\{\bigl|y(s)\bigr| \colon 0 \leqslant s \leqslant \jmath\bigr\}$$

since $(\mathcal{Q}_g'', \|\cdot\|_\jmath)$ is a Banach space.

Set $\mathscr{Q}_p = \{y \in \mathcal{Q}_g'' \colon \|y\|_\jmath \leqslant p\}$ for some $p > 0$, then $\mathscr{Q}_p \subseteq \mathcal{Q}_g''$ is uniformly bounded, and for $y \in \mathscr{Q}_p$, by using Lemma 1 one can get

$$\begin{aligned} \|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} &\leqslant \|y_t\|_{\mathcal{Q}_g} + \|\widehat{\phi}_t\|_{\mathcal{Q}_g} \\ &\leqslant l\bigl[p + \widehat{P}\bigl(\bigl|\phi(0)\bigr| + \mathcal{P}_{\mathcal{E}}\bigr) + \widehat{P}|z_1|\bigr] + \|\widehat{\phi}\|_{\mathcal{Q}_g} \leqslant p'. \end{aligned} \tag{3}$$

Define $\Omega : \mathcal{Q}_g'' \to \mathcal{Q}_g''$ by

$$\Omega y(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ \int_0^t K_r(t-s) \int_0^s h(s, \nu, y_\nu + \widehat{\phi}_\nu)\, \mathrm{d}\nu\, \mathrm{d}s \\ \quad + \int_0^t K_r(t-s) f(s, y_s + \widehat{\phi}_s, (K(y_s + \widehat{\phi}_s)), (W(y_s + \widehat{\phi}_s)))\, \mathrm{d}s \\ \quad + \int_0^t K_r(t-s) B(s) u(s)\, \mathrm{d}s \\ \quad + \sum_{j=1}^n S_r(t-t_j) m_j + \sum_{j=1}^n Q_r(t-t_j) \widetilde{m}_j, \quad t \in V. \end{cases}$$

Moreover, if Θ has a fixed point, which is equivalent to Ω having a fixed point, then we can show that Ω is continuous and completely continuous.

Also, Ω mapping from bounded sets into bounded sets in $\mathcal{Q}_g''$.

In fact, it is sufficient to prove that there exists a positive constant χ such that for every $y \in \mathscr{Q}_p$, one has $\|\Omega y\| \leqslant \chi$. Consider that $y \in \mathscr{Q}_p$. From (3), for every $t \in V$, we get

$$\Omega y(t) = \int_0^t K_r(t-s) \int_0^s h(s,\, \nu,\, y_\nu + \widehat{\phi}_\nu)\, \mathrm{d}\nu\, \mathrm{d}s + \int_0^t K_r(t-s) f\bigl(s,\, y_s + \widehat{\phi}_s,\, \bigl(K(y_s + \widehat{\phi}_s)\bigr),\, \bigl(W(y_s + \widehat{\phi}_s)\bigr)\bigr)\, \mathrm{d}s$$

$$+\int_0^t K_r(t-s)B(s)u(s)\,\mathrm{d}s$$
$$+\sum_{j=1}^{n} S_r(t-t_j)m_j+\sum_{j=1}^{n} Q_r(t-t_j)\widetilde{m}_j, \quad t\in V.$$

From the hypotheses, for every $t \in V$, we obtain

$$\begin{aligned}
&\big\|\Omega y(t)\big\| \\
&\leqslant \left\|\int_0^t K_r(t-s)\int_0^s h(s,\nu,\, y_\nu+\widehat{\phi}_\nu)\,\mathrm{d}\nu\,\mathrm{d}s\right\| \\
&\quad+\left\|\int_0^t K_r(t-s) f\big(s, y_s+\widehat{\phi}_s, \big(K(y_s+\widehat{\phi}_s)\big), \big(W(y_s+\widehat{\phi}_s)\big)\big)\,\mathrm{d}s\right\| \\
&\quad+\left\|\int_0^t K_r(t-s)B(s)u(s)\,\mathrm{d}s\right\|+\left\|\sum_{j=1}^{n} S_r(t-t_j)m_j\right\|+\left\|\sum_{j=1}^{n} Q_r(t-t_j)\widetilde{m}_j\right\| \\
&\leqslant \widehat{P}\int_0^t \big[\zeta_h\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\zeta}_h\big]\,\mathrm{d}s \\
&\quad+\widehat{P}\int_0^t \big[E_f\big(\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\eta_1\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\eta}_1+\eta_2\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\eta}_2\big)+\widehat{E}_f\big]\,\mathrm{d}s \\
&\quad+\widehat{P}\|B\|_\infty\int_0^t \big\|u(s)\big\|\,\mathrm{d}s+\widehat{P}\sum_{j=1}^{n}\|m_j\|+\widehat{P}\sum_{j=1}^{n}\|\widetilde{m}_j\| \\
&\leqslant \widehat{P}\jmath\big[\zeta_h p'+\widehat{\zeta}_h\big]+\widehat{P}\jmath\big[E_f(p'+\eta_1 p'+\widehat{\eta}_1+\eta_2 p'+\widehat{\eta}_2)+\widehat{E}_f\big] \\
&\quad+\widehat{P}\|B\|_\infty \jmath^{(q-1)/q}\|u\|_{L^q(V,\mathcal{X})}+\widehat{P}\sum_{j=1}^{n}\|m_j\|+\widehat{P}\sum_{j=1}^{n}\|\widetilde{m}_j\|=\chi.
\end{aligned}$$

Hence, we get $\|\Omega y(t)\| \leqslant \chi$.

We check that Ω mapping from bounded sets into equicontinuous sets of $\mathscr{Q}_p$.

Suppose that $0<t_1<t_2\leqslant \jmath$. In addition, $\mathscr{Q}_p$ is bounded set of $\mathcal{Q}_g''$. We get

$$\begin{aligned}
&\big\|\Omega y(t_2)-\Omega y(t_1)\big\| \\
&=\bigg\|\int_0^{t_2} K_r(t_2-s)\int_0^s h(s,\,\nu,\,y_\nu+\widehat{\phi}_\nu)\,\mathrm{d}\nu\,\mathrm{d}s \\
&\quad+\int_0^{t_2} K_r(t_2-s) f\big(s,\,y_s+\widehat{\phi}_s, \big(K(y_s+\widehat{\phi}_s)\big), \big(W(y_s+\widehat{\phi}_s)\big)\big)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t_2} K_r(t_2-s)B(s)u(s)\,\mathrm{d}s+\sum_{0<t_j<t_2} S_r(t_2-t_j)m_j+\sum_{0<t_j<t_2} Q_r(t_2-t_j)\widetilde{m}_j \\
&-\int_0^{t_1} K_r(t_1-s)\int_0^{s} h\big(s,\,\nu,\,y_\nu+\widehat{\phi}_\nu\big)\,\mathrm{d}\nu\,\mathrm{d}s \\
&-\int_0^{t_1} K_r(t_1-s)f\big(s,\,y_s+\widehat{\phi}_s,\big(K(y_s+\widehat{\phi}_s)\big),\big(W(y_s+\widehat{\phi}_s)\big)\big)\,\mathrm{d}s \\
&-\int_0^{t_1} K_r(t_1-s)B(s)u(s)\,\mathrm{d}s-\sum_{0<t_j<t_1} S_r(t_1-t_j)m_j-\sum_{0<t_j<t_1} Q_r(t_1-t_j)\widetilde{m}_j\Bigg\| \\
\leqslant{}&\int_{t_1}^{t_2}\Bigg\|K_r(t_2-s)\int_0^{s} h\big(s,\,\nu,\,y_\nu+\widehat{\phi}_\nu\big)\,\mathrm{d}\nu\Bigg\|\,\mathrm{d}s \\
&+\int_0^{t_1}\Bigg\|\big[K_r(t_2-s)-K_r(t_1-s)\big]\int_0^{s} h\big(s,\,\nu,\,y_\nu+\widehat{\phi}_\nu\big)\,\mathrm{d}\nu\Bigg\|\,\mathrm{d}s \\
&+\int_{t_1}^{t_2}\Big\|K_r(t_2-s)f\big(s,\,y_s+\widehat{\phi}_s,\big(K(y_s+\widehat{\phi}_s)\big),\big(W(y_s+\widehat{\phi}_s)\big)\big)\Big\|\,\mathrm{d}s \\
&+\int_0^{t_1}\Big\|\big[K_r(t_2-s)-K_r(t_1-s)\big]f\big(s,\,y_s+\widehat{\phi}_s,\big(K(y_s+\widehat{\phi}_s)\big),\big(W(y_s+\widehat{\phi}_s)\big)\big)\Big\|\,\mathrm{d}s \\
&+\int_{t_1}^{t_2}\big\|K_r(t_2-s)B(s)u(s)\big\|\,\mathrm{d}s+\int_0^{t_1}\big\|\big[K_r(t_2-s)-K_r(t_1-s)\big]B(s)u(s)\big\|\,\mathrm{d}s \\
&+\sum_{t_1<t_j<t_2}\big\|S_r(t_2-t_j)m_j\big\|+\sum_{0<t_j<t_1}\big\|\big[S_r(t_2-t_j)-S_r(t_1-t_j)\big]m_j\big\| \\
&+\sum_{t_1<t_j<t_2}\big\|Q_r(t_2-t_j)\widetilde{m}_j\big\|+\sum_{0<t_j<t_1}\big\|\big[Q_r(t_2-t_j)-Q_r(t_1-t_j)\big]\widetilde{m}_j\big\| \\
\leqslant{}&\widehat{P}\int_{t_1}^{t_2}\big[\zeta_h\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\zeta}_h\big]\,\mathrm{d}s \\
&+\int_0^{t_1}\big\|K_r(t_2-s)-K_r(t_1-s)\big\|\big[\zeta_h\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\zeta}_h\big]\,\mathrm{d}s \\
&+\widehat{P}\int_{t_1}^{t_2}\big[E_f\big(\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\eta_1\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\eta}_1+\eta_2\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\eta}_2\big)+\widehat{E}_f\big]\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_1} \big\|K_r(t_2-s)-K_r(t_1-s)\big\| \big[E_f\big(\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\eta_1\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g} \\
&\qquad +\widehat{\eta}_1+\eta_2\|y_s+\widehat{\phi}_s\|_{\mathcal{Q}_g}+\widehat{\eta}_2\big)+\widehat{E}_f\big]\,\mathrm{d}s \\
&+\widehat{P}\|B\|_\infty \int_{t_1}^{t_2} \big\|u(s)\big\|\,\mathrm{d}s+\|B\|_\infty \int_0^{t_1} \big\|K_r(t_2-s)-K_r(t_1-s)\big\|\big\|u(s)\big\|\,\mathrm{d}s \\
&+\sum_{t_1<t_j<t_2} \big\|S_r(t_2-t_j)m_j\big\|+\sum_{0<t_j<t_1} \big\|\big[S_r(t_2-t_j)-S_r(t_1-t_j)\big]m_j\big\| \\
&+\sum_{t_1<t_j<t_2} \big\|Q_r(t_2-t_j)\widetilde{m}_j\big\|+\sum_{0<t_j<t_1} \big\|\big[Q_r(t_2-t_j)-Q_r(t_1-t_j)\big]\widetilde{m}_j\big\| \\
\leqslant{}& \widehat{P}[\zeta_h p'+\widehat{\zeta}_h](t_2-t_1)+[\zeta_h p'+\widehat{\zeta}_h]\int_0^{t_1} \big\|K_r(t_2-s)-K_r(t_1-s)\big\|\,\mathrm{d}s \\
&+\widehat{P}\big[E_f(p'+\eta_1 p'+\widehat{\eta}_1+\eta_2 p'+\widehat{\eta}_2)+\widehat{E}_f\big](t_2-t_1) \\
&+\big[E_f(p'+\eta_1 p'+\widehat{\eta}_1+\eta_2 p'+\widehat{\eta}_2)+\widehat{E}_f\big]\int_0^{t_1} \big\|K_r(t_2-s)-K_r(t_1-s)\big\|\,\mathrm{d}s \\
&+\widehat{P}\|B\|_\infty(t_2-t_1)^{(q-1)/q}\|u\|_{L^q(V,\mathcal{X})} \\
&+\|B\|_\infty\|x\|_{L^q(V,\mathcal{X})}\int_0^{t_1} \big\|K_r(t_2-s)-K_r(t_1-s)\big\|\,\mathrm{d}s \\
&+\sum_{t_1<t_j<t_2} \big\|S_r(t_2-t_j)m_j\big\|+\sum_{0<t_j<t_1} \big\|\big[S_r(t_2-t_j)-S_r(t_1-t_j)\big]m_j\big\| \\
&+\sum_{t_1<t_j<t_2} \big\|Q_r(t_2-t_j)\widetilde{m}_j\big\|+\sum_{0<t_j<t_1} \big\|\big[Q_r(t_2-t_j)-Q_r(t_1-t_j)\big]\widetilde{m}_j\big\|.
\end{aligned}
$$

By using the continuity of functions $t \to \|S_r(t)\|$, $t \to \|Q_r(t)\|$, and $t \to \|K_r(t)\|$, the right-hand side of the above inequalities tends to zero as $t_2 \to t_1$. Therefore, Ωy is equicontinuous. From the Arzelà–Ascoli theorem it is obvious that Ω is continuous and completely continuous. Then we verity that there exists an open set $\mathbb{U} \subseteq \mathscr{Q}_p$ with $y \neq \Lambda\Omega y$ for $\Lambda \in (0,1)$ and $y \in \partial\mathbb{U}$.

Let $y \in \mathscr{Q}_p$ and $y = \Lambda\Omega y$ for some $\Lambda \in (0,1)$. For any $t \in V$, we get

$$
\begin{aligned}
y(t) ={}& \int_0^t K_r(t-s)\int_0^s h(s,\,\nu,\,y_\nu+\widehat{\phi}_\nu)\,\mathrm{d}\nu\,\mathrm{d}s \\
&+\int_0^t K_r(t-s)f\big(s,\,y_s+\widehat{\phi}_s,\,\big(K(y_s+\widehat{\phi}_s)\big),\,\big(W(y_s+\widehat{\phi}_s)\big)\big)\,\mathrm{d}s
\end{aligned}
$$

$$+ \int_0^t K_r(t-s)B(s)u(s)\,\mathrm{d}s + \sum_{j=1}^n S_r(t-t_j)m_j + \sum_{j=1}^n Q_r(t-t_j)\widetilde{m}_j, \quad t \in V.$$

Then we get

$$\begin{aligned}
\|y(t)\| &= \left\| \int_0^t K_r(t-s) \int_0^s h(s,\, \nu,\, y_\nu + \widehat{\phi}_\nu)\,\mathrm{d}\nu\,\mathrm{d}s \right\| \\
&\quad + \left\| \int_0^t K_r(t-s) f\bigl(s,\, y_s + \widehat{\phi}_s, \bigl(K(y_s + \widehat{\phi}_s)\bigr), \bigl(W(y_s + \widehat{\phi}_s)\bigr)\bigr)\,\mathrm{d}s \right\| \\
&\quad + \left\| \int_0^t K_r(t-s)B(s)u(s)\,\mathrm{d}s \right\| + \left\| \sum_{j=1}^n S_r(t-t_j)m_j \right\| + \left\| \sum_{j=1}^n Q_r(t-t_j)\widetilde{m}_j \right\| \\
&\leqslant \widehat{P} \int_0^t \bigl[\zeta_h \|y_s + \widehat{\phi}_s\|_{\mathcal{Q}_g} + \widehat{\zeta}_h\bigr]\,\mathrm{d}s \\
&\quad + \widehat{P} \int_0^t \bigl[E_f\bigl(\|y_s + \widehat{\phi}_s\|_{\mathcal{Q}_g} + \eta_1 \|y_s + \widehat{\phi}_s\|_{\mathcal{Q}_g} + \widehat{\eta}_1 + \eta_2 \|y_s \\
&\qquad + \widehat{\phi}_s\|_{\mathcal{Q}_g} + \widehat{\eta}_2\bigr) + \widehat{E}_f\bigr]\,\mathrm{d}s \\
&\quad + \widehat{P}\|B\|_\infty \int_0^t \bigl\|u(s)\bigr\|\,\mathrm{d}s + \widehat{P}\sum_{j=1}^n \|m_j\| + \widehat{P}\sum_{j=1}^n \|\widetilde{m}_j\| \\
&\leqslant \widehat{P}\jmath\zeta_h \|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} + \widehat{P}\jmath\widehat{\zeta}_h + \widehat{P}\jmath\bigl[E_f\bigl(\|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} \\
&\quad + \eta_1 \|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} + \widehat{\eta}_1 + \eta_2 \|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} + \widehat{\eta}_2\bigr) + \widehat{E}_f\bigr] \\
&\quad + \widehat{P}\|B\|_\infty \jmath^{(q-1)/q} \|u\|_{L^q(V,\mathcal{X})} + \widehat{P}\sum_{j=1}^n \|m_j\| + \widehat{P}\sum_{j=1}^n \|\widetilde{m}_j\|.
\end{aligned}$$

Now, we obtain

$$\|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} \leqslant l \sup_{s\in[0,t]} \bigl|y(s)\bigr| + l\widehat{P}\bigl(\bigl|\phi(0)\bigr| + \mathcal{P}_{\mathcal{E}}\bigr) + l\widehat{P}|z_1| + \|\widehat{\phi}\|_{\mathcal{Q}_g}.$$

If $\delta(t)$ the right-hand side of above inequality, then we get

$$\|y_t + \widehat{\phi}_t\|_{\mathcal{Q}_g} \leqslant \delta(t).$$

Let $t^* \in [0,t]$ be such that

$$\delta(t) = l\bigl\|y(t^*)\bigr\| + l\widehat{P}\bigl(\bigl\|\phi(0)\bigr\| + \mathcal{P}_{\mathcal{E}}\bigr) + l\widehat{P}\|z_1\| + \|\widehat{\phi}\|_{\mathcal{Q}_g}.$$

Since from the above inequality, for any $t \in V$,

$$\begin{aligned}\delta(t) &= l\widehat{P}\jmath\zeta_h\delta(t) + l\widehat{P}\jmath\widehat{\zeta}_h \\ &\quad + l\widehat{P}\jmath\big[E_f\big(\delta(t) + \eta_1\delta(t) + \widehat{\eta}_1 + \eta_2\delta(t) + \widehat{\eta}_2\big) + \widehat{E}_f\big] \\ &\quad + l\widehat{P}\|B\|_\infty \jmath^{(q-1)/q}\|u\|_{L^q(V,\mathcal{X})} + l\widehat{P}\sum_{j=1}^{n}\|m_j\| + l\widehat{P}\sum_{j=1}^{n}\|\widetilde{m}_j\| \\ &\quad + l\widehat{P}\big(\big|\phi(0)\big| + \mathcal{P}_{\mathcal{E}}\big) + l\widehat{P}|z_1| + \|\widehat{\phi}\|_{\mathcal{Q}_g},\end{aligned}$$

then by taking norm we get

$$\begin{aligned}\big\|\delta(t)\big\| &\leqslant= l\widehat{P}\jmath\zeta_h\big\|\delta(t)\big\| \\ &\quad + l\widehat{P}\jmath\widehat{\zeta}_h + l\widehat{P}\jmath\big[E_f\big(\big\|\delta(t)\big\| + \eta_1\big\|\delta(t)\big\| + \widehat{\eta}_1 + \eta_2\big\|\delta(t)\big\| + \widehat{\eta}_2\big) + \widehat{E}_f\big] \\ &\quad + l\widehat{P}\|B\|_\infty \jmath^{(q-1)/q}\|u\|_{L^q(V,\mathcal{X})} + l\widehat{P}\sum_{j=1}^{n}\|m_j\| + l\widehat{P}\sum_{j=1}^{n}\|\widetilde{m}_j\| \\ &\quad + l\widehat{P}\big(\big\|\phi(0)\big\| + \mathcal{P}_{\mathcal{E}}\big) + l\widehat{P}\|z_1\| + \|\widehat{\phi}\|_{\mathcal{Q}_g},\end{aligned}$$

which implies

$$\frac{[1 - l\widehat{P}\jmath(\zeta_h + E_f(1+\eta_1+\eta_2))]\|\delta(t)\|}{l\widehat{P}[\zeta_h\jmath + \jmath E_f(\widehat{\eta}_1 + \widehat{\eta}_2) + \widehat{E}_f\jmath + \mathcal{O}]} \leqslant 1, \tag{4}$$

where

$$\begin{aligned}\mathcal{O} &= \|B\|_\infty \jmath^{(q-1)/q}\|u\|_{L^q(V,\mathcal{X})} + \sum_{j=1}^{n}\|m_j\| + \sum_{j=1}^{n}\|\widetilde{m}_j\| \\ &\quad + \big\|\phi(0)\big\| + \mathcal{P}_{\mathcal{E}} + \|z_1\| + \|\widehat{\phi}\|_{\mathcal{Q}_g}.\end{aligned}$$

By (4) and (H7) $\|\delta\| \neq \mathcal{I}$. Let

$$\mathbb{U} = \big\{y \in \mathcal{Q}_g'' \colon \|y\| < \mathcal{I} + 1\big\}.$$

By the choose of $\mathbb{U}$ there is no $y \in \partial\mathbb{U}$ such that $y = \varLambda\varTheta y$ for some $\varLambda \in (0,1)$. From the nonlinear alternative of Leray–Schauder-type theorem we obtain that $\varOmega$ has a fixed point. Hence, $\varTheta$ has a fixed point, which is a solution to system (1). □

4 Existence of optimal control results

In this case, we consider the Lagrange problem: discuss $(z^0, u^0) \in C(V, \mathbb{Y}) \times \lambda_{\mathrm{ad}}$ such that

$$\mathcal{J}\big(z^0, u^0\big) \leqslant \mathcal{J}(z^u, u) \quad \forall\, u \in \lambda_{\mathrm{ad}},$$

where $\mathcal{J}(z^u, u) = \int_0^\jmath \mathscr{L}(t, z_t^u, u(t))\,\mathrm{d}t$, and z^u denotes the mild solution of system (1) corresponding to the control $u \in \lambda_{\mathrm{ad}}$. In order to the existence of mild solution for system (1), we assume that the following hypothesis holds:

(H8) (i) The Borel measurable functional $\mathscr{L} : V \times \mathcal{Q}_g \times \mathbb{Y} \times \mathcal{X} \to \mathbb{R} \cup \{\infty\}$.
(ii) The sequentially lower semicontinuous functional $\mathscr{L}(t, \cdot, \cdot)$ on $\mathcal{Q}_g \times \mathbb{Y} \times \mathcal{X}$ for a.e. $t \in V$.
(iii) The convex $\mathscr{L}(t, z, \cdot)$ on $\mathcal{X}$ for any $z \in \mathcal{Q}_g$ and for every $t \in V$.
(iv) There exist $\tau \geqslant 0$, $d > 0$, a nonnegative α, $\alpha \in L^1(V, \mathbb{R})$, such that

$$\mathscr{L}(t, z, u) \geqslant \alpha(t) + \tau\|z\|_{\mathcal{Q}_g} + d\|u\|^q_{\mathcal{X}}.$$

Lemma 2. (*See* [12].) *Consider that A is sectorial operator of type (P, κ, r, γ) on $\mathbb{Y}$. In addition, the $\aleph : L^q(V, \mathcal{X}) \to C(V, \mathcal{X})$ with $q > 1$, presented by*

$$(\aleph f)(\cdot) = \int_0^{\cdot} K_r(\cdot - s) f(s)\,\mathrm{d}s,$$

is strongly continuous.

Next, we present the results regarding the existence of the optimal controls for (1).

Theorem 5. *Suppose A is sectorial operator of type (P, κ, r, γ) in $\mathbb{Y}$. If all the assumptions of Theorem* 4 *and* (H8) *hold, then the Lagrange problem has at least one optimal pair.*

Proof. If $\inf\{\mathcal{J}(z^u, u),\ u \in \lambda_{\mathrm{ad}}\} = +\infty$, there will be nothing to verify. Without loss of generality, one can assume that $\inf\{\mathcal{J}(z^u, u)\colon\ u \in \lambda_{\mathrm{ad}}\} = \varsigma < +\infty$. By (H8) we get $\varsigma > -\infty$. From the definition of inf there exists a minimizing sequence feasible pair

$$\big\{(z^\ell, u^\ell)\big\} \subset \mathbb{A}_{\mathrm{ad}} \equiv \big\{(z, u)\colon\ z \text{ is a mild solution of system (1)}, \\ \text{obviously it depends on} u \text{ in} \lambda_{\mathrm{ad}}\big\},$$

$\mathcal{J}(z^\ell, u^\ell) \to \varsigma$ as $\ell \to +\infty$. Since $\{u^\ell\} \subseteq \lambda_{\mathrm{ad}}$, $\{u^\ell\}$ is a bounded subset of $L^q(V, \mathcal{X})$, there exists a subsequence, relabeled as $\{u^\ell\}$, and u^0 in $L^q(V, \mathcal{X})$ such that $u^\ell \xrightarrow{w} u^0$ in $L^q(V, \mathcal{X})$. Since λ_{ad} is closed and convex by reason of Marzur lemma, $u^0 \in \lambda_{\mathrm{ad}}$.

Let z^ℓ be the mild solution of system (1) according to u^ℓ, $\ell = 0, 1, \dots$, z^ℓ satisfies the following integral equation:

$$\begin{aligned} z^\ell(t) = {}& S_r(t)\big(\phi(0) + \mathcal{E}\big(z^\ell_{t_1}, z^\ell_{t_2}, z^\ell_{t_3}, \dots, z^\ell_{t_i}\big)(0)\big) + Q_r(t) z_1 \\ &+ \int_0^t K_r(t-s) \int_0^s h\big(s, \nu, z^\ell_\nu\big)\,\mathrm{d}\nu\,\mathrm{d}s \\ &+ \int_0^t K_r(t-s) f\big(s, z^\ell_s, \big(K z^\ell_s\big), \big(W z^\ell_s\big)\big)\,\mathrm{d}s + \int_0^t K_r(t-s) B(s) u^\ell(s)\,\mathrm{d}s \\ &+ \sum_{j=1}^n S_r(t-t_j) m_j + \sum_{j=1}^n Q_r(t-t_j)\widetilde{m}_j, \quad t \in V. \end{aligned}$$

Assume that $f_s^\ell = f(s, z_s^\ell, (Kz_s^\ell), (Wz_s^\ell))$. From (H2) we have that f^ℓ is a bounded continuous operator from V into $\mathbb{Y}$ since f_s^ℓ in $L^q(V, \mathbb{Y})$. Moreover, $\{f_s^\ell\} \subseteq \mathbb{Y}$, $\{f_s^\ell\}$ is bounded in $L^q(V, \mathbb{Y})$, there exists a subsequence, relabeled as $\{f_s^\ell\}$, and $\widehat{f}_s$ in $L^q(V, \mathbb{Y})$ such that

$$f_s^\ell \xrightarrow{w} \widehat{f}_s \quad \text{in } L^q(V, \mathbb{Y}).$$

From Lemma 2 we get

$$\aleph f_s^\ell \xrightarrow{s} \aleph \widehat{f}_s \quad \text{in } C(V, \mathbb{Y}).$$

Consider the following system:

$$\begin{aligned} &{}^C D_t^r z(t) = Az(t) + \int_0^t h(t, s, z_s)\, \mathrm{d}s + \widehat{f}_t + B(t)u(t), \quad t \in V,\ t \neq t_j, \\ &\Delta z(t_j) = m_j, \quad \Delta z'(t_j) = \widetilde{m}_j, \quad j = 1, 2, \dots, n, \\ &\quad z(0) = \phi(0) + \mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i}) \in \mathcal{Q}_g, \quad t \in (-\infty, 0],\ z'(0) = z_1. \end{aligned} \tag{5}$$

We know that (5) has a mild solution

$$\begin{aligned} \widehat{z}(t) = {} & S_r(t)\big(\phi(0) + \mathcal{E}(\widehat{z}_{t_1}, \widehat{z}_{t_2}, \widehat{z}_{t_3}, \dots, \widehat{z}_{t_i})(0)\big) + Q_r(t)z_1 \\ & + \int_0^t K_r(t-s) \int_0^s h(s, \nu, \widehat{z}_\nu)\, \mathrm{d}\nu\, \mathrm{d}s \\ & + \int_0^t K_r(t-s)\widehat{f}_s\, \mathrm{d}s + \int_0^t K_r(t-s)B(s)u^0(s)\, \mathrm{d}s \\ & + \sum_{j=1}^n S_r(t-t_j)m_j + \sum_{j=1}^n Q_r(t-t_j)\widetilde{m}_j, \quad t \in V. \end{aligned}$$

Then

$$\begin{aligned} &\big\|z^\ell(t) - \widehat{z}(t)\big\| \\ &\quad \leqslant \big\|S_r(t)\big[\mathcal{E}\big(z_{t_1}^\ell, z_{t_2}^\ell, z_{t_3}^\ell, \dots, z_{t_i}^\ell\big)(0) - \mathcal{E}(\widehat{z}_{t_1}, \widehat{z}_{t_2}, \widehat{z}_{t_3}, \dots, \widehat{z}_{t_i})(0)\big]\big\| \\ &\qquad + \int_0^t \left\|K_r(t-s)\left(\int_0^s h\big(s, \nu, z_\nu^\ell\big)\, \mathrm{d}\nu - \int_0^s h(s, \nu, \widehat{z}_\nu)\, \mathrm{d}\nu\right)\right\| \mathrm{d}s \\ &\qquad + \int_0^t \big\|K_r(t-s)\big[f_s^\ell - \widehat{f}_s\big]\big\|\, \mathrm{d}s + \int_0^t \big\|K_r(t-s)\big(B(s)u^\ell(s) - B(s)u^0(s)\big)\big\|\, \mathrm{d}s \\ &\quad \leqslant \widehat{P} \sum_{i=1}^k M_i(\mathcal{E})\big\|z^\ell - \widehat{z}\big\|_{\mathcal{Q}_g} \end{aligned}$$

$$+ \widehat{P} \int_0^t \left\| \int_0^s \left[h\left(s, \nu, z_\nu^\ell\right) - h\left(s, \nu, \widehat{z}_\nu\right)\right] \mathrm{d}\nu \right\| \mathrm{d}s \quad \text{(denoted by } \mathcal{G}_1\text{)}$$
$$+ \widehat{P} \int_0^t \left\|B(s)u^\ell(s) - B(s)u^0(s)\right\| \mathrm{d}s \quad \text{(denoted by } \mathcal{G}_2\text{)},$$

which implies

$$\left\|z^\ell - \widehat{z}\right\| \leqslant \frac{\mathcal{G}_1 + \mathcal{G}_2}{1 - \sum_{\jmath=1}^{i} M_\jmath(\mathcal{E})} \leqslant \delta_\ell.$$

Applying Lemma 2 next, we obtain

$$\delta_\ell \to 0 \in C(V, \mathbb{R}) \quad \text{as } \ell \to \infty.$$

Moreover, we get

$$z^\ell \to \widehat{z} \in C(V, \mathbb{Y}) \quad \text{as } \ell \to \infty.$$

Moreover, applying hypotheses (H2) and (H3), we have

$$f_s^\ell \to f\left(s, \widehat{z}_s, (K\widehat{z}_s), (W\widehat{z}_s)\right) \quad \text{in } C(V, \mathbb{Y}) \text{ as } \ell \to \infty.$$

Applying the uniqueness of limit, we obtain

$$\widehat{f}_t = f\left(t, \widehat{z}_t, (K\widehat{z}_t), (W\widehat{z}_t)\right).$$

Thus, $\widehat{z}$ can be presented as

$$\widehat{z}(t) = S_r(t)\left(\phi(0) + \mathcal{E}\left(\widehat{z}_{t_1}, \widehat{z}_{t_2}, \widehat{z}_{t_3}, \dots, \widehat{z}_{t_i}\right)(0)\right) + Q_r(t)z_1$$
$$+ \int_0^t K_r(t-s) \int_0^s h(s, \nu, \widehat{z}_\nu)\, \mathrm{d}\nu\, \mathrm{d}s$$
$$+ \int_0^t K_r(t-s) f\left(s, \widehat{z}_s, (K\widehat{z}_t), (W\widehat{z}_t)\right) \mathrm{d}s + \int_0^t K_r(t-s)B(s)u^0(s)\, \mathrm{d}s$$
$$+ \sum_{j=1}^n S_r(t-t_j)m_j + \sum_{j=1}^n Q_r(t-t_j)\widetilde{m}_j, \quad t \in V.$$

is only a mild solution of equations (5) that corresponds to u^0. Since $C(V, \mathbb{Y}) \to L^1(V, \mathbb{Y})$, using (H8) and Balder's theorem, we obtain

$$\varsigma = \lim_{\ell \to \infty} \int_0^\jmath \mathscr{L}\left(t, z_t^\ell, u^\ell(t)\right) \mathrm{d}t \geqslant \int_0^\jmath \mathscr{L}\left(t, \widehat{z}_t, u^0(t)\right) \mathrm{d}t = \mathcal{J}(\widehat{z}, u^0) \geqslant \varsigma.$$

This shows that $\mathcal{J}$ fulfills its minimum at $(\widehat{z}, u^0) \in C(V, \mathbb{Y}) \times \lambda_{\mathrm{ad}}$. The proof is finished. □

5 Application

Derive the fractional delay integrodifferential equations with impulses of the form

$$\begin{aligned}
\frac{\partial^r}{\partial t^r} z(t,\rho) &= \frac{\partial^2}{\partial \rho^2} z(t,\rho) + \int_0^t \int_{-\infty}^s \beta(s-\nu) M\bigl(z(\nu,\rho)\bigr)\,\mathrm{d}\nu\,\mathrm{d}s + \int_0^1 \mathcal{G}(\rho,s) u(s,t)\,\mathrm{d}s \\
&\quad + \widetilde{f}\Biggl(t, \int_{-\infty}^t \xi_1(s-t) z(s,\rho)\,\mathrm{d}s, \int_0^t \int_{-\infty}^s \xi_2(s,\,\rho,\,\iota-s) z(\iota,\rho)\,\mathrm{d}\iota\,\mathrm{d}s, \\
&\qquad \int_0^t \int_{-\infty}^s \xi_3(s,\,\rho,\,\iota-s) z(\iota,\rho)\,\mathrm{d}\iota\,\mathrm{d}s\Biggr), \qquad (6)\\
&\quad t \in V = [0,1],\ \rho \in [0,\pi],\ t \neq t_j,\ j = 1,2,\dots,n, \\
z(t,0) &= z(t,\pi) = 0, \quad t \in V, \\
z\bigl(t_j^+,\rho\bigr) - z\bigl(t_j^-,\rho\bigr) &= m_j, \quad z'\bigl(t_j^+,\rho\bigr) - z'\bigl(t_j^-,\rho\bigr) = \widetilde{m}_j, \quad j = 1,2,\dots,n, \\
z(t,\rho) &= \phi(t,\rho) + \sum_{\jmath=0}^{i} M_\jmath z(t_\jmath + \rho), \quad t \in (-\infty, 0], \\
z'(0,\rho) &= z_1(\rho), \quad \rho \in [0,\pi],
\end{aligned}$$

where $\partial^{3/2}/\partial t^{3/2}$ represents the fractional partial derivative of order $r = 3/2$. Consider $0 < t_1 < t_2 < \cdots < t_n < \jmath$, $n \in \mathbb{N}$, $0 = t_1 < t_1 < t_2 < \cdots < t_j < t_{j+1} = \jmath$ are prefixed points; $z(t_j^+) = \lim_{(\epsilon^+,\rho)\to(0^+,\rho)} z(t_j+\epsilon,\rho)$ and $z(t_j^-) = \lim_{(\epsilon^-,\rho)\to(0^-,\rho)} z(t_j+\epsilon,\rho)$; $q : [0,\pi] \times [0,1] \to \mathbb{R}$ is continuous, $u \in L^2(V \times [0,\pi])$.

Let $\mathbb{Y} = \mathcal{X} = L^2([0,\pi])$, and consider $A : D(A) \subset \mathbb{Y} \to \mathbb{Y}$ given by $Az = z''$ with domain $D(A)$, which is

$$D(A) = \bigl\{ z \in \mathbb{Y}\colon\ z, z' \text{ are absolutely continuous},\ z'' \in \mathbb{Y},\ z(0) = z(\pi) = 0 \bigr\}.$$

It is well known that A denotes infinitesimal generator of an analytic semigroup $\{K(t),\ t \geqslant 0\}$ on $\mathbb{Y}$. In addition, A has discrete spectrum with eigenvalues $-\hbar^2$, $\hbar$ in $\mathbb{N}$ and corresponding normalized eigenfunctions given by $y_\hbar(z) = \sqrt{2/\pi}\sin(\hbar\pi z)$. Then $y_\hbar$ is an orthonormal basic of $\mathbb{Y}$ (for more details, we refer to [20]),

$$K(t) = \sum_{\hbar=1}^{\infty} \mathrm{e}^{-\hbar^2 t} \langle z, y_\hbar \rangle y_\hbar, \quad z \in \mathbb{Y},$$

$K(t)$ is compact for all $t > 0$, and $K(t) \leqslant \mathrm{e}^{-t}$ for every $t \geqslant 0$ [23] based on this representation.

$A = \partial^2/\partial\rho^2$ represents sectorial operator of type (P, κ, r, γ) and produces compact r-resolvent operators $S_r(t)$, $Q_r(t)$, and $K_r(t)$ for $t \geqslant 0$. Since $A = \partial^2/\partial\rho^2$ is an m-

accretive operator on $\mathbb{Y}$ with (H1) satisfied,

$$Az = \sum_{\hbar=1}^{\infty} \hbar^2 \langle z, y_\hbar \rangle y_\hbar, \quad z \in D(A).$$

The controls are functions $u : Kz([0,\pi]) \to \mathbb{R}$ such that $u \in L^2(Kz([0,\pi]))$. This claim is that $t \to u(\cdot,t)$ going from V into $\mathcal{X}$ is measurable. Set $\lambda(t) = \{u \in \mathcal{X}: \|u\|_{\mathcal{X}} \leqslant \vartheta\}$, where $\vartheta \in L^2(V, \mathbb{R}^+)$. We restrict the admissible controls λ_{ad} to be all $u \in L^2(Kz([0,\pi]))$ such that $\|u(\cdot,t)\|_{L^2([0,\pi])} \leqslant \vartheta$ a.e.

For phase space, we take $g = \mathrm{e}^{2s}$, $s < 0$, in addition $l = \int_{-\infty}^0 g(s)\,\mathrm{d}s = 1/2 < \infty$, for every $t \in (-\infty, 0]$,

$$\|\phi\|_{\mathcal{Q}_g} = \int\limits_{-\infty}^{0} g(s) \sup_{s \leqslant \theta \leqslant 0} \big\|\phi(\theta)\big\|_{L^2}\,\mathrm{d}s.$$

Define

$$z(t)(\rho) = z(t,\rho),$$

$$(Hs)(\rho) = \int\limits_0^t h(t,\varpi,s)(\rho)\,\mathrm{d}\varpi = \int\limits_0^t \int\limits_{-\infty}^0 \beta(\varpi - \hbar) M\big(\phi(\iota)(\varpi)\big)\,\mathrm{d}\iota\,\mathrm{d}\varpi,$$

$$k(t,\theta)(\rho) = \int\limits_{-\infty}^t \xi_2(t,\rho,s)\theta(s)(\rho)\,\mathrm{d}s, \qquad w(t,\theta)(\rho) = \int\limits_{-\infty}^t \xi_3(t,\rho,s)\theta(s)(\rho)\,\mathrm{d}s,$$

$$f\Bigg(t,\, \theta,\, \int\limits_0^t k(s,\theta)\,\mathrm{d}s,\, \int\limits_0^t w(s,\theta)\,\mathrm{d}s\Bigg)$$
$$= \widetilde{f}\Bigg(t,\, \int\limits_{-\infty}^0 \xi_1(s)\theta(s)(\rho)\,\mathrm{d}s,\, \int\limits_0^t k(s,\theta)(\rho)\,\mathrm{d}s,\, \int\limits_0^t w(s,\theta)(\rho)\,\mathrm{d}s\Bigg),$$

$$\mathcal{E}(z_{t_1}, z_{t_2}, z_{t_3}, \dots, z_{t_i})(\rho) = \sum_{\mathfrak{z}=0}^{i} M_{\mathfrak{z}} z(t_{\mathfrak{z}} + \rho), \qquad B(t)u(t)\rho = \int\limits_0^1 \mathcal{G}(\rho,s)u(s,t)\,\mathrm{d}s.$$

Then system (1) can be abstracted as problem (6).

Suppose

$$\big|(Hs)(\rho)\big|_{L^2} = \left[\int\limits_0^{\pi} \Bigg(\int\limits_0^t \int\limits_{-\infty}^0 \beta(\nu - \iota) M\big(\phi(\iota)(\rho)\big)\,\mathrm{d}\iota\,\mathrm{d}s\Bigg)^2 \mathrm{d}\rho\right]^{1/2}$$
$$\leqslant \left[\int\limits_0^{\pi} \Bigg(\int\limits_0^t \int\limits_{-\infty}^0 \beta(\nu - \iota) \wp\Bigg(\int\limits_{-\infty}^0 \mathrm{e}^{2s} \big\|\phi(s)(\cdot)\big\|_{L^2}\,\mathrm{d}s\Bigg)\,\mathrm{d}\iota\,\mathrm{d}\nu\Bigg)^2 \mathrm{d}\rho\right]^{1/2}$$

$$= \left[\int_0^{\pi} \left(\int_0^{t} \int_{-\infty}^{0} \beta(\nu - \iota)\, \mathrm{d}\iota\, \mathrm{d}s \right)^2 \mathrm{d}\rho \right]^{1/2} \wp\big(\|\phi\|_{\mathcal{Q}_g}\big)$$

$$= \left[\int_0^{\pi} (u^*(t))^2\, \mathrm{d}\rho \right]^{1/2} \wp\big(\|\phi\|_{\mathcal{Q}_g}\big) = \sqrt{\pi}\,\overline{u}(t)\wp\big(\|\phi\|_{\mathcal{Q}_g}\big).$$

Hence, $\wp : [0, +\infty) \to (0, +\infty)$ is continuous increasing function, and we can take $w(t) = \overline{u}(t)$ with $c = \sqrt{\pi}$ and $\Pi_1(p) = \wp(p)$ in (H3).

Now, consider the following cost function:

$$\mathcal{J}(u) = \int_0^{\jmath} \mathscr{L}\big(t, z_t^u, u(t)\big)\, \mathrm{d}t,$$

where

$$\mathscr{L}\big(t, z_t^u, u(t)\big)(\rho) = \int_0^{\pi} \int_{-\infty}^{0} \big|z^u(t+s,\, \rho)\big|^2\, \mathrm{d}s\, \mathrm{d}\rho + \int_0^{1} \int_0^{\pi} \big|z(t,\rho)\big|^2\, \mathrm{d}\rho\, \mathrm{d}t + \int_0^{1} \int_0^{\pi} \big|u(t,\rho)\big|^2\, \mathrm{d}\rho\, \mathrm{d}t.$$

It is easy to see that all the assumptions in Theorem 5 are satisfied: therefore, problem (6) has at least one optimal pair.

6 Conclusion

In this paper, we have investigated the existence results for impulsive fractional delay integrodifferential systems of mixed type of order $r \in (1, 2)$ with nonlocal conditions by using the fractional calculations, sectorial operators, fixed point approach, and appropriate analysis. Furthermore, we developed the optimal control results for the given systems. Then an example of the theory explaining the significant findings is developed. In the future, we will discuss the null controllability results for fractional stochastic differential systems of order $r \in (1, 2)$ with delay. Then we will extend to the relative controllability results for the Hilfer fractional differential systems with impulses by using semigroup theory, measure of noncompactness, multivalued map, and mild solutions.

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