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Fuzzy quantities and their ranking: a new proposal

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ABSTRACT

We deal with the problem of evaluating and ranking fuzzy quantities. We call fuzzy quantity any non-normal and non-convex fuzzy set, defined as the union of two, or more, generalized fuzzy numbers. For this purpose we suggest an evaluation defined by a pair index based on “value” & “ambiguity”. Either value or ambiguity depend on two parameters connected the first with the optimistic/pessimistic point of view of the decision maker and the second on an additive measure that can be used to express the decision maker’s preferences.

Keywords: Fuzzy sets; Generalized fuzzy numbers; Evaluation function; Ranking; Defuzzification; Ambiguity.

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1 Introduction

Either in many fuzzy optimization or in decision making problems, evaluation and/or ranking definitions of fuzzy numbers play an important role. Several proposals of different kind have appeared in literature [1–36]. Following the line of “utility function” definition in decision making problems one wide group of them proposes to define a real function on the fuzzy numbers set to obtain a real value associated to the fuzzy set useful for its evaluation and ranking too. This approach has produced several proposals with different characteristics. Some of them have chosen to obtain a value into the support of fuzzy set. This is the idea we have decided to follow even in the field of fuzzy quantities. A fuzzy quantity is a fuzzy set obtained by the union of two or more fuzzy sets not necessarily normal, called generalized fuzzy numbers. These complex sets are non convex and non normal fuzzy sets. Our choice is due even by the fact that these types of fuzzy sets are the typical output of inference fuzzy systems and the necessity of a way to produce a “defuzzification method” (that is the transformation into a crisp number to obtain the final system output). This idea, like any others in literature, produces equivalent classes very wide, so, to reduce their size, we propose a new definition that uses a lexicographic order based on two index, value and ambiguity. The “value” definition we use is proposed in [37] where the authors present a definition of evaluation of a fuzzy quantity based on α -cut levels and depending on two parameters: a real number connected with the optimistic/pessimistic point of view of the decision maker and an additive measure that allows the decision maker to attribute different weights to each level, according to his preference. In this paper we add a notion of ambiguity of a fuzzy quantity. Ambiguity is a measure of the vagueness, that is the lack of precision in determining the exact value of a magnitude. Index of ambiguity was suggested for fuzzy numbers to characterize the global spread of the membership function of a fuzzy number [38]. We provide some numerical examples to illustrate the applicability of the proposed method.

In Section 2 we give basic definitions and notations. In Section 3 we deal with the concepts of value and ambiguity for normal fuzzy numbers. The evaluation of generalized fuzzy numbers is presented in Section 4. For them we give a definition of ambiguity and investigate some of its properties. In Section 5 we deal with the fuzzy quantities evaluation introduced in [37, 39] presenting a review of their definition of evaluation. Furthermore, we propose a definition of ambiguity of a fuzzy quantity and provide some results. In Section 6 we propose a ranking method for fuzzy quantities based on the value-ambiguity pair and discuss some of its properties. Some numerical examples illustrate our method. Finally, in Section 7 we give an alternative procedure to compute the value of a fuzzy quantity using fuzzy arithmetic; to illustrate this method a numerical example is presented.

2 Preliminaries and notation

Let X denote a universe of discourse. A fuzzy set A in X is defined by a membership function $\mu_A : X \rightarrow [0, 1]$ which assigns to each element of X a grade of membership to the set A . The support and the core of A are defined, respectively, as the crisp sets $supp(A) = \{x \in X; \mu_A(x) > 0\}$ and $core(A) = \{x \in X; \mu_A(x) = 1\}$. A fuzzy set A is normal if its core is nonempty. A fuzzy number A is a fuzzy set of the real line \mathbb{R} with a normal, convex and upper-semicontinuous membership function of bounded support. From the definition given above there exist four numbers $a_1, a_2, a_3, a_4 \in \mathbb{R}$, with $a_1 \leq a_2 \leq a_3 \leq a_4$, and two functions

$f_A, g_A : \mathbb{R} \rightarrow [0, 1]$ called the left side and the right side of A , respectively, where f_A is nondecreasing and g_A is nonincreasing, such that

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \\ f_A(x) & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ g_A(x) & a_3 < x \leq a_4 \\ 0 & a_4 < x. \end{cases}$$

The α -cut of A , $0 \leq \alpha \leq 1$, is defined as the crisp set $A_\alpha = \{x \in X; \mu_A(x) \geq \alpha\}$. According to the definition of a fuzzy number every α -cut of a fuzzy number is a closed interval $A_\alpha = [a_L(\alpha), a_R(\alpha)]$ where $a_L(\alpha) = \inf A_\alpha$ and $a_R(\alpha) = \sup A_\alpha$.

A generalized fuzzy number $A = (a_1, a_2, a_3, a_4; w_A)$ is a fuzzy set of the real line with membership function

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \\ f_A(x) & a_1 \leq x < a_2 \\ w_A & a_2 \leq x \leq a_3 \\ g_A(x) & a_3 < x \leq a_4 \\ 0 & a_4 < x \end{cases}$$

where $0 < w_A \leq 1$. The difference between a fuzzy number and a generalized fuzzy number is that the height of a fuzzy number is equal to one, but the height w of a generalized fuzzy number is between zero and one. We consider generalized fuzzy numbers such that f_A is a continuous and strictly increasing function and g_A is a continuous and strictly decreasing function. A generalized fuzzy number $A = (a_1, a_2, a_3, a_4; w_A)$ is said to be a generalized trapezoidal fuzzy number if its membership function is given by

$$\mu_A(x) = \begin{cases} 0 & x < a_1 \\ \frac{x - a_1}{a_2 - a_1} w_A & a_1 \leq x < a_2 \\ w_A & a_2 \leq x \leq a_3 \\ \frac{a_4 - x}{a_4 - a_3} w_A & a_3 < x \leq a_4 \\ 0 & a_4 < x. \end{cases}$$

If $a_2 = a_3$ the trapezoidal fuzzy number reduces to a generalized triangular fuzzy number. We call fuzzy quantity any non-normal and non-convex fuzzy set, defined as the union of two, or more, generalized fuzzy numbers [37].

3 Evaluation of normal fuzzy numbers

In this section we recall the concepts of value and ambiguity of a normal fuzzy number A . The evaluation interval of A with respect to an additive measure S on $[0, 1]$ is $[V_*(A; S), V^*(A; S)]$ where

$$V_*(A; S) = \int_0^1 a_L(\alpha) dS(\alpha), \quad V^*(A; S) = \int_0^1 a_R(\alpha) dS(\alpha).$$

The decision maker chooses S according to a subjective assignation of weights related to the importance of each level α . We assume that S is a normalized Stieltjes measure on $[0, 1]$

defined through the function s , i.e.

$$S([a, b]) = s(b) - s(a) \quad 0 \leq a < b \leq 1,$$

where $s : [0, 1] \rightarrow [0, 1]$ is a strictly increasing and continuous function such that $s(0) = 0$ and $s(1) = 1$.

The value of A with respect to the additive measure S on $[0, 1]$ and the parameter $\lambda \in [0, 1]$ is the real number

$$V_\lambda(A; S) = \int_0^1 \phi_\lambda(A_\alpha) dS(\alpha) = (1 - \lambda)V_*(A; S) + \lambda V^*(A; S),$$

where

$$\phi_\lambda([x_1, x_2]) = (1 - \lambda)x_1 + \lambda x_2, \quad x_1 \leq x_2,$$

is an evaluation function. The parameter λ is an optimistic/pessimistic degree. The ambiguity of the normal fuzzy number A is

$$Amb(A; S) = \int_0^1 \frac{a_R(\alpha) - a_L(\alpha)}{2} dS(\alpha).$$

It results

$$Amb(A; S) = \frac{V^*(A; S) - V_*(A; S)}{2}. \quad (1)$$

Hence the ambiguity of A is equal to one-half of the length of the evaluation interval $[V_*(A; S), V^*(A; S)]$.

Remark 3.1. When the Stieltjes measure S is generated by $s(\alpha) = \alpha^r$, $r > 0$, we will denote the value $V_\lambda(A; S)$ by $V_\lambda(A; r)$ and the ambiguity $Amb(A; S)$ by $Amb(A; r)$, that is

$$V_\lambda(A; r) = r \int_0^1 \phi_\lambda(A_\alpha) \alpha^{r-1} d\alpha, \quad Amb(A; r) = r \int_0^1 \frac{a_R(\alpha) - a_L(\alpha)}{2} \alpha^{r-1} d\alpha.$$

The choice of parameter r allows decision maker, according to his preferences, to give more weight to the high values of α ($r > 1$) or more weight to the low values of α ($r < 1$).

4 Evaluation of generalized fuzzy numbers

4.1 Value of generalized fuzzy numbers

Let A be a generalized fuzzy number with *height* $A = w_A \leq 1$, and α -cuts $A_\alpha = [a_L(\alpha), a_R(\alpha)]$, $\alpha \in [0, w_A]$.

Definition 4.1. We define the lower and upper values of A as

$$V_*(A; S) = \frac{1}{s(w_A)} \int_0^{w_A} a_L(\alpha) dS(\alpha), \quad V^*(A; S) = \frac{1}{s(w_A)} \int_0^{w_A} a_R(\alpha) dS(\alpha),$$

and the value of A [37, 26]

$$V_\lambda(A; S) = \frac{1}{s(w_A)} \int_0^{w_A} \phi_\lambda(A_\alpha) dS(\alpha) = (1 - \lambda)V_*(A; S) + \lambda V^*(A; S) \quad (2)$$

with $\lambda \in [0, 1]$. When the Stieltjes measure S is generated by $s(\alpha) = \alpha^r$, $r > 0$, we denote the value of A as

$$V_\lambda(A; r) = \frac{r}{w_A^r} \int_0^{w_A} \phi_\lambda(A_\alpha) \alpha^{r-1} d\alpha.$$

Remark 4.2. $V_*(A; S)$, $V^*(A; S)$ and $V_\lambda(A; S)$ belong to the support of A .

Example 4.3. If $A = (a_1, a_2, a_3, a_4; w_A)$ is a generalized trapezoidal fuzzy number we have [37]

$$V_\lambda(A; S) = \lambda a_3 + (1 - \lambda)a_2 + K(S, w_A)(\lambda(a_4 - a_3) - (1 - \lambda)(a_2 - a_1))$$

where $K(S, w_A) = \frac{1}{s(w_A)w_A} \int_0^{w_A} s(\alpha) d\alpha$. Note that

$$K(S, w_A) = \frac{w_A - E(S, w_A)}{w_A},$$

where $E(S, w_A)$ is the preference index [40]

$$E(S, w_A) = \frac{1}{s(w_A)} \int_0^{w_A} \alpha dS(\alpha).$$

In the special case of $s(\alpha) = \alpha^r$ we have $K(S, w_A) = 1/(r + 1)$ and thus the value of A is

$$V_\lambda(A; r) = \lambda a_3 + (1 - \lambda)a_2 + \frac{\lambda(a_4 - a_3) - (1 - \lambda)(a_2 - a_1)}{r + 1} \quad (3)$$

or, equivalently,

$$V_\lambda(A; r) = (1 - \lambda) \left[a_1 + \frac{r}{r + 1}(a_2 - a_1) \right] + \lambda \left[a_4 - \frac{r}{r + 1}(a_4 - a_3) \right].$$

Hence when $s(\alpha) = \alpha^r$ the evaluation of a generalized trapezoidal fuzzy number A does not depend on w_A .

Remark 4.4. If A is the generalized fuzzy number defined by $\mu_A(x) = w_A$ if $x = a$ and $\mu_A(x) = 0$ otherwise then $V_\lambda(A; S) = a$.

The following properties hold:

Proposition 4.5. Let A, B be two generalized fuzzy numbers with the same height $w_A = w_B$ and let $k \in \mathbb{R}$. Then, for $\lambda \in [0, 1]$,

$$(i) \quad V_\lambda(kA; S) = \begin{cases} k V_\lambda(A; S) & k > 0 \\ k V_{1-\lambda}(A; S) & k < 0 \end{cases}$$

$$(ii) \quad V_\lambda(A \oplus B; S) = V_\lambda(A; S) + V_\lambda(B; S)$$

$$(iii) \quad V_\lambda(A \ominus B; S) = V_\lambda(A; S) - V_{1-\lambda}(B; S)$$

where the operations addition of fuzzy numbers and multiplication of a real number by a fuzzy number are defined by Zadeh's extension principle [42].

4.2 Ambiguity of generalized fuzzy numbers

We now introduce the definition of ambiguity for a generalized fuzzy number and investigate some of its properties.

Definition 4.6. We define the ambiguity of a generalized fuzzy number A with respect to S as

$$Amb(A; S) = \int_0^{w_A} \frac{a_R(\alpha) - a_L(\alpha)}{2} dS(\alpha).$$

If $s(\alpha) = \alpha^r$, $r > 0$, we denote $Amb(A; r) = Amb(A; S)$, that is

$$Amb(A; r) = r \int_0^{w_A} \frac{a_R(\alpha) - a_L(\alpha)}{2} \alpha^{r-1} d\alpha.$$

Proposition 4.7. Let A, B be two generalized fuzzy numbers. The following property holds:

$$\text{if } A \subset B \text{ then } Amb(A; S) \leq Amb(B; S).$$

Proof. Since $A \subset B$ we have $[a_L(\alpha), a_R(\alpha)] = A_\alpha \subseteq B_\alpha = [b_L(\alpha), b_R(\alpha)]$ and $w_A \leq w_B$. Then

$$\begin{aligned} Amb(A; S) &= \int_0^{w_A} \frac{a_R(\alpha) - a_L(\alpha)}{2} dS(\alpha) \leq \\ &\leq \int_0^{w_A} \frac{b_R(\alpha) - b_L(\alpha)}{2} dS(\alpha) \leq \int_0^{w_B} \frac{b_R(\alpha) - b_L(\alpha)}{2} dS(\alpha) = Amb(B; S). \quad \square \end{aligned}$$

In the next result we extend property (1) to generalized fuzzy numbers.

Proposition 4.8.

$$Amb(A; S) = \frac{V^*(A; S) - V_*(A; S)}{2} s(w_A). \quad (4)$$

Remark 4.9. If A is a generalized fuzzy number it can easily be seen that

- (i) $(V_\lambda(A; S) - V_*(A; S)) s(w_A) = 2\lambda Amb(A; S)$;
- (ii) $(V^*(A; S) - V_\lambda(A; S)) s(w_A) = 2(1 - \lambda) Amb(A; S)$.

In particular, when $\lambda = 1/2$ and A is a normal fuzzy number, according to [43, p. 207], we get

$$V_{1/2}(A; S) = V_*(A; S) + Amb(A; S) = V^*(A; S) - Amb(A; S).$$

Moreover

$$(V_\lambda(A; S) - V_{1-\lambda}(A; S)) s(w_A) = 2(2\lambda - 1) Amb(A; S).$$

Example 4.10. Let $A = (a_1, a_2, a_3, a_4; w_A)$ be a generalized trapezoidal fuzzy number. For $\alpha \in [0, w_A]$, α -cuts of A are

$$A_\alpha = \left[a_1 + (a_2 - a_1) \frac{\alpha}{w_A}, a_4 - (a_4 - a_3) \frac{\alpha}{w_A} \right]$$

and thus

$$Amb(A; S) = \left(\frac{a_4 - a_1}{2} - \frac{a_4 - a_3 + a_2 - a_1}{2w_A} E(S, w_A) \right) s(w_A).$$

If $s(\alpha) = \alpha^r$, $r > 0$, we have $E(S, w_A) = w_A r / (r + 1)$ and so

$$Amb(A; r) = \left(\frac{a_4 - a_1}{2} - \frac{(a_4 - a_3 + a_2 - a_1)r}{2(r+1)} \right) w_A^r.$$

If $A = (a_1, a_2, a_3; w_A)$ is a generalized triangular fuzzy number we obtain

$$Amb(A; r) = \frac{(a_3 - a_1)w_A^r}{2(r+1)}.$$

Proposition 4.11. *Let A, B be two generalized fuzzy numbers with the same height $w_A = w_B$ and let $k \in \mathbb{R}$. Then*

$$(i) \quad Amb(A \oplus B; S) = Amb(A; S) + Amb(B; S),$$

$$(ii) \quad Amb(kA; S) = |k| Amb(A; S),$$

where the operations addition and multiplication of a real number by a fuzzy number are defined by Zadeh's extension principle.

The following proposition generalizes a result given in [44] concerning the value of normal fuzzy numbers. We extend this result to value and ambiguity of generalized fuzzy numbers.

Proposition 4.12. *Let A^s be the generalized fuzzy number with membership function*

$$\mu_{A^s}(x) = s(\mu_A(x)).$$

Then we have

$$[V_*(A; S), V^*(A; S)] = [V_*(A^s; 1), V^*(A^s; 1)]$$

and

$$V_\lambda(A; S) = V_\lambda(A^s; 1), \quad Amb(A; S) = Amb(A^s; 1).$$

Proof. First, we observe that

$$\begin{aligned} [a_L^s(t), a_R^s(t)] &= A_t^s = \{x; \mu_{A^s}(x) \geq t\} = \\ &= \{x; \mu_A(x) \geq s^{-1}(t)\} = A_{s^{-1}(t)} = [a_L(s^{-1}(t)), a_R(s^{-1}(t))]. \end{aligned}$$

Then, by making the substitution $t = s(\alpha)$ and noting that $w_{A^s} = s(w_A)$ we get

$$\begin{aligned} V_*(A; S) &= \frac{1}{s(w_A)} \int_0^{w_A} a_L(\alpha) dS(\alpha) = \frac{1}{s(w_A)} \int_0^{s(w_A)} a_L(s^{-1}(t)) dt \\ &= \frac{1}{w_{A^s}} \int_0^{w_{A^s}} a_L^s(t) dt = V_*(A^s; 1). \end{aligned}$$

In a similar way we obtain $V^*(A; S) = V^*(A^s; 1)$. It follows $V_\lambda(A; S) = V_\lambda(A^s; 1)$ and

$$Amb(A; S) = \frac{V^*(A^s; 1) - V_*(A^s; 1)}{2} w_{A^s} = Amb(A^s; 1). \quad \square$$

From the previous result the value and ambiguity of a generalized fuzzy number A with respect to a measure S are the same of those of A^s with respect to the Lebesgue measure \mathcal{L} . If $s(\alpha) = \alpha^r$, $r > 0$, we denote $A^r = A^s$. Then the value and ambiguity of A with respect to S are those of A^r with respect to the Lebesgue measure, that is $V_\lambda(A; r) = V_\lambda(A^r; 1)$ and $Amb(A; r) = Amb(A^r; 1)$.

Remark 4.13. In [45] the operator of concentration/dilation of a fuzzy set is introduced. For a fuzzy set A the fuzzy set $\mathcal{S}_r(A)$, $r > 0$, is defined by the membership function

$$\mu_{\mathcal{S}_r(A)}(x) = (\mu_A(x))^r.$$

If $r = 1$ then $\mathcal{S}_1(A) = A$. If $r > 1$ the modified fuzzy set is a concentration of A , that is the reduction in the magnitude of the grade of membership is small for those elements which have a high grade of membership in A and large for the elements with low membership. If $0 < r < 1$ the modified fuzzy set is a dilation of A . The effect of dilation is the opposite of that of concentration. Concentration by $r = 2$ is interpreted as the linguistic hedge *very* and dilation by $r = 0.5$ as *more or less*.

Observe that if the grade of membership of x in A is 1, then the same is true for $\mathcal{S}_r(A)$, that is, $\mu_A(x) = 1$ implies $\mu_{\mathcal{S}_r(A)}(x) = 1$. In particular ($r = 2$)

$$\mu_A(x) = 1 \implies \mu_{\text{very}A}(x) = 1. \quad (5)$$

About this Zadeh wrote: “if the grade of membership of John in the class of old men is 1, then the same is true of the grade of membership of John in the class of very old men. Is this in accord with our intuition? This basic question does not appear to have a clear-cut answer on purely intuitive grounds. It is easy to show, however, that Eq. (5) can be deduced as a consequence of the following two assumptions:

(a) *very* distributes over the union (e.g., *very* (tall or fat) = *very* tall or *very* fat) (b) *very* $A = A$ if A is non-fuzzy (e.g., *very* square = square).” [45]

See also [46, p. 488] for more discussion on this point.

Note that if A is a generalized fuzzy number then $\mathcal{S}_r(A)$ is also a generalized fuzzy number and $\mathcal{S}_r(A) = A^r$.

If A is a subnormal fuzzy number then $height A^r < height A$ if $r > 1$ and $height A^r > height A$ if $r < 1$ (see Fig. 1). Moreover for a generalized fuzzy number A we have

- (i) $r \rightarrow 0^+ \implies A^r \rightarrow \text{supp}(A)$;
- (ii) $r \rightarrow +\infty \implies A^r \rightarrow [a_2, a_3]$ if A is normal (i.e. $w_A = 1$),
 $r \rightarrow +\infty \implies A^r \rightarrow \emptyset$ if A is subnormal (i.e. $w_A < 1$).

The following properties will be used later.

Proposition 4.14.

- (i) $(A \cup B)^s = A^s \cup B^s$,
- (ii) $(A \cap B)^s = A^s \cap B^s$.

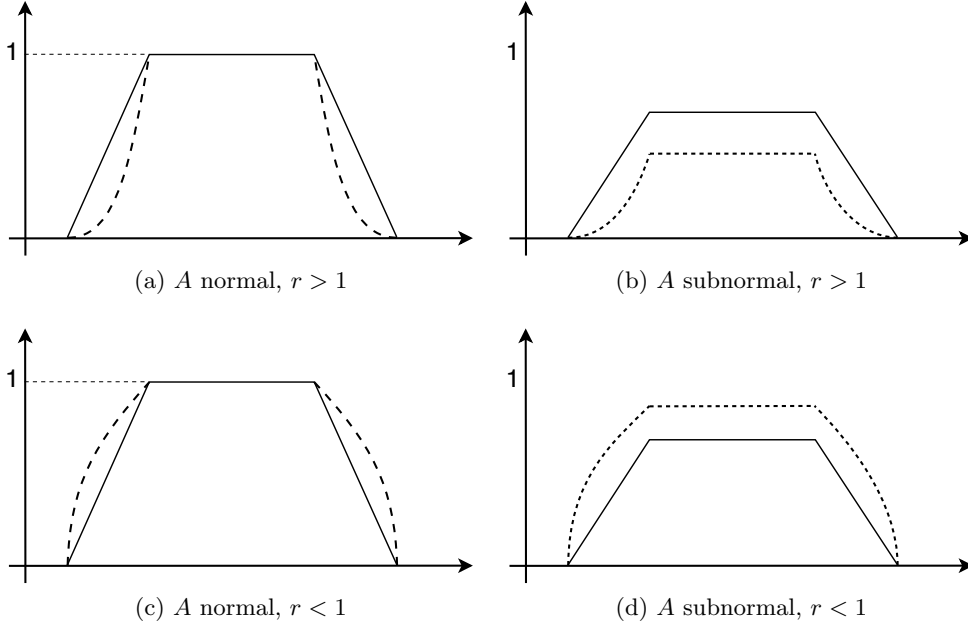


Figure 1: Membership functions of A (continuous line) and A^r (dashed line)

Proof. Let us prove (i). Since $s(\alpha)$ is an increasing function we get

$$\begin{aligned} \mu_{(A \cup B)^s}(x) &= s(\max\{\mu_A(x), \mu_B(x)\}) = \\ &= \max\{s(\mu_A(x)), s(\mu_B(x))\} = \max\{\mu_A^s(x), \mu_B^s(x)\} = \mu_{A^s \cup B^s}(x). \end{aligned}$$

In a similar way we prove (ii). □

5 Evaluation of fuzzy quantities

In this section we deal with the case of fuzzy quantities, that is the union of two or more generalized fuzzy numbers, which is not in general a generalized fuzzy number. We introduce the concepts of value and ambiguity of a fuzzy quantity and investigate some of their properties.

5.1 Value of fuzzy quantities

From now on we denote by $\overline{\text{supp}} B$ the closure of the support of the generalized fuzzy number B .

Definition 5.1. ([37])

Let B, C be two generalized fuzzy numbers with height w_B and w_C , respectively, such that $\overline{\text{supp}} B \cap \overline{\text{supp}} C \neq \emptyset$. The value of the fuzzy quantity $B \cup C$ is the real number

$$\mathcal{V}_\lambda(B \cup C; S) = \sigma_1 V_\lambda(B; S) + \sigma_2 V_\lambda(C; S) - \sigma_3 V_\lambda(B \cap C; S) \quad (6)$$

where $\sigma_i = \sigma_i(w_B, w_C, w_{B \cap C}) = \psi_i(s(w_B), s(w_C), s(w_{B \cap C}))$, $i = 1, 2, 3$, with

$$\psi_i(z_1, z_2, z_3) = \frac{z_i}{z_1 + z_2 - z_3}, \quad z_1 + z_2 - z_3 \neq 0. \quad (7)$$

Note that $\sigma_i \geq 0$ and $\sigma_1 + \sigma_2 - \sigma_3 = 1$.

Remark 5.2. If B, C are two generalized fuzzy numbers with height w_B and w_C , respectively, then $B \cap C$ is a generalized fuzzy number with height $w_{B \cap C}$. Thus the previous definition is well-posed.

Remark 5.3. If A is a generalized fuzzy number, noting that $A = A \cup A = A \cap A$, from (6) we have $\mathcal{V}_\lambda(A; S) = V_\lambda(A; S)$ and thus, in the following, we denote the value of A by $\mathcal{V}_\lambda(A; S)$.

Proposition 5.4. *The following properties hold:*

(i) $\mathcal{V}_\lambda(B \cup C; S)$ belongs to the support of $B \cup C$;

(ii) if A is a fuzzy quantity and B, C, D, E are generalized fuzzy numbers such that $A = B \cup C = D \cup E$ then

$$\mathcal{V}_\lambda(A; S) = \mathcal{V}_\lambda(B \cup C; S) = \mathcal{V}_\lambda(D \cup E; S).$$

Let $A = B \cup C$ be a fuzzy quantity. Two different cases may occur: (i) $A = B \cup C$ has convex α -cuts, hence it is a fuzzy number; (ii) $A = B \cup C$ has non-convex α -cuts, hence it is a fuzzy quantity. In the case (ii) there exists a decomposition of the support of A in such a way that on each subinterval α -cuts of A be convex, that is, we can write $A = A_1 \cup A_2$ where A_1 and A_2 are generalized fuzzy numbers such that $\text{supp } A_1 = (a_1, \bar{x}]$ and A_1 has convex α -cuts $[a_L^{(1)}(\alpha), a_R^{(1)}(\alpha)]$, $\text{supp } A_2 = [\bar{x}, a_4]$ and A_2 has convex α -cuts $[a_L^{(2)}(\alpha), a_R^{(2)}(\alpha)]$, where $a_1 = \min\{b_1, c_1\}$ and $a_4 = \max\{b_4, c_4\}$. The heights of A_1 and A_2 are, respectively, $w_{A_1} = w_B$ and $w_{A_2} = w_C$ if $b_3 < c_2$ or $w_{A_1} = w_C$ and $w_{A_2} = w_B$ if $c_2 < b_3$ [37]. The hypographs of the membership functions μ_{A_1} and μ_{A_2} are not disjoint since their intersection is the vertical segment T between points $P_1 = (\bar{x}, 0)$ and $P_2 = (\bar{x}, w_T)$ where $w_T = w_{B \cap C}$.

Then if $A = B \cup C$ is a fuzzy quantity there exist A_1, A_2 generalized fuzzy numbers such that

$$A = A_1 \cup A_2 \quad \text{and} \quad A_1 \cap A_2 = T \quad (8)$$

where T is a generalized fuzzy number defined by $\mu_T(x) = w_T$ if $x = \bar{x}$ and $\mu_T(x) = 0$ otherwise. If we compute the value of the fuzzy quantity $A = B \cup C$ by using the decomposition (8) we obtain

$$\mathcal{V}_\lambda(A; S) = \mathcal{V}_\lambda(A_1 \cup A_2; S) = \beta_1 \mathcal{V}_\lambda(A_1; S) + \beta_2 \mathcal{V}_\lambda(A_2; S) - \beta_3 \mathcal{V}_\lambda(T; S)$$

where $\beta_i = \psi_i(s(w_{A_1}), s(w_{A_2}), s(w_{w_T}))$, $i = 1, 2, 3$, and ψ_i is defined by (7). Note that, by Remark 4.4, $\mathcal{V}_\lambda(T; S) = \bar{x}$.

It is convenient to introduce the notion of lower and upper values of a fuzzy quantity.

Definition 5.5. We define the lower and upper values, respectively, as

$$\mathcal{V}_*(B \cup C; S) = \sigma_1 V_*(B; S) + \sigma_2 V_*(C; S) - \sigma_3 V_*(B \cap C; S) \quad (9)$$

$$\mathcal{V}^*(B \cup C; S) = \sigma_1 V^*(B; S) + \sigma_2 V^*(C; S) - \sigma_3 V^*(B \cap C; S) \quad (10)$$

Proposition 5.6. *We have*

$$\mathcal{V}_\lambda(B \cup C; S) = (1 - \lambda) \mathcal{V}_*(B \cup C; S) + \lambda \mathcal{V}^*(B \cup C; S), \quad \lambda \in [0, 1].$$

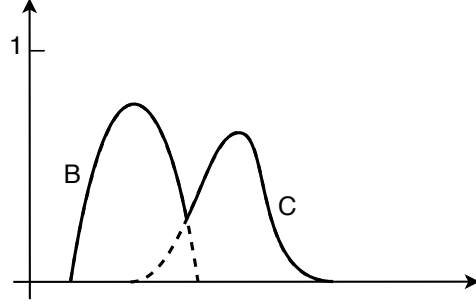


Figure 2: Fuzzy quantity $A = B \cup C$

Proof. From (2) we get

$$\begin{aligned}\mathcal{V}_\lambda(B; S) &= (1 - \lambda)V_*(B; S) + \lambda V^*(B; S), \\ \mathcal{V}_\lambda(C; S) &= (1 - \lambda)V_*(C; S) + \lambda V^*(C; S), \\ \mathcal{V}_\lambda(B \cap C; S) &= (1 - \lambda)V_*(B \cap C; S) + \lambda V^*(B \cap C; S).\end{aligned}$$

Substituting into the equation

$$\mathcal{V}_\lambda(B \cup C; S) = \sigma_1 \mathcal{V}_\lambda(B; S) + \sigma_2 \mathcal{V}_\lambda(C; S) - \sigma_3 \mathcal{V}_\lambda(B \cap C; S)$$

and taking into account (9) and (10) we obtain the assertion. \square

5.2 Ambiguity of fuzzy quantities

We now extend the notion of ambiguity to the case of fuzzy quantities.

Definition 5.7. We call ambiguity of the fuzzy quantity A the real number

$$\mathcal{A}(A; S) = \frac{1}{2} \int_0^{w_A} m(A_\alpha) dS(\alpha) \quad (11)$$

where w_A is the height of A and $m(\cdot)$ is the Lebesgue measure on the real line.

Remark 5.8. If A is a generalized fuzzy number we have $A_\alpha = [a_L(\alpha), a_R(\alpha)]$ for $\alpha \in [0, w_A]$ and then $Amb(A; S) = \mathcal{A}(A; S)$. So, from now on we denote by $\mathcal{A}(A; S)$ the ambiguity of A .

Proposition 5.9. *The ambiguity of the fuzzy quantity $A = B \cup C$ is*

$$\mathcal{A}(B \cup C; S) = \mathcal{A}(B; S) + \mathcal{A}(C; S) - \mathcal{A}(B \cap C; S). \quad (12)$$

Proof. Taking into account that $(B \cup C)_\alpha = B_\alpha \cup C_\alpha$ we obtain $m(A_\alpha) = m(B_\alpha \cup C_\alpha) = m(B_\alpha) + m(C_\alpha) - m(B_\alpha \cap C_\alpha)$ and the claim follows from (11) by using the linearity of the Riemann-Stieltjes integral. \square

Proposition 5.10. *Let A and B be two fuzzy quantities. Then*

$$A \subset B \implies \mathcal{A}(A; S) \leq \mathcal{A}(B; S).$$

Proof. Since $A \subset B$ we have $A_\alpha \subseteq B_\alpha$ and $w_A \leq w_B$. Then $m(A_\alpha) \leq m(B_\alpha)$ and

$$\int_0^{w_A} m(A_\alpha) dS(\alpha) \leq \int_0^{w_A} m(B_\alpha) dS(\alpha) \leq \int_0^{w_B} m(B_\alpha) dS(\alpha) = \mathcal{A}(B; S)$$

from which it follows that $\mathcal{A}(A; S) \leq \mathcal{A}(B; S)$. □

The next result extends property (4) to fuzzy quantities.

Proposition 5.11.

$$\mathcal{A}(B \cup C; S) = \frac{\mathcal{V}^*(B \cup C; S) - \mathcal{V}_*(B \cup C; S)}{2} h(w_B, w_C, w_{B \cap C}; S)$$

where

$$h(w_B, w_C, w_{B \cap C}; S) = s(w_B) + s(w_C) - s(w_{B \cap C}).$$

Proof. From (12), by using (4), we have

$$\begin{aligned} \mathcal{A}(B \cup C; S) &= \text{Amb}(B; S) + \text{Amb}(C; S) - \text{Amb}(B \cap C; S) \\ &= \frac{V^*(B; S) - V_*(B; S)}{2} s(w_B) + \frac{V^*(C; S) - V_*(C; S)}{2} s(w_C) \\ &\quad - \frac{V^*(B \cap C; S) - V_*(B \cap C; S)}{2} s(w_{B \cap C}) \end{aligned}$$

and then, applying (9) and (10),

$$\begin{aligned} &= \frac{1}{2} \left[(\sigma_1 V^*(B; S) + \sigma_2 V^*(C; S) - \sigma_3 V^*(B \cap C; S)) + \right. \\ &\quad \left. + (\sigma_1 V_*(B; S) + \sigma_2 V_*(C; S) - \sigma_3 V_*(B \cap C; S)) \right] (s(w_B) + s(w_C) - s(w_{B \cap C})) \\ &= \frac{\mathcal{V}^*(B \cup C; S) - \mathcal{V}_*(B \cup C; S)}{2} h(w_B, w_C, w_{B \cap C}; S). \end{aligned}$$

□

Note that from previous result the ambiguity of a fuzzy quantity $A = B \cup C$ is equal to the area of a triangle with base the interval $[\mathcal{V}_*(B \cup C; S), \mathcal{V}^*(B \cup C; S)]$ and height $h(w_B, w_C, w_{B \cap C}; S)$.

We now prove that the value and ambiguity of a fuzzy quantity $A = B \cup C$ with respect to a measure S are the same of those of the fuzzy quantity $\mathcal{I}_r(A) = (B \cup C)^r = B^r \cup C^r$ with respect to the Lebesgue measure.

Proposition 5.12. *We have*

$$(i) \quad \mathcal{V}_\lambda(B \cup C; r) = \mathcal{V}_\lambda(B^r \cup C^r; 1) = \mathcal{V}_\lambda((B \cup C)^r; 1);$$

$$(ii) \quad \mathcal{A}_\lambda(B \cup C; r) = \mathcal{A}_\lambda(B^r \cup C^r; 1) = \mathcal{A}_\lambda((B \cup C)^r; 1).$$

Proof. Since (by Proposition 4.14) $(B \cap C)^r = B^r \cap C^r$ we have $w_{B^r} = (w_B)^r$, $w_{C^r} = (w_C)^r$, $w_{B^r \cap C^r} = w_{(B \cap C)^r} = (w_{B \cap C})^r$ and thus, by defining $\gamma_i = \psi_i(w_{B^r}, w_{C^r}, w_{B^r \cap C^r})$ for $i = 1, 2, 3$, we get

$$\begin{aligned}\gamma_i &= \psi_i(w_{B^r}, w_{C^r}, w_{B^r \cap C^r}) = \psi_i((w_B)^r, (w_C)^r, (w_{B \cap C})^r) \\ &= \psi_i(s(w_B), s(w_C), s(w_{B \cap C})) = \sigma_i.\end{aligned}$$

From Proposition 4.12, noting that $(B \cup C)^r = B^r \cup C^r$, we get

$$\begin{aligned}\mathcal{V}_\lambda((B \cup C)^r; 1) &= \mathcal{V}_\lambda(B^r \cup C^r; 1) \\ &= \gamma_1 \mathcal{V}_\lambda(B^r; 1) + \gamma_2 \mathcal{V}_\lambda(C^r; 1) - \gamma_3 \mathcal{V}_\lambda(B^r \cap C^r; 1) \\ &= \sigma_1 \mathcal{V}_\lambda(B; r) + \sigma_2 \mathcal{V}_\lambda(C; r) - \sigma_3 \mathcal{V}_\lambda((B \cap C)^r; 1) \\ &= \sigma_1 \mathcal{V}_\lambda(B; r) + \sigma_2 \mathcal{V}_\lambda(C; r) - \sigma_3 \mathcal{V}_\lambda(B \cap C; r) \\ &= \mathcal{V}_\lambda(B \cup C; r)\end{aligned}$$

and thus (i) is proved. In a similar way we obtain (ii). \square

In the following we compute value and ambiguity of a particular class of fuzzy quantities.

Example 5.13. Let us consider the fuzzy quantity

$$A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7; w_1, w_2, w_3)$$

defined by the membership function

$$\mu_A(x) = \begin{cases} \frac{w_1}{a_2 - a_1}(x - a_1) & a_1 \leq x \leq a_2 \\ w_1 & a_2 \leq x \leq a_3 \\ \frac{w_1 - w_2}{a_4 - a_3}(a_4 - x) + w_2 & a_3 \leq x \leq a_4 \\ \frac{a_4 - a_3}{a_5 - a_4}(x - a_4) + w_2 & a_4 \leq x \leq a_5 \\ w_3 & a_5 \leq x \leq a_6 \\ \frac{w_3}{a_7 - a_6}(a_7 - x) & a_6 \leq x \leq a_7 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

with $w_2 < \min\{w_1, w_3\}$. The fuzzy quantity A is shown in Fig. 3.

Proposition 5.14. *If $A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7; w_1, w_2, w_3)$ is the fuzzy quantity defined by (13) then the value and ambiguity of A are given by, respectively,*

$$\begin{aligned}\mathcal{V}_\lambda(A; r) &= (1 - \lambda)\mathcal{V}_*(A; r) + \lambda\mathcal{V}^*(A; r) \\ \mathcal{A}(A; r) &= \frac{\mathcal{V}^*(A; r) - \mathcal{V}_*(A; r)}{2} h(w_1, w_3, w_2; r)\end{aligned}$$

where $h(w_1, w_3, w_2; r) = w_1^r + w_3^r - w_2^r$,

$$\begin{aligned}\mathcal{V}_*(A; r) &= \gamma_1 \left[a_1 + \frac{r}{r+1}(a_2 - a_1) \right] - \gamma_2 a_4 + \gamma_3 \left[a_4 + \frac{r}{r+1}g(w_3; w_2, r)(a_5 - a_4) \right] \\ \mathcal{V}^*(A; r) &= \gamma_1 \left[a_4 - \frac{r}{r+1}g(w_1; w_2, r)(a_4 - a_3) \right] - \gamma_2 a_4 + \gamma_3 \left[a_7 - \frac{r}{r+1}(a_7 - a_6) \right]\end{aligned}$$

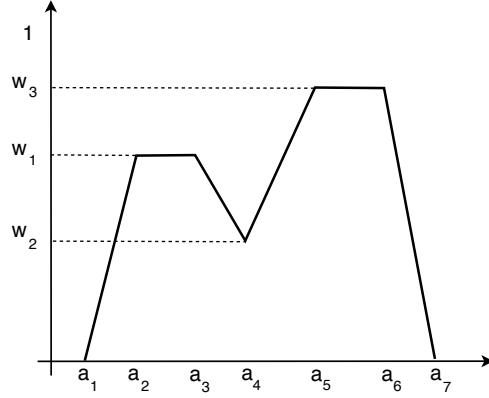


Figure 3: Fuzzy quantity $A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7; w_1, w_2, w_3)$

with

$$g(w; w_2, r) = 1 - \frac{w_2}{r(w - w_2)} \left[1 - \left(\frac{w_2}{w} \right)^r \right], \quad w = w_1, w_3$$

and $\gamma_i = \frac{w_i^r}{h(w_1, w_3, w_2; r)}$, $i = 1, 2, 3$.

Proof. We can write $A = B \cup C$ where B and C are generalized fuzzy numbers defined, respectively, by

$$\mu_B(x) = \begin{cases} \frac{w_1}{a_2 - a_1}(x - a_1) & a_1 \leq x \leq a_2 \\ w_1 & a_2 \leq x \leq a_3 \\ \frac{w_1 - w_2}{a_4 - a_3}(a_4 - x) + w_2 & a_3 \leq x \leq a_4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu_C(x) = \begin{cases} \frac{w_3 - w_2}{a_5 - a_4}(x - a_4) + w_2 & a_4 \leq x \leq a_5 \\ w_3 & a_5 \leq x \leq a_6 \\ \frac{w_3}{a_7 - a_6}(a_7 - x) & a_6 \leq x \leq a_7 \\ 0 & \text{otherwise} \end{cases}.$$

By observing that $B_\alpha = [b_L(\alpha), b_R(\alpha)]$, $0 \leq \alpha \leq w_1$, where

$$b_L(\alpha) = a_1 + \frac{a_2 - a_1}{w_1} \alpha \quad 0 \leq \alpha \leq w_1,$$

$$b_R(\alpha) = \begin{cases} a_4 & 0 \leq \alpha \leq w_2 \\ a_4 - \frac{a_4 - a_3}{w_1 - w_2}(\alpha - w_2) & w_2 \leq \alpha \leq w_1 \end{cases}$$

and $C_\alpha = [c_L(\alpha), c_R(\alpha)]$, $0 \leq \alpha \leq w_3$, where

$$\begin{aligned} c_L(\alpha) &= \begin{cases} a_4 & 0 \leq \alpha \leq w_2 \\ a_4 + \frac{a_5 - a_4}{w_3 - w_2}(\alpha - w_2) & w_2 \leq \alpha \leq w_3, \end{cases} \\ c_R(\alpha) &= a_7 - \frac{a_7 - a_6}{w_3}\alpha \quad 0 \leq \alpha \leq w_3 \end{aligned}$$

from Definition 4.1 it follows by simple calculation that

$$\begin{aligned} \mathcal{V}_\lambda(B; r) &= (1 - \lambda) \left[a_1 + \frac{r}{r+1}(a_2 - a_1) \right] + \lambda \left[a_4 - \frac{r}{r+1}g(w_1; w_2, r)(a_4 - a_3) \right] \\ \mathcal{V}_\lambda(C; r) &= (1 - \lambda) \left[a_4 + \frac{r}{r+1}g(w_3; w_2, r)(a_5 - a_4) \right] + \lambda \left[a_7 - \frac{r}{r+1}(a_7 - a_6) \right] \end{aligned}$$

with

$$g(w; w_2, r) = \frac{w}{w - w_2} \left[1 - \left(\frac{w_2}{w} \right)^{r+1} \right] - \frac{(r+1)w_2}{r(w - w_2)} \left[1 - \left(\frac{w_2}{w} \right)^r \right], \quad w = w_1, w_3.$$

The last expression can be rewritten as

$$g(w; w_2, r) = 1 - \frac{w_2}{r(w - w_2)} \left[1 - \left(\frac{w_2}{w} \right)^r \right], \quad w = w_1, w_3.$$

Moreover, noting that $B \cap C$ is the generalized fuzzy number defined by $\mu_{B \cap C}(x) = w_2$ if $x = a_4$ and $\mu_{B \cap C}(x) = 0$ otherwise, the value of $B \cap C$ is $\mathcal{V}_\lambda(B \cap C; r) = a_4$. From (6), taking into account that $\sigma_1 = \gamma_1$, $\sigma_2 = \gamma_3$ and $\sigma_3 = \gamma_2$ since $w_B = w_1$, $w_C = w_3$ and $w_{B \cap C} = w_2$, we easily get the assertion. \square

Remark 5.15. If $r = 1$ we obtain for the fuzzy quantity defined by (13)

$$\begin{aligned} \mathcal{V}_*(A; 1) &= \frac{1}{2h} [(a_1 + a_2)w_1 + (a_4 + a_5)(w_3 - w_2)] \\ \mathcal{V}^*(A; 1) &= \frac{1}{2h} [(a_3 + a_4)(w_1 - w_2) + (a_6 + a_7)w_3] \end{aligned}$$

where $h = h(w_1, w_3, w_2; 1) = w_1 + w_3 - w_2$, and then, by denoting $\mathcal{V}_\lambda(A) = \mathcal{V}_\lambda(A; 1)$ and $\mathcal{A}(A) = \mathcal{A}(A; 1)$, the value of A is

$$\mathcal{V}_\lambda(A) = (1 - \lambda)\mathcal{V}_*(A; 1) + \lambda\mathcal{V}^*(A; 1)$$

and the ambiguity

$$\mathcal{A}(A) = \frac{1}{4} [(a_3 + a_4 - a_1 - a_2)w_1 + (a_5 - a_3)w_2 + (a_6 + a_7 - a_4 - a_5)w_3].$$

A simple calculation gives the following properties

$$(i) \quad \frac{\partial \mathcal{V}_\lambda(A)}{\partial w_1} < 0, \quad \frac{\partial \mathcal{V}_\lambda(A)}{\partial w_3} > 0,$$

$$(ii) \frac{\partial \mathcal{V}_\lambda(A)}{\partial w_2} > 0 \text{ iff}$$

$$\lambda(a_6 + a_7 - a_3 - a_4)w_3 > (1 - \lambda)(a_4 + a_5 - a_1 - a_2)w_1, \quad (14)$$

$$(iii) \frac{\partial \mathcal{A}(A)}{\partial w_i} > 0, i = 1, 2, 3.$$

Hence the evaluation of A is decreasing with respect to w_1 and increasing with respect to w_3 . Moreover, it is increasing with respect to w_2 if condition (14) is satisfied. Observe that if in (14) equality holds then the evaluation is independent of w_2 .

Let $\bar{\lambda}$ be such that in (14) equality holds. The effect of the parameter λ on the monotonicity of the value is that when $\lambda < \bar{\lambda}$ (pessimist decision maker) then evaluation is decreasing with respect to w_2 , whereas in the case of $\lambda > \bar{\lambda}$ (optimist decision maker) it is increasing with respect to w_2 .

Also, the ambiguity is increasing with respect to w_1, w_2, w_3 .

Example 5.16. As an application, we consider an example proposed in [26]. Let A be the fuzzy quantity shown in Fig. 4. In [26] the authors give an evaluation of A equal to 5.15 by using the area compensation method. In [37], by decomposing the fuzzy quantity A into the union of two generalized fuzzy numbers B and C , the value of A is calculated by applying (6).

We observe that A is a fuzzy quantity of the type (13), that is

$$A = (1, 3, 3, 4, 6, 7, 9; 0.6, 0.4, 0.8),$$

and then we can compute the value and the ambiguity of A by using Proposition 5.14. For $\lambda = 0.5$ and $r = 1$ we obtain $\mathcal{V}(A) = 5.15$ and $\mathcal{A}(A) = 1.95$.

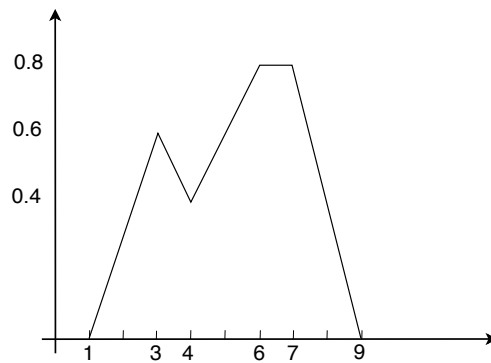


Figure 4: Fuzzy quantity

6 Ranking of fuzzy quantities

We propose a lexicographic ranking procedure based on the value-ambiguity pair. We use the ambiguity as the degree of ordering in the case that the values of the two fuzzy quantities are

equal. Ambiguity is a measure of the vagueness, that is the lack of precision in determining the exact value of a magnitude. A crisp number has zero ambiguity. Therefore a fuzzy quantity is smaller as its ambiguity is greater.

Let us denote by $\mathcal{V}(A)$ and $\mathcal{A}(A)$, respectively, the value and ambiguity of A with respect to a parameter λ and a measure S (fixed).

Our ranking method can be summarized into the following steps:

1. For two fuzzy quantities A and B , compare $\mathcal{V}(A)$ and $\mathcal{V}(B)$ and rank A and B according to the relative position of $\mathcal{V}(A)$ and $\mathcal{V}(B)$, i.e.
 - if $\mathcal{V}(A) > \mathcal{V}(B)$ then $A \succ B$;
 - if $\mathcal{V}(A) < \mathcal{V}(B)$ then $A \prec B$;
 - if $\mathcal{V}(A) = \mathcal{V}(B)$ then go to the next step.
2. Compare $\mathcal{A}(A)$ and $\mathcal{A}(B)$:
 - if $\mathcal{A}(A) < \mathcal{A}(B)$ then $A \succ B$;
 - if $\mathcal{A}(A) > \mathcal{A}(B)$ then $A \prec B$;
 - if $\mathcal{A}(A) = \mathcal{A}(B)$ then $A \sim B$, that is A and B are indifferent.

In [47] the authors proposed the following axioms as reasonable properties for the rationality of a ranking method \mathcal{R} for the ordering of fuzzy quantities belong to a set \mathcal{S} :

- A1. For any arbitrary finite subset \mathcal{A} of \mathcal{S} and $A \in \mathcal{A}$, $A \preceq A$ on \mathcal{A} .
- A2. For any arbitrary finite subset \mathcal{A} of \mathcal{S} and $(A, B) \in \mathcal{A}^2$, $A \succeq B$ and $B \succeq A$ on \mathcal{A} , we should have $A \sim B$ on \mathcal{A} .
- A3. For any arbitrary finite subset \mathcal{A} of \mathcal{S} and $(A, B, C) \in \mathcal{A}^3$, $A \succeq B$ and $B \succeq C$ on \mathcal{A} , we should have $A \succeq C$ on \mathcal{A} .
- A4. For any arbitrary finite subset \mathcal{A} of \mathcal{S} and $(A, B) \in \mathcal{A}^2$, $\inf \text{supp}(A) > \sup \text{supp}(B)$, we should have $A \succeq B$ on \mathcal{A} .
- A5. Let \mathcal{S} and \mathcal{S}' be two arbitrary finite sets of fuzzy quantities in which \mathcal{R} can be applied and A and B are in $\mathcal{S} \cap \mathcal{S}'$. We obtain the ranking order $A \succeq B$ on \mathcal{S}' iff $A \succeq B$ on \mathcal{S} .
- A6. Let $A, B, A \oplus C, B \oplus C$ be elements of \mathcal{S} . If $A \succeq B$ on $\{A, B\}$ then $A \oplus C \succeq B \oplus C$ on $\{A \oplus C, B \oplus C\}$.

The proposed ranking method satisfies properties $A_1 - A_5$ but it does not satisfy property A_6 as shown in the following example.

Example 6.1. Let us consider three generalized triangular fuzzy numbers $A = (2, 3, 4; 0.5)$, $B = (1, 2, 4; 1)$ and $C = (5, 14, 15; 1)$.

To calculate $A \oplus C$ and $B \oplus C$, where the operator \oplus is the sum of fuzzy numbers defined through Zadeh's extension principle [41, 42] we apply the following result of [48]: the sum of two generalized trapezoidal fuzzy numbers $A = (a_1, a_2, a_3, a_4; w_A)$ and $C = (c_1, c_2, c_3, c_4; w_C)$ is the generalized trapezoidal fuzzy number $D = A \oplus C = (d_1, d_2, d_3, d_4; w_D)$ given by

$$\begin{aligned}
 w_D &= \min\{w_A, w_C\} \\
 d_1 &= a_1 + c_1 \\
 d_2 &= a_1 + c_1 + (a_2 - a_1)w_D/w_A + (c_2 - c_1)w_D/w_C \\
 d_3 &= a_4 + c_4 - (a_4 - a_3)w_D/w_A - (c_4 - c_3)w_D/w_C \\
 d_4 &= a_4 + c_4.
 \end{aligned}$$

So we get $A \oplus C = (7, 12.5, 17.5, 19; 0.5)$ (generalized trapezoidal fuzzy number) and $B \oplus C = (6, 16, 19; 1)$ (generalized triangular fuzzy number). For $s(\alpha) = \alpha$ and $\lambda = 0.5$ we obtain (by using (3)) $\mathcal{V}(A) = 3 > 2.25 = \mathcal{V}(B)$ and thus $A \succ B$, but $A \oplus C \prec B \oplus C$ since $\mathcal{V}(A \oplus C) = 14 < 14.25 = \mathcal{V}(B \oplus C)$.

As an application, we use eight sets of fuzzy quantities to illustrate the working of the proposed ranking method. The eight sets of fuzzy quantities are shown in Fig. 5. We assume an optimism/pessimism coefficient λ equal to 0.5.

The results of ranking are shown in Table 1 and Table 2 for $s(\alpha) = \alpha$ and $s(\alpha) = \alpha^2$, respectively.

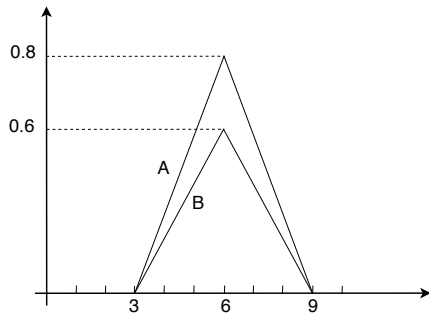
Note that the fuzzy quantities shown in Set 7 and in Set 8 are of the type (13).

Table 1: Results of ranking for $\lambda = 0.5$ and $r = 1$

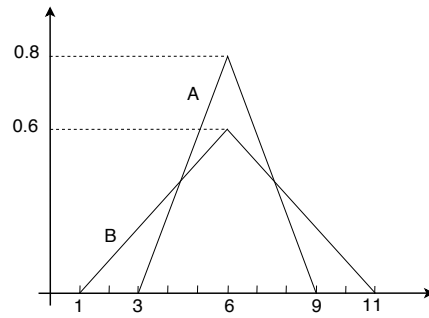
<i>Sets</i>	<i>A</i>		<i>B</i>		<i>Results</i>
	<i>Value</i>	<i>Ambiguity</i>	<i>Value</i>	<i>Ambiguity</i>	
<i>Set 1</i>	6.00	1.20	6.00	0.90	$B \succ A$
<i>Set 2</i>	6.00	1.20	6.00	1.50	$A \succ B$
<i>Set 3</i>	6.00	1.35	6.00	0.00	$B \succ A$
<i>Set 4</i>	6.00	1.35	3.00	0.00	$A \succ B$
<i>Set 5</i>	3.00	1.00	4.00	2.00	$B \succ A$
<i>Set 6</i>	4.00	2.50	6.50	2.00	$B \succ A$
<i>Set 7</i>	6.25	2.63	6.10	2.13	$A \succ B$
<i>Set 8</i>	6.39	3.55	6.32	2.65	$A \succ B$

Table 2: Results of ranking for $\lambda = 0.5$ and $r = 2$

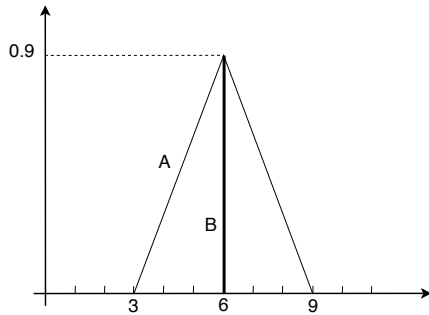
<i>Sets</i>	<i>A</i>		<i>B</i>		<i>Results</i>
	<i>Value</i>	<i>Ambiguity</i>	<i>Value</i>	<i>Ambiguity</i>	
<i>Set 1</i>	6.00	0.64	6.00	0.36	$B \succ A$
<i>Set 2</i>	6.00	0.64	6.00	0.60	$B \succ A$
<i>Set 3</i>	6.00	0.81	6.00	0.00	$B \succ A$
<i>Set 4</i>	6.00	0.81	3.00	0.00	$A \succ B$
<i>Set 5</i>	3.00	0.67	3.67	1.33	$B \succ A$
<i>Set 6</i>	3.83	2.17	6.83	1.50	$B \succ A$
<i>Set 7</i>	6.44	1.50	6.29	1.16	$A \succ B$
<i>Set 8</i>	6.67	2.81	6.51	1.71	$A \succ B$



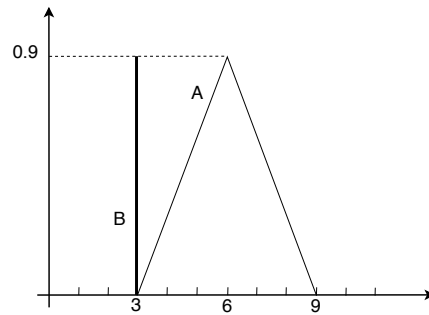
(a) *Set 1* $A = (3, 6, 9; 0.8)$
 $B = (3, 6, 9; 0.6)$



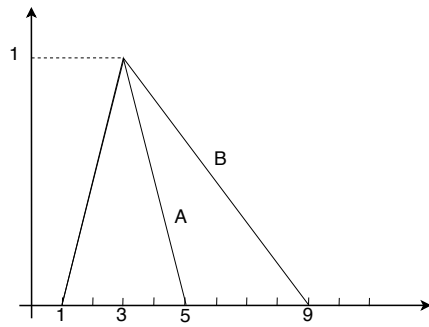
(b) *Set 2* $A = (3, 6, 9; 0.8)$
 $B = (1, 6, 11; 0.6)$



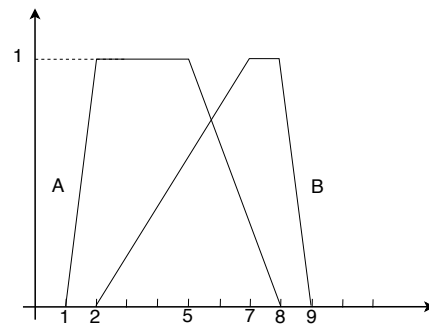
(c) *Set 3* $A = (3, 6, 9; 0.9)$
 $B = (6, 6, 6; 0.9)$



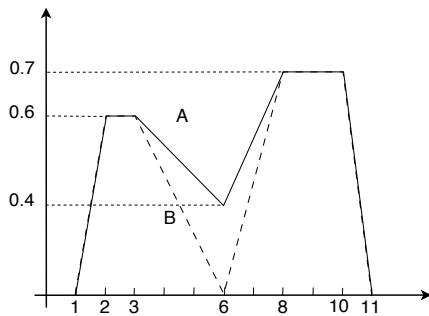
(d) *Set 4* $A = (3, 6, 9; 0.9)$
 $B = (3, 3, 3; 0.9)$



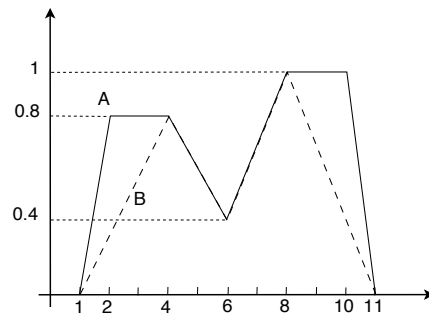
(e) *Set 5* $A = (1, 3, 5; 1)$
 $B = (1, 3, 9; 1)$



(f) *Set 6* $A = (1, 2, 5, 8; 1)$
 $B = (2, 7, 8, 9; 1)$



(g) *Set 7*
 $A =$
 $(1, 2, 3, 6, 8, 10, 11; 0.6, 0.4, 0.7)$
 $B =$
 $(1, 2, 3, 6, 8, 10, 11; 0.6, 0, 0.7)$



(h) *Set 8*
 $A =$
 $(1, 2, 4, 6, 8, 10, 11; 0.8, 0.4, 1)$
 $B =$
 $(1, 4, 4, 6, 8, 8, 11; 0.8, 0.4, 1)$

Figure 5: Sets of fuzzy quantities

7 Normalization

In this section we propose an alternative procedure to compute the value of a fuzzy quantity using fuzzy arithmetic. We illustrate an application of this method in Example 7.5.

Proposition 7.1. *Let A be a generalized fuzzy number with $0 < w_A \leq 1$ and let \tilde{A} be the normal fuzzy number defined by the membership function*

$$\mu_{\tilde{A}}(x) = \frac{\mu_A(x)}{w_A}.$$

Then we have

- (i) $V_*(A; r) = V_*(\tilde{A}; r)$, $V^*(A; r) = V^*(\tilde{A}; r)$, $\mathcal{V}_\lambda(A; r) = \mathcal{V}_\lambda(\tilde{A}; r)$;
- (ii) $\mathcal{A}(A; r) = w_A^r \mathcal{A}(\tilde{A}; r)$.

Proof. First we observe that for $t \in [0, 1]$

$$\tilde{A}_t = \{x \in X; \mu_{\tilde{A}}(x) \geq t\} = \{x \in X; \mu_A(x) \geq w_A t\} = A_{w_A t}$$

and so we have $\tilde{a}_L(t) = \inf \tilde{A}_t = \inf A_{w_A t} = a_L(w_A t)$. Then, by using the substitution $\alpha = w_A t$, we get

$$\begin{aligned} V_*(A; r) &= \frac{r}{w_A^r} \int_0^{w_A} a_L(\alpha) \alpha^{r-1} d\alpha = \frac{r}{w_A^r} \int_0^1 a_L(w_A t) w_A^{r-1} t^{r-1} w_A dt \\ &= r \int_0^1 \tilde{a}_L(t) t^{r-1} dt = V_*(\tilde{A}; r). \end{aligned}$$

In a similar way we obtain the remaining assertions. □

Remark 7.2. Note that previous result does not hold for a general S .

Corollary 7.3. *We have*

- (i) $V_*(A; S) = V_*(\tilde{A}^s; 1)$, $V^*(A; S) = V^*(\tilde{A}^s; 1)$, $\mathcal{V}_\lambda(A; S) = \mathcal{V}_\lambda(\tilde{A}^s; 1)$,
- (ii) $\mathcal{A}(A; S) = s(w_A) \mathcal{A}(\tilde{A}^s; 1)$,

where \tilde{A}^s is defined by

$$\mu_{\tilde{A}^s}(x) = \frac{\mu_{A^s}(x)}{s(w_A)} = \frac{s(\mu_A(x))}{s(w_A)}.$$

Proof. (i) From Proposition 4.12 we have $\mathcal{V}_\lambda(A; S) = \mathcal{V}_\lambda(A^s; 1)$. Moreover, by applying previous proposition to generalized fuzzy number A^s with $r = 1$ we get $\mathcal{V}_\lambda(A^s; 1) = \mathcal{V}_\lambda(\tilde{A}^s; 1)$.

(ii) From Proposition 4.12 we have $\mathcal{A}(A; S) = \mathcal{A}(A^s; 1)$. Again, by applying previous proposition (with $r = 1$) to A^s we get $\mathcal{A}(A^s; 1) = w_{A^s} \mathcal{A}(\tilde{A}^s; 1) = s(w_A) \mathcal{A}(\tilde{A}^s; 1)$. □

Proposition 7.4. *If X is the normal fuzzy number defined as*

$$X = \sigma_1 \tilde{B} \oplus \sigma_2 \tilde{C} \ominus \sigma_3 \widetilde{B \cap C} \quad (15)$$

then

$$\mathcal{V}_{1/2}(B \cup C; r) = \mathcal{V}_{1/2}(X; r).$$

Proof. From (6) and Proposition 7.1 we get

$$\begin{aligned} \mathcal{V}_{1/2}(B \cup C; r) &= \sigma_1 \mathcal{V}_{1/2}(B; r) + \sigma_2 \mathcal{V}_{1/2}(C; r) - \sigma_3 \mathcal{V}_{1/2}(B \cap C; r) \\ &= \sigma_1 \mathcal{V}_{1/2}(\tilde{B}; r) + \sigma_2 \mathcal{V}_{1/2}(\tilde{C}; r) - \sigma_3 \mathcal{V}_{1/2}(\widetilde{B \cap C}; r) \\ &= \mathcal{V}_{1/2}(X; r) \end{aligned}$$

where the last equality follows from Proposition 4.5 noting that for $\lambda = 1/2$ we have $\mathcal{V}_{1/2}(kA; S) = k\mathcal{V}_{1/2}(A; S)$ for any real number k . \square

Example 7.5. As an application of this result, let us consider again the fuzzy quantity A of Example 5.16. The fuzzy quantity A can be viewed as the union of two generalized fuzzy numbers B and C , i.e. $A = B \cup C$, where $B = (1, 3, 6; 0.6)$ is triangular and $C = (2, 6, 7, 9; 0.8)$ is trapezoidal. Their intersection is the triangular generalized fuzzy number $B \cap C = (2, 4, 6; 0.4)$.

We now compute the value of A by using fuzzy arithmetic. First we calculate the normal trapezoidal fuzzy number $X = (x_1, x_2, x_3, x_4; 1)$ defined in (15):

$$\begin{aligned} x_1 &= (0.6) \cdot 1 + (0.8) \cdot 2 - (0.4) \cdot 6 = -1/5 \\ x_2 &= (0.6) \cdot 3 + (0.8) \cdot 6 - (0.4) \cdot 4 = 5 \\ x_3 &= (0.6) \cdot 3 + (0.8) \cdot 7 - (0.4) \cdot 4 = 29/5 \\ x_4 &= (0.6) \cdot 6 + (0.8) \cdot 9 - (0.4) \cdot 2 = 10 \end{aligned}$$

and thus $X = (-1/5, 5, 29/5, 10; 1)$. Then, by applying Proposition 7.4 and using (3) we obtain for $\lambda = 0.5$ and $r = 1$

$$\mathcal{V}(A) = \mathcal{V}(B \cup C) = \mathcal{V}(X) = \frac{103}{20} = 5.15.$$

8 Concluding Remarks

In this article we studied the problem of evaluating and ranking fuzzy quantities, where a fuzzy quantity is any non-normal and non-convex fuzzy set, defined as the union of two, or more, generalized fuzzy numbers. To this aim we introduced a definition of ambiguity of non-normal and non-convex fuzzy membership functions. Relations between value and ambiguity were also investigated.

In our view, this framework can be also employed to other types of fuzzy sets characterized by complex shaped membership functions. For instance, our procedure can be used for the evaluation and ranking of non convex and non normal intuitionistic fuzzy sets. This will be a topic of our future research work.

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