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Whittaker functions on  $GL(4, \mathbb{R})$  and archimedean zeta integrals (Automorphic form, automorphic  $L$ -functions and related topics)

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# Whittaker functions on $GL(4, \mathbb{R})$ and archimedean zeta integrals

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## 1 Introduction

Let  $F = \mathbb{R}$  or  $\mathbb{C}$ . Explicit formulas of Whittaker functions on  $GL(2, F)$  and  $GL(3, F)$  and related archimedean zeta integrals have been studied by several authors (cf. [2]). In this note we give explicit formulas of Whittaker functions on  $G = GL(4, \mathbb{R})$  and its application to archimedean zeta integrals. Since the case of principal series are already done in [4], we consider the remaining cases  $-P_{(2,1,1)}$  and  $P_{(2,2)}$ -principal series. As in our previous work [2], we derive a system of partial differential equations satisfied by Whittaker functions, and give Mellin-Barnes integral representations of moderate growth solutions. By using these explicit formulas, we compute the archimedean parts of Bump-Friedberg zeta integrals ([1]) to give test vectors.

## 2 Basic notation

We define subgroups  $N$ ,  $A$  and  $K$  of  $G = GL(4, \mathbb{R})$  by

$$\begin{aligned} N &= \{x = (x_{i,j}) \in G \mid x_{i,i} = 1 \ (1 \leq i \leq 4), \ x_{j,k} = 0 \ (1 \leq k < j \leq 4)\}, \\ A &= \{y = \text{diag}(y_1 y_2 y_3 y_4, \ y_2 y_3 y_4, \ y_3 y_4, \ y_4) \mid y_1, y_2, y_3, y_4 > 0\}, \\ K &= O(4). \end{aligned}$$

Then we have an Iwasawa decomposition  $G = NAK$ . Let  $\mathfrak{g}$ ,  $\mathfrak{n}$ ,  $\mathfrak{a}$  and  $\mathfrak{k}$  be the associated Lie algebras of  $G$ ,  $N$ ,  $A$  and  $K$ , respectively. Let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $\mathfrak{g}$  with respect to the Killing form, that is,  $\mathfrak{p} = \{X \in \mathfrak{g} \mid {}^t X = X\}$ . For  $1 \leq i, j \leq 4$ , the symbol  $E_{i,j}$  denotes the matrix in  $\mathfrak{g}$  with 1 at the  $(i, j)$ -th entry and 0 at other entries. We set  $E_{i,j}^{\mathfrak{k}} = E_{i,j} - E_{j,i}$  and  $E_{i,j}^{\mathfrak{p}} = E_{i,j} + E_{j,i}$  ( $1 \leq i, j \leq 4$ ). Then  $\{E_{i,j} \mid 1 \leq i, j \leq 4\}$ ,  $\{E_{i,j}^{\mathfrak{k}} \mid 1 \leq i < j \leq 4\}$  and  $\{E_{i,j}^{\mathfrak{p}} \mid 1 \leq i \leq j \leq 4\}$  are bases of  $\mathfrak{g}$ ,  $\mathfrak{k}$  and  $\mathfrak{p}$ , respectively.

Let  $\mathfrak{g}_{\mathbb{C}}$  be the complexification of  $\mathfrak{g}$ , and  $U(\mathfrak{g}_{\mathbb{C}})$  the universal enveloping algebra of  $\mathfrak{g}_{\mathbb{C}}$ . We define a matrix  $\mathcal{E} = (\mathcal{E}_{i,j})_{1 \leq i,j \leq 4}$  of size 4 with entries in  $U(\mathfrak{g}_{\mathbb{C}})$  by

$$\mathcal{E}_{i,j} = \begin{cases} E_{i,i} - \frac{5-2i}{2} & \text{if } i = j, \\ E_{i,j} & \text{if } i \neq j. \end{cases}$$

We define the Capelli elements  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$  by the identity

$$\text{Det}(t1_4 + \mathcal{E}) = t^4 + \mathcal{C}_1 t^3 + \mathcal{C}_2 t^2 + \mathcal{C}_3 t + \mathcal{C}_4$$

in a variable  $t$ . Here  $\text{Det}$  means the vertical determinant defined by

$$\text{Det}(X) = \sum_{w \in \mathfrak{S}_4} \text{sgn}(w) X_{1,w(1)} X_{2,w(2)} X_{3,w(3)} X_{4,w(4)}, \quad X = (X_{i,j})_{1 \leq i,j \leq 4}$$

with the symmetric group  $\mathfrak{S}_4$  of degree 4. It is known that the Capelli elements  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$  generate the center  $Z(\mathfrak{g}_{\mathbb{C}})$  of  $U(\mathfrak{g}_{\mathbb{C}})$  as a  $\mathbb{C}$ -algebra.

We define a character  $\psi_{(c_1, c_2, c_3)}$  of  $N$  by

$$\psi_{(c_1, c_2, c_3)}(x) = \exp\{2\pi\sqrt{-1}(c_1 x_{1,2} + c_2 x_{2,3} + c_3 x_{3,4})\} \quad (x = (x_{i,j}) \in N)$$

for  $(c_1, c_2, c_3) \in \mathbb{R}^3$ . Then unitary characters of  $N$  are exhausted by the characters of this form. We say that  $\psi_{(c_1, c_2, c_3)}$  is non-degenerate if  $(c_1, c_2, c_3) \in (\mathbb{R}^\times)^3$ . For  $c \in \mathbb{R}$ , we denote  $\psi_{(c,c,c)}$  simply by  $\psi_c$ .

We regard  $C^\infty(G)$  as a  $G$ -module via the right translation. For a non-degenerate character  $\psi$  of  $N$ , let  $C^\infty(N \backslash G; \psi)$  be the subspace of  $C^\infty(G)$  consisting of all functions  $f$  satisfying  $f(xg) = \psi(x)f(g)$  ( $x \in N, g \in G$ ). For an admissible representation  $(\Pi, H_\Pi)$  of  $G$ , let

$$\mathcal{I}_{\Pi, \psi} = \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(H_{\Pi, K}, C^\infty(N \backslash G; \psi)_K).$$

Here  $H_{\Pi, K}$  and  $C^\infty(N \backslash G; \psi)_K$  are the subspaces of  $H_\Pi$  and  $C^\infty(N \backslash G; \psi)$  consisting of all  $K$ -finite vectors, respectively. We define the subspace  $\mathcal{I}_{\Pi, \psi}^{\text{mg}}$  of  $\mathcal{I}_{\Pi, \psi}$  consisting of all homomorphisms  $\Phi$  such that  $\Phi(f)$  ( $f \in H_{\Pi, K}$ ) are moderate growth functions. We define the space  $\text{Wh}(\Pi, \psi)$  of Whittaker functions for  $(\Pi, \psi)$  by

$$\text{Wh}(\Pi, \psi) = \mathbb{C}\text{-span}\{\Phi(f) \mid f \in H_{\Pi, K}, \Phi \in \mathcal{I}_{\Pi, \psi}\},$$

and define the subspace  $\text{Wh}(\Pi, \psi)^{\text{mg}}$  of  $\text{Wh}(\Pi, \psi)$  by

$$\text{Wh}(\Pi, \psi)^{\text{mg}} = \mathbb{C}\text{-span}\{\Phi(f) \mid f \in H_{\Pi, K}, \Phi \in \mathcal{I}_{\Pi, \psi}^{\text{mg}}\}.$$

Let  $\varphi: V_\tau \rightarrow \text{Wh}(\Pi, \psi)$  be a  $K$ -embedding with a  $K$ -type  $(\tau, V_\tau)$  of  $\Pi$ . By definition, we have

$$\varphi(v)(xgk) = \psi(x)\varphi(\tau(k)v)(g) \quad (v \in V_\tau, x \in N, g \in G, k \in K).$$

Because of the Iwasawa decomposition  $G = NAK$ ,  $\varphi$  is characterized by its restriction  $v \mapsto \varphi(v)|_A$  to  $A$ . We call  $v \mapsto \varphi(v)|_A$  the radial part of  $\varphi$ .

Assume that  $\Pi$  is irreducible. Then the multiplicity one theorem tells that the intertwining space  $\mathcal{I}_{\Pi, \psi}^{\text{mg}}$  is at most one dimensional. It is known that  $\mathcal{I}_{\Pi, \psi} \neq 0$  if and only if  $\Pi$  is large in the sense of Vogan [5].

For  $(c_1, c_2, c_3) \in (\mathbb{R}^\times)^3$ , there is a  $G$ -isomorphism

$$\Xi_{(c_1, c_2, c_3)}: C^\infty(N \backslash G; \psi_1) \rightarrow C^\infty(N \backslash G; \psi_{(c_1, c_2, c_3)})$$

defined by  $\Xi_{(c_1, c_2, c_3)}(f)(g) = f(\text{diag}(c_1 c_2 c_3, c_2 c_3, c_3, 1)g)$  ( $g \in G$ ). Hence, it suffices to consider the case of  $\psi_1$ . In this note, we give explicit formulas of the radial part of a  $K$ -embedding  $\varphi: V_\tau \rightarrow \text{Wh}(\Pi, \psi_1)^{\text{mg}}$  for an irreducible admissible large representation  $\Pi$  of  $G$  and the minimal  $K$ -type  $(\tau, V_\tau)$  of  $\Pi$ .

### 3 Representation theory of $K$

Let us briefly explain a way of construction of irreducible representation of  $K$ . We define a representation  $(\tau_{\text{st}}, V_{\text{st}})$  of  $K$  by  $V_{\text{st}} = M_{4,1}(\mathbb{C}) \simeq \mathbb{C}^4$  and  $\tau_{\text{st}}(h)v = hv$  ( $h \in K$ ,  $v \in V_{\text{st}}$ ). Here  $hv$  is the ordinal product of matrices  $h$  and  $v$ . For  $1 \leq i \leq 4$ , the symbol  $\xi_i$  denotes the matrix unit in  $V_{\text{st}} = M_{4,1}(\mathbb{C})$  with 1 at  $(i, 1)$ -th entry and 0 at other entries. For  $1 \leq i, j \leq 4$ , we define  $\xi_{ij} = \xi_i \wedge \xi_j \in V_{\text{st}} \wedge_{\mathbb{C}} V_{\text{st}}$ . We define the graded  $\mathbb{C}$ -algebra  $\mathcal{R} = \bigoplus_{\lambda_1 \geq \lambda_2 \geq 0} \mathcal{R}_{(\lambda_1, \lambda_2)}$  by

$$\mathcal{R} = \text{Sym}(V_{\text{st}}) \otimes_{\mathbb{C}} \text{Sym}(V_{\text{st}} \wedge_{\mathbb{C}} V_{\text{st}}), \quad \mathcal{R}_\lambda = \text{Sym}^{\lambda_1 - \lambda_2}(V_{\text{st}}) \otimes_{\mathbb{C}} \text{Sym}^{\lambda_2}(V_{\text{st}} \wedge_{\mathbb{C}} V_{\text{st}}).$$

Here  $\text{Sym}(V) = \bigoplus_{m \geq 0} \text{Sym}^m(V)$  is the symmetric algebra on  $V$  with the usual grading for a  $\mathbb{C}$ -vector space  $V$ . We regard  $\mathcal{R}$  as a  $K$ -module via the action  $\mathcal{T}$  which is induced from  $\tau_{\text{st}}$ . Then  $\mathcal{R}_{(\lambda_1, \lambda_2)}$  is a  $K$ -submodule of  $\mathcal{R}$ .

For  $v \in V_{\text{st}}$  and  $v' \in V_{\text{st}} \wedge_{\mathbb{C}} V_{\text{st}}$ , the elements  $v \otimes 1$  and  $1 \otimes v'$  of  $\mathcal{R}$  are denoted simply by  $v$  and  $v'$ , respectively. Then we note that  $\{\xi_i \mid 1 \leq i \leq 4\} \cup \{\xi_{jk} \mid 1 \leq j < k \leq 4\}$  is a system of generators of  $\mathcal{R}$  as a  $\mathbb{C}$ -algebra.

For  $1 \leq i, j, k \leq 4$ , we define elements  $\widehat{\xi}, \widehat{\xi}_i, \widehat{\xi}_{ijk}, \widehat{\xi}_{ij}, \widehat{\xi}_{1234}$  of  $\mathcal{R}$  by

$$\begin{aligned} \widehat{\xi} &= (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 + (\xi_4)^2, & \widehat{\xi}_{ijk} &= \xi_i \xi_{jk} - \xi_j \xi_{ik} + \xi_k \xi_{ij}, \\ \widehat{\xi}_i &= \xi_1 \xi_{i1} + \xi_2 \xi_{i2} + \xi_3 \xi_{i3} + \xi_4 \xi_{i4}, & \widehat{\xi}_{1234} &= \xi_{12} \xi_{34} - \xi_{13} \xi_{24} + \xi_{14} \xi_{23}. \\ \widehat{\xi}_{ij} &= \xi_{i1} \xi_{j1} + \xi_{i2} \xi_{j2} + \xi_{i3} \xi_{j3} + \xi_{i4} \xi_{j4}, \end{aligned}$$

Let  $I_{\mathcal{R}}$  be the ideal of  $\mathcal{R}$  generated by

$$\{\widehat{\xi}, \widehat{\xi}_{1234}\} \cup \{\widehat{\xi}_i \mid 1 \leq i \leq 4\} \cup \{\widehat{\xi}_{ij} \mid 1 \leq i \leq j \leq 4\} \cup \{\widehat{\xi}_{ijk} \mid 1 \leq i < j < k \leq 4\}.$$

We can show that the ideal  $I_{\mathcal{R}}$  is  $K$ -invariant. The action of  $K$  on  $\mathcal{R}/I_{\mathcal{R}}$  induced from  $\mathcal{T}$  is denoted by  $\widehat{\mathcal{T}}$ . Let  $\mathfrak{q}_{\mathcal{R}}: \mathcal{R} \ni r \mapsto r + I_{\mathcal{R}} \in \mathcal{R}/I_{\mathcal{R}}$  be the natural surjection. Let  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^2 \times \{0, 1\}$  with  $\lambda_1 \geq \lambda_2 \geq 0$ . We define a representation  $(\tau_\lambda, V_\lambda)$  of  $K$  by

$$\tau_\lambda(k) = (\det k)^{\lambda_3} \widehat{\mathcal{T}}(k) \quad (k \in K), \quad V_\lambda = \mathfrak{q}_{\mathcal{R}}(\mathcal{R}_{(\lambda_1, \lambda_2)}).$$

Then we can show the following:

**Proposition 1.** *Let*

$$\Lambda_K = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^2 \times \{0, 1\} \mid \lambda_1 \geq \lambda_2 \geq 0, \lambda_2 \lambda_3 = 0\}.$$

*The correspondence  $\lambda \leftrightarrow \tau_\lambda$  gives a bijection between  $\Lambda_K$  and the set of equivalence classes of irreducible representations of  $K$ .*

Let  $S_\lambda$  be the set of  $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in (\mathbb{Z}_{\geq 0})^{10}$  satisfying

$$l_1 + l_2 + l_3 + l_4 = \lambda_1 - \lambda_2, \quad l_{12} + l_{13} + l_{14} + l_{23} + l_{24} + l_{34} = \lambda_2.$$

For  $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_\lambda$ , we set

$$u_l = \mathfrak{qR} \left( \prod_{1 \leq i \leq 4} (\xi_i)^{l_i} \prod_{1 \leq j < k \leq 4} (\xi_{jk})^{l_{jk}} \right).$$

We note that  $\{u_l\}_{l \in S_\lambda}$  forms a system of generators of  $V_\lambda$  as a  $\mathbb{C}$ -vector space. It is convenient to set  $u_l = 0$  if  $l \notin (\mathbb{Z}_{\geq 0})^{10}$ . We set  $\mathbf{0} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$  and

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0, 0, 0, 0, 0, 0), & e_2 &= (0, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\ e_3 &= (0, 0, 1, 0, 0, 0, 0, 0, 0, 0), & e_4 &= (0, 0, 0, 1, 0, 0, 0, 0, 0, 0), \\ e_{12} = e_{21} &= (0, 0, 0, 0, 1, 0, 0, 0, 0, 0), & e_{13} = e_{31} &= (0, 0, 0, 0, 0, 1, 0, 0, 0, 0), \\ e_{14} = e_{41} &= (0, 0, 0, 0, 0, 0, 1, 0, 0, 0), & e_{23} = e_{32} &= (0, 0, 0, 0, 0, 0, 0, 1, 0, 0), \\ e_{24} = e_{42} &= (0, 0, 0, 0, 0, 0, 0, 0, 1, 0), & e_{34} = e_{43} &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 1). \end{aligned}$$

For  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda_K$ , we have the following relations.

- When  $\lambda_1 - \lambda_2 \geq 2$ , for  $l \in S_{\lambda - (2, 0, 0)}$ , we have

$$u_{l+2e_1} + u_{l+2e_2} + u_{l+2e_3} + u_{l+2e_4} = 0.$$

- When  $\lambda_1 > \lambda_2 > 0$ , for  $l \in S_{\lambda - (2, 1, 0)}$ , we have

$$\begin{aligned} \sum_{1 \leq j \leq 4, j \neq i} \operatorname{sgn}(j - i) u_{l+e_j+e_{ij}} &= 0 \quad (1 \leq i \leq 4), \\ u_{l+e_i+e_{jk}} - u_{l+e_j+e_{ik}} + u_{l+e_k+e_{ij}} &= 0 \quad (1 \leq i < j < k \leq 4). \end{aligned}$$

- When  $\lambda_2 \geq 2$ , for  $l \in S_{\lambda - (2, 2, 0)}$ , we have

$$\begin{aligned} \sum_{1 \leq k \leq 4, k \notin \{i, j\}} \operatorname{sgn}((k - i)(k - j)) u_{l+e_{ik}+e_{jk}} &= 0 \quad (1 \leq i, j \leq 4), \\ u_{l+e_{12}+e_{34}} - u_{l+e_{13}+e_{24}} + u_{l+e_{14}+e_{23}} &= 0. \end{aligned}$$

## 4 Generalized principal series representations of $G$

We recall the definition of generalized principal series representations of  $G$  and the associated  $L$ - and  $\varepsilon$ -factors. We specify certain representations of  $G_1 = \mathrm{GL}(1, \mathbb{R})$  and  $G_2 = \mathrm{GL}(2, \mathbb{R})$  as follows:

- For  $\nu \in \mathbb{C}$  and  $\delta \in \{0, 1\}$ , we define a character  $\chi_{(\nu, \delta)}$  of  $G_1$  by  $\chi_{(\nu, \delta)}(t) = \mathrm{sgn}(t)^\delta |t|^\nu$  ( $t \in G_1$ ).
- For  $\nu \in \mathbb{C}$  and  $\kappa \in \mathbb{Z}_{\geq 2}$ , let  $D_{(\nu, \kappa)}$  be an irreducible Hilbert representation of  $G_2$  such that  $D_{(\nu, \kappa)}(t1_2) = t^{2\nu}$  ( $t \in \mathbb{R}_+$ ) and  $D_{(\nu, \kappa)} \simeq D_\kappa^+ \oplus D_\kappa^-$  as  $(\mathfrak{sl}(2, \mathbb{R}), \mathrm{SO}(2))$ -modules, where  $D_\kappa^\pm$  is the discrete series representations of  $\mathrm{SL}(2, \mathbb{R})$  with the minimal  $\mathrm{SO}(2)$ -type:  $\mathrm{SO}(2) \ni \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mapsto e^{\pm \sqrt{-1} \kappa \theta} \in \mathbb{C}^\times$ .

For  $\mathbf{n} \in \{(1, 1, 1, 1), (2, 1, 1), (2, 2)\}$ , we associate the block upper triangular parabolic subgroup  $P_{\mathbf{n}} = N_{\mathbf{n}} M_{\mathbf{n}}$  of  $G$  as usual manner. Here  $N_{\mathbf{n}}$  and  $M_{\mathbf{n}}$  are the unipotent radical and the Levi part of  $P_{\mathbf{n}}$ , respectively. Because of Vogan's characterization, any irreducible admissible large representation  $\Pi$  of  $G$  is infinitesimally equivalent to some  $\Pi_\sigma := \mathrm{Ind}_{P_{\mathbf{n}}}^G(\sigma)$ , which is induced from one of the following representations:

- Case 1 ( $\mathbf{n} = (1, 1, 1, 1)$ ):  
 $\sigma = \chi_{(\nu_1, \delta_1)} \boxtimes \chi_{(\nu_2, \delta_2)} \boxtimes \chi_{(\nu_3, \delta_3)} \boxtimes \chi_{(\nu_4, \delta_4)}$  where  $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{C}$  and  $\delta_1, \delta_2, \delta_3, \delta_4 \in \{0, 1\}$  with  $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4$ .
- Case 2 ( $\mathbf{n} = (2, 1, 1)$ ):  
 $\sigma = D_{(\nu_1, \kappa_1)} \boxtimes \chi_{(\nu_2, \delta_2)} \boxtimes \chi_{(\nu_3, \delta_3)}$  where  $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$ ,  $\kappa_1 \in \mathbb{Z}_{\geq 2}$  and  $\delta_2, \delta_3 \in \{0, 1\}$  with  $\delta_2 \geq \delta_3$ .
- Case 3 ( $\mathbf{n} = (2, 2)$ ):  
 $\sigma = D_{(\nu_1, \kappa_1)} \boxtimes D_{(\nu_2, \kappa_2)}$  where  $\nu_1, \nu_2 \in \mathbb{C}$  and  $\kappa_1, \kappa_2 \in \mathbb{Z}_{\geq 2}$  with  $\kappa_1 \geq \kappa_2$ .

We call  $\Pi_\sigma$  a generalized principal series representation of  $G$  and the representation space of  $\Pi_\sigma$  is denoted by  $H(\sigma)$ . To discuss three kinds of the generalized principal series representations simultaneously, we set  $\kappa_1, \kappa_2 \in \mathbb{Z}$ ,  $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$  and  $\nu'_1 \in \mathbb{C}$  as follows:

- Case 1:  $\kappa_1 := \delta_1 - \delta_4$ ,  $\kappa_2 := \delta_2 - \delta_3$ ,  $\nu'_1 := \kappa_1 \nu_1 + \kappa_2 \nu_4$ .
- Case 2:  $\delta_1 \equiv \kappa_1 \pmod{2}$ ,  $\kappa_2 := \delta_2 - \delta_3$ ,  $\nu'_1 := \nu_1$ .
- Case 3:  $\delta_1 \equiv \kappa_1 \pmod{2}$ ,  $\delta_2 \equiv \kappa_2 \pmod{2}$ ,  $\delta_3 := 0$ ,  $\nu'_1 := \nu_1$ .

Then we know  $\tau_{(\kappa_1, \kappa_2, \delta_3)}$  is the minimal  $K$ -type of  $\Pi_\sigma$ .

Let us recall the  $L$ - and  $\varepsilon$ -factors of  $\Pi_\sigma$ . See [2, §5.1, §5.2] for the precise. The equivalence classes of irreducible representations of the Weil group  $W_{\mathbb{R}}$  is exhausted by the characters  $\phi_\nu^\delta$  ( $\nu \in \mathbb{C}, \delta \in \{0, 1\}$ ) and the two dimensional representations  $\phi_{\nu, \kappa}$  ( $\nu \in \mathbb{C}, \kappa \in \mathbb{Z}_{\geq 1}$ ). The associated  $L$ - and  $\varepsilon$ -factors are

$$L(s, \phi_\nu^\delta) = \Gamma_{\mathbb{R}}(s + \nu + \delta), \quad \epsilon(s, \phi_\nu^\delta, \psi_1) = (\sqrt{-1})^\delta,$$

$$L(s, \phi_{\nu, \kappa}) = \Gamma_{\mathbb{C}}(s + \nu + \frac{\kappa}{2}), \quad \epsilon(s, \phi_{\nu, \kappa}, \psi_1) = (\sqrt{-1})^{\kappa+1}.$$

Here we set  $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$  and  $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ . For a finite dimensional semisimple representation  $\phi$  of  $W_{\mathbb{R}}$ , we define the corresponding  $L$ - and  $\epsilon$ -factors by

$$L(s, \phi) = \prod_{i=1}^m L(s, \phi_i), \quad \epsilon(s, \phi, \psi_1) = \prod_{i=1}^m \epsilon(s, \phi_i, \psi_1),$$

where  $\phi \simeq \bigoplus_{i=1}^m \phi_i$  is the irreducible decomposition of  $\phi$ . The local Langlands correspondence is a bijection between the set of infinitesimal equivalence classes of irreducible admissible representations of  $\mathrm{GL}(n, \mathbb{R})$  and the set of equivalence classes of  $n$ -dimensional semisimple representations of  $W_{\mathbb{R}}$ . For an irreducible admissible representation  $\Pi$  of  $\mathrm{GL}(n, \mathbb{R})$ , the corresponding representation  $\phi[\Pi]$  of  $W_{\mathbb{R}}$  is called Langlands parameter of  $\Pi$ . We define the local  $L$ -factors  $L(s, \Pi)$ ,  $L(s, \Pi, \wedge^2)$  and  $\epsilon$ -factors  $\epsilon(s, \Pi, \psi_1)$ ,  $\epsilon(s, \Pi, \wedge^2, \psi_1)$  by

$$\begin{aligned} L(s, \Pi) &= L(s, \phi[\Pi]), & L(s, \Pi, \wedge^2) &= L(s, \wedge^2(\phi[\Pi])), \\ \epsilon(s, \Pi, \psi_1) &= \epsilon(s, \phi[\Pi], \psi_1), & \epsilon(s, \Pi, \wedge^2, \psi_1) &= \epsilon(s, \wedge^2(\phi[\Pi]), \psi_1). \end{aligned}$$

Here  $\wedge^2 : \mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{GL}(\frac{n(n-1)}{2}, \mathbb{C})$  is the exterior square representation.

Since the Langlands parameter  $\phi[\Pi_{\sigma}]$  of irreducible generalized principal series  $\Pi_{\sigma}$  is given by

$$\phi[\Pi_{\sigma}] = \begin{cases} \phi_{\nu_1}^{\delta_1} \oplus \phi_{\nu_2}^{\delta_2} \oplus \phi_{\nu_3}^{\delta_3} \oplus \phi_{\nu_4}^{\delta_4} & \text{case 1,} \\ \phi_{\nu_1, \kappa_1-1} \oplus \phi_{\nu_2}^{\delta_2} \oplus \phi_{\nu_3}^{\delta_3} & \text{case 2,} \\ \phi_{\nu_1, \kappa_1-1} \oplus \phi_{\nu_2, \kappa_2-1} & \text{case 3,} \end{cases}$$

we know that

$$\begin{aligned} L(s, \Pi_{\sigma}) &= \begin{cases} \prod_{1 \leq i \leq 4} \Gamma_{\mathbb{R}}(s + \nu_i + \delta_i) & \text{case 1,} \\ \Gamma_{\mathbb{C}}(s + \nu_1 + \frac{\kappa_1-1}{2}) \Gamma_{\mathbb{R}}(s + \nu_2 + \delta_2) \Gamma_{\mathbb{R}}(s + \nu_3 + \delta_3) & \text{case 2,} \\ \Gamma_{\mathbb{C}}(s + \nu_1 + \frac{\kappa_1-1}{2}) \Gamma_{\mathbb{C}}(s + \nu_2 + \frac{\kappa_2-1}{2}) & \text{case 3,} \end{cases} \\ L(s, \Pi_{\sigma}, \wedge^2) &= \begin{cases} \prod_{1 \leq i < j \leq 4} \Gamma_{\mathbb{R}}(s + \nu_i + \nu_j + |\delta_i - \delta_j|) & \text{case 1,} \\ \Gamma_{\mathbb{C}}(s + \nu_1 + \nu_2 + \frac{\kappa_1-1}{2}) \Gamma_{\mathbb{C}}(s + \nu_1 + \nu_3 + \frac{\kappa_1-1}{2}) \\ \quad \times \Gamma_{\mathbb{R}}(s + 2\nu_1 + \delta_1) \Gamma_{\mathbb{R}}(s + \nu_2 + \nu_3 + |\delta_2 - \delta_3|) & \text{case 2,} \\ \Gamma_{\mathbb{C}}(s + \nu_1 + \nu_2 + \frac{\kappa_1-\kappa_2}{2}) \Gamma_{\mathbb{C}}(s + \nu_1 + \nu_2 + \frac{\kappa_1+\kappa_2-2}{2}) \\ \quad \times \Gamma_{\mathbb{R}}(s + 2\nu_1 + \delta_1) \Gamma_{\mathbb{R}}(s + 2\nu_2 + \delta_2) & \text{case 3,} \end{cases} \\ \epsilon(s, \Pi_{\sigma}, \psi) &= \begin{cases} (\sqrt{-1})^{\delta_1+\delta_2+\delta_3+\delta_4} & \text{case 1,} \\ (\sqrt{-1})^{\kappa_1+\delta_2+\delta_3} & \text{case 2,} \\ (\sqrt{-1})^{\kappa_1+\kappa_2} & \text{case 3,} \end{cases} \\ \epsilon(s, \Pi_{\sigma}, \wedge^2, \psi) &= \begin{cases} (\sqrt{-1})^{\delta_1, 2+\delta_1, 3+\delta_1, 4+\delta_2, 3+\delta_2, 4+\delta_3, 4} & \text{case 1,} \\ (\sqrt{-1})^{\delta_1+\delta_2, 3+2\kappa_1} & \text{case 2,} \\ (\sqrt{-1})^{\delta_1+\delta_2+2\kappa_1} & \text{case 3.} \end{cases} \end{aligned}$$

Here we define  $\delta_{i,j} \in \{0, 1\}$  by  $\delta_{i,j} \equiv \delta_i + \delta_j \pmod{2}$ .

## 5 Differential equations

We give a system of partial differential equations satisfied by Whittaker functions belonging to minimal  $K$ -type  $\tau_{(\kappa_1, \kappa_2, \delta_3)}$  of  $\Pi_\sigma$ . For  $1 \leq i \leq 4$ , let  $S_i(a_1, a_2, a_3, a_4)$  be the  $i$ -th elementary symmetric function:

$$\begin{aligned} S_1(a_1, a_2, a_3, a_4) &= \sum_{1 \leq i \leq 4} a_i, & S_2(a_1, a_2, a_3, a_4) &= \sum_{1 \leq i < j \leq 4} a_i a_j, \\ S_3(a_1, a_2, a_3, a_4) &= \sum_{1 \leq i < j < k \leq 4} a_i a_j a_k, & S_4(a_1, a_2, a_3, a_4) &= a_1 a_2 a_3 a_4. \end{aligned}$$

For  $1 \leq i \leq 4$ , we set

$$\gamma_i = \begin{cases} S_i(\nu_1, \nu_2, \nu_3, \nu_4) & \text{case 1,} \\ S_i(\nu_1 + \frac{\kappa_1 - 1}{2}, \nu_1 - \frac{\kappa_1 - 1}{2}, \nu_2, \nu_3) & \text{case 2,} \\ S_i(\nu_1 + \frac{\kappa_1 - 1}{2}, \nu_1 - \frac{\kappa_1 - 1}{2}, \nu_2 + \frac{\kappa_2 - 1}{2}, \nu_2 - \frac{\kappa_1 - 1}{2}) & \text{case 3.} \end{cases}$$

Then we have

$$\Pi_\sigma(\mathcal{C}_i)f = \gamma_i f \quad \text{for } f \in H(\sigma)_K. \quad (5.1)$$

As in [2], we can find that  $\text{Hom}_K(V_{(\kappa_1, \kappa_2, \delta_3)}, H(\sigma)_K) = \mathbb{C}\hat{\eta}_\sigma$  and show the following:

- Assume that  $\kappa_1 > \kappa_2$ . For  $l \in S_{(\kappa_1 - 1, \kappa_2, \delta_3)}$  and  $1 \leq i \leq 4$ , we have

$$2\nu'_1 \hat{\eta}_\sigma(u_{l+e_i}) = \sum_{k=1}^4 \Pi_\sigma(E_{i,k}^{\text{p}}) \hat{\eta}_\sigma(u_{l+e_k}), \quad (5.2)$$

$$\text{where } \nu'_1 = \begin{cases} (\delta_1 - \delta_2)\nu_1 + (\delta_3 - \delta_4)\nu_4 & \text{case 1,} \\ \nu_1 & \text{cases 2,3.} \end{cases}$$

- Assume that  $\kappa_2 \geq 1$ . For  $l \in S_{(\kappa_1 - 1, \kappa_2 - 1, 0)}$  and  $1 \leq i < j \leq 4$ , we have

$$\begin{aligned} 2(\nu_1 + \nu_2) \hat{\eta}_\sigma(u_{l+e_{ij}}) &= \Pi_\sigma(E_{i,i}^{\text{p}} + E_{j,j}^{\text{p}}) \hat{\eta}_\sigma(u_{l+e_{ij}}) \\ &+ \sum_{1 \leq k \leq 4, k \notin \{i,j\}} \{ \text{sgn}(j-k) \Pi_\sigma(E_{i,k}^{\text{p}}) \hat{\eta}_\sigma(u_{l+e_{kj}}) + \text{sgn}(k-i) \Pi_\sigma(E_{j,k}^{\text{p}}) \hat{\eta}_\sigma(u_{l+e_{ik}}) \}. \end{aligned} \quad (5.3)$$

Let  $\varphi : V_{(\kappa_1, \kappa_2, \delta_3)} \rightarrow \text{Wh}(\Pi_\sigma, \psi_1)$  be a  $K$ -homomorphism. For  $l \in S_{(\kappa_1, \kappa_2, \delta_3)}$ , we give a system of partial differential equations for  $\varphi(u_l)(y)$  ( $y = \text{diag}(y_1 y_2 y_3 y_4, y_2 y_3 y_4, y_3 y_4, y_4) \in A$ ). Let  $\partial_i = y_i \frac{\partial}{\partial y_i}$ . Since (5.1) with  $i = 1$  implies that

$$(\partial_4 - \gamma_1)\varphi(u_l)(y) = 0,$$

we can define a function  $\hat{\varphi}_l$  on  $(\mathbb{R}_+)^3$  by

$$\varphi(u_l)(y) = (\sqrt{-1})^{-l_1 + l_3 - l_{13} + l_{24}} (-1)^{l_2 + l_{14} + l_{23}} y_1^{3/2} y_2^2 y_3^{3/2 - \kappa_2} y_4^{\gamma_1} \hat{\varphi}_l(y_1, y_2, y_3)$$

for  $l \in S_{(\kappa_1, \kappa_2, \delta_3)}$ . From (5.1), (5.2) and (5.3) we get the following system of partial differential equations for  $\hat{\varphi}_l$ .



**Proposition 2.** (i) For  $l \in S_{(\kappa_1, \kappa_2, \delta_3)}$ , we have

$$\{\Delta_2 - (2\pi y_1)\mathfrak{K}_{12} - (2\pi y_2)\mathfrak{K}_{23} - (2\pi y_3)\mathfrak{K}_{34}\}\hat{\varphi}_l = 0,$$

$$\begin{aligned} &\{\Delta_3 + (2\pi y_1)(\partial_2 - \gamma_1)\mathfrak{K}_{12} + (2\pi y_2)(-\partial_1 + \partial_3 - \gamma_1 - \kappa_2)\mathfrak{K}_{23} \\ &- (2\pi y_3)\partial_2\mathfrak{K}_{34} + (2\pi y_1)(2\pi y_2)\mathfrak{K}_{13} + (2\pi y_2)(2\pi y_3)\mathfrak{K}_{24}\}\hat{\varphi}_l = 0 \end{aligned}$$

and

$$\begin{aligned} &[\Delta_4 - (2\pi y_1)\{(-\partial_2 + \partial_3 - \kappa_2)(-\partial_3 + \gamma_1 + \kappa_2) + (2\pi y_3)^2\}\mathfrak{K}_{12} \\ &- (2\pi y_2)\partial_1(-\partial_3 + \gamma_1 + \kappa_2)\mathfrak{K}_{23} - (2\pi y_3)\{\partial_1(-\partial_1 + \partial_2) + (2\pi y_1)^2\}\mathfrak{K}_{34} \\ &+ (2\pi y_1)(2\pi y_2)(-\partial_3 + \gamma_1 + \kappa_2)\mathfrak{K}_{13} + (2\pi y_2)(2\pi y_3)\partial_1\mathfrak{K}_{24} \\ &+ (2\pi y_1)(2\pi y_2)(2\pi y_3)\mathfrak{K}_{14} + (2\pi y_1)(2\pi y_3)\mathfrak{K}_{12,34}]\hat{\varphi}_l = 0, \end{aligned}$$

where  $\Delta_2$ ,  $\Delta_3$  and  $\Delta_4$  are the differential operators defined by

$$\begin{aligned} \Delta_2 = &-\partial_1^2 - \partial_2^2 - (\partial_3 - \kappa_2)^2 + \partial_1\partial_2 + \partial_2(\partial_3 - \kappa_2) + \gamma_1(\partial_3 - \kappa_2) \\ &+ (2\pi y_1)^2 + (2\pi y_2)^2 + (2\pi y_3)^2 - \gamma_2, \end{aligned}$$

$$\begin{aligned} \Delta_3 = &\partial_2(\partial_1 - \partial_3 + \kappa_2)(\partial_1 - \partial_2 + \partial_3 - \kappa_2) - \gamma_1(\partial_1^2 + \partial_2^2 - \partial_1\partial_2 - \partial_2(\partial_3 - \kappa_2)) \\ &+ (2\pi y_1)^2(-\partial_2 + \gamma_1) + (2\pi y_2)^2(\partial_1 - \partial_3 + \gamma_1 + \kappa_2) + (2\pi y_3)^2\partial_2 - \gamma_3 \end{aligned}$$

and

$$\begin{aligned} \Delta_4 = &\partial_1(\partial_2 - \partial_1)(\partial_3 - \partial_2 - \kappa_2)(-\partial_3 + \gamma_1 + \kappa_2) \\ &+ (2\pi y_1)^2(\partial_3 - \partial_2 - \kappa_2)(-\partial_3 + \gamma_1 + \kappa_2) + (2\pi y_2)^2\partial_1(-\partial_3 + \gamma_1 + \kappa_2) \\ &+ (2\pi y_3)^2\partial_1(\partial_2 - \partial_1) + (2\pi y_1)^2(2\pi y_3)^2 - \gamma_4, \end{aligned}$$

respectively. Here

$$\begin{aligned} \mathfrak{K}_{12}\hat{\varphi}_l &= l_1\hat{\varphi}_{l-e_1+e_2} + l_2\hat{\varphi}_{l-e_2+e_1} \\ &\quad + l_{13}\hat{\varphi}_{l-e_{13}+e_{23}} + l_{14}\hat{\varphi}_{l-e_{14}+e_{24}} + l_{23}\hat{\varphi}_{l-e_{23}+e_{13}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{14}}, \\ \mathfrak{K}_{23}\hat{\varphi}_l &= l_2\hat{\varphi}_{l-e_2+e_3} + l_3\hat{\varphi}_{l-e_3+e_2} \\ &\quad + l_{12}\hat{\varphi}_{l-e_{12}+e_{13}} + l_{13}\hat{\varphi}_{l-e_{13}+e_{12}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{34}} + l_{34}\hat{\varphi}_{l-e_{34}+e_{24}}, \\ \mathfrak{K}_{34}\hat{\varphi}_l &= l_3\hat{\varphi}_{l-e_3+e_4} + l_4\hat{\varphi}_{l-e_4+e_3} \\ &\quad + l_{13}\hat{\varphi}_{l-e_{13}+e_{14}} + l_{14}\hat{\varphi}_{l-e_{14}+e_{13}} + l_{23}\hat{\varphi}_{l-e_{23}+e_{24}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{23}}, \\ \mathfrak{K}_{13}\hat{\varphi}_l &= l_1\hat{\varphi}_{l-e_1+e_3} + l_3\hat{\varphi}_{l-e_3+e_1} \\ &\quad - l_{12}\hat{\varphi}_{l-e_{12}+e_{23}} + l_{14}\hat{\varphi}_{l-e_{14}+e_{34}} + l_{23}\hat{\varphi}_{l-e_{23}+e_{12}} - l_{34}\hat{\varphi}_{l-e_{34}+e_{14}}, \\ \mathfrak{K}_{24}\hat{\varphi}_l &= l_2\hat{\varphi}_{l-e_2+e_4} - l_4\hat{\varphi}_{l-e_4+e_2} \\ &\quad + l_{12}\hat{\varphi}_{l-e_{12}+e_{14}} - l_{14}\hat{\varphi}_{l-e_{14}+e_{12}} - l_{23}\hat{\varphi}_{l-e_{23}+e_{34}} + l_{34}\hat{\varphi}_{l-e_{34}+e_{23}}, \\ \mathfrak{K}_{14}\hat{\varphi}_l &= -l_1\hat{\varphi}_{l-e_1+e_4} - l_4\hat{\varphi}_{l-e_4+e_1} \\ &\quad + l_{12}\hat{\varphi}_{l-e_{12}+e_{24}} + l_{13}\hat{\varphi}_{l-e_{13}+e_{34}} + l_{24}\hat{\varphi}_{l-e_{24}+e_{12}} + l_{34}\hat{\varphi}_{l-e_{34}+e_{13}} \end{aligned}$$

and

$$\begin{aligned}
& \mathfrak{K}_{12,34}\hat{\varphi}_l \\
&= l_1 l_3 \hat{\varphi}_{l-e_1-e_3+e_2+e_4} + l_1 l_4 \hat{\varphi}_{l-e_1-e_4+e_2+e_3} + l_2 l_3 \hat{\varphi}_{l-e_2-e_3+e_1+e_4} + l_2 l_4 \hat{\varphi}_{l-e_2-e_4+e_1+e_3} \\
&+ l_1 (l_{13} \hat{\varphi}_{l-e_1+e_2-e_{13}+e_{14}} + l_{14} \hat{\varphi}_{l-e_1+e_2-e_{14}+e_{13}} + l_{23} \hat{\varphi}_{l-e_1+e_2-e_{23}+e_{24}} + l_{24} \hat{\varphi}_{l-e_1+e_2-e_{24}+e_{23}}) \\
&+ l_2 (l_{13} \hat{\varphi}_{l-e_2+e_1-e_{13}+e_{14}} + l_{14} \hat{\varphi}_{l-e_2+e_2-e_{14}+e_{13}} + l_{23} \hat{\varphi}_{l-e_2+e_1-e_{23}+e_{24}} + l_{24} \hat{\varphi}_{l-e_2+e_1-e_{24}+e_{23}}) \\
&+ l_3 (l_{13} \hat{\varphi}_{l-e_3+e_4-e_{13}+e_{23}} + l_{14} \hat{\varphi}_{l-e_3+e_4-e_{14}+e_{24}} + l_{23} \hat{\varphi}_{l-e_3+e_4-e_{23}+e_{13}} + l_{24} \hat{\varphi}_{l-e_3+e_4-e_{24}+e_{14}}) \\
&+ l_4 (l_{13} \hat{\varphi}_{l-e_4+e_3-e_{13}+e_{23}} + l_{14} \hat{\varphi}_{l-e_4+e_3-e_{14}+e_{24}} + l_{23} \hat{\varphi}_{l-e_4+e_3-e_{23}+e_{13}} + l_{24} \hat{\varphi}_{l-e_4+e_3-e_{24}+e_{14}}) \\
&+ l_{13} (l_{14} + l_{23} + 1) \hat{\varphi}_{l-e_{13}+e_{24}} + l_{24} (l_{14} + l_{23} + 1) \hat{\varphi}_{l-e_{24}+e_{13}} \\
&+ l_{14} (l_{13} + l_{24} + 1) \hat{\varphi}_{l-e_{14}+e_{23}} + l_{23} (l_{13} + l_{24} + 1) \hat{\varphi}_{l-e_{23}+e_{14}} \\
&+ l_{13} l_{24} (\hat{\varphi}_{l-e_{13}-e_{24}+2e_{14}} + \hat{\varphi}_{l-e_{13}-e_{24}+2e_{23}}) + l_{14} l_{23} (\hat{\varphi}_{l-e_{14}-e_{23}+2e_{13}} + \hat{\varphi}_{l-e_{14}-e_{23}+2e_{24}}) \\
&+ l_{13} (l_{13} - 1) \hat{\varphi}_{l-2e_{13}+e_{14}+e_{23}} + l_{24} (l_{24} - 1) \hat{\varphi}_{l-2e_{24}+e_{14}+e_{23}} \\
&+ l_{14} (l_{14} - 1) \hat{\varphi}_{l-2e_{14}+e_{13}+e_{24}} + l_{23} (l_{23} - 1) \hat{\varphi}_{l-2e_{23}+e_{13}+e_{24}}.
\end{aligned}$$

(ii) Assume that  $\kappa_1 > \kappa_2$ . For  $l \in S_{(\kappa_1-1, \kappa_2, \delta_3)}$ , we have

$$(\partial_1 - \nu'_1 - \frac{\kappa_1-1}{2})\hat{\varphi}_{l+e_1} + 2\pi y_1 \hat{\varphi}_{l+e_2} = 0,$$

$$\begin{aligned}
& (-\partial_1 + \partial_2 - \nu'_1 - \frac{\kappa_1-1}{2} + l_1) \hat{\varphi}_{l+e_2} - 2\pi y_1 \hat{\varphi}_{l+e_1} + 2\pi y_2 \hat{\varphi}_{l+e_3} \\
&+ l_2 \hat{\varphi}_{l-e_2+2e_1} + l_{13} \hat{\varphi}_{l+e_1-e_{13}+e_{23}} + l_{14} \hat{\varphi}_{l+e_1-e_{14}+e_{24}} \\
&+ l_{23} \hat{\varphi}_{l+e_1-e_{23}+e_{13}} + l_{24} \hat{\varphi}_{l+e_1-e_{24}+e_{14}} = 0,
\end{aligned}$$

$$\begin{aligned}
& (-\partial_2 + \partial_3 - \nu'_1 + \frac{\kappa_1-1}{2} - \kappa_2 - l_4) \hat{\varphi}_{l+e_3} - 2\pi y_2 \hat{\varphi}_{l+e_2} + 2\pi y_3 \hat{\varphi}_{l+e_4} \\
&- l_3 \hat{\varphi}_{l-e_3+2e_4} - l_{13} \hat{\varphi}_{l+e_4-e_{13}+e_{14}} - l_{14} \hat{\varphi}_{l+e_4-e_{14}+e_{13}} \\
&- l_{23} \hat{\varphi}_{l+e_4-e_{23}+e_{24}} - l_{24} \hat{\varphi}_{l+e_4-e_{24}+e_{23}} = 0,
\end{aligned}$$

$$(-\partial_3 + \gamma_1 - \nu'_1 + \frac{\kappa_1-1}{2} + \kappa_2) \hat{\varphi}_{l+e_4} - 2\pi y_3 \hat{\varphi}_{l+e_3} = 0.$$

(iii) Assume that  $\kappa_2 \geq 1$ . For  $l \in S_{(\kappa_1-1, \kappa_2-1, 0)}$ , we have

$$(\partial_2 - \nu_1 - \nu_2 - \frac{\kappa_1+\kappa_2}{2} + 1) \hat{\varphi}_{l+e_{12}} + 2\pi y_2 \hat{\varphi}_{l+e_{13}} = 0,$$

$$\begin{aligned}
& (\partial_1 - \partial_2 + \partial_3 - \nu_1 - \nu_2 - \frac{\kappa_1+\kappa_2}{2} - l_{13} - l_{14} - l_{23} - l_{24}) \hat{\varphi}_{l+e_{13}} \\
&+ 2\pi y_1 \hat{\varphi}_{l+e_{23}} - 2\pi y_2 \hat{\varphi}_{l+e_{12}} + 2\pi y_3 \hat{\varphi}_{l+e_{14}} \\
&+ l_2 \hat{\varphi}_{l-e_2+e_3+e_{12}} + l_3 \hat{\varphi}_{l-e_3+e_2+e_{12}} + l_{13} \hat{\varphi}_{l-e_{13}+2e_{12}} + l_{24} \hat{\varphi}_{l-e_{24}+e_{12}+e_{34}} = 0,
\end{aligned}$$

$$\begin{aligned}
& (\partial_1 - \partial_3 + \gamma_1 - \nu_1 - \nu_2 - \frac{\kappa_1-3\kappa_2}{2} + l_4) \hat{\varphi}_{l+e_{14}} \\
&+ 2\pi y_1 \hat{\varphi}_{l+e_{24}} - 2\pi y_3 \hat{\varphi}_{l+e_{13}} + l_2 \hat{\varphi}_{l-e_2+e_4+e_{12}} + l_3 \hat{\varphi}_{l-e_3+e_4+e_{13}} = 0,
\end{aligned}$$

$$(-\partial_1 + \partial_3 - \nu_1 - \nu_2 + \frac{\kappa_1 - 3\kappa_2}{2} - l_4)\hat{\varphi}_{l+e_{23}} - 2\pi y_1 \hat{\varphi}_{l+e_{13}} + 2\pi y_3 \hat{\varphi}_{l+e_{24}} - l_2 \hat{\varphi}_{l-e_2+e_4+e_{34}} - l_3 \hat{\varphi}_{l-e_3+e_4+e_{24}} = 0,$$

$$(-\partial_1 + \partial_2 - \partial_3 + \gamma_1 - \nu_1 - \nu_2 + \frac{\kappa_1 + \kappa_2}{2} + l_{13} + l_{14} + l_{23} + l_{24})\hat{\varphi}_{l+e_{24}} - 2\pi y_1 \hat{\varphi}_{l+e_{14}} + 2\pi y_2 \hat{\varphi}_{l+e_{34}} - 2\pi y_3 \hat{\varphi}_{l+e_{23}} - l_2 \hat{\varphi}_{l-e_2+e_3+e_{34}} - l_3 \hat{\varphi}_{l-e_3+e_2+e_{34}} - l_{13} \hat{\varphi}_{l-e_{13}+e_{12}+e_{34}} - l_{24} \hat{\varphi}_{l-e_{24}+2e_{34}} = 0,$$

$$(-\partial_2 + \gamma_1 - \nu_1 - \nu_2 + \frac{\kappa_1 + \kappa_2}{2} - 1)\hat{\varphi}_{l+e_{34}} - 2\pi y_2 \hat{\varphi}_{l+e_{24}} = 0.$$

## 6 Explicit formulas of Whittaker functions

We shall give moderate growth solutions of the system above. Since the case 1 is already done in [4], we treat the cases 2 and 3. Our key observation is that the function  $\hat{\varphi}_{(0,0,0,l_4,l_{12},0,0,0,l_{34})}(y)$  satisfies similar system of partial differential equations as that for the class one Whittaker functions (case 1 with  $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$ ). Then we know Mellin-Barnes integral representation of  $\hat{\varphi}_{(0,0,0,l_4,l_{12},0,0,0,l_{34})}(y)$  and the other  $\hat{\varphi}_l(y)$  can be determined by the equations (ii) and (iii) in Proposition 2.

Here are our main results. In the following, the path of integration is a vertical line in the complex plane, of sufficiently large real part to keep the poles of integrand on its left.

**Theorem 3.** (Case 2) Let  $\sigma = D_{(\nu_1, \kappa_1)} \boxtimes \chi_{(\nu_2, \delta_2)} \boxtimes \chi_{(\nu_3, \delta_3)}$  with  $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$ ,  $\kappa_1 \in \mathbb{Z}_{\geq 2}$ ,  $\delta_2, \delta_3 \in \{0, 1\}$  and  $\delta_2 \geq \delta_3$  such that  $\Pi_\sigma$  is irreducible. There exists a  $K$ -homomorphism  $\varphi_\sigma : V_{(\kappa_1, \delta_2 - \delta_3, \delta_3)} \rightarrow \text{Wh}(\Pi_\sigma, \psi_1)^{\text{mg}}$  whose radial part is given by

$$\begin{aligned} \varphi_\sigma(u_l)(y) &= y_1^{3/2} y_2^2 y_3^{3/2} y_4^{2\nu_1 + \nu_2 + \nu_3} \cdot (\sqrt{-1})^{-l_1 + l_3 - l_{13} + l_{24}} (-1)^{l_2 + l_{14} + l_{23}} \\ &\quad \times \frac{1}{(4\pi\sqrt{-1})^3} \int_{s_3} \int_{s_2} \int_{s_1} V_l(s_1, s_2, s_3) y_1^{-s_1} y_2^{-s_2} y_3^{-s_3} ds_1 ds_2 ds_3 \end{aligned}$$

with  $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_{(\kappa_1, \delta_2 - \delta_3, \delta_3)}$ . Here

$$\begin{aligned} &V_l(s_1, s_2, s_3) \\ &= \Gamma_{\mathbb{C}}(s_1 + \nu_1 + \frac{\kappa_1 - 1}{2}) \Gamma_{\mathbb{R}}(s_2 + 2\nu_1 + l_3 + l_4 + l_{12} + l_{34}) \\ &\quad \times \Gamma_{\mathbb{R}}(s_2 + \nu_2 + \nu_3 + l_1 + l_2 + l_{12} + l_{34}) \Gamma_{\mathbb{C}}(s_3 + \nu_1 + \nu_2 + \nu_3 + \frac{\kappa_1 - 1}{2}) \\ &\quad \times \sum_{i_{14}=0}^{l_{14}} \sum_{i_{23}=0}^{l_{23}} \sum_{i_{13}=0}^{l_{13}} \sum_{i_{24}=0}^{l_{24}} \\ &\quad \times \frac{1}{4\pi\sqrt{-1}} \int_q \frac{\Gamma_{\mathbb{R}}(s_1 - q + l_1 + i_{14} + i_{23} + i_{13} + i_{24}) \Gamma_{\mathbb{C}}(s_2 - q + \nu_1 + \frac{\kappa_1 - 1}{2})}{\Gamma_{\mathbb{R}}(s_1 + s_2 - q + 2\nu_1 + l_1 + l_3 + l_4 + l_{12} + l_{34} + i_{14} + i_{23} + i_{13} + i_{24})} \\ &\quad \times \frac{\Gamma_{\mathbb{R}}(s_3 - q + 2\nu_1 + l_4 + l_{14} + l_{23} - i_{14} - i_{23} + i_{13} + i_{24})}{\Gamma_{\mathbb{R}}(s_2 + s_3 - q + 2\nu_1 + \nu_2 + \nu_3 + l_1 + l_2 + l_4 + l_{12} + l_{34} + l_{14} + l_{23} - i_{14} - i_{23} + i_{13} + i_{24})} \\ &\quad \times \Gamma_{\mathbb{R}}(q + \nu_2 + l_{34} + l_{24} + l_{23} - i_{23} + i_{14} + i_{13} - i_{24}) \\ &\quad \times \Gamma_{\mathbb{R}}(q + \nu_3 + l_{12} + l_{13} + l_{14} - i_{14} + i_{23} - i_{13} + i_{24}) dq. \end{aligned}$$

**Theorem 4.** (Case 3) Let  $\sigma = D_{(\nu_1, \kappa_1)} \boxtimes D_{(\nu_2, \kappa_2)}$  with  $\nu_1, \nu_2 \in \mathbb{C}$ ,  $\kappa_1, \kappa_2 \in \mathbb{Z}_{\geq 2}$  and  $\kappa_1 \geq \kappa_2$  such that  $\Pi_\sigma$  is irreducible. There exists a  $K$ -homomorphism  $\varphi_\sigma : V_{(\kappa_1, \kappa_2, 0)} \rightarrow \text{Wh}(\Pi_\sigma, \psi_1)^{\text{mg}}$  whose radial part is given by

$$\begin{aligned} \varphi_\sigma(u_i)(y) &= y_1^{3/2} y_2^2 y_3^{3/2} y_4^{2\nu_1+2\nu_2} \cdot (\sqrt{-1})^{-l_1+l_3-l_{13}+l_{24}} (-1)^{l_2+l_{14}+l_{23}} \\ &\quad \times \frac{1}{(4\pi\sqrt{-1})^3} \int_{s_3} \int_{s_2} \int_{s_1} V_l(s_1, s_2, s_3) y_1^{-s_1} y_2^{-s_2} y_3^{-s_3} ds_1 ds_2 ds_3 \end{aligned}$$

with  $l = (l_1, l_2, l_3, l_4, l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}) \in S_{(\kappa_1, \kappa_2, 0)}$ . Here

$$\begin{aligned} &V_l(s_1, s_2, s_3) \\ &= \Gamma_{\mathbb{C}}(s_1 + \nu_1 + \frac{\kappa_1-1}{2}) \Gamma_{\mathbb{R}}(s_2 + 2\nu_1 + l_3 + l_4 + l_{12} + l_{34}) \\ &\quad \times \Gamma_{\mathbb{R}}(s_2 + 2\nu_2 + l_1 + l_2 + l_{12} + l_{34}) \Gamma_{\mathbb{C}}(s_3 + \nu_1 + 2\nu_2 + \frac{\kappa_1-1}{2}) \\ &\quad \times \sum_{i=0}^{l_{14}+l_{23}} \sum_{j=0}^{l_{13}+l_{24}} \binom{l_{14}+l_{23}}{i} \binom{l_{13}+l_{24}}{j} \\ &\quad \times \frac{1}{4\pi\sqrt{-1}} \int_q \frac{\Gamma_{\mathbb{R}}(s_1 - q + l_1 + i + j) \Gamma_{\mathbb{C}}(s_2 - q + \nu_1 + \frac{\kappa_1-1}{2})}{\Gamma_{\mathbb{R}}(s_1 + s_2 - q + 2\nu_1 + l_1 + l_3 + l_4 + l_{12} + l_{34} + i + j)} \\ &\quad \times \frac{\Gamma_{\mathbb{R}}(s_3 - q + 2\nu_1 + l_4 + l_{14} + l_{23} - i + j) \Gamma_{\mathbb{C}}(q + \nu_2 + \frac{\kappa_2-1}{2})}{\Gamma_{\mathbb{R}}(s_2 + s_3 - q + 2\nu_1 + 2\nu_2 + l_1 + l_2 + l_4 + l_{12} + l_{14} + l_{23} + l_{34} - i + j)} dq. \end{aligned}$$

## 7 Archimedean Bump-Friedberg zeta integrals

As an application of our explicit formulas, we evaluate the archimedean part of Bump-Friedberg zeta integral ([1]) which represents the product of the standard and the exterior square  $L$ -functions on  $\text{GL}(4)$  at the unramified non-archimedean places. Let  $G_2 = \text{GL}(2, \mathbb{R})$ ,  $N_2 = \{(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) \mid x \in \mathbb{R}\}$ ,  $K_2 = \text{O}(2)$ . We define an embedding  $\tilde{i} : G_2 \times G_2 \rightarrow G$  by

$$\left( g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \mapsto \tilde{i}(g_1, g_2) := \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

Let  $\mathcal{S}(\mathbb{R}^2)$  be the space of Schwartz functions on  $\mathbb{R}^2$ . For  $s_1, s_2 \in \mathbb{C}$ ,  $\Phi \in \mathcal{S}(\mathbb{R}^2)$  and  $W \in \text{Wh}(\Pi_\sigma, \psi_1)^{\text{mg}}$ , we consider the following archimedean zeta integral:

$$\begin{aligned} Z(s_1, s_2, W, \Phi) &= \int_{N_2 \backslash G_2} \int_{N_2 \backslash G_2} W(\tilde{i}(g_1, g_2)) \Phi((0, 1)g_2) \\ &\quad \times |\det g_1|^{s_1 - \frac{1}{2}} |\det g_2|^{-s_1 + s_2 + \frac{1}{2}} d\dot{g}_1 d\dot{g}_2. \end{aligned}$$

Here  $d\dot{g}$  is the right  $G_2$ -invariant measure on  $N_2 \backslash G_2$  normalized so that

$$\int_{N_2 \backslash G_2} f(g) d\dot{g} = \int_0^\infty \int_0^\infty \int_{K_2} f(\text{diag}(y_1 y_2, y_2) k) dk \frac{2dy_1}{y_1^2} \frac{2dy_2}{y_2}$$

for any compactly supported continuous function  $f$  on  $N_2 \backslash G_2$ . Here  $dk$  is the normalized Haar measure on  $K_2$  such that  $\int_{K_2} dk = 1$ . By using our explicit formulas we can compute the integral  $Z(s_1, s_2, W, \Phi)$  to get the following:

**Theorem 5.** *For each  $\Pi_\sigma$ , there exists  $(W, \Phi) \in \text{Wh}(\Pi_\sigma, \psi_1)^{\text{mg}} \times \mathcal{S}(\mathbb{R}^2)$  such that*

$$\begin{aligned} Z(s_1, s_2, W, \Phi) &= L(s_1, \Pi_\sigma) L(s_2, \Pi_\sigma, \Lambda^2), \\ Z(1 - s_1, 1 - s_2, \widetilde{W}, \widehat{\Phi}) &= \varepsilon(s_1, \Pi_\sigma, \psi_1) \varepsilon(s_2, \Pi_\sigma, \psi_1, \Lambda^2) L(1 - s_1, \widetilde{\Pi}_\sigma) L(1 - s_2, \widetilde{\Pi}_\sigma, \Lambda^2). \end{aligned}$$

Here  $\widetilde{W}$  is the contragredient Whittaker function of  $W$  defined by  $\widetilde{W}(g) = W\left(\begin{pmatrix} & & & 1 \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix} g^{-1}\right)$  and  $\widehat{\Phi}$  is the Fourier transform of  $\Phi$ .

Here are test vectors, that is, pairs of  $(W, \Phi)$  such that the above identities hold. The symbol  $R$  means the right differential. Since the case 1 is done in [3], we treat the cases 2 and 3. We define  $\delta_1, \delta_2, \delta_3$  as before, and define  $\delta \in \{0, 1\}$  by  $\delta \equiv \delta_1 + \delta_2 + \delta_3 \pmod{2}$ .

- Case 2:  $\sigma = D_{(\nu_1, \kappa_1)} \boxtimes \chi_{(\nu_2, \delta_2)} \boxtimes \chi_{(\nu_3, \delta_3)}$  with  $\delta_2 \geq \delta_3$ . We have

$$\Phi(x_1, x_2) = (\sqrt{-1}x_1 + x_2)^\delta \exp\{-\pi(x_1^2 + x_2^2)\}$$

and

$$\begin{aligned} W &= (-\sqrt{-1})^{\kappa_1} \\ &\times \begin{cases} \varphi_\sigma(w_0) & (\delta_1, \delta_2, \delta_3) = (0, 0, 0), \\ \varphi_\sigma(w_2 + \sqrt{-1}w_4) & (\delta_1, \delta_2, \delta_3) = (1, 0, 0), \\ (4\pi)^{-1} \{R(E_{1,2}^p) \varphi_\sigma(w_{3,4}) - R(E_{2,3}^p) \varphi_\sigma(w_{1,4}) \\ \quad + R(E_{3,4}^p) \varphi_\sigma(w_{1,2}) - R(E_{1,4}^p) \varphi_\sigma(w_{2,3})\} & (\delta_1, \delta_2, \delta_3) = (0, 1, 1), \\ (4\pi)^{-1} \{R(-E_{3,4}^p + \sqrt{-1}E_{2,3}^p) \varphi_\sigma(w_1) \\ \quad + R(E_{1,4}^p - \sqrt{-1}E_{1,2}^p) \varphi_\sigma(w_3)\} & (\delta_1, \delta_2, \delta_3) = (1, 1, 1), \\ \varphi_\sigma(-\sqrt{-1}w_{2,24} + w_{4,24}) & (\delta_1, \delta_2, \delta_3) = (0, 1, 0), \\ (4\pi)^{-1} \{R(E_{1,2}^p) \varphi_\sigma(w_{12}) - R(E_{2,3}^p) \varphi_\sigma(w_{23}) \\ \quad + R(E_{3,4}^p) \varphi_\sigma(w_{34}) + R(E_{1,4}^p) \varphi_\sigma(w_{14})\} & (\delta_1, \delta_2, \delta_3) = (1, 1, 0). \end{cases} \end{aligned}$$

Here we define  $w_0, w_p, w_{p,q}, w_{pq}, w_{p,qr} \in V_{(\kappa_1, \delta_2, \delta_3)}$  ( $1 \leq p, q, r \leq 4$ ) by

$$\begin{aligned} w_0 &= \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{\kappa_1/2}), & w_p &= \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1-1)/2} \xi_p), \\ w_{p,q} &= \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1-2)/2} \xi_p \xi_q), & w_{pq} &= \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1-1)/2} \xi_{pq}), \\ & & w_{p,qr} &= \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1-2)/2} \xi_p \xi_{qr}). \end{aligned}$$

- Case 3:  $\sigma = D_{(\nu_1, \kappa_1)} \boxtimes D_{(\nu_2, \kappa_2)}$  with  $\kappa_1 \geq \kappa_2$ . We have

$$\Phi(x_1, x_2) = (\sqrt{-1}x_1 + x_2)^\delta \exp\{-\pi(x_1^2 + x_2^2)\}$$

and

$$W = \begin{cases} \varphi_\sigma(w_0) & (\delta_1, \delta_2) = (0, 0), \\ R(E_{1,2}^p)\varphi_\sigma(w_{12}) - R(E_{2,3}^p)\varphi_\sigma(w_{23}) \\ \quad + R(E_{3,4}^p)\varphi_\sigma(w_{34}) + R(E_{1,4}^p)\varphi_\sigma(w_{14}) & (\delta_1, \delta_2) = (1, 1), \\ \varphi_\sigma(w_2 + \sqrt{-1}w_4) & \delta_1 \neq \delta_2. \end{cases}$$

Here we define  $w_0, w_p, w_{pq} \in V_{(\kappa_1, \kappa_2, 0)}$  ( $1 \leq p, q \leq 4$ ) by

$$w_0 = \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1 - \kappa_2)/2} \xi_{24}^{\kappa_2}), \quad w_p = \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1 - \kappa_2 - 1)/2} \xi_p \xi_{24}^{\kappa_2}), \\ w_{pq} = \mathfrak{qR}((\xi_1^2 + \xi_3^2)^{(\kappa_1 - \kappa_2)/2} \xi_{pq} \xi_{24}^{\kappa_2 - 1}).$$

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