The Tarksi Theorems, Extensions to Group Rings and Logical Rigidity

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Abstract

The famous Tarski theorems state that all free groups heve the same elementary theory. In 2019 I gave a talk at the Kobe conference explaining the Tarski theorems and the accompanying language. Subsequently in [FGRS 1,2,3] and [FGKRS] the relationship between the universal and elementary theory of a group ring $R[G]$ and the corresponding universal and elementary theory of the associated group G and ring R was examined. These are relative to an appropriate logical language L_0, L_1, L_2 for groups, rings and group rings respectively. Axiom systems for these were provided in [FGRS 1]. In [FGRS 1] it was proved that if $R[G]$ is elementarily equivalent to $S[H]$ with respect to L_2, then simultaneously the group G is elementarily equivalent to the group H with respect to L_0, and the ring R is elementarily equivalent to the ring S with respect to L_1. We then let F be a rank 2 free group and Z be the ring of integers. Examining the universal theory of the free group ring $Z[F]$ the hazy conjecture was proved that the universal sentences true in $Z[F]$ are precisely the universal sentences true in F modified appropriately for group ring theory and the converse that the universal sentences true in F are the universal sentences true in $Z[F]$ modified appropriately for group theory. Finally we mention logical group rigidity. A group G is **logically rigid** if being elementary equivalent to G is equivalent to being isomorphic to G. In this paper we survey all of these findings.

Keywords: Group ring, elementary equivalent, universally equivalent, discriminates, axiomatic systems, quasi-identity

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1 Introduction

Around 1945, Alfred Tarski proposed several questions concerning the elementary theory of non-abelian free groups. These questions then became well-known conjectures but remained open for 60 years. They were proved in the period 1996-2006 independently by O. Kharlampovich and A. Myasnikov [KhM 1-5] and by Z. Sela [Se 1-5]. The proofs, by both sets of authors, were monumental, and involved the development of several new areas of infinite group theory. Because of the tremendous amount of material developed and used in the two different proofs, the details of the solution are largely unknown, even to the general group theory population. The book [FGMRS], presented an introductory guide through the material. In this paper and the talk presented we first provide, for a general mathematical audience, an introduction to both the Tarksi theorems and the vast new ideas that went into the proof. These ideas straddle the line between algebra and mathematical logic and hence most group theorists don't know enough logic to fully understand the details while in the other direction most logicians don't understand enough infinite group theory. Details and an explanation of the proof can be found in the book *The Elementary Theory of Groups* by B.Fine, A. Gaglione, A. Myasnikov, G. Rosenberger and D. Spellman.

In [FGRS 1-2] and [FGKRS] the relationship was studied between the universal and elementary theory of a group ring $R[G]$ and the corresponding universal and elementary theory of the associated group G and ring R where we assume that R is a commutative ring with identity $1 \neq 0$. These are relative to an appropriate logical language L_0, L_1, L_2 for groups, rings and group rings respectively. Axiom systems for these were provided in [FGRS 1]. In [FGRS 1-2] it was then proved that if $R[G]$ is elementarily equivalent to $S[H]$ with respect to L_2 then simultaneously the group G is elementarily equivalent to the group H with respect to L_0 and the ring R is elementarily equivalent to the ring S with respect to L_1. If we let F be a rank 2 free group and Z be the ring of integers we call the group ring $Z[F]$ a free group ring. It is easy to prove that all free group rings for non-abelian free groups have the same universal theory. A **Kaplansky group** G is a group G where the group ring $K[G]$ with K a field has no zero divisors. It was proved in FGR-1] that the class of Kaplansky groups is universally axiomatizable. In [BM] Bakulin and Myasnikov establish a set of axioms for the universal theory of the Kaplansky Groups

Myasnikov and Remeslennikov [MR] have given axiom systems for the universal theory of non-abelian free groups. In particular they proved that if F is a non-abelian free group then the universal theory of F is axiomatized by (see section 2 for relevant definition) the diagram of F, the strict universal Horn sentences of $L_0[F]$ true in F and group commutative transitivity (see sections 3 and 4 for relevant definitons). In [FGKRS] we extended this to axiom systems for free group rings and prove that the universal theory of a free group ring $Z[F]$ is axiomatized by the diagram of $Z[F]$, the strict universal Horn sentences of $L_2[Z[F]]$ true in $Z[F]$ and ring commutative transitivity when the models are restricted to group rings. Hence if $R[G]$ satisfies the diagram of $Z[F]$ and the strict universal Horn sentences true in $Z[F]$ and ring commutative transitivity then $R[G]$ is universally equivalent to $Z[F]$.

In the next section we give the necessary preliminaries on group theory, logic and axiom systems. In section 3 we go over the Tarski theorems. In section 4 we look at the extensions to group rings while in section 5 we look axiom systems for free group rings. Finally in section 6 we briefly discuss what is called logical rigidity.

2 Basic Preliminaries

For a general algebraic structure, for example a group, a ring, a field or an algebra, A, its **elementary theory** is the set of all first-order sentences in a logical language appropriate for that structure, true in A. Hence if F is a given free group, its elementary theory consists of all first-order sentences in a language appropriate for group theory that are true in F. Two algebraic structures are **elementary equivalent** or **elementarily equivalent** if they have the same elementary theory. The Tarski theorems proved by Kharlampovich and Myasnikov and independently by Sela (see [FGMRS]) say that all non-abelian free groups satisfy the same elementary theory. Kharlampovich and Myasnikov also showed that the elementary theory of free groups is decidable, that is, there is an algorithm to decide if any elementary sentence is true in all free groups or not. For a group ring they have proved that the first-order theory (in the language of ring theory) is not decidable and have studied equations over group rings especially for torsion-free hyperbolic groups.

The set of universal sentences in an algebraic structure A that are true in A is its **universal theory** while two structures are **universally equivalent** if they have the same universal theory. It is straightforward to show that all non-abelian free groups have the same universal theory (see [FGMRS]). As part of the general solution to the Tarski theorems it was shown that a finitely generated non-abelian group is **universally free** (that is has the same universal theory as a non-abelian free group) if and only if it is a limit group (see [FGMRS).

We start with a first-order language appropriate for group theory. This language, which we denote by L_0, is the first-order language with equality containing a binary operation symbol $\bullet$, a unary operation symbol $^{-1}$ and a constant symbol 1. A **universal sentence** of L_0 is one of the form $\forall \overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ is a tuple of distinct variables, $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing at most the variables of $\overline{x}$. Similarly an **existential sentence** is one of the form $\exists \overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ and $\phi(\overline{x})$ are as above.

If G is a group then the **universal theory** of G consists of the set of all universal sentences of L_0 true in G. We denote the universal theory of a group G by $Th_\forall(G)$. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same **existential theory**. The set of all sentences of L_0 true in G is called the **first-order theory** or the **elementary theory** of G. We denote this by $Th(G)$. We note that being **first-order** or **elementary** means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values only individual group elements - never, for example, subsets of, nor functions on, the group in which they are interpreted.

We say that two groups G and H are **elementarily equivalent** (symbolically $G \equiv H$) if they have the same first-order theory, that is $Th(G) = Th(H)$.

3 The Tarski Problems and Elementary Free Groups

Alfred Tarski in 1940 made three well-known conjectures concerning nonabelian free groups. We call these the **Tarski Problems** or **Tarski Conjectures** and they asked, among other things, whether all nonabelian free groups satisfy the same first-order or elementary theory.

We say that two groups G and H are **elementarily equivalent** (symbolically $G \equiv H$) if they have the same first-order theory, that is $Th(G) = Th(H)$.

Group monomorphisms which preserve the truth of first-order formulas are called elementary embeddings. Specifically, if H and G are groups and

$$f : H \to G$$

is a monomorphism then f is an **elementary embedding** provided whenever $\phi(x_0, ..., x_n)$ is a formula of L_0 containing free at most the distinct variables $x_0, ..., x_n$ and $(h_0, ..., h_n) \in H^{n+1}$ then $\phi(h_0, , ..., h_n)$ is true in H if and only if

$$\phi(f(h_0), , ..., f(h_n))$$

is true in G. If H is a subgroup of G and the inclusion map $i : H \to G$ is an elementary embedding then we say that G is an **elementary extension** of H.

Two very important concepts in the elementary theory of groups, are **completeness** and **decidability**. Given a nonempty class of groups $\mathcal{X}$ closed under isomorphism we say that its first-order theory is **complete** if given a sentence ϕ of L_0 either ϕ is true in every group in $\mathcal{X}$ or ϕ is false in every group in $\mathcal{X}$. The first-order theory of $\mathcal{X}$ is **decidable** if there exists a recursive algorithm which, given a sentence ϕ of L_0 decides whether or not ϕ is true in every group in $\mathcal{X}$.

The positive solution to the Tarski Problems, given by Kharlampovich and Myasnikov (see [KhM 1-9] and independently by Sela (see [Se 1-6]) is given in the next three theorems:

Theorem 3.1 *(Tarski 1) Any two nonabelian free groups are elementarily equivalent. That is any two nonabelian free groups satisfy exactly the same first-order theory.*

Theorem 3.2 *(Tarski 2) If the nonabelian free group H is a free factor in the free group G then the inclusion map $H \to G$ is an elementary embedding.*

In addition to the completeness of the theory of the nonabelian free groups the question of its **decidability** also arises. The **decidability** of the theory of nonabelian free groups means the question of whether there exists a recursive algorithm which, given a sentence ϕ of L_0, decides whether or not ϕ is true in every nonabelian free group. Kharlampovich and Myasnikov, in addition to proving the two above Tarski conjectures also proved the following.

Theorem 3.3 *(Tarski 3) The elementary theory of the nonabelian free groups is decidable.*

Prior to the solution of the Tarski problems it was asked whether there exist non-free **elementary free groups**, that is whether there exists non-free groups that have exactly the same first-order theory as the class of nonabelian free groups. The answer was yes, and both the Kharlampovich-Myasnikov solution and the Sela solution provide a complete characterization of the finitely generated elementary free groups. In the Kharlampovich-Myasnikov formulation these are given as a special class of what are termed NTQ groups (see [KhM 1-9]) The primary examples of non-free elementary free groups are the orientable surface groups of genus $g \geq 2$ and the nonorientable surface groups of genus $g \geq 4$. Recall that a **surface group** is the fundamental group of a compact surface. If the surface is orientable it is an orientable surface group otherwise a nonorientable surface group.

If S_g denotes the orientable surface group of genus g then S_g has a one-relator presentation with a quadratic relator.

$$S_g = \langle a_1, b_1, ..., a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1\rangle.$$

Groups with presentations similar to this play a major role in the structure theory of fully residually free groups and NTQ groups (see [KhM 1-6]).

Further if N_g denotes the nonorientable surface group of genus g then N_g has a one-relator presentation with a quadratic relator.

$$N_g = \langle a_1, ..., a_g; a_1^2 \cdots a_g^2 = 1\rangle.$$

We note that the solution to the Tarski Problems implies that any first-order theorem holding in the class of nonabelian free groups must also hold in most surface groups. In many cases proving these results directly is very nontrivial.

Theorem 3.4 *(see [KhM 1-9] Sela [1-6]) An orientable surface group of genus $g \geq 2$ is elementary free, that is has the same elementary theory as the class of nonabelian free groups. Further the nonorientable surface groups N_g for $g \geq 4$ are also elementary free.*

We need several other concepts. Let $\mathcal{X}$ be a class of groups. Then a group G is **residually** $\mathcal{X}$ if given any nontrivial element $g \in G$ there is a homomorphism $\phi : G \to H$ where H is a group in $\mathcal{X}$ such that $\phi(g) \neq 1$. A group G is *fully residually* $\mathcal{X}$ if given finitely many nontrivial elements $g_1, ..., g_n$ in G there is a homomorphism $\phi : G \to H$, where H is a group in $\mathcal{X}$, such that $\phi(g_i) \neq 1$ for all $i = 1, ..., n$. Fully residually free groups have played a crucial role in the study of equations and first-order formulas over free groups. In Sela's solution to the Tarski problems finitely generated fully resiudally free groups are called **limit groups**. The **universal theory** of a group G consists of all universal sentences true in G. All nonabelian free groups share the same universal theory and a group G is called **universally free** if it shares the same universal theory as the class of nonabelian free groups.

A group G is **commutative transitive** or **CT** if commutativity is transitive on the set of nontrivial elements of G. That is if $[x, y] = 1$ and $[y, z] = 1$ for nontrivial elements $x, y, z \in G$ then $[x, z] = 1$. A subgroup H of a group G is **malnormal** if $x^{-1}Hx \cap H = \{1\}$ if $x \notin H$. A group G is **CSA** if maximal abelian subgroups are malnormal. CSA implies commutative transitivity but there exist CT groups that are not CSA. For example it can be shown that a noncyclic one-relator group G with torsion is CT but not CSA if G has elements of order 2 (see [FMgrRR]). Another example of a CT group that is not CSA is the infinite dihedral group $G = \langle a, b; a^2 = b^2 = 1 \rangle$. It is straightforward that free products of abelian groups are CT and hence G is CT. On the other hand the commutator subgroup G' is the cyclic subgroup of G generated by ab. A nonabelian CSA group cannot have a nontrivial abelian normal subgroup and hence G is not CSA.

Remeslennikov [Re] and independently Gaglione and Spellman [GS 1] proved the following remarkable theorem which became one of the cornerstones in the proof of the Tarski problems (see [Kh 1]and [Se 1].)

Theorem 3.5 *Suppose G is nonabelian and residually free. Then the following are equivalent:*
(1) G is fully residually free,
(2) G is commutative transitive,
(3) G is universally free.

Therefore the class of nonabelian fully residually free groups coincides with the class of residually free universally free groups. The equivalence of (1) and (2) in the theorem above was proved originally by Benjamin Baumslag ([BB]), where he introduced the concept of fully residually free. Any finitely generated elementary free group being universally free must satisfy this theorem and hence be fully residually free.

In [FGRS 3] a study was done on elementary free groups. It was shown that such grous have a wide array of properties many of which are non-first order. For example they are hyperbolci and stably hyperbolic and satisfy Turner's retraction theorem. We summarie some of these

Theorem 3.6 *Let G be a finitely generated elementary free group then*
(1)G satisfies Magnus' theorem: If $R, S \in F$ then if $N(R) = N(S)$,it follows that R is conjugate to either S or S^{-1}.
(2) G has cyclic centralizers

(3) G is hyperbolic and stably hyperbolic
(4) G satisfies Turner's Retract theorem classifying the test words in G
(5) The automorphism group of G is tame
(6) G admits a faithful representation into $PSL(2, C)$

4 Extensions to Group Rings

In order to extend some of these results to group rings we introduced two additional first-order languages L_1 and L_2 whose models are rings and group rings respectively. Formal axiom systems for L_0, $L - 1$ and L_2 can be found in [FGRS 1].

With regard to these languages we attempt to extend the Tarski results to group rings, in particular to group rings of free groups. We start by considering the universal theory of a group ring $R[G]$ where R is a commutative ring with an identity. Let F be a nonabelian finitely generated free group and Z the integers. If F_1 is any other nonabelian countable free group then we have the same snake eating its tail situation $F_1 \leq F \leq F_1$ as before. Since every subring of Z with an identity is Z itself the same argument as for groups shows that for any language we use

$$Th_\forall Z[F] \subset Th_\forall Z[F_1] \subset Th_\forall Z[F].$$

It follows as for groups that all integral group rings of finitely generated nonabelian free groups are universally equivalent. Here we will use the universal theory with the axioms T_2 and language L_2 for group rings.

Theorem 4.1 *All integral group rings for nonabelian countable free groups are universally equivalent.*

In this context we call any group ring universally equivalent to $Z[F]$ for a nonabelian countable free group a **universally free group ring**. We now consider the question of classifying the universally free group rings in a manner similar to the Gaglione-Spellman-Remeslennikov theorem.

In [FGRS 1-2] the following more general results were proved answering this.

Theorem 4.2 *Let R be a commutative ring with identity $1 \neq 0$ and let G and H be groups. If G and H are universally equivalent with respect to L_0, then $R[G]$ and $R[H]$ are universally equivalent with respect to L_1.*

Theorem 4.3 *Let R and S be commutative rings with identity $1 \neq 0$ ad let G be a group. If R and S are universally equivalent with respect to L_1, then $R[G]$ and $S[G]$ are universally equivalent with respect to L_1.*

Combining these two theorems and using the transitivity of universal equivalence with respect to L_1, we immediately deduce

Theorem 4.4 *Let G and H be groups and let R and S be commutative rings with identity $1 \neq 0$. If G and H are universally equivalent with respect to L_0 and R and S are universally equivalent with respect to L_1, then $R[G]$ and $S[H]$ are universally equivalent with respect to L_1.*

We now examine the elementary equivalence of group rings. We need the following proposition that can be found in the book of Chang and Keisler ([CK]).

(Keisler-Shelah [S]) Let L be a first order language with equality and A and B be L-structures. If $A \equiv_L B$, then there is a nonempty set I and an ultrafilter D in I such that the ultrapowers ${}^*A = A^I/D$ and ${}^*B = B^I/D$ are isomorphic.

This proposition was first proven by Keisler using the Generalized Continuum Hypothesis and subsequently reproven by Shelah without need of that assumption.

Using this we get the following which says that if two group rings are elementary equivalent with respect to L_2 then the groups are elementary equivalent with respect to L_0 and the rings are elementary equivalent with respect to L_1.

Theorem 4.5 *Let G and H be groups and R and S be commutative rings with $1 \neq 0$. View the group rings $R[G]$ and $S[H]$ as standard models of T_2, A and B respectively. If $A \equiv_{L_2} B$, then $G \equiv_{L_0} H$ and $R \equiv_{L_1} S$.*

The converse is not true in general, that is whether or not $G \equiv_{L_0} H$ and $R \equiv_{L_1} S$ imply that $R[G] \equiv_{L_2} S[H]$.

5 Axiomatics for the Universal Theory of Free Group Rings

Myasnikov and Remeslennikov [MR] proved the following. A group is CSA or conjugately separated abelian if maximal abelian subgroups are malnormal. A subgroup M of a group G is malnormal if $g^{-1}Mg \cap M \neq \{1\}$ implies that $g \in M$. The CSA property implies CT.

Theorem 5.1 *Let G be a non-abelian CSA group which is equationally Noetherian. Then the universal theory of G with respect to $L_0[G]$ is axiomatizbale by the set Q of quasi-identities of $L_0[G]$ true in G together with CT when the models are restricted to be G-groups.*

It was subsequently shown by Fine, Gaglione and Spellman [FGS] that the equationally Noetherian condition is not necessary and hence we have the following.

Theorem 5.2 *Let F be a non-abelian free group. Then any F-group H which is a model of the set Q of quasi-identities of $L_0[F]$ true in F together with CT is already a model of the universal theory of F with respect to $L_0[F]$.*

It is known that that the elementary theory of a CT group G is axiomatized by the set $H(G)$ of Horn sentences true in G together with CT (see [MR] for terminology)). Further Myasnikov and Remeslennikov have proved [MR] the universal theory of a CSA group G is given by the diagram of G, the strict universal Horn sentences of $L_0[G]$ true in G and commutative transitivity. Myasnikov and Remeslennikov required G to be equationally Noetherian but it was shown in [FGS] that equationally Noetherian is superfluous. In light of this result and the examples of universal sentences in free group rings the hazy conjecture was made that the universal theory of a free group rings consisted of the universal theory of free groups appropriately modified to group ring theory and vice versa. The main result in [FGKRS] is the following which shows this to be true in terms of axiom systems. We obtain a result similar to the theorem on elementary theory.

Theorem 5.3 *Let G be a group and suppose that the group ring $R[G]$ satisfies the diagram of the free group ring $Z[F]$, the strict universal Horn sentences of $L_2[Z[F]]$ true in $Z[F]$ and ring commutative transitivity. Then $R[G] \equiv_\forall Z[F]$ with respect to $L_2[Z[F]]$.*

6 Logical Rigidity

If G i a group then any group isomorphic to G must be elementarily equivalent to G. If the converse is true, that is being elementarily equivalent to G implies isomorphic to G then G is called **logically rigid.** The survey by I. Kazachkov [K] describes some examples and properties of logically rigid groups. It is difficult to come up with examples of such groups but it is known that a free solvable group is logically rigid. Relative to group rings the following can be shown.

Theorem 6.1 *Let G be a logically rigid group and Z the integers. Then the group ring $Z[G]$ is logically rigid.*

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