Simultaneous actuator and sensor fault reconstruction of singular delayed linear parameter varying systems in the presence of unknown time varying delays and inexact parameters

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Summary

In this article, robust fault diagnosis of a class of singular delayed linear parameter varying systems is considered. The considered system has delayed dynamics with unknown time varying delays and also it is affected by noise, disturbance and faults in both actuators and sensors. Moreover, *in addition to the aforementioned* unknown inputs and uncertainty, another source of uncertainty related to inexact measures of the scheduling parameters is present in the system. Making use of the descriptor system approach, sensor faults in the system are added as additional states into the original state vector to obtain an augmented system. Then, by designing a suitable proportional double integral unknown input observer (PDIUIO), the states, actuator, and sensor faults are estimated. The uncertainty due to the mismatch between the inexact parameters that schedule the observer and the real parameters that schedule the original system is formulated with an uncertain system approach. In the PDIUIO, the uncertainty induced by unknown inputs (disturbance, noise and actuator, and sensor faults), unknown delays, and inexact parameter measures are attenuated in H_∞ sense with different weights. The constraints regarding the existence and the robust stability of the designed PDIUIO are formulated using linear matrix inequalities. The efficiency of the proposed method is verified using an application example based on an electrical circuit.

KEYWORDS

actuator and sensor fault reconstruction, inexact parameters, multiple unknown time varying delays, proportional double integral unknown input observer, singular delayed linear parameter varying systems

1 | INTRODUCTION

Fault diagnosis has become an active area of research during the past decades. Although this branch of control system theory was born in the realm of safety-critical systems such as airplanes and nuclear reactors; nowadays it has become an essential part of many industrial systems due to the need for high efficiency in addition to safety and reliability. Although faults can happen in different places in a control loop, sensor and actuator faults are more common and the diagnosis of sensor and actuator faults has been the subject of many researches. Many methods have been proposed for fault diagnosis. However, in the control systems community, the model-based fault diagnosis methods have been attracted much attention due to their formulation similarity with the kind of mathematics which is involved in classical control systems designs. Observer-based methods are the most popular methods in the model-based fault diagnosis strategies. Unknown input observers (UIOs)[1,2] have the ability to decouple the effect of unknown inputs in the estimation error of the observer. On the other hand, proportional-integral (PI) observers[3–5] can reconstruct the actuator faults in addition to decouple the other unknown input signals such as disturbance and noise. Also, some other types of observers such as proportional-integral-derivative (PID) observer, adaptive observer, learning observer and various types of sliding mode observers (SMOs) have been designed and utilized in the literature for fault detection, isolation and reconstruction tasks. The interested reader is referred to the recent survey papers in the topic.[6,7] Some methods perform fault detection, fault isolation and fault reconstruction in different stages. But recently, a new strategy has become prevalent which performs the three stages in a direct fault diagnosis manner in which the faults are detected and isolated based on their estimated values.[8,9] On the other hand, in the last years, the descriptor approach has been used to estimate some variables in addition to the system states by augmenting the original system states with the additional states and then by designing a suitable observer.[8,10,11] This method can be applied for fault reconstruction by considering the faults as additional states.

The systems under consideration in the current study are singular delayed linear parameter varying (SDLPV) systems.[12] These systems have a great potential to model a broad class of real applications.[13,14] It is known that most real systems, either natural or man-made, are described by means of nonlinear models. During the last decades, these nonlinear models have been widely approximated by linear ones with the assumption that the system works near an operating point. Based on this linearization method, various methods have been applied for analysis and control of these systems. However, the assumption of small operating range is violated in some applications where the operation of the system is not confined in a single operating point. In these applications, the linear modeling strategy does not lead to satisfactory results. Linear parameter varying (LPV) systems which have been proposed by Shamma[15] as a generalization of gain-scheduling systems are a remedy for this challenge. The LPV approach transforms the nonlinear system to a model with linear structure by embedding the nonlinearities in some varying parameters; thus, the obtained model is valid in a wide area of operating conditions. Thus, LPV systems have a linear structure but the matrices of the model are varying according to the parameters changes. In recent years, many researchers have focused to solve various open problems for this type of systems. The interested reader is refereed to review papers[16,17] and references therein. Singular LPV systems are a class of LPV systems which can model the algebraic constraints between the states in addition to their dynamic relations. These systems can model the behavior of many applications, such as distillation columns and bioreactors.[11,18] The fault diagnosis of singular LPV systems has just been recently studied. Constant actuator fault detection and isolation via proportional integral observer has been addressed.[5] These results have been extended to fault estimation of time-varying actuator faults by a suitable adaptive observer.[19] The same authors have designed SMO for fault diagnosis in these systems.[20] A robust observer for sensor fault diagnosis in the presence of unmeasurable gain scheduling functions has been designed.[11] Various problems in the domain of fault diagnosis and fault tolerant control of singular LPV systems are still unsolved and need to be addressed by researchers.

Time delay which occurs in dynamics of many systems is a source of instability and poor performance. The analysis of time-delay systems is different in comparison to the systems without delay.[21–23] These systems do not have a characteristic equation in a polynomial form. Their characteristic equation is transcendental so there are no finite number of poles in these systems. Thus, some approaches like those which are based on classical pole placement cannot be applied. Recently, singular LPV systems with time delay have attracted the attention of researchers[12] because of the powerful modeling in their structure. Some results have been published recently related to fault diagnosis for this type of systems. Sensor fault diagnosis is addressed using a UIO[8] and actuator fault diagnosis is considered using proportional integral observer[9] and using an adaptive observer.[24] However, there are still some problems unsolved. One of the problems which has not been addressed yet is the simultaneous actuator and sensor fault diagnosis in these systems. Because of the importance of this problem, it is one of the hot topics in the fault diagnosis community and to the best of authors' knowledge, it is not addressed for SDLPV systems and currently is an open issue. Also, the existence of various unknown inputs such as disturbance and noise and uncertainties like inexactness of measured scheduling parameters and unknown delays that may exist in these especial type of systems increases the complexity of this problem.

The problem which is considered in this article is the simultaneous actuator and sensor fault diagnosis in SDLPV systems. The problem of simultaneous actuator and sensor fault diagnosis has been tackled with defining a filtered version of the system measurements and by designing a suitable robust adaptive observer in LPV systems[25] and by designing a suitable proportional integral observer for quasi-LPV (qLPV) systems.[26] This problem has been addressed by constructing an augmented state variable and designing an appropriate observer for descriptor systems[27] and by designing a descriptor reduced order SMO in time-delayed systems.[28] In our study, sensor faults are considered as additional states and by using descriptor approach, the original system plus the static relations is converted to a higher order SDLPV system. Then, by designing a suitable proportional double integral unknown input observer (PDIUIO), the actuator fault vector and the augmented state vector which includes the sensor fault vector are robustly estimated in the presence of unknown inputs and uncertainties. Adding more integral terms to design proportional multiple integral (PMI) observers can help to estimate unknown input signals with the polynomial form of any degree[4] but at the price of increasing design complexity. Recently, this kind of observers has been applied for unknown input estimation in Takagi–Sugeno fuzzy systems.[29] The idea of adding multiple integrals is also applied in SMO design for LPV systems.[30] The designed PDIUIO in this article has the advantage of not needing to assume that the actuator faults to be piecewise constant which is normally needed in the design of traditional proportional integral unknown input observers (PIUIOs); thus, it can be applied for diagnosis of a broader class of faults. The assumption which is needed for designing the PDIUIO is that the second derivative of actuator faults should be almost zero that is satisfied for piecewise constant faults (including abrupt faults), incipient faults and slow time-varying faults. The presented method is also easily extendable to design proportional multiple integral unknown input observer (PMIUIO) for SDLPV systems. The selection of the double integral case in the current study provides a suitable tradeoff between the class of actuator faults that can be reconstructed and the complexity of the observer. Another advantage of the presented method is that by making use of both descriptor approach and integral terms, direct diagnosis of simultaneously actuator and sensor faults has been made possible.

In our study, it is assumed that the system has unknown time-varying state delays. Designing observers for systems with time-varying delays has attracted the attention of researchers because of its application in systems such as communication systems and network control systems. Functional interval observer has been designed for fractional-order systems with time-varying delays[31] and distributed observers in the presence of time-varying delays are designed for large-scale LTI systems.[32] Designing PDIUIO in the case considered in this article presents additional challenges because the knowledge about the delay values is needed in formulating the delayed terms in the observer structure and also it is needed in constructing the related Lyapunov–Krasovskii functional for proving the stability of observer error dynamics. State estimation in the presence of unknown delays has been achieved by a functional SMO in linear systems[23] and by a nonlinear delayed observer in nonlinear systems.[22] Different with these approaches; in this article, the uncertainty induced by unknown delays is modeled as a disturbance signal and it is attenuated in H_∞ sense. The problem of simultaneous actuator and sensor fault diagnosis in SDLPV systems is addressed with the assumption that the available information on the scheduling parameter signals is inexact. This additional challenge is overcome by converting the original system into an uncertain system. The induced uncertainty due to inexact parameters is robustly attenuated. The design procedure of PDIUIO for SDLPV systems with the mentioned assumptions is provided as a solution to a convex optimization problem with a set of linear matrix inequalities (LMIs) constraints.

The main contributions of this work are summarized as follows:

- Designing PDIUIO for SDLPV systems having multiple unknown time varying delays in the presence of disturbance, noise and inexact scheduling parameters.
- Simultaneous reconstruction of actuator and sensor faults with the designed PDIUIO.
- Verifying the obtained results in simultaneous reconstruction of actuator and sensor faults in an electrical circuit example modeled as an SDLPV system.

The rest of this article is organized as follows. The problem formulation is presented in Section 2. A suitable PDIUIO is proposed in Section 3. Robust exponential stability of SDLPV systems is presented in Section 4. The design of the PDIUIO for SDLPV systems and also the state and fault estimation based on this observer is presented in Section 5. An electrical circuit example is used to illustrate the efficiency of the proposed method in Section 6. Section 7 draws the main conclusions and presents future research paths.

Notations: A standard notation is used in this article. $\mathbb{R}$ is the set of real numbers. $\mathbb{C}$ is the set of complex numbers. For a matrix X, X^T indicates its transpose. X^{-1} is the inverse and X^+ is the pseudo inverse of X. I_n is the n-dimensional identity matrix. 0 as a sub-block of matrices means a zero block with appropriate dimension. * is used to show the elements induced by symmetry in a symmetric matrix. $sym\{X\}$ is a short notation for $X + X^T$. For a symmetric matrix X, $X > 0$

$(X < 0)$ shows that it is positive (negative) definite. For a square integrable function $x(t)$, its L_2-norm is defined as $\|x(t)\|_2 = \sqrt{\int_0^\infty x(t)^T x(t) dt}$.

2 | PROBLEM FORMULATION

In this article, a class of SDLPV systems is considered as follows:

$$\begin{cases} E\dot{x}(t) = \left(\sum_{k=0}^{s} A_k(\theta(t))x(t-\tau_k(t))\right) + B(\theta(t))u(t) + R(\theta(t))d(t) + F(\theta(t))f_a(t), \\ y(t) = Cx(t) + D_f f_s(t) + D_n n(t), \\ \tau_0(t) = 0, \\ 0 \leq \tau_k(t) \leq \tau_{km}, \quad k = 1, \ldots, s, \\ \dot{\tau}_k(t) \leq \mu_k < 1, \quad k = 1, \ldots, s, \\ x(t) = \phi(t), \qquad -\tau_m < t < 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_0}$, $u(t) \in \mathbb{R}^{k_u}$, $y(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^{k_d}$, $n(t) \in \mathbb{R}^{k_n}$, $f_a(t) \in \mathbb{R}^{k_{f_a}}$, and $f_s(t) \in \mathbb{R}^{k_{f_s}}$ are the vectors of state variables, input signals, output signals, exogenous disturbances, output noises, actuator faults and sensor faults, respectively. In (1), $E \in \mathbb{R}^{n_0 \times n_0}$ is a constant square matrix that may be rank deficient ($rank(E) = r \leq n_0$). $A_k(\theta(t))$ for $k = 0, \ldots, s$, $B(\theta(t))$, $R(\theta(t))$, and $F(\theta(t))$ are matrices with appropriate dimensions which depend affinely on the time varying parameter $\theta(t) \in \mathbb{R}^l$ that is assumed to be real time measurable. C, D_f, and D_n are constant matrices with appropriate dimensions. $\tau_k(t)$ for $k = 1, \ldots, s$ are unknown time varying delays and $\tau_0(t) = 0$ is related to the undelayed part of dynamics in (1). τ_{km} and μ_k for $k = 1, \ldots, s$ are the upper bounds on delay and delay derivative values, respectively. $\tau_{0m} = 0$ is considered in this article for having simple notation. $\tau_m = \max_k \tau_{km}$ is the maximum of all delay upper bounds. $\phi(t)$ is a continuous vector-valued initial function.

Assumption 1. The time varying parameter vector belongs to the following hyperbox:

$$\Omega = \left\{ \theta(t) | \theta_{k_\theta}^m \leq \theta_{k_\theta}(t) \leq \theta_{k_\theta}^M \text{ for } k_\theta = 1, \ldots, l \right\}, \tag{2}$$

in which $\theta_{k_\theta}^m$ and $\theta_{k_\theta}^M$ define the minimum and maximum bounds of the parameter $\theta_{k_\theta}(t)$.

Definition 1 (33). The matrix pencil (E, A) is regular if $\det(sE - A)$ is not identically zero.

Definition 2 (33). The matrix pencil (E, A) is impulse-free if $\deg(\det(sE - A)) = \text{rank}(E)$.

Definition 3 (12). System (1) is regular and impulse-free if all the 2^s matrix pencils generated from $\left(E, A_0(\theta(t)) + \sum_{k=1}^{s} \alpha_k A_k(\theta(t))\right)$ in which each α_k can be 0 or 1 are regular and impulse-free for the all domains of $\theta(t)$ defined in (2).

Definition 4 (12). System (1) is admissible if it is regular, impulse free and stable.

Assumption 2 System (1) is assumed to be admissible.

The matrices of SDLPV system (1) depend on the time varying parameter vector $\theta(t)$. In this article, this system is transformed to a polytopic representation as follows:

$$\begin{cases} E\dot{x}(t) = \sum_{i=1}^{h} \rho_i(\theta(t)) \left[\left(\sum_{k=0}^{s} A_{ki} x(t - \tau_k(t)) \right) + B_i u(t) + R_i d(t) + F_i f_a(t) \right], \\ y(t) = Cx(t) + D_f f_s(t) + D_n n(t). \end{cases} \tag{3}$$

In (3), the system (1) is represented as a weighted summation of $h = 2^l$ singular delayed LTI subsystems which are defined in the vertices of the hyperbox (2).

A_{ki}, B_i, R_i, and F_i are matrices related to the ith subsystem ($i = 1, \dots, h$) located in the ith vertex of the hyperbox and the corresponding weights of the subsystems, $\rho_i(\theta(t))$ satisfy the following constraints:

$$0 \le \rho_i(\theta(t)) \le 1, \quad i = 1, \dots, h, \tag{4}$$

$$\sum_{i=1}^{h} \rho_i(\theta(t)) = 1. \tag{5}$$

In order to reconstruct the sensor faults in (3), the state vector is augmented with the sensor fault vector to constitute a new state vector:

$$\widetilde{x}(t) = \begin{bmatrix} x(t) \\ f_s(t) \end{bmatrix}. \tag{6}$$

By using the new state vector (6), system (3) can be transformed to:

$$\begin{cases} \widetilde{E}\dot{\widetilde{x}}(t) = \sum_{i=1}^{h} \rho_i(\theta(t)) \left[\left(\sum_{k=0}^{s} \widetilde{A}_{ki} \widetilde{x}(t - \tau_k(t)) \right) + \widetilde{B}_i u(t) + \widetilde{R}_i d(t) + \widetilde{F}_{1i} f_a(t) + \widetilde{F}_{2i} f_s(t) \right], \\ y(t) = \widetilde{C}\widetilde{x}(t) + D_n n(t), \end{cases} \tag{7}$$

where

$$\widetilde{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{A}_{0i} = \begin{bmatrix} A_{0i} & 0 \\ 0 & -I_{k_{f_s}} \end{bmatrix}, \quad \widetilde{A}_{ki} = \begin{bmatrix} A_{ki} & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{for } k = 1, \dots, s),$$
$$\widetilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \widetilde{R}_i = \begin{bmatrix} R_i \\ 0 \end{bmatrix}, \quad \widetilde{F}_{1i} = \begin{bmatrix} F_i \\ 0 \end{bmatrix}, \quad \widetilde{F}_{2i} = \begin{bmatrix} 0 \\ I_{k_{f_s}} \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & D_f \end{bmatrix}.$$

Remark 1. The order of augmented system (7) is $n = n_0 + k_{f_s}$ and all the matrices are with appropriate dimensions.

The following conditions are assumed for the augmented system (7):

Assumption 3. The triple matrix $\left(\widetilde{E}, \widetilde{A}_{0i}, \widetilde{C}\right)$ is R-observable for $i = 1, \dots, h$[24]:

$$Rank \begin{bmatrix} s\widetilde{E} - \widetilde{A}_{0i} \\ \widetilde{C} \end{bmatrix} = n, \quad \forall s \in \mathbb{C}. \tag{8}$$

Assumption 4. The triple matrix $\left(\widetilde{E}, \widetilde{A}_{0i}, \widetilde{C}\right)$ is impulse-observable for $i = 1, \dots, h$[24]:

$$Rank \begin{bmatrix} \widetilde{E} & \widetilde{A}_{0i} \\ 0 & \widetilde{E} \\ 0 & \widetilde{C} \end{bmatrix} = n + Rank(\widetilde{E}). \tag{9}$$

3 | PDIUIO FORMULATION AND DESIGN

In this section, for the augmented system (7), a suitable PDIUIO is proposed:

$$\begin{cases} \dot{z}(t) = \sum_{j=1}^{h} \rho_j(\widehat{\theta}(t)) \left[\left(\sum_{k=0}^{s} N_{kj} z(t - \tau_{km}) + L_{kj} y(t - \tau_{km}) \right) + G_j u(t) + W_j \widehat{f}_{a0}(t) \right], \\ \hat{\widetilde{x}}(t) = z(t) + H_2 y(t), \\ \widehat{y}(t) = \widetilde{C}\hat{\widetilde{x}}(t), \\ \dot{\widehat{f}}_{a0}(t) = \sum_{j=1}^{h} \rho_j(\widehat{\theta}(t)) \Lambda_{0j}(y(t) - \widehat{y}(t)) + \widehat{f}_{a1}(t), \\ \dot{\widehat{f}}_{a1}(t) = \sum_{j=1}^{h} \rho_j(\widehat{\theta}(t)) \Lambda_{1j}(y(t) - \widehat{y}(t)), \\ z(t) = 0, \quad -\tau_m < t < 0, \end{cases} \tag{10}$$

where $\hat{\tilde{x}}(t) \in \mathbb{R}^n$, $\hat{y}(t) \in \mathbb{R}^m$, and $z(t) \in \mathbb{R}^n$ are the vectors of augmented state (including the original states and the sensor faults) estimate, output estimate and observer state, respectively. $\hat{f}_{a0}(t)$ and $\hat{f}_{a1}(t)$ are the estimates of the actuator faults and the estimates of their derivatives, respectively. Since the exact knowledge of the scheduling parameters may not be available in real applications, the proposed PDIUIO is scheduled according to the weights calculated based on $\hat{\theta}(t)$ (the inexact measure of the parameter vector). N_{ki}, L_{ki}, G_i, W_i, Λ_{0i}, Λ_{1i}, and H_2 are PDIUIO matrices with appropriate dimensions and obtained using the design procedure presented in the following.

Remark 2. In the proposed observer the maximum bounds of delays are utilized for delayed terms since their real values are not known.

Remark 3. The strategy presented in this article can be extended for the design of PMIUIO. However, for the simplification of presentation, the case of double integral is presented here.

The uncertainty caused by the mismatch between the inexact measured parameters and the real parameters is taken into account by extending the method proposed by Ichalal et al.[34] and converting system (7) to the following uncertain system:

$$\begin{cases} \tilde{E}\dot{\tilde{x}}(t) = \sum_{i=1}^{h}\sum_{j=1}^{h} \rho_i(\theta(t))\rho_j(\hat{\theta}(t)) \left[\left(\sum_{k=0}^{s} \breve{A}_{kij}\tilde{x}(t-\tau_k(t)) \right) + \breve{B}_{ij}u(t) + \tilde{R}_i d(t) + \breve{F}_{1ij} f_a(t) + \tilde{F}_{2i} f_s(t) \right], \\ y(t) = \tilde{C}\tilde{x}(t) + D_n n(t), \end{cases} \tag{11}$$

where the following notation is used

$$\breve{A}_{kij} = \tilde{A}_{kj} + \Delta\tilde{A}_{kij}, \quad \Delta\tilde{A}_{kij} = \tilde{A}_{ki} - \tilde{A}_{kj}, \tag{12}$$

$$\breve{B}_{ij} = \tilde{B}_j + \Delta\tilde{B}_{ij}, \quad \Delta\tilde{B}_{ij} = \tilde{B}_i - \tilde{B}_j, \tag{13}$$

$$\breve{F}_{1ij} = \tilde{F}_{1j} + \Delta\tilde{F}_{1ij}, \quad \Delta\tilde{F}_{1ij} = \tilde{F}_{1i} - \tilde{F}_{1j}. \tag{14}$$

The state estimation error is:

$$e(t) = \tilde{x}(t) - \hat{\tilde{x}}(t), \tag{15}$$

which according to (10) and (11) is reformulated as:

$$e(t) = \tilde{x}(t) - z(t) - H_2\tilde{C}\tilde{x}(t) - H_2D_n n(t) = \left(I_n - H_2\tilde{C}\right)\tilde{x}(t) - z(t) - H_2D_n n(t). \tag{16}$$

By introducing $H_1 \in \mathbb{R}^n$ satisfying the following constraint:

$$H_1\tilde{E} = I_n - H_2\tilde{C}. \tag{17}$$

Equation (16) is converted into:

$$e(t) = H_1\tilde{E}\tilde{x}(t) - z(t) - H_2D_n n(t) \tag{18}$$

and subsequently the error dynamics is described by:

$$\dot{e}(t) = H_1\tilde{E}\dot{\tilde{x}}(t) - \dot{z}(t) - H_2D_n\dot{n}(t). \tag{19}$$

Substituting (10) and (11) in (19) results:

$$\begin{aligned} \dot{e}(t) = \sum_{i=1}^{h}\sum_{j=1}^{h} \rho_i(\theta(t))\rho_j(\hat{\theta}(t)) \Bigg[\sum_{k=0}^{s} \Bigg(& H_1\left(\tilde{A}_{kj} + \Delta\tilde{A}_{kij}\right)\tilde{x}(t-\tau_k(t)) - N_{kj}z(t-\tau_{km}) \\ & - L_{kj}y(t-\tau_{km}) \Bigg) + H_1\left(\tilde{B}_j + \Delta\tilde{B}_{ij}\right)u(t) + H_1\tilde{R}_i d(t) + H_1\left(\tilde{F}_{1j} + \Delta\tilde{F}_{1ij}\right)f_a(t) \\ & + H_1\tilde{F}_{2i}f_s(t) - W_j\hat{f}_{a0}(t) - G_j u(t) \Bigg] - H_2D_n\dot{n}(t) \end{aligned} \tag{20}$$

and after some manipulations, the following relation is derived:

$$\begin{aligned}\dot{e}(t) = &\sum_{i=1}^{h}\sum_{j=1}^{h}\rho_i(\theta(t))\rho_j(\widehat{\theta}(t))\Bigg[\sum_{k=0}^{s}\Bigg(N_{kj}e\left(t-\tau_{km}\right)+\left(H_1\widetilde{A}_{kj}-L_{kj}\widetilde{C}-N_{kj}H_1\widetilde{E}\right)\widetilde{x}\left(t-\tau_{km}\right)\\ &+\left(N_{kj}H_2-L_{kj}\right)D_n n\left(t-\tau_{km}\right)+H_1\Delta\widetilde{A}_{kij}\widetilde{x}\left(t-\tau_{km}\right)\Bigg)\\ &+\sum_{k=1}^{s}\left(H_1\left(\widetilde{A}_{kj}+\Delta\widetilde{A}_{kij}\right)\left(\widetilde{x}\left(t-\tau_k(t)\right)-\widetilde{x}\left(t-\tau_{km}\right)\right)\right)+\left(H_1\widetilde{B}_j-G_j\right)u(t)\\ &+H_1\Delta\widetilde{B}_{ij}u(t)+H_1\widetilde{R}_i d(t)+H_1\widetilde{F}_{1j}f_a(t)-W_j\widehat{f}_{a0}(t)+H_1\Delta\widetilde{F}_{1ij}f_a(t)+H_1\widetilde{F}_{2i}f_s(t)\Bigg]-H_2D_n\dot{n}(t).\end{aligned}\tag{21}$$

By imposing the following constraints on (21):

$$H_1\widetilde{A}_{kj}-L_{kj}\widetilde{C}-N_{kj}H_1\widetilde{E}=0,\tag{22}$$

$$G_j=H_1\widetilde{B}_j,\tag{23}$$

$$W_j=H_1\widetilde{F}_{1j},\tag{24}$$

the error dynamics is transformed into:

$$\begin{aligned}\dot{e}(t) = &\sum_{i=1}^{h}\sum_{j=1}^{h}\rho_i(\theta(t))\rho_j(\widehat{\theta}(t))\Bigg[\sum_{k=0}^{s}\Bigg(N_{kj}e\left(t-\tau_{km}\right)+\left(N_{kj}H_2-L_{kj}\right)D_n n\left(t-\tau_{km}\right)\\ &+H_1\Delta\widetilde{A}_{kij}\widetilde{x}\left(t-\tau_{km}\right)\Bigg)+\sum_{k=1}^{s}\left(H_1\left(\widetilde{A}_{kj}+\Delta\widetilde{A}_{kij}\right)\left(\widetilde{x}\left(t-\tau_k(t)\right)-\widetilde{x}\left(t-\tau_{km}\right)\right)\right)\\ &+H_1\Delta\widetilde{B}_{ij}u(t)+H_1\widetilde{R}_i d(t)+W_j e_{f_{a0}}(t)+H_1\Delta\widetilde{F}_{1ij}f_a(t)+H_1\widetilde{F}_{2i}f_s(t)\Bigg]-H_2D_n\dot{n}(t),\end{aligned}\tag{25}$$

where $e_{f_{a0}}(t)$ is the actuator fault estimation error defined as:

$$e_{f_{a0}}(t)=f_a(t)-\widehat{f}_{a0}(t).\tag{26}$$

Similarly, the estimation error of actuator fault derivative is defined as:

$$e_{f_{a1}}(t)=\dot{f}_a(t)-\widehat{f}_{a1}(t).\tag{27}$$

Assuming that the actuator fault second derivative is almost zero ($\ddot{f}_a(t)\cong 0$), according to (10) and (27), the dynamics of actuator fault derivative estimation error is obtained:

$$\dot{e}_{f_{a1}}(t)=-\sum_{j=1}^{h}\rho_j(\widehat{\theta}(t))\Lambda_{1j}(y(t)-\widehat{y}(t))=-\sum_{j=1}^{h}\rho_j(\widehat{\theta}(t))\Lambda_{1j}\left(\widetilde{C}e(t)+D_n n(t)\right).\tag{28}$$

The dynamics of actuator fault estimation error using (10) and (26) is obtained:

$$\dot{e}_{f_{a0}}(t)=e_{f_{a1}}(t)-\sum_{j=1}^{h}\rho_j(\widehat{\theta}(t))\Lambda_{0j}(y(t)-\widehat{y}(t))=e_{f_{a1}}(t)-\sum_{j=1}^{h}\rho_j(\widehat{\theta}(t))\Lambda_{0j}\left(\widetilde{C}e(t)+D_n n(t)\right).\tag{29}$$

The constraints (17) and (22)–(24) will be considered in the design procedure of the proposed PDIUIO. The constraint (17) is formulated as the following matrix equation:

$$\begin{bmatrix}H_1 & H_2\end{bmatrix}\begin{bmatrix}\widetilde{E}\\ \widetilde{C}\end{bmatrix}=I_n.\tag{30}$$

Assumption 5. The following rank is assumed to be satisfied for unknown input decoupling

$$Rank \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix} = n.$$

Under Assumption 5, the solution of matrix equation (30) is:

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+ + K \left(I_{n+m} - \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix} \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+ \right), \tag{31}$$

in which $\begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+$ is the pseudo inverse of $\begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}$ calculated by:

$$\begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+ = \left(\begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^T \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix} \right)^{-1} \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^T \tag{32}$$

and $K \in \mathbb{R}^{n \times (n+m)}$ is a factor that adds additional degree of freedom in designing the PDIUIO which can be freely chosen to satisfy other restrictions on the problem. H_1 and H_2 in (31) are partitioned as follows:

$$H_1 = H_{10} + KX_1, \tag{33}$$

$$H_2 = H_{20} + KX_2, \tag{34}$$

where

$$H_{10} = \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+ T_1, \tag{35}$$

$$H_{20} = \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+ T_2, \tag{36}$$

$$X_1 = XT_1, \tag{37}$$

$$X_2 = XT_2, \tag{38}$$

considering that $T_1 = \begin{bmatrix} I_n \\ 0_{m \times n} \end{bmatrix}$, $T_2 = \begin{bmatrix} 0_{n \times m} \\ I_m \end{bmatrix}$, and $X = I_{n+m} - \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix} \begin{bmatrix} \widetilde{E} \\ \widetilde{C} \end{bmatrix}^+$.

Now, some new variables are introduced in order to facilitate the design of PDIUIO:

$$K_{kj} = L_{kj} - N_{kj} H_2. \tag{39}$$

Thus, (22) can be formulated as:

$$N_{kj} = H_1 \widetilde{A}_{kj} - K_{kj} \widetilde{C}, \tag{40}$$

which by substituting (33) is transformed into:

$$N_{kj} = H_{10} \widetilde{A}_{kj} + KX_1 \widetilde{A}_{kj} - K_{kj} \widetilde{C}. \tag{41}$$

By substituting (39) in (25), the PDIUIO error dynamics becomes

$$\begin{aligned} \dot{e}(t) = & \sum_{i=1}^{h} \sum_{j=1}^{h} \rho_i(\theta(t)) \rho_j(\hat{\theta}(t)) \Bigg[\sum_{k=0}^{s} \left(N_{kj} e(t - \tau_{km}) - K_{kj} D_n n(t - \tau_{km}) + H_1 \Delta \widetilde{A}_{kij} \widetilde{x}(t - \tau_{km}) \right) \\ & + \sum_{k=1}^{s} \left(H_1 \left(\widetilde{A}_{kj} + \Delta \widetilde{A}_{kij} \right) \left(\widetilde{x}(t - \tau_k(t)) - \widetilde{x}(t - \tau_{km}) \right) \right) \\ & + H_1 \Delta \widetilde{B}_{ij} u(t) + H_1 \widetilde{R}_i d(t) + W_j e_{f_{a0}}(t) + H_1 \Delta \widetilde{F}_{1ij} f_a(t) + H_1 \widetilde{F}_{2i} f_s(t) \Bigg] - H_2 D_n \dot{n}(t). \end{aligned} \tag{42}$$

In order to analyze the convergence of the error dynamics computed in (28)–(29) and (42) which includes uncertainty due to both unknown delays and inexact measured parameters, the following augmented system is constructed:

$$\begin{cases} \widehat{E}\dot{\zeta}(t) = \sum_{i=1}^{h}\sum_{j=1}^{h}\rho_i(\theta(t))\rho_j(\widehat{\theta}(t))\left[\sum_{k=0}^{s}\widehat{N}_{kij}\zeta(t-\tau_{km}) + \widehat{R}_{wij}\widehat{w}(t)\right], \\ \widehat{e} = \widehat{C}\zeta(t), \end{cases} \tag{43}$$

in which the following notations are used:

$$\zeta(t) = \begin{bmatrix} e(t)^T & e_{f_{a0}}(t)^T & e_{f_{a1}}(t)^T & x(t)^T \end{bmatrix}^T, \tag{44}$$

$$\widehat{w}(t) := \begin{bmatrix} \widehat{w}_1(t) \\ \widehat{w}_2(t) \end{bmatrix}, \quad \widehat{w}_1(t) = \begin{bmatrix} u(t) \\ d(t) \\ f_a(t) \\ f_s(t) \\ n(t) \\ n(t-\tau_{1m}) \\ \vdots \\ n(t-\tau_{sm}) \\ \dot{n}(t) \end{bmatrix}, \quad \widehat{w}_2(t) = \begin{bmatrix} \widetilde{x}(t-\tau_1(t)) - \widetilde{x}(t-\tau_{1m}) \\ \vdots \\ \widetilde{x}(t-\tau_s(t)) - \widetilde{x}(t-\tau_{sm}) \end{bmatrix}, \tag{45}$$

$$\widehat{e} = \begin{bmatrix} e(t)^T & e_{f_{a0}}(t)^T \end{bmatrix}^T, \tag{46}$$

and the coefficient matrices are

$$\widehat{E} = \begin{bmatrix} I_{n+2k_{f_a}} & 0 \\ 0 & \widetilde{E} \end{bmatrix}, \tag{47}$$

$$\widehat{N}_{0ij} = \left[\begin{array}{ccc|c} N_{0j} & W_j & 0 & H_1\Delta\widetilde{A}_{0ij} \\ -\Lambda_{0j}\widetilde{C} & 0 & I & 0 \\ -\Lambda_{1j}\widetilde{C} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \widetilde{A}_{0i} \end{array}\right], \quad \widehat{N}_{kij} = \left[\begin{array}{ccc|c} N_{kj} & 0 & 0 & H_1\Delta\widetilde{A}_{kij} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \widetilde{A}_{ki} \end{array}\right] \quad (\text{for } k = 1, \ldots, s), \tag{48}$$

$$\widehat{C} = \begin{bmatrix} I_{n+k_{f_a}} & 0 \end{bmatrix}, \tag{49}$$

$$\widehat{R}_{wij} = \begin{bmatrix} \widehat{R}^1_{wij} & \widehat{R}^2_{wij} \end{bmatrix}, \tag{50}$$

where

$$\widehat{R}^1_{wij} = \begin{bmatrix} H_1\Delta\widetilde{B}_{ij} & H_1\widetilde{R}_i & H_1\Delta\widetilde{F}_{1ij} & H_1\widetilde{F}_{2i} & -K_{0j}D_n & -K_{1j}D_n & \cdots & -K_{sj}D_n & -H_2D_n \\ 0 & 0 & 0 & 0 & -\Lambda_{0j}D_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Lambda_{1j}D_n & 0 & \cdots & 0 & 0 \\ \widetilde{B}_i & \widetilde{R}_i & \widetilde{F}_{1i} & \widetilde{F}_{2i} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\widehat{R}^2_{wij} = \begin{bmatrix} H_1\left(\widetilde{A}_{1j} + \Delta\widetilde{A}_{1ij}\right) & \cdots & H_1\left(\widetilde{A}_{sj} + \Delta\widetilde{A}_{sij}\right) \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}.$$

For the stability analysis of the uncertain SDLPV system (43), the following theorem is presented:

Theorem 1. *The following SDLPV system is considered:*

$$\begin{cases} E\dot{x}(t) = \sum_{i=1}^{h} \rho_i(\theta(t)) \left[\left(\sum_{k=0}^{s} A_{ki} x\left(t - \tau_k(t)\right) \right) + R_{wi} w(t) \right], \\ z(t) = Cx(t) + D_w w(t), \end{cases} \tag{51}$$

in which $w(t)$ is a L_2-norm bounded input signal and $z(t)$ is the measured output signal. All the matrices are with compatible dimension and the other assumptions are similar to the definition of system (3). For a given $\gamma > 0$, if there exist matrices P and $Q_k > 0$ for $k = 1, \ldots, s$ such that the following conditions hold for $i = 1, \ldots, h$:

$$P^T E = E^T P \geq 0, \tag{52}$$

$$\sum^{i} = \begin{bmatrix} \sum_{11}^{i} & P^T A_{1i} & \cdots & P^T A_{si} & P^T R_{wi} & C^T \\ * & -\left(1-\mu_1\right) e^{-2\alpha\tau_{1m}} Q_1 & \cdots & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & -\left(1-\mu_s\right) e^{-2\alpha\tau_{sm}} Q_s & 0 & 0 \\ * & * & * & * & -\gamma^2 I & D_w^T \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{53}$$

where $\sum_{11}^{i} = P^T A_{0i} + A_{0i}^T P + 2\alpha P^T E + Q_1 + \cdots + Q_s$, then, system (51) is exponentially stable with the decay rate of $\alpha \geq 0$ for $w(t) = 0$ and the attenuation condition $\|z(t)\|_2 < \gamma \|w(t)\|_2$ holds for zero initial conditions.

Proof. The following Lyapunov–Krasovskii functional is considered:

$$V\left(t, x_t\right) = x^T(t) P^T E x(t) + \sum_{k=1}^{s} \int_{t-\tau_k(t)}^{t} x^T(\lambda) Q_k e^{2\alpha(\lambda - t)} x(\lambda) d\lambda, \tag{54}$$

in which $P^T E = E^T P \geq 0$, $Q_k = Q_k^T > 0$ for $k = 1, \ldots, s$ and $x_t := x(t + \omega)$ where $\omega \in [-\tau_m, 0]$. The following index is considered:

$$J = \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t) \right] dt. \tag{55}$$

Showing $J < 0$ in the case of zero initial conditions proves $\|z(t)\|_2 < \gamma \|w(t)\|_2$. By adding the term $\int_0^\infty \dot{V}\left(t, x_t\right) dt + V\left(t, x_t\right)|_{t=0} - V\left(t, x_t\right)|_{t=\infty}$ that equals to zero and also adding and subtracting $2\alpha V(t, x_t)$; the index J in (55) is reformulated as:

$$\begin{aligned} J = & \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t) + \dot{V}\left(t, x_t\right) + 2\alpha V\left(t, x_t\right) \right] dt \\ & - \int_0^\infty 2\alpha V\left(t, x_t\right) dt + V\left(t, x_t\right)|_{t=0} - V\left(t, x_t\right)|_{t=\infty}. \end{aligned} \tag{56}$$

Since $V\left(t, x_t\right)|_{t=0} = 0$ and $V\left(t, x_t\right)|_{t=\infty} \geq 0$, the following inequality is deduced:

$$J \leq \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t) + \dot{V}\left(t, x_t\right) + 2\alpha V\left(t, x_t\right) \right] dt. \tag{57}$$

The time derivative of the Lyapunov–Krasovskii functional (54) can be calculated as follows:

$$\dot{V}(t,x_t) = \sum_{i=1}^{h}\rho_i(\theta(t))Sym\left\{x^T(t)P^TA_{0i}x(t) + x^T(t)P^TR_{wi}w(t) + \sum_{k=1}^{s}\left(x^T(t)P^TA_{ki}x(t-\tau_k(t))\right)\right\}$$
$$+\sum_{k=1}^{s}\left\{x^T(t)Q_kx(t) - (1-\dot{\tau}_k(t))x^T(t-\tau_k(t))Q_ke^{-2\alpha\tau_k(t)}x(t-\tau_k(t)) - 2\alpha\int_{t-\tau_k(t)}^{t}x^T(\lambda)Q_ke^{2\alpha(\lambda-t)}x(\lambda)d\lambda\right\}. \tag{58}$$

By considering the upper bounds on delay values and delay rates and the convex property of the parameters, respectively, as stated in (1) and (5), the inequality (57) is converted into:

$$J \le \int_0^\infty \sum_{i=1}^{h}\rho_i(\theta(t))\xi(t)^T\Xi^i\xi(t)dt, \tag{59}$$

where $\xi(t) = \begin{bmatrix} x(t)^T & x(t-\tau_1(t))^T & \cdots & x(t-\tau_s(t))^T & w(t)^T\end{bmatrix}^T$ and

$$\Xi^i = \begin{bmatrix} \Xi_{11}^i & P^TA_{1i} & \cdots & P^TA_{si} & P^TR_{wi}+C^TD_w \\ * & -(1-\mu_1)e^{-2\alpha\tau_{1m}}Q_1 & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots \\ * & * & * & -(1-\mu_s)e^{-2\alpha\tau_{sm}}Q_s & 0 \\ * & * & * & * & D_w^TD_w-\gamma^2I \end{bmatrix}, \tag{60}$$

where $\Xi_{11}^i = P^TA_{0i} + A_{0i}^TP + 2\alpha P^TE + Q_1 + \cdots + Q_s + C^TC$. The inequalities $\Xi^i < 0$ for $i = 1, \ldots, h$ assures $J < 0$. Reformulating the inequalities $\Xi^i < 0$ by applying Schur complement lemma will result in inequalities (53), thus the attenuation condition $\|z(t)\|_2 < \gamma\|w(t)\|_2$ for system (51) is proved which is analogous to H_∞ performance. Now, the following submatrix of $\sum^i$ is considered

$$\begin{bmatrix} \sum_{11}^i & P^TA_{1i} & \cdots & P^TA_{si} \\ * & -(1-\mu_1)e^{-2\alpha\tau_{1m}}Q_1 & \cdots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & -(1-\mu_s)e^{-2\alpha\tau_{sm}}Q_s \end{bmatrix}, \tag{61}$$

which is negative definite due to negative definiteness of $\sum^i$ and is equivalent to $\dot{V}(t,x_t) + 2\alpha V(t,x_t) < 0$ for the non-actuated dynamics. Thus, the exponential stability is also proved. ■

The following two results are special cases of Theorem 1:

Corollary 1. *If in system (51), $D_w = 0$, then LMI (53) in Theorem 1, becomes*

$$\bar{\Xi}^i = \begin{bmatrix} \bar{\Xi}_{11}^i & P^TA_{1i} & \cdots & P^TA_{si} & P^TR_{wi} \\ * & -(1-\mu_1)e^{-2\alpha\tau_{1m}}Q_1 & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots \\ * & * & * & -(1-\mu_s)e^{-2\alpha\tau_{sm}}Q_s & 0 \\ * & * & * & * & -\gamma^2I \end{bmatrix} < 0, \tag{62}$$

where

$$\bar{\Xi}_{11}^i = P^TA_{0i} + A_{0i}^TP + 2\alpha P^TE + Q_1 + \cdots + Q_s + C^TC, \tag{63}$$

which is obtained by applying $D_w = 0$ in (60).

Corollary 2. *If $D_w = 0$ and $E = I$ in system (51) which reduces to a delayed LPV system, then the constraint (52) in Theorem 1 reduces to P being a symmetric positive definite matrix and LMI (53) becomes similar to LMI (62) just differing in the (1, 1) block which is formulated as follows*:

$$\overline{\Xi}_{11}^{i} = P^T A_{0i} + A_{0i}^T P + 2\alpha P + Q_1 + \cdots + Q_s + C^T C. \tag{64}$$

5 | MAIN RESULT

In this section, the stability of the proposed PDIUIO is addressed with the aid of the stability criteria obtained for SDLPV systems in the previous section.

Theorem 2. *Considering system (7), an exponential decay rate α and an attenuation weighting factor η, if there exist symmetric positive definite matrices P_1, Q_{1k}, and Q_{2k} for $k = 1, \ldots, s$ and matrices P_2, M, and M_{kj} for $k = 0, \ldots, s$, $j = 1, \ldots, h$ and positive scalar $\overline{\gamma}$ obtained as the solution to the following optimization problem:*

$$\min_{P_1, P_2, Q_{1k}, Q_{2k}, M, M_{kj}} \overline{\gamma} \tag{65}$$

subject to the following LMIs for $i = 1, \ldots, h$ and $j = 1, \ldots, h$:

$$P_2^T \widetilde{E} = \widetilde{E}^T P_2 \geq 0, \tag{66}$$

$$\overline{\Omega}^{ij} = \begin{bmatrix} \overline{\Omega}_{11}^{ij} & \cdots & \overline{\Omega}_{16}^{ij} \\ * & \ddots & \vdots \\ * & * & \overline{\Omega}_{66}^{ij} \end{bmatrix} < 0, \tag{67}$$

where

$$\overline{\Omega}_{11}^{ij} = Sym\left\{ P_1 \begin{bmatrix} H_{10}\widetilde{A}_{0j} & H_{10}\widetilde{F}_{1j} & 0 \\ 0 & 0 & I_{k_{f_a}} \\ 0 & 0 & 0 \end{bmatrix} + M \begin{bmatrix} X_1\widetilde{A}_{0j} & X_1\widetilde{F}_{1j} & 0 \end{bmatrix} - M_{0j} \begin{bmatrix} \widetilde{C} & 0 & 0 \end{bmatrix} \right\}$$
$$+\ 2\alpha P_1 + Q_{11} + \cdots + Q_{1s} + \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{k_{f_a}} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\Omega}_{12}^{ij} = P_1 \begin{bmatrix} H_{10}\Delta\widetilde{A}_{0ij} \\ 0 \\ 0 \end{bmatrix} + M X_1 \Delta\widetilde{A}_{0ij},$$

$$\overline{\Omega}_{13}^{ij} = \begin{bmatrix} \Gamma_{kij}|_{k=1} & \cdots & \Gamma_{kij}|_{k=s} \end{bmatrix},$$

$$\Gamma_{kij} = \begin{bmatrix} P_1 \begin{bmatrix} H_{10}\widetilde{A}_{kj} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + M \begin{bmatrix} X_1\widetilde{A}_{kj} & 0 & 0 \end{bmatrix} - M_{kj} \begin{bmatrix} \widetilde{C} & 0 & 0 \end{bmatrix} & P_1 \begin{bmatrix} H_{10}\Delta\widetilde{A}_{kij} \\ 0 \\ 0 \end{bmatrix} + M X_1 \Delta\widetilde{A}_{kij} \end{bmatrix},$$

$$\overline{\Omega}_{14}^{ij} = \begin{bmatrix} P_1 \begin{bmatrix} H_{10}\Delta\widetilde{B}_{ij} \\ 0 \\ 0 \end{bmatrix} + M X_1 \Delta\widetilde{B}_{ij} & P_1 \begin{bmatrix} H_{10}\widetilde{R}_i \\ 0 \\ 0 \end{bmatrix} + M X_1 \widetilde{R}_i & P_1 \begin{bmatrix} H_{10}\Delta\widetilde{F}_{1ij} \\ 0 \\ 0 \end{bmatrix} + M X_1 \Delta\widetilde{F}_{1ij} & P_1 \begin{bmatrix} H_{10}\widetilde{F}_{2i} \\ 0 \\ 0 \end{bmatrix} + M X_1 \widetilde{F}_{2i} \end{bmatrix},$$

$$\overline{\Omega}_{15}^{ij} = \begin{bmatrix} -M_{0j}D_n & -M_{1j}D_n & \cdots & -M_{sj}D_n & -P_1\begin{bmatrix} H_{20}D_n \\ 0 \\ 0 \end{bmatrix} - MX_2D_n \end{bmatrix},$$

$$\overline{\Omega}_{16}^{ij} = \begin{bmatrix} P_1\begin{bmatrix} H_{10}\widetilde{A}_{1i} \\ 0 \\ 0 \end{bmatrix} + MX_1\widetilde{A}_{1i} & \cdots & P_1\begin{bmatrix} H_{10}\widetilde{A}_{si} \\ 0 \\ 0 \end{bmatrix} + MX_1\widetilde{A}_{si} \end{bmatrix},$$

$$\overline{\Omega}_{22}^{ij} = Sym\left\{P_2^T\widetilde{A}_{0i}\right\} + 2\alpha P_2^T\widetilde{E} + Q_{21} + \cdots + Q_{2s},$$

$$\overline{\Omega}_{23}^{ij} = \begin{bmatrix} \overline{\Gamma}_{kij}\Big|_{k=1} & \cdots & \overline{\Gamma}_{kij}\Big|_{k=s} \end{bmatrix}, \quad \overline{\Gamma}_{kij} = \begin{bmatrix} 0 & P_2^T\widetilde{A}_{ki} \end{bmatrix},$$

$$\overline{\Omega}_{24}^{ij} = \begin{bmatrix} P_2^T\widetilde{B}_i & P_2^T\widetilde{R}_i & P_2^T\widetilde{F}_{1i} & P_2^T\widetilde{F}_{2i} \end{bmatrix}, \quad \overline{\Omega}_{25}^{ij} = 0, \quad \overline{\Omega}_{26}^{ij} = 0,$$

$$\overline{\Omega}_{33}^{ij} = diag\left\{\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_s\right\}, \quad \widetilde{\Gamma}_k = \begin{bmatrix} -Q_{1k}e^{-2\alpha\tau_{km}} & 0 \\ 0 & -Q_{2k}e^{-2\alpha\tau_{km}} \end{bmatrix},$$

$$\overline{\Omega}_{34}^{ij} = 0, \quad \overline{\Omega}_{35}^{ij} = 0, \quad \overline{\Omega}_{36}^{ij} = 0,$$

$$\overline{\Omega}_{44}^{ij} = diag\left\{-\overline{\gamma}I_{k_u}, -\overline{\gamma}I_{k_d}, -\overline{\gamma}I_{k_{f_a}}, -\overline{\gamma}I_{k_{f_s}}\right\}, \quad \overline{\Omega}_{45}^{ij} = 0, \quad \overline{\Omega}_{46}^{ij} = 0,$$

$$\overline{\Omega}_{55}^{ij} = diag\left\{-\overline{\gamma}I_{k_n}, \ldots, -\overline{\gamma}I_{k_n}\right\}, \quad \overline{\Omega}_{56}^{ij} = 0,$$

$$\overline{\Omega}_{66}^{ij} = diag\left\{-\eta\overline{\gamma}I_n, \ldots, -\eta\overline{\gamma}I_n\right\}.$$

Then, the robust state and fault estimator (10) with exponential decay rate α and the best achievable attenuation level $\gamma = \sqrt{\overline{\gamma}}$ for attenuating disturbance, noise, faults and the uncertainty induced by inexact measured parameters and with attenuation level $\sqrt{\eta\overline{\gamma}}$ for attenuating the uncertainty induced by unknown delays exists. The matrices Λ_{0j} and Λ_{1j} for $j = 1, \ldots, h$ that are the integrator gain and double integrator gain of the PDIUIO (10) are calculated from:

$$\Lambda_{0j} = \begin{bmatrix} 0_{k_{f_a}\times n} & I_{k_{f_a}} & 0_{k_{f_a}} \end{bmatrix}\left(P^{-1}M_{0j}\right), \tag{68}$$

$$\Lambda_{1j} = \begin{bmatrix} 0_{k_{f_a}\times n} & 0_{k_{f_a}} & I_{k_{f_a}} \end{bmatrix}\left(P^{-1}M_{0j}\right). \tag{69}$$

Also, the matrices K and K_{kj} for $k = 0, \ldots, s$ and $j = 1, \ldots, h$ can be determined as follows:

$$K = \left(P\begin{bmatrix} I_n & 0_{n\times k_{f_a}} & 0_{n\times k_{f_a}} \end{bmatrix}^T\right)^+ M, \tag{70}$$

$$K_{0j} = \begin{bmatrix} I_n & 0_{n\times k_{f_a}} & 0_{n\times k_{f_a}} \end{bmatrix}\left(P^{-1}M_{0j}\right), \tag{71}$$

$$K_{kj} = \left(P\begin{bmatrix} I_n & 0_{n\times k_{f_a}} & 0_{n\times k_{f_a}} \end{bmatrix}^T\right)^+ M_{kj} \quad (for\ k = 1, \ldots, s), \tag{72}$$

and the matrices N_{kj}, G_j, W_j, H_2, and L_{kj} are calculated from (41), (23), (24), (34), and (39), respectively.

Proof. Corollary 1 is applied to system (43). To do so, the following block diagonal matrices for constructing the Lyapunov–Krasovskii functional (54) is used:

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \tag{73}$$

$$Q_k = \begin{bmatrix} Q_{1k} & 0 \\ 0 & Q_{2k} \end{bmatrix} \quad (\text{for } k = 1, \ldots, s), \tag{74}$$

where $P_1, Q_{1k} \in \mathbb{R}^{(n+2k_{f_a})\times(n+2k_{f_a})}$ and $P_2, Q_{2k} \in \mathbb{R}^{n\times n}$. According to (47), the condition (52) is equivalent with $P_1 = P_1^T \geq 0$ and $P_2^T\widetilde{E} = \widetilde{E}^T P_2 \geq 0$ for system (43). Then, the state space matrices of system (43) as stated in (47)–(50) are substituted in (62) and (63) and next some reformulations are applied on them. The upper left blocks of the augmented matrices defined in (48) are reformulated as follows:

$$\begin{bmatrix} N_{0j} & W_j & 0 \\ -\Lambda_{0j}\widetilde{C} & 0 & I \\ -\Lambda_{1j}\widetilde{C} & 0 & 0 \end{bmatrix} = \begin{bmatrix} H_{10}\widetilde{A}_{0j} & H_{10}\widetilde{F}_{1j} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X_1\widetilde{A}_{0j} & X_1\widetilde{F}_{1j} & 0 \end{bmatrix} - \begin{bmatrix} K_{0j} \\ \Lambda_{0j} \\ \Lambda_{1j} \end{bmatrix} \begin{bmatrix} \widetilde{C} & 0 & 0 \end{bmatrix}, \tag{75}$$

$$\begin{bmatrix} N_{kj} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} H_{10}\widetilde{A}_{kj} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X_1\widetilde{A}_{kj} & 0 & 0 \end{bmatrix} - \begin{bmatrix} K_{kj} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \widetilde{C} & 0 & 0 \end{bmatrix} \quad (\text{for } k = 1, \dots, s). \tag{76}$$

Also, $\widehat{R}_{wij}$ is reformulated in a similar manner. Then, by considering γ and $\gamma\sqrt{\eta}$ as attenuation levels for attenuating $\widehat{w}_1(t)$ and $\widehat{w}_2(t)$, respectively; a set of non-LMIs are obtained since there exist multiplicative terms of some unknown variables. To resolve the nonlinearities in the obtained matrix inequalities, the following change of variables are applied:

$$M = P\begin{bmatrix} K^T & 0 & 0 \end{bmatrix}^T, \tag{77}$$

$$M_{0j} = P\begin{bmatrix} K_{0j}^T & \Lambda_{0j}^T & \Lambda_{1j}^T \end{bmatrix}^T, \tag{78}$$

$$M_{kj} = P\begin{bmatrix} K_{kj}^T & 0 & 0 \end{bmatrix}^T \quad (\text{for } k = 1, \dots, s), \tag{79}$$

$$\overline{\gamma} = \gamma^2, \tag{80}$$

which results in the set of LMIs (67). In system (43), the delay values are constant (τ_{km}), so $\mu_k = 0$ is considered while applying (62) to system (43). According to Corollary 1, the robust exponential convergence of the state and fault estimator is guaranteed. When the optimization problem (65) under LMI conditions (66) and (67) is solved; according to (77)–(79), Λ_{0j} and Λ_{1j} are calculated from (68)–(69), K and K_{kj} are calculated from (70)–(72), respectively. Then, the matrices N_{kj}, G_j, W_j, H_2, and L_{kj} of PDIUIO are calculated from (41), (23), (24), (34), and (39), respectively. ■

Remark 4. The attenuation weighting factor η in Theorem 2 provides additional degree of freedom to consider different weights on attenuating the uncertainty induced by unknown delays versus attenuating other unknown inputs.

Remark 5. Theorem 2 involves a non-strict LMI since it contains the equality constraint (66). This may cause numerical problems and can be avoided by parameterizing P_2 as $\mathcal{P}_2 = P_2\widetilde{E} + SV$ where $P_2 > 0$ and $V \in \mathbb{R}^{(n-r)\times n}$ are the parameters and $S \in \mathbb{R}^{n\times(n-r)}$ is any full column rank matrix which satisfies $\widetilde{E}^T S = 0$.[33]

Corollary 3. *Theorem 2 may be simplified to consider the case which the scheduling parameters of the system (1) are exactly measured. So, there will be no mismatch between the scheduling parameters of PDIUIO (10) and system (1) in this situation. Following the material presented before Theorem 2 and in its proof, the following simplification in this case should be applied: the LMI (66) is omitted and in LMI (67); $\overline{\Omega}_{12}^{ij}$, the second block in Γ_{kij} (of the block $\overline{\Omega}_{13}^{ij}$), the first and third blocks in $\overline{\Omega}_{14}^{ij}$, $\overline{\Omega}_{22}^{ij}$, $\overline{\Omega}_{23}^{ij}$, $\overline{\Omega}_{24}^{ij}$, $\overline{\Omega}_{25}^{ij}$, $\overline{\Omega}_{26}^{ij}$ and the second block in block diagonal matrix $\widetilde{\Gamma}_k$ (of the block $\overline{\Omega}_{33}^{ij}$) will be removed. Obviously, the corresponding blocks symmetric to these blocks will be removed and the zero blocks' dimensions will be modified correspondingly. After these simplifications, optimization problem (65), subject to the simplified version of LMIs (67) for $j = 1, \dots, h$ is solved and according to other unchanged parts of Theorem 2, the unknown matrices of the PDIUIO may be calculated.*

Remark 6. Theorem 2 states PDIUIO design conditions for SDLPV systems with inexact scheduling parameters. A similar proposition can be established for PDIUIO design conditions for SDLPV systems with unmeasurable parameters. This is done by replacing $\rho_i(\theta(t))$ and $\rho_j(\widehat{\theta}(t))$ with $\rho_i(x(t))$ and $\rho_j(\widehat{x}(t))$, respectively, in the corresponding formulations.

Algorithm 1. Design of PDIUIO for simultaneous state estimation and both actuator and sensor fault reconstruction in SDLPV systems with multiple unknown time varying delays in the presence of inexact parameters

Step 1. Convert system (1) to polytopic form (3), construct the augmented system (7) and then convert system (7) to the uncertain system (11).
Step 2. Check Assumptions 2–5.
Step 3. Calculate H_{10}, H_{20}, X_1, and X_2 from (35)–(38), respectively.
Step 4. Solve the optimization problem (65) under constraint (66) and the set of LMIs (67) and obtain the scalar $\overline{\gamma}$ and the matrices P_1, P_2, Q_{1k}, and Q_{2k} (for $k = 1, \dots, s$) and matrices M and M_{kj} (for $k = 0, \dots, s, j = 1, \dots, h$).
Step 5. Calculate Λ_{0j} and Λ_{1j} (for $j = 1, \dots, h$) from (68) and (69), respectively.
Step 6. Calculate K and K_{kj} (for $k = 0, \dots, s$ and $j = 1, \dots, h$) from (70)–(72), respectively.
Step 7. Calculate H_1 and H_2 from (33) and (34), respectively.
Step 8. Calculate matrices N_{kj} and L_{kj} (both for $k = 0, \dots, s$ and $j = 1, \dots, h$) from (41) and (39), respectively.
Step 9. Calculate G_j and W_j (both for $j = 1, \dots, h$) from (23) and (24).

6 | EXAMPLE

6.1 | Description

In this section, an electrical circuit example[19] is used to evaluate the efficiency of the proposed methods. The circuit is depicted in Figure 1. This circuit has four meshes and the currents inside the meshes are shown with $i_1(t)$, $i_2(t)$, $i_3(t)$, and $i_4(t)$. The circuit has two voltage sources, eight resistors and two inductors. The numerical values of the elements are listed in Table 1 which are similar to the selection of Rodrigues et al.[19] As expressed in Table 1, the resistors R_1 and R_6 are variable resistors which their resistances vary subject to the variation of exogenous parameters $\theta_1(t)$ and $\theta_2(t)$, respectively. These two parameters' range of variations are $\theta_1(t) \in [-0.5, 0.5]$ and $\theta_2(t) \in [-1, 1]$. The circuit model is obtained by applying Kirchhoff voltage law (KVL) to its meshes. Thus, the following equations are obtained:

$$\begin{cases} L_1 \frac{di_1}{dt} + R_1 i_1 + R_3 (i_1 - i_3) + R_5 (i_1 - i_4) = 0, \\ L_2 \frac{di_2}{dt} + R_7 (i_2 - i_4) + R_4 (i_2 - i_3) + R_6 i_2 = 0, \\ R_2 i_3 - v_1(t) + R_4 (i_3 - i_2) + R_3 (i_3 - i_1) = 0, \\ R_8 i_4 - v_2(t) + R_5 (i_4 - i_1) + R_7 (i_4 - i_2) = 0. \end{cases} \tag{81}$$

It is considered that there are some delays in transmitting the input commands to the voltage sources due to communication channel. The voltage source vector is $v(t) = \begin{bmatrix} v_1(t) & v_2(t) \end{bmatrix}^T$ and considering the communication delay $\tau(t)$, it relates to input vector as $v(t) = u(t - \tau(t))$. The delay is time-varying and unknown but it is known that $\tau(t) \in [0.4, 1]$. The

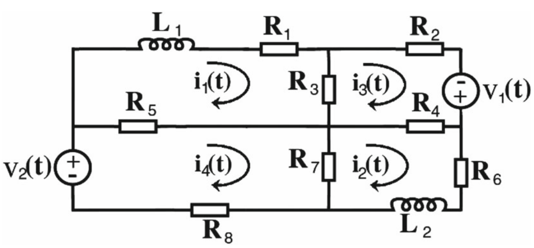

FIGURE 1 Electrical circuit example

TABLE 1 Numerical values of the circuit elements

Element	L_1	L_2	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
Value	0.3	0.65	$10 + \theta_1(t)$	17	3	5	2	$27 + \theta_2(t)$	8	10
Unit	H	H	Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω

state vector is selected as $x(t) = \begin{bmatrix} i_1(t) & i_2(t) & i_3(t) & i_4(t) \end{bmatrix}^T$. The voltages across the resistors R_5, R_7, and R_4 are measured. Thus, applying Ohm's Law the following set of output equations is obtained:

$$\begin{cases} y_1(t) = v_{R_5} = R_5\left(i_4 - i_1\right), \\ y_2(t) = v_{R_7} = R_7\left(i_4 - i_2\right), \\ y_3(t) = v_{R_4} = R_4\left(i_2 - i_3\right). \end{cases} \tag{82}$$

The output vector is $y(t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \end{bmatrix}^T$. The circuit's model can be summarized as:

$$\begin{cases} E\dot{x}(t) = A(\theta(t))x(t) + Bu(t-\tau(t)) + Rd(t) + Ff_a(t), \\ y(t) = Cx(t) + D_f f_s(t) + D_n n(t), \end{cases} \tag{83}$$

where the matrices of the model are.

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0.3 \\ 0.2 \\ -0.3 \\ 0 \end{bmatrix},$$

$$A(\theta(t)) = \begin{bmatrix} -\frac{R_{11}+\theta_1(t)}{L_1} & 0 & \frac{R_3}{L_1} & \frac{R_5}{L_1} \\ 0 & -\frac{R_{22}+\theta_2(t)}{L_2} & \frac{R_4}{L_2} & \frac{R_7}{L_2} \\ R_3 & R_4 & -R_{33} & 0 \\ R_5 & R_7 & 0 & -R_{44} \end{bmatrix},$$

$$C = \begin{bmatrix} -R_5 & 0 & 0 & R_5 \\ 0 & -R_7 & 0 & R_7 \\ 0 & R_4 & -R_4 & 0 \end{bmatrix}, \quad D_f = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad D_n = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.2 \end{bmatrix},$$

and

$$R_{11} = \overline{R}_1 + R_3 + R_5,\ R_{22} = R_4 + \overline{R}_6 + R_7,\ R_{33} = R_2 + R_3 + R_4,\ R_{44} = R_5 + R_7 + R_8.$$

In system (83), a fault on the first actuator (voltage source) and another fault on the second sensor is considered. Also, as it is seen in (83) the system is affected by disturbances and noise. System (83) is a singular LPV system with delayed inputs. By choosing the inputs as auxiliary states and also using them as additional outputs like

$$\overline{x}(t) = \begin{bmatrix} i_1(t) & i_2(t) & i_3(t) & i_4(t) & u_1(t) & u_2(t) \end{bmatrix}^T, \tag{84}$$

$$\overline{y}(t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) & u_1(t) & u_2(t) \end{bmatrix}^T, \tag{85}$$

the following singular LPV system with state delay is obtained:

$$\begin{cases} \overline{E}\dot{\overline{x}}(t) = \overline{A}_0(\theta(t))\overline{x}(t) + \overline{A}_1\overline{x}(t-\tau(t)) + \overline{B}u(t) + \overline{R}d(t) + \overline{F}f_a(t), \\ \overline{y}(t) = \overline{C}\overline{x}(t) + \overline{D}_n n(t) + \overline{D}_f f_s(t), \end{cases} \tag{86}$$

where

$$\overline{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_0(\theta(t)) = \begin{bmatrix} A(\theta(t)) & 0 \\ 0 & -I_{k_u} \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} 0 \\ I_{k_u} \end{bmatrix}, \quad \overline{R} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \quad \overline{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} C & 0 \\ 0 & I_{k_u} \end{bmatrix}, \quad \overline{D}_n = \begin{bmatrix} D_n \\ 0 \end{bmatrix}, \quad \overline{D}_f = \begin{bmatrix} D_f \\ 0 \end{bmatrix}.$$

System (86) is a SDLPV system in the form of (1). It should be converted to polytopic representation (3) in order to apply the methodologies proposed in this article. There are two varying parameters in the system (86) and hence there are four subsystems in the polytopic representation. Matrices $\overline{A}_{0i}$ and weights $\rho_i(\theta(t))$ of different subsystems (for $i = 1, \dots, 4$) are calculated as follows

$$\begin{aligned}
\overline{A}_{01} &= \overline{A}_0(\theta(t))\Big|_{\substack{\theta_1(t)=\theta_1^m\\ \theta_2(t)=\theta_2^m}},\\
\overline{A}_{02} &= \overline{A}_0(\theta(t))\Big|_{\substack{\theta_1(t)=\theta_1^M\\ \theta_2(t)=\theta_2^m}},\\
\overline{A}_{03} &= \overline{A}_0(\theta(t))\Big|_{\substack{\theta_1(t)=\theta_1^m\\ \theta_2(t)=\theta_2^M}},\\
\overline{A}_{04} &= \overline{A}_0(\theta(t))\Big|_{\substack{\theta_1(t)=\theta_1^M\\ \theta_2(t)=\theta_2^M}}
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
\rho_1(\theta(t)) &= \alpha_1(t)\alpha_2(t),\\
\rho_2(\theta(t)) &= (1-\alpha_1(t))\,\alpha_2(t),\\
\rho_3(\theta(t)) &= \alpha_1(t)\,(1-\alpha_2(t)),\\
\rho_4(\theta(t)) &= (1-\alpha_1(t))\,(1-\alpha_2(t)),
\end{aligned} \tag{88}$$

where $\alpha_1(t) = \frac{\theta_1^M - \theta_1(t)}{\theta_1^M - \theta_1^m}$ and $\alpha_2(t) = \frac{\theta_2^M - \theta_2(t)}{\theta_2^M - \theta_2^m}$.

6.2 | Results

Now, a PDIUIO is designed for system (86) following the steps of Algorithm 1. Assumptions 2–5 that are needed for the design of PDIUIO are satisfied for system (86). Solving the optimization problem is carried out via YALMIP toolbox[35] using SeDuMi solver.[36] The parameters $\alpha = 0$ and $\eta = 100$ are chosen. The minimum value obtained for attenuation level is $\gamma_{opt} = 0.5073$. The matrices Λ_{0j} and Λ_{1j} for $j = 1, \dots, 4$ are obtained as follows.

$$\begin{aligned}
\Lambda_{01} &= [1.2570\text{e}{+}05 \quad 7.2151\text{e}{+}03 \quad -7.7269\text{e}{+}04 \quad -4.1997\text{e}{-}02 \quad -3.9494\text{e}{-}02],\\
\Lambda_{11} &= [7.5318\text{e}{+}01 \quad 4.3234\text{e}{+}00 \quad -4.6300\text{e}{+}01 \quad -2.4676\text{e}{-}05 \quad -2.4525\text{e}{-}05],\\
\Lambda_{02} &= [1.2413\text{e}{+}05 \quad 5.9794\text{e}{+}03 \quad -7.9550\text{e}{+}04 \quad 7.8762\text{e}{-}02 \quad -9.3033\text{e}{-}02],\\
\Lambda_{12} &= [7.4379\text{e}{+}01 \quad 3.5829\text{e}{+}00 \quad -4.7666\text{e}{+}01 \quad 4.7683\text{e}{-}05 \quad -5.6606\text{e}{-}05],\\
\Lambda_{03} &= [1.2405\text{e}{+}05 \quad 5.9165\text{e}{+}03 \quad -7.9665\text{e}{+}04 \quad 8.6080\text{e}{-}02 \quad 7.6668\text{e}{-}02],\\
\Lambda_{13} &= [7.4331\text{e}{+}01 \quad 3.5452\text{e}{+}00 \quad -4.7736\text{e}{+}01 \quad 5.2068\text{e}{-}05 \quad 4.5080\text{e}{-}05],\\
\Lambda_{04} &= [1.2408\text{e}{+}05 \quad 5.9398\text{e}{+}03 \quad -7.9622\text{e}{+}04 \quad 2.8771\text{e}{-}02 \quad -8.5414\text{e}{-}02],\\
\Lambda_{14} &= [7.4349\text{e}{+}01 \quad 3.5591\text{e}{+}00 \quad -4.7710\text{e}{+}01 \quad 1.7728\text{e}{-}05 \quad -5.2041\text{e}{-}05].
\end{aligned}$$

The obtained matrices K_{0j} and K_{1j} for $j = 1, \dots, 4$ are

$$K_{01} = \begin{bmatrix}
-1.9183\text{e}{+}02 & -1.7855\text{e}{+}01 & 1.2786\text{e}{+}02 & 9.9988\text{e}{-}05 & 4.7380\text{e}{-}05\\
-3.2520\text{e}{+}02 & -1.7994\text{e}{+}01 & 1.9977\text{e}{+}02 & 8.6689\text{e}{-}05 & 1.2534\text{e}{-}04\\
7.1971\text{e}{+}02 & 4.8137\text{e}{+}01 & -4.5259\text{e}{+}02 & -2.6460\text{e}{-}04 & -2.0620\text{e}{-}04\\
-1.5328\text{e}{+}03 & -9.0677\text{e}{+}01 & 9.4595\text{e}{+}02 & 5.3577\text{e}{-}04 & 4.7576\text{e}{-}04\\
-9.4533\text{e}{-}05 & 1.7314\text{e}{-}05 & 2.1281\text{e}{-}05 & 1.0110\text{e}{+}00 & 8.1350\text{e}{-}07\\
-3.5037\text{e}{-}05 & 1.4082\text{e}{-}05 & -3.6779\text{e}{-}06 & 4.9364\text{e}{-}06 & 1.0110\text{e}{+}00\\
7.3551\text{e}{+}02 & 7.8556\text{e}{+}01 & -5.0390\text{e}{+}02 & -4.1469\text{e}{-}04 & -1.5378\text{e}{-}04
\end{bmatrix},$$

$$K_{02} = \begin{bmatrix} -1.8776e{+}02 & -1.4468e{+}01 & 1.2942e{+}02 & -1.1112e{-}04 & 1.4050e{-}04 \\ -3.3225e{+}02 & -1.2816e{+}01 & 2.0894e{+}02 & -2.4713e{-}04 & 2.7374e{-}04 \\ 7.1607e{+}02 & 3.9205e{+}01 & -4.6572e{+}02 & 4.3179e{-}04 & -5.1429e{-}04 \\ -1.5064e{+}03 & -7.7065e{+}01 & 9.7321e{+}02 & -9.5369e{-}04 & 1.1358e{-}03 \\ -2.2912e{-}05 & 3.7276e{-}06 & 5.9093e{-}06 & 1.0110e{+}00 & 8.1349e{-}07 \\ -7.2040e{-}06 & 4.0183e{-}06 & -2.5780e{-}06 & 3.0095e{-}07 & 1.0110e{+}00 \\ 7.2172e{+}02 & 7.1150e{+}01 & -5.1914e{+}02 & 3.7370e{-}04 & -5.0057e{-}04 \end{bmatrix},$$

$$K_{03} = \begin{bmatrix} -1.8883e{+}02 & -1.4187e{+}01 & 1.2987e{+}02 & -1.1937e{-}04 & -1.6878e{-}04 \\ -3.3205e{+}02 & -1.2501e{+}01 & 2.0904e{+}02 & -2.7350e{-}04 & -1.8196e{-}04 \\ 7.1512e{+}02 & 3.9091e{+}01 & -4.6652e{+}02 & 4.6970e{-}04 & 4.7568e{-}04 \\ -1.5068e{+}03 & -7.6032e{+}01 & 9.7485e{+}02 & -1.0415e{-}03 & -9.6456e{-}04 \\ -2.4822e{-}05 & 3.9372e{-}06 & 6.5134e{-}06 & 1.0110e{+}00 & 8.1349e{-}07 \\ -1.1739e{-}05 & 4.8977e{-}06 & -1.6110e{-}06 & -5.5348e{-}07 & 1.0110e{+}00 \\ 7.2459e{+}02 & 7.0658e{+}01 & -5.2120e{+}02 & 3.9382e{-}04 & 6.7722e{-}04 \end{bmatrix},$$

$$K_{04} = \begin{bmatrix} -1.8768e{+}02 & -1.4421e{+}01 & 1.2956e{+}02 & -1.9787e{-}05 & 1.3479e{-}04 \\ -3.3232e{+}02 & -1.2531e{+}01 & 2.0898e{+}02 & -1.1375e{-}04 & 2.4513e{-}04 \\ 7.1538e{+}02 & 3.9196e{+}01 & -4.6627e{+}02 & 1.4102e{-}04 & -4.7675e{-}04 \\ -1.5058e{+}03 & -7.6560e{+}01 & 9.7411e{+}02 & -3.3515e{-}04 & 1.0462e{-}03 \\ -2.2941e{-}05 & 3.6979e{-}06 & 5.9521e{-}06 & 1.0110e{+}00 & 8.1350e{-}07 \\ -6.7810e{-}06 & 3.9295e{-}06 & -2.7337e{-}06 & 1.6177e{-}06 & 1.0110e{+}00 \\ 7.2003e{+}02 & 7.1697e{+}01 & -5.2011e{+}02 & 2.5784e{-}05 & -4.9149e{-}04 \end{bmatrix},$$

$$K_{11} = \begin{bmatrix} 6.4397e{-}07 & -3.1480e{-}07 & 1.5031e{-}07 & 1.4898e{-}01 & 6.2216e{-}01 \\ -1.6038e{-}06 & 5.2095e{-}07 & 2.0366e{-}08 & -9.2219e{-}01 & -5.8183e{-}01 \\ -1.5270e{-}06 & 5.0233e{-}07 & 9.9185e{-}09 & -7.4435e{-}01 & -5.0169e{-}01 \\ 2.7023e{-}06 & -5.8891e{-}07 & -4.6769e{-}07 & -7.3212e{-}02 & 5.2211e{-}01 \\ 1.4203e{-}07 & -3.7757e{-}08 & -1.4377e{-}08 & -2.6454e{-}07 & 2.0176e{-}06 \\ -1.0469e{-}07 & 3.9136e{-}08 & -6.3750e{-}09 & -2.0874e{-}06 & 5.3501e{-}07 \\ -1.6277e{-}05 & 4.7977e{-}06 & 9.4111e{-}07 & -4.0643e{+}00 & -4.7169e{+}00 \end{bmatrix},$$

$$K_{12} = \begin{bmatrix} 7.9867e{-}06 & -1.0667e{-}06 & -2.3933e{-}06 & 1.4900e{-}01 & 6.2213e{-}01 \\ 1.3110e{-}06 & -1.3861e{-}07 & -4.4752e{-}07 & -9.2218e{-}01 & -5.8183e{-}01 \\ 2.2123e{-}06 & -3.6600e{-}07 & -5.5710e{-}07 & -7.4435e{-}01 & -5.0170e{-}01 \\ 6.8617e{-}06 & -8.3633e{-}07 & -2.1763e{-}06 & -7.3192e{-}02 & 5.2209e{-}01 \\ -5.9537e{-}07 & 8.2953e{-}08 & 1.7326e{-}07 & -6.1417e{-}07 & 2.3989e{-}07 \\ 7.5425e{-}08 & 5.0202e{-}08 & -1.1302e{-}07 & -7.2216e{-}07 & 4.5679e{-}07 \\ -2.5564e{-}05 & 3.2152e{-}06 & 7.9590e{-}06 & -4.0644e{+}00 & -4.7169e{+}00 \end{bmatrix},$$

$$K_{13} = \begin{bmatrix} -2.9773e{-06} & -1.0001e{-06} & 2.9893e{-06} & 1.4900e{-01} & 6.2215e{-01} \\ -1.4060e{-06} & 2.2654e{-08} & 6.6895e{-07} & -9.2219e{-01} & -5.8182e{-01} \\ -2.7625e{-07} & -1.6635e{-07} & 3.8759e{-07} & -7.4434e{-01} & -5.0168e{-01} \\ -2.3229e{-06} & -1.0363e{-06} & 2.7163e{-06} & -7.3194e{-02} & 5.2210e{-01} \\ -3.1376e{-07} & -4.3269e{-08} & 2.2179e{-07} & 1.6983e{-06} & 9.4272e{-07} \\ -1.2613e{-07} & 3.2784e{-08} & 1.3889e{-08} & -7.2930e{-07} & -1.0203e{-06} \\ 7.8706e{-06} & 4.4730e{-06} & -1.0647e{-05} & -4.0644e{+00} & -4.7169e{+00} \end{bmatrix},$$

$$K_{14} = \begin{bmatrix} 1.5461e{-06} & -5.6470e{-07} & 7.3658e{-08} & 1.4898e{-01} & 6.2216e{-01} \\ 1.0377e{-06} & -2.1378e{-07} & -1.9812e{-07} & -9.2220e{-01} & -5.8183e{-01} \\ 1.2298e{-06} & -2.9405e{-07} & -1.7378e{-07} & -7.4436e{-01} & -5.0169e{-01} \\ 1.9122e{-06} & -6.3836e{-07} & 1.1197e{-09} & -7.3213e{-02} & 5.2211e{-01} \\ -1.0603e{-06} & 1.4844e{-07} & 3.0749e{-07} & -1.0247e{-06} & 3.9948e{-07} \\ 5.8641e{-07} & -4.9540e{-08} & -2.1891e{-07} & 2.9828e{-07} & -1.9213e{-07} \\ -2.9603e{-06} & 1.5596e{-06} & -8.5753e{-07} & -4.0643e{+00} & -4.7170e{+00} \end{bmatrix}.$$

The other matrices of polytopic PDIUIO are calculated based on these values following the steps of Algorithm 1 and due to space limitation are not presented.

6.3 | Simulation

The presented electrical circuit and the designed polytopic PDIUIO has been simulated. In the simulation, the parameter variations are $\theta_1(t) = 0.5\sin(0.3t)$ and $\theta_2(t) = \cos(0.8t)$. The inputs change according to $u_1(t) = 5 + \cos(0.03t)$ and $u_2(t) = 5 + \sin(0.02t)$. The measurement noise which also corrupts the scheduling parameter measurements is a zero-mean noise with maximum amplitude of 0.1. The disturbance which acts on the actuators (voltage sources) is a zero-mean noise with maximum amplitude of 0.2. Different scenarios are applied to evaluate the performance of the proposed method. In the first scenario, two abrupt actuator and sensor faults are considered as follows:

$$f_a(t) = \begin{cases} 0, & t < 120, \\ 1, & 120 \le t < 320, \\ 0, & 320 \le t, \end{cases} \quad f_s(t) = \begin{cases} 0, & t < 220, \\ 0.8, & 220 \le t < 420, \\ 0, & 420 \le t. \end{cases}$$

The state estimation and fault reconstruction results are depicted in Figures 2 and 3, respectively. From Figure 2, it can be observed that robust state estimation has been done successfully despite the presence of various kinds of unknown inputs such as disturbance, noise, actuator and sensor faults and also the uncertainty due to inexact measures of parameters and unknown delays in the system. Figure 3 shows that the simultaneous actuator and sensor faults have been successfully reconstructed despite the presence of various sources of unknown inputs and uncertainty. It should be noted that even though the two faults are overlapping, the designed polytopic PDIUIO has achieved a robust reconstruction of the two faults with no cross sensitivity between the faults which is very important in the estimation of simultaneous occurring faults.

In the second scenario, two incipient faults occur in the system as follows:

$$f_a(t) = \begin{cases} 0, & t < 120, \\ \frac{t-120}{200}, & 120 \le t < 320, \\ 0, & 320 \le t, \end{cases} \quad f_s(t) = \begin{cases} 0, & t < 220, \\ \frac{t-220}{200}, & 220 \le t < 420, \\ 0, & 420 \le t. \end{cases}$$

The fault reconstruction results are depicted in Figure 4. It can be seen that the incipient faults are robustly estimated despite the presence of unknown inputs and the uncertainty caused by unknown delay and inexact measures of

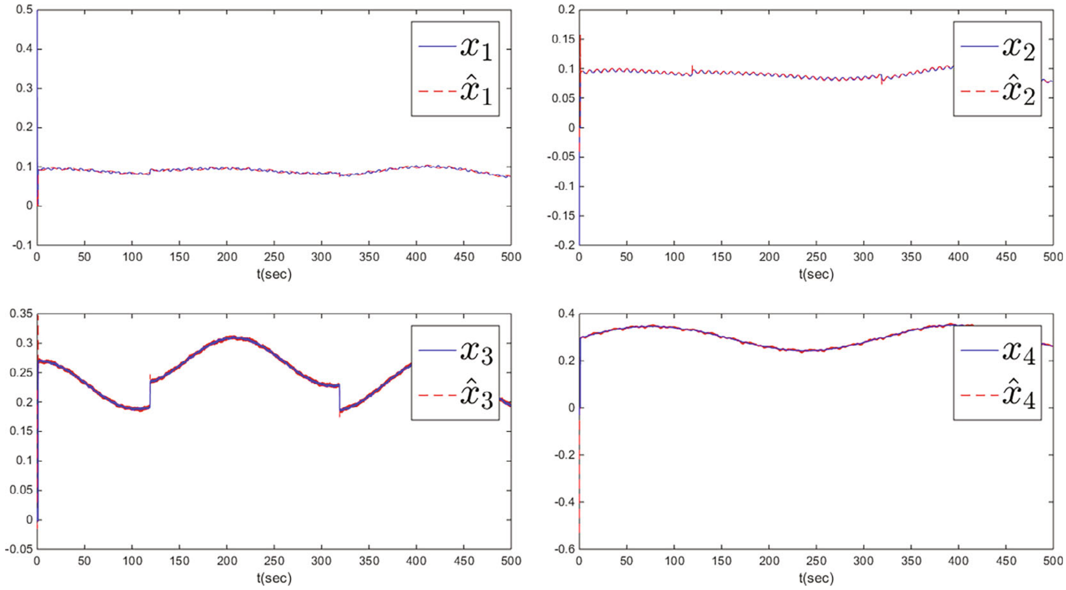

FIGURE 2 System states and their estimates in the first scenario

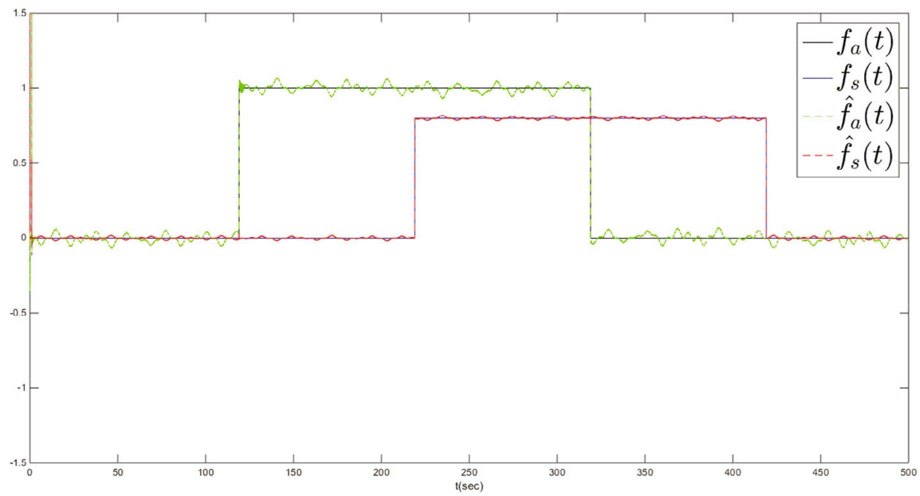

FIGURE 3 Actuator and sensor faults and their estimates in the first scenario

parameters. The estimate of each fault is decoupled from the other fault which is observed in the moments when the two faults are both present in the system. Fault diagnosis (including three phases of detection, isolation and identification) is achieved directly via the provided fault estimates. The early diagnosis of incipient faults as achieved in this scenario is a principal requirement for the successful operation of an active fault tolerant controller.

In the third scenario, two time-varying faults happen in the system as follows:

$$f_a(t) = \begin{cases} 0, & t < 120, \\ \sin(0.05t), & 120 \leq t < 320, \\ 0, & 320 \leq t, \end{cases} \quad f_s(t) = \begin{cases} 0, & t < 220, \\ \sin(0.08t), & 220 \leq t < 420, \\ 0, & 420 \leq t. \end{cases}$$

The fault reconstruction results are depicted in Figure 5. It is seen that the faults have been reconstructed robustly with an acceptable performance in the presence of unknown inputs and uncertainty due to unknown delay and

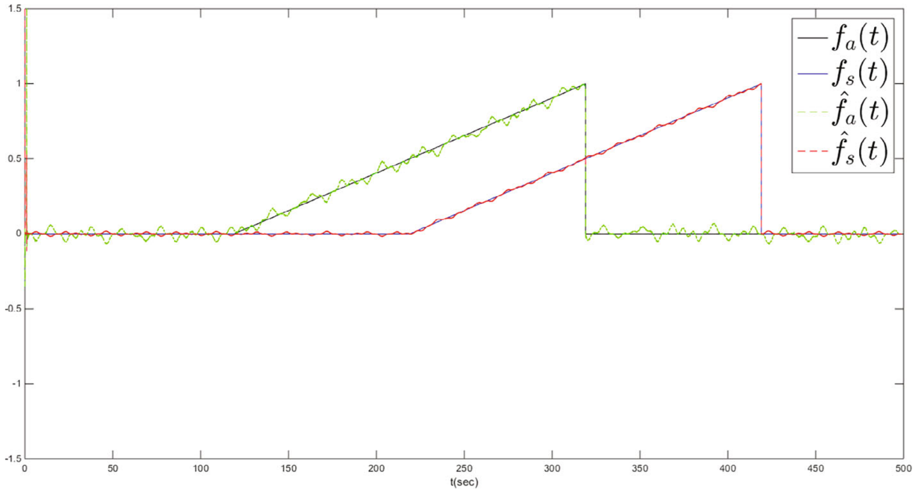

FIGURE 4 Actuator and sensor faults and their estimates in the second scenario

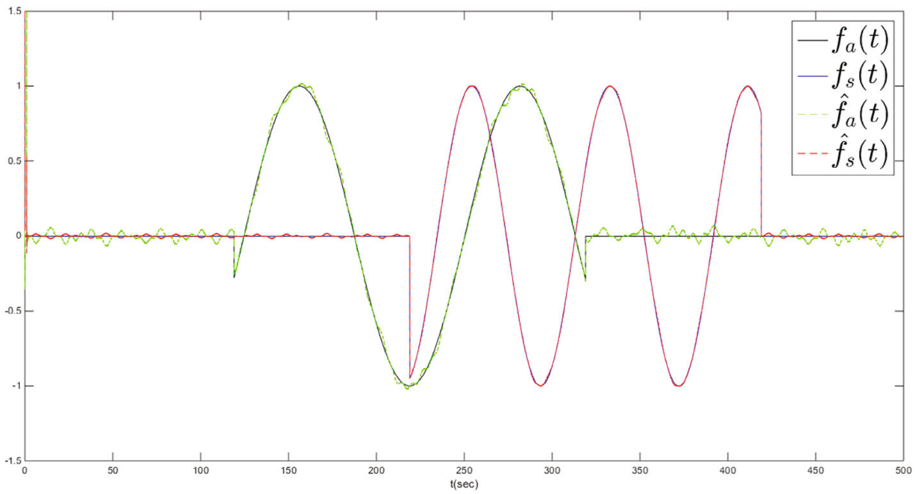

FIGURE 5 Actuator and sensor faults and their estimates in the third scenario

inexactness of parameter measures even in the interval of 220–320 s when the two faults exist simultaneously in the system.

As a fourth scenario, a comparison is done to evaluate the proposed scheme in attenuating the uncertainty caused by inexactness of the scheduling parameters measurements. In this scenario, a PDIUIO is designed based on Corollary 3 for the case of exact measurements of the scheduling parameters but it is applied to a setup similar to the first scenario in which the scheduling parameters measurements are noisy. The result of the actuator and sensor faults reconstruction is depicted in Figure 6. Comparing Figure 6 with Figure 3 reveals that the uncertainty due to inexactness of the scheduling parameters measurements is not attenuated in the fault estimates in Figure 6 and the fault estimates are very noisy reflecting the existent noise in the scheduling parameters measurements. However, these noisy effects are attenuated in Figure 3 and the successfulness of the proposed scheme (Theorem 2) to cope with inexactness of scheduling parameters measurements is evident. The obtained results in the fourth and first scenario are also compared numerically with mean square error (MSE) measure. The MSE of actuator fault estimation in the fourth scenario and in the first scenario are 0.0199 and 0.0017, respectively, that clearly shows the successfulness of Theorem 2 over Corollary 3 in reduction of MSE of actuator fault estimation in the first scenario compared to the fourth scenario. The MSE of sensor fault estimation in the fourth and first scenario are 0.0012 and 0.0002, respectively, that reveals the reduction of MSE in the first scenario compared to the fourth scenario. It should be highlighted that in the case of inexactness of parameter measurements; in addition to lower robustness achieved by a PDIUIO designed based on Corollary 3, the stability will not be guaranteed.

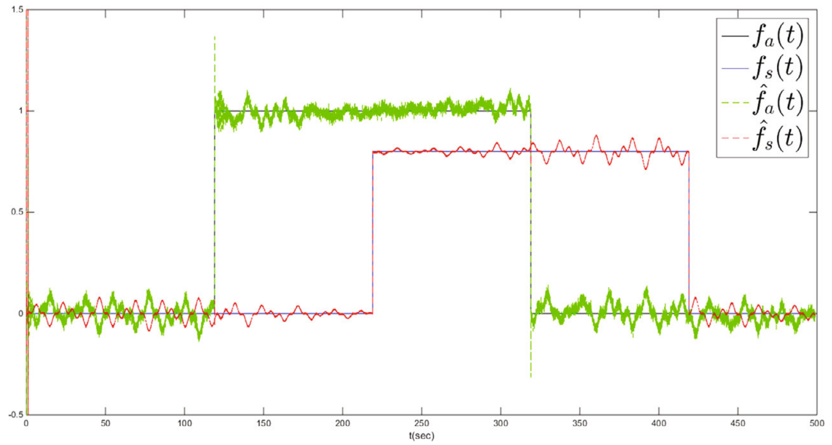

FIGURE 6 Actuator and sensor faults and their estimates in the fourth scenario

7 | CONCLUSION

In this article, a class of SDLPV systems was considered. The considered system is a singular LPV system which has delayed dynamics derived by multiple unknown time varying state delays. Simultaneous state estimation and actuator and sensor fault diagnosis in the presence of disturbance and noise in these systems was achieved by constructing a suitable PDIUIO. To do this, the sensor faults have been used as additional states to construct an augmented system making use of descriptor system approach. The delayed parts of the observer were run by the delay upper bounds instead of the real values which are not available. In the considered problem, scheduling parameters of the system were assumed to be measured inexactly. So, the observer is scheduled with these inexact measured parameters and to deal with this situation, the original system is converted to an uncertain system in order to address the uncertainty due to inexactness of parameter measures. The uncertainty due to unknown delays and inexact parameters plus other unknown inputs is attenuated using H_∞ theory and the design of PDIUIO is formulated as an optimization problem with LMI constraints. The efficiency of the results was demonstrated by an electrical circuit example. Faults in the system are tolerable until the system works in the polytope that the system has been designed in it. Outside this region, the system may become unstable. The characterization of the tolerable fault size for this kind of system is one of the future research trends. Applying the obtained results in this article to design an active fault tolerant controller is part of the future research.

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