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Hans M. Amman
Marco P. Tucci

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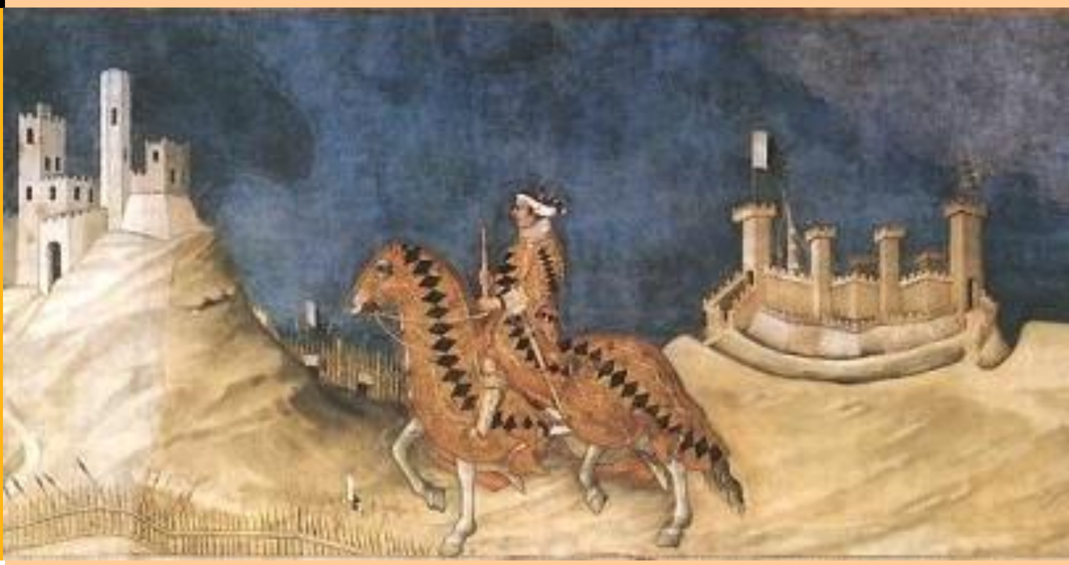

THE DUAL APPROCH IN AN INFINITE HORIZON MODEL

WITH A TIME-VARYING PARAMETER

by

Hans M. Amman and Marco P. Tucci

In a previous paper Amman and Tucci (2017) discuss the DUAL control method, based on Tse and Bar-Shalom (1973) and (Kendrick, 1981) seminal works, applied to the BMW infinite horizon model with an unknown but constant parameter. In these pages the DUAL solution to the BMW infinite horizon model with one time-varying parameter is reported. The special case where the desired path for the state and control are set equal to 0 and the linear system has no constant is considered. The appropriate Riccati quantities for the augmented system are derived and the time-invariant feedback rule are defined following the same steps as in Amman and Tucci (2017). Finally the new approximate cost-to-go is presented. Two cases are considered. In the first one the optimal control is selected using the updated estimate of the time-varying parameter in the model. In the second one only an old estimate of that parameter is available at the time the decision maker chooses her/his control. For the reader's sake, most of the technical derivations are confined to a number of short appendices.

1. Introduction

In a previous paper Amman and Tucci (2017) discuss the DUAL control method, based on Tse and Bar-Shalom (1973) and (Kendrick, 1981) seminal works, applied to the BMW infinite horizon model with an unknown but constant parameter. Building on their results, in these pages the DUAL solution to the BMW infinite horizon model with one time-varying parameter associated to the control variable is reported. The special case where the desired path for the state and control are set equal to 0 and the linear system has no constant is considered. Two scenarios are studied. In the first one the optimal control is selected using the updated estimate of the time-varying parameter in the model. In the second one only an old estimate of that parameter is available at the time the decision maker chooses her/his control. It is as if the estimate of the time-varying parameter available to the decision maker at the time the control is selected, or the decision is made, is going to be old by the time the control is applied. This situation is fairly common when deciding fiscal policy for next period.

The paper is organized as follows. The problem is stated in Section 2 and the one-period ahead projection of the mean and variance of the augmented state vector is discussed in Section 3. Section 4 is devoted to the compution of the nominal path for the state and control. The Riccati equations and the updating of the covariances of the augmented system are then considered (Section 5 and 6). In Section 7 the approximate cost-to-go is derived for the case where the updated estimate of the time-varying parameter in the model is used. Finally the appropriate derivations for the case where only an old estimate of that parameter is available at the time the decision maker chooses her/his control are reported (Section 8). A number of short appendices contain most of the technical derivations.

2. Statement of the Problem

Amman and Tucci (2017) consider an infinite horizon model in which the policy maker wants find the set of controls u_t for $t = 0, 1, ..., \infty$, where $t = 0$ denotes the current period, which minimizes the linear functional

$$J = E_0\left\{(1/2)\sum_{t=0}^{\infty}\left(x_t^2 w_t + u_t^2 \lambda_t\right)\right\} \tag{2-1}$$

where E_0 is the expectation operator conditional on the information available at time 0, $\lambda_t = \rho^t \lambda$ and $w_t = \rho^t w$ where ρ is the discount factor between 0 and 1, subject to

$$x_{t+1} = \alpha x_t + \beta u_t + \varepsilon_{t+1} \qquad \text{for } t = 0, 1, ..., \infty \tag{2-2}$$

with x_t and u_t the state and control variables, respectively. The parameters of the system equation are α and β with the latter assumed constant but unknown with mean, at time *t*, b_t and variance $\sigma_{t|t}^{\beta\beta}$. The error term ε_{t+1} is assumed identically and independently distributed (i.i.d.) normal with mean zero and variance *q*. Finally, the initial state x_0 and the penalty weights *w*'s and λ's are given constants. Also, the state is measured without error.[1]

[1] This is equivalent to setting **H**=**I** and **R**=**O** in Kendrick (1981, 2002, Ch. 10 -11) or Tucci (2004, Ch. 2-5).

Following Tse and Bar-Shalom (1973) methods for solving active learning stochastic control problem, Amman and Tucci (2017) compute, for each time period, the approximate cost-to-go at different values of the control and then choose that value which yields the minimum approximate cost.[2] This approximate cost-to-go is decomposed into three terms and, for the present problem, written as

$$J_N = J_{D,N} + J_{C,N} + J_{P,N} \tag{2-3}$$

where J_N is the total cost-to-go with *N* periods remaining and $J_{D,N}$, $J_{C,N}$ and $J_{P,N}$ are the deterministic, cautionary and probing component, respectively. The deterministic component includes only terms which are not stochastic. The cautionary one includes uncertainty only in the next time period and the probing term contains uncertainty in all future time periods. Thus the probing term includes the motivation to perturb the controls in the present time period in order to reduce future uncertainty about parameter values.[3]

In the following pages, this model is rewritten to allow for a time-varying parameter β, i.e.

$$x_{t+1} = \alpha x_t + \beta_t u_t + \varepsilon_{t+1} \qquad \text{for } t = 0, 1, ..., \infty \tag{2-4a}$$

with

$$\beta_{t+1} = \phi\left(\beta_t - \beta\right) + \beta + \eta_{t+1}. \tag{2-4b}$$

The parameters of the system equation are α and β_t with the latter assumed evolving over time according to a mean-reverting, or return to normality, model with β its unconditional mean, ϕ the transition parameter and the stochastic term η_{t+1} assumed i.i.d. normal with mean zero and variance σ_η^2. For simplicity sake it is here assumed that the hyperstructural parameters β, ϕ and σ_η^2, as well as α, are known with certainty.[4] Furthermore, the constraint $|\phi| < 1$ is imposed for stationarity reasons.

The control problem (2-2) and (2-4) is solved treating the stochastic parameters as additional state variables (Kendrick, 1981, 2002, Ch. 10) and restating it in terms of an augmented state vector $\mathbf{z}_t$ as: find the controls u_t for *t* = 0, 1, ..., ∞ minimizing

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[2] See Kendrick (1981, 2002, Ch. 9-10) or Tucci (2004, Ch. 2) for details.

[3] See Kendrick (1981, 2002, pp. 97-98) for an introduction to this decomposition.

[4] See, e.g., Tucci (2004, Ch. 2) for details. The case where the hyperstructural parameters are assumed known with uncertainty is discussed in Chapter 3 and 4 of the same reference.

$$J = E_0\left\{(1/2)\sum_{t=0}^{\infty}\left(\mathbf{z}_t'\mathbf{W}_t^*\mathbf{z}_t + u_t^2\lambda_t\right)\right\} \tag{2-5}$$

with $\mathbf{W}_t^*$ having w_t on the top left corner and zeros elsewhere, subject to the discrete-time system equations, with no measurement equation,

$$\mathbf{z}_{t+1} = \mathbf{f}\left(\mathbf{z}_t, u_t\right) + \varepsilon_{t+1}^{\mathbf{z}} \tag{2-6}$$

with the arrays defined as

$$\mathbf{z}_t = \begin{bmatrix} x \\ \beta \end{bmatrix}_t,\ \mathbf{f}\left(\mathbf{z}_t, u_t\right) = \begin{bmatrix} \alpha x_t + \beta_t u_t \\ \phi(\beta_t - \beta) + \beta \end{bmatrix},\ \varepsilon_t^{\mathbf{z}} = \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}_t. \tag{2-7}$$

Problems (2-2) and (2-4) and (2-5)-(2-7) are equivalent "however the first is described as a linear quadratic problem with random coefficients and the second as a non linear (in x, u and β) stochastic control problem" as noted in Kendrick (1981, 2002, p. 94).

3. One-period ahead projection of the mean and variance of the augmented state vector z

For this simple model the one-period ahead projection of the mean of the augmented state vector **z,** after control at time zero is applied, is

$$\hat{x}_{1|0} = \alpha x_0 + \beta_{0|0} u_0^{\tau} \tag{3-1}$$

$$\beta_{1|0} = \phi\left(\beta_{0|0} - \beta\right) + \beta \tag{3-2}$$

where x_0 is the initial condition for the state, u_0^{τ} is the search control at iteration τ, with the Certainty Equivalence (CE) solution being the first search control, i.e. $u_0^1 \equiv u_0^{CE}$ from now on simply u_0 to save on notation, and $\beta_{1|0}$ is the estimate of the unknown parameter at time 1 given its estimated value at time zero, i.e. $\beta_{0|0}$ with estimated variance $\sigma_{0|0}^{\beta\beta}$. For the BMW problem with no measurement error, the projected variances in this case look like[5]

[5] See, e.g., Kendrick (1981, 2002, Ch. 10, pp. 102) or Tucci (2004, Ch. 2, pp. 21-2) for details.

$$\sigma_{1|0}^{xx} = \left(u_0^{\tau}\right)^2 \sigma_{0|0}^{\beta\beta} + q$$
$$\sigma_{1|0}^{\beta x} = \phi\sigma_{0|0}^{\beta\beta} u_0^{\tau} \tag{3-3}$$
$$\sigma_{1|0}^{\beta\beta} = \phi^2\sigma_{0|0}^{\beta\beta} + \sigma_{\eta}^2$$

4. The nominal path for the state and control

At this point the nominal, or CE, path for state and control are needed. This is done by solving the CE problem for the unaugmented system from time 1 on, using $\hat{x}_{1|0} \equiv x_{o,1}$ as initial condition and the nominal path for the time-varying parameter generated using Eq. (3-2). Then the nominal control for a generic period j in the time-horizon can be expressed as

$$u_{o,j} = G_j x_{o,j} = G_j \left[\prod_{i=1}^{j-1}\left(\alpha + \beta_{i|0} G_i\right)\hat{x}_{o,1}\right] \qquad \text{for} \qquad j = 1, \ldots, \infty \,. \tag{4-1}$$

As explained in Appendix A, in this case G_j is not time-invariant as in Amman and Tucci (2017, p. 17) and is defined as

$$G_i = \left(-\frac{\alpha\beta_{i|0}k_{i+1}}{\lambda_i + \beta_{i|0}^2 k_{i+1}}\right) \text{ and } \left(\alpha + \beta_{i|0}G_i\right) = \left(\alpha - \frac{\alpha\beta_{i|0}^2 k_{i+1}}{\lambda_i + \beta_{i|0}^2 k_{i+1}}\right) \tag{4-2}$$

where $\beta_{i|0} \equiv E_0\left(\beta_i\right) = \phi^i\left(\beta_{0|0} - \beta\right) + \beta$. However for $j\to\infty$ the estimate of the unknown time-varying parameter β_j at time 0, i.e. $\beta_{j|0}$, and G_j converge to β and G_n, respectively, and Eq. (4-1) simplifies to

$$u_{o,j} = G_n\left(\alpha + \beta G_n\right)^{j-n} x_{o,n} \qquad \text{for } j = n, n+1, \ldots \tag{4-3}$$

with

$$x_{o,n} = \prod_{i=1}^{n-1}\left(\alpha + \beta_{i|0}G_i\right)x_{o,1} \tag{4-4}$$

where n indicates the first period in which $\beta_{j|0} = \beta$.

Therefore when the conditions for the existence of an infinite horizon solution are satisfied,[6] the results in Amman and Tucci (2017, App. A) remain valid after convergence of the projected value of the time-varying parameter to its unconditional mean when $x_{o,n}$ is used as initial nominal state.

[6] See, e.g., De Koning (1982) and Hansen and Sargent (2007) and the results in Appendix A.

In this case, when $\lambda_j = \rho^j \lambda$ and $w_j = \rho^j w$, the Riccati equations are defined as[7]

$$k_j \equiv k_j^{CE} = w_j + \alpha^2 k_{j+1} - \left(\alpha k_{j+1}\beta_{j|0}\right)^2 \left(\lambda_j + k_{j+1}\beta_{j|0}\right)^{-1} \qquad \text{for } j = n-1, n-2, \ldots \quad (4\text{-}5)$$

and

$$k_n \equiv k_n^{CE} = w_n + \alpha^2 \rho k_n - \left(\alpha\rho k_n \beta\right)^2 \left(\lambda_n + \rho k_n \beta^2\right)^{-1} \quad , \qquad (4\text{-}6)$$

with k_n the fixed point solution to the usual Riccati recursions, for $j = n, n+1, n+2, \ldots$.

5. Riccati equations for the arrays of the augmented system

The **K** Riccati array of the augmented system is partitioned as

$$\mathbf{K}_j = \begin{bmatrix} k^{xx} & k^{x\beta} \\ k^{\beta x} & k^{\beta\beta} \end{bmatrix}_j \qquad (5\text{-}1)$$

where the quantity k^{xx} corresponds to k^{CE} discussed in the previous section. When the condition for stabilizability hold the quantity $k^{x\beta} = k^{\beta x}$ and $k^{\beta\beta}$ reduces to, when $\ell < n$,

$$\begin{aligned} k_\ell^{\beta x} &= D^{-1}\rho k_n^{xx}\left(\alpha + \beta G_n\right)\left[\prod_{j=\ell}^{n-1}\left(\alpha + \beta_{j|0} G_j\right)\right] G_{o,n} x_{o,1} \\ &+ \sum_{i=\ell}^{n-1} k_{i+1}^{xx}\left(\alpha + \beta_{i|0} G_i\right)\left[\prod_{j=\ell}^{i-1}\left(\alpha + \beta_{j|0} G_j\right)\right] G_{o,i} x_{o,1} = \tilde{k}_{\ell,\infty}^{\beta x} x_{o,1} + \tilde{k}_{\ell,n}^{\beta x} x_{o,1} = \tilde{k}_\ell^{\beta x} x_{o,1} \end{aligned} \qquad (5\text{-}2)$$

with

$$G_{o,i} = \left(-\frac{\alpha\beta_{i|0} k_{i+1}^{xx}}{\lambda_i + \beta_{i|0}^2 k_{i+1}^{xx}}\right)\left[\prod_{j=1}^{i-1}\left(\alpha + \beta_{j|0} G_j\right)\right] \qquad \text{for } i = 1, \ldots, n-1 \qquad (5\text{-}3)$$

where it is understood that when the lower limit of a summation is higher than its upper limit the summation is zero and when the lower limit of a product is higher than its upper limit the product is one. The finite summation in Eq. (5-2) disappears when $\ell \geq n$. Then $k_\ell^{\beta x}$ simplifies to

$$k_\ell^{\beta x} = D^{-1}\rho k_\ell^{xx}\left(\alpha + \beta G_n\right) G_n x_{o,\ell} = \left[\rho\left(\alpha + \beta G_n\right)\right]^{\ell - n} k_n^{\beta x} \qquad (5\text{-}4)$$

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[7] As pointed out in Amman and Tucci (2017), in this case the Riccati equation is scalar function and can easily be solved. The multi-dimensional case can be more complicated to solve. See, e.g., Amman and Neudecker(1997).

with $k_n^{\beta x} = D^{-1}\rho k_n^{xx}\left(\alpha+\beta G_n\right)G_n x_{o,n}$ and $D=\left[1-\rho\left(\alpha+\beta G_n\right)^2\right]$ as shown in Appendix B.

Finally, $k^{\beta\beta}$ in Eq. (5-1) looks like

$$k_j^{\beta\beta} = \left(\tilde{k}_{j,\infty}^{\beta\beta} + \tilde{k}_{j,n}^{\beta\beta}\right)x_{o,1}^2 \equiv \tilde{k}_j^{\beta\beta} x_{o,1}^2, \tag{5-5}$$

for $j<n$, where the infinite summation from n to ∞ is denoted by

$$\tilde{k}_{j,\infty}^{\beta\beta} = D^{-2}\rho k_n^{xx} G_{o,n}^2\left\{1+\rho\left(\alpha+\beta G_n\right)^2 - \rho k_n^{xx} D^{-1}\beta^2\left(\lambda_n + \rho k_n^{xx}\beta^2\right)^{-1}\right\} \tag{5-6a}$$

and the summation of the first $n-1$ terms takes the form

$$\tilde{k}_{j,n}^{\beta\beta} = \sum_{i=j}^{n-1}\left[k_{i+1}^{xx}G_{o,i}^2 + 2\tilde{k}_{i+1}^{\beta x}G_{o,i} - \left(k_{i+1}^{xx}G_{o,i} + \tilde{k}_{i+1}^{\beta x}\right)^2 \beta_{i|0}^2\left(\lambda_i + k_{i+1}^{xx}\beta_{i|0}^2\right)^{-1}\right]. \tag{5-6b}$$

For $j \geq n$ the finite summation disappears and $k_j^{\beta\beta}$ looks like

$$k_j^{\beta\beta} = \left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n}\tilde{k}_n^{\beta\beta}x_{o,1}^2 \tag{5-7}$$

with $\tilde{k}_n^{\beta\beta}$ identical to $\tilde{k}_{j,\infty}^{\beta\beta}$ in Eq. (5-6a) as shown in Appendix C.

6. Updating the covariances of the augmented system

For the BMW problem the updating equations for the covariances of the augmented system look like[8]

$$\boldsymbol{\Sigma}_{j|j} = \begin{bmatrix} O & O \\ -\sigma_{j|j-1}^{\beta x}\left(\sigma_{j|j-1}^{xx}\right)^{-1} & 1 \end{bmatrix}\boldsymbol{\Sigma}_{j|j-1}, \tag{6-1}$$

then the elements of the updated covariance matrix are defined as

$$\sigma_{j|j}^{xx} = 0\ ,\ \sigma_{j|j}^{x\beta} \equiv \sigma_{j|j}^{\beta x} = 0,\ \sigma_{j|j}^{\beta\beta} = \sigma_{j|j-1}^{\beta\beta} - \sigma_{j|j-1}^{\beta x}\left(\sigma_{j|j-1}^{xx}\right)^{-1}\sigma_{j|j-1}^{x\beta} \tag{6-2}$$

where the projected covariances take the form in (3-3) when j and $j-1$ replace 1 and 0, respectively.

[8] See, e.g., Kendrick (1981, 2002, Ch. 10, pp. 103) or Tucci (2004, Ch. 2, pp. 27-8) for details.

Combining (6-2) and (3-3), it yields, for j = 1,

$$\sigma_{1|1}^{\beta\beta} = \left(\phi^2\sigma_{0|0}^{\beta\beta} + \sigma_\eta^2\right) - \left(\phi\sigma_{0|0}^{\beta\beta}u_0\right)^2\left(u_0^2\sigma_{0|0}^{\beta\beta} + q\right)^{-1} = \phi^2\sigma_{0|0}^{\beta\beta}\left(u_0^2\sigma_{0|0}^{\beta\beta}q^{-1} + 1\right)^{-1} + \sigma_\eta^2 \tag{6-3}$$

and in general it can be shown that (Appendix D)

$$\sigma_{j|j}^{\beta\beta} = \phi^{2(j-1)}\sigma_{1|1}^{\beta\beta}\overline{A}_j^{-1} + \sigma_\eta^2\left[\left(\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}\right] \qquad \text{for } j > 1 \tag{6-4}$$

with

$$\overline{A}_j = 1 + \sum_{i=1}^{j-1}\phi^{2(i-1)}S_i + \sigma_\eta^2 q^{-1}x_{o,1}^2\sum_{m=2}^{j-1}\overline{A}_m\left(\sum_{i=m}^{j-1}\phi^{2(i-m)}G_{o,i}^2\right), \qquad \text{for } j > 1 \tag{6-5}$$

with $\overline{A}_1 = 1$, $G_{o,i}$ as before and

$$S_i = u_{o,i}^2\sigma_{1|1}^{\beta\beta}q^{-1} = G_{o,i}^2 x_{o,1}^2\sigma_{1|1}^{\beta\beta}q^{-1} \qquad \text{for } i < n \tag{6-6a}$$

which simplifies to

$$S_i = u_{o,i}^2\sigma_{1|1}^{\beta\beta}q^{-1} = \left(\alpha + \beta G_n\right)^{2(i-n)}G_{o,n}^2 x_{o,1}^2\sigma_{1|1}^{\beta\beta}q^{-1}, \qquad \text{for } i \geq n \tag{6-6b}$$

and $u_{o,0} \equiv u_0$.

7. The approximate cost-to-go

As in Kendrick (1981, 2002, Ch. 10) the approximate cost-to-go associated with the 'search' control u_t^τ is decomposed into three parts: deterministic (J_D), cautionary (J_C) and probing (J_P). The deterministic component for the control at time 0 is, see, e.g., Eq. 10.49 in the cited reference,

$$J_{D,T-t} = \frac{1}{2}\lambda_t u_t^2 + \frac{1}{2}k_T^{CE}\hat{x}_T^2 + \frac{1}{2}\sum_{j=t+1}^{T-1}\left(x_{o,j}^2 K_j^{CE} + \lambda_j u_{o,j}^2\right) \tag{7-1}$$

with CE indicating the Certainty Equivalence value associated with the non-augmented model. As shown in Appendix E, in this case Eq. (7-1) can be rewritten as

$$J_{D,\infty} = \psi_1 u_0^2 + \psi_2 u_0 + \psi_3 \tag{7-2}$$

with

$$\begin{aligned}
\psi_1 &= (1/2)\left[\lambda_0 + \beta_{0|0}^2\left(\tilde{\psi}_n + \tilde{\psi}_\infty\right)\right] \\
\psi_2 &= \alpha\beta_{0|0}x_0\left(\tilde{\psi}_n + \tilde{\psi}_\infty\right) \\
\psi_3 &= (1/2)\left(\alpha x_0\right)^2\left(\tilde{\psi}_n + \tilde{\psi}_\infty\right).
\end{aligned} \tag{7-3}$$

where $\tilde{\psi}_n$ is the sum of $n-1$ terms and $\tilde{\psi}_\infty$ the sum of an infinite number of terms defined as

$$\tilde{\psi}_n = \sum_{j=1}^{n-1}\left\{\left[\prod_{i=1}^{j-1}\left(\alpha+\beta_{i|0}G_i\right)\right]^2\left(k_j^{CE}+\lambda_j G_j^2\right)\right\}$$

$$\tilde{\psi}_\infty = \left[\prod_{i=1}^{n-1}\left(\alpha+\beta_{i|0}G_i\right)\right]^2\left[\left(k_n^{CE}+\lambda_n G_n^2\right)\right]\left[1-\rho\left(\alpha+\beta G_n\right)^2\right]^{-1}. \tag{7-4}$$

It is understood that the product term in square brackets is one when its lower limit is larger than its upper limit.

The cautionary component looks like

$$J_{C,\infty} = (1/2)\left(k_1^{xx}\sigma_{1|0}^{xx}+k_1^{\beta\beta}\sigma_{1|0}^{\beta\beta}\right)+k_1^{x\beta}\sigma_{1|0}^{x\beta}+(1/2)\sum_{j=1}^{\infty}\left(k_{j+1}^{xx}q+k_{j+1}^{\beta\beta}\sigma_\eta^2\right) \tag{7-5}$$

with the k_j^{xx}'s, $\tilde{k}_1^{\beta x}$ and the $\tilde{k}_j^{\beta\beta}$'s defined as above. By using the results in Appendix F it yields

$$J_{C,\infty} = \delta_1 u_0^2+\delta_2 u_0+\delta_3 \tag{7-6}$$

with

$$\delta_1 = (1/2)\left[k_1^{xx}\sigma_{0|0}^{\beta\beta}+\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)\beta_{0|0}^2+2\tilde{k}_1^{\beta x}\phi\sigma_{0|0}^{\beta\beta}\beta_{0|0}\right]$$

$$\delta_2 = \left[\tilde{k}_1^{\beta x}\phi\sigma_{0|0}^{\beta\beta}+\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)\beta_{0|0}\right]\alpha x_0 \tag{7-7}$$

$$\delta_3 = (1/2)\sum_{j=0}^{n-1}k_{j+1}^{xx}q+(1/2)\rho k_n^{xx}q(1-\rho)^{-1}+(1/2)\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)\left(\alpha x_0\right)^2$$

and

$$\tilde{\delta}_n = \tilde{k}_1^{\beta\beta}\left(\phi^2\sigma_{0|0}^{\beta\beta}+\sigma_\eta^2\right)+\sum_{j=1}^{n-1}\tilde{k}_{j+1}^{\beta\beta}\sigma_\eta^2$$

$$= \tilde{k}_{1,\infty}^{\beta\beta}\left(\phi^2\sigma_{0|0}^{\beta\beta}+n\sigma_\eta^2\right)+\tilde{k}_{1,n}^{\beta\beta}\left(\phi^2\sigma_{0|0}^{\beta\beta}+\sigma_\eta^2\right)+\sigma_\eta^2\sum_{j=1}^{n-1}\tilde{k}_{j+1,n}^{\beta\beta} \tag{7-8}$$

$$\tilde{\delta}_\infty = \sum_{j=n}^{\infty}\tilde{k}_{j+1}^{\beta\beta}\sigma_\eta^2 = \sigma_\eta^2\tilde{k}_{1,\infty}^{\beta\beta}\rho\left(\alpha+\beta G_n\right)^2\left[1-\rho\left(\alpha+\beta G_n\right)^2\right]^{-1}.$$

Finally, the probing component takes the form

$$J_{P,\infty} = (1/2)\sum_{j=1}^{\infty}\left[u_{o,j}k_{j+1}^{xx}\beta_{j|0}+k_{j+1}^{\beta x}\beta_{j|0}\right]^2\left(\lambda_j+k_{j+1}^{xx}\beta_{j|0}^2\right)^{-1}\sigma_{j|j}^{\beta\beta}. \tag{7-9}$$

As discussed in Appendix G, using a relevant approximation to compute the finite summation it yields an expression which looks like

$$J_{P,\infty} \simeq \frac{1}{2}\frac{g(u_0)}{h(u_0)} + \frac{1}{2}v + \frac{1}{2}f(u_0) \tag{7-10}$$

with

$$g(u_0) = \tilde{k}_{1,-}^{\beta\beta} x_{o,1}^2, \tag{7-11a}$$

$$h(u_0) = \left(\sigma_{1|1}^{\beta\beta}\right)^{-1}, \tag{7-11b}$$

$$v = \tilde{k}_{n/4+1,-}^{\beta\beta}\left[\sigma_{1|1}^{\beta\beta}\left(\tilde{\bar{A}}_{n/4+1}\right)^{-1}\right]\phi^2\left(1-\phi^2\right)^{-1}, \tag{7-11c}$$

$$\begin{aligned} f(u_0) = \sigma_\eta^2 \Bigg\{ & \frac{n-2}{2}\left[\tilde{k}_{n/4+1,-}^{\beta\beta}\left(\tilde{\bar{A}}_{n/4+1}^{*}\right)^{-1} + \tilde{k}_{3n/4+1,-}^{\beta\beta}\left(\tilde{\bar{A}}_{3n/4+1}^{*}\right)^{-1}\right] \\ & + \left[1-\rho\left(\alpha+\beta G_n\right)^2\right]^{-1}\tilde{k}_{n,-}^{\beta\beta}\Bigg\} x_{o,1}^2 \end{aligned} \tag{7-11d}$$

where $\tilde{k}_{j,-}^{\beta\beta}$ stands for the 'minus' portion in $\tilde{k}_j^{\beta\beta}$ as defined in (5-6), $\bar{A}_j^*$ is the quantity in square brackets in Equation (6-4) and $\tilde{\bar{A}}_j$ denotes an approximation to the original term of the form $\bar{A}_j \simeq \tilde{\bar{A}}_j x_{o,1}^2$. Equation (7-10) is slightly different from the formulation of the probing component usually found in the literature, see e.g. Amman and Kendrick (1995), Tucci et al. (2010) and Amman and Tucci (2017). The familiar portion can be rewritten as usual, i.e. [9]

$$\frac{g(u_0)}{h(u_0)} = \frac{\phi_1\left(\phi_2 u_0 + \phi_3\right)^2}{\left[\phi^2\sigma_{0|0}^{\beta\beta} q\left(u_0^2\sigma_{0|0}^{\beta\beta}+q\right)^{-1} + \sigma_\eta^2\right]^{-1}}, \tag{7-12}$$

with

$$\begin{aligned} \phi_1 &= \tilde{k}_{1,-}^{\beta\beta} \\ \phi_2 &= \beta_{0|0} \\ \phi_3 &= \alpha x_0 \end{aligned} \tag{7-13}$$

Then, two new terms v and $f(u_0)$ appear. The former is largely independent of $\sigma_{1|1}^{\beta\beta}$ and $x_{o,1}^2$. The latter, largely independent of $\sigma_{1|1}^{\beta\beta}$ as well, takes into account the penalty associated with the variance of the stochastic parameter σ_η^2. It is interesting to notice that the component (7-11d) can be rearranged as

$$f(u_0) = \phi_4\left(\phi_2 u_0 + \phi_3\right)^2 \tag{7-14}$$

with

[9] The reader should be aware of the fact that the parameter ϕ included in $h(u_0)$ has nothing in common with the parameters ϕ_1, ϕ_2 and ϕ_3 appearing in the function $g(u_0)$. The former is the transition parameter in the law of motion of the time-varying parameter β_t, while ϕ_1, ϕ_2 and ϕ_3 are coefficients used to define the function $g(u_0)$ in the probing component of the approximate cost-to-go.

$$\phi_4 =$$
$$\sigma_\eta^2 \left\{ \frac{n-2}{2} \left[\tilde{k}^{\beta\beta}_{n/4+1,-} \left(\tilde{\tilde{A}}^*_{n/4+1} \right)^{-1} + \tilde{k}^{\beta\beta}_{3n/4+1,-} \left(\tilde{\tilde{A}}^*_{3n/4+1} \right)^{-1} \right] + \left[1 - \rho \left(\alpha + \beta G \right)^2 \right]^{-1} \tilde{k}^{\beta\beta}_{n,-} \right\} \tag{7-15}$$

and ϕ_2, ϕ_3 as in (G-13) as shown in Appendix G. At this point by substituting (7-2), (7-6) and (7-10) into (7-1) yields

$$J_\infty = \left(\psi_1 + \delta_1\right) u_0^2 + \left(\psi_2 + \delta_2\right) u_0 + \left(\psi_3 + \delta_3\right)$$
$$+ \left(\frac{1}{2}\right) \frac{\phi_1 \left(\phi_2 u_0 + \phi_3\right)^2}{\left[\phi^2 \sigma^{\beta\beta}_{0|0} q \left(u_0^2 \sigma^{\beta\beta}_{0|0} + q \right)^{-1} + \sigma_\eta^2 \right]^{-1}} + \frac{1}{2} v + \frac{1}{2} \phi_4 \left(\phi_2 u_0 + \phi_3\right)^2 \tag{7-13}$$

with the parameters defined as in (7-3)-(7-4), (7-7)-(7-8) and (7-11)-(7-15). As shown in the Appendices, these new definitions collapse to those associated to the infinite horizon model discussed in Amman and Tucci (2017) when $\phi = 1$ and $\sigma_\eta^2 = 0$.

8. The dual control in the infinite horizon model with an old estimate of the time-varying parameter

In this section it considered the case where the estimate of the time-varying parameter available to the decision maker at the time the control is selected, or the decision is made, is going to be old by the time the control is applied. It is as if the control is selected using old information, say the information available at time −1, instead of the information at time 0 as assumed in the previous sections. Then, the one-period ahead projection of the mean of the augmented state vector **z,** after control at time zero is applied, is

$$x_{o,1|-1} = \alpha x_0 + \beta_{0|-1} u_0, \tag{8-1}$$

$$\beta_{0|-1} = \phi \left(\beta_{-1|-1} - \beta \right) + \beta. \tag{8-2}$$

The projected variances in the absence of observation ‘0’ are

$$\begin{aligned} \sigma^{xx}_{0|-1} &= u_{-1}^2 \sigma^{\beta\beta}_{-1|-1} + q \\ \sigma^{\beta x}_{0|-1} &= \phi \sigma^{\beta\beta}_{-1|-1} u_{-1} \\ \sigma^{\beta\beta}_{0|-1} &= \phi^2 \sigma^{\beta\beta}_{-1|-1} + \sigma_\eta^2 \end{aligned} \tag{8-3}$$

and

$$\begin{aligned} \sigma^{xx}_{1|-1} &= u_0^2 \left(\phi^2 \sigma^{\beta\beta}_{-1|-1} + \sigma_\eta^2 \right) + q \\ \sigma^{\beta x}_{1|-1} &= \phi \left(\phi^2 \sigma^{\beta\beta}_{-1|-1} + \sigma_\eta^2 \right) u_0 \\ \sigma^{\beta\beta}_{1|-1} &= \phi^2 \left(\phi^2 \sigma^{\beta\beta}_{-1|-1} + \sigma_\eta^2 \right) + \sigma_\eta^2. \end{aligned} \tag{8-4}$$

In this case the updated variance of the stochastic parameter for $j = 1$ is

$$\sigma^{\beta\beta}_{1|1,-1} = \phi^2 \sigma^{\beta\beta}_{0|-1} q \left(u_0^2 \sigma^{\beta\beta}_{0|-1} + q \right)^{-1} + \sigma^2_\eta \qquad (8\text{-}5)$$

which is identical to Eq. (6-3) when $\sigma^{\beta\beta}_{0|0}$ is replaced by $\sigma^{\beta\beta}_{0|-1}$. The more complicated notation $\sigma^{\beta\beta}_{1|1,-1}$ is here preferred to stress the fact that the nominal value of u_1, say $u_{o,1|-1}$, is obtained by replacing G_1 by

$$G_{1|-1} = \left(-\frac{\alpha \beta_{1|-1} k_{2|-1}}{\lambda_1 + \beta^2_{1|-1} k_{2|-1}} \right) \qquad (8\text{-}6)$$

in (4-1). Analogously, Eqs. (4-2)-(4-4) should be rewritten with the G_i's and $\beta_{i|0}$'s substituted by $G_{i|-1}$ and $\beta_{i|-1}$, respectively. In this case the Riccati array is labeled $k_{j|-1} \equiv k^{xx}_{j|-1}$, $n-1$ denotes the first period in which $\beta_{j|-1} = \beta$, G_j converges to G, say $G_{n-1|-1}$, and Eqs. (4-5) and (4-6), take the form

$$u_{o,j|-1} = G_{n-1|-1} \left(\alpha + \beta G_{n-1|-1} \right)^{j-n+1} x_{o,n-1|-1} \quad \text{and} \quad x_{o,n-1|-1} = \prod_{i=1}^{n-2} \left(\alpha + \beta_{i|-1} G_{i|-1} \right) x_{o,1|-1} \qquad (8\text{-}7)$$

for $j \geq n-1$. The fixed point solution to the usual Riccati recursions, say $k^{xx}_{n-1|-1}$, is obtained from Eq. (A-10) in Appendix A with w_n replaced by w_{n-1}.

The quantity $k^{\beta x}_\ell$ in the Riccati array for the augmented system, say $k^{\beta x}_{\ell|-1}$, now looks like

$$\begin{aligned} k^{\beta x}_{\ell|-1} &= D^{-1}_{|-1} \rho k^{xx}_{n-1|-1} \left(\alpha + \beta G_{n-1|-1} \right) \left[\prod_{j=\ell}^{n-2} \left(\alpha + \beta_{j|-1} G_{j|-1} \right) \right] G_{o,n-1|-1} x_{o,1|-1} \\ &+ \sum_{i=\ell}^{n-2} k^{xx}_{i+1|-1} \left(\alpha + \beta_{i|-1} G_{i|-1} \right) \left[\prod_{j=\ell}^{i-1} \left(\alpha + \beta_{j||-1} G_{j|-1} \right) \right] G_{o,i|-1} x_{o,1|-1} \\ &= \tilde{k}^{\beta x}_{\ell,\infty|-1} x_{o,1|-1} + \tilde{k}^{\beta x}_{\ell,n|-1} x_{o,1|-1} = \tilde{k}^{\beta x}_{\ell|-1} x_{o,1|-1} \end{aligned} \qquad (8\text{-}8)$$

for $\ell < n-1$ with $D_{|-1} = \left[1 - \rho \left(\alpha + \beta G_{n-1|-1} \right)^2 \right]$, $G_{o,i|-1}$ similar to $G_{o,i}$ in Eq. (5-3) but based on the parameter estimate at time '−1', see Appendix H, and

$$k^{\beta x}_{\ell|-1} = \left[\rho \left(\alpha + \beta G_{n-1|-1} \right) \right]^{\ell-n+1} D^{-1}_{|-1} \rho k^{xx}_{n-1|-1} \left(\alpha + \beta G_{n-1|-1} \right) G_{n-1|-1} x_{o,n-1|-1} \qquad (8\text{-}9)$$

for $\ell \geq n-1$. Analogously, $k^{\beta\beta}$ in Eq. (5-1), say $k^{\beta\beta}_{j|-1}$, looks like

$$k^{\beta\beta}_{j|-1} = \left(\tilde{k}^{\beta\beta}_{j,\infty|-1} + \tilde{k}^{\beta\beta}_{j,n-1|-1}\right) x^2_{o,1|-1} \equiv \tilde{k}^{\beta\beta}_{j|-1} x^2_{o,1|-1} \tag{8-10}$$

for $j < n-1$, where the infinite summation from $n-1$ to ∞ is denoted by

$$\tilde{k}^{\beta\beta}_{j,\infty|-1} = D^{-2}_{|-1} \rho k^{xx}_{n-1|-1} G^2_{o,n-1|-1} \left\{ 1 + \rho\left(\alpha + \beta G_{n-1|-1}\right)^2 - \rho k^{xx}_{n-1|-1} D^{-1}_{|-1} \beta^2 \left(\lambda_{n-1} + \rho k^{xx}_{n-1|-1} \beta^2\right)^{-1} \right\} \tag{8-11}$$

and the finite summation takes the form

$$\tilde{k}^{\beta\beta}_{j,n-1|-1} = \sum_{i=j}^{n-2} \left[k^{xx}_{i+1|-1} G^2_{o,i|-1} + 2\tilde{k}^{\beta x}_{i+1|-1} G_{o,i|-1} - \left(k^{xx}_{i+1|-1} G_{o,i|-1} + \tilde{k}^{\beta x}_{i+1|-1}\right)^2 \beta^2_{i|-1} \left(\lambda_i + k^{xx}_{i+1|-1} \beta^2_{i|-1}\right)^{-1} \right] \tag{8-12}$$

When $j \geq n-1$, Eq. (8-10) simplifies to $k^{\beta\beta}_{j|-1} = \left[\rho\left(\alpha + \beta G_{n-1|-1}\right)^2\right]^{j-n+1} \tilde{k}^{\beta\beta}_{n-1|-1} x^2_{o,1|-1}$ with $\tilde{k}^{\beta\beta}_{n-1|-1}$ identical to $\tilde{k}^{\beta\beta}_{j,\infty|-1}$ in Eq. (8-11) as shown in Appendix H.

In this case, the updated variance of the stochastic parameter for a generic period j looks like

$$\sigma^{\beta\beta}_{j|j,-1} = \left\{ \phi^{2(j-1)} \sigma^{\beta\beta}_{1|1,-1} + \sigma^2_\eta \sum_{m=2}^{j} \phi^{2(j-m)} \overline{A}_{m|-1} \right\} \overline{A}^{-1}_{j|-1} \tag{8-13}$$

with

$$\begin{aligned} \overline{A}_{j|-1} &= 1 + \sum_{i=1}^{j-1} \phi^{2(i-1)} S_{i|-1} + \sigma^2_\eta q^{-1} \sum_{m=2}^{j-1} \overline{A}_{m|-1} \left(\sum_{i=m}^{j-1} \phi^{2(i-m)} G^2_{o,i|-1} x^2_{o,1|-1} \right) \\ &= \overline{A}_{j-1|-1} + \phi^{2(j-2)} S_{j-1|-1} + \sigma^2_\eta q^{-1} G^2_{o,j-1|-1} x^2_{o,1|-1} \sum_{m=2}^{j-1} \phi^{2(j-1-m)} \overline{A}_{m|-1} \end{aligned} \qquad \text{for } j > 1 \tag{8-14}$$

where $\overline{A}_{1|-1} = 1$, $S_{i|-1} = G^2_{o,i|-1} x^2_{o,1|-1} \sigma^{\beta\beta}_{1|1,-1} q^{-1}$ for $j < n-1$ and $S_{i|-1} = \left(\alpha + \beta G_{n-1|-1}\right)^{2(i-n+1)} G^2_{o,n-1|-1} x^2_{o,1|-1} \times \sigma^{\beta\beta}_{1|1,-1} q^{-1}$ when $j \geq n-1$ (Appendix H).

Then, using the results in Appendix I, the approximate cost-to-go can be rearranged as

$$\begin{aligned} J_\infty &= \left(\psi_{1|-1} + \delta_{1|-1}\right) u_0^2 + \left(\psi_{2|-1} + \delta_{2|-1}\right) u_0 + \left(\psi_{3|-1} + \delta_{3|-1}\right) \\ &+ \left(\frac{1}{2}\right) \frac{\phi_{1|-1}\left(\phi_{2|-1} u_0 + \phi_{3|-1}\right)^2}{\left[\phi^2 \sigma^{\beta\beta}_{0|-1} q \left(u_0^2 \sigma^{\beta\beta}_{0|-1} + q\right)^{-1} + \sigma^2_\eta\right]^{-1}} + \frac{1}{2} v_{|-1} + \frac{1}{2} \phi_{4|-1} \left(\phi_{2|-1} u_0 + \phi_{3|-1}\right)^2 \end{aligned} \tag{8-15}$$

with these parameters being the exact counterpart of those appearing in (7-13) and defined as in (7-3)-(7-4), (7-7)-(7-8) and (7-11)-(7-15).

9. Conclusion

In these pages the DUAL solution to the BMW infinite horizon model with one time-varying parameter on the control variable is reported. This may be useful, e.g., when the decision maker faces time-varying expenditure multipliers or economic agents with 'moody' preferences. The special case where the desired path for the state and control are set equal to 0 and the linear system has no constant is considered. The appropriate Riccati quantities for the augmented system are derived and the time-invariant feedback rule defined following the same steps as in Amman and Tucci (2017). Finally the new approximate cost-to-go is presented. Two cases are considered. In the first one the optimal control is selected using the updated estimate of the time-varying parameter in the model. In the second one only an old estimate of that parameter is available at the time the decision maker chooses her/his control. In this case the observation at time zero of the time-varying parameter is treated as missing and the updated variance of the stochastic parameter for j =1 is computed starting out from the projected variance at time '-1'.

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Appendix A. Deriving the nominal path for control as a function of the projected state in the infinite horizon model with a time-varying parameter

Given a certain control at time 0, say u_0, the nominal, or Certainty Equivalence (CE), value of x_1, denoted by $x_{o,1}$, is given by

$$x_{o,1} = \alpha x_0 + \beta_{0|0} u_0, \tag{A-1}$$

with $\beta_{0|0} \equiv E_0(\beta_0)$ the estimate of the time-varying parameter at time 0 based on all the information available at time 0, when the other system parameter is assumed constant and known and there is no intercept. When this parameter is assumed to evolve over time as in the text, i.e.

$$\beta_{j+1} = \phi(\beta_j - \beta) + \beta + \eta_{j+1}$$

with β its unconditional mean, and the desired path for the state and control is zero, the nominal or CE value of u_1, $u_{o,1}$, is given by[1]

$$u_{o,1} = G_1 x_{o,1} = \left(-\frac{\alpha \beta_{1|0} k_2}{\lambda_1 + \beta_{1|0}^2 k_2} \right) x_{o,1}, \tag{A-2}$$

with $\beta_{1|0} \equiv E_0(\beta_1) = \phi(\beta_{0|0} - \beta) + \beta$ the projected value of the mean-reverting time-varying parameter at time 1 based on all the information available at time 0.

By repeating this procedure, it is then apparent that the nominal control for a generic period j in the infinite horizon problem can be written as

$$\begin{aligned} u_{o,j} &= G_j x_{o,j} = G_j \left[\prod_{i=1}^{j-1} \left(\alpha + \beta_{i|0} G_i \right) x_{o,1} \right] \\ &= G_j \left[\prod_{i=m}^{j-1} \left(\alpha + \beta_{i|0} G_i \right) \prod_{i=1}^{m-1} \left(\alpha + \beta_{i|0} G_i \right) x_{o,1} \right] = G_j \left[\prod_{i=m}^{j-1} \left(\alpha + \beta_{i|0} G_i \right) x_{o,m} \right] \end{aligned} \tag{A-3}$$

with

$$G_i = \left(-\frac{\alpha \beta_{i|0} k_{i+1}}{\lambda_i + \beta_{i|0}^2 k_{i+1}} \right) \text{ and } \left(\alpha + \beta_{i|0} G_i \right) = \left(\alpha - \frac{\alpha \beta_{i|0}^2 k_{i+1}}{\lambda_i + \beta_{i|0}^2 k_{i+1}} \right) \tag{A-4}$$

where $\beta_{i|0} \equiv E_0(\beta_i) = \phi^i(\beta_{0|0} - \beta) + \beta$. In this case G is not time-invariant as in Amman and Tucci (2017, p. 17). However for $j \to \infty$ the estimate of the unknown time-varying parameter β_j at time 0, $\beta_{j|0}$, and G_j converge to β and G_n, respectively, and Eq. (A-3) simplifies to

$$u_{o,j} = G_j \prod_{i=1}^{j-1} \left(\alpha + \beta_{i|0} G_i \right) x_{o,1} = G_n x_{o,j} = G_n \left(\alpha + \beta G_n \right)^{j-n} x_{o,n} \qquad \text{for } j = n, n+1, ... \tag{A-5}$$

with

$$x_{o,n} = \prod_{i=1}^{n-1} \left(\alpha + \beta_{i|0} G_i \right) x_{o,1} \tag{A-6}$$

---

[1] See, e.g., Tucci et al. (2010).

where n indicates the first period in which $\beta_{j|0} = \beta$.

The time-varying parameter has an unconditional distribution with known constant statistics, i.e. $E(\beta_j) = \beta$ and $\text{var}(\beta_j) = \phi^2(1-\phi^2)^{-1}\sigma_\eta + \sigma_\eta$. The necessary and sufficient condition for the mean-square (ms) stability of the model stated in De Koning (1982, p. 451, Th. 6.1) then applies. In the present case this translates to the condition

$$\tilde{\pi} = \alpha^2 \left\{1 - \beta^2 \left[\beta^2 + \phi^2(1-\phi^2)^{-1}\sigma_\eta + \sigma_\eta\right]^{-1}\right\} < 1$$

which is always true as long as $|\alpha| \le 1$. When the conditional distribution of the β_j's at time 0 is considered the mean is as in Eq. (A-4) and $\text{var}(\beta_{j|0}) == (1+\phi^{2j})(1-\phi^2)^{-1}\sigma_\eta + \phi^{2j}\sigma_{\beta_0}$ with σ_{β_0} the variance associated with the initial estimate $\beta_{0|0}$.[2] These quantities are not constant but for $j \to \infty$ they converge to $E(\beta_{j|0}) = \beta$ and $\text{var}(\beta_{j|0}) = (1-\phi^2)^{-1}\sigma_\eta$, respectively. The relevant condition for ms-stability in the 'not constant' trajectory looks like

$$\tilde{\pi}_j = \alpha^2 - \alpha^2\left[\phi^j(\beta_{0|0} - \beta) + \beta\right]^2$$
$$\times \left\{\left[\phi^j(\beta_{0|0} - \beta) + \beta\right]^2 + \left[(1-\phi^2)^{-1}\sigma_\eta + \phi^{2j}(1-\phi^2)^{-1}\sigma_\eta + \phi^{2j}\sigma_{\beta_0}\right]\right\}^{-1}$$

and is less than 1 for all j's, as long as $|\alpha| \le 1$. Moreover, this sequence quickly converges, not unexpectedly given the presence of $2j$ as exponent, to $\tilde{\pi}$ with $0 < \tilde{\pi} < \ldots < \tilde{\pi}_2 < \tilde{\pi}_1 < 1$.[3]

Therefore, the results in Amman and Tucci (2017, App. A) remain valid after convergence of the projected value of the time-varying parameter to its unconditional mean when $x_{o,n}$ is used as initial nominal state. This is due to the fact that under these conditions, when $\lambda_j = \rho^j \lambda$ and $w_j = \rho^j w$ with ρ the discount factor, the optimal control law looks like

$$G_j = \left(-\frac{\alpha\beta_{j|0}k_{j+1}}{\lambda_j + \beta_{j|0}^2 k_{j+1}}\right), \tag{A-7}$$

with

$$k_j \equiv k_j^{CE} = w_j + \alpha^2 k_{j+1} - (\alpha k_{j+1}\beta_{j|0})^2 (\lambda_j + k_{j+1}\beta_{j|0})^{-1} \qquad \text{for } j = n-1, n-2, \ldots \tag{A-8}$$

and

---

[2] See, e.g., Cowpertwait and Metcalfe (2009) Equations 12.1, 12.6 and 12.7.

[3] For example when $\beta_{0|0} = -.3, \beta = -.5, \phi = .9, \sigma_\eta = .1$ and $\sigma_{\beta_0} = .1$ the sequence starts with $\tilde{\pi}_1 = .91$ and quickly converges to $\tilde{\pi} = .678$. At $j = 8$, $\tilde{\pi}_j$ is equal to .789.

$$G_j \equiv G_n = -\alpha\beta\rho k_n\left(\lambda_n + \beta^2\rho k_n\right)^{-1}, \tag{A-9}$$

with k_n the fixed point solution to the usual Riccati recursions

$$k_n \equiv k_n^{CE} = w_n + \alpha^2\rho k_n - \left(\alpha\rho k_n\beta\right)^2\left(\lambda_n + \rho k_n\beta^2\right)^{-1} \quad , \tag{A-10}$$

for $j = n, n+1, n+2, \ldots$.

Appendix B. Deriving submatrix $k^{\beta x}$ of the augmented system in the infinite horizon model with a time-varying parameter

In the BMW model with desired paths for the state and control set to zero, no intercept and a time-varying parameter following a mean-reverting model as in the previous appendix, the general formula for $k^{\beta x}$, see e.g. Kendrick (1981, 2002, Eq. 10-40) or Tucci (2004, Eq. 2.56), specializes to

$$k_1^{\beta x} = k_2^{xx}\left(\alpha + \beta_{1|0} G_1\right) u_{o,1} + k_2^{\beta x}\left(\alpha + \beta_{1|0} G_1\right) \tag{B-1}$$

with $\beta_{1|0}$ the estimate of the unknown time-varying parameter β_1 at time 0, $k_2^{xx} \equiv k_2^{CE}$, G_1 and $u_{o,1}$ defined as in App. A and

$$k_2^{\beta x} = k_3^{xx}\left(\alpha + \beta_{2|0} G_2\right) u_{o,2} + k_3^{\beta x}\left(\alpha + \beta_{2|0} G_2\right). \tag{B-2}$$

Then, by repeated substitution, it can be shown that

$$k_1^{\beta x} = \sum_{i=1}^{\infty} k_{i+1}^{xx}\left[\prod_{j=1}^{i}\left(\alpha + \beta_{j|0} G_j\right)\right] u_{o,i}\,. \tag{B-3}$$

As observed in the previous appendix, when the system is stable and $\rho < 1$, for $j \to \infty$ the estimate of the unknown time-varying parameter β_j at time 0, $\beta_{j|0}$, and G_j converge to β and G_n, respectively, and Eq. (A-5) holds. Then Eq. (B-3) can be rewritten as

$$k_1^{\beta x} = \sum_{i=n}^{\infty} k_n^{xx}\left[\rho\left(\alpha + \beta G_n\right)\right]^{i-n+1}\left[\prod_{j=1}^{n-1}\left(\alpha + \beta_{j|0} G_j\right)\right] G_i x_{o,i} + \sum_{i=1}^{n-1} k_{i+1}^{xx}\left[\prod_{j=1}^{i}\left(\alpha + \beta_{j|0} G_j\right)\right] G_i x_{o,i} \tag{B-4}$$

with

$$G_i x_{o,i} = \left(\alpha + \beta G_n\right)^{i-n} G_n x_{o,n} = \left(\alpha + \beta G\right)^{i-n} G_{o,n} x_{o,1} \qquad \text{for } i = n, n+1, \ldots \tag{B-5a}$$

$$G_i x_{o,i} = \left(-\frac{\alpha \beta_{i|0} k_{i+1}^{xx}}{\lambda_i + \beta_{i|0}^2 k_{i+1}^{xx}}\right)\left[\prod_{j=1}^{i-1}\left(\alpha + \beta_{j|0} G_j\right)\right] x_{o,1} = G_{o,i} x_{o,1} \qquad \text{for } i = 1, \ldots, n-1 \tag{B-5b}$$

$G_{o,i}$ and $G_{o,n}$ implicitly defined and $x_{o,n}$ as in (A-6). When the system is ms-stable and $\left|\alpha\right| \le 1$

$$G_i = \left(-\frac{\alpha \beta_{i|0} k_{i+1}^{xx}}{\lambda_i + \beta_{i|0}^2 k_{i+1}^{xx}}\right) < \infty, \qquad \alpha + \beta_{i|0} G_i = \alpha\left(1 - \frac{\beta_{i|0}^2 k_{i+1}^{xx}}{\lambda_i + \beta_{i|0}^2 k_{i+1}^{xx}}\right) < 1 \tag{B-6a}$$

$$G_n = \left(-\frac{\alpha \beta \rho k_n^{xx}}{\lambda_n + \beta^2 \rho k_n^{xx}}\right) < \infty, \qquad \alpha + \beta G_n = \alpha\left(1 - \frac{\beta^2 \rho k_n^{xx}}{\lambda_i + \beta^2 \rho k_n^{xx}}\right) < 1 \tag{B-6b}$$

and $k_1^{\beta x}$ looks like

$$
\begin{aligned}
k_1^{\beta x} &= D^{-1}\rho k_n^{xx}\left(\alpha+\beta G_n\right)\left[\prod_{j=1}^{n-1}\left(\alpha+\beta_{j|0}G_j\right)\right]G_{o,n}x_{o,1} \\
&+\sum_{i=1}^{n-1}k_{i+1}^{xx}\left(\alpha+\beta_{i|0}G_i\right)\left[\prod_{j=1}^{i-1}\left(\alpha+\beta_{j|0}G_j\right)\right]G_{o,i}x_{o,1}=\tilde{k}_{1,\infty}^{\beta x}x_{o,1}+\tilde{k}_{1,n-1}^{\beta x}x_{o,1}=\tilde{k}_1^{\beta x}x_{o,1}
\end{aligned}
\tag{B-7}
$$

with $G_{o,i}$ and $G_{o,n}$ as above, $D=\left[1-\rho\left(\alpha+\beta G_n\right)^2\right]$ and k_n^{xx} the fixed point solution to the Riccati quantity described in Appendix A, when there is no intercept, the desired paths are set to zero, the system is stabilizable and the discount factor is less than 1. By repeating the same procedure it can be shown that a generic $k_\ell^{\beta x}$ with $\ell \leq n-1$ is defined as

$$
\begin{aligned}
k_\ell^{\beta x} &= D^{-1}\rho k_n^{xx}\left(\alpha+\beta G_n\right)\left[\prod_{j=\ell}^{n-1}\left(\alpha+\beta_{j|0}G_j\right)\right]G_{o,n}x_{o,1} \\
&+\sum_{i=\ell}^{n-1}k_{i+1}^{xx}\left(\alpha+\beta_{i|0}G_i\right)\left[\prod_{j=\ell}^{i-1}\left(\alpha+\beta_{j|0}G_j\right)\right]G_{o,i}x_{o,1}=\tilde{k}_{\ell,\infty}^{\beta x}x_{o,1}+\tilde{k}_{\ell,n}^{\beta x}x_{o,1}=\tilde{k}_\ell^{\beta x}x_{o,1}
\end{aligned}
\tag{B-8}
$$

where it is understood that when the lower limit of a summation is higher than its upper limit the summation is zero and when the lower limit of a product is higher than its upper limit the product is one. It follows that

$$
k_\ell^{\beta x}=D^{-1}\rho k_\ell^{xx}\left(\alpha+\beta G_n\right)G_n x_{o,\ell}=\left[\rho\left(\alpha+\beta G_n\right)\right]^{\ell-n}k_n^{\beta x}
\tag{B-9}
$$

for $\ell \geq n$ with $k_n^{\beta x}=D^{-1}\rho k_n^{xx}\left(\alpha+\beta G_n\right)G_n x_{o,n}$. These results are fully consistent with Eq. (B-28) in Amman and Tucci (2017, App. B).

Appendix C. Deriving submatrix $k^{\beta\beta}$ of the augmented system in the infinite horizon model with a time-varying parameter

In the BMW model specified as in App. B, the general formula for $k^{\beta\beta}$, see e.g. Kendrick (1981, 2002, Eq. 10-42) or Tucci (2004, Eq. 2.57), specializes to

$$k_j^{\beta\beta} = \left(u_{o,j}^2 k_{j+1}^{xx} + u_{o,j} k_{j+1}^{\beta x}\right) + \left(u_{o,j} k_{j+1}^{x\beta} + k_{j+1}^{\beta\beta}\right) - \left[u_{o,j} k_{j+1}^{xx} \beta_{j|0} + k_{j+1}^{\beta x} \beta_{j|0}\right]^2 \left(\lambda_j + k_{j+1}^{xx} \beta_{j|0}^2\right)^{-1} \quad \text{(C-1)}$$

with $\beta_{j|0}$ the estimate of the unknown time-varying parameter β_j at time 0, k_{j+1}^{xx}, $k_{j+1}^{\beta x}$ and $u_{o,j}$ defined as in App. B. Then, by repeated substitution, it yields for $j = 1$

$$k_1^{\beta\beta} = \sum_{i=1}^{\infty} k_{i+1}^{xx} u_{o,i}^2 + 2\sum_{i=1}^{\infty} k_{i+1}^{\beta x} u_{o,i} - \sum_{i=1}^{\infty} \left\{u_{o,i} k_{i+1}^{xx} \beta_{i|0} + k_{i+1}^{\beta x} \beta_{i|0}\right\}^2 \left(\lambda_i + k_{i+1}^{xx} \beta_{i|0}^2\right)^{-1}. \quad \text{(C-2)}$$

By proceeding as in the previous appendix, the first infinite summation in (C-2) looks like

$$\sum_{i=1}^{\infty} k_{i+1}^{xx} u_{o,i}^2 = \sum_{i=n}^{\infty} k_{i+1}^{xx} \left(G_i x_{o,i}\right)^2 + \sum_{i=1}^{n-1} k_{i+1}^{xx} \left(G_i x_{o,i}\right)^2 = D^{-1} \rho k_n^{xx} G_{o,n}^2 x_{o,1}^2 + \sum_{i=1}^{n-1} k_{i+1}^{xx} G_{o,i}^2 x_{o,1}^2 \quad \text{(C-3)}$$

with D, $G_{o,i}$ and $G_{o,n}$ as in Eq. (B-7) and k_n^{xx} the fixed point solution to the Riccati quantity described in Appendix A, when there is no intercept, the desired paths are set to zero, the system is stabilizable and the discount factor is less than 1. It is understood that when the upper limit of the summation is lower than the lower limit the corresponding term is zero and when the same occurs for a product the product term is one. Similarly, the second term can be written as

$$\begin{aligned} 2\sum_{i=1}^{\infty} k_{i+1}^{\beta x} u_{o,i} &= 2\left[\sum_{i=n}^{\infty} k_{i+1}^{\beta x} \left(G_i x_{o,i}\right) + \sum_{i=1}^{n-1} k_{i+1}^{\beta x} \left(G_i x_{o,i}\right)\right] \\ &= 2\left[\rho\left(\alpha + \beta G_n\right)^2\right] D^{-2} \rho k_n^{xx} G_{o,n}^2 x_{o,1}^2 + 2\sum_{i=1}^{n-1} \left(\tilde{k}_{i+1,\infty}^{\beta x} + \tilde{k}_{i+1,n}^{\beta x}\right) G_{o,i} x_{o,1}^2 \end{aligned} \quad \text{(C-4)}$$

where the results in Eqs. (B-8) and (B-9) are used. Finally, the squared portion is

$$\begin{aligned} &\sum_{i=1}^{\infty} \left[u_{o,i} k_{i+1}^{xx} \beta_{i|0} + k_{i+1}^{\beta x} \beta_{i|0}\right]^2 \left(\lambda_i + k_{i+1}^{xx} \beta_{i|0}^2\right)^{-1} \\ &= \sum_{i=n}^{\infty} \left[u_{o,i} k_{i+1}^{xx} + k_{i+1}^{\beta x}\right]^2 \beta^2 \left(\lambda_i + k_{i+1}^{xx} \beta^2\right)^{-1} + \sum_{i=1}^{n-1} \left[u_{o,i} k_{i+1}^{xx} + k_{i+1}^{\beta x}\right]^2 \beta_{i|0}^2 \left(\lambda_i + k_{i+1}^{xx} \beta_{i|0}^2\right)^{-1} \end{aligned} \quad \text{(C-5)}$$

with

$$u_{o,i} k_{i+1}^{xx} = G_n \left(\alpha + \beta G_n\right)^{i-n} \rho^{i-n+1} k_n^{xx} x_{o,n} = \left[\rho\left(\alpha + \beta G_n\right)\right]^{i-n} \rho k_n^{xx} G_{o,n} x_{o,1} \quad \text{(C-6a)}$$

and

$$k_{i+1}^{\beta x} = \left[\rho\left(\alpha + \beta G_n\right)\right]^{(i+1)-n} D^{-1} \left(\rho k_n^{xx}\right)\left(\alpha + \beta G_n\right) G_{o,n} x_{o,1} \quad \text{(C-6b)}$$

for $i = n, n+1, \ldots$ when Eq. (A-6) is used.

Then the summation from n to infinity in (C-2) can be rewritten as

$$k_{1,\infty}^{\beta\beta} = D^{-2}\rho k_n^{xx} G_{o,n}^2 x_{o,1}^2 \left\{1+\rho\left(\alpha+\beta G_n\right)^2 - \rho k_n^{xx} D^{-1}\beta^2\left(\lambda_n + \rho k_n^{xx}\beta^2\right)^{-1}\right\} \equiv \tilde{k}_{1,\infty}^{\beta\beta} x_{o,1}^2 \qquad \text{(C-7)}$$

where the appropriate portions of Eq. (C-3), (C-4) and (C-5) are used and after some tedious algebra the summation of the first $n-1$ terms takes the form

$$k_{1,n}^{\beta\beta} = \sum_{i=1}^{n-1}\left[k_{i+1}^{xx}G_{o,i}^2 + 2\tilde{k}_{i+1}^{\beta x}G_{o,i} - \left(k_{i+1}^{xx}G_{o,i} + \tilde{k}_{i+1}^{\beta x}\right)^2 \beta_{i|0}^2\left(\lambda_i + k_{i+1}^{xx}\beta_{i|0}^2\right)^{-1}\right]x_{o,1}^2 = k_{1,n}^{\beta\beta}x_{o,1}^2 \qquad \text{(C-8)}$$

with $G_{o,i}$ defined as above and $\tilde{k}_{i+1}^{\beta x}$ as in (B-8). Consequently, $k_1^{\beta\beta}$ looks like

$$k_1^{\beta\beta} = \left(\tilde{k}_{1,\infty}^{\beta\beta} + \tilde{k}_{1,n}^{\beta\beta}\right)x_{o,1}^2 \equiv \tilde{k}_1^{\beta\beta}x_{o,1}^2. \qquad \text{(C-9)}$$

Similarly, under these conditions,

$$k_2^{\beta\beta} = u_{o,2}^2 k_3^{xx} + 2u_{o,2}k_3^{\beta x} + k_3^{\beta\beta} - \left[u_{o,2}k_3^{xx}\beta_{2|0} + k_3^{\beta x}\beta_{2|0}\right]^2\left(\lambda_2 + k_3^{xx}\beta_{2|0}^2\right)^{-1} \qquad \text{(C-10)}$$

and by repeated substitution it yields

$$k_2^{\beta\beta} = \sum_{i=2}^{\infty} k_{i+1}^{xx}u_{o,i}^2 + 2\sum_{i=2}^{\infty} k_{i+1}^{\beta x}u_{o,i} - \sum_{i=2}^{\infty}\left[u_{o,i}k_{i+1}^{xx}\beta_{i|0} + k_{i+1}^{\beta x}\beta_{i|0}\right]^2\left(\lambda_i + k_{i+1}^{xx}\beta_{i|0}^2\right)^{-1}. \qquad \text{(C-11)}$$

The only difference with respect to $k_1^{\beta\beta}$ lies in the shorter finite summation for $i = 2, \ldots, n-1$ and putting all pieces together it yields

$$k_2^{\beta\beta} = \tilde{k}_{1,\infty}^{\beta\beta}x_{o,1}^2 + \sum_{i=2}^{n-1}\left[k_{i+1}^{xx}G_{o,i}^2 + 2\tilde{k}_{i+1}^{\beta x}G_{o,i} - \left(k_{i+1}^{xx}G_{o,i} + \tilde{k}_{i+1}^{\beta x}\right)^2\beta_{i|0}^2\left(\lambda_i + k_{i+1}^{xx}\beta_{i|0}^2\right)^{-1}\right]x_{o,1}^2 \; . \qquad \text{(C-12)}$$

By repeating this procedure for the various j's it is apparent that for $j = n$

$$k_n^{\beta\beta} = D^{-2}\rho k_n^{xx}G_{o,n}^2 x_{o,1}^2\left\{1+\rho\left(\alpha+\beta G_n\right)^2 - \rho k_n^{xx}D^{-1}\beta^2\left(\lambda_n + \rho k_n^{xx}\beta^2\right)^{-1}\right\} \equiv \tilde{k}_n^{\beta\beta}x_{o,1}^2, \qquad \text{(C-13)}$$

with $G_{o,n}$ defined as above, which is identical to $k_{1,\infty}^{\beta\beta}$. It follows that for $j > n$

$$k_j^{\beta\beta} = \left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n} k_n^{\beta\beta} = \left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n}\tilde{k}_{1,\infty}^{\beta\beta}x_{o,1}^2 \equiv \tilde{k}_j^{\beta\beta}x_{o,1}^2 \qquad \text{(C-14)}$$

given that $k_{j+1} = \rho k_j \;\; \forall j \geq n$ and $x_{o,n+1} = \left(\alpha+\beta G_n\right)x_{o,n}$.

Appendix D. Updating the variance of the augmented system in the infinite horizon model when the updated estimate of the time-varying parameter is available

By combining (3.3') and (6.2), it follows that the updated variance of the stochastic parameter β in the BMW model for period 1 is given by

$$\begin{aligned}\sigma_{1|1}^{\beta\beta} &= \left(\phi^2\sigma_{0|0}^{\beta\beta}+\sigma_\eta^2\right)-\left(\phi\sigma_{0|0}^{\beta\beta}u_0\right)^2\left[u_0^2\sigma_{0|0}^{\beta\beta}+q\right]^{-1} = \phi^2\sigma_{0|0}^{\beta\beta}\left(u_0^2\sigma_{0|0}^{\beta\beta}q^{-1}+1\right)^{-1}+\sigma_\eta^2 \\ &= \phi^2\sigma_{0|0}^{\beta\beta}qA_1^{-1}+\sigma_\eta^2\end{aligned} \quad \text{(D-1)}$$

with $A_1 = u_0^2\sigma_{0|0}^{\beta\beta}+q = q\left(u_0^2\sigma_{0|0}^{\beta\beta}q^{-1}+1\right)$, and the updated variance for period 2 can be rewritten as

$$\sigma_{2|2}^{\beta\beta} = \phi^2\sigma_{1|1}^{\beta\beta}\left(u_{o,1}^2\sigma_{1|1}^{\beta\beta}q^{-1}+1\right)^{-1}+\sigma_\eta^2 = \left(\phi^4\sigma_{0|0}^{\beta\beta}q+\sigma_\eta^2\phi^2A_1\right)A_2^{-1}+\sigma_\eta^2$$

with $A_2 = \left(u_{o,1}^2\phi^2+u_0^2\right)\sigma_{0|0}^{\beta\beta}+u_{o,1}^2q^{-1}A_1\sigma_\eta^2+q.$

By repeating this procedure, it can be shown that for a generic term j it yields

$$\begin{aligned}\sigma_{j|j}^{\beta\beta} &= \left\{\phi^{2j}\sigma_{0|0}^{\beta\beta}q+\phi^{2(j-1)}\sigma_\eta^2A_1+\phi^{2(j-2)}\sigma_\eta^2A_2+\ldots+\phi^2\sigma_\eta^2A_{j-1}\right\}A_j^{-1}+\sigma_\eta^2 \\ &= \left\{\phi^{2j}\sigma_{0|0}^{\beta\beta}q+\sigma_\eta^2\left(\sum_{l=1}^{j-1}\phi^{2(j-l)}A_l\right)\right\}A_j^{-1}+\sigma_\eta^2\end{aligned} \quad \text{(D-2)}$$

with the A's defined as

$$A_j = \sigma_{0|0}^{\beta\beta}\sum_{i=0}^{j-1}\phi^{2i}u_{o,i}^2+\sigma_\eta^2q^{-1}\sum_{l=1}^{j-1}\left[A_l\sum_{i=l}^{j-1}\phi^{2(i-l)}u_{o,i}^2\right]+q. \quad \text{(D-3)}$$

Equation (D-2) reduces to Eq. (D-4) in Amman and Tucci (2017) when $\phi=1$ and $\sigma_\eta^2=0$.

Using the formulae in App. A for the nominal path of the state and control, i.e. Eqs. (A-1)- (A-6), and, after some tedious algebra, it can be shown that Eq. (D-2) can be rewritten in the terms of $\sigma_{1|1}^{\beta\beta}$ as

$$\begin{aligned}\sigma_{j|j}^{\beta\beta} &= \left\{\phi^{2(j-1)}\sigma_{1|1}^{\beta\beta}+\phi^{2(j-2)}\sigma_\eta^2\bar{A}_2+\phi^{2(j-3)}\sigma_\eta^2\bar{A}_3+\ldots+\phi^2\sigma_\eta^2\bar{A}_{j-1}\right\}\bar{A}_j^{-1}+\sigma_\eta^2 \\ &= \left\{\phi^{2(j-1)}\sigma_{1|1}^{\beta\beta}+\sigma_\eta^2\sum_{m=2}^{j}\phi^{2(j-m)}\bar{A}_m\right\}\bar{A}_j^{-1}\end{aligned} \quad \text{for } j>1 \quad \text{(D-4)}$$

with

$$\begin{aligned}\bar{A}_j &= 1+\sum_{i=1}^{j-1}\phi^{2(i-1)}S_i+\sigma_\eta^2q^{-1}\sum_{m=2}^{j-1}\bar{A}_m\left(x_{o,1}^2\sum_{i=m}^{j-1}\phi^{2(i-m)}G_{o,i}^2\right), \\ &= \bar{A}_{j-1}+\phi^{2(j-2)}S_{j-1}+\sigma_\eta^2q^{-1}G_{o,j-1}^2x_{o,1}^2\sum_{m=2}^{j-1}\phi^{2(j-1-m)}\bar{A}_m\end{aligned} \quad \text{for } j>1 \quad \text{(D-5)}$$

with $\bar{A}_1=1$, $G_{o,i}$ as in the previous appendices, and

$$S_i = u_{o,i}^2\sigma_{1|1}^{\beta\beta}q^{-1} = G_{o,i}^2x_{o,1}^2\sigma_{1|1}^{\beta\beta}q^{-1} \quad \text{(D-6a)}$$

which simplifies to

$$S_i = u_{o,i}^2 \sigma_{1|1}^{\beta\beta} q^{-1} = \left(\alpha + \beta G_n\right)^{2(i-n)} G_{o,n}^2 x_{o,1}^2 \sigma_{1|1}^{\beta\beta} q^{-1}, \qquad \text{(D-6b)}$$

with $u_{o,i}$ and $x_{o,n}$ as in Eq. (A-5) and (A-6), respectively, when $i \geq n$. It is understood that when the upper limit of the summation in (D-5) is lower than the lower limit the corresponding term is zero and the term $\phi^{2(j-2)} S_{j-1}$ vanishes for $i > (n/2)$. Finally, notice that the term in braces multiplying the $\bar{A}_m$'s looks like

$$x_{o,1}^2 \sum_{i=m}^{j-1} \phi^{2(i-m)} G_{o,i}^2 \qquad \text{for } m < j < n \qquad \text{(D-7a)}$$

$$x_{o,1}^2 \sum_{i=m}^{n-1} \phi^{2(i-m)} G_{o,i}^2 + G_{o,n}^2 x_{o,1}^2 \sum_{i=n}^{j-1} \phi^{2(i-m)} \left(\alpha + \beta G\right)^{2(i-n)} \qquad \text{for } m < n < j \qquad \text{(D-7b)}$$

$$G_{o,n}^2 x_{o,1}^2 \sum_{i=m}^{j-1} \phi^{2(i-m)} \left(\alpha + \beta G\right)^{2(i-n)} \qquad \text{for } n \leq m < j. \qquad \text{(D-7c)}$$

Equation (D-4) reduces to Eq. (D-9) in Amman and Tucci (2017) when $\phi = 1$ and $\sigma_\eta^2 = 0$.

Appendix E. The deterministic component in the presence of updated estimates of the time-varying parameter

The deterministic component of the approximate cost-to-go can be written as in Kendrick (1981, 2002, Eqt. 10.49), i.e.

$$J_{D,T-t} = \frac{1}{2}\lambda_t u_t^2 + \frac{1}{2}k_T^{CE}\hat{x}_T^2 + \frac{1}{2}\sum_{j=t+1}^{T-1}\left(x_{o,j}^2 K_j^{CE} + \lambda_j u_{o,j}^2\right) \quad \text{(E-1)}$$

when there is no constant term and the desired path for the state and control are zero, with *CE* indicating the Certainty Equivalence value associated with the non-augmented model. In the infinite horizon model with an updated estimate of the time-varying parameters Eq. (E-1) looks like

$$J_{D,\infty} = \frac{1}{2}\lambda_0 u_0^2 + \frac{1}{2}\left(\alpha x_0 + \beta_{0|0}u_0\right)^2 \tilde{\psi}_n + \frac{1}{2}\left(\alpha x_0 + \beta_{0|0}u_0\right)^2 \tilde{\psi}_\infty \quad \text{(E-2)}$$

with $\tilde{\psi}_n$ the sum of a finite number of terms and $\tilde{\psi}_\infty$ the sum of an infinite number of terms defined as

$$\begin{aligned}\tilde{\psi}_n &= \sum_{j=1}^{n-1}\left\{\left[\prod_{i=1}^{j-1}\left(\alpha + \beta_{i|0}G_i\right)\right]^2 \left(k_j^{CE} + \lambda_j G_j^2\right)\right\} \\ \tilde{\psi}_\infty &= \left[\prod_{i=1}^{n-1}\left(\alpha + \beta_{i|0}G_i\right)\right]^2 \left[\left(k_n^{CE} + \lambda_n G_n^2\right)\right]\left[1 - \rho\left(\alpha + \beta G_n\right)^2\right]^{-1}\end{aligned} \quad \text{(E-3)}$$

when the results and definitions of Appendix A are used and it is understood that the product term in square brackets is one when its lower limit is larger than its upper limit. It follows that Eq. (E-1) can be rearranged as

$$J_{D,\infty} = \psi_1 u_0^2 + \psi_2 u_0 + \psi_3 \quad \text{(E-4)}$$

where

$$\begin{aligned}\psi_1 &= (1/2)\left[\lambda_0 + \beta_{0|0}^2\left(\tilde{\psi}_n + \tilde{\psi}_\infty\right)\right] \\ \psi_2 &= \alpha\beta_{0|0}x_0\left(\tilde{\psi}_n + \tilde{\psi}_\infty\right) \\ \psi_3 &= (1/2)\left(\alpha x_0\right)^2\left(\tilde{\psi}_n + \tilde{\psi}_\infty\right).\end{aligned} \quad \text{(E-5)}$$

Appendix F. The cautionary component in the presence of updated estimates of the time-varying parameter

The general formula for the cautionary component of the approximate cost-to-go, see e.g. Kendrick (1981; 2002, equation 10.50) or Tucci (2004, equation 2.68), for $t=0$ and $T=\infty$ looks like

$$J_{C,\infty}=\left(1/2\right)\left(k_1^{xx}\sigma_{1|0}^{xx}+k_1^{\beta\beta}\sigma_{1|0}^{\beta\beta}\right)+k_1^{x\beta}\sigma_{1|0}^{x\beta}+\left(1/2\right)\sum_{j=1}^{\infty}\left(k_{j+1}^{xx}q+k_{j+1}^{\beta\beta}\sigma_\eta^2\right) \tag{F-1}$$

with the k_j^{xx}'s, $\tilde{k}_1^{\beta x}$ and the $\tilde{k}_j^{\beta\beta}$'s defined as in App. A, B and C, respectively. Given that in this case the projected variances are defined as in (3.3), i.e. $\sigma_{1|0}^{xx}=u_0^2\sigma_{0|0}^{\beta\beta}+q$, $\sigma_{1|0}^{\beta x}=\phi\sigma_{0|0}^{\beta\beta}u_0$ and $\sigma_{1|0}^{\beta\beta}=\phi^2\sigma_{0|0}^{\beta\beta}+\sigma_\eta^2$, after some simple but tedious manipulations Eq. (F-1) can be rewritten as

$$J_{C,\infty}=\left(1/2\right)\sum_{j=0}^{\infty}k_{j+1}^{xx}q+\left(1/2\right)k_1^{xx}\sigma_{0|0}^{\beta\beta}u_0^2+\left(1/2\right)\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)x_{o,1}^2+\tilde{k}_1^{\beta x}\phi\sigma_{0|0}^{\beta\beta}u_0x_{o,1} \tag{F-2}$$

where $x_{o,1}\equiv x_{1|0}$, $\tilde{\delta}_n$ is the sum of a finite number of terms and $\tilde{\delta}_\infty$ the sum of an infinite number of terms defined as

$$\begin{aligned}\tilde{\delta}_n&=\tilde{k}_1^{\beta\beta}\left(\phi^2\sigma_{0|0}^{\beta\beta}+\sigma_\eta^2\right)+\sum_{j=1}^{n-1}\tilde{k}_{j+1}^{\beta\beta}\sigma_\eta^2\\&=\tilde{k}_{1,\infty}^{\beta\beta}\left(\phi^2\sigma_{0|0}^{\beta\beta}+n\sigma_\eta^2\right)+\tilde{k}_{1,n}^{\beta\beta}\left(\phi^2\sigma_{0|0}^{\beta\beta}+\sigma_\eta^2\right)+\sigma_\eta^2\sum_{j=1}^{n-1}\tilde{k}_{j+1,n}^{\beta\beta}\\\tilde{\delta}_\infty&=\sum_{j=n}^{\infty}\tilde{k}_{j+1}^{\beta\beta}\sigma_\eta^2=\sigma_\eta^2\tilde{k}_{1,\infty}^{\beta\beta}\rho\left(\alpha+\beta G_n\right)^2\left[1-\rho\left(\alpha+\beta G_n\right)^2\right]^{-1}\end{aligned} \tag{F-3}$$

with $\tilde{k}_{1,\infty}^{\beta\beta}$ defined as in Eq. (C-7), the quantities $\tilde{k}_{j,n}^{\beta\beta}$'s as in (C-8) and the other results of App. C used. After some additional steps the cautionary component looks like

$$J_{C,\infty}=\delta_1u_0^2+\delta_2u_0+\delta_3 \tag{F-4}$$

with

$$\begin{aligned}\delta_1&=\left(1/2\right)\left[k_1^{xx}\sigma_{0|0}^{\beta\beta}+\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)\beta_{0|0}^2+2\tilde{k}_1^{\beta x}\phi\sigma_{0|0}^{\beta\beta}\beta_{0|0}\right]\\\delta_2&=\left[\tilde{k}_1^{\beta x}\phi\sigma_{0|0}^{\beta\beta}+\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)\beta_{0|0}\right]\alpha x_0\\\delta_3&=\left(1/2\right)\sum_{j=0}^{n-1}k_{j+1}^{xx}q+\left(1/2\right)\rho k_n^{xx}q\left(1-\rho\right)^{-1}+\left(1/2\right)\left(\tilde{\delta}_n+\tilde{\delta}_\infty\right)\left(\alpha x_0\right)^2\end{aligned} \tag{F-5}$$

where the first *n* quantities k^{xx}'s are as in Eq. (A-8), k_n^{xx} is the fixed point solution to Eq. (A-10) and $\tilde{k}_1^{\beta x}$ is defined as in (B-7).

Appendix G. The probing component in the presence of updated estimates of the time-varying parameter

In this context, when the desired paths for the state and control are zero and there is no intercept the BMW model, the general formula for the probing component of the approximate cost-to-go, see e.g. Kendrick (1981, 2002, Eqt. 10.51) or Tucci (2004, Eqt. 2.69), for $t=0$ and $T=\infty$ looks like

$$J_{P,\infty}=(1/2)\sum_{j=1}^{\infty}\left[u_{o,j}k_{j+1}^{xx}\beta_{j|0}+k_{j+1}^{\beta x}\beta_{j|0}\right]^2\left(\lambda_j+k_{j+1}^{xx}\beta_{j|0}^2\right)^{-1}\sigma_{j|j}^{\beta\beta} \qquad \text{(G-1)}$$

where the unknown parameter time-varying parameter β_j is replaced by its estimate at time 0, i.e. $\beta_{j|0}$. As noticed in DEPS 766, the *j*–th term multiplying the updated variance corresponds to the 'minus term'(C-5), say $k_{j,-}^{\beta\beta}$, in the formula for $k_j^{\beta\beta}$.[4] As shown in App. C it can be written as

$$k_{j,-}^{\beta\beta}=\left(\rho k_n^{xx}\right)^2 D^{-3}\beta^2\left(\lambda_n+\rho k_n^{xx}\beta^2\right)^{-1}G_{o,n}^2x_{o,1}^2+\sum_{i=j}^{n-1}\left(G_{o,i}k_{i+1}^{xx}+\tilde{k}_{i+1}^{\beta x}\right)^2\beta_{i|0}^2\left(\lambda_i+k_{i+1}^{xx}\beta_{i|0}^2\right)^{-1}x_{o,1}^2$$
$$=\tilde{k}_{j,\infty-}^{\beta\beta}x_{o,1}^2+\tilde{k}_{j,n-}^{\beta\beta}x_{o,1}^2=\tilde{k}_{j,-}^{\beta\beta}x_{o,1}^2$$

for $j<n$, with $G_{o,n}$, $G_{o,i}$ as defined there, $k_t^{\beta x}$ as in Eq. (B-8) and

$$\tilde{k}_{j,-}^{\beta\beta}=\left[\rho\left(\alpha+bG_n\right)^2\right]^{j-n}\tilde{k}_{n,-}^{\beta\beta}$$

for $j\geq n$, with $\tilde{k}_{n,-}^{\beta\beta}\equiv\tilde{k}_{1,\infty-}^{\beta\beta}$. Then the probing component can be rewritten as

$$J_{P,\infty}=(1/2)\tilde{k}_{1,-}^{\beta\beta}\sigma_{1|1}^{\beta\beta}x_{o,1}^2+(1/2)\left\{\sum_{j=2}^{n-1}\tilde{k}_{j,-}^{\beta\beta}\sigma_{j|j}^{\beta\beta}+\sum_{j=n}^{\infty}\left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n}\tilde{k}_{n,-}^{\beta\beta}\sigma_{j|j}^{\beta\beta}\right\}x_{o,1}^2 \qquad \text{(G-2)}$$

By replacing the updated variances with Eq. (D-4) in App. D, the infinite sum in (G-2) looks like

$$\tilde{k}_{n,-}^{\beta\beta}\sum_{j=n}^{\infty}\left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n}\left[\phi^{2(j-1)}\sigma_{1|1}^{\beta\beta}\bar{A}_j^{-1}+\sigma_\eta^2\left(\sum_{m=2}^{j}\phi^{2(j-m)}\bar{A}_m\right)\bar{A}_j^{-1}\right]x_{o,1}^2$$
$$=\tilde{k}_{n,-}^{\beta\beta}\sigma_\eta^2x_{o,1}^2\sum_{j=n}^{\infty}\left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n}\left(\sum_{m=2}^{j}\phi^{2(j-m)}\bar{A}_m\right)\bar{A}_j^{-1} \qquad \text{(G-3)}$$
$$=\tilde{k}_{n,-}^{\beta\beta}\sigma_\eta^2x_{o,1}^2\left[1-\rho\left(\alpha+\beta G_n\right)^2\right]^{-1}$$

with $\bar{A}_j$ as in (D-5). The first equality sign is due to the fact that the term $\phi^{2(j-1)}\sigma_{1|1}^{\beta\beta}\bar{A}_j^{-1}=0$ for $j\geq n$, given that $\phi^{2(j-1)}=0$ and $\sigma_{1|1}^{\beta\beta}$ and $\bar{A}_n^{-1}$ are finite quantities. The second one follows from the fact that

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[4] The 'minus term' was defined as $k_{j,2}^{\beta\beta}$ in Amman and Tucci (2017).

$$\tilde{k}_{n,-}^{\beta\beta}\sigma_\eta^2\sum_{j=n}^{\infty}\left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-n}\left(\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}$$
$$=\tilde{k}_{n,-}^{\beta\beta}\sigma_\eta^2\left\{\left(\sum_{m=2}^{n}\phi^{2(n-m)}\overline{A}_m\right)\overline{A}_n^{-1}+\left[\rho\left(\alpha+\beta G_n\right)^2\right]\left(\sum_{m=2}^{n+1}\phi^{2(n+1-m)}\overline{A}_m\right)\overline{A}_{n+1}^{-1}\right. \tag{G-4}$$
$$\left.+\left[\rho\left(\alpha+\beta G_n\right)^2\right]^2\left(\sum_{m=2}^{n+2}\phi^{2(n+2-m)}\overline{A}_m\right)\overline{A}_{n+2}^{-1}+\ldots\right\}$$

with $\lim_{j\to\infty}\left[\rho\left(\alpha+\beta G\right)^2\right]^{j-n}=0$ when the system is stabilizable, and

$$1<\lim_{j\to\infty}\left[\left(\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}\right]=\left(1-\phi^2\right)^{-1}<\infty\ , \tag{G-5}$$

because

$$\left(\sum_{m=2}^{\infty}\phi^{2(\infty-m)}\overline{A}_m\right)\overline{A}_\infty^{-1}=$$
$$\left[\phi^{2(\infty-2)}\overline{A}_2+\phi^{2(\infty-3)}\overline{A}_3+\ldots+\phi^{2\left[\infty-(\infty-n/2+1)\right]}\overline{A}_{\infty-n/2+1}+\ldots+\phi^2\overline{A}_{\infty-1}+\overline{A}_\infty\right]\overline{A}_\infty^{-1}$$
$$=\left[\phi^{2(n/2-1)}\overline{A}_{\infty-n/2+1}+\phi^{2(n/2-2)}\overline{A}_{\infty-n/2+2}+\ldots+\phi^2\overline{A}_{\infty-1}+\overline{A}_\infty\right]\overline{A}_\infty^{-1}$$

and from (D-5) follows that $\overline{A}_{\infty-n/2}=\overline{A}_{\infty-n/2+1}=\ldots=\overline{A}_\infty$. Then by using the limiting ratio approach, it can be shown that

$$\lim_{j\to\infty}\left|\frac{s_j}{s_{j-1}}\right|=\lim_{j\to\infty}\left|\frac{\left[\rho\left(\alpha+\beta G_n\right)^2\right]^j\left(\sum_{m=2}^{j-1}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}}{\left[\rho\left(\alpha+\beta G_n\right)^2\right]^{j-1}\left(\sum_{m=2}^{j-2}\phi^{2(j-1-m)}\overline{A}_m\right)\overline{A}_{j-1}^{-1}}\right|=\rho\left(\alpha+\beta G_n\right)^2. \tag{G-6}$$

Analogously, the finite sum in (G-2) can be rewritten as

$$\sigma_{1|1}^{\beta\beta}\sum_{j=2}^{n-1}\phi^{2(j-1)}\tilde{k}_{j,-}^{\beta\beta}\overline{A}_j^{-1}+\sigma_\eta^2\sum_{j=2}^{n-1}\tilde{k}_{j,-}^{\beta\beta}\left[\left(\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}\right]$$
$$=\sigma_{1|1}^{\beta\beta}\sum_{j=2}^{n/2}\phi^{2(j-1)}\tilde{k}_{j,-}^{\beta\beta}\overline{A}_j^{-1}+\sigma_\eta^2\sum_{j=2}^{n-1}\tilde{k}_{j,-}^{\beta\beta}\left[\left(\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}\right] \tag{G-7}$$

Is there a computationally fast way to approximate these two summations? For the first one, going from 2 to $n/2$, a possibility is to compute $\overline{A}_{n/4+1}^{-1}$ and $\tilde{k}_{n/4+1,-}^{\beta\beta}$. As far as the second one is concerned, notice that

$$\sigma_\eta^2\sum_{j=2}^{n-1}\tilde{k}_{j,-}^{\beta\beta}\left[\left(\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_m\right)\overline{A}_j^{-1}\right]\equiv\sigma_\eta^2\sum_{j=2}^{n-1}\tilde{k}_{j,-}^{\beta\beta}\left(\overline{A}_j^{*}\right)^{-1}$$
$$=\sigma_\eta^2\left[\tilde{k}_{2,-}^{\beta\beta}+\tilde{k}_{3,-}^{\beta\beta}\left(\phi^2\overline{A}_2+\overline{A}_3\right)\overline{A}_3^{-1}+\ldots+\tilde{k}_{n/2,-}^{\beta\beta}\left(\phi^{2(n/2-2)}\overline{A}_2+\phi^{2(n/2-3)}\overline{A}_3+\ldots+\overline{A}_{n/2}\right)\overline{A}_{n/2}^{-1}\right. \tag{G-8}$$
$$+\tilde{k}_{n/2+1,-}^{\beta\beta}\left(\phi^{2(n/2-1)}\overline{A}_2+\phi^{2(n/2-2)}\overline{A}_3+\ldots+\overline{A}_{n/2+1}\right)\overline{A}_{n/2+1}^{-1}+\ldots$$
$$\left.+\tilde{k}_{n-1,-}^{\beta\beta}\left(\phi^{2[n-1-n/2]}\overline{A}_{n-n/2}+\ldots+\phi^{2[n-1-(n-2)]}\overline{A}_{n-2}+\overline{A}_{n-1}\right)\overline{A}_{n-1}^{-1}\right]$$

Given this structure, a reasonable approximation of the mean can be obtained computing only three quantities $\left(\overline{A}^{*}_{n/4+1}\right)^{-1}$, $\left(\overline{A}^{*}_{3n/4+1}\right)^{-1}$ $\tilde{k}^{\beta\beta}_{3n/4,-}$, i.e.

$$\sigma_\eta^2 \sum_{j=2}^{n-1} \tilde{k}^{\beta\beta}_{j,-} \left(\overline{A}^{*}_{j}\right)^{-1} \simeq \sigma_\eta^2 \frac{n-2}{2} \left[\tilde{k}^{\beta\beta}_{n/4+1,-} \left(\overline{A}^{*}_{n/4+1}\right)^{-1} + \tilde{k}^{\beta\beta}_{3n/4+1,-} \left(\overline{A}^{*}_{3n/4+1}\right)^{-1} \right]. \quad \text{(G-9)}$$

By defining $\tilde{\overline{A}}_j x^2_{o,1} = \overline{A}_j - 1$ and using the relevant approximation $\overline{A}_j \simeq \tilde{\overline{A}}_j x^2_{o,1}$ in the finite summation and putting all pieces together it yields

$$J_{P,\infty} \simeq \frac{1}{2} \frac{g(u_0)}{h(u_0)} + \frac{1}{2} v + \frac{1}{2} f(u_0) \quad \text{(G-10)}$$

with

$$g(u_0) = \tilde{k}^{\beta\beta}_{1,-} x^2_{o,1}, \quad \text{(G-11a)}$$

$$h(u_0) = \left(\sigma^{\beta\beta}_{1|1}\right)^{-1}, \quad \text{(G-11b)}$$

$$v = \tilde{k}^{\beta\beta}_{n/4+1,-} \left[\sigma^{\beta\beta}_{1|1} \left(\tilde{\overline{A}}_{n/4+1}\right)^{-1} \right] \phi^2 \left(1-\phi^2\right)^{-1}, \quad \text{(G-11c)}$$

$$f(u_0) = \sigma_\eta^2 \left\{ \frac{n-2}{2} \left[\tilde{k}^{\beta\beta}_{n/4+1,-} \left(\tilde{\overline{A}}^{*}_{n/4+1}\right)^{-1} + \tilde{k}^{\beta\beta}_{3n/4+1,-} \left(\tilde{\overline{A}}^{*}_{3n/4+1}\right)^{-1} \right] + \left[1 - \rho\left(\alpha + \beta G_n\right)^2\right]^{-1} \tilde{k}^{\beta\beta}_{n,-} \right\} x^2_{o,1} \quad \text{(G-11d)}$$

Notice that $\sigma^{\beta\beta}_{1|1} \left(\tilde{\overline{A}}_{n/4}\right)^{-1} = \left[\tilde{\overline{A}}_{n/4} \left(\sigma^{\beta\beta}_{1|1}\right)^{-1}\right]^{-1}$, which appears in v, is largely independent of $\sigma^{\beta\beta}_{1|1}$ and consequently the ratios $\tilde{\overline{A}}_m \tilde{\overline{A}}_j^{-1}$ behind the terms $\left(\tilde{\overline{A}}^{*}_{j}\right)^{-1}$, which appear in $f(u_0)$, are largely independent of $\sigma^{\beta\beta}_{1|1}$.

Equation (G-10) is slightly different from the formulation of the probing component usually found in the literature, see e.g. Amman and Kendrick (1995), Tucci et al. (2010) and Amman and Tucci (2017). The familiar portion can be rewritten as usual, i.e.

$$\frac{g(u_0)}{h(u_0)} = \frac{\phi_1 \left(\phi_2 u_0 + \phi_3\right)^2}{\left[\phi^2 \sigma^{\beta\beta}_{0|0} q \left(u_0^2 \sigma^{\beta\beta}_{0|0} + q\right)^{-1} + \sigma_\eta^2\right]^{-1}}, \quad \text{(G-12)}$$

with

$$\phi_1 = \tilde{k}^{\beta\beta}_{1,-}$$
$$\phi_2 = \beta_{0|0} \quad \text{(G-13)}$$
$$\phi_3 = \alpha x_0$$

Then, two new terms v and $f(u_0)$ appear. The fomer is an approximation of the first summation in (G-7) largely independent of $\sigma^{\beta\beta}_{1|1}$ and $x^2_{o,1}$ as pointed out above. The latter, largely independent of $\sigma^{\beta\beta}_{1|1}$ as well, takes into account the penalty associated with the variance of the stochastic parameter σ_η^2. It is interesting to notice that the component (G-11d) can be rearranged as

$$f\left(u_0\right) = \phi_4 \left(\phi_2 u_0 + \phi_3\right)^2 \qquad \text{(G-14)}$$

with

$$\phi_4 = \sigma_\eta^2 \left\{ \frac{n-2}{2} \left[\tilde{k}_{n/4+1,-}^{\beta\beta} \left(\tilde{\tilde{A}}_{n/4+1}^{*} \right)^{-1} + \tilde{k}_{3n/4+1,-}^{\beta\beta} \left(\tilde{\tilde{A}}_{3n/4+1}^{*} \right)^{-1} \right] + \left[1 - \rho\left(\alpha + \beta G_n\right)^2 \right]^{-1} \tilde{k}_{n,-}^{\beta\beta} \right\} \qquad \text{(G-15)}$$

and ϕ_2, ϕ_3 as in (G-13).

Appendix H. The dual control in the infinite horizon model with an old estimate of the time-varying parameter

When the estimate of the time-varying parameter is based on old information, say the information available at time −1, the value of $x_{o,1}$ in Eq. (A-1) of Appendix A, say $x_{o,1|-1}$, is computed using $\beta_{0|-1}=\phi\left(\beta_{-1|-1}-\beta\right)+\beta$, i.e.

$$x_{o,1|-1}=\alpha x_0+\beta_{0|-1}u_0,$$

and the nominal value of u_1, say $u_{o,1|-1}$, is obtained by replacing G_1 with

$$G_{1|-1}=\left(-\frac{\alpha\beta_{1|-1}k_{2|-1}}{\lambda_1+\beta_{1|-1}^2k_{2|-1}}\right) \tag{H-1}$$

in (A-2). Analogously, Eqs. (A-3)-(A-4) should be rewritten with the G_i's and $\beta_{i|0}$'s substituted by $G_{i|-1}$ and $\beta_{i|-1}$, respectively. In this case the Riccati array is labeled $k_{j|-1}\equiv k_{j|-1}^{xx}$, $n-1$ denotes the first period in which $\beta_{j|-1}=\beta$, G_j converge to G, say $G_{n-1|-1}$, Eqs. (A-5) and (A-6) look like

$$\begin{aligned}u_{o,j|-1}&=G_{j|-1}\prod_{i=1}^{j-1}\left(\alpha+\beta_{i|-1}G_{i|-1}\right)x_{o,1|-1}=G_{n-1|-1}x_{o,j|-1}\\&=G_{n-1|-1}\left(\alpha+\beta G_{n-1|-1}\right)^{j-n+1}x_{o,n-1|-1}\end{aligned}\qquad\text{for } j\geq n-1 \tag{H-2}$$

and

$$x_{o,n-1|-1}=\prod_{i=1}^{n-2}\left(\alpha+\beta_{i|-1}G_{i|-1}\right)x_{o,1|-1}. \tag{H-3}$$

The fixed point solution to the usual Riccati recursions, say $k_{n-1|-1}^{xx}$, is obtained from (A-10) with w_n replaced by w_{n-1}.

In Appendix B, Eq. (B-1) should be computed using $\beta_{1|-1}$, $k_{2|-1}^{xx}$, $G_{1|-1}$ and $u_{o,1|-1}$ and the new Riccati array labeled as $k_{1|-1}^{\beta x}$. Then Eq. (B-2) is obtained using $\beta_{2|-1}$, $k_{3|-1}^{xx}$, $G_{2|-1}$ and $u_{o,2|-1}$ and Eq. (B-3) accordingly. In this context, Eq. (B-4) should be rewritten as

$$\begin{aligned}k_{1|-1}^{\beta x}&=\sum_{i=n-1}^{\infty}k_{n-1|-1}^{xx}\left[\rho\left(\alpha+\beta G_{n-1|-1}\right)\right]^{i-n+2}\left[\prod_{j=1}^{n-2}\left(\alpha+\beta_{j|-1}G_{j|-1}\right)\right]Gx_{o,i|-1}\\&+\sum_{i=1}^{n-2}k_{i+1|-1}^{xx}\left[\prod_{j=1}^{i}\left(\alpha+\beta_{j|-1}G_{j|-1}\right)\right]G_{i|-1}x_{o,i|-1}\end{aligned} \tag{H-4}$$

with $G_i x_{o,i|-1} = \left(\alpha + \beta G_{n-1|-1}\right)^{i-n+1} G_{n-1|-1} x_{o,n-1|-1} = \left(\alpha + \beta G_{n-1|-1}\right)^{i-n+1} G_{o,n-1|-1} x_{o,1|-1}$ for $i = n-1, n, \ldots$ and

$$G_{i|-1} x_{o,i|-1} = \left(-\frac{\alpha\beta_{i|-1} k^{xx}_{i+1|-1}}{\lambda_i + \beta^2_{i|-1} k^{xx}_{i+1|-1}}\right) \left[\prod_{j=1}^{i-1}\left(\alpha + \beta_{j|-1} G_{j|-1}\right)\right] x_{o,1|-1} = G_{o,i|-1} x_{o,1|-1} \quad \text{(H-5)}$$

for $i = 1, \ldots, n-2$. It follows that a generic $k^{\beta x}_{\ell}$ with $\ell \leq n-2$ is defined as

$$\begin{aligned} k^{\beta x}_{\ell|-1} &= D^{-1}_{|-1} \rho k^{xx}_{n-1|-1} \left(\alpha + \beta G_{n-1|-1}\right) \left[\prod_{j=\ell}^{n-2}\left(\alpha + \beta_{j|-1} G_{j|-1}\right)\right] G_{o,n-1|-1} x_{o,1|-1} \\ &+ \sum_{i=\ell}^{n-2} k^{xx}_{i+1|-1} \left(\alpha + \beta_{i|-1} G_{i|-1}\right) \left[\prod_{j=\ell}^{i-1}\left(\alpha + \beta_{j||-1} G_{j|-1}\right)\right] G_{o,i|-1} x_{o,1|-1} \\ &= \tilde{k}^{\beta x}_{\ell,\infty|-1} x_{o,1|-1} + \tilde{k}^{\beta x}_{\ell,n|-1} x_{o,1|-1} = \tilde{k}^{\beta x}_{\ell|-1} x_{o,1|-1} \end{aligned} \quad \text{(H-6)}$$

with $D_{|-1} = \left[1 - \rho\left(\alpha + \beta G_{n-1|-1}\right)^2\right]$ and

$$\begin{aligned} k^{\beta x}_{\ell|-1} &= D^{-1}_{|-1} \rho k^{xx}_{\ell|-1} \left(\alpha + \beta G_{n-1|-1}\right) G_{n-1|-1} x_{o,\ell} \\ &= \left[\rho\left(\alpha + \beta G_{n-1|-1}\right)\right]^{\ell-n+1} D^{-1}_{|-1} \rho k^{xx}_{n-1|-1} \left(\alpha + \beta G_{n-1|-1}\right) G_{n-1|-1} x_{o,n-1|-1} \end{aligned} \quad \text{(H-7)}$$

for $\ell \geq n-1$.

Following the same steps, the derivation of the Riccati array labeled $k^{\beta\beta}_{1|-1}$ can be carried out. Equation (C-1) on Appendix C should be computed using $\beta_{j|-1}$ as the estimate of β_j, $k^{xx}_{j+1|-1}$, $k^{\beta x}_{j+1|-1}$ and $u_{o,j|-1}$. Then Eq. (C-2) is rewritten accordingly. As noticed above, in this context $n-1$ denotes the first period in which $\beta_{j|-1} = \beta$, G_j converge to G. Then the infinite summations in Eq. (C-3), (C-4) and (C-5) should be written separating the first $n-2$ from the rest and the formulae for $u_{o,i|-1} k^{xx}_{i+1|-1}$, $k^{\beta x}_{i+1|-1}$ and $k^{\beta\beta}_{1,\infty|-1}$ valid for $i = n-1, n, \ldots$ are derived from (C-6) and (C-7), respectively, with $x_{o,n}$, ρk^{xx}_{n}, $G_{o,n}$ and $x_{o,1}$ replaced by $x_{o,n-1|-1}$, $\rho k^{xx}_{n-1|-1}$, $G_{o,n-1|-1}$ and $x_{o,1|-1}$, respectively. Then the infinite and finite summations look like

$$\begin{aligned} k^{\beta\beta}_{j,\infty|-1} &= D^{-2}_{|-1} \rho k^{xx}_{n-1|-1} G^2_{o,n-1|-1} x^2_{o,1|-1} \left\{1 + \rho\left(\alpha + \beta G_{n-1|-1}\right)^2 - \rho k^{xx}_{n-1|-1} D^{-1}_{|-1} \beta^2 \left(\lambda_{n-1} + \rho k^{xx}_{n-1|-1} \beta^2\right)^{-1}\right\} \\ &\equiv \tilde{k}^{\beta\beta}_{j,\infty|-1} x^2_{o,1|-1} \end{aligned} \quad \text{(H-8)}$$

and

$$k_{j,n-1|-1}^{\beta\beta}=\sum_{i=j}^{n-2}\left[k_{i+1|-1}^{xx}G_{o,i|-1}^{2}+2\tilde{k}_{i+1|-1}^{\beta x}G_{o,i|-1}-\left(k_{i+1|-1}^{xx}G_{o,i|-1}+\tilde{k}_{i+1|-1}^{\beta x}\right)^{2}\beta_{i|-1}^{2}\left(\lambda_{i}+k_{i+1|-1}^{xx}\beta_{i|-1}^{2}\right)^{-1}\right] \\ \times x_{o,1|-1}^{2}=\tilde{k}_{j,n-1|-1}^{\beta\beta}x_{o,1|-1}^{2}, \tag{H-9}$$

respectively, with $G_{o,i|-1}$ and $\tilde{k}_{i+1}^{\beta x}$ as above. Consequently, $k_{j|-1}^{\beta\beta}$ looks like

$$k_{j|-1}^{\beta\beta}=\left(\tilde{k}_{j,\infty|-1}^{\beta\beta}+\tilde{k}_{j,n-1|-1}^{\beta\beta}\right)x_{o,1|-1}^{2}\equiv\tilde{k}_{j|-1}^{\beta\beta}x_{o,1|-1}^{2} \tag{H-10}$$

for $j<n-1$. By repeating this procedure for the various j's it follows that for $j\geq n-1$

$$\begin{aligned}k_{j|-1}^{\beta\beta}&=\left[\rho\left(\alpha+\beta G_{n-1|-1}\right)^{2}\right]^{j-n+1}k_{n-1|-1}^{\beta\beta}\\&=\left[\rho\left(\alpha+\beta G_{n-1|-1}\right)^{2}\right]^{j-n+1}\tilde{k}_{1,\infty|-1}^{\beta\beta}x_{o,1|-1}^{2}\equiv\tilde{k}_{j|-1}^{\beta\beta}x_{o,1|-1}^{2}\end{aligned}. \tag{H-11}$$

The projected variances in the absence of observation '0' are as in the text Eqs. (8.3)-(8.4). It follows that the updated variance of the stochastic parameter for $j=1$ is

$$\sigma_{1|1}^{\beta\beta}\equiv\sigma_{1|1,-1}^{\beta\beta}=\sigma_{1|-1}^{\beta\beta}-\sigma_{1|-1}^{\beta x}\left(\sigma_{1|-1}^{xx}\right)^{-1}\sigma_{1|-1}^{\beta x}=\phi^{2}\sigma_{0|-1}^{\beta\beta}qA_{1|-1}^{-1}+\sigma_{\eta}^{2} \tag{H-12}$$

with $A_{1|-1}=u_{0}^{2}\sigma_{0|-1}^{\beta\beta}+q=q\left(u_{0}^{2}\sigma_{0|-1}^{\beta\beta}q^{-1}+1\right)$ which is identical to Eq. (D-1) in Appendix D with $\sigma_{0|0}^{\beta\beta}$ replaced by $\sigma_{0|-1}^{\beta\beta}$. The more complicated notation $\sigma_{1|1,-1}^{\beta\beta}$ is here preferred to stress the fact that this, and the following, updated variance(s) are obtained treating the observation at time '0' on the time-varying parameter as missing. After repeated substitutions, it yields

$$\sigma_{j|j,-1}^{\beta\beta}=\left\{\phi^{2(j-1)}\sigma_{1|1,-1}^{\beta\beta}+\sigma_{\eta}^{2}\sum_{m=2}^{j}\phi^{2(j-m)}\overline{A}_{m|-1}\right\}\overline{A}_{j|-1}^{-1} \tag{H-13}$$

for a generic term j with

$$\begin{aligned}\overline{A}_{j|-1}&=1+\sum_{i=1}^{j-1}\phi^{2(i-1)}S_{i|-1}+\sigma_{\eta}^{2}q^{-1}\sum_{m=2}^{j-1}\overline{A}_{m|-1}\left(\sum_{i=m}^{j-1}\phi^{2(i-m)}G_{o,i|-1}^{2}x_{o,1|-1}^{2}\right)\\&=\overline{A}_{j-1|-1}+\phi^{2(j-2)}S_{j-1|-1}+\sigma_{\eta}^{2}q^{-1}G_{o,j-1|-1}^{2}x_{o,1|-1}^{2}\sum_{m=2}^{j-1}\phi^{2(j-1-m)}\overline{A}_{m|-1}\end{aligned}\quad\text{for } j>1 \tag{H-14}$$

where $\overline{A}_{1|-1}=1$ and

$$S_{i|-1}=u_{o,i|-1}^{2}\sigma_{1|1,-1}^{\beta\beta}q^{-1}=G_{o,i|-1}^{2}x_{o,1|-1}^{2}\sigma_{1|1,-1}^{\beta\beta}q^{-1}, \tag{H-15a}$$

which simplifies to

$$S_{i|-1} = u_{o,i|-1}^2 \sigma_{1|1,-1}^{\beta\beta} q^{-1} = \left(\alpha + \beta G_{n-1|-1}\right)^{2(i-n+1)} G_{o,n-1|-1}^2 x_{o,1|-1}^2 \sigma_{1|1,-1}^{\beta\beta} q^{-1}, \qquad \text{(H-15b)}$$

with $u_{o,i|-1}$, $x_{o,n-1|-1}$ and $G_{o,i|-1}$ as above, when $j \geq n-1$. Again, when the upper limit of the summation in (A4.5a) is lower than the lower limit the corresponding term is zero and the term $\phi^{2(j-2)} S_{j-1|-1}$ vanishes for $i > (n/2) - 1$. Finally, notice that the term in braces multiplying the $\overline{A}_{m|-1}$'s looks like

$$x_{o,1|-1}^2 \sum_{i=m}^{j-1} \phi^{2(i-m)} G_{o,i|-1}^2 \qquad \text{for } m < j < n-1 \qquad \text{(H-16a)}$$

$$x_{o,1|-1}^2 \sum_{i=m}^{n-2} \phi^{2(i-m)} G_{o,i|-1}^2 + G_{o,n|-1}^2 x_{o,1|-1}^2 \sum_{i=n-1}^{j-1} \phi^{2(i-m)} \left(\alpha + \beta G_{n-1|-1}\right)^{2(i-n+1)} \quad \text{for } m < n-1 < j \qquad \text{(H-16b)}$$

$$G_{o,n-1|-1}^2 x_{o,1|-1}^2 \sum_{i=m}^{j-1} \phi^{2(i-m)} \left(\alpha + \beta G_{n-1|-1}\right)^{2(i-n+1)} \qquad \text{for } n-1 \leq m < j. \qquad \text{(H-16c)}$$

Appendix I. The deterministic, cautionary and probing components when using an old estimate of the time-varying parameter

In the infinite horizon model with an old estimate of the time-varying parameter, the deterministic component of the approximate cost-to-go looks like

$$J_{D,\infty} = \frac{1}{2}\lambda_0 u_0^2 + \frac{1}{2}\left(\alpha x_0 + \beta_{0|-1}u_0\right)^2 \tilde{\psi}_{n|-1} + \frac{1}{2}\left(\alpha x_0 + \beta_{0|-1}u_0\right)^2 \tilde{\psi}_{\infty|-1}, \qquad \text{(I-1)}$$

because $x_{o,1|-1} = \alpha x_0 + \beta_{0|-1}u_0$, with $\tilde{\psi}_{n|-1}$ the sum of a finite number of terms and $\tilde{\psi}_{\infty|-1}$ the sum of an infinite number of terms defined as

$$\begin{aligned}
\tilde{\psi}_{n|-1} &= \sum_{j=1}^{n-2}\left\{\left[\prod_{i=1}^{j-1}\left(\alpha + \beta_{i|-1}G_{i|-1}\right)\right]^2 \left(k_{j|-1}^{CE} + \lambda_j G_{j|-1}^2\right)\right\} \\
\tilde{\psi}_{\infty|-1} &= \left[\prod_{i=1}^{n-2}\left(\alpha + \beta_{i|-1}G_{i|-1}\right)\right]^2 \left[\left(k_{n-1}^{CE} + \lambda_{n-1}G^2\right)\right]\left[1-\rho\left(\alpha+\beta G\right)^2\right]^{-1}
\end{aligned} \qquad \text{(I-2)}$$

when the results and definitions of Appendix H are used and it is understood that the product term in square brackets is one when its lower limit is larger than its upper limit. It follows that Eq. (I-1) can be rearranged as

$$J_{D,\infty|-1} = \psi_{1|-1}u_0^2 + \psi_{2|-1}u_0 + \psi_{3|-1} \qquad \text{(I-3)}$$

where

$$\begin{aligned}
\psi_{1|-1} &= (1/2)\left[\lambda_0 + \beta_{0|-1}^2\left(\tilde{\psi}_{n|-1} + \tilde{\psi}_{\infty|-1}\right)\right] \\
\psi_{2|-1} &= \alpha\beta_{0|-1}x_0\left(\tilde{\psi}_{n|-1} + \tilde{\psi}_{\infty|-1}\right) \\
\psi_{3|-1} &= (1/2)\left(\alpha x_0\right)^2\left(\tilde{\psi}_{n|-1} + \tilde{\psi}_{\infty|-1}\right).
\end{aligned} \qquad \text{(I-4)}$$

The cautionary component takes the form

$$J_{C,\infty|-1} = (1/2)\left(k_{1|-1}^{xx}\sigma_{1|-1}^{xx} + k_{1|-1}^{\beta\beta}\sigma_{1|-1}^{\beta\beta}\right) + k_{1|-1}^{x\beta}\sigma_{1|-1}^{x\beta} + (1/2)\sum_{j=1}^{\infty}\left(k_{j+1|-1}^{xx}q + k_{j+1|-1}^{\beta\beta}\sigma_\eta^2\right) \qquad \text{(I-5)}$$

with the $k_{j|-1}^{xx}$'s, $\tilde{k}_{1|-1}^{\beta x}$ and the $\tilde{k}_{j|-1}^{\beta\beta}$'s defined as in App. H. Given that in this case the projected variances are defined as in (3.3')-(3.3"), i.e. $\sigma_{1|-1}^{xx} = u_0^2\left(\phi^2\sigma_{-1|-1}^{\beta\beta} + \sigma_\eta^2\right) + q$, $\sigma_{1|-1}^{\beta x} = \phi\left(\phi^2\sigma_{-1|-1}^{\beta\beta} + \sigma_\eta^2\right)u_0$ and $\sigma_{1|-1}^{\beta\beta} = \phi^2\left(\phi^2\sigma_{-1|-1}^{\beta\beta} + \sigma_\eta^2\right) + \sigma_\eta^2$, after some simple but tedious manipulations Eq. (I-5) can be rewritten as

$$\begin{aligned}
J_{C,\infty|-1} &= (1/2)k_{1|-1}^{xx}\left(\phi^2\sigma_{-1|-1}^{\beta\beta} + \sigma_\eta^2\right)u_0^2 + \tilde{k}_{1|-1}^{\beta x}\phi\left(\phi^2\sigma_{-1|-1}^{\beta\beta} + \sigma_\eta^2\right)u_0 x_{o,1|-1} \\
&+ (1/2)\left(\tilde{\delta}_{n|-1} + \tilde{\delta}_{\infty|-1}\right)x_{o,1|-1}^2 + (1/2)\sum_{j=0}^{\infty}k_{j+1|-1}^{xx}q
\end{aligned} \qquad \text{(I-6)}$$

where $\tilde{\delta}_{n|-1}$ is the sum of a finite number of terms and $\tilde{\delta}_{\infty|-1}$ the sum of an infinite number of terms defined as

$$\tilde{\delta}_{n|-1} = \tilde{k}^{\beta\beta}_{1|-1}\left[\phi^4\sigma^{\beta\beta}_{-1|-1} + \phi^2\sigma^2_\eta + \sigma^2_\eta\right] + \sum_{j=1}^{n-2}\tilde{k}^{\beta\beta}_{j+1|-1}\sigma^2_\eta$$

$$= \tilde{k}^{\beta\beta}_{1,\infty|-1}\left[\phi^4\sigma^{\beta\beta}_{-1|-1} + \phi^2\sigma^2_\eta + (n-1)\sigma^2_\eta\right] + \tilde{k}^{\beta\beta}_{1,n}\left[\phi^4\sigma^{\beta\beta}_{-1|-1} + \phi^2\sigma^2_\eta + \sigma^2_\eta\right] + \sigma^2_\eta\sum_{j=1}^{n-2}\tilde{k}^{\beta\beta}_{j+1,n|-1} \quad \text{(I-7)}$$

$$\tilde{\delta}_{\infty|-1} = \sum_{j=n-1}^{\infty}\tilde{k}^{\beta\beta}_{j+1|-1}\sigma^2_\eta = \sigma^2_\eta\tilde{k}^{\beta\beta}_{1,\infty|-1}\rho\left(\alpha+\beta G_{n+1|-1}\right)^2\left[1-\rho\left(\alpha+\beta G_{n+1|-1}\right)^2\right]^{-1}$$

with $\tilde{k}^{\beta\beta}_{1,\infty|-1}$ defined as in Eq. (H-8), the quantities $\tilde{k}^{\beta\beta}_{j,n-1|-1}$'s as in (H-9) and the other results of App. H are used. After some additional steps Eq. (I-6) can be rearranged as

$$J_{C,\infty|-1} = \delta_{1|-1}u_0^2 + \delta_{2|-1}u_0 + \delta_{3|-1} \quad \text{(I-8)}$$

with

$$\delta_{1|-1} = (1/2)\left[k^{xx}_{1|-1}\left(\phi^2\sigma^{\beta\beta}_{-1|-1} + \sigma^2_\eta\right) + \left(\tilde{\delta}_{n|-1} + \tilde{\delta}_{\infty|-1}\right)\beta^2_{0|-1} + 2\tilde{k}^{\beta x}_{1|-1}\phi\left(\phi^2\sigma^{\beta\beta}_{-1|-1} + \sigma^2_\eta\right)\beta_{0|-1}\right]$$

$$\delta_{2|-1} = \left[\tilde{k}^{\beta x}_{1|-1}\phi\left(\phi^2\sigma^{\beta\beta}_{-1|-1} + \sigma^2_\eta\right) + \left(\tilde{\delta}_{n|-1} + \tilde{\delta}_{\infty|-1}\right)\beta_{0|-1}\right]\alpha x_0 \quad \text{(I-9)}$$

$$\delta_{3|-1} = (1/2)\sum_{j=0}^{n-2}k^{xx}_{j+1|-1}q + (1/2)\rho k^{xx}_{n-1|-1}q(1-\rho)^{-1} + (1/2)\left(\tilde{\delta}_{n|-1} + \tilde{\delta}_{\infty|-1}\right)(\alpha x_0)^2$$

where the first $n-1$ quantities $k^{xx}_{j|-1}$'s are as in Appendix H, k^{xx}_{n-1} is the fixed point solution to Eq. (A-10) when an old estimate of the time-varying parameter is available and $\tilde{k}^{\beta x}_{1|-1}$ is defined as in (H-4).

In this context, the probing component is written as

$$J_{P,\infty|-1} = (1/2)\sum_{j=1}^{\infty}\left[u_{o,j|-1}k^{xx}_{j+1|-1}\beta_{j|-1} + k^{\beta x}_{j+1|-1}\beta_{j|-1}\right]^2\left(\lambda_j + k^{xx}_{j+1|-1}\beta^2_{j|-1}\right)^{-1}\sigma^{\beta\beta}_{j|j,-1} \quad \text{(I-10)}$$

where the unknown parameter time-varying parameter β_j is replaced by its estimate $\beta_{j|-1}$. Using the results in Appendix H the 'minus term' in (H-8)-(H-9), say $k^{\beta\beta}_{j,-|-1}$, can be written as

$$k^{\beta\beta}_{j,-|-1} = \left(\rho k^{xx}_{n-1|-1}\right)^2 D^{-3}\beta^2\left(\lambda_{n-1} + \rho k^{xx}_{n-1|-1}\beta^2\right)^{-1}G^2_{o,n-1|-1}x^2_{o,1|-1}$$

$$+\sum_{i=j}^{n-2}\left(G_{o,i|-1}k^{xx}_{i+1|-1} + \tilde{k}^{\beta x}_{i+1|-1}\right)^2\beta^2_{i|-1}\left(\lambda_i + k^{xx}_{i+1}\beta^2_{i|-1}\right)^{-1}x^2_{o,1|-1}$$

$$= \tilde{k}^{\beta\beta}_{j,\infty-|-1}x^2_{o,1|-1} + \tilde{k}^{\beta\beta}_{j,n-|-1}x^2_{o,1|-1} = \tilde{k}^{\beta\beta}_{j,-|-1}x^2_{o,1|-1}$$

for $j < n-1$, with $G_{o,n-1|-1}$, $G_{o,i|-1}$, $k^{xx}_{i+1|-1}$ and $\tilde{k}^{\beta x}_{i+1|-1}$, as defined in there and

$$\tilde{k}^{\beta\beta}_{j,-|-1} = \left[\rho\left(\alpha + bG_{n+1|-1}\right)^2\right]^{j-n+1}\tilde{k}^{\beta\beta}_{n,-|-1}$$

for $j \geq n-1$, with $\tilde{k}^{\beta\beta}_{n,-|-1} \equiv \tilde{k}^{\beta\beta}_{1,\infty-|-1}$. Then Eq. (I-10) can be rewritten as

$$J_{P,\infty|-1} = (1/2)\,\tilde{k}^{\beta\beta}_{1,-|-1}\sigma^{\beta\beta}_{1|1,-1}x^2_{o,1|-1}$$
$$+(1/2)\left\{\sum_{j=2}^{n-2}\tilde{k}^{\beta\beta}_{j,-|-1}\sigma^{\beta\beta}_{j|j,-1}+\sum_{j=n-1}^{\infty}\left[\rho\left(\alpha+\beta G_{n+1|-1}\right)^2\right]^{j-n+1}\tilde{k}^{\beta\beta}_{n-1,-|-1}\sigma^{\beta\beta}_{j|j,-1}\right\}x^2_{o,1|-1} \quad \text{(I-11)}$$

Proceding as in Appendix G, defining $\tilde{\bar{A}}_{j|-1}x^2_{o,1|-1}=\bar{A}_{j|-1}-1$ and using the relevant approximation $\bar{A}_{j|-1}\simeq\tilde{\bar{A}}_{j|-1}x^2_{o,1|-1}$ in the finite summation and putting all pieces together it yields

$$J_{P,\infty|-1}\simeq\frac{1}{2}\frac{g_{|-1}(u_0)}{h_{|-1}(u_0)}+\frac{1}{2}v_{|-1}+\frac{1}{2}f_{|-1}(u_0) \quad \text{(I-12)}$$

with

$$g_{|-1}(u_0)=\tilde{k}^{\beta\beta}_{1,-|-1}x^2_{o,1|-1}, \quad \text{(I-13a)}$$

$$h_{|-1}(u_0)=\left(\sigma^{\beta\beta}_{1|1,-1}\right)^{-1}, \quad \text{(I-13b)}$$

$$v_{|-1}=\tilde{k}^{\beta\beta}_{n/4+1,-|-1}\left[\sigma^{\beta\beta}_{1|1,-1}\left(\tilde{\bar{A}}_{n/4+1|-1}\right)^{-1}\right]\phi^2\left(1-\phi^2\right)^{-1}, \quad \text{(I-13c)}$$

$$f_{|-1}(u_0)=\sigma^2_\eta\left\{\frac{n-2}{2}\left[\tilde{k}^{\beta\beta}_{n/4+1,-|-1}\left(\tilde{\bar{A}}^*_{n/4+1|-1}\right)^{-1}+\tilde{k}^{\beta\beta}_{3n/4+1,-|-1}\left(\tilde{\bar{A}}^*_{3n/4+1|-1}\right)^{-1}\right]\right.$$
$$\left.+\left[1-\rho\left(\alpha+\beta G_{n+1|-1}\right)^2\right]^{-1}\tilde{k}^{\beta\beta}_{n,-|-1}\right\}x^2_{o,1}. \quad \text{(I-13d)}$$

Again $\sigma^{\beta\beta}_{1|1,-1}\left(\tilde{\bar{A}}_{n/4|-1}\right)^{-1}=\left[\tilde{\bar{A}}_{n/4|-1}\left(\sigma^{\beta\beta}_{1|1,-1}\right)^{-1}\right]^{-1}$, in $v_{|-1}$, is largely independent of $\sigma^{\beta\beta}_{1|1,-1}$ and consequently the ratios $\tilde{\bar{A}}_{m|-1}\tilde{\bar{A}}^{-1}_{j|-1}$ behind the terms $\left(\tilde{\bar{A}}^*_{j|-1}\right)^{-1}$, approximations to $\left(\bar{A}^*_{j|-1}\right)^{-1}$ defined as in (G-8) and included in $f_{|-1}(u_0)$, are largely independent of $\sigma^{\beta\beta}_{1|1,-1}$. The familiar portion can be rewritten as

$$\frac{g_{|-1}(u_0)}{h_{|-1}(u_0)}=\frac{\phi_{1|-1}\left(\phi_{2|-1}u_0+\phi_{3|-1}\right)^2}{\left[\phi^2\sigma^{\beta\beta}_{0|-1}q\left(u_0^2\sigma^{\beta\beta}_{0|-1}+q\right)^{-1}+\sigma^2_\eta\right]^{-1}}, \quad \text{(I-14)}$$

with

$$\phi_{1|-1}=\tilde{k}^{\beta\beta}_{1,-|-1}$$
$$\phi_{2|-1}=\beta_{0\|-1} \quad \text{(I-15)}$$
$$\phi_{3|-1}\equiv\phi_3=\alpha x_0.$$

Similarly to Appendix G, $f_{|-1}(u_0)$ can be rearranged as

$$f_{|-1}(u_0)=\phi_{4|-1}\left(\phi_{2|-1}u_0+\phi_{3|-1}\right)^2 \quad \text{(I-16)}$$

with

$$\phi_{4|-1}=$$
$$\sigma^2_\eta\left\{\frac{n-2}{2}\left[\tilde{k}^{\beta\beta}_{n/4+1,-|-1}\left(\tilde{\bar{A}}^*_{n/4+1|-1}\right)^{-1}+\tilde{k}^{\beta\beta}_{3n/4+1,-|-1}\left(\tilde{\bar{A}}^*_{3n/4+1|-1}\right)^{-1}\right]+\left[1-\rho\left(\alpha+\beta G_{n+1|-1}\right)^2\right]^{-1}\tilde{k}^{\beta\beta}_{n,-|-1}\right\} \quad \text{(I-17)}$$

and $\phi_{2|-1}$, $\phi_{3|-1}$ as in (I-15).

At this point by adding the three components of the approximate cost-to-go, i.e. Eqs. (I-3), (I-8) and (I-12), it yields

$$J_{\infty} = \left(\psi_{1|-1} + \delta_{1|-1}\right)u_0^2 + \left(\psi_{2|-1} + \delta_{2|-1}\right)u_0 + \left(\psi_{3|-1} + \delta_{3|-1}\right)$$
$$+\left(\frac{1}{2}\right)\frac{\phi_{1|-1}\left(\phi_{2|-1}u_0 + \phi_{3|-1}\right)^2}{\left[\phi^2\sigma_{0|-1}^{\beta\beta}q\left(u_0^2\sigma_{0|-1}^{\beta\beta} + q\right)^{-1} + \sigma_\eta^2\right]^{-1}} + \frac{1}{2}v_{|-1} + \frac{1}{2}\phi_{4|-1}\left(\phi_{2|-1}u_0 + \phi_{3|-1}\right)^2 \qquad \text{(I-18)}$$

with the parameters defined as in (I-4), (I-9), (I-14) , (I-15) and (I-16).