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Compactness in Groups of Group-Valued Mappings

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Abstract: We introduce the concepts of extended equimeasurability and extended uniform quasiboundedness in groups of group-valued mappings endowed with a topology that generalizes the topology of convergence in measure. Quantitative characteristics modeled on these concepts allow us to estimate the Hausdorff measure of noncompactness in such a contest. Our results extend and encompass some generalizations of Fréchet–Šmulian and Ascoli–Arzelà compactness criteria found in the literature.

Keywords: group; pseudonorm; convergence (and local convergence) in measure; measure of noncompactness; equimeasurability; uniform quasiboundedness

MSC: 54D30; 54C35; 47H08

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1. Introduction

Let us start by recalling the Fréchet–Šmulian criterion of compactness. Consider a Lebesgue-measurable subset E of $\mathbb{R}^n$, the σ-algebra Σ of all Lebesgue-measurable subsets of E, and the Lebesgue measure λ. The criterion (Fréchet [1] if $\lambda(E) < +\infty$, and Šmulian [2] if $\lambda(E) = +\infty$) states that a subset M of the space of all real-valued Lebesgue totally measurable functions defined on E is relatively compact with respect to convergence in measure if and only if given $\varepsilon > 0$ there is a finite partition $\{A_1, \ldots, A_n\}$ of E in Σ, a number $a > 0$ and , for each $f \in M$, there is a set D_f in Σ with $\lambda(D_f) \leq \varepsilon$ such that $\sup\{|f(x) - f(y)| : x, y \in A_i \setminus D_f\} \leq \varepsilon$ for $i = 1, \ldots, n$ and $\sup\{|f(x)| : x \in E \setminus D_f\} \leq a$. In other words, M is relatively compact with respect to convergence in measure if and only if it is equimeasurable and uniformly quasibounded. In the literature, the introduction of quantitative characteristics measuring the lack of the above two properties has allowed many authors to obtain inequalities that estimate the classical, Hausdorff or Kuratowski, measures of noncompactness and include the Fréchet–Šmulian criterion and its extensions to more general spaces of functions (see, for example, Refs. [3–8]). Following such an approach, the aim of this paper is to obtain quantitative versions of theorems about compactness in pseudonormed groups of mappings defined on a given set Ω with values in a normed group G. The paper is organized as follows. In Section 2, we introduce notations, definitions, and preliminary facts that are used throughout the paper. Given a submeasure μ defined on an algebra $\mathcal{A}$ in the power set of Ω and taking values in $[0, +\infty]$, we consider the group $(\mathcal{F}(\Omega), \|\cdot\|_\eta)$ of all G-valued mappings defined on Ω with the topology generated by the group pseudonorm

$$\|f\|_\eta = \inf\{a > 0 : \eta(\{x \in \Omega : \|f(x)\| \geq a\}) \leq a\},$$

where η is obtained by extending μ to the power set of Ω. We introduce new equimeasurability-type and uniform quasiboundedness-type concepts for subsets M of a given subgroup $(\mathcal{H}(\Omega), \|\cdot\|_\eta)$ of $(\mathcal{F}(\Omega), \|\cdot\|_\eta)$, and associate to them the quantitative characteristics

$\tilde{\omega}_{\mathcal{H}_\eta}(M)$ and $\tilde{\sigma}_{\mathcal{H}_\eta}(M)$, respectively. In Section 3 the Hausdorff measure of noncompactness is estimated in our general setting by means of the new quantitative characteristics. We prove the following inequalities:

$$\max\left\{\tilde{\sigma}_{\mathcal{H}_\eta}(M), \frac{1}{2}\tilde{\omega}_{\mathcal{H}_\eta}(M)\right\} \le \gamma_{\mathcal{H}_\eta}(M) \le 4(\tilde{\sigma}_{\mathcal{H}_\eta}(M) + \tilde{\omega}_{\mathcal{H}_\eta}(M)), \tag{1}$$

which contains a Fréchet–Šmulian-type compactness criterion in the group $(\mathcal{H}(\Omega), \|\cdot\|_\eta)$. Moreover, we show that in the group of totally $\mathcal{A}$-measurable mappings (see Ref. [9]) the quantitative characteristics $\tilde{\omega}_{\mathcal{H}_\eta}$ and $\tilde{\sigma}_{\mathcal{H}_\eta}$ reduce to those introduced in Ref. [4]; thus, our results on compactness extend the analogous ones obtained in Ref. [4]. Then, inequalities (1) are applied to obtain a compactness criterion in a general group $\mathcal{H}(\Omega)$ endowed with the topology of local convergence in measure. In Section 4, we examine the case of groups of G-valued mappings endowed with the standard supremum seminorm $\|\cdot\|_\infty$, which is obtained as a particular case of $\|\cdot\|_\eta$. It is worthwhile mentioning that Nussbaum [6], generalizing a criterion of compactness of Ambrosetti [10], has proved a quantitative version of the Ascoli–Arzelà-type theorem in the space of continuous mappings from a compact metric space Ω into an arbitrary metric space X, obtaining that a bounded subset of that space is relatively compact if and only if it is equicontinuous and pointwise relatively compact. Nussbaum's result has been extended (see Refs. [4,11]) to the space of totally bounded mappings from a general topological space Ω into an arbitrary metric space X. In such a situation, the estimates provide as a special case also the Bartle compactness criterion [12], by virtue of which a bounded subset of the space of real-bounded and continuous functions defined on a topological space Ω is relatively compact if and only if the following condition holds: for any positive ε there is a finite partition $\{A_1, \dots, A_n\}$ of Ω such that if x, y belong to the same A_i, then $|f(x) - f(y)| \le \varepsilon$ for all $f \in M$. On the basis of the above considerations, in Section 4, we estimate, in our general setting, the Hausdorff measure of noncompactness of a given subset M of a group $(\mathcal{H}(\Omega), \|\cdot\|_\infty)$ by means of $\tilde{\omega}_{\mathcal{H}_\infty}(M)$ and the classical quantitative characteristic $\mu_{\gamma_G}(M)$, which is related to pointwise total boundedness. In such a way, we obtain a compactness criterion that generalizes, among others, the compactness criteria we have just mentioned.

2. Preliminaries

Throughout the paper we will only consider commutative additive groups and real linear spaces. We denote by $\mathcal{P}(T)$ the power set of a set T and we assume $\inf\emptyset = +\infty$. Now, if $L = (L, +)$ is a group with zero element θ, a group pseudonorm on L is a mapping

$$\|\cdot\|_L : L \to [0, +\infty]$$

such that $\|\theta\|_L = 0$, $\|-x\|_L = \|x\|_L$ and $\|x + y\|_L \le \|x\|_L + \|y\|_L$, for all $x, y \in L$. A group norm is a group pseudonorm that also satisfies $f = \theta$ if $\|f\|_L = 0$. If L is a pseudonormed group, given a subset X of L, then the symbol $\mathrm{diam}(X)$ stands for the diameter of X. Moreover, given $x \in L$ and $r > 0$, the symbol $B_L(x, r)$ will denote the closed ball centered at x with radius r. Further, we will say that a set function $\nu : \mathcal{P}(L) \to [0, +\infty]$ is a measure of noncompactness in the sense of Ref. [13] (where it is defined on bounded subsets of a complete metric space) if it satisfies the following properties:

(i) $\nu(M) = 0$ if and only if M is totally bounded (regularity);
(ii) $\nu(M) = \nu(\overline{M})$ (invariance under closure);
(iii) $\nu(M \cup N) = \max\{\nu(M), \nu(N)\}$ (semi-additivity).

The following properties can be deduced by these axioms:

(iv) $M \subseteq N$ implies $\nu(M) \le \nu(N)$ (monotonicity);
(v) $\nu(S) = 0$ for every one-element set S in L (non-singularity).

Moreover, we require, having in mind that the pseudonorm group L is assumed to be additive, the following additional properties:

(vi) $\nu(M+N) \le \nu(M) + \nu(N)$ (algebraic semi-additivity);
(vii) $\nu(f+M) = \nu(M)$ for any $f \in L$ (invariance under translations).

We recall that for a subset M of L, the Hausdorff measure of noncompactness $\gamma_L(M)$ of M is the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in L, i.e.,

$$\gamma_L(M) = \inf\{\varepsilon > 0 : \ M \subseteq F + B_L(\theta, \varepsilon), \ F \subseteq L, \ F \text{ finite}\}.$$

For more details on measures of noncompactness, we refer to Ref. [13] and also Ref. [14]. In the following, we assume $G = (G, \|\cdot\|)$ to be a normed group. If Ω is a nonempty set, $\mathcal{F}(\Omega) = \mathcal{F}(\Omega, G)$ will denote the group of all G-valued mappings defined on Ω, and $\mathcal{A}$ will be an algebra in $\mathcal{P}(\Omega)$. For $f \in \mathcal{F}(\Omega)$ and $A \in \mathcal{A}$, we define $f\chi_A : \Omega \to G$ by setting $f\chi_A(x) = f(x)$ if $x \in A$ and $f\chi_A(x) = \theta$ if $x \in \Omega \setminus A$. A mapping $s \in \mathcal{F}(\Omega)$ is called $\mathcal{A}$-simple if there are $z_1, ..., z_n \in G$ such that $s(\Omega) = \{z_1, \ldots, z_n\}$ and $s^{-1}(z_i) \in \mathcal{A}$, for $i = 1, ..., n$. We denote by $\mathcal{S}(\Omega) = \mathcal{S}(\Omega, G)$ the group of all $\mathcal{A}$-simple mappings. Now, let $\mu : \mathcal{A} \to [0, +\infty]$ be a submeasure (i.e., a monotone, subadditive function with $\mu(\emptyset) = 0$) and $\eta : \mathcal{P}(\Omega) \to [0, +\infty]$ be the submeasure defined by

$$\eta(B) = \inf\{\mu(A) : \ A \in \mathcal{A} \text{ and } B \subseteq A\}. \tag{2}$$

Then, we consider a natural generalization of the topology of convergence in measure (see ref. [7]); that is, the topology generated on the group $\mathcal{F}(\Omega)$ by the group pseudonorm $\|\cdot\|_\eta : \mathcal{F}(\Omega) \to [0, +\infty]$ defined by

$$\|f\|_\eta = \inf\{a > 0 : \eta(\{x \in \Omega : \|f(x)\| \ge a\}) \le a\}.$$

We will use the notation $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$ for the pseudonormed group $(\mathcal{F}(\Omega), \|\cdot\|_\eta)$. Throughout, $\mathcal{H}(\Omega)$ will stand for a subgroup of $\mathcal{F}(\Omega)$, possibly the group $\mathcal{F}(\Omega)$ itself. In particular, $\mathcal{TM}(\Omega) = \mathcal{TM}(\Omega, \mathcal{A}, \eta, G)$ will denote the subgroup of all totally $\mathcal{A}$-measurable mappings; that is, the closure in $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$ of the group $\mathcal{S}(\Omega)$ of all $\mathcal{A}$- simple mappings (Ref. [9]). Finally, we will write $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$ for $(\mathcal{H}(\Omega), \|\cdot\|_\eta)$. Now, we introduce new equimeasurability-type and uniform quasiboundedness-type concepts in our general setting. To this end, for M in $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$ and $\varepsilon > 0$, we denote by $\Phi_{\eta,\varepsilon}(M)$ the set of all multimappings $\varphi : \ M \to \mathcal{P}(\Omega)$ such that $\eta(\varphi(f)) \le \varepsilon$ for all $f \in M$.

Definition 1. *A subset M of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$ is said to be extended equimeasurable if for any $\varepsilon > 0$ there are a finite partition $\{A_1, \ldots, A_n\}$ of Ω in $\mathcal{A}$, a finite set $\{f_1, \ldots, f_m\}$ in $\mathcal{H}(\Omega)$ and a multimapping $\varphi \in \Phi_{\eta,\varepsilon}(M)$ such that, for all $f \in M$, there is $j \in \{1, \ldots, m\}$.*

$$\operatorname{diam}\big((f - f_j)(A_i \setminus \varphi(f))\big) \le \varepsilon, \quad \textit{for } i = 1, \ldots, n.$$

A subset M of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$ is said to be extended uniformly quasibounded if for any $\varepsilon > 0$ there are a set G_0 in G with $\gamma(G_0) \le \varepsilon$, a finite set $\{g_1, \ldots, g_k\}$ in $\mathcal{H}(\Omega)$ and a multimapping $\psi \in \Phi_{\eta,\varepsilon}(M)$ such that, for all $f \in M$, there is $s \in \{1, \ldots, k\}$ with

$$(f - g_s)(\Omega \setminus \psi(f)) \subseteq G_0.$$

Now, apart from the Hausdorff measure of noncompactness in $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$, which we will simply denote by $\gamma_{\mathcal{H}_\eta}$, we consider the set quantitative characteristics $\tilde{\omega}_{\mathcal{H}_\eta}, \tilde{\sigma}_{\mathcal{H}_\eta}$: $\mathcal{P}(\mathcal{H}_{\|\cdot\|_\eta}(\Omega)) \to [0, +\infty]$, defined by setting

$$\begin{aligned} \tilde{\omega}_{\mathcal{H}_\eta}(M) = \inf\{\varepsilon > 0 : \ & \text{there are a finite partition } \{A_1, \ldots, A_n\} \text{ of } \Omega \text{ in } \mathcal{A}, \text{ a finite set} \\ & \{f_1, \ldots, f_m\} \text{ in } \mathcal{H}(\Omega) \text{ and a multimapping } \varphi \in \Phi_{\eta,\varepsilon}(M) \text{ such that,} \\ & \text{for all } f \in M, \text{ there is } j \in \{1, \ldots, m\} \text{ with } \operatorname{diam}((f - f_j)(A_i \setminus \varphi(f))) \le \varepsilon \\ & \text{for } i = 1, \ldots, n\}. \end{aligned} \tag{3}$$

$$\tilde{\sigma}_{\mathcal{H}_\eta}(M) = \inf\{\varepsilon > 0 : \text{ there are a set } G_0 \text{ in } G \text{ with } \gamma(G_0) \leq \varepsilon, \text{ a finite set } \{g_1, \ldots, g_k\} \text{ in } \mathcal{H}(\Omega) \text{ and a multimapping } \psi \in \Phi_{\eta,\varepsilon}(M) \text{ such that, for all } f \in M, \text{ there is } s \in \{1, \ldots, k\} \text{ with } (f - g_s)(\Omega \setminus \psi(f)) \subseteq G_0\}. \quad (4)$$

In such a way, a subset M of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$ is extended equimeasurable if and only if $\tilde{\omega}_{\mathcal{H}}(M) = 0$, and extended uniformly quasibounded if and only if $\tilde{\sigma}_{\mathcal{H}}(M) = 0$.

Remark 1. *It is worth noting for the sequel that the quantitative characteristics $\tilde{\omega}_{\mathcal{H}_\eta}$ and $\tilde{\sigma}_{\mathcal{H}_\eta}$ satisfy conditions (ii)–(vii) of a measure of noncompactness.*

3. Compactness in Pseudonormed Subgroups of $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$

The main result of this section provides estimates for the Hausdorff measure of noncompactness in terms of the simpler quantitative characteristics we have introduced.

Theorem 1. *Let M be a subset of $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$, then*

$$\gamma_{\mathcal{F}_\eta}(M) \leq 2(\tilde{\sigma}_{\mathcal{F}_\eta}(M) + \tilde{\omega}_{\mathcal{F}_\eta}(M)). \quad (5)$$

Let M be a subset of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$, then

$$\max\left\{\tilde{\sigma}_{\mathcal{H}_\eta}(M), \frac{1}{2}\tilde{\omega}_{\mathcal{H}_\eta}(M)\right\} \leq \gamma_{\mathcal{H}_\eta}(M) \leq 4(\tilde{\sigma}_{\mathcal{H}_\eta}(M) + \tilde{\omega}_{\mathcal{H}_\eta}(M)). \quad (6)$$

Proof. Let us prove (5). Let M be a subset of $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$. Since $\gamma_{\mathcal{F}_\eta}(M) \leq \eta(\Omega)$, if either $\tilde{\sigma}_{\mathcal{F}_\eta}(M) = \eta(\Omega)$ or $\tilde{\omega}_{\mathcal{F}_\eta}(M) = \eta(\Omega)$, the inequality holds true. Assume $\tilde{\sigma}_{\mathcal{F}_\eta}(M) < \eta(\Omega)$ and $\tilde{\omega}_{\mathcal{F}_\eta}(M) < \eta(\Omega)$. Let $a > \tilde{\omega}_{\mathcal{F}_\eta}(M)$ and $b > \tilde{\sigma}_{\mathcal{F}_\eta}(M)$. Then, choose a finite partition $\{A_1, \ldots, A_n\}$ of Ω in $\mathcal{A}$, a finite set $\{f_1, \ldots, f_m\}$ in $\mathcal{F}(\Omega)$ and a multimapping $\varphi \in \Phi_{\eta,a}(M)$ such that, for all $f \in M$, there is $j \in \{1, \ldots, m\}$ with $\text{diam}((f - f_j)(A_i \setminus \varphi(f))) \leq a$ for $i = 1, \ldots, n$. Choose also a set G_0 in G with $\gamma_G(G_0) \leq b$, a finite set $\{g_1, \ldots, g_k\}$ in $\mathcal{F}(\Omega)$ and a multimapping $\psi \in \Phi_{\eta,b}(M)$ such that, for all $f \in M$, there is $s \in \{1, \ldots, k\}$ with $(f - g_s)(\Omega \setminus \psi(f)) \subseteq G_0$. Moreover, let $\{z_1, \ldots, z_\ell\}$ be a $\|\cdot\|$-b-net for G_0 in G. Now, set

$$M_j = \{f \in M : \text{diam}((f - f_j)(A_i \setminus \varphi(f))) \leq a, \text{ for } i = 1, \ldots, n\}$$

and

$$M_{j,s} = \{f \in M_j : \ (f - g_s)(\Omega \setminus \psi(f)) \subseteq G_0\},$$

for all $j \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, k\}$, so that $M = \bigcup_{j=1}^m M_j = \bigcup_{j=1}^m \bigcup_{s=1}^k M_{j,s}$. Then, fix a mapping $f_{j,s} \in M_{j,s}$, and denote by $T_{j,s}$ the finite set of all mappings $h : \Omega \to G$, defined as follows

$$h(x) = f_{j,s}(x) + z_{p_i} - z_{q_i} \text{ if } x \in A_i, \quad \text{for } i = 1, \ldots, n,$$

where the $2n$-tuples $z_{p_1}, \ldots, z_{p_n}, z_{q_1}, \ldots z_{q_n}$ vary in $\{z_1, \ldots, z_\ell\}$. We will show that, for all $j \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, k\}$, the set $T_{j,s}$ is a finite $\|\cdot\|_\eta$- $2(a+b)$-net for $M_{j,s}$ in $\mathcal{F}(\Omega)$. To this end, let $f \in M_{j,s}$ be arbitrarily fixed and set $D_f = \varphi(f) \cup \psi(f)$; then, fix $x_i \in A_i$, for $i \in \{1, \ldots, n\}$ such that $x_i \notin D_f$ if $A_i \setminus D_f \neq \emptyset$. Further fix z_{p_i} such that $\|(f - g_s)(x_i) - z_{p_i}\| \leq b$, and z_{q_i} such that $\|(f_{j,s} - g_s)(x_i) - z_{q_i}\| \leq b$. Finally, define the mapping $h_f : \Omega \to G$ by setting

$$h_f(x) = \begin{cases} f_{j,s}(x) + z_{p_i} - z_{q_i} & \text{if } x \in A_i \text{ and } A_i \setminus D_f \neq \emptyset \\ f_{j,s}(x) & \text{if } x \in A_i \text{ and } A_i \setminus D_f = \emptyset \end{cases} \quad \text{for } i = 1, \ldots, n.$$

Then, $h_f \in T_{j,s}$. Moreover, for $x \in A_i \setminus D_f$ we have

$$\begin{aligned}\|f(x) - h_f(x)\| &= \|f(x) - f_{j,s}(x) - z_{p_i} + z_{q_i}\| \\ &= \|f(x) - f_{j,s}(x) - f_{j,s}(x_i) + f_{j,s}(x_i) - f(x_i) + f(x_i) - z_{p_i} \\ &\quad + z_{q_i} + g_s(x_i) - g_s(x_i)\| \\ &\leq \|(f - f_{j,s})(x) - (f - f_{j,s})(x_i)\| + \|(f - g_s)(x_i) - z_{p_i}\| \\ &\quad + \|(f_{j,s} - g_s)(x_i) - z_{q_i}\| \\ &\leq \|(f - f_j)(x) - (f - f_j)(x_i)\| + \|(f_{j,s} - f_j)(x) - (f_{j,s} - f_j)(x_i)\| + 2b \\ &\leq 2a + 2b.\end{aligned}$$

Since $D_f = \varphi(f) \cup \psi(f)$, with $\varphi \in \Phi_{\eta,a}(M)$ and $\psi \in \Phi_{\eta,b}(M)$, we have $\eta(D_f) \leq a + b$. Therefore, we find $\|f - h_f\|_\eta \leq 2(a + b)$. Now, having in mind $M = \bigcup_{j=1}^m \bigcup_{s=1}^k M_{j,s}$, we find $\gamma_{\mathcal{F}_\eta}(M) = \max_{j=1}^m \max_{s=1}^k \gamma_{\mathcal{F}_\eta}(M_{j,s}) \leq 2(a + b)$, and by the arbitrariness of a and b we obtain $\gamma_{\mathcal{F}_\eta}(M) \leq 2(\tilde{\sigma}_{\mathcal{F}_\eta}(M) + \tilde{\omega}_{\mathcal{F}_\eta}(M))$, as desired.

Now, we prove (6). Let M be a subset of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$. The right inequality follows from (5) taking into account that $\gamma_{\mathcal{H}_\eta}(M) \leq 2\gamma_{\mathcal{F}_\eta}(M)$, $\tilde{\omega}_{\mathcal{F}_\eta}(M) \leq \tilde{\omega}_{\mathcal{H}_\eta}(M)$, and $\tilde{\sigma}_{\mathcal{F}_\eta}(M) \leq \tilde{\sigma}_{\mathcal{H}_\eta}(M)$. Now, we prove the left inequality. Since $\tilde{\sigma}_{\mathcal{H}_\eta}(M) \leq \eta(\Omega)$ and $\tilde{\omega}_{\mathcal{H}_\eta}(M) \leq \eta(\Omega)$, the inequality is true if $\gamma_{\mathcal{H}_\eta}(M) = \eta(\Omega)$. Assume $\gamma_{\mathcal{H}_\eta}(M) < \eta(\Omega)$. Let $a > \gamma_{\mathcal{H}_\eta}(M)$ and let $\{f_1, \ldots, f_m\}$ be a $\|\cdot\|_\eta$-a-net for M in $\mathcal{H}(\Omega)$. For $f \in M$, choose $j \in \{1, \ldots, m\}$, such that $\|f - f_j\|_\eta \leq a$, and set $D_f = \{x \in \Omega : \|f(x) - f_j(x)\| > a\}$. Then, by the definition of $\|\cdot\|_\eta$, we have $\eta(D_f) \leq a$. Hence, the multimapping $\psi : M \to \mathcal{P}(\Omega)$, which is defined for each $f \in M$ by $\psi(f) = D_f$, belongs to $\Phi_{\eta,a}(M)$. Then, on the one hand, since for all $x \in \Omega \setminus \psi(f)$ we have $\|f(x) - f_j(x)\| \leq a$, it follows that

$$(f - f_j)(\Omega \setminus \psi(f)) \subseteq B_G(\theta, a).$$

Choosing $G_0 = B_G(\theta, a)$, $\{f_1, \ldots, f_m\}$ as a finite set in $\mathcal{H}(\Omega)$ and ψ as a multimapping in $\Phi_{\eta,a}(M)$, we find $\tilde{\sigma}_{\mathcal{H}_\eta}(M) \leq a$.Hence, the arbitrariness of a implies $\tilde{\sigma}_{\mathcal{H}_\eta}(M) \leq \gamma_{\mathcal{H}_\eta}(M)$. On the other hand, considering $\{\Omega\}$ as a partition of Ω in $\mathcal{A}$, $\{f_1, \ldots, f_m\}$ as a finite set in $\mathcal{H}(\Omega)$ and ψ in $\Phi_{\eta,a}(M)$, we have

$$\operatorname{diam}((f - f_j)(\Omega \setminus \psi(f)) = \sup_{x,y \in \Omega \setminus \psi(f)} \|(f - f_j)(x) - (f - f_j)(y)\| \leq 2a,$$

which gives $\tilde{\omega}_{\mathcal{H}_\eta}(M) \leq 2\gamma_{\mathcal{H}_\eta}(M)$. The proof is completed. □

As a corollary, we obtain the following Fréchet–Šmulian-type compactness criterion.

Corollary 1. *A subset M of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$ is totally bounded if and only*

$$\tilde{\omega}_{\mathcal{H}_\eta}(M) = \tilde{\sigma}_{\mathcal{H}_\eta}(M) = 0;$$

that is, if and only if M is extended equimeasurable and uniformly quasibounded.

The above corollary says that $\tilde{\sigma}_{\mathcal{H}_\eta} + \tilde{\omega}_{\mathcal{H}_\eta}$ satisfies property (i) of a measure of noncompactness, which together with Remark 1 gives that $\tilde{\sigma}_{\mathcal{H}_\eta} + \tilde{\omega}_{\mathcal{H}_\eta}$ is indeed a measure of noncompactness in $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$. Moreover, due to the inequalities (6), we have that it is equivalent to the Hausdorff measure of noncompactness. Now, notice that Theorem 1 provides estimates for the Hausdorff measure of noncompactness in any given group in $\mathcal{F}_{\|\cdot\|_\eta}(\Omega)$. While in Ref. [4], Theorem 2.1, analogous estimates have been proved in the group $\mathcal{TM}_{\|\cdot\|_\eta}(\Omega)$ of totally $\mathcal{A}$-measurable mappings (denoted, in Ref. [4], by $L_0(\Omega, \mathcal{A}, \eta, G)$) using the quantitative characteristics $\omega(M)$ and $\sigma(M)$, defined as follows

$$\omega(M) = \inf\{\varepsilon > 0 : \text{ there are a finite partition } \{A_1, \ldots, A_n\} \text{ of } \Omega \text{ in } \mathcal{A} \text{ and a multimapping } \varphi \in \Phi_{\eta,\varepsilon}(M) \text{ such that, for all } f \in M, \operatorname{diam}(f(A_i \setminus \varphi(f))) \leq \varepsilon \text{ for } i = 1, \ldots, n\},$$

$$\sigma(M) = \inf\{\varepsilon > 0 : \text{ there are a set } G_0 \text{ in } G \text{ with } \gamma_G(G_0) \leq \varepsilon \text{ and a multimapping } \psi \in \Phi_{\eta,\varepsilon}(M) \text{ such that, for all } f \in M,\ f(\Omega \setminus \psi(f)) \subseteq G_0\}.$$

We observe that, on the one hand, the quantitative characteristics $\omega(M)$ and $\sigma(M)$ do not allow us to estimate the Hausdorff measure of noncompactness when M is not a subset of $\mathcal{TM}_{\|\cdot\|_\eta}(\Omega)$. To see this, it is enough to consider M as a singleton set whose element is a not-totally $\mathcal{A}$-measurable mapping. On the other hand, we have that the results on compactness of Ref. [4] can be seen as a particular case of Corollary 3, since the following proposition proves that in $\mathcal{TM}_{\|\cdot\|_\eta}(\Omega)$ the quantitative characteristics we have introduced reduce to those of Ref. [4].

Proposition 1. *Let $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$ be a group in $\mathcal{TM}_{\|\cdot\|_\eta}(\Omega)$. Then, for every subset M of $\mathcal{H}_{\|\cdot\|_\eta}(\Omega)$, we have $\tilde{\omega}_{\mathcal{H}_\eta}(M) = \omega(M)$ and $\tilde{\sigma}_{\mathcal{H}_\eta}(M) = \sigma(M)$.*

Proof. Let f_0 denote the null mapping in $\mathcal{H}(\Omega)$. Choosing $\{f_0\}$ as a finite subset of $\mathcal{H}(\Omega)$ in both the definitions of $\tilde{\sigma}_{\mathcal{H}_\eta}(M)$ and $\tilde{\omega}_{\mathcal{H}_\eta}(M)$, we find $\tilde{\sigma}_{\mathcal{H}_\eta}(M) \leq \sigma(M)$ and $\tilde{\omega}_{\mathcal{H}_\eta}(M) \leq \omega(M)$. Now, we prove the reverse inequalities. Since $\sigma(\Omega) \leq \eta(\Omega)$ and $\omega(\Omega) \leq \eta(\Omega)$, the inequalities will hold true, respectively, if either $\tilde{\sigma}_{\mathcal{H}_\eta}(M) = \eta(\Omega)$ or $\tilde{\omega}_{\mathcal{H}_\eta}(M) = \eta(\Omega)$. Therefore, we assume $\tilde{\sigma}_{\mathcal{H}_\eta}(M) < \eta(\Omega)$ and $\tilde{\omega}_{\mathcal{H}_\eta}(M) < \eta(\Omega)$. At first, let $a > \tilde{\sigma}_{\mathcal{H}_\eta}(M)$, let $G_0 \subseteq G$ with $\gamma(G_0) < a$, $\psi \in \Phi_{\eta,a}(M)$ and let $\{g_1, \ldots, g_k\}$ be a finite set in $\mathcal{H}(\Omega)$ such that, for all $f \in M$, there is $s \in \{1, \ldots, k\}$ such that $(f - g_s)(\Omega \setminus \psi(f)) \subseteq G_0$. Next, given $\delta > 0$, choose $\mathcal{A}$-simple mappings $s_1, \ldots s_k \in S(\Omega)$ such that $\|g_s - s_s\|_\eta \leq \delta$. Define the multimapping $\bar{\psi} : M \to \mathcal{P}(\Omega)$ by setting

$$\bar{\psi}(f) = \psi(f) \cup \{x \in \Omega : \|g_s(x) - s_s(x)\| > \delta\}.$$

Then, since $\eta(\bar{\psi}(f)) \leq \eta(\psi(f)) + \eta(\{x \in \Omega : \|g_s(x) - s_s(x)\| > \delta\}) \leq a + \delta$ for all $f \in M$, we have $\bar{\psi} \in \Phi_{\eta,a+\delta}(M)$. Now, for $f \in M$, choose $s \in \{1, \ldots, k\}$ such that $(f - g_s)(\Omega \setminus \psi(f)) \subseteq G_0$. Then, for all $x \in \Omega \setminus \bar{\psi}(f)$, we have

$$f(x) - s_s(x) = f(x) - g_s(x) + g_s(x) - s_s(x) \in G_0 + B_G(\theta, \delta).$$

Therefore, $f(x) \in s_s(x) + G_0 + B_G(\theta, \delta)$. Setting $\bar{G} = \bigcup_{s=1}^k (s_s(x) + G_0 + B_G(\theta, \delta))$, we have $\gamma(\bar{G}) \leq a + \delta$ and $f(\Omega \setminus \bar{\psi}(f)) \subseteq \bar{G}$, for all $f \in M$. By the arbitrariness of a and δ, we obtain $\sigma(M) \leq \tilde{\sigma}_{\mathcal{H}_\eta}(M)$, as desired.

Now, let $a > \tilde{\omega}_{\mathcal{H}_\eta}(M)$. Choose a finite partition $\{A_1, \ldots, A_n\}$ of Ω in $\mathcal{A}$, a finite set $\{f_1, \ldots, f_m\}$ in $\mathcal{H}(\Omega)$ and a multimapping $\varphi \in \Phi_{\eta,a}(M)$ such that, for all $f \in M$, there is $j \in \{1, \ldots, m\}$ with $\operatorname{diam}((f - f_j)(A_i \setminus \varphi(f))) \leq a$ for $i = 1, \ldots, n$. Given $\delta > 0$, choose $s_1, \ldots, s_m \in S(\Omega)$ such that $\|f_j - s_j\|_\eta \leq \delta$. Let $\{B_1, \ldots, B_k\}$ be a finite partition of Ω in $\mathcal{A}$, such that each restriction $s_j|_{B_p}$ is constant for $j = 1, \ldots, m$, and $p = 1, \ldots, k$, and let $\{C_1, \ldots, C_r\}$ be the finite partition of Ω in $\mathcal{A}$ that is generated by the partitions $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_k\}$. Further, for $f \in M$, fix $j \in \{1, \ldots, m\}$ such that $\operatorname{diam}((f - f_j)(A_i \setminus \varphi(f))) \leq a$ for $i = 1, \ldots, n$, and define the multimapping $\bar{\varphi} : M \to \mathcal{P}(\Omega)$ as follows

$$\bar{\varphi}(f) = \varphi(f) \cup \{x \in \Omega : \|f_j(x) - s_j(x)\| > \delta\}.$$

Then, $\bar{\varphi} \in \Phi_{\eta,a+\delta}(M)$. Consequently, for all $f \in M$, we can choose $j \in \{1, \ldots, m\}$ in such a way to obtain

$$
\begin{aligned}
\operatorname{diam}(f(C_\ell \setminus \bar{\varphi}(f))) &= \sup_{x,y\in C_\ell\setminus\bar{\varphi}(f)} \|f(x)-f(y)\| \\
&\le \sup_{x,y\in C_\ell\setminus\bar{\varphi}(f)} \|(f-f_j)(x)-(f-f_j)(y)\| + \sup_{x\in C_\ell\setminus\bar{\varphi}(f)} \|f_j(x)-s_j(x)\| \\
&+ \sup_{y\in C_\ell\setminus\bar{\varphi}(f)} \|f_j(y)-s_j(y)\| + \sup_{x,y\in C_\ell\setminus\bar{\varphi}(f)} \|s_j(x)-s_j(y)\| \\
&\le a+2\delta,
\end{aligned}
$$

for all $\ell \in \{1,\dots,r\}$. Therefore, $\omega(M) \le a+2\delta$. By virtue of the arbitrariness of a and δ, we obtain $\omega(M) \le \tilde{\omega}_{\mathcal{H}_\eta}(M)$. So, the proof is completed. □

Remark 2. *Observe that if M is a subset of $\mathcal{TM}_{\|\cdot\|_\eta}(\Omega)$, then the best possible estimates for $\gamma_{\mathcal{TM}_\eta}(M)$ are obtained in Ref. [4], Theorem 2.1, precisely*

$$\max\left\{\tilde{\sigma}_{\mathcal{TM}_\eta}(M), \frac{1}{2}\tilde{\omega}_{\mathcal{TM}_\eta}(M)\right\} \le \gamma_{\mathcal{TM}_\eta}(M) \le \tilde{\sigma}_{\mathcal{TM}_\eta}(M) + \tilde{\omega}_{\mathcal{TM}_\eta}(M).$$

We devote the remainder of this section to derive a compactness criterion in groups of G-valued mappings endowed with the topology of local convergence in measure. To this end, we restrict ourselves to the family $\mathcal{A}_0 = \{A \in \mathcal{A} : \ \eta(A) < +\infty\}$. For any $A \in \mathcal{A}_0$, $\|f\|_{\eta,A} = \inf\{a > 0 : \eta(\{x \in A : \|f(x)\| \ge a\}) \le a\}$ is a group pseudonorm on $\mathcal{F}(\Omega)$, and the topology τ_η generated by the family of group pseudonorms

$$\{\|\cdot\|_{\eta,A} : \ A \in \mathcal{A}_0\} \tag{7}$$

generalizes the classical topology of local convergence in measure. We denote by $\mathcal{F}_{\tau_\eta}(\Omega)$ the topological group $(\mathcal{F}(\Omega), \tau_\eta)$, and for a subgroup $\mathcal{H}(\Omega)$ of $\mathcal{F}(\Omega)$ we write $\mathcal{H}_{\tau_\eta}(\Omega)$ for $(\mathcal{H}(\Omega), \tau_\eta)$.

Remark 3. *Let us observe that $f_n \xrightarrow{\|\cdot\|_\eta} f \Rightarrow f_n \xrightarrow{\tau_\eta} f$ holds for every sequence $\{f_n\}$ in $\mathcal{F}(\Omega)$ while the viceversa fails to hold. Indeed, enough to consider $f_n : [0,+\infty) \to \mathbb{R}$, for $n = 1,2,\dots,$ defined for all $x \in [0,+\infty)$ by setting $f_n(x) = \left(1-\frac{1}{n}\right)x$.*

We recall that the generalized Hausdorff measures of noncompactness $\gamma_{\mathcal{H}_{\tau_\eta}}(M)$ of a set M in a topological group $\mathcal{H}_{\tau_\eta}(\Omega)$ is that generated by the family of group pseudonorms given in (7). Precisely, $\gamma_{\mathcal{H}_{\tau_\eta}}(M)$, following Ref. [15], Definition 1.2.1, is the set function $\gamma_{\mathcal{H}\tau_\eta}(M) : \ \mathcal{A}_0 \to [0,+\infty)$ where $\gamma_{\mathcal{H}_{\tau_\eta}}(M)(A) = \gamma_{\|\cdot\|_{\eta,A}}(M)$, that is, the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in $\mathcal{H}(\Omega)$ with respect to the group pseudonorm $\|\cdot\|_{\eta,A}$. Then, the quantitative characteristics we are dealing with can be defined accordingly. We define them as set functions $\tilde{\omega}_{\mathcal{H}_{\tau_\eta}}(M)$, $\tilde{\sigma}_{\mathcal{H}_{\tau_\eta}}(M) : \mathcal{A}_0 \to [0,+\infty)$ by setting $\tilde{\omega}_{\mathcal{H}_{\tau_\eta}}(M)(A) = \tilde{\omega}_{\mathcal{H}_{\|\cdot\|_{\eta,A}}}(M)$ and $\tilde{\sigma}_{\mathcal{H}_{\tau_\eta}}(M)(A) = \tilde{\sigma}_{\mathcal{H}_{\|\cdot\|_{\eta,A}}}(M)$, where

$$
\begin{aligned}
\tilde{\omega}_{\mathcal{H}_{\|\cdot\|_{\eta,A}}}(M) = \inf\{&\varepsilon > 0 : \text{ there are a finite partition } \{A_1,\dots,A_n\} \text{ of } A \text{ in } \mathcal{A}_0, \text{ a finite set} \\
&\{f_1,\dots,f_m\} \text{ in } \mathcal{H}(\Omega) \text{ and a multimapping } \varphi \in \Phi_{\eta,\varepsilon}(M) \text{ such that, for all } f \in M, \\
&\text{there is } j \in \{1,\dots,m\} \text{ with } \operatorname{diam}((f-f_j)(A_i \setminus \varphi(f))) \le \varepsilon \text{ for } i = 1,\dots,n\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\sigma}_{\mathcal{H}_{\|\cdot\|_{\eta,A}}}(M) = \inf\{&\varepsilon > 0 : \text{ there are a set } G_A \text{ in } G \text{ with } \gamma(G_A) \le \varepsilon, \text{a finite set } \{g_1,\dots,g_k\} \\
&\text{in } \mathcal{H}(\Omega) \text{ and a multimapping } \psi \in \Phi_{\eta,\varepsilon}(M) \text{ such that, for all } f \in M, \\
&\text{there is } s \in \{1,\dots,k\} \text{ with } (f-g_s)(A \setminus \psi(f)) \subseteq G_A\}.
\end{aligned}
$$

The following result follows from Theorem 1.

Theorem 2. *Let M be a subset of a given subgroup $\mathcal{H}_{\tau_\eta}(\Omega)$ of $\mathcal{F}_{\tau_\eta}(\Omega)$, then*

$$\max\left\{\tilde{\sigma}_{\mathcal{H}_{\tau_\eta}}(M), \frac{1}{2}\tilde{\omega}_{\mathcal{H}_{\tau_\eta}}(M)\right\} \leq \gamma_{\mathcal{H}_{\tau_\eta}}(M) \leq 4(\tilde{\sigma}_{\tau_\eta}(M) + \tilde{\omega}_{\mathcal{H}_{\tau_\eta}}(M)).$$

Consequently, the set M is τ_η-totally bounded if and only if $\tilde{\omega}_{\mathcal{H}_{\tau_\eta}}(M) = \tilde{\sigma}_{\mathcal{H}_{\tau_\eta}}(M) = 0$. Finally, we want to mention that the group $(\mathcal{M}(\Omega, G), \tau_\eta)$ of all $\mathcal{A}$-measurable mappings, that is, of all mappings $f \in \mathcal{F}(\Omega)$ such that $f\chi_A$ is totally $\mathcal{A}$-measurable for any $A \in \mathcal{A}_0$, endowed with the topology τ_η (see Ref. [9], Chapter III), can indeed be considered as a special subgroup of $\mathcal{F}_{\tau_\eta}(\Omega)$. Therefore, we can say that a subset M of $(\mathcal{M}(\Omega, G), \tau_\eta)$ is τ_η-totally bounded if and only if it is extended equimeasurable and extended uniformly quasibounded in the sense of the τ_η-topology.

4. Compactness in Seminormed Subgroups of $\mathcal{F}_{\|\cdot\|_\infty}(\Omega)$

In this section on particularizing the submeasure μ, we will deal with groups of G-valued mappings defined on Ω and endowed with the standard supremum seminorm. Precisely, we consider the submeasure $\mu_\infty : \mathcal{A} \to [0, +\infty]$ defined by $\mu_\infty(\emptyset) = 0$ and $\mu_\infty(A) = +\infty$ if $\emptyset \neq A \in \mathcal{A}$. Then, we will denote by η_∞ the submeasure defined in (2) for $\mu = \mu_\infty$, that is, $\eta_\infty(\emptyset) = 0$ and $\eta_\infty(B) = +\infty$ if $\emptyset \neq B \in \mathcal{P}(\Omega)$. Therefore, the pseudonorm $\|\cdot\|_{\eta_\infty}$ coincides with the standard supremum seminorm $\|\cdot\|_\infty$. We will use the notation $\mathcal{F}_{\|\cdot\|_\infty}(\Omega)$ for $\mathcal{F}_{\|\cdot\|_{\eta_\infty}}(\Omega)$. As seminormed subgroups we can consider $\mathcal{B}_{\|\cdot\|_\infty}(\Omega)$ and $\mathcal{T}B_{\|\cdot\|_\infty}(\Omega)$ consisting, respectively, of all bounded and totally bounded mappings belonging to $\mathcal{F}_{\|\cdot\|_\infty}(\Omega)$, with the seminorm $\|\cdot\|_\infty$. Finally we observe that $\mathcal{TM}(\Omega, \mathcal{A}, \eta_\infty, G)) \subseteq \mathcal{TM}(\Omega, \mathcal{A}, \eta, G))$ for any given η, and that $(\mathcal{TM}(\Omega, \mathcal{P}(\Omega), \eta_\infty, G), \|\cdot\|_{\eta_\infty}) = \mathcal{T}B_{\|\cdot\|_\infty}(\Omega)$.

Now let $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$ be a seminormed subgroup of $\mathcal{F}_{\|\cdot\|_\infty}(\Omega)$, and M a subset of $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$. We will use the symbols $\gamma_{\mathcal{H}_\infty}$, $\tilde{\omega}_{\mathcal{H}_\infty}$ and $\tilde{\sigma}_{\mathcal{H}_\infty}$ for $\gamma_{\mathcal{H}_{\eta_\infty}}$, $\tilde{\omega}_{\mathcal{H}_{\eta_\infty}}$ and $\tilde{\sigma}_{\mathcal{H}_{\eta_\infty}}$, respectively. Then, let us observe that the infimum in the definition of $\tilde{\omega}_{\mathcal{H}_\infty}(M)$ is obtained by taking $\varphi(f) = \emptyset$ in (3), for each $f \in M$, and, in parallel, the infimum in the definition of $\tilde{\sigma}_{\mathcal{H}_\infty}(M)$ is obtained with $\psi(f) = \emptyset$ in (4), for each $f \in M$. Therefore, we will have:

$$\begin{aligned}\tilde{\omega}_{\mathcal{H}_\infty}(M) = \inf\{\varepsilon > 0 :\ & \text{there are a finite partition } \{A_1, \ldots, A_n\} \text{ of } \Omega \text{ in } \mathcal{A} \text{ and a finite set} \\ & \{f_1, \ldots, f_m\} \text{ in } \mathcal{H}(\Omega) \text{ such that, for all } f \in M, \text{ there is } j \in \{1, \ldots, m\} \text{ with} \\ & \operatorname{diam}((f - f_j)(A_i)) \leq \varepsilon \text{ for } i = 1, \ldots, n\},\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{\mathcal{H}_\infty}(M) = \inf\{\varepsilon > 0 :\ & \text{there is a set } G_0 \text{ in } G \text{ with } \gamma_G(G_0) \leq \varepsilon \text{ and a finite set } \{g_1, \ldots, g_k\} \\ & \text{in } \mathcal{H}(\Omega) \text{ such that, for all } f \in M, \text{ there is } s \in \{1, \ldots, k\} \text{ with } (f - g_s)(\Omega) \subseteq G_0\}.\end{aligned}$$

Such a formulation of $\tilde{\omega}_{\mathcal{H}_\infty}$ has been introduced in Ref. [16], to study the compactness of bounded sets in Banach space-valued spaces of bounded mappings defined on a general set Ω endowed with the standard supremum norm. Now, according to Ref. [11], one can say that the quantitative characteristic $\tilde{\omega}_{\mathcal{H}_\infty}$ generalizes the "measure of non-equicontinuity" of Nussbaum [6] to more general settings than that of spaces of continuous functions. Therefore, it is natural, in the setting of this section, to estimate the Hausdorff measure of noncompactness of a given set M by means of $\tilde{\omega}_{\mathcal{H}_\infty}(M)$ and the classical quantitative characteristic $\mu_{\gamma_G}(M)$ (see Ref. [10]), which measures the lack of pointwise totally boundedness, given by

$$\mu_{\gamma_G}(M) = \sup_{x \in \Omega} \gamma_G(M(x)),$$

where $M(x) = \{f(x) : f \in M\}$. In such a way, we will be able to generalize some classical and more recent compactness results (see Refs. [4,6,10–12,16], among others). To this end, we have the following result, which estimates the extended uniformly quasiboundedness of a given set M by means of $\tilde{\omega}_{\mathcal{H}_\infty}(M)$ and $\mu_{\gamma_G}(M)$.

Proposition 2. *Let $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$ be a subgroup of $\mathcal{F}_{\|\cdot\|_\infty}(\Omega)$ and let M be a subset of $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$. Then*

$$\mu_{\gamma_G}(M) \le \tilde{\sigma}_{\mathcal{H}_\infty}(M) \le \mu_{\gamma_G}(M) + \tilde{\omega}_{\mathcal{H}_\infty}(M). \tag{8}$$

Proof. We prove the left inequality. First, we observe that $\mu_{\gamma_G}(M) \le \gamma_G(M(\Omega))$; thus, if $\tilde{\sigma}_{\mathcal{H}_\infty}(M) \ge \gamma_G(M(\Omega))$, the inequality is immediate. Now, assume $\tilde{\sigma}_{\mathcal{H}_\infty}(M) < \gamma_G(M(\Omega))$. Let $a > \tilde{\sigma}_{\mathcal{H}_\infty}(M)$ and choose G_0 in G with $\gamma_G(G_0) \le a$ and a finite set $\{g_1, \dots, g_k\}$ in $\mathcal{H}(\Omega)$ such that, for all $f \in M$, there is $s \in \{1, \dots, k\}$ with $(f - g_s)(\Omega) \subseteq G_0$. We set, for $s \in \{1, \dots, k\}$,

$$M_s = \{f \in M : (f - g_s)(\Omega) \subseteq G_0\}.$$

Then, given $x \in \Omega$, for all $f \in M_s$ we have $f(x) \in g_s(x) + G_0$; therefore, $M_s(x) \subseteq g_s(x) + G_0$. Moreover, since $M(x) = \bigcup_{s=1}^k M_s(x) \subseteq \bigcup_{s=1}^k (g_s(x) + G_0)$, it follows

$$\gamma_G(M(x)) = \max_{s=1}^k \gamma_G(M_s(x)) \le \gamma_G\left(\bigcup_{s=1}^k (g_s(x) + G_0)\right) = \gamma_G(G_0) \le a.$$

Then, $\mu_{\gamma_G}(M) = \sup_{x\in\Omega} \gamma_G(M(x)) \le a$. The arbitrariness of a implies $\mu_{\gamma_G}(M) \le \tilde{\sigma}_{\mathcal{H}_\infty}(M)$.

Now, we prove the right inequality. Since $\tilde{\sigma}_{\mathcal{H}_\infty}(M) \le \eta(\Omega)$, if either $\mu_{\gamma_G}(M) = \eta(\Omega)$ or $\tilde{\omega}_{\mathcal{H}_\infty}(M) = \eta(\Omega)$ the inequality holds true. Assume $\tilde{\sigma}_{\mathcal{F}_\eta}(M) < \eta(\Omega)$ and $\tilde{\omega}_{\mathcal{F}_\eta}(M) < \eta(\Omega)$. Let $a > \tilde{\omega}_{\mathcal{H}_\infty}(M)$, choose a partition $\{A_1, \dots, A_n\}$ of Ω in $\mathcal{A}$ and a finite set $\{f_1, \dots, f_m\}$ in $\mathcal{H}(\Omega)$ such that, for all $f \in M$, there is $j \in \{1, \dots, m\}$ with $\mathrm{diam}((f - f_j)(A_i)) \le a$ for $i = 1, \dots, n$. Moreover, let $b > \mu_{\gamma_G}(M)$ and fix $x_i \in A_i$ for $i = 1, \dots, n$. Then, since $\gamma_G(\bigcup_{i=1}^n M(x_i)) \le b$, there are $z_1, \dots, z_k \in G$ such that

$$\bigcup_{i=1}^n M(x_i) \subseteq \bigcup_{s=1}^k B_G(z_s, b).$$

Set, for $j = 1, \dots, m$,

$$M_j = \{f \in M : \mathrm{diam}((f - f_j)(A_i)) \le a \text{ for } i = 1, \dots, n\}. \tag{9}$$

Further, for $f \in M$ and for $i \in \{1, \dots, n\}$, choose z_s such that

$$\|f(x_i) - z_s\| \le b. \tag{10}$$

Then, for each $f \in M_j$ and for $x \in A_i$, we have

$$(f - f_j)(x) - z_s = (f - f_j)(x) - (f - f_j)(x_i) + (f - f_j)(x_i) - z_s.$$

Using (9) and (10), we obtain $(f - f_j)(x) \in z_s - f_j(x_i) + B_G(\theta, a + b)$. Hence,

$$(f - f_j)(A_i) \subseteq \bigcup_{s=1}^k \big(z_s - f_j(x_i) + B_G(\theta, a + b)\big).$$

Consequently,

$$(f - f_j)(\Omega) = \bigcup_{i=1}^n (f - f_j)(A_i) \subseteq \bigcup_{i=1}^n \bigcup_{s=1}^k \big(z_s - f_j(x_i) + B_G(\theta, a + b)\big).$$

Setting $G_0 = \bigcup_{i=1}^n \bigcup_{s=1}^k \big(z_s - f_j(x_i) + B_G(\theta, a + b)\big)$, we have $\gamma_G(G_0) \le a + b$ and, hence $\tilde{\sigma}_{\mathcal{H}_\infty}(M) \le a + b$. The arbitrariness of a and b completes the proof of the right inequality; therefore, we have proved (8). □

Now, from Theorem 1 and Proposition 2, given a subset M of $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$, we obtain

$$\max\left\{\mu_{\gamma_G}(M), \frac{1}{2}\tilde{\omega}_{\mathcal{H}_\infty}(M)\right\} \leq \gamma_{\mathcal{H}_\infty}(M) \leq 4(\mu_{\gamma_G}(M) + 2\tilde{\omega}_{\mathcal{H}_\infty}(M)).$$

Corollary 2. *A subset M of $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$ is totally bounded if and only if*

$$\tilde{\omega}_{\mathcal{H}_\infty}(M) = \mu_{\gamma_G}(M) = 0,$$

that is, if and only if M is extended equimeasurable and pointwise totally bounded.

One can verify that μ_{γ_G} satisfies properties (ii)–(vii) of a measure of noncompactness, as $\tilde{\omega}_{\mathcal{H}_\infty}$ clearly does. Then, the above results ensure that $\mu_{\gamma_G} + 2\tilde{\omega}_{\mathcal{H}_\infty}$ is a measure of noncompactness equivalent to the Hausdorff measure of noncompactness. Let us observe that if $\mathcal{H}_{\|\cdot\|_\infty}(\Omega) = \mathcal{B}_{\|\cdot\|_\infty}(\Omega)$, the previous result generalizes Ref. [16], Corollary 3.1, from the case of spaces of Banach space-valued mappings to the case of spaces of G-valued mappings. Finally, if $\mathcal{H}_{\|\cdot\|_\infty}(\Omega)$ is a subgroup of $\mathcal{T}B_{\|\cdot\|_\infty}(\Omega)$, in view of Proposition 1, the quantitative characteristics $\tilde{\omega}_{\mathcal{H}_\infty}(M)$ and $\tilde{\sigma}_{\mathcal{H}_\infty}(M)$ coincide with the corresponding ones given in Ref. [4]. Precisely,

$$\begin{aligned}\tilde{\omega}_{\mathcal{H}_\infty}(M) = \inf\{\varepsilon > 0 :\ &\text{there is a finite partition } \{A_1, \ldots, A_n\} \text{ of } \Omega \text{ in } \mathcal{A} \\ &\text{such that, for all } f \in M,\ \operatorname{diam}((f)(A_i)) \leq \varepsilon \text{ for } i = 1, \ldots, n\},\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{\mathcal{H}_\infty}(M) = \inf\{\varepsilon > 0 :\ &\text{there is a set } G_0 \text{ in } G \text{ with } \gamma_G(G_0) \leq \varepsilon \text{ such that,} \\ &\text{for all } f \in M,\ f(\Omega) \subseteq G_0\}.\end{aligned}$$

Therefore, Theorem 2 generalizes Ref. [4], Theorem 3.1, which is proved in spaces of totally bounded mappings from a general set Ω into a pseudometric space (see also Ref. [11]).

Remark 4. *Whenever Ω is a topological space, Corollary 2 extends the Bartle compactness criterion to the seminormed group $(\mathcal{B}C(\Omega, G), \|\cdot\|_\infty)$ of all G-valued bounded and continuous mappings defined on Ω. Therefore, if Ω is compact, it extends the Ascoli–Arzelà compactness criterion.*

5. Conclusions

The degree of noncompactness of sets in groups constituted by mappings from a general set into an arbitrary additive normed group and endowed with a pseudonorm that induces the topology of convergence in measure is estimated by means of two new quantitative characteristics. The sum of those quantitative characteristics is a regular measure of noncompactness, i.e., such a measure vanishes on all totally bounded subsets.

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