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THE MAXIMAL GENUS OF SPACE CURVES IN THE RANGE A

EDOARDO BALLICO

*Dipartimento di Matematica,
Università di Trento,
Via Sommarive 14, 38123 Povo, Trento, Italy.*

PHILIPPE ELLIA

*Dipartimento di Matematica e Informatica,
Università degli Studi di Ferrara,
Via Machiavelli 30, 44121, Ferrara, Italy.*

ABSTRACT. Fix positive integers d, m such that $\frac{m^2+4m+6}{6} \le d < \frac{m^2+4m+6}{3}$ (the so-called Range A for space curves). Let $G(d,m)$ be the maximal genus of a smooth and connected degree d curve $C \subset \mathbb{P}^3$ such that $h^0(\mathcal{I}_C(m-1)) = 0$. Here we prove that $G(d,m) = 1 + (m-1)d - \binom{m+2}{3}$ if $m \ge 13.8 \cdot 10^5$. The case $\frac{m^2+4m+6}{4} \le d < \frac{m^2+4m+6}{3}$ was known by work of Fløystad [14, 15] and joint work of Ballico, Bolondi, Ellia, Mirò-Roig; see [2]. To prove the case $\frac{m^2+4m+6}{6} \le d < \frac{m^2+4m+6}{4}$ we show that in this range for large d every integer between 0 and $1 + (m-1)d - \binom{m+2}{3}$ is the genus of some degree d smooth and connected curve $C \subset \mathbb{P}^3$ such that $h^0(\mathcal{I}_C(m-1)) = 0$.

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1. Introduction

Fix integers $m \ge 2$ and $d \ge 3$. Let $G(d,m)$ be the maximal genus of a smooth and connected curve, of degree d, $C \subset \mathbb{P}^3$ with $h^0(\mathcal{I}_C(m-1)) = 0$. A classical problem which goes back to Halphen [17], is the computation of the integer $G(d,m)$ ([18, Problem 3.1], [19]). For this problem the set of all $(d,m) \subset \mathbb{N}^2$ was divided in the

E-mail addresses: `edoardo.ballico@unitn.it, phe@unife.it`.
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following 4 regions (ranges $\emptyset$, A, B and C) ([2, 14, 18, 19]), because both the integer $G(d,m)$ and the geometric properties of the curve with maximal genera are very different in different regions. Set

$$G_A(d,m) := 1 + (m-1)d - \binom{m+2}{3}.$$

If $d < \frac{m^2+4m+6}{6}$, then no such curve exists ([18, Theorem 3.3]). Hence this is called the Range $\emptyset$. Range A is when

$$\frac{m^2+4m+6}{6} \le d < \frac{m^2+4m+6}{3} \tag{1}$$

In Range A it is easy to see that $G(d,m) \le G_A(d,m)$ ([18, Theorem 3.3]) and it was conjectured that equality holds ([22, page 364]). This conjecture is known to be true for all d, m such that $\frac{m^2+4m+6}{4} \le d < \frac{m^2+4m+6}{3}$ ([14], [2, Corollary 2.4]). We also mention [15] and [22] which settle a few cases just on the right of the last inequality of (1). In this paper we prove the following result.

Theorem 1. *Fix an integer $m \ge 13.8 \cdot 10^5$. Let d be any integer satisfying (1). Then $G(d,m) = G_A(d,m)$.*

Fix m and d in the Range A, i.e. satisfying (1). By [18, first part of the proof of Theorem 3] any curve $C \subset \mathbb{P}^3$ with degree d and genus g with $g = G_A(d,m)$ has $h^1(\mathcal{O}_C(m-1)) = 0$, i.e. $h^2(\mathcal{I}_C(m-1)) = 0$. Since $h^0(\mathcal{I}_C(m-1)) = 0$ and $(m-1)d + 1 - g = \binom{m+2}{3}$, Riemann-Roch gives $h^0(\mathcal{O}_C(m-1)) = \binom{m+2}{3}$ and hence $h^i(\mathcal{I}_C(m-1)) = 0$ for all $i \in \mathbb{N}$. Thus the curves proved to exist in Theorem 1 have some nice cohomological properties. Now we add the stronger assumption that $h^1(\mathcal{O}_C(m-2)) = 0$, i.e. $h^2(\mathcal{I}_C(m-2)) = 0$. Since $\dim C = 1$, the exact sequence

$$0 \to \mathcal{I}_C(t) \to \mathcal{O}_{\mathbb{P}^3}(t) \to \mathcal{O}_C(t) \to 0$$

gives $h^i(\mathcal{I}_C(m-i)) = 0$ for all $i \ge 3$. Thus the Castelnuovo-Mumford's lemma gives $h^1(\mathcal{I}_C(t)) = 0$ for all $t \ge m$, that $\mathcal{I}_C(m))$ is globally generated and that the homogeneous ideal of C is generated by forms of degree m ([26, p. 99], [4, §3]). Thus with this additional assumption we would get strong geometrical properties of C. In the next theorem we will prove that $h^1(\mathcal{O}_C(m-2)) = 0$ and hence the homogeneous ideal of C is generated by degree m forms.

For any scheme $X \subset \mathbb{P}^3$ let N_X denote its normal sheaf. Our main result is the following one.

Theorem 2. *Fix positive integers m and d such that $\frac{m^2+4m+6}{6} \le d < \frac{m^2+4m+6}{4}$ and $G_A(d,m) \ge 0.34 \cdot 10^{15}$. Then $G(d,m) = G_A(d,m)$ and there is a smooth and connected curve $C \subset \mathbb{P}^3$ of degree d and genus $G(d,m)$ such that $h^1(\mathcal{O}_C(m-2)) = 0$, $h^i(\mathcal{I}_C(m-1)) = 0$, $i = 0,1$, and $h^1(N_C(-1)) = 0$.*

We will get Theorem 1 from Theorem 2 with a small argument.

In the statement of Theorem 2 we assumed that $d < \frac{m^2+4m+6}{4}$, because the range $\frac{m^2+4m+6}{4} \le d < \frac{m^2+4m+6}{3}$ is covered by [2, Proposition 2.2 and Corollary 2.4]. In the range of [2, Corollary 2.4] our proof of Theorem 2 is very bad (it gives examples of nice curves C, but not enough to cover all d). We stated in Theorem 2 that the solution C satisfies $h^1(N_C(-1)) = 0$, because this vanishing has the following interesting geometrical consequences. Since $h^1(N_C(-1)) = 0$, we have $h^1(N_C) = 0$ and hence the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^3)$ is smooth and of dimension $4d$ at $[C]$ ([27,

§1]). Let Γ be the unique irreducible component of $\mathrm{Hilb}(\mathbb{P}^3)$ containing $[C]$. Fix a a plane $H \subset \mathbb{P}^3$. Since $h^1(N_C(-1)) = 0$, for a general $S \subset H$ with cardinality d there is $X \in \Gamma$ such that $X \cap \Gamma = S$ ([25], [27, Theorem 1.5]).

Take (d, m) in the Range A and an integer g such that $0 \le g < G_A(d, m)$. Is there a smooth and connected curve $C \subset \mathbb{P}^3$ of degree d and genus g with $h^0(\mathcal{I}_C(m-1)) = 0$? In the upper half of Range A we know it only if $G_A(d, m) - g$ is small ([2, Proposition 4.3]). In the lower half of the Range A we adapt the proof of Theorem 2 to prove the following result.

Theorem 3. *Fix integers m, d, g such that $m \ge 13.8 \cdot 10^5$, $\frac{m^2+4m+6}{6} \le d < \frac{m^2+4m+6}{4}$ and $0 \le g \le G_A(d, m)$. Then there is a smooth and connected curve $C \subset \mathbb{P}^3$ of degree d and genus g such that $h^0(\mathcal{I}_C(m-1)) = 0$ and $h^1(N_C(-1)) = 0$.*

As an easy corollary of Theorem 3 we get the following result.

Corollary 1. *Fix integers d, m such that $m \ge 13.8 \cdot 10^5$ and $d \ge \frac{m^2+4m+6}{4}$. Set $\delta := \lfloor \frac{m^2+4m+5}{4} \rfloor$. Fix an integer g such that $0 \le g \le G_A(\delta, m)$. Then there is a smooth and connected curve $C \subset \mathbb{P}^3$ of degree d and genus g such that $h^0(\mathcal{I}_C(m-1)) = 0$ and $h^1(N_C(-1)) = 0$.*

There are a few natural questions related to the topic of this paper. Fix positive integers d, m. We recall that if $d > m(m-1)$ (the so-called Range C) we have $G(d, m) = 1 + [d(d + m^2 - 4m) - r(m-r)(m-1)]/2m$, where r is the only integer such that $0 \le r \le m-1$ and $d + r \equiv 0 \pmod{m}$, and equality holds if and only if the curve is linked to a plane curve of degree r by the complete intersection of a surface of degree m and a surface of degree $d + r$ ([16]). For $d \gg m$ we have $G_C(d, m) \sim d^2/2m$ and hence $G(d, m) = G_C(d, m) \gg G_A(d, m)$. Now assume

$$\frac{m^2 + 4m + 6}{3} \le d \le m^2 - m$$

(the Range B). R. Hartshorne and A. Hirschowitz proved that $G(d, m) \ge G_B(d, m)$, where $G_B(d, m)$ is a complicated explicit function ([22, Th. 5.4]); the integer $G_B(d, m) - G_A(d, m)$ is clearly described as a sum of two terms in [22, Th. 5.4]. For this, using reflexive sheaves with prescribed Hilbert function ([20, 24]), they constructed curves C achieving this bound. Moreover they conjectured that $G(d, m) = G_B(d, m)$ in this range. There are some partials results ([10, 11, 20, 29, 30]), but this conjecture is still widely open.

Question 1. Take d, m in the Range B. Is every integer g such that $0 \le g \le G_B(d, m)$ the genus of a smooth curve $C \subset \mathbb{P}^3$ such that $\deg(C) = d$ and $h^0(\mathcal{I}_C(m-1)) = 0$?

Remark 1. Theorem 3 says that for the pair (d, m) listed in its statement all the genera up to the maximal one are realized. In the range C for the pairs (d, m) very near to the maximum genus $G(d, m)$ there are well-known gaps (called Halphen's gaps) for the genera of degree d smooth space curve C with $h^0(\mathcal{I}_C(m-1)) = 0$ ([9], [5, Theorem 3.3]). If we use the unknown integer $G(d, m)$ instead of $G_B(d, m)$, then Question 1 asks a proof of the non-existence of Halphen's gaps in the Range B. We also conjecture that Halphen's gaps do not arise in the Range A. We conjecture that Question 1 has a positive answer, except at most for a tiny set of pairs (d, m) near the Range C. For the Range C we conjecture that all (d, g, m) with (d, m) in the Range C and $0 \le g \le G_C(d, m+1)$ are realized by some smooth curve $C \subset \mathbb{P}^3$.

We explained before and after Theorem 2 the geometric properties (Hilbert function, Hilbert function of its general hyperplane section and the smoothness of the Hilbert scheme) satisfied by any curve C as in the statement of Theorem 2. All curves C realizing $G(d,s)$ in the range C are arithmetically Cohen-Macaulay and in particular they have maximal rank and the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^3)$ is smooth at $[C]$ of known dimension $h^0(N_C)$ ([12]). In almost all cases $h^0(N_C) > 4d$. In general we may ask:

Question 2. Is it true that any curve, C, of degree d, genus $G(d,s)$, with $h^0(\mathcal{I}_C(s-1)) = 0$, is of maximal rank and is a smooth point of the Hilbert scheme ?

For given d, s, do these curves belong to the same irreducible component of the Hilbert scheme ?

See [6] for partial results. In section 2 we briefly describe the proof, describe the main novelties of this paper, e.g. the difference with [3] and the main numerical obstructions arising for small m. The proofs of our theorems use an inductive proof, called the Horace Method, first used by A. Hirschowitz ([23]).

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2. A roadmap

In section 3 we describes curves $C_t \subset \mathbb{P}^3$ and the union $C_{t,k}$, $t \geq k$, of a curve C_t and a curve C_k. Their main properties is that $h^i(\mathcal{I}_{C_{k,t}}(k+t-1)) = 0$ for all $i \geq 0$. Curves C_t are used in [2, 8, 14, 15]. The curves $C_{t,t}$ and $C_{t,t-1}$ are the starting point of the inductive proof of the existence of curves with maximal rank given in [3] for ranges of pairs (d,g) with g of order $d^{3/2}$, i.e. very far from the Brill-Noether range. In the present paper we use all curves $C_{t,k}$ with, say, $t \leq 200k$. In Example 1 we explain why a key inductive step does not work if, for a fixed k, we try to use the curves $C_{t,k}$ with $t \gg k$.

Then we add inductively non-special curves in the following way. For all integers $s \geq k+t+1$ such that $s \equiv k+t-1 \pmod 2$ we construct a non-special curve $Y_s \subset \mathbb{P}^3$ with a certain degree $a(s,t,k)$ and genus $g(s,t,k)$ such that $Y_s \cap C_{t,k} = \emptyset$ and $h^i(\mathcal{I}_{C_{t,k}\cup Y_s}(s)) = 0$ for all $i \geq 0$. Setting $g_t := h^1(\mathcal{O}_{C_t})$ and $g_{t,k} := g_t + g_k$ the smooth curve $C_{t,k} \cup Y_s$ has 3 connected components and $h^1(\mathcal{O}_{C_{t,k}\cup Y}) = g_{t,k} + g(s,t,k)$. Since $\lim_{s\to+\infty} g(s,t,k) = +\infty$ (Lemma 4), for each fixed genus g we may find $s \gg 0$ with $g_{t,k} + g(s,t,k) > g$.

The novel part of this paper are essentially sections 7 and 10 (or, rather that section 7 allows us to conclude the proof of all theorems in section 10). We need to prove the theorems of the introduction for a fixed very large m. We fix m and d and take $g := G_A(d,m)$. For large d,m we may assume that $g \geq g_{1000,1000}$. To be sure that for large m and hence (since we are in the Range A a large d) we may take a very large g (g with order $m^3 \gg d$) we use [3]. We take t,k such that $g_{t,k} \leq g \leq g_{t,k+2}$ and $t+k \equiv m \pmod 2$. Then we get a maximal integer $y \equiv t+k-1 \pmod 2$ such that $g_{t,k} + g(y,t,k) \leq g$. A key numerical lemma is that for a large m we have $y \leq m-7$ (Lemma 26). Then for $x \geq y+2$ and $x \equiv y \pmod 2$ we get a smooth connected curve X_x such that $X_x \cap C_{t,k} = \emptyset$, $p_a(X_s) + g_{t,k} = g$ and $h^1(\mathcal{I}_{C_{t,k}\cup X_x}(x)) = 0$ (Assertion $B(x,t,k)$ and its proof in Lemmas 27 and 28). We use a modification $B'(m-3,t,k)$ of $B(m-3,t,k)$ to get in one step a smooth and connected curve X such that $h^i(\mathcal{I}_X(m-1)) = 0$ for all i and X has degree

d and genus $g = G_A(d,m)$. Small modifications of this last step give all theorems stated in the introduction.

In the construction we allow a parameter α and so the reader will met $a(s,t,k)_\alpha$ and $g(s,t,k)_\alpha$. From section 7 on we just take $\alpha := 202$ and the integers $a(s,t,k)$ and $g(s,t,k)$ are just the integers $a(s,t,k)_\alpha$ and $g(s,t,k)_\alpha$ with $\alpha = 202$.

3. Preliminaries

We work over an algebraically closed base field with characteristic zero. By [2, Corollary 2.4] we only look at (d,m) in the Range A and with $d < (m^2+4m+6)/4$.

For any $o \in \mathbb{P}^3$ let $2o$ denote the zero-dimensional subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_o)^2$ as its ideal sheaf. The scheme $2o \subset \mathbb{P}^3$ is a zero-dimensional scheme with $\deg(2o) = 4$ and $2o_{\mathrm{red}} = \{o\}$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric. For any scheme $X \subset \mathbb{P}^3$ the residual scheme $\mathrm{Res}_Q(X)$ of X with respect to Q is the closed subscheme of $\mathbb{P}^3$ with $\mathcal{I}_X : \mathcal{I}_Q$ as its ideal sheaf. For any $t \in \mathbb{Z}$ we have an exact sequence (often called the residual sequence of X and Q):

$$0 \to \mathcal{I}_{\mathrm{Res}_Q(X)}(t-2) \to \mathcal{I}_X(t) \to \mathcal{I}_{X\cap Q,Q}(t) \to 0$$

Now assume $o \in X_{\mathrm{red}} \cap Q$ and the existence of a neighborhood $\mathcal{U}$ of o in $\mathbb{P}^3$ such that $X \cap \mathcal{U} = 2o \cup T$ with T a nodal curve, $T \subset Q$, and o a singular point of T. Then $\mathrm{Res}_Q(X) \cap \mathcal{U} = \{o\}$; see [23] for many pictures, which explain how to use the Horace lemma with respect to Q. We do a similar proof in several key lemmas (with T a union of $e \le 202$ lines in a ruling of Q and $\delta - e$ lines in the other ruling of Q).

Fix an integer $t > 0$. Let $C_t \subset \mathbb{P}^3$ denote any curve fitting in an exact sequence

$$0 \to t\mathcal{O}_{\mathbb{P}^3}(-t-1) \to (t+1)\mathcal{O}_{\mathbb{P}^3}(-t) \to \mathcal{I}_{C_t} \to 0 \tag{2}$$

Each C_t is arithmetically Cohen-Macaulay and in particular $h^0(\mathcal{O}_{C_t}) = 1$. By taking the Hilbert function in (2) we get

$$\deg(C_t) = t(t+1)/2, \quad p_a(C_t) = 1 + \frac{t(t+1)(2t-5)}{6},$$

$h^1(\mathcal{O}_{C_t}(t-2)) > 0$ and $h^1(\mathcal{O}_C(t-1)) = 0$. Hence $h^i(\mathcal{I}_{C_t}(t-1)) = 0$, $i = 0,1,2$. Note that $p_a(C_t) = 1 + \deg(C_t)(t-1) - \binom{t+2}{3}$. The set of all curves C_t fitting in (2) is an irreducible variety and its general member is smooth and connected. Among them there are the stick-figures called $\mathbf{K}_t$ in [2, 14, 15] (see [14, Lemma 2.11]), but we only use the smooth C_t. Since $h^1(N_{C_t}(-2)) = 0$ ([8, proof of Proposition 2], [14], [2, page 4592]), for a general $S \subset Q$ with $\#(S) = t(t+1)$ we may find C_t with $C_t \cap Q = S$ ([27, Theorem 1.5]). Set $C_{t,0} := C_t$. For all positive integers t, k let $C_{t,k}$ denote any disjoint union of a curve C_t and a curve C_k. If $t > 0$ and $k > 0$, then $\deg(C_{t,k}) = t(t+1)/2 + k(k+1)/2$, $h^0(\mathcal{O}_{C_{t,k}}) = 2$ and $h^1(\mathcal{O}_{C_{t,k}}) = 2 + t(t+1)(2t-5)/6 + k(k+1)(2k-5)/6$. Since $h^1(N_{C_{t,k}}(-2)) = 0$, for a general $S \subset Q$ with $\#(S) = t(t+1) + k(k+1)$ there is some $C_{t,k}$ with $C_{t,k} \cap Q = S$. Equivalently, for a general $C_{t,k}$ the set $C_{t,k} \cap Q$ is general in Q. We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$ ([3]). For reader's sake we reproduce the statement and the proof from [3].

Lemma 1. *([3, Lemma 2.1]) We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0,1,2$.*

Proof. ([3, Lemma 2.1]) Since $C_t \cap C_k = \emptyset$, we have $Tor^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{I}_{C_t}, \mathcal{I}_{C_k}) = 0$ and $\mathcal{I}_{C_t} \otimes \mathcal{I}_{C_k} = \mathcal{I}_{C_{t,k}}$. Therefore tensoring (2) with $\mathcal{I}_{C_k}(t+k-1)$ we get

$$0 \to t\mathcal{I}_{C_k}(k-2) \to (t+1)\mathcal{I}_{C_k}(k-1) \to \mathcal{I}_{C_{t,k}}(t+k-1) \to 0 \tag{3}$$

We have $h^2(\mathcal{I}_{C_k}(x)) = h^1(\mathcal{O}_{C_k}(x))$ and the latter integer is zero for all integers $x \ge k-2$, because $k-3$ is the maximal integer z with $h^1(\mathcal{O}_{C_k}(z)) > 0$. We have $h^1(\mathcal{I}_{C_k}(k-1)) = 0$, because C_k is arithmetically Cohen-Macaulay. Taking k instead of t in (2) we get $h^0(\mathcal{I}_{C_k}(k-1)) = 0$. Hence (3) gives $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$, $i = 0, 1, 2$. □

If $t > 0$ and $k > 0$ set

$$g_t := p_a(C_t) = 1 + \frac{t(t+1)(2t-5)}{6},$$
$$g_{t,k} := g_t + g_k = h^1(\mathcal{O}_{C_{t,k}}),$$
$$d_{t,k} := \deg(C_{t,k}) = t(t+1) + k(k+1).$$

Remark 2. Let $D_0 \subset \mathbb{P}^3$ be a smooth curve such that $h^1(\mathcal{O}_{D_0}(1)) = 0$. Fix distinct lines D_i, $1 \le i \le k$, such that $D_0 \cup D_1 \cup \cdots \cup D_k$ is nodal and $1 \le \#((D_0 \cup \cdots \cup D_{i-1}) \cap D_i) \le 2$ for each $i = 1, \ldots, k$. Then $D_0 \cup \cdots \cup D_k$ is smoothable ([21], [28]). A Mayer-Vietoris exact sequence gives $h^1(\mathcal{O}_{D_0 \cup \cdots \cup D_k}(1)) = 0$ and hence $D_0 \cup \cdots \cup D_k$ may be deformed to a non-special smooth curve.

We fix integers $m \ge 5$, $\alpha > 0$ and take positive integers t, k such that $t + k \equiv m \pmod 2$ and $t \ge k$. For all integers $s \ge t + k - 1$ with $s \equiv t + k - 1 \pmod 2$ we define the integers $a(s,t,k)_\alpha$, $b(s,t,k)_\alpha$ and $g(s,t,k)_\alpha$ in the following way. Set $a(t+k-1,t,k)_\alpha = b(t+k-1,t,k)_\alpha = g(t+k-1,t,k)_\alpha = 0$. Define the integers $a(t+k+1,t,k)_\alpha$ and $b(t+k+1,t,k)_\alpha$ by the relations

$$\begin{aligned} &(t+k+1)(d_{t,k} + a(t+k+1,t,k)_\alpha) + 3 - g_{t,k} \\ &+ b(t+k+1,t,k)_\alpha = \binom{t+k+4}{3},\ 0 \le b(t+k+1,t,k)_\alpha \le t+k \end{aligned} \tag{4}$$

Set $g(t+k+1,t,k)_\alpha := 0$. Hence if $s \in \{t+k-1, t+k+1\}$ the integers $a(s,t,k)_\alpha$, $b(s,t,k)_\alpha$ and $g(s,t,k)_\alpha$ do not depend on α. Fix an integer $s \ge t+k+3$ with $s \equiv t+k+1 \pmod 2$ and assume defined the integers $a(s-2,t,k)_\alpha$, $b(s-2,t,k)_\alpha$ and $g(s-2,t,k)_\alpha$. Define the integers $a(s,t,k)_\alpha$ and $b(s,t,k)_\alpha$ by the relations

$$\begin{aligned} &2(d_{t,k} + a(s-2,t,k)_\alpha) + (s-1)(a(s,t,k)_\alpha - a(s-2,t,k)_\alpha) + \alpha + \\ &+ b(s,t,k)_\alpha - b(s-2,t,k)_\alpha = (s+1)^2,\ 0 \le b(s,t,k) \le s-2 \end{aligned} \tag{5}$$

Set $g(s,t,k)_\alpha = g(s-2,t,k)_\alpha + a(s,t,k)_\alpha - a(s-2,t,k)_\alpha - \alpha$. We claim that

$$s(d_{t,k} + a(s,t,k)_\alpha) + 3 - g_{t,k} - g(s,t,k)_\alpha + b(s,t,k)_\alpha = \binom{s+3}{3} \tag{6}$$

To prove the claim use induction on s; add the equation in (5) to the case $s' = s-2$ of (6); start the inductive assumption with the case $s = t+k+1$ true by (4), (7) and (8)). We have $b(t+k+1,t,k)_\alpha \le t+k$, $b(s,t,k)_\alpha \le s-2$ if $s \ge t+k+3$ and $g(s,t,k)_\alpha = a(s,t,k)_\alpha - a(t+k+1,t,k)_\alpha - \alpha(s-t-k-1)/2$ for all $s \ge t+k+3$.

Later (from Lemma 16 on), we will assume $\alpha = 202$ and write $a(s,t,k)$, $b(s,t,k)$ and $g(s,t,k)$ instead of $a(s,t,k)_{202}$, $b(s,t,k)_{202}$ and $g(s,t,k)_\alpha$.

All our constructions work for any $\alpha \ge 105$, but we would get far worst bounds in some key numerical lemma taking $\alpha = 105$. The culprits are the numerical lemmas, which give upper bounds for a certain integer e. We always need to take $\alpha \ge e+1$. To get $e \le 104$ (resp. $e \le 201$) in Lemmas 14 and 15 (resp. Lemmas 8 and 10) we need $t + k \ge 1113636$ and $s \ge 1157520$ (resp. $t + k \ge 42040$ and $s \ge 42674$).

Taking $\alpha = 202$ we get far better bounds for all results stated in the introduction (e.g. if we used $\alpha = 105$ in Theorem 1 we would need roughly $m \geq 5 \cdot 10^6$), but it is certainly not an optimal choice. It is not necessary to quote [3] to obtain our results (e.g. a very weak form of [1] would suffice), but the bounds would be far worse (roughly you square all lower bounds assumptions); here the culprit is the last part of the proof of Theorem 2.

Remark 3. Under suitable assumptions on t, k and α we have $g(s,t,k)_\alpha \geq 26$ for all $s \geq t+k+3$ with $s \equiv t+k-1 \pmod 2$ (Lemma 13). Hence a general curve Y of genus $g(s,t,k)_\alpha$ and degree $a(s,t,k)_\alpha$ satisfies $h^1(N_Y(-2)) = 0$ ([27, page 67, inequality $DP(g) \leq g+3$]). We have $g(t+k+1,t,k)_\alpha = 0$. A general smooth rational space curve T of degree $x \geq 3$ has balanced normal bundle, i.e. N_T is a direct sum of two line bundles of degree $2x-1$ ([7, Proposition 6]), i.e. $h^1(N_T(-2)) = 0$. Fix integers q and $z \geq q+3$ such that a general non-special smooth curve $C \subset \mathbb{P}^3$ of degree z and genus q satisfies $h^1(N_C(-2)) = 0$ and hence (since $\chi(N_C(-2)) = 0$) $h^0(N_C(-2)) = 0$. Fix a smooth quadric $Q \subset \mathbb{P}^3$ and take a general $B \subset Q$ with $\#(B) = 2z$. By [27, Theorem 5.12] there is a non-special smooth curve $Y \subset \mathbb{P}^3$ of degree z and genus q such that $B = Y \cap Q$. The assumption "$g(s,t,k)_\alpha \geq 26$ " was the assumption made in [3]. After [3] was completed E. Larson proved a far better result ([25, Theorem 1.4], [31, Theorem 1.4]). Using it we could get slightly weaker numerical assumptions in all results in [3] (and hence in the results of this paper), but without lowering the bounds in the assumption by an order of magnitude.

Remark 4. Lemma 19 shows that to carry over all our steps we cannot have $t \gg k$, say we need $t \leq 200k$. Lemma 18 shows that to carry over the last few steps we also need $t \geq 30k$.

4. ASSERTIONS $A(s,t,k)_\alpha$ AND $A(s,t,k)$

As in section 3 we fix an integer $m \geq 5$, a positive integer α and take positive integers t,k such that $t+k \equiv m \pmod 2$.

For any integer $s \geq t+k+1$ with $s \equiv t+k+1 \pmod 2$ we define the following Assertion $A(s,t,k)_\alpha$ (we use the lemmas in the next section to make sense of it). We will prove $A(s,t,k)_\alpha$ for the quadruples (α,s,t,k) we need in section 6.

Let Q be a smooth quadric and $W \subset \mathbb{P}^3$ a reduced curve such that $Q \cap W$ is formed by $2\deg(W)$ points and no line of Q contains two or more points of W. Let $E \subset Q$ be a finite set. Take positive integers a, b. An *(a,b)-grid of Q adapted to* (W,E) is a union $T \subset Q$ of $a+b$ distinct lines of Q, a of them of bidegree $(1,0)$, b of them of bidegree $(0,1)$, each line of T contains a point of $W \cap Q$, no two lines of T contain the same point of $W \cap Q$ and $T \cap E = \emptyset$. Obviously the existence on an (a,b)-grid for (W,E) implies $2\deg(W) \geq a+b$. We have $\mathrm{Sing}(T) = (a-1)(b-1)$. Since no two lines of T contain the same point of $W \cap Q$, we have $\mathrm{Sing}(T) \cap W = \emptyset$.

Assertion $A(s,t,k)_\alpha$, $s \geq t+k+1$ with $s \equiv t+k+1 \pmod 2$: Set $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha$. Let e be the maximal positive integer such that $b(s,t,k)_\alpha > (e-1)(\delta-e-1)$ and $e \leq \delta/2$. Let Q be a smooth quadric. Fix $C_{t,k}$ intersecting transversally Q and such that $Q \cap C_{t,k}$ is formed by $2d_{t,k}$ general points of Q. We call $A(s,t,k)_\alpha$ the existence of a triple (Y,T,S) with the following properties:

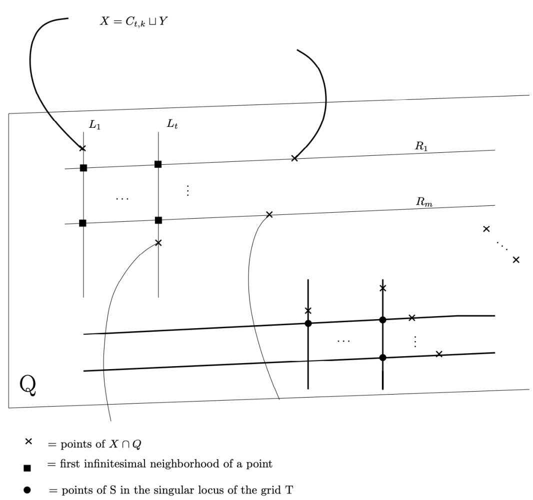
(1) Y is a smooth and connected curve of degree $a(s,t,k)_\alpha$ and genus $g(s,t,k)_\alpha$ such that $Y \cap C_{t,k} = \emptyset$, Y intersects transversally Q and $(C_{t,k} \cup Y) \cap Q$ is general in Q.
(2) T is a grid of type $(e, \delta - e)$ for (Y, E), where $E := C_{t,k} \cap Q$. $S \subset \mathrm{Sing}(T)$.
(3) $h^i(\mathcal{I}_{Y \cup C_{t,k} \cup S}(s)) = 0$, $i = 0, 1$.

In the next section we collect the numerical lemmas needed to prove $A(s,t,k)_\alpha$ for certain quadruples (α, s, t, k).

The proof is by induction on s. We may give a rough outline with the following two pictures in the next page. Look at the first one.

We consider $Z = X \cup (\cup_i L_i) \cup (\cup_j R_j) \cup \chi \cup S$. Here χ denotes the set of nilpotents (pictured with a black square). Assume F is a form of degree $s+2$ vanishing on Z. We look at $F|Q$: it vanishes on $D = (\cup_i L_i) \cup (\cup_j R_j)$, on the points of $X \cap (Q \setminus D)$ and on the points of S. We check that this implies $F|Q = 0$. It follows that $F = QG$, where G is a form of degree s vanishing on the residual scheme $Res_Q(Z)$. This residual scheme is given in the second picture.

Observe that the set of points χ_{red} lies on a grid (build by the lines L_i, R_j) on Q. Since we have taken $X \cup \chi_{red}$ satisfying $A(s,t,k)$, we get $G = 0$. Hence $F = 0$. Then we show that $X \cup (\cup_i L_i) \cup (\cup_j R_j) \cup \chi$ is smoothable. By semi-continuity this shows $A(s,t,k) \Rightarrow A(s+2,t,k)$.



We explain here the numerical restriction needed just to know that $A(s,t,k)_\alpha$ is well-defined (e.g. we need $\delta \geq 0$ and if $b(s,t,k)_\alpha > 0$ we also need $e \leq \delta/2$.

(1) For $s \geq t+k+3$ we need the existence of a non-special curve Y with degree $d(s,t,k)_\alpha$ and genus $g(s,t,k)_\alpha$ such that $Y \cap Q$ is formed by $2a(s,t,k)_\alpha$ general points of Q. To get this property it is sufficient to use that $g(s,t,k)_\alpha \geq 26$ (Remark 3) and quote [27, page 67, inequality $DP(g) \leq g+3$]). Since $(Y \cup C_{t,k} \cap Q$ is general in Q, no two points of it are contained in the same line of Q.
(2) We need $\delta \geq 0$ and this is true by Lemma 4.
(3) When $b(s,t,k)_\alpha > 0$ we need $e \leq \delta/2$. For this we use Lemmas 4, 14 and 8.

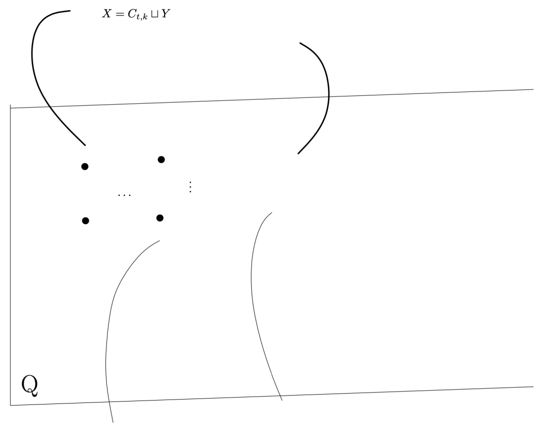


FIGURE 1. The residue scheme with respect to Q.

(4) We need $2a(s,t,k)_\alpha \ge \delta$ to get a degree δ grid $T \subset Q$ such that each irreducible component of T contains a point of $T \cap Q$.

Now we explain with examples why we need lower bounds on k to have a chance that $A(s,t,k)_\alpha$ is true when $s-t-k$ is small. In Lemma 19 we will show that it is important that t is not too large with respect to k.

Example 1. First take $k = 1$. We have $d_{t,1} = d_t + 2$ and $g_{t,1} = g_t$. Since $td_{t,1} + 2 - g_{t,1} = \binom{t+3}{3}$ and $\binom{t+5}{3} - \binom{t+3}{3} = (t+1)^2$, (4) gives $2d_{t,1} + (t+2)a(t+2,t,1)_\alpha + b(t+2,t,1)_\alpha) = (t+3)^2$ and hence $a(t+2,t,1)_\alpha \le 5$, which is not enough to attach any grid to $Y \cap Q$ when $\deg(Y) = a(t+2,t,1)_\alpha$.

Now take k arbitrary. We show that if $t \gg k$ we cannot find a grid $T \subset Q$ with $\deg(T) = \delta$ and with $\#(\mathrm{Sing}(T)) \ge s-2$ and thus if $b(s,t,k)$ is large there is no $S \subseteq \mathrm{Sing}(T)$ with $\#(S) = b(s,t,k)$. The maximum integer $\#(\mathrm{Sing}(T))$ for grids of bidegree $(e, \delta - e)$ is $\lfloor \delta/2 \rfloor \lceil \delta/2 \rceil$. We take $s = t+k+3$ and any fixed α. For $t \gg k$ we get $a(t+k+1,t,k)_\alpha \sim 4k$ and $\delta := a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha \sim 8k \ll s$. There are other numerical problems if $t \gg k$, most of them arising in the part in which g is used (Lemma 19).

5. NUMERICAL LEMMAS, I

For any integer $s \ge t+k-1$ set $I_{t,k}(s) := h^0(\mathcal{I}_{C_{t,k}}(s))$. Since $\deg(C_{t,k}) = d_{t,k}$, $h^1(\mathcal{O}_{C_{t,k}}) = g_{t,k}$ and $h^1(\mathcal{I}_{C_{t,k}}(s)) = 0$ for $s \ge t+k-1$, we have

$$I_{t,k}(s) = \frac{(s-t-k+1)}{6}[(s-t-k+3)(s-t-k+2)+3(t+k)(s-t-k+2)+6kt] \tag{7}$$

Note that $I_{t,k}(t+k-1) = 0$. By (4) we have

$$(s-1)a(s,t,k)_\alpha + b(s,t,k)_\alpha = I_{t,k}(s) - 4 - \alpha(s-t-k-1)/2 \tag{8}$$

Lemma 2. *We have*

$$I_{t,k}(s+2) - I_{t,k}(s) = (s+3)^2 - (t^2 + k^2 + t + k)$$

and if $u \in \mathbb{N}$ *and* $s = t + k + 1 + 2u$ *then*

$$I_{t,k}(s) = \frac{4}{3}u^3 + (6 + 2(t + k))u^2 + (\frac{26}{3} + 5(t + k) + 2tk)u + 4 + 3(t + k) + 2kt$$

Proof. Let Q be a general quadric surface and consider the exact sequence: $0 \to \mathcal{I}_C(s) \to \mathcal{I}_C(s+2) \to \mathcal{I}_{C\cap Q}(s+2) \to 0$ (where $C := C_{t,k}$). Since $h^i(\mathcal{I}_C(m)) = 0$, if $m \geq t + k - 1, 1 \leq i \leq 2$, we get: $I(s + 2) - I(s) = h^0(\mathcal{I}_{C\cap Q}(s + 2)) = h^0(\mathcal{O}_Q(s + 2)) - 2d_{t,k} = (s + 3)^2 - (t^2 + t + k^2 + k)$.

For the second assertion one can make a direct computation or proceed by induction using: $0 \to \mathcal{I}_C(s) \to \mathcal{I}_C(s + 1) \to \mathcal{I}_{C\cap H}(s + 1) \to 0$, where H is a general plane, and taking into account that $h^0(\mathcal{I}_{C\cap H}(s + 1)) = h^0(\mathcal{O}_H(s + 1)) - d_{t,k}$. □

We fix an integer $\alpha \geq 0$. Write $s = t + k + 1 + 2u$ with $u \in \mathbb{N}$. By (8) we have

$$a(s, t, k)_\alpha(2u + t + k) + b(s, t, k)_\alpha =$$
$$\frac{4u^3}{3} + [6 + 2(t + k)]u^2 + [\frac{26}{3} + 5(t + k) + 2kt]u + 4 + 3(t + k) + 2kt - \alpha u$$

By (5) we have

$$(9) \qquad (s - 1)a(s, t, k)_\alpha + b(s, t, k)_\alpha = I_{t,k}(s) - 4 - \alpha(s - t - k - 1)/2$$

We compare $a(s, t, k)_\alpha$, $s = t + k + 1 + 2u$, with the function

$$\psi(u) := \frac{2u^2}{3} + [\frac{2(t + k)}{3} + 3]u + t + k$$

Lemma 3. *If* $t \geq 4$ *and* $k \geq 4$, *then* $a(s, t, k)_\alpha \leq \psi(u)$, *i.e.*

$$a(s, t, k)_\alpha \leq \frac{s^2 + 7s}{6} - \frac{(t + k)^2}{6} - \frac{3(t + k)}{6} - \frac{8}{6}$$

Proof. Note that

$$(2u + t + k)\psi(u) = \frac{4u^3}{3} + [2(t + k) + 6]u^2 + [\frac{2(t + k)^2}{3} + 5(t + k)]u + (t + k)^2$$

Since $b(s, t, k)_\alpha \geq 0$ and $\alpha u \geq 0$, we get:

$$(10) \quad (2u+t+k)(\psi(u)-a(s, t, k)_\alpha \geq [\frac{2(t + k)^2}{3} - 2kt - \frac{26}{3}]u + (t+k)^2 - 3(t+k) - 2kt$$

We have $\frac{2(t+k)^2}{3} - 2kt - \frac{26}{3} \geq 0 \Leftrightarrow 2(t + k)^2 \geq 6kt + 26 \Leftrightarrow (t - k)^2 + t^2 + k^2 \geq 26$, which is satisfied if $t \geq 4$ and $k \geq 4$. We have $(t + k)^2 - 3(t + k) - 2kt \geq 0 \Leftrightarrow t^2 + k^2 - 3(t + k) \geq 0 \Leftrightarrow t(t - 3) + k(k - 3) \geq 0$, which is true if $k \geq 3$ and $t \geq 3$. Thus the right hand side of (10) is non-negative. For the last assertion we use that $\psi(u) = \frac{(s-t-k-1)}{6}[(s-k-t-1)+2(t+k)+9]+(t+k) = \frac{(s-t-k-1)(s+t+k+8)}{6}+t+k$. □

Lemma 4. *Fix integers* $\alpha \geq 0$, $u \geq 0$, *and set* $s := t + k + 1 + 2u$ *and* $\delta := a(s + 2, t, k)_\alpha - a(s, t, k)_\alpha$. *Assume* $t \geq k \geq t/200$ *(resp.* $t \geq k \geq t/3$ *and* $k \geq 4$*). Then*

$$\delta > \frac{s}{102} - \frac{\alpha}{s + 1}$$

(resp. $\delta > \frac{s}{3} - \frac{\alpha}{s+1}$*). In particular, if* $s + 1 > \alpha$, *then* $\delta > -1 + \frac{s}{102}$ *(resp.* $\delta > -1 + \frac{s}{3}$*).*

Proof. Taking the difference of (9) and the same equation for the integer $s' = s+2$, we get

(11) $(s+1)\delta + 2a(s,t,k)_\alpha + b(s+2,t,k)_\alpha - b(s,t,k)_\alpha = I_{t,k}(s+2) - I_{t,k}(s) - \alpha$

By Lemma 2 we have

$(s+1)\delta + 2a(s,t,k)_\alpha + b(s+2,t,k)_\alpha - b(s,t,k)_\alpha = (s+3)^2 - (t^2+k^2+t+k) - \alpha$

We have $b(s,t,k)_\alpha \geq 0$, $b(s+2,t,k)_\alpha \leq s$ and $2a(s,t,k)_\alpha \leq [(s^2+7s) - (t+k)^2 - 3(t+k) - 8]/3$ (Lemma 3). Thus

$$(s+1)\delta \geq (s+3)^2 - (t^2+k^2+t+k) - \frac{2(s^2+7s)}{3} + \frac{2(t+k)^2}{3} + 2(t+k) + \frac{16}{3} - \alpha,$$

i.e.

$$(s+1)\delta \geq \frac{(s^2+s)}{3} - \frac{(t^2+k^2-4tk-3t-3k-1)}{3} - \alpha.$$

We obviously have $3t+3k+1 \geq 0$. If $t \leq 3k$, then $(t^2+k^2-4tk-3t-3k-1) < 0$ (more precisely, it is sufficient to assume $t < \frac{4k+3+\sqrt{12k^2+36k+73}}{2}$) and hence we conclude in this case. Now assume only $t \leq 200k$. Since $s \geq t+k$, it is sufficient to prove that $t^2+k^2-4tk \leq \frac{33}{34}(t+k)^2$, i.e. $t^2+k^2-202tk \leq 0$. This is true if $t \geq k > 0$ and $t/k \leq 101 + \sqrt{101^2-1}$. □

Note that $a(t+k+1,t,k)_\alpha$ and $b(t+k+1,t,k)_\alpha$ do not depend from α.

Lemma 5. *Let A and B be positive rational numbers. Assume $t+k \geq 4A+2B$ and $(s-1)d_{t,k}+I_{t,k}(s) \geq As^2+Bs$, for some $s \geq t+k+1$. Then $(s+1)d_{t,k}+I_{t,k}(s+2) \geq A(s+2)^2+B(s+2)$.*

Proof. By Lemma 2, we have $I(s+2) - I(s) = (s+3)^2 - (t^2+k^2+t+k)$ (where $I(m) := I_{t,k}(m)$). It follows that:

$$\begin{aligned}(s+1)d_{t,k} + I(s+2) &= (s-1)d_{t,k} + I(s) + 2d_{t,k} + (s+3)^2 \\ -(t^2+k^2+t+k) &\geq As^2 + Bs + 2d_{t,k} + (s+3)^2 - (t^2+k^2+t+k)\end{aligned} \tag{12}$$

So it is enough to prove:

$$2d_{t,k} + (s+3)^2 - (t^2+k^2+t+k) = (s+3)^2 \geq 4As + 4A + 2B \tag{13}$$

But $s^2+6s \geq 4As+4A+2B \Leftrightarrow s+6 \geq 4A + \frac{4A+2B)}{s}$ and this last inequality follows from: $s+6 > s \geq 4A + \frac{4A+2B}{s}$, since $s \geq t+k+1 > 4A+2B$. □

Lemma 6. *Let $C > 0$ and D be rational numbers. Assume $t \geq k \geq 2C$ and $t+k \geq 4C + 2|D-C+\alpha+2|$. Then we have $a(s,t,k) + d_{t,k} \geq Cs + D$ for any $s \geq t+k+1$, $s \equiv t+k+1 \pmod 2$.*

Proof. By definition $a(s)(s-1) + b(s) = I(s) - 4 - \alpha(s-t-k-1)/2$, with $0 \leq b(s) \leq s-2$ (we drop the indices t,k in a,b,I). Using $b(s) \leq s-2$, we get:

$$a(s)(s-1) \geq I(s) - 2 - \frac{\alpha(s-t-k-1)}{2} - s \geq I(s) - \frac{s(\alpha+2}{2} \tag{14}$$

By dividing by $s-1$ and using $s(\alpha+2)/2(s-1) \leq \alpha+2$, we get:

$$a(s) \geq I(s)/(s-1) - (\alpha+2)$$

So it is enough to show: $d_{t,k} + I(s)/(s-1) \geq Cs + D + \alpha + 2$, which is equivalent to:

$$(s-1)d_{t,k} + I(s) \geq As^2 + Bs\ , \tag{15}$$

where $A = C$ and where $B = D - C + \alpha + 2$. Note that B can be negative so we will consider the following inequality:

$$(s-1)d_{t,k} + I(s) \geq As^2 + |B|s\ . \tag{16}$$

Clearly (16) implies (15). According to Lemma 5, if $t+k \geq 4A+2|B|$, it is enough to prove this inequality for $s = t+k+1$. Since $I(t+k+1) = 2tk+3(t+k)+4$, for $s = t+k+1$ (16) reads like:

$$\begin{aligned} t^2(t+1)/2 + k^2(k+1)/2 + tk(t+k+2)/2 + 2tk + 3(t+k) + 4 \\ \geq At^2 + Ak^2 + 2Atk + (t+k+1)(2A+|B|) - A \end{aligned} \tag{17}$$

We have $t^2(t/2) \geq At^2$, $k^2(k/2) \geq Ak^2$ (because $t \geq k \geq 2A$). For the same reason, we also have $tk(t+k+2)/2 \geq tk(2A+1)$. It remains to show the inequality

$$(t^2+k^2)/2 + 3tk + 3(t+k+1) + 1 \geq (t+k+1)(2A+|B|) - A.$$

Note that

$$(t^2+k^2)/2 + 3tk + 3(t+k+1) + 1 = (t+k+1)^2/2 + 2tk + 2t + 2k + 7/2$$

Since $(t+k+1)/2 \geq 2A + |B| + 1/2$ by assumption, we get inequality (16), hence also inequality (15). □

Corollary 2. *Let M be a positive rational number. Assume $t \geq k \geq M+6$ and $t+k \geq 2M+\alpha+18$, then $\delta := a(s+2) - a(s) < s - M$, $\forall s \geq t+k+1$, $s \equiv t+k+1 \pmod 2$.*

Proof. We have: $2(d_{t,k} + a(s)) + (s+1)\delta + \alpha + b(s+2) - b(s) = (s+3)^2$ by (5). If $\delta \geq s - M$, since $b(s+2) \geq 0$ and $b(s) \leq s-2$, we get: $(s+3)^2 \geq 2(d_{t,k}+a(s)) + (s+1)(s-M) + \alpha - s + 2$, i.e. $s(M+6) + M - \alpha + 7 \geq 2(d_{t,k}+a(s))$. Using Lemma 6 with $C = \frac{M}{2} + 3$ and $D = \frac{M-\alpha}{2} + 4$, we get a contradiction. □

Lemma 7. *Assume $t \geq k \geq 4$ and $t \leq 200k$ (resp. $t \leq 3k$). Then $a(t+k+1, t, k)_\alpha \geq \frac{t+k}{102}$ (resp. $a(t+k+1,t,k)_\alpha \geq \frac{t+k}{3}$). If $\alpha \leq -1 + \frac{t+k}{102}$ (resp. $\alpha \leq -1 + \frac{t+k}{3}$), then*

$$a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha \leq 1 + 2a(t+k+1,t,k)_\alpha.$$

Let e be the minimal integer x such that

$$1 \leq x \leq \delta := a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha$$
$$\text{and } b(t+k+1,t,k)_\alpha \leq (x-1)(\delta - x - 1).$$

Assume $\alpha \leq -1 + \frac{t+k}{102}$ and $t+k \geq 1113636$ (resp. $\alpha \leq -1 + \frac{t+k}{3}$ and $t+k > 78$). Then $e \leq 104$ (resp. $e \leq 5$).

Proof. Since $(t+k+2)^2 - (t^2+k^2+t+k) = 2tk+3t+3k+4$, Lemma 2 for $s = t+k-1$ gives

$$(t+k+1)a(t+k+1,t,k)_\alpha + 1 + b(t+k+1,t,k)_\alpha = 2tk+3t+3k+4.$$

Since $b(t+k+1,t,k)_\alpha \leq t+k$, we get

$$(t+k+1)a(t+k+1,t,k)_\alpha \geq 2tk+2t+2k+3 \tag{18}$$

If $a(t+k+1,t,k)_\alpha \leq \frac{t+k-1}{102}$ (resp. $a(t+k+1,t,k)_\alpha \leq \frac{t+k-1}{3}$, then $(t+k+1)(t+k-1) \geq 102(2tk+2t+2k+3)$ (resp. $(t+k+1)(t+k-1) \geq 6tk+12t+12k+18$), which is false for all positive k if $t \leq 200k$ (resp. $t \leq 3k$).

Now assume $\alpha \le -1 + \frac{t+k}{102}$ (resp. $\alpha \le -1 + \frac{t+k}{3}$). To prove the second assertion of the lemma it is sufficient to prove that $a(k+t+3,t,k)_\alpha + 1 \le 3a(k+t+1,t,k)_\alpha$. We have

$$I_{t,k}(t+k+3) = \frac{4}{6}[(30+15(t+k)+6kt] = 20+10(t+k)+4kt$$

and

$$a(t+k+3,t,k)_\alpha \le \frac{(I_{t,k}(t+k+3)+4)}{t+k+2} = \frac{20+10(t+k)+4kt}{t+k+2},$$

while $a(t+k+1)_\alpha \le \frac{I_{t,k}(t+k+1)-1-t-k}{t+k+1} = 2\frac{t+k+tk}{t+k+1}$.

Since $b(t+k+1,t,k)_\alpha \le t+k$, to prove the last assertion of the lemma it is sufficient to observe that $103(\delta - 202) \ge t+k$ (resp. $4\delta \ge 24+t+k$) by the case $s = t+k+1$ of Lemma 4 and the assumption $t+k \ge 1113636 = 106 \cdot 103 \cdot 102$, which gives that $103(-1+\frac{t+k+1}{102} - 202) \ge t+k$. □

Lemma 8. *Assume $200k \ge t \ge k \ge 4$, $\alpha \le -1 + \frac{t+k}{102}$ and $t+k \ge 42040$. Let e be the minimal integer x such that $1 \le x \le \delta := a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha$ and $b(t+k+1,t,k)_\alpha \le (x-1)(\delta - x - 1)$. Then $e \le 201$.*

Proof. Assume $e \ge 202$. Since $b(t+k+1,t,k)_\alpha \le t+k$, we get $201(\delta - 203)$ $ek+t-1$. Since $\delta \le -1+\frac{t+k}{102}$ by Lemma 14, we get $99(k+t) \le 203\cdot 201\cdot 102 - 102$, which is false if $k+t \ge 42040$. □

Lemma 9. *Fix integers $\alpha \ge 0$, $u \ge 0$, and set $s := t+k+1+2u$ and $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha$. Assume $s+1 > \alpha$. Assume $t \ge k \ge \frac{t}{200}$, $k \ge 4$ and $s \ge 1157520$ (resp. $s > 78$, $t \ge k \ge \frac{t}{3}$ and $k \ge 4$). Let e be the minimal integer x such that $1 \le x \le \delta$ and $b(s,t,k)_\alpha \le (x-1)(\delta - x - 1)$. Then $e \le 104$ (resp. $e \le 5$).*

Proof. The integer e is defined by Lemma 4. Since $b(s,t,k)_\alpha \le s-2$, it is enough to check that $s-2 \le 103(\delta - 202)$ (resp. $s-2 \le 4(\delta-6)$, which is true, because Lemma 4 gives $\delta \ge -1+\frac{s}{102}$ (resp. $\delta > -1+\frac{s}{3}$) and we assumed $s \ge (2+103\cdot 106)\cdot 106 = 1157520$ (resp. $s > 78$). □

Lemma 10. *Fix integers $\alpha \ge 0$, $u \ge 0$, and set $s := t+k+1+2u$ and $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha$. Assume $s+1 > \alpha$, $t \ge k \ge \frac{t}{200}$, $k \ge 4$ and $s \ge 42674$. Let e be the minimal integer x such that $1 \le x \le \delta$ and $b(s,t,k)_\alpha \le (x-1)(\delta - x - 1)$. Then $e \le 201$.*

Proof. The integer e is defined by Lemma 4. Since $b(s,t,k)_\alpha \le s-2$, it is enough to check that $s-2 \le 98(\delta - 203)$, which is true, because Lemma 4 gives $\delta \ge -1+\frac{s}{102}$ and we assumed $s \ge 42674$. □

Lemma 11. *If $t \ge k$ and $k^2 \ge (\alpha+5)(k+2)$, then $2tk \ge (\alpha+5)(t+k+4)$.*

Proof. Set $\gamma(t,k) := 2tk - (\alpha+5)(t+k+4)$. We have $\gamma(k,k) \ge 0$ if $k^2 \ge (\alpha+5)(k+2)$. To get the lemma use that $\partial_t\gamma(t,k) = 2k - \alpha - 5$. □

Lemma 12. *Assume $k \le t \le 200k$, $2tk \ge (\alpha+5)(t+k+4)$, $\alpha \le -1+\frac{t+k}{102}$ and take an integer $s \ge t+k+1$ with $s \equiv t+k+1 \pmod 2$. Then:*

(a) $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha \le 2a(s,t,k)_\alpha - \alpha$;
(b) $\tau := a(s+4,t,k)_\alpha - a(s+2)_\alpha \le 2a(s,t,k)_\alpha + \alpha - 1$.

Proof. Assume $\delta \ge 2a(s,t,k)_\alpha - \alpha + 2$. Since $b(s+2,t,k)_\alpha \ge 0$ and $b(s,t,k)_\alpha \le s-1$, (21) gives

$$2d_{t,k} + (2s+4)a(s,t,k)_\alpha \le (s+3)^2 + \alpha(s+1) - s - 3 \tag{19}$$

First assume $s = t+k+1$. In this case (19) is equivalent to

$$(2t+2k+6)a(t+k+1,t,k)_\alpha \le \alpha(t+k+2) + 2tk + 6t + 6k + 12 \tag{20}$$

From (18) we get $4a(t+k+1,t,k)_\alpha + 2tk \le \alpha(t+k+2) + 2t + 2k + 6$, which is false if $2tk \ge (\alpha+2)(t+k+2)$.

Now assume $s \ge t+k+3$ and that (19) is false for the integer $s-2$. Since $(s+3)^2 + \alpha(s+1) - (s+1)^2 - \alpha(s-1) = 4s + 2\alpha + 8$, it is sufficient to use that $(2s+4)a(s,t,k)_\alpha - 2s\cdot a(s-2,t,k)_\alpha = 4a(s-2,t,k)_\alpha + (2s+4)(a(s,t,k)_\alpha - a(s-2,t,k)_\alpha) \ge 4(k+3) + 4s$ by Lemmas 14 and 4.

Now assume $\tau \ge 2a(s,t,k)_\alpha + 2 - \alpha$. From (5) for the integer $s' = s+4$ and using that $b(s+4,t,k)_\alpha \le s+2$ and $b(s+2,t,k)_\alpha \ge 0$ and $2a(s+2,t,k) \ge 2a(s,t,k) - 2 + s/51$, we get

$$2d_{t,k} - 2 + \frac{s}{51} + (2s+8)a(s,t,k)_\alpha + 2s + 6 - (s+3)\alpha \le (s+5)^2 \tag{21}$$

First assume $s = t+k+1$. In this case (21) is equivalent to

$$(2t+2k+10)a(t+k+1,t,k)_\alpha \le \alpha(t+k+4) + 2tk + 9t + 9k + 30 - \frac{t+k+1}{51} \tag{22}$$

From (22) we get

$$8a(t+k+1,t,k)_\alpha + 2tk \le \alpha(t+k+4) + 5t + 5k + 24 - \frac{t+k+1}{51},$$

which is false if $2tk \ge (\alpha+5)(t+k+4)$.

Now assume $s \ge t+k+3$ and that (21) is false for the integer $s-2$. Since $(s+5)^2 - (s+3)^2 = 4s + 16$ and $2s + 8 - 2(s-2) - 8 = 4$, to get that (21) is false for the integer s it is sufficient to use that $a(s,t,k)_\alpha - a(s-2,t,k)_\alpha \ge 2 + \alpha$, which is true by Lemma 4 and the assumption $s \ge t+k+3$, our assumptions on t, k and α. $\square$

Lemma 13. *Assume $t \ge k \ge \frac{t}{200}$ and $t + k \ge 102(\alpha + 27)$. Then we have $g(s,t,k)_\alpha \ge 26$.*

Proof. We first do the case $s = t+k+3$. We have $g(t+k+3,t,k)_\alpha = a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha - \alpha$. Lemma 4 gives $a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha > -1 + \frac{t+k+1}{102}$ (resp. $a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha > -1 + \frac{t+k+1}{3}$). Now assume $s \ge t+k+5$. By induction on s it is sufficient to prove that $g(s,t,k)_\alpha \ge g(s-2,t,k)_\alpha$, i.e. $a(s,t,k)_\alpha - a(s-2,t,k)_\alpha \ge \alpha$, which is true by Lemma 4. $\square$

Lemma 14. *Assume $t \ge k \ge 4$ and $t \le 200k$. Then $a(t+k+1,t,k)_\alpha \ge \frac{t+k}{102}$. If $\alpha \le -1 + \frac{t+k}{102}$, then*

$$a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha \le 1 + 2a(t+k+1,t,k)_\alpha.$$

Let e be the minimal integer x such that $1 \le x \le \delta := a(t+k+3,t,k)_\alpha - a(t+k+1,t,k)_\alpha$ and $b(t+k+1,t,k)_\alpha \le (x-1)(\delta - x - 1)$. Assume $\alpha \le -1 + \frac{t+k}{102}$ and $t + k \ge 1113636$. We have $e \le 104$.

Proof. Since $(t+k+2)^2-(t^2+k^2+t+k) = 2tk+3t+3k+4$, Lemma 2 for $s = t+k-1$ gives $(t+k+1)a(t+k+1,t,k)_\alpha + 1 + b(t+k+1,t,k)_\alpha = 2tk+3t+3k+4$. Since $b(t+k+1,t,k)_\alpha \le t+k$, we get $(t+k+1)a(t+k+1,t,k)_\alpha \ge 2tk+2t+2k+3$. If $a(t+k+1,t,k)_\alpha \le (t+k-1)/102$, then $(t+k+1)(t+k-1) \ge 102(2tk+2t+2k+3)$, which is false for all positive integers k if $t \le 200k$.

Now assume $\alpha \le -1+(t+k)/102$. To prove the second assertion of the lemma it is sufficient to prove that $a(k+t+3,t,k)_\alpha + 1 \le 3a(k+t+1,t,k)_\alpha$. We have $I_{t,k}(t+k+3) = \frac{4}{6}[(30+15(t+k)+6kt] = 20+10(t+k)+4kt$ and $a(t+k+3,t,k)_\alpha \le (I_{t,k}(t+k+3)+4)/(t+k+2) = (20+10(t+k)+4kt)/(t+k+2)$, while $a(t+k+1)_\alpha \le (I_{t,k}(t+k+1)-1-t-k)/(t+k+1) = 2(t+k+tk)/(t+k+1)$.

Since $b(t+k+1,t,k)_\alpha \le t+k$, to prove the last assertion of the lemma it is sufficient to observe that $103(\delta-202) \ge t+k$ by the case $s = t+k+1$ of Lemma 4 and the assumption $t+k \ge 1113636 = 106\cdot 103\cdot 102$, which gives that $103(-1+\frac{t+k+1}{102}-202) \ge t+k$. □

Lemma 15. *Fix integers $\alpha \ge 0$, $u \ge 0$, and set $s := t+k+1+2u$ and $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha$. Assume $s+1 > \alpha$, $t \ge k \ge t/200$, $k \ge 4$ and $s \ge 1157520$. Let e be the minimal integer x such that $1 \le x \le \delta$ and $b(s,t,k)_\alpha \le (x-1)(\delta-x-1)$. Then $e \le 104$.*

Proof. The integer e is defined by Lemma 4. Since $b(s,t,k)_\alpha \le s-2$, it is enough to check that $s-2 \le 103(\delta-202)$, which is true, because Lemma 4 gives $\delta \ge -1+\frac{s}{102}$ and we assumed $s \ge (2+103\cdot 106)\cdot 106 = 1157520$. □

6. PROOFS OF $A(s,t,k)_\alpha$ AND $A(s,t,k)$

Remark 5. Assume $b(s,t,k)_\alpha > 0$. Assume that $A(s,t,k)_\alpha$ is true and take (Y,Q,T_1) satisfying it. Set $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha$. Let e be the maximal positive integer such that $b(s-2,t,k) > (e-1)(\delta-e-1)$ and $e \le \delta/2$. Write $T_1 = R_1\cup\cdots\cup R_e\cup M_1\cup\cdots\cup M_{\delta-e}$ with $R_j \in |\mathcal{O}_Q(1,0)|$ and $M_h \in |\mathcal{O}_Q(0,1)|$. Now we modify T_1 to a new grid $T \in |\mathcal{O}_Q(e',\delta-e')|$, $e' \in \{e,e+1\}$, in which some of the lines of T does not meet $Y\cap Q$. In each case we also describe a very specific set $S \subseteq \mathrm{Sing}(T_1)$ with $\#(S) = b(s-2,t,k)_\alpha$ and show why we have $h^1(\mathcal{I}_{C_{t,k}\cup Y\cup S}(s)) = 0$, $i = 0,1$, although S is not a subset of $\mathrm{Sing}(T_1)$. In each case we will say that T is the grid adapted for $A(t,s,k)$. We have $\deg(Y\cup T) = a(s+2,t,k)_\alpha$. In all cases we will check that we have $\chi(\mathcal{O}_{Y\cup T\cup\chi}) = g(s+2,t,k)_\alpha - g_{t,k}$.

(a) Assume $(e-1)(\delta-e-1)+\alpha-e \le b(s-2,t,k)_\alpha \le e(\delta-e-1)$. In this case we will have $e' = e+1$. Take distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le \delta-e-1$, such that $L_i = M_i$ if $\alpha-e \le i \le b(s,t,k)_\alpha-(e-1)(\delta-e-1)$, while $L_i\cap Y = \emptyset$ in all other cases. Take $R_0 \in |\mathcal{O}_Q(1,0)|$ containing a point of $Y\cap(Q\setminus T_1)$. We take $T := R_0\cup\cdots\cup R_e\cup L_1\cup\cdots\cup L_{\delta-e-1}$. We take as S the union of all points $R_j\cap L_i$ with either $j>1$ or $j=1$ and $1 \le i \le b(s-2,t,k)_\alpha-(e-1)(\delta-e-1)$. Each L_j moves in a family of lines of Q with M_j in its limit. In this degeneration of some of the lines of T to some of the lines of T_1 the set S degenerate to a subset $S_1 \subseteq \mathrm{Sing}(T_1)$ (although T and T_1 are grids with different bidegrees). By $A(s,t.k)_\alpha$ we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup S_1}(s)) = 0$, $i = 0,1$. By the semicontinuity theorem we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup S_1}(s)) = 0$, $i = 0,1$. We have $T \in |\mathcal{O}_Q(e+1,\delta-e-1)|$, $\#(\mathrm{Sing}(T)) = (e+1)(\delta-e-1)$ and $\#(Y\cap T)) = 1+\#(Y\cap(T\setminus R_0)) = 1+e+b(s,t,k)-(e-1)(\delta-e-1)-\alpha+1+e = b(s,t,k)-(e-1)\delta+e^2+2e-\alpha+1$. Hence $1-\chi(\mathcal{O}_{Y'}) = p_a(Y)-\#(S)+\#(\mathrm{Sing}(T\cup R_0))+\#(Y\cap(T\cup R_0))-\deg(R_0\cup T) =$

$g(s,t,k) - b(s,t,k) + b(s,t,k) - (e-1)\delta + e^2 + 2e - \alpha + 1 + (e+1)(\delta - e - 1) - \delta = g(s,t,k)_\alpha + \delta - \alpha = g(s+2,t,k)_\alpha$.

(b) Assume $(e-1)(\delta-e-1)+e-1 \le b(s-2,t,k)_\alpha \le (e-1)(\delta-e-1)+\alpha-1-e$. We have $(e-1)(\delta-e) \le b(s-2,t,k)_\alpha \le (e-1)(\delta-e)+\alpha-2$. We take $e' = e$ with $L_i = M_i$ if either $i > \alpha - e$ or $1 \le i \le b(s,t,k)_\alpha - (e-1)(\delta-e)$, while for the other indices i's L_i is a general deformation of M. We take as S the union of all points $R_j \cap L_i$ with either $j > 1$ or $j = 1$ and $1 \le i \le b(s-2,t,k)_\alpha - (e-1)(\delta-e)$. In this degeneration of some of the lines of T to some of the lines of T_1 (L_j degenerates to M_j) S moves to a subset $S_1 \subseteq \mathrm{Sing}(T_1)$. By $A(s,t,k)_\alpha$ we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup S_1}(s)) = 0$, $i = 0,1$. By the semicontinuity theorem we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup S_1}(s)) = 0$, $i = 0,1$. Set $Y' := Y \cup \chi \cup T$. We have $\deg(T) = \delta$, $\#(\mathrm{Sing}(T)) = e(\delta-e)$ and $\#(Y\cap T) = b(s,t,k)_\alpha - (e-1)(\delta-e) + \delta - \alpha + e = b(s,t,k)_\alpha - (e-2)\delta + e^2 - \alpha$. Thus $1 - \chi(\mathcal{O}_{Y'}) = p_a(Y) - b(s,t,k)_\alpha + e(\delta-e) + b(s,t,k)_\alpha - (e-2)\delta + e^2 - \alpha - \delta = g(s,t,k)_\alpha + \delta - \alpha = g(s+2,t,k)_\alpha$.

(c) Assume $(e-1)(\delta-e-1) < b(s-2,t,k)_\alpha \le (e-1)(\delta-e-1) + e - 2$ and hence $e \ge 3$. We have $b(s-2,t,k)_\alpha < (e-1)(\delta-e)$. Since $e \le \alpha - 1$, Lemma 4 gives $\delta \ge e$ and $\delta \ge 2\alpha - 2 \ge 2e - 4$ and hence $(e-1)(\delta-e-1)+1 \ge (e-2)(\delta-e)$. Thus $b(s-2,t,k)_\alpha \ge (e-2)(\delta-e)$. Take $L_i = M_i$ if $1 \le i \le \delta - e$ and L_i a small deformation of M_i not intersecting $Y \cap Q$. Let S be the union of all points $R_j \cap L_i$ with either $j > 1$ or $j = 1$ and $1 \le b(s,t,k)_\alpha - (e-2)(\delta-e)$. In this degeneration of some of the lines of T to some of the lines of T_1 the set S degenerates to a subset $S_1 \subseteq \mathrm{Sing}(T_1)$. By $A(s-2,t.k)_\alpha$ we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup S_1}(s-2)) = 0$, $i = 0,1$. By the semicontinuity theorem we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup S_1}(s-2)) = 0$, $i = 0,1$. Set $Y' := Y \cup T \cup R_0 \cup \chi$. We have $R_0 \cup T \in |\mathcal{O}_Q(e,\delta-e)|$, $\deg(R_0\cup T) = \delta$, $\#(\mathrm{Sing}(R_0\cup T)) = e(\delta-e)$ and $\#(Y\cap(R_0\cup T)) = 1 + \#(Y\cap T) = 1 + e - 1 + b(s,t,k)_\alpha - (e-2)(\delta-e) - \alpha + e = b(s,t,k)_\alpha - (e-2)\delta + e^2 - \alpha$. Thus $1 - \chi(\mathcal{O}_{Y'}) = p_a(Y) - b(s,t,k) + e(\delta-e) + b(s,t,k)_\alpha - (e-2)\delta - \alpha + e^2 - \delta = g(s,t,k)_\alpha + \delta - \alpha = g(s+2,t,k)_\alpha$

From now on we take $\alpha = 202$ and write $a(s,t,k)$, $b(s,t,k)$ and $A(s,t,k)$ instead of $a(s,t,k)_{202}$, $b(s,t,k)_{202}$ and $A(s,t,k)_{202}$.

Lemma 16. *Assume $k \le t \le 200k$, $t + k > 102\cdot 229$ and $k^2 \ge 207\cdot(k+2)$.*

(i) $A(t+k+1,t,k)$ is true.

(ii) Fix an integer $s \ge t+k+1$ such that $s \equiv t+k+1 \pmod 2$ and assume that $A(s,t,k)$ is true. Then $A(s+2,t,k)$ is true.

Proof. We first prove (ii). We will show in step (d) the small modification needed to get $A(t+k+1,t,k)$. Set $\delta := a(s+2,t,k) - a(s,t,k)$. By assumption we have $t+k-1 \ge 102\cdot 229$. We take Y of degree $d(s,t,k)$ and genus $g(t,s,k)$ such that for all $\tilde{S}$ in some grid and with cardinality $b(s,t,k)$ we have $h^i(\mathcal{I}_{C_{t,k}\cup Y\cup\tilde{S}}(s)) = 0$, $i = 0,1$. We will only use a very specific $\tilde{S}$ described separately in each case. In steps (a) and (b) we will construct the curve Y' appearing (as Y) in the statement of $A(s+2,t,k)$. In step (c) we will construct the grid T_1 such that for all $S' \subseteq \mathrm{Sing}(T_1)$ with $\#(S') = b(s+2,t,k)$ we have $h^i(\mathcal{I}_{C_{t,k}\cup Y'\cup S'}(s+2)) = 0$, $i = 0,1$.

(a) Assume $b(s,t,k) > 0$.Take $e' \in \{e, e+1\}$ as in Remark 5 so that T is union of e' lines $R_j \in |\mathcal{O}_Q(1,0)|$ and $\delta - e'$ lines $L_j \in |\mathcal{O}_Q(0,1)|$. Take as $\tilde{S}$ the set $S \subseteq \mathrm{Sing}(T)$, the one considered in Remark 5. Set $\chi := \cup_{o\in S} 2o$; to make the construction of Remark 5 we need that $2a(s,t,k) = \#(Y\cap Q)$ is at least the number of lines R_j and L_i containing a point of $Y \cap Q$; since the latter number is at most

δ, it is sufficient to quote Lemma 12. In all cases we have $Y' := Y \cup T \cup \chi$ with $T \in |\mathcal{O}_Q(e', \delta - e')|$ and $T \cap C_{t,k} = \emptyset$. Set $\Psi := Y \cap (Q \setminus T_1)$.

Claim 1: We have $2d_{t,k} + \#(\Psi) + b(s+2,t,k) = (s+3-e')(s+3-\delta+e')$.

Proof of Claim 1: First assume that we are in case (a) of Remark 5. In this case we have $e' = e+1$ and $\#(\Psi) = 2a(s,t,k) - b(s,t,k) + (e-1)\delta - e^2 - 2e + 201$. By (5) we have $(s+2-e)(s+4-\delta+e) = (s+3)^2 - 1 - (s+2-e)\delta - 2e - e^2 = 2d_{t,k} + 2a(s,t,k) + (s+1)\delta + 202 + b(s+2,t,k) - b(s,t,k) - 1 - (s+2-e)\delta - 2e - e^2 = 2d_{t,k} + \#(\Psi) + b(s+2,t,k)$.

Now assume that we are in case (b) of Remark 5. We have $e' = e$ and $\#(\Psi) = 2a(s,t,k) - b(s,t,k) + (e-2)\delta - e^2 + 202$. By (5) we have $(s+3-e)(s+3-\delta+e) = (s+3)^2 - (s+3-e)\delta - e^2 = 2d_{t,k} + 2a(s,t,k) + (s+1)\delta + 202 + b(s+2,t,k) - b(s,t,k) - (s+3-e)\delta - e^2 = 2d_{t,k} + b(s+2,t,k) + \#(\Psi)$.

Now assume that we are in case (c) of Remark 5. We have $a' = e$ and $\#(\Psi) = 2a(s,t,k) - b(s,t,k) + (e-2)\delta - e^2 + 202$. By (5) we have $(s+3-e)(s+3-\delta+e) = (s+3)^2 - (s+3-e)\delta - e^2 = 2d_{t,k} + 2a(s,t,k) + 202 + (s+1)\delta + b(s+2,t,k) - b(s,t,k) - (s+3-e)\delta - e^2 = 2d_{t,k} + b(s+2,t,k) + \#(\Psi)$. □

By Claim 1 and the generality of $\Psi \cup (C_{t,k} \cap Q)$ we have $h^i(Q, \mathcal{I}_{C_{t,k} \cup Y' \cup S', Q}(s+2)) = 0$. Since $\mathrm{Res}_Q(C_{t,k} \cup Y' \cup S') = C_{t,k} \cup Y \cup S$, the residual sequence of Q gives $h^i(\mathcal{I}_{C_{t,k} \cup Y' \cup S'}(s+2)) = 0$, $i = 0, 1$. In step (c) we will prove that we may find S' and a deformation of Y' to get $A(s+2,t,k)$.

Claim 2: Y' is a flat limit of a family of connected smooth curves of degree $a(s+2,t,k)$ and genus $g(s+2,t,k)$.

Proof of Claim 2: Fix any two skew lines $D_1, D_2 \subset \mathbb{P}^3$ and any $p \in \mathbb{P}^3 \setminus (D_1 \cup D_2)$. The linear projection from p shows that there is a unique line L with $p \in L$, $L \cap D_1 \neq \emptyset$ and $L \cap D_2 \neq \emptyset$. If D_1, D_2 and p depend continuously from certain parameters, then the line L depends continuously from the same parameters.

We assume that $A(s,t,k)$ is in case (a) of Remark 5 (the cases described in (b) and (c) of Remark 5 are done in the same way). Take as a parameter space an integral affine curve Δ and fix $o \in \Delta$. Set $Y'(o) := Y'$, $R_j(o) := R_j$ and $L_i(o) := L_i$. Take an algebraic family $\{R_j(z)\}_{z \in \Delta}$ of lines of $\mathbb{P}^3$ with $R_j(z)$ transversal to Q if $z \neq o$ and $R_j \cap Y \in R_j(z)$ for all z, and an algebraic family $\{L_i(z)\}_{z \in \Delta}$, $202 - e \le i \le b(s,t,k) - (e-1)(\delta - e - 1)$, of lines of $\mathbb{P}^3$ with $L_i(z)$ transversal to Q if $z \neq o$, $L_i \cap Y \in L_i(z)$ for all z and $L_i(z) \cap R_j(z) \neq \emptyset$ if and only if $j = 0, 1$. Changing if necessary Δ we may find an algebraic family $\{L_i(z)\}_{z \in \Delta}$, $1 \le i \le 202 - 1 - e$, of lines with $L_i(z) \cap R_j(z) \neq \emptyset$, $z \in \Delta \setminus \{o\}$, if and only $j = 0$, and an algebraic family $\{L_i(z)\}_{z \in \Delta}$, $b(s,t,k) - (e-1)(\delta - e - 1) < i \le \delta - e - 1$, of lines with $L_i(z) \cap R_j(z) \neq \emptyset$ if any only if $j = 0, 1$. For any $z \in \Delta \setminus \{o\}$ set $Y'(z) := Y \cup \bigcup R_j(z) \cup \bigcup L_i(z)$. The family $\{Y'(z)\}_{z \in \Delta}$ is flat. Then we use Remark 2 to smooth $Y \cup \bigcup R_j(z) \cup \bigcup L_i(z)$ for some $z \in \Delta \setminus \{o\}$. □

(b) Assume $b(s,t,k) = 0$, i.e. $S = \emptyset$. Instead of lines R_j and L_j we take a line $R_0 \in |\mathcal{O}_Q(1,0)|$ with $R_0 \cap Y \neq \emptyset$ and $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le \delta - 1$, such that $L_i \cap Y \neq \emptyset$ if and only if $i \ge 202$; we are using that $\delta \ge 202$ (Lemma 12). We assume $L_i \cap C_{t,k} = \emptyset$ for all i. In this case we have $a_1 = 1$, $b_1 = \delta - 1$ and $\#(\Psi) = 2a(s,t,k) - \delta + 201$. By (5) we have $(s+2)(s+4-\delta) = (s+3)^2 - 1 - (s+2)\delta = 2d_{t,k} + 2a(s,t,k) + (s+1)\delta + 201 + b(s+2,t,k) - (s+2)\delta = 2d_{t,k} + \#(\Psi) + b(s+2,t,k)$. The union Y' of Y and all lines R_j and L_i is smoothable by Remark 2.

Claim 3: Fix $S'' \subset Q$ such that $\#(S'') = b(s+2,t,k)$ and $S'' \cap Q \cap (T \cup Y \cup C_{t,k}) = \emptyset$. If $h^1(Q, \mathcal{I}_{S'',Q}(s+2-e', s+2-\delta+e')) = 0$, then $h^i(\mathcal{I}_{C_{t,k}\cup Y'\cup S''}(s+2)) = 0$, $i = 0,1$.

Proof of Claim 3: We have $\mathrm{Res}_Q(C_{t,k}\cup Y'\cup S'') = C_{t,k}\cup Y'\cup S$ and $Q\cap(C_{t,k}\cup Y'\cup S'') = (Q\setminus T)\cap(C_{t,k}\cup Y)\cup S''$. By $A(s,t,k)$ we have $h^i(\mathcal{I}_{C_{t,k}\cup Y'\cup S}(s)) = 0$. Therefore the residual exact sequence of Q shows that it is sufficient to prove that $h^i(Q, \mathcal{I}_{(Q\setminus T)\cap(C_{t,k}\cup Y)\cup S''\cup T}(s+2,s+2)) = 0$, $i = 0,1$, i.e., to prove that we have $h^i(Q, \mathcal{I}_{(Q\setminus T)\cap(C_{t,k}\cup Y)\cup S''\cup T}(s+2-e', s+2-\delta+e')) = 0$, $i = 0,1$ for some $C_{t,k}$ and Y for a fixed S''. We may deform $C_{t,k}\cup Y$ (keeping fixed S'') so that $(Q\setminus T)\cap(C_{t,k}\cup Y)$ are general. Thus it is sufficient to observe that in parts (a) and (b) we proved that $\#((Q\setminus T)\cap(C_{t,k}\cup Y)) = (s+3-e')(s+3-\delta+e') - b(s+2,t,k)$. □

(c) If $b(s+2,t,k) = 0$, then $S' = \emptyset$ and hence parts (a) and (b) prove $A(s+2,t,k)$, because we proved that Y' is smoothable (Claim 2 for the case $b(s,t,k) > 0$) and Claim 3 with $S'' = \emptyset$ gives $h^i(\mathcal{I}_{C_{t,k}\cup Y'}(s+2)) = 0$, $i = 0,1$. Now assume $b(s+2,t,k) > 0$. We prove $A(s+2,t,k)$, but with exchanged the two rulings of Q. Set $\tau := a(s+4,t,k) - a(s+2,t,k)$. Let f be the maximal positive integer such that $(f-1)(\tau-f-1) < b(s+2,t,k)$. $A(s+2,t,k)$ is as in one of the cases (a), (b) or (c) of Remark 5 with τ instead of δ and f instead of e. We call T_1 the grid of bidegree $(\tau-f', f')$, $f' \in \{f, f+1\}$ called T in in Remark 5, but with exchanged the 2 rulings of Q, i.e. in all cases we take $S' \subseteq \mathrm{Sing}(T_1)$ with T_1 unions of $\tau - f'$ distinct lines of bidegree $(1,0)$ and f' distinct lines of bidegree $(0,1)$. $h^1(Q, \mathcal{I}_{\mathrm{Sing}(T_1),Q}(s+2-e', s+2-\delta+e')) = 0$. If $A(s+2,t,k)_\alpha$ is in case (a) (resp. (b) or (c) of Remark 5 we take $f' = f+1$ (resp. $f' = f$). We have $h^1(Q, \mathcal{I}_{\mathrm{Sing}(T_2),Q}(s+2-f', s+2-\delta+e')) = 0$ (and hence $h^1(Q, \mathcal{I}_{S',Q}(s+2-e', s+2-\delta+e')) = 0$ for each $S' \subseteq \mathrm{Sing}(T_1)$), because $e'+\delta-e' \le s+2$ and $f'1+\delta \le s+2$, (we use that $e+1+\tau \le \tau+\alpha+1 \le s+2$ and $f+1+\delta-1 \le s+2$ use Corollary 2). Claim 3 gives $A(s+2,t,k)$.

(d) Now we prove $A(k+t+1,t,k)$. We have $h^i(\mathcal{I}_{C_{t,k}}(t+k-1)) = 0$. Since $g(t+k+1,t,k) = 0$, we need to add a smooth rational curve of degree $a(t+k+1,t,k)$. We start with a general $F \in |\mathcal{O}_Q(a(t+k+1,t,k)-1,1)|$. Thus $F \cap C_{t,k} = \emptyset$ and F is a smooth rational curve. Since $C_{t,k}\cap Q$ is general in Q, for any set $S \subset Q\setminus(F\cup(C_{t,k}\cap Q))$ such that $\#(S) = b(k+t+1,t,k)$ and $h^1(Q, \mathcal{I}_S(t+k, t+k+2-a(t+k+1,t,k)+2)) = 0$, the residual exact sequence of Q gives $h^i(\mathcal{I}_{C_{t,k}\cup F\cup S}(t+k+1)) = 0$, $i = 0,1$. If $b(t+k+1,t,k) = 0$, it is sufficient to deform F to a general smooth rational curve Y of degree $a(t+k+1,t,k)$ and use that $h^1(N_Y(-2)) = 0$, because $a(t+k+1,t,k) \ge 3$. Now assume $b(t+k+1,t,k) > 0$. We may take S in a grid T of bidegree $(e, \delta-e)$ as in the statement of $A(t+k+1,t,k)$, because we may deform F to a curve transversal to Q and fixing one point for each irreducible component of the grid T. We use that $\delta \le a(t+k+1,t,k)$ by Lemma 14. □

Remark 6. The inductive proof of Lemma 16 gives the following statement stronger that $A(s,t,k)$ but that we proved to be equivalent to it. Set $\delta := a(s+2,t,k)_\alpha - a(s,t,k)_\alpha$. Let e be the maximal positive integer such that $b(s,t,k) > (e-1)(\delta-e-1)$ and $e \le \delta/2$. Let Q be a smooth quadric. Fix $C_{t,k}$ intersecting transversally Q and such that $Q\cap C_{t,k}$ is formed by $2d_{t,k}$ general points of Q. There is a pair (Y,T) with the following properties. Y is a smooth and connected curve of degree $a(s,t,k)_\alpha$ and genus $g(s,t,k)_\alpha$ such that $Y\cap C_{t,k} = \emptyset$, Y intersects transversally Q

and $(C_{t,k}\cup Y)\cap Q$ is general in Q and T is a grid of Q adapted to $(Y, C_{t,k}\cap Q)$ such that for every $S\subseteq \mathrm{Sing}(T)$ we have $h^0(\mathcal{I}_{C_{t,k}\cup Y\cup S}(s)) = \max\{0, b(s,t,k) - \#(S)\}$ and $h^1(\mathcal{I}_{C_{t,k}\cup Y\cup S}(s)) = \max\{0, -b(s,t,k) + \#(S)\}$.

7. The genus enters into the playground

Now we fix m, d and take $g := 1+d(m-1)-\binom{m+2}{3}$. Recall that we take $\alpha = 202$.

Remark 7. Since $g = 1+(m-1)d-\binom{m+2}{3}$ and $d < \frac{m^2+4m+6}{4}$, we have $g < \frac{m^3+3m^2+2m-18}{12}$.

From now on we assume $g \ge g_{1000,1000} = 1+1000\cdot 1001\cdot 1995/3$. Let t be the maximal integer such that $g_t \le \frac{999999}{1000000}g$. The maximality of t gives $g_{t+1} > \frac{999999}{1000000}g$ and so $\frac{999999}{1000000}g-(t+1)(3t+1)/3 < g_t \le \frac{999999}{1000000}g$. Since $g \ge g_{1000,1000}$, we have $t \ge 1000$. Let k be the maximal positive integer such that $g_{t,k} \le g$ and $t+k \equiv m \pmod 2$; k exists, because $g_2 = 2 \le g/1000000 \le g-g_t$. Since $2g_t \ge 2\frac{999999}{1000000}g - 2\frac{(t+1)(3t+1)}{3} \ge g$, we have $k\le t$. The minimality of k gives $g_{t,k+2} > g \ge g_{t,k}$, i.e.

$$g_{t,k} \le g \le g_{t,k} + 2k^2 + 2k \tag{23}$$

We use $A(s,t,k)$ for these integers t,k. A key tool that we need to cover all interval for the genus and the degrees in Theorems 1, 2 and 3 (and not just prove for many (d,g) the existence curves C with $h^0(\mathcal{I}_C(m-1)) = 0$ and (degree, genus) = (d,g)), is that in (23) the interval depends only on k and it is quadratic on k, while we take $t \gg k$, say $t \ge 30k$ (Lemma 18).

Let y be the maximal integer $s \ge t+k+1$, $s \equiv t+k+1 \pmod 2$ such that $g(s,t,k)+g_{t,k} \le g$; y exists because $g(t+k+1,t,k) = 0$, $g_{t,k}\le g$ and $g(s,t,k) > g(s-2,t,k)$ for all $s \ge t+k+3$ by Lemma 4 and the inequality $t+k+3 \ge 202\times 102+2$. Note that $y \equiv m-1 \pmod 2$. For all integers x,y such that $x \ge y+2$ and $x\equiv y \pmod 2$ we define the integers $u(x,t,k)$ and $v(x,t,k)$ by the relations

$$x(d_{t,k}+u(x,t,k))+3-g+v(x,t,k) = \binom{x+3}{3},\ 0 \le v(x,t,k) \le x-1 \tag{24}$$

The integers y, $u(x,t,k)$ and $v(x,t,k)$ depend on g, but we do not put g in their symbols.

Remark 8. The main actor of this section is a a general smooth curve Y of genus $g-g_{t,k}$ and of some degree $z \ge g-g_{t,k}+3$ with $z = u(x,t,k)$ for some $x \ge y+2$ with $x\equiv y \pmod 2$. We need to check that $h^i(N_Y(-2)) = 0$, $i = 0,1$. If $g-g_{t,k} \ge 26$, then this is true by [27, page 67, inequality $DP(g) \le g+3$]. If $g-g_{t,k} = 0$, then we just quote [7, Proposition 6]. Now assume $1 \le g-g_{t,k} \le 25$. In our case we have $z = u(x,t,k) \ge u(y+2,t,k) \ge a(y,t,k)+202 \ge 202$ (Lemmas 12 and 13). See [27, page 106-107] for the best published results (we only need them for $z \ge 202$). Fix a general $A \subset Q$ such that $\#(A) = 2a$. Since $h^1(N_C(-2)) = 0$ for a general smooth curve of genus $g-g_{t,k}$ and degree z, we may find a smooth curve Y of genus $g-g_{t,k}$ and degree z intersecting transversally Q and with $A = Y\cap Q$.

For any integer $x \ge y+2$ with $x\equiv y \pmod 2$ define the following Assertion $B(x,t,k)$.

Assertion $B(x,t,k)$: Let Q be a smooth quadric. Fix $C_{t,k}$ intersecting transversally Q and such that $Q\cap C_{t,k}$ is formed by $2d_{t,k}$ general points of Q.

Set $\delta := u(x+2) - u(x,t,k)$. Let e be the maximal positive integer such that $v(x,t,k) > (e-1)(\delta - e)$ and $e < \delta/2$. We call $B(x,t,k)$ the existence of a pair (Y,T) with the following properties:

(1) Y is a smooth and connected curve of degree $u(x,t,k)$ and genus $g - g_{t,k}$ such that $Y \cap C_{t,k} = \emptyset$, Y intersects transversally Q and $(C_{t,k} \cup Y) \cap Q$ is general in Q;
(2) T is a grid $T \in |\mathcal{O}_Q(e, \delta - e)|$;
(3) for each $S \subseteq \mathrm{Sing}(T)$ with $\#(S) = v(x,t,k)$ we have $h^i(\mathcal{I}_{C_{t,k} \cup Y \cup S}(x)) = 0$, $i = 0, 1$.

Remark 9. We use Remark 8 for the existence of Y. Lemmas 20 shows that e exists and that $e \le 201$.

Remark 10. As in Remark 6 proving inductively $B(x,t,k)$ we will also prove that for all $S \subseteq \mathrm{Sing}(T)$ we have $h^0(\mathcal{I}_{C_{t,k} \cup Y \cup S}(v)) = \max\{0, v(x,t,k) - \#(S)\}$ and $h^1(\mathcal{I}_{C_{t,k} \cup Y \cup S}(x)) = \max\{0, -v(x,t,k) + \#(S)\}$.

8. NUMERICAL LEMMAS, II

In this section we collect the numerical lemmas related to section 7 and used in the next sections. We take m, d and g as in section 7 and in particular we are forced to assume $t \ge 1000$.

The next lemma only use that $g_t < g$.

Lemma 17. *If $t \ge 250$, then $m > 1.58t$.*

Proof. We have $\sqrt[3]{4} \ge 1.587401$, $4 - 1.58^3 = 0.055688$ and $3 \cdot 1.58^2 = 7.46892$. Remark 7 gives $g < 1 + \frac{m^3+3m^2+2m-30}{12}$. Assume $m \le 1.58t$. Since $g > g_t$, we get $1 + \frac{1.58^3t^3+7.46892t^2+3.16t-30}{12} > 1 + \frac{t(t+1)(2t-5)}{6}$, i.e. $13.46892t^2 + 13.16t - 30 > 0.055688t^3$, which is false if $t \ge 250$. $\square$

Lemma 18. *If $t \ge 1000^5$, then $k \le \frac{t}{30}$.*

Proof. Since $g_{t+1} > \frac{999999}{1000000}g$, we have $g_k \le g - g_t < g/1000000 + \frac{(t+1)(3t+1)}{3} < g_{t+1}/999999 + \frac{(t+1)(3t+1)}{3}$. Assume $k > \frac{t}{30}$. We have $30^3 = 27000$. Even when $t/30 \notin \mathbb{N}$ we get $999999[6 \cdot 27000 + t(t+30)(2t-150)] < 27000[6 + (t+1)(t+2)(2t-3) + 2(t+1)(3t+1)]$, which is false if $t \ge 100$. $\square$

Lemma 19. *If $t \ge 4000$, then $k > \frac{t}{200}$.*

Proof. By (23) we have $g_k \ge g/1000000 - 2k^2 - 2k \ge g_t/999999 - 2k^2 - 2k$. If $k \le \frac{t}{200}$ we get (even it $\frac{t}{200} \notin \mathbb{N}$, because $t \ge 1200$) $1 + \frac{t(t+201)(2t-1000)}{48000000} \ge 1/999999 + \frac{t(t+1)(2t-5)}{5999994} - \frac{2t^2}{40000} - \frac{2t}{200}$, a contradiction. If $t \le 100k$ we get $999999(1 + \frac{k(k+1)(2k-5)}{6}) + 1999998k^2 + 1999998k) \ge 1 + \frac{100k(100k+1)(200k-5)}{6}$ which is false for $k \ge 4000$. $\square$

From now on in this section we take $\alpha = 202$.

Lemma 20. *Assume $k \le t \le 200k$, $t + k \ge 102 \cdot 229$ and $y \ge 111 \cdot 210$. Fix an integer $x \ge y + 2$ such that $x \equiv y \pmod 2$. Then $u(x+2,t,k) - u(x,t,k) \ge -2 + \frac{x(x+1)}{102(x+2)} \ge 202$.*

Proof. From (24) for $x+2$ and x we get

$$\begin{aligned} &2d_{t,k}+2u(x,t,k)+(x+2)(u(x+2,t,k)-u(x,t,k))+\\ (25)\qquad &v(x+2,t,k)-v(x,t,k)=(x+3)^2 \end{aligned}$$

Since $u(x,t,k)\le a(x,t,k)$ and $v(x,t,k)\ge b(x,t,k)$ if $u(x,t,k)=v(x,t,k)$, (21) and (25) give $u(x+2,t,k)-u(x,t,k)\ge (x+1)(a(x+2,t,k)-a(x,t,k))/(x+2)-1$. Use Lemma 4. The second inequality follows from the first one, because $x(x+1)\ge 102\cdot 204(x+2)$ if $x\ge 111\cdot 210$. $\square$

Lemma 21. *Assume $k\le t\le 200k$, $t+k\ge 102\cdot 229$. Then $u(y+2,t,k)-a(y,t,k)\ge -2+\frac{y(y+1)}{102(y+2)}\ge 202$.*

Proof. Recall that $(y+2)a(y+2,t,k)+b(y+2,t,k)-g(y+2,t,k)-g_{t,k}=(y+2)u(y+2,t,k)+v(y+2,t,k)-g$ with $g(y+2,t,k)-g_{t,k}>g\ge g_{t,k}+g(y,t,k)$, $g(y+2,t,k)-g(y,t,k)=a(y+2,t,k)-a(y,t,k)-202$ and hence $(y+1)(a(y+2,t,k)-a(y,t,k))\le (y+2)(u(y+2,t,k)-u(y,t,k))$. Use Lemma 4 to get the first inequality and the inequality $y\ge 111\cdot 210$ to get the second inequality. $\square$

Lemma 22. *Assume $k\le t\le 200k$, $t+k\ge 102\cdot 229$. Fix an integer $x\ge y+2$ such that $x\equiv y \pmod 2$ and $x\ge 210\cdot 111$. Set $\delta:=u(x+2,t,k)-u(x,t,k)$. Let e be the minimal integer z with $(z-1)(\delta-z)\le v(x+2,t,k)$. Then e exists and $e\le 201$.*

Proof. Since $v(x+2,t,k)\le x+1$ and $\delta>-2+\frac{x(x+1)}{102(x+2)}$ (Lemma 20) it is sufficient to check that $200(-203+\frac{x(x+1)}{102(x+2)})\ge x+1$, i.e. $198x^2\ge (203\cdot 102\cdot 200-200)x+400\cdot 203\cdot 102+102$, which is true if $x\ge 210\cdot 111$. $\square$

Lemma 23. *Assume $x\ge y+2$, $k\le t\le 200k$, $t+k\ge 102\cdot 229$ and $y\ge 111\cdot 210$. Fix an integer $x\ge y+2$ such that $x\equiv y \pmod 2$. Then*

$$2u(x,t,k)\ge u(x+4,t,k)-u(x+2,t,k)+202 \text{ and}$$
$$2u(x,t,k)\ge u(x+2,t,k)-u(x,t,k)+202.$$

Proof. Assume $2u(x,t,k)\le u(x+2,t,k)-u(x,t,k)+201$. Since $v(x+2,t,k)\ge 0$ and $v(x,t,k)\le x-1$, (25) give

$$(26)\qquad 2d_{t,k}+(2x+4)u(x,t,k)-201(2x+4)-x+1\le (x+3)^2$$

Since $(x+3)^2-(x+1)^2=2x+8$, $201(2x+4)-x-201(2(x-2)+4)-x+2=802$ and $u(x,t,k)\ge u(x-2,t,k)+202$ by Lemma 20, it is sufficient to disprove (26) when $x=y+2$. Since $u(y+2,t,k)\ge a(y,t,k)+202$ by Lemma 21, it is sufficient to prove that

$$(27)\qquad 2d_{t,k}+(2y+8)a(y,t,k)>y^2+9y+22$$

See the contradiction coming from the case $y=s$ of (21).

Now assume $2u(x,t,k)\le u(x+4,t,k)-u(x+2,t,k)+201$. Since $v(x+4,t,k)\ge 0$, $v(x+2,t,k)\le x+1$ and $u(x+2,t,k)\ge u(x,t,k)-2+\frac{x(x+1}{102(x+2)}$, the case $x':=x+2$ of (25) gives

$$\begin{aligned} &2d_{t,k}+(2x+10)u(x,t,k)\le (x+5)^2+\\ (28)\qquad &x+1+201(2x+4)+2(x+4)-\frac{x(x+1)(x+4)}{102(x+2)} \end{aligned}$$

As in the first part of the proof we reduce to prove an inequality weaker than (27). □

Lemma 24. *Assume $x \geq y+2$, $k \leq t \leq 200k$, $k \leq t \leq 200k$, $t+k \geq 102 \cdot 229$ and $y \geq 111 \cdot 210$. Then $2a(y,t,k) \geq u(y+2,t,k) - a(y,t,k) + 202$ and $2a(y,t,k) \geq u(y+4,t,k) - u(y+2) + 202$.*

Proof. The first inequality is true by Lemma 12, because $u(y{+}2,t,k) \leq a(y{+}2,t,k)$. From (eqov4) for $x' := x+2$ and (5) for $s = y+2$ we get $(y+4)(u(y+4,t,k) - u(y+2,t,k)) + v(u+4,t,k) - v(u+2,t,k) = (y+4)(a(u+4,t,k) - a(u+2,t,k) + b(u+4,t,k) - b(u+2,t,k) + g(y+4,t,k) - g(y+2,t,k)$. Use Lemma 12 and that $g(y+4,t,k) - g(y+2,t,k) \geq -106 + (y+2)/102$. □

Lemma 25. *Assume $k \leq t \leq 200k$, $t+k \geq 102 \cdot 229$ and $y \geq 211 \cdot 210$. Set $\delta := u(y+2,t,k) - a(y,t,k)$ and $\tau := u(y+4,t,k) - u(y+2,t,k)$. Let e (resp f) be the minimal positive integer such that $(e-1)(\delta - e) \leq v(y+2,t,k)$ (resp. $(f-1)(\tau - f) \leq v(y+4,t,k)$). Then $e \leq 201$ and $f \leq 201$.*

Proof. The assertion on f is true by the case $x = y+2$ of Lemma 22. Since $v(y+2,t,k) \leq y+1$, to prove the assertion on e it is sufficient to prove that $200\delta \geq 200 \cdot 201 + y + 1$. By Lemma 21 it is sufficient to prove that $-400 + 200\frac{y(y+1)}{102(y+2)} \geq 200 \cdot 201 + y + 1$, i.e. $200y(y+1) \geq 200 \cdot 203 \cdot 102(y+2) + 102y(y+1)$, contradicting our assumption on y. □

Lemma 26. *If $k \leq t \leq 200k$ and $t \geq 3 \cdot 10^4$, then $y \leq m - 7$.*

Proof. Since $y \equiv m-1 \pmod 2$, it is sufficient to prove that $y \leq m-6$. Lemma 17 gives $m > 1.58t$. Thus it is sufficient to prove that $y - t - k - 1 \leq 0.58t - k - 7$. Assume $y - t - k - 1 > 0.58t - k - 7$. Recall that for $s = t+k+3, \dots, y$ we have $g(s,t,k) - g(s-2,t,k) \geq -1 + s/102$ (Lemma 4). Hence $g(y,t,k) \geq -202(y-t-k-1)/2 + (y-t-k-1)(y+t+k+3)/204 = (y-t-k-1)(y+t+k-10707)/204 > (0.58t-k-8)(1.58t-10714)/204$. Since $g(y,t,k) \leq 2k^2+2k$ by (23), we get $(0.58t-k-8)(1.58t-10714) < 408k(k+1)$. Lemma 18 gives $t \geq 30k$. Hence $k \leq \frac{t}{30}$. We get $30(17.4t-8)(1.58t-10714) < 408t(t+30)$, which is false if $t \geq 3 \cdot 10^4$. □

9. PROOF $B(x,t,k)$

Lemma 27. *Fix integers t,k such that $k \leq t \leq 200k$, $t+k \geq 102 \cdot 229$ and $k^2 \geq 207(k+2)$. Fix an integer $x \geq y+2$ such that $x \equiv y \pmod 2$. If $B(x,t,k)$ is true, then $B(x+2,t,k)$ is true.*

Proof. Fix Q, $C_{t,k}$, and (Y,T_1) satisfying $B(x,t,k)$ with (if $v(x,t,k) > 0$) a grid T_1. Set $\delta := u(x+2,t,k) - u(x,t,k)$.

(a) Assume $v(x,t,k) = 0$. Fix $p \in Y \cap Q$ and take a general $E \in |\mathcal{I}_{p,Q}(1,\delta-1)|$. We have $E \cap C_{t,k} = \emptyset$ and $E \cap Y = \{p\}$. Since E intersects almost transversally Y and at a unique point, the nodal curve $Y \cup E$ is smoothable ([21], [28]) and $p_a(Y \cup E) = g - g_{t,k}$. Fix any $S' \subset Q \setminus E$ such that $\#(S') = v(x+2,t,k)$, S' contains no point of $(C_{t,k} \cup Y) \cap Q$. We assume that $h^1(Q, \mathcal{I}_{S',Q}(x+1, x+3-\delta)) = 0$. Set $\Sigma := ((C_{t,k} \cup Y) \cap Q) \setminus \{p\}$. Since Σ is general in Q and $\#(\Sigma \cup S') = (x+2)(x+4-\delta)$ by (25) and the equality $(x+3)^2 - (x+2)(x+4) = 1$, we have $h^i(Q, \mathcal{I}_{E \cup \Sigma \cup S',Q}(x+2,x+2)) = 0$, $i = 0,1$. Since $h^i(\mathcal{I}_{C_{t,k} \cup Y}(x)) = 0$, $i = 0,1$, the residual exact sequence of Q gives $h^i(\mathcal{I}_{C_{t,k} \cup Y \cup E \cup S'}(x+2)) = 0$, $i = 0,1$.

(b) Assume $v(x,t,k) > 0$. Let e be the maximal positive integer such that $v(x,t,k) > (e-1)(\delta-e)$ and $e < \delta/2$. Lemma 22 gives that e exists and that $e \le 201$. The definition of e gives $v(x,t,k) \le e(\delta-e-1)$. The grid T_1 has e lines $R_j \in |\mathcal{O}_Q(1,0)|$ and $\delta-e$ lines $M_i \in |\mathcal{O}_Q(0,1)|$. Each irreducible component of T_1 contains a point of $Y\cap Q$. We deform T_1 to a grid $T \in |\mathcal{O}_Q(e,\delta-e)|$ with the same irreducible component R_j of bidegree $(1,0)$ and with $\delta-e$ irreducible component L_i of bidegree $(0,1)$ with $L_i = M_i$ for $1 \le i \le v(x,t,k)-(e-1)(\delta-e)$, while the other lines L_i are small deformations of M_i not intersecting $Y\cap Q$. By $B(x,t,k)$ the set S is the union of all points $R_j \cap L_i$ with either $j \ge 2$ or $j = 1$ and $1 \le i \le v(x,t,k)-(e-1)(\delta-e)$; we have enough points of $Y\cap Q$ to link all lines R_j and the prescribed lines L_i, because we only need $2u(x,t,k) \ge \delta$ and we use Lemma 23. Let T be the union of all lines R_j and L_i. We have $\#(\mathrm{Sing}(T)) = e(\delta-e)$. Set $\chi := \cup_{o\in S} 2o$ and $Y' := Y\cup T\cup \chi$. We have $\deg(Y'_{\mathrm{red}}) = u(x+2,t,k)$ and $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_Y)+\deg(T)+\#(S)-\#(\mathrm{Sing}(T))+\#(T\cap Y) = 1-p_a(Y) = 1-g+g_{t,k}$. We have $\mathrm{Res}_Q(C_{t,k}\cup Y') = C_{t,k}\cup Y\cup S$. Fix any $S' \subset Q\setminus T$ containing no point of $Q\cap(C_{t,k}\cup Y)$, with $\#(S') = v(x+2,t,k)$ and with $h^1(Q,\mathcal{I}_{S',Q}(x+2-e,x+2-\delta+e)) = 0$. Since $(x+3)^2-(x+3-e)(x+3-\delta+e) = v(x+2,t,k)+2d_{t,k}+2u(x,t,k)-\#(T\cap Y)$ by (25) and $(Y\cap C_{t,k})\cap Q$ are general in Q, we have $h^i(Q,\mathcal{I}_{S'\cup(Y'\cup C_{t,k})\cap Q,Q}(x+2,x+2)) = 0$, $i = 0,1$. The residual sequence of Q gives $h^i(\mathcal{I}_{C_{t,k}\cup Y'\cup S'}(x+2,x+2)) = 0$, $i = 0,1$. We claim that Y' is a flat limit of a family of smooth and connected curves of genus $g-g_{t,k}$ and degree $u(x+2,t,k)$. Take a smooth and connected affine curve Δ and $o\in\Delta$. Set $\{o_j\} := R_j\cap Y$, $1\le j\le e$, $\{q_i\} := L_i\cap Y$, $1 \le i \le v(x,t,k)-(e-1)(\delta-e)$, and $\{e_h\} := R_1\cap L_h$, $v(x,t,k)-(e-1)(\delta-e) < h \le \delta-e$. We may find flat families $\{R_j(z)\}_{z\in\Delta}$, $1\le j\le e$, and $\{L_i(z)\}_{z\in\Delta}$, $1 \le j \le \delta-e$, of lines with the following properties. For all i,j and all $z \in \Delta\setminus\{o\}$ we have $R_j(o) = R_j$, $R_j(z) \nsubseteq Q$, $o_j \in R_j(z)$, $L_i(o) = L_i$, $L_i(z) \nsubseteq Q$, $q_i \in L_i(z)$ if $1 \le i \le v(x,t,k)-(e-1)(\delta-e)$, $L_i(z)\cap R_j(z) = \emptyset$ for all j if $1 \le i \le v(x,t,k)-(e-1)(\delta-e)$ and $L_i(z)\cap R_j(z) \ne \emptyset$, $v(x,t,k)-(e-1)(\delta-e) < i \le \delta-e$, if and only if $j = 1$. For all $z \in \Delta\setminus\{o\}$ set $Y'_z := Y \cup \bigcup R_j \cup \bigcup L_i$. The family $\{Y'_z\}_{z\in\Delta}$ is flat. Each Y'_z, $z \in \Delta\setminus\{o\}$ is smoothable (Remark 2). Hence Y'_z, $z \ne o$, is a flat limit of a family of smooth curves of genus $g-g_{t,k}$ and degree $u(x+2,t,k)$ (Remark 2). Hence there is a flat family $\{Y''_z\}_{z\in\Gamma}$, with Γ an integral affine curve, $o\in\Gamma$, $Y''_o = Y'$, and Y''_z smooth and of genus $g-g_{t,k}$ for all $z \ne o$.

(c) Now we check that we may take S' in steps (a) and (b) to get a solution of $B(x+2,t,k)$, except that if $v(x+2,t,k) > 0$ we shift the two rulings of Q. Assume $v(x+2,t,k) > 0$ and take Y, S, e, R_j, L_i as in step (b). Set $\tau := u(x+4,t,k)-u(x+2,t,k)$. Let f be the maximal positive integer such that $v(x+2,t,k) > (f-1)(\tau-f)$ and $f < \tau/2$. Lemma 20 gives that f exists and that $f \le 201$. Note that $v(x+2,t,k) \le f(\tau-f-1)$. Fix distinct lines $R'_j \in |\mathcal{O}_Q(0,1)|$, $1 \le j \le f$, each of them containing a point of $Y\cap(Q\setminus T)$. Fix distinct lines $L'_i \in |\mathcal{O}_Q(1,0)|$, $1 \le i \le \tau-f$, such that L'_j contains a point of $Y\cap Q$ if and only if $1 \le i \le v(x+4,t,k)-(f-1)(\tau-f)$. We impose that no line L'_i contain a point of $C_{t,k}\cap Q$. We impose $R'_j \ne L_h$ and $L'_i \ne R_z$ for all i,j,h,z. So if $p \in Y\cap Q$ we allow that it is contained in a line of T and in one of the lines R'_j, L'_i, but in this case we assume that they are in different rulings of Q. So we may assume that $R_j \ne L'_i$ and $R'_j \ne L_i$ for all i,j. Since $\tau \ge \delta$ (use (25 for x and the integer $x' := x+2$) and $e \le 201$, it is sufficient to use that $\#(Y\cap Q) = 2u(x,t,k) \ge \tau+201$ and that $2u(x,t,k) \ge \delta+201$ (Lemma 23). We take as S' the union of all points

$R'_j \cap L'_i$ with either $j > 1$ or $j = 1$ and $1 \le i \le v(x+2t,k) - (f-1)(\tau - f)$. We have $h^1(Q, \mathcal{I}_{S',Q}(x+2-e, x+2-\delta+e)) = 0$, because $\tau - 1 \le x+3-e$ (since $\tau \le x - 197$ by Lemma 25). and $f \le 201 \le x+2-\delta+e$ (Lemma 20). Let N' (resp. N'') be the set of all points of $Y \cap Q$ contained in some line R'_j (resp. L'_i). Set $N := N' \cup N''$. Note that $N' \cap N'' = \emptyset$. Now take the last deformation made in step (b) with Γ as its parameter space. Since Y is transversal to Q, restricting Γ to a neighborhood of o and then taking a finite covering we may assume that $\{Y''_z\}_{z\in\Gamma}$ has $\#(N)$ sections m_p, $p \in N$, with $m_p(o) = p$ and $m_p(z) \in Q \cap Y''_z$ for all z. For each $z \in \Gamma$ and any $p \in N'$, say $p \in Y \cap R'_j$ (resp $p \in N''$, say $p \in L'_i \cap Y$), let $R'_j(z))$ (resp. $L'_i(z))$ be the line of bidegree $(0,1)$ (resp. $(1,0)$) containing the point $m_p(z)$. If $L'_i \cap Y = \emptyset$, then set $L'_i(z) := L_i$. Taking the union of all these lines we get a grid $T_1(z) \in |\mathcal{O}_Q(\tau - \#(N'), \#(N')|$ union of $\#(N')$ lines of bidegree $(0,1)$ containing a point of $Y \cap Q$ and $\tau - \#(N')$ lines of bidegree $(1,0)$, $\#(N'')$ of them containing a point of $Y \cap Q$. Set $S_o := S'$. For each $z \in \Gamma \setminus \{o\}$ we get a set $S'_z \subset Q$ taking the union of all $R'_j(z) \cap L'_i(z))$ according to the rules of the cases $(f,1)$, $(f,0,+)$ or $(f,0,-)$. Note that $\#(S'_z) = v(x+2,t,k)$. The family $\{S'_z\}_{z\in\Delta}$ is flat. The semicontinuity theorem for cohomology gives $h^i(\mathcal{I}_{C_{t,k}\cup Y''_z \cup S'_z}(x+2)) = 0$, $i = 0,1$, for a general $z \in \Gamma$.

If $v(x,t,k) = 0$, then we take $S = \emptyset$. In this case the same construction work, because $\tau \le x+2$ and $f \le x+3-u(x+2,t,k)+u(x,t,k)$ and $2u(x-2,t,k) \ge \delta$ (Lemma 23). $\square$

Lemma 28. *Assume $k \le t \le 200k$, $t+k \ge 42040$, that $A(y,t,k)$ is true and that either $y = t+k+1$ or $A(y-2,t,k)$ is true. Then $B(y+2,t,k)$ is true.*

Proof. Remember that $y \ge t+k+1$ and that $y \equiv t+k+1 \pmod 2$.

First assume $y \ge t+k+3$. Hence $A(y-2,t,k)$ is true. Taking the difference between (24) for $x = y+2$ and (6) for $s = y$ we get

$$\begin{aligned}&2(d_{t,k} + a(y,t,k)) + (y+2)(u(y+2,t,k) - a(y,t,k))\\ &+ v(y+2,t,k) - b(y,y,k) - g + g_{t,k} + g(y,t,k) = (y+3)^2\end{aligned} \tag{29}$$

Set $\delta := u(y+2,t,k) - a(y,t,k)$, $\gamma := g - g_{t,k} - g(y,t,k)$ and $\mu := \delta - \gamma$. Since $g \le g_{t,k} + g(y+2,t,k)$ we have $a(y+2,t,k) \ge u(y+2,t,k)$. By the definition of $u(y+2,t,k)$ we get

$$\begin{aligned}&(y+2)(a(y+2,t,k) - u(y+2,t,k)) =\\ &v(y+2,t,k) - b(y+2,t,k) + g_{t,k} + g(y+2,t,k) - g\end{aligned} \tag{30}$$

Claim: We have $\delta - \gamma \ge 201$.

Proof of the Claim: The integers $u(y+2,t,k)$, $v(y+2,t,k)$, δ, γ and μ depend on g and we write $u(y+2,t,k)(g)$, $v(y+2,t,k)(g)$, $\delta(g)$ and $\mu(g)$ to stress their dependence on g. They are defined for all g with $g_{t,k} + g(y,t,k) \le y < g(y+2,t,k)$ by the definition of y, but we may also define them for $g = g_{t,k} + g(y+2,t,k)$, writing $u(y+2,g,t,k)(g_{t,k} + g(y+2,t,k)) := a(y+2,t,k)$ and $v(y+2,g,t,k)(g_{t,k} + g(y+2,t,k)) := b(y+2,t,k)$ and still have (29) and (30). The Claim is true with strict inequality for $g = g_{t,k} + g(y+2,t,k)$ by Lemma 21. Use (30) both for g and $g-1$. When we decrease by one the genus in these two equations we decrease by at most one the integer $u(y+2,t,k)(g)$. Hence we get the Claim. $\square$

We introduce the following assertion.

Assertion $A'(y,t,k)$: There is a pair (Y,T_1) with the following properties. Y is a smooth and connected curve of degree $a(y,t,k)$ and genus $g(y,t,k)$ such that $Y \cap C_{t,k} = \emptyset$, Y intersects transversally Q and $(C_{t,k} \cup Y) \cap Q$ is general in Q. In particular no line of Q contains two or more points of $Y \cup C_{t,k}$. If $b(y,t,k) = 0$ we take as T any grid. Now assume $b(y,t,k) > 0$. Let e be the maximal positive integer such that $b(y,t,k) > (e-1)(\delta - e - 1)$ and $e < \delta/2$. Lemma 25 gives that e exists and that $e \le 201$. We take a grid $T \subset Q$ of bidegree $(e, \delta - e)$ adapted to $(Y, C_{t,k} \cap Q)$ such that $h^i(\mathcal{I}_{C_{t,k} \cup Y \cup S}(y)) = 0$, $i = 0,1$, for each $S \subseteq \mathrm{Sing}(T_1)$ with $\#(S) = b(y,t,k)$.

Assertion $A'(y,t,k)$ is proved as in Lemma 16, with the distinction of the 3 cases introduced in Remark 5, except for the following modification in one of them, which we now discuss as step (a).

(a) Assume $(e-1)(\delta - e - 1) < b(y,t,k) \le (e-1)(\delta - e - 1) + e - 2$ and $\delta < \mu + e$. Since $e \le 202$, we get $\gamma \le 201$. Let f be the maximal integer such that $f(\delta - f) \le \gamma + b(y,t,k)$ and $f \le \delta/2$. We say that $A'(y,t,k)$ is in case $(f,2)$. We assume the existence of distinct lines $R_j \in |\mathcal{O}_Q(1,0)|$, $1 \le j \le f$, with $R_j \cap (Y \cap Q) \ne \emptyset$ for all j and distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le \delta - f$, such that $L_i \cap (Y \cap Q) \ne \emptyset$ if and only if $1 \le i \le b(y,t,k) + \gamma - f(\delta - f)$, $L_i \cap (C_{t,k} \cap Q) = R_j \cap (C_{t,k} \cap Q) = \emptyset$ for all i, j, and let S be the union of all points $R_j \cap L_i$ with either $j \ge 2$ or $j = 1$ and $1 \le i \le b(y,t,k) - f(\delta - f)$.

By Lemma 24 we have enough points of $\#(Y \cap Q) \ge \delta$ to find the lines R_j and L_i as described in Remark 5; also we use this observation and the same in the analogous of step (c) of the proof of Lemma 28.

(b) Note that in the case we have $\gamma = g - g(y,t,k) - g_{t,k} \le 201$. In the case $\gamma = 0$, the proof of Lemma 28 (case (c) of Remark 5) would work verbatim, while in the case $0 < \gamma \le 201$ it only requires the modifications outlined in (d), which explains exactly which lines L_i must intersect $Y \cap Q$. Part (c) of the proof of Lemma 27 with $x = y - 2$ shows how to prove that $A(y-2,t,k)$ implies $A'(y,t,k)$. As in Remark 5 and Lemmas 27 and 28 $A'(y,t,k)$ implies $B(y+2,t,k)$. □

10. Proofs of Theorems 1, 2, and 3

Finishing the proof of Theorem 2: Lemma 26 gives $y \le m-7$. Hence $B(m-5,t,k)$ and $B(m-3,t,k)$ are true. Since $1 + (m-1)d + 1 - g = \binom{m+1}{3}$, we have $u(m-1,t,k) = d-1$ and $v(m-1,t,k) = m-3$. Take a solution of $B(m-5,t,k)$ with respect to Q and use the proof of Lemma 27 to prove the existence of a solution of the following modification $B'(m-3,t,k)$ of $B(m-3,t,k)$:

Assertion $B'(m-3,t,k)$: Let Q be a smooth quadric. Fix $C_{t,k}$ intersecting transversally Q and such that $Q \cap C_{t,k}$ is formed by $2d_{t,k}$ general points of Q. We call $B'(m-3,t,k)$ the existence of a pair (Y,T_1) with the following properties. Y is a smooth and connected curve of degree $u(m-3,t,k)$ and genus $g - g_{t,k}$ such that $Y \cap C_{t,k} = \emptyset$, Y intersects transversally Q and $(Y \cup C_{t,k}) \cap Q$ is general in Q. In particular no line of Q contains two or more points of $Y \cup C_{t,k}$. T_1 is a grid adapted to $(Y, C_{t,k} \cap Q)$, which we now describe. In all cases we assume that for all $S \subseteq \mathrm{Sing}(T_1)$ with $\#(S) = v(m-3,t,k)$ we have $h^i(\mathcal{I}_{C_{t,k} \cup Y \cup S}(x)) = 0$, $i = 0,1$. If $v(m-3,t,k) = 0$, then take as T_1 any adapted grid. Now assume $v(m-3,x,t) > 0$. Set $\delta := d - u(m-3,t,k)$. Let e be the maximal positive integer such that $v(x,t,k) > (e-1)(\delta - e)$ and $e \le \delta/2$. Since $d = u(m-1,t,k) + 1$, Lemma 25 gives that e exists and that $e \le 201$. We assume that the grid T_1 has bidegree $(e, \delta - e)$.

If $v(m-3,t,k) > 0$ we write R_j, $1 \le j \le e$, for the lines of bidegree $(1,0)$ of T_1 and M_i, $1 \le i \le \delta - e$, for the one of bidegree $(0,1)$. As in the proof of Lemma 16 we deform T_1 to another grid T with the same lines of bidegree $(1,0)$ and with lines L_i of bidegree $(0,1)$ only some of them containing a point of $Y \cap Q$.

The definition of e gives $v(m-3,t,k) \le e(d-u(m-3,t,k)-e-1)$. Fix distinct lines $R_j \in |\mathcal{O}_Q(1,0)|$, $1 \le j \le e$, with $R_j \cap (Y \cap Q) \ne \emptyset$ for all j and distinct lines $L_i \in |\mathcal{O}_Q(0,1)|$, $1 \le i \le \delta - e$, such that $L_i \cap (Q \cap (Y \cup C_{t,k})) \ne \emptyset$ if and only if $1 \le i \le v(x,t,k) - (e-1)(\delta - e)$. We assume $Y \cap R_j \cap L_i = \emptyset$ for all i, j. We assume $R_j \cap Y = \emptyset$ for all j , $L_i \cap C_{t,k} = \emptyset$ if $i \le \delta - e - 2$, $L_i \cap C_t \ne \emptyset$ if and only if $i = \delta - e - 1$ and $L_i \cap C_k \ne \emptyset$ if and only if $i = \delta - e$. Let S be the union of the points $R_j \cap L_i$ with either $j \ge 2$ or $j = 1$ and $1 \le i \le v(x,t,k) - (e-1)(\delta - e)$. About $B'(m-3,t,k)$ we only use that $h^i(\mathcal{I}_{C_{t,k} \cup Y \cup S}(x)) = 0$, $i = 0, 1$, for this specific set $S \subset \mathrm{Sing}(T)$. Set $\chi := \cup_{o \in S} 2o$. Let Y' be the union of Y, χ and all lines R_j and L_i. Y' is smoothable to a smooth and connected curve Y'' of genus $g - g_{t,k}$ and we may find a smoothing family fixing the two points of $Y' \cap C_{t,k}$. For this choice of Y'' the curve $M := Y'' \cup C_{t,k}$ is a nodal and connected curve with arithmetic genus g and with exactly 2 nodes. Since $t \ge k$, we have $h^1(\mathcal{O}_M(t)) = 0$ and hence $h^1(\mathcal{O}_M(m-2)) = 0$. The vector bundle $N_M|C_t$ (resp. $N_M|C_k$, resp. $N_M|Y''$) is obtained from N_{C_t} (resp. N_{C_k}, resp. $N_{Y''}$) making a positive elementary transformation at the point $Y'' \cap C_t$ (resp. the point $Y'' \cap C_k$, resp. each of the two points $Y'' \cap C_{t,k}$) in the direction corresponding to the tangent line of Y'' (resp. Y'', resp. $C_{t,k}$) at the point $Y'' \cap C_t$ (resp. the point $Y'' \cap C_k$, resp. each of the two points of $Y'' \cap C_{t,k}$). Since $h^1(N_{Y''}(-2)) = 0$, $h^1(N_{C_{t,k}}(-2)) = 0$ and $\#(\mathrm{Sing}(M)) = 2$, the Mayer-Vietoris exact sequence of Y'' and $C_{t,k}$ gives $h^1(N_M(-1)) = 0$. Hence M is smoothable ([14, Corollary 1.2]). By the semicontinuity theorem for cohomology a smoothing of M proves Theorem 2 for the pair (d,m), except that we must discuss the bounds on m and g assumed in Theorem 2. We need g, m for which we may take $t \ge 10^5$ and some k with $t/30 \ge k \ge t/200$ (these bounds are sufficient to use Remarks 3 and 8). For m we need $m \ge k + t + 7$ and we do not need other assumptions on m if the pair (g,m) allows us to do the construction, i.e. the burden is shifted to g. It is sufficient to have $g \ge g_{t,k} + g(t+k+1,t,k)$. We have $g_x = 1 + x(x+1)(2x-5)/6 \le x^3/3$ for all $x \ge 10$. Since $k \le t/30$ for very large t (Lemma 18), we assume $g \ge g_{10^5} + g_{\lfloor 10^4/3 \rfloor}$. Hence it is sufficient to assume $g \ge 10^{15}/3 + 10^{12}/27$. Hence it is sufficient to assume $g \ge 0.34 \cdot 10^{15}$. □

Proof of Theorem 1: Take (d,m) in the range A and set $g := 1 + m(d-1) - \binom{m+2}{3}$. By [2, Corollary 2.4] we may assume $d < \frac{m^2+4m+6}{4}$. Since $m \ge 13.8 \cdot 10^5$, and $d > \frac{m^2+4m+6}{6}$, we have $d \ge 31.3 \cdot 10^{10}$. Hence $0.02 d^{3/2} \ge 0.34 \cdot 10^{15}$. We claim that for these integers d we always have $Kd^{3/2} - 6\epsilon d \ge 0.02 d^{3/2}$, where $K := \frac{2}{3}\frac{1}{10}^{3/2}$ and $\epsilon := \frac{11}{20} + 4(\frac{1}{20})^{3/2}$. Indeed, $K \ge 0.021$ and $6\epsilon \le 0.6$ and for $d \ge 31.3 \cdot 10^{10}$ we have $0.001\sqrt{d} \ge 0.6$. Hence if $g \le 0.02 d^{3/2}$, then we apply [3, Corollary 1.3]. If $g \ge 0.34 \cdot 10^{15}$, then we apply Theorem 2. □

Proof of Theorem 3: By Theorem 1 we may assume that $g < G_A(d,m)$. We prove Theorem 3 for the fixed integer m by induction on the integer d.

(a) First assume that $(d-1,m)$ is not in the Range A, i.e. assume $d-1 \le \frac{m^2+4m+6}{6}$. Since $d \le \frac{m^2+4m+6}{6}+1$, we have $G_A(d,m) \le 1+m-1+\frac{(m-1)(m^2+4m+6)}{6}-\binom{m+2}{3} = m-1$. Since $d > \frac{m^2+4m+6}{6}$, we have $g \le d-3$ and so it is sufficient to quote [1] and that in this range of degrees and genera a general non-special curve C has $h^1(N_C(-2)) = 0$ (Remark 3).

(b) Now assume that $(d-1,m)$ is in the Range A. By the inductive assumption for each integer q such that $0 \le q \le G_A(d-1,m)$ there is a smooth and connected curve $Y \subset \mathbb{P}^3$ of genus q and degree $d-1$ such that $h^1(N_Y(-1)) = 0$ and $h^0(\mathcal{I}_Y(m-1)) = 0$. Let $L \subset \mathbb{P}^3$ be a general line intersecting quasi-transversally Y at a unique point p. The vector bundle $N_{Y\cup L|Y}(-1)$ is obtained from $N_Y(-1)$ making a positive elementary transformation at p ([21]). The vector bundle $N_{Y\cup L|L}$ is obtained from $N_L(-1)$ making a positive transformation at p ([21]) and hence it is a direct sum of a line bundle of degree 1 and a line bundle of degree 0. A Mayer-Vietoris exact sequence gives $h^1(N_{Y\cup L}(-1)) = 0$ and hence $Y \cup L$ is smoothable ([14, Corollary 1.2]). If $g \le G_A(d-1,m)$, then it is sufficient to take $q := g$. Now assume $G_A(d-1,m) < g < G_A(d,m)$. In this range any solution C must have $h^0(\mathcal{I}_C(m-1)) = 0$ and hence $h^1(\mathcal{I}_C(m-1)) = G_A(d,m) - g$.

Assume for the moment that $g \ge 0.34 \cdot 10^{15}$. We repeat the construction of Theorem 2 for this integer g. Remember that the integers t, k, y, $u(x,t,k)$ and $v(x,t,k)$ only depends on g and the parity of m. We have $1+(m-1)d-g = \binom{m+2}{3} + G_A(d,m) - g$. Since $G_A(d,m) - G_A(d-1,m) = m-1$, we have $1 \le G_A(d,m) - g \le m-2$. Since $3-g+(m-1)u(m-1,t,k)+v(m-1,t,k) = \binom{m+2}{3}$ and $0 \le v(m-1,t,k) \le m-2$, we have $u(m-1,t,k)-1 \le d \le u(m-1,t,k)+2$ and the first inequality holds only if $g = G_A(d,m)-1$ and $v(m-1,t,k) = 0$. Hence it is sufficient to adapt $B'(m-3,t,k)$ with the new value of d.

Now assume $g < 0.34 \cdot 10^{15}$. Since $m \ge 13.8 \cdot 10^5$ and $d > \frac{m^2+4m+6}{6}$, we have $d \ge 31.3 \cdot 10^{10}$. Hence $0.02 \cdot d^{3/2} \ge 0.34 \cdot 10^{15}$. Thus as in the proof of Theorem 1 it is sufficient to quote [3, Corollary 1]. □

Remark 11. Fix positive integers m, d, g, $m \ge 3$, such that there is a smooth, connected and non-degenerate curve $C \subset \mathbb{P}^3$ with degree d, genus g, $h^0(\mathcal{I}_C(m-1)) = 0$ and $h^1(N_C(-1)) = 0$. The latter condition implies that asymptotically for $d \gg 0$ we are not far from the *generalized Range A* $\frac{m^2+4m+6}{6} \le d \le D_m := m(m+1)/2$ of [2, Proposition 4.2]. These conditions are satisfied with $g = G_A(d,m)$ if $m \gg 0$ and (d,m) is in the Range A (Theorem 1) or for the (m,d,g) covered by [2, Proposition 4.2] or if $m \gg 0$ and $g \le G_A(d,m)$ (Theorem 3).

Claim: For each integer $d_1 > d$ there is a smooth and connected curve $X \subset \mathbb{P}^3$ with degree d_1, genus g, $h^0(\mathcal{I}_X(m-1)) = 0$ and $h^1(N_X(-1)) = 0$.

Proof of the Claim: By induction on d_1 we reduce to the case $d_1 = d+1$. Fix $p \in C$. Let $L \subset \mathbb{P}^3$ be a general line containing p. We have $\#(C \cap L) = 1$ and L is not tangent to C at p. Obviously $h^0(\mathcal{I}_{C\cup L}(m-1)) = 0$. As in step (b) of the proof of Theorem 3 we get $h^1(N_{C\cup L}(-1)) = 0$. Hence $C \cup L$ is smoothable ([14, Corollary 1.2]). Use the semicontinuity theorem. □.

When $d_1 \gg d$ we may cover some pairs (d_1, g') with $g' > g$ taking in the proof of the Claim instead of a line a smooth rational curve D of degree $d_1 - d$. with $\#(D \cap C) = g' - g + 1$ and D intersecting quasi-transversally C. Since any two quintuples of points of $\mathbb{P}^3$ in linearly general position are projectively equivalent, for $g' - g \le 4$ we may see D as a general rational space curve of degree $d_1 - d$ and hence if $d_1 - d \geq 3$ we may assume that the normal bundle of D is a direct sum of two line bundles of degree $2d_1 - 2d - 1$.

Proof of Corollary 1: By Theorem 3 there is a smooth, connected and non-degenerate curve $Y \subset \mathbb{P}^3$ with degree δ, genus g, $h^0(\mathcal{I}_Y(m-1)) = 0$ and $h^1(N_Y(-1)) = 0$. Apply the Claim in Remark 11. $\square$

Now we discuss the weak parts of our proof of Theorem 2, since any improvement of these parts would give huge improvements for the lower bounds assumed in Theorem 2 and hence for the assumption of the results in the introduction. A key point was using [2, Corollary 2.4], since for d large with respect to m our proof is far less efficient (for a fixed m it leaves gaps in the set of all d satisfying (1). Hence if there is some other construction of good curves covering, say, the range $\frac{m^2+4m+6}{4.5} \leq d < \frac{m^2+4m+6}{4}$, then it would be a very good help and we could use pairs (t,k) with far lower t/k; Lemmas 4, 14 and 15 show how better are the bounds and simplified the proofs if it is sufficient to take all (t,k) with $k \leq t \leq 3k$. Then (as observed at the end of Remark 3) a further improvement may be obtained by sharpening the results of [3]. The very first step for the latter project (as observed at the end of Remark 3) is to use [25] instead of [27].

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