ON DELIGNE'S CONJECTURE FOR SYMMETRIC FIFTH L-FUNCTIONS AND QUADRUPLE PRODUCT L-FUNCTIONS OF MODULAR FORMS

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1. Introduction and main results

This report is based on a talk given at the RIMS conference "Automorphic form, automorphic L-functions and related topics" which was held online in January, 2022.

In [Del79], Deligne proposed a remarkable conjecture on the algebraicity of critical values of L-functions of motives, in terms of the periods obtained by comparing the Betti and de Rham realizations of the motives. As special cases, we consider the conjecture for symmetric power L-functions and tensor product L-functions of modular forms.

1.1. **Symmetric power L-functions.** Let

$$f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_\kappa(N,\omega), \quad q = e^{2\pi\sqrt{-1}\,\tau}$$

be a normalized elliptic modular newform of weight $\kappa \geqslant 2$, level N, and nebentypus ω. For each prime $p \nmid N$, denote by α_p, β_p the Satake parameters of f at p and put

$$A_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}.$$

Recall that α_p, β_p are the roots of the Hecke polynomial $X^2 - a_f(p)X + p^{\kappa_i - 1}\omega(p)$. For $n \geqslant 1$, the symmetric n-th power L-function $L(s, \mathrm{Sym}^n(f))$ is defined by an Euler product

$$L(s,\mathrm{Sym}^n(f)) = \prod_p L_p(s,\mathrm{Sym}^n(f)), \quad \mathrm{Re}(s) > 1 + \tfrac{n(\kappa-1)}{2}.$$

Here the Euler factors are defined by

$$L_p(s,\mathrm{Sym}^n(f)) = \det\left(\mathbf{1}_{n+1} - \mathrm{Sym}^n(A_p)\cdot p^{-s}\right)^{-1}$$

for $p \nmid N$, where $\mathrm{Sym}^n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$ is the symmetric n-th power representation. By the result of Barnet-Lamb, Geraghty, Harris, and Taylor [BLGHT11, Theorem B], the symmetric power L-functions admit meromorphic continuation to the whole complex plane and satisfy functional equations relating $L(s, \mathrm{Sym}^n(f))$ to $L(1 + n(\kappa-1) - s, \mathrm{Sym}^n(f^\vee))$, where $f^\vee \in S_\kappa(N, \omega^{-1})$ is the normalized newform dual to f. The archimedean local factors are defined by

$$L_\infty(s,\mathrm{Sym}^n(f)) = \begin{cases} \Gamma_{\mathbb{R}}(s - \frac{n(\kappa-1)}{2}) \prod_{i=0}^{r-1} \Gamma_{\mathbb{C}}(s - i(\kappa-1)) & \text{if } n = 2r \text{ and } r(\kappa-1) \text{ is even,} \\ \Gamma_{\mathbb{R}}(s - \frac{n(\kappa-1)}{2} + 1) \prod_{i=0}^{r-1} \Gamma_{\mathbb{C}}(s - i(\kappa-1)) & \text{if } n = 2r \text{ and } r(\kappa-1) \text{ is odd,} \\ \prod_{i=0}^{r} \Gamma_{\mathbb{C}}(s - i(\kappa-1)) & \text{if } n = 2r+1. \end{cases}$$

Here

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2}), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

A critical point for $L(s, \mathrm{Sym}^n(f))$ is an integer m such that $L_\infty(s, \mathrm{Sym}^n(f))$ and $L_\infty(1+n(\kappa-1)-s, \mathrm{Sym}^n(f^\vee))$ are holomorphic at $s = m$. Associated to f, we have a pure motive M_f over $\mathbb{Q}$ of rank 2 with coefficients in $\mathbb{Q}(f)$, which was constructed by Deligne [Del71] and Scholl [Sch90], such that

$$L(M_f, s) = \left(L(s, {}^\sigma f)\right)_{\sigma:\mathbb{Q}(f)\to\mathbb{C}}.$$

We have the Hodge decomposition

$$H_B(M_f) \otimes_{\mathbb{Q}} \mathbb{C} = H_B^{0,\kappa-1}(M_f) \oplus H_B^{\kappa-1,0}(M_f)$$

as well as the Hodge filtration

$$H_{dR}(M_f) = F^0(M_f) \supsetneq F^{\kappa-1}(M_f) \supsetneq 0.$$

The comparison isomorphism

$$I_\infty : H_B(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H_{dR}(M_f) \otimes_{\mathbb{Q}} \mathbb{C}$$

induces

$$I_\infty^\pm : H_B^\pm(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \hookrightarrow H_B(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H_{dR}(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \twoheadrightarrow H_{dR}(M_f)/F^{\kappa-1}(M_f) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The Deligne's periods of M_f are elements in $(\mathbb{Q}(f) \otimes_{\mathbb{Q}} \mathbb{C})^\times / \mathbb{Q}(f)^\times$ defined by

$$\delta(M_f) := \det(I_\infty), \quad c^\pm(M_f) := \det(I_\infty^\pm),$$

where the determinants are computed with respect to $\mathbb{Q}(f)$-rational bases on both sides. Consider the symmetric power motive $\mathrm{Sym}^n(M_f)$. We have

$$L(\mathrm{Sym}^n(M_f), s) = (L(s, \mathrm{Sym}^n({}^\sigma f)))_{\sigma:\mathbb{Q}(f)\to\mathbb{C}}.$$

In [Del79, Proposition 7.7], Deligne computed the periods of $\mathrm{Sym}^n(M_f)$. More precisely, we have

$$c^\pm(\mathrm{Sym}^n(M_f)) = \begin{cases} \delta(M_f)^{r(r\pm1)/2}(c^+(M_f)c^-(M_f))^{r(r+1)/2} & \text{if } n = 2r, \\ \delta(M_f)^{r(r+1)/2}c^\pm(M_f)^{(r+1)(r+2)/2}c^\mp(M_f)^{r(r+1)/2} & \text{if } n = 2r+1. \end{cases}$$

As a special case of the conjecture in [Del79, Conjecture 2.8], we have the following:

Conjecture 1.1 (Deligne)**.** *Let $m \in \mathbb{Z}$ be a critical point for $\mathrm{Sym}^n(M_f)$. We have*

$$\frac{L(\mathrm{Sym}^n(M_f), m)}{(2\pi\sqrt{-1})^{d^{(-1)^m}(\mathrm{Sym}^n(M_f))m} \cdot c^{(-1)^m}(\mathrm{Sym}^n(M_f))} \in \mathbb{Q}(f),$$

where $d^+(\mathrm{Sym}^n(M_f)) = r+1$, $d^-(\mathrm{Sym}^n(M_f)) = r$ if $n = 2r$, and $d^\pm(\mathrm{Sym}^n(M_f)) = r+1$ if $n = 2r+1$.

The conjecture holds if f is a CM-form. For general f, as explained in [Del79, § 7], the conjecture is known if $n = 1$. It was then considered by various authors when $n = 2, 3, 4, 6$ listed as follows:

- $n = 2$: Sturm [Stu80], [Stu89].
- $n = 3$: Garrett–Harris [GH93] and C.- [Che21a].
- $n = 4, 6$: Morimoto [Mor21] and C.- [Che21b], [Che21c].

In these cases, the conjecture was proved using the integral representations of automorphic L-functions and their algebraic/cohomological interpretations. When $n = 2$, we have the integral representation discovered by Shimura [Shi75]. When $n = 3$, the symmetric cube L-function appears as a factor of the triple product L-function $L(s, f\otimes f\otimes f)$ for which we have the integral representation due to Garrett [Gar87]. For $n = 2, 3$, the ideas for the proof of algebraicity of these integral representations are similar to the ones in the pioneering work of Shimura [Shi76]. The authors consider holomorphic Eisenstein series integrated against complex conjugation of elliptic modular forms. In [Mor21], Morimoto observed that (twisted) symmetric even power L-functions are factors of adjoint L-functions of unitary groups. In [GL21], Grobner and Lin proved a period relation between the Betti–Whittaker periods of cohomological conjugate self-dual cuspidal automorphic representations of GL_N over CM-fields and certain special values of adjoint L-functions of unitary groups. On the other hand, we have the result of Raghuram [Rag10], [Rag16] which expressed the algebraicity of critical values of Rankin–Selberg L-functions for $\mathrm{GL}_N \times \mathrm{GL}_{N-1}$ in terms of product of Betti–Whittaker periods. Therefore, Conjecture 1.1 for $n = 4, 6$ (under some assumptions) then follows from the algebraicity results of Morimoto [Mor14], [Mor18] for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ and Garrett–Harris [GH93] for $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. In [Che21b], based on the same idea, we show that Conjecture 1.1 holds for $n = 4$ when $\kappa \geqslant 3$ by generalizing and refining the results of Grobner–Lin [GL21] to essentially conjugate self-dual representations in the case $\mathrm{GL}_3 \times \mathrm{GL}_2$. In [Che21c], we show that Conjecture 1.1 holds for $n = 6$ when $\kappa \geqslant 6$. We extend the result of Morimoto based on a different approach. The observation is that the (twisted) symmetric sixth power L-function is a factor of the adjoint L-function of the Kim–Ramakrishnan–Shahidi lift of f to GSp_4. We define the de Rham–Whittaker periods associated to globally generic cohomological cuspidal automorphic representations of GSp_4. In the case of the Kim–Ramakrishnan–Shahidi lift, we establish some periods relations between the de Rham–Whittaker periods and powers of Petersson norms of f. The conjecture then follows from our previous results [CI19], [Che22a]. Following is our main result for $n = 5$ (see also Remark 1.3 for higher n):

Theorem 1.2 ([Che22c]). *If $\kappa \geqslant 6$, then Conjecture 1.1 holds.*

Remark 1.3. Recently, we have proved Conjecture 1.1 in [Che22b, Theorem 5.11] when n is odd, κ is odd, and $\kappa \geqslant 5$. It's an ongoing project of the author to prove Conjecture 1.1 when n is even under the same assumptions on κ.

1.2. **Quadruple product L-functions.** As another example of Deligne's conjecture, we consider quadruple product L-functions of modular forms. Let $f_i \in S_{\kappa_i}(N_i, \omega_i)$ be normalized elliptic newform for $i = 1, 2, 3, 4$. Define the quadruple product L-function $L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4)$ by an Euler product

$$L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4) = \prod_p L_p(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4), \quad \mathrm{Re}(s) > 1 + \sum_{i=1}^4 \tfrac{\kappa_i - 1}{2}.$$

Here the Euler factors are given by

$$L_p(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4) = \det\left(\mathbf{1}_{16} - A_{1,p} \otimes A_{2,p} \otimes A_{3,p} \otimes A_{4,p} \cdot p^{-s}\right)^{-1}$$

for $p \nmid N_1N_2N_3N_4$. By the results of Jacquet–Shalika [JS81a], [JS81b] and Ramakrishnan [Ram00], the quadruple product L-function admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating $L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4)$ to $L(1 + \sum_{i=1}^4 (\kappa_i - 1) - s, f_1^\vee \otimes f_2^\vee \otimes f_3^\vee \otimes f_4^\vee)$. For $1 \leqslant i \leqslant 4$, let $G(\omega_i)$ be the Gauss sum of ω_i and $\|f_i\|$ the Petersson norm of f_i defined by

$$\|f_i\| = \mathrm{vol}(\Gamma_0(N_i)\backslash\mathfrak{H})^{-1} \int_{\Gamma_0(N_i)\backslash\mathfrak{H}} |f_i(\tau)|^2 y^{\kappa_i - 2}\, d\tau.$$

Assume $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \kappa_4$. We have three types of critical ranges:

$$\begin{cases} \kappa_1 + \kappa_4 - 1 > \kappa_1 - \kappa_4 + 1 > \kappa_2 + \kappa_3 - 1 > \kappa_2 - \kappa_3 + 1 & \text{Case 1,} \\ \kappa_1 + \kappa_4 - 1 > \kappa_2 + \kappa_3 - 1 > \kappa_1 - \kappa_4 + 1 > \kappa_2 - \kappa_3 + 1 & \text{Case 2,} \\ \kappa_2 + \kappa_3 - 1 > \kappa_1 + \kappa_4 - 1 > \kappa_1 - \kappa_4 + 1 > \kappa_2 - \kappa_3 + 1 & \text{Case 3.} \end{cases}$$

In [Bla87], Blasius explicitly computed Deligne's periods of tensor product motives for GL_2. In particular, we have the following refinement of Deligne's conjecture for the quadruple product L-function:

Conjecture 1.4 (Blasius). *Let $m \in \mathbb{Z}$ be a critical point for $L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4)$. We have*

$$\sigma\left(\frac{L(m, f_1 \otimes f_2 \otimes f_3 \otimes f_4)}{(2\pi\sqrt{-1})^{8m} \cdot c(f_1 \otimes f_2 \otimes f_3 \otimes f_4)}\right) = \frac{L(m, {}^\sigma f_1 \otimes {}^\sigma f_2 \otimes {}^\sigma f_3 \otimes {}^\sigma f_4)}{(2\pi\sqrt{-1})^{8m} \cdot c({}^\sigma f_1 \otimes {}^\sigma f_2 \otimes {}^\sigma f_3 \otimes {}^\sigma f_4)}, \quad \sigma \in \mathrm{Aut}(\mathbb{C}).$$

Here

$$c(f_1 \otimes f_2 \otimes f_3 \otimes f_4) = (2\pi\sqrt{-1})^{4\sum_{i=1}^4 (1-\kappa_i)} \cdot \prod_{i=1}^4 G(\omega_i)^4 \cdot (\pi \cdot \|f_i\|)^{t_i}$$

with

$$(t_1, t_2, t_3, t_4) = \begin{cases} (4,0,0,0) & \textit{Case 1,} \\ (3,1,1,1) & \textit{Case 2,} \\ (2,2,2,0) & \textit{Case 3.} \end{cases}$$

When two of the f_i's are CM by the same imaginary quadratic extension, the quadruple product L-function decomposes into product of triple product L-functions. In this special case, Conjecture 1.4 reduces to Deligne's conjecture for triple product L-functions. For the general case, recently we were able to prove the conjecture under certain parity and regularity conditions on the weights. Following theorem is a special case of [Che22b, Theorem 5.8] ($n = 4$):

Theorem 1.5. *Conjecture 1.4 holds under the following conditions:*

(1) $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$ *is even.*
(2) $|\sum_{i=1}^4 (\varepsilon_i - \varepsilon_i')(\kappa_i - 1)| \geqslant 6$ *for all* $(\varepsilon_1, \cdots, \varepsilon_4)$ *and* $(\varepsilon_1', \cdots, \varepsilon_4')$ *in* $\{\pm 1\}^4$.

2. Sketch of proof

2.1. **Sketch of proof of Theorem 1.2.** Let Π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, where $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$. We say Π is regular algebraic if the infinitesimal character of Π_∞ is regular and belongs to $(\mathbb{Z}+\frac{n+1}{2})^n$. We say Π is tamely isobaric if it is isobaric and the exponents of the summands are the same. First we recall the following theorem which is a consequence of (a variant of) the result of Raghuram [Rag10]. It is an algebraicity result on the ratio of product of critical values of Rankin–Selberg L-functions of regular algebraic tamely isobaric automorphic representations.

Theorem 2.1. *Let Σ, Σ' (resp. Π, Π') be regular algebraic tamely isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ (resp. $\mathrm{GL}_{n'}(\mathbb{A})$) satisfying the following conditions:*

(1) *Σ and Σ' are cuspidal.*
(2) *$n' = n-1$ and $(\Sigma_\infty, \Pi_\infty)$ is balanced.*
(3) *$\Sigma_\infty = \Sigma'_\infty$ and $\Pi_\infty = \Pi'_\infty$.*

Let $m_0 \in \mathbb{Z}+\frac{n+n'}{2}$ be a critical point for $L(s, \Sigma\times\Pi)$ such that $L(m_0, \Sigma\times\Pi')\cdot L(m_0, \Sigma'\times\Pi) \neq 0$. Then, for $\sigma\in\mathrm{Aut}(\mathbb{C})$, we have

$$\sigma\left(\frac{L(m_0, \Sigma\times\Pi)\cdot L(m_0, \Sigma'\times\Pi')}{L(m_0, \Sigma\times\Pi')\cdot L(m_0, \Sigma'\times\Pi)}\right) = \frac{L(m_0, {}^\sigma\Sigma\times{}^\sigma\Pi)\cdot L(m_0, {}^\sigma\Sigma'\times{}^\sigma\Pi')}{L(m_0, {}^\sigma\Sigma\times{}^\sigma\Pi')\cdot L(m_0, {}^\sigma\Sigma'\times{}^\sigma\Pi)}.$$

Remark 2.2. In practice, conditions (1) and (2) are too strong for application. In [Che22b, Theorem 1.2], we remove conditions (1) and (2). Instead, we impose some parity and regularity conditions on Σ_∞ and Π_∞.

Back to our normalized newform $f \in S_\kappa(N, \omega)$. We may assume that f is not a CM-form. Let $\Pi(f)$ be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f (it is unique up to twisting by integral powers of the adelic absolute value $|\ |_\mathbb{A}$). For $n \geqslant 1$, let $\mathrm{Sym}^n\Pi(f)$ be the functorial lift of $\Pi(f)$ to $\mathrm{GL}_{n+1}(\mathbb{A})$ with respect to the symmetric n-th power representation of $\mathrm{GL}_2(\mathbb{C})$. The existence of the lifts was proved by Newton and Thorne [NT21a], [NT21b] (see also [GJ72], [KS02], [Kim03], [CT15], [CT17] for $n \leqslant 8$). It is easy to see that $\mathrm{Sym}^n\Pi(f)$ is regular algebraic and tamely isobaric. Since we assumed that f is not a CM-form, $\mathrm{Sym}^n\Pi(f)$ is cuspidal. To prove Conjecture 1.1 for $n=5$, we apply Theorem 2.1 in the case $\mathrm{GL}_4\times\mathrm{GL}_3$. More precisely, let Σ and Π be regular algebraic cuspidal automorphic representations of $\mathrm{GL}_4(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$ respectively defined by

$$\Sigma = \mathrm{Sym}^3\Pi(f), \quad \Pi = \mathrm{Sym}^2\Pi(f).$$

One can verify easily that $(\Sigma_\infty, \Pi_\infty)$ is balanced (cf. [Rag10, Theorem 5.3]). For a cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$, let $\Pi(f)\boxtimes\tau$ be the functorial lift of the Rankin–Selberg convolution of $\Pi(f)$ and τ to $\mathrm{GL}_4(\mathbb{A})$. The existence of the lift was proved by Ramakrishnan in [Ram00]. We assume further τ is chosen so that:

- τ is regular algebraic and non-CM.
- $(\Pi(f)_\infty\boxtimes\tau_\infty)\otimes|\ |_\infty^{-1/2} = \Sigma_\infty$.

We also choose an algebraic Hecke character of $\mathbb{A}^\times$ such that

$$(\tau_\infty\otimes|\ |_\infty^{-1/2})\boxplus\chi_\infty = \Pi_\infty.$$

Let Σ' and Π' be isobaric automorphic representations of $\mathrm{GL}_4(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$ respectively defined by

$$\Sigma' = (\Pi(f)\boxtimes\tau)\otimes|\ |_\mathbb{A}^{-1/2}, \quad \Pi' = (\tau\otimes|\ |_\mathbb{A}^{-1/2})\boxplus\chi.$$

By our assumptions on τ and χ, it is easy to see that Σ' (resp. Π') is regular algebraic and cuspidal (resp. tamely isobaric), and $\Sigma_\infty = \Sigma'_\infty$, $\Pi_\infty = \Pi'_\infty$. Therefore, by Theorem 2.1, for all non-central critical points $m+\frac{1}{2}\in\mathbb{Z}+\frac{1}{2}$ for $L(s, \Sigma\times\Pi)$, we have

$$L(m+\tfrac{1}{2}, \Sigma\times\Pi) \sim \frac{L(m+\frac{1}{2}, \Sigma\times\Pi')\cdot L(m+\frac{1}{2}, \Sigma'\times\Pi)}{L(m+\frac{1}{2}, \Sigma'\times\Pi')}. \tag{2.1}$$

Here $\sim$ means the ratio of left-hand side by right-hand side is equivariant under $\mathrm{Aut}(\mathbb{C})$. On the other hand, we have the following factorizations of L-functions:

$$
\begin{aligned}
L(s, \Sigma \times \Pi) &= L(s, \mathrm{Sym}^5 \Pi(f)) \cdot L(s, \mathrm{Sym}^3 \Pi(f) \otimes \omega_{\Pi(f)}) \cdot L(s, \Pi(f) \otimes \omega^2_{\Pi(f)}),\\
L(s, \Sigma \times \Pi') &= L(s - \tfrac{1}{2}, \mathrm{Sym}^3 \Pi(f) \times \tau) \cdot L(s, \mathrm{Sym}^3 \Pi(f) \otimes \chi),\\
L(s, \Sigma' \times \Pi) &= L(s - \tfrac{1}{2}, \mathrm{Sym}^3 \Pi(f) \times \tau) \cdot L(s - \tfrac{1}{2}, \Pi(f) \times \tau \otimes \omega_{\Pi(f)}),\\
L(s, \Sigma' \times \Pi') &= L(s-1, \Pi(f) \times \tau \times \tau) \cdot L(s - \tfrac{1}{2}, \Pi(f) \times \tau \otimes \chi).
\end{aligned} \tag{2.2}
$$

Here $\omega_{\Pi(f)}$ is the central character of $\Pi(f)$. By the result of Shimura [Shi76], Deligne's conjecture holds for $L(s, \Pi(f) \times \tau \otimes \omega_{\Pi(f)})$ and $L(s, \Pi(f) \times \tau \otimes \chi)$. By the results of Garrett–Harris [GH93] and the author [Che21a], Deligne's conjecture holds for the triple product L-function $L(s, \Pi(f) \times \tau \times \tau)$. When $\kappa \geqslant 3$, Deligne's conjecture also holds for $L(s, \mathrm{Sym}^3 \Pi(f) \otimes \omega_{\Pi(f)})$ (cf. [Che21a, Theorem 1.6]). Consider the descent of $\mathrm{Sym}^3 \Pi(f)$ to $\mathrm{GSp}_4(\mathbb{A})$. By the results of Morimoto [Mor14], [Mor18] and the author [Che21b], we see that Deligne's conjecture holds for $L(s, \mathrm{Sym}^3 \Pi(f) \times \tau)$ when $\kappa \geqslant 6$. We then conclude from (2.1) that Conjecture 1.1 for $n = 5$ holds for non-central critical points. Indeed, it is easy to deduce from (2.1) and Deligne's conjecture for the L-functions on the right-hand sides of (2.2) (except for $\mathrm{Sym}^5 \Pi(f)$) that $L^{(\infty)}(m + \frac{1}{2}, \mathrm{Sym}^5 \Pi(f))$ is equivalent to some integral powers of $2\pi\sqrt{-1}$, $\delta(f)$, and $c^{\pm}(f)$. A straightforward computation shows that the exponents do coincide with the expected ones. For the central critical point, Conjecture 1.1 follows from the non-central critical points together with the result of Harder–Raghuram [HR20].

2.2. **Sketch of proof of Theorem 1.5.** The idea of the proof is similar as above. We apply Theorem 2.1 in the case $\mathrm{GL}_4 \times \mathrm{GL}_4$ (cf. Remark 2.2). Let Π_i be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f_i for $i = 1, 2, 3, 4$. Let Σ and Π be the regular algebraic tamely isobaric automorphic representations of $\mathrm{GL}_4(\mathbb{A})$ defined by

$$\Sigma = (\Pi_1 \boxtimes \Pi_2) \otimes |\ |_{\mathbb{A}}^{1/2}, \quad \Pi = (\Pi_3 \boxtimes \Pi_4) \otimes |\ |_{\mathbb{A}}^{-1/2}.$$

Let $\Pi_1', \Pi_2', \Pi_3', \Pi_4'$ be auxiliary regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ such that

$$\Pi'_{1,\infty} \boxplus \Pi'_{2,\infty} = \Sigma_\infty, \quad \Pi'_{3,\infty} \boxplus \Pi'_{4,\infty} = \Pi_\infty.$$

Let Σ' and Π' be the regular algebraic tamely isobaric automorphic representations of $\mathrm{GL}_4(\mathbb{A})$ defined by

$$\Sigma' = \Pi_1' \boxplus \Pi_2', \quad \Pi' = \Pi_3' \boxplus \Pi_4'.$$

The assumptions (1) and (2) in Theorem 1.5 then implies that Theorem 2.1 holds in our case. Therefore, we have

$$L(m, \Sigma \times \Pi) \sim \frac{L(m, \Sigma \times \Pi') \cdot L(m, \Sigma' \times \Pi)}{L(m, \Sigma' \times \Pi')} \tag{2.3}$$

for all critical points $m \in \mathbb{Z}$ for $L(s, \Sigma \times \Pi)$. On the other hand, we have the following factorizations of L-functions:

$$
\begin{aligned}
L(s, \Sigma \times \Pi) &= L(s, \Pi_1 \times \Pi_2 \times \Pi_3 \times \Pi_4),\\
L(s, \Sigma \times \Pi') &= L(s + \tfrac{1}{2}, \Pi_1 \times \Pi_2 \times \Pi_3') \cdot L(s + \tfrac{1}{2}, \Pi_1 \times \Pi_2 \times \Pi_4'),\\
L(s, \Sigma' \times \Pi) &= L(s - \tfrac{1}{2}, \Pi_3 \times \Pi_4 \times \Pi_1') \cdot L(s - \tfrac{1}{2}, \Pi_3 \times \Pi_4 \times \Pi_2'),\\
L(s, \Sigma' \times \Pi') &= L(s, \Pi_1' \times \Pi_3') \cdot L(s, \Pi_1' \times \Pi_4') \cdot L(s, \Pi_2' \times \Pi_3') \cdot L(s, \Pi_2' \times \Pi_4').
\end{aligned} \tag{2.4}
$$

By the result of Shimura [Shi76], we known that Deligne's conjecture holds for the Rankin–Selberg L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Therefore, by (2.3), we are reduced to show that Deligne's conjecture holds for the triple product L-functions appear on the right-hand sides of (2.4). For these triple product L-functions, we can play the same trick as above. This time apply Theorem 2.1 in the case $\mathrm{GL}_4 \times \mathrm{GL}_2$.

Acknowledgement. The authors would like to thank Professor Kazuki Morimoto and Professor Tadashi Miyazaki for giving him the opportunity to give a talk at the conference.

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