An effective study of continued fractions and applications (Theory and Applications of Proof and Computation)

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An effective study of continued fractions and applications

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1 Introduction

We have worked on computable aspects of real functions which are not necessarily continuous. [1] In [6], we developed a theory of IB(irrational based)-computability, which yields a unifying method to deal with such a theme. IB-computability means roughly that effectivity of acquiring function values as well as modulus of continuity is required only for computable sequences of irrational numbers.

Basing the notion of computability of real numbers and functions on computable sequences of irrational numbers (not on recursive rational sequences) was a natural arrival point due to the fact that a computable sequence of irrational numbers can be uniquely represented with a recursive (double) sequence of positive integers (except the first one) in terms of continued fractions, and so various properties of computable objects can be shown by dealing with recursive sequences of positive integers. It is therefore important for us to study some effective properties of recursive continued fractions and their applications to mathematics.

There are several characterizations of computability of a real number, including the the one in terms of continued fraction representation. For a good survey, the reader is referred to [1].

Here in this article, we first review recursive continued fraction representations of computable irrational sequences (Section 2). Then, we introduce the notion of 'initially computable sequences of irrational numbers' whose continued fraction representations are recursive and, based on it, re-define the computability structure on the continuum (CF-computability), showing

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[1]General surveys are seen in [4] and [5]. A list of references is provided in [6] and [7].

that CF-computability is, as a family of sequences, identical with traditional computability (Section 3).

Applications of continued fractions in [2] involve mostly irrational numbers alone. So, in relation to Sections 2 and 3, we present effectivizations of some theorems in [2]: Liouville's theorem (Section 4), Quadratic irrational number theorem (Section 5) and Kuz'min's result (Sections 6 and 7). [2]

2 Recursive continued fraction representation and computability of irrational sequence

We employ technical terms, notations and theorems concerning continued fractions from [2]. Some frequently used notations and terminologies are listed below. $A \equiv [a_0; a_1, a_2, \cdots, a_k, \cdots]$ will denote an infinite sequence, where a_0 is an arbitrary integer and a_i for each $i \geq 1$ is a positive integer. A provides a regular continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

and hence A will be identified with this continued fraction.

It is known that A represents an irrational number, say α, and that, conversely, any irrational number is uniquely represented with an infinite continued fraction. A will be called the continued fraction representation of α. There is a one-to-one correspondence between irrational numbers and the representations in the form of A. We will hereafter identify them.

Given an A as above, $s_k \equiv [a_0; a_1, a_2, \cdots, a_k]$ will be called the k-th initial segment of A, and will be regarded as a finite continued fraction which represents a rational number, say γ_k. For $k \geq 1$, γ_k can be expressed in terms of the fraction of a pair of positive coprime numbers, say $\frac{p_k}{q_k}$, which is called the k-th order convergent of A. We will mostly deal with the case where $a_0 = 0$. p_k and q_k are then positive and $q_k > p_k$.

$\{\frac{p_k}{q_k}\}$, called the 'sequence of convergents associated with A', converges to α, and $\{q_k\}$ is strictly increasing (Section 4 of [2]). $r_k \equiv [a_k; a_{k+1}, a_{k+2}, \cdots, a_i, \cdots]$ is called the k-th remainder of A.

[2]For details, refer to [7], a longer version of this article, in which various proofs are presented without omissions.

Definition 2.1 (Recursive continued fraction) A as above is called recursive if the sequence $\{a_0, a_1, a_2, \cdots, a_i, \cdots\}$ is recursive.

Proposition 2.1 If A is recursive, then also $\{p_k\}$ and $\{q_k\}$ are, and hence $\{\frac{p_k}{q_k}\}$ is, recursive.

Proposition 2.2 The convergence of the associated convergents $\frac{p_k}{q_k}$ of a recursive A to the number α is effective. This property can be extended to a sequence or a multiple sequence.

Proof Note that $\{q_k\}$ is recursive and is strictly increasing. It also holds that $|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k^2}$ (Theorem 9 of [2]). Given p, one can effectively find a k_p such that $q_{k_p}^2 > 2^p$. $\{k_p\}$ is recursive and can serve as a recursive modulus of convergence.

In what follows, the word 'computable' without any modifier will mean the traditional 'computable', of real numbers, real sequences, and so on. See, for example, [3].

Proposition 2.3 Any recursive sequence of continued fractions determines a computable sequence of irrational numbers.

Proof By Propositions 2.1 and 2.2, $\{\frac{p_k}{q_k}\}$ is recursive and converges effectively to the irrational number α, and hence α is computable. This reasoning can be extended immediately to a recursive sequence of continued fractions.

Proposition 2.4 Let $\{\alpha_m\}$ be a computable sequence of irrational numbers. Then it can be represented by a recursive double sequence of positive integers $\{A_m\}$, where $A_m \equiv [a_{m0}; a_{m1}, a_{m2}, \cdots, a_{mk}, \cdots]$, in a manner that $\{\alpha_m\}$ can be approximated effectively by the recursive sequence of irreducible fractions $\{\frac{p_{mk}}{q_{mk}}\}$, the convergents associated with $\{A_m\}$.

Proof For a computable irrational number α, the process of determing A for α in [2] (the proof of Theorem 14, Section 5) is itself effective, since the integral part of a computable irrational number can be effectively calculated. It is hence extendible to a sequence of computable irrational numbers.

Putting Propositions 2.3 and 2.4 together, we obtain the following equivalence.

Proposition 2.5 Given a sequence of irrational numbers $\{\alpha_m\}$. It is computable if and only if its continued fraction representation $\{A_m\}$ is recursive.

Since those are in one-to-one correspondence as mentioned above, they will henceforth be identified.

Proposition 2.5 suggests that it should be natural to define computability structure on the continuum based on computable sequences of irrational numbers.

3 Computability of number sequences: CF-computability

Definition 3.1 (Computability of real number sequences: [3]) (i) A recursive sequence of rational numbers is *computable*.

(ii) A sequence of real numbers is *computable* if it is effectively approximated by a (double) sequence of recursive rational numbers. (Effective approximation means that there is a recursive modulus of convergence.)

Remark 1 1) Integers are included in (i) above. An integer sequence is computable if and only if it is recursive.

2) (ii) above creates non-recursive but computable rational sequences (cf. Example 4 in Chapter 0 of [3]).

3) It is proved in [3] that the effective limit of a computable (double) sequence is a computable number (sequence).

Definition 3.2 (Irrational sequence) A sequence of real numbers whose terms are all irrational will be called an *irrational sequence*.

Proposition 3.1 A recursive sequence of rational numbers is effectively approached by a computable double irrational sequence.

Proof Put

$$y_{nk} = s_n + \frac{1}{2^k\sqrt{2}}.$$

$\{y_{nk}\}$ is a computalbe double irrational sequence, which effectively converges to $\{s_n\}$ with a recursive modulus of convergence $\alpha(n,p) = p$.

Proposition 3.2 Let $\{x_n\}$ be any computable sequence of real numbers. Then there is a computable double irrational sequence $\{w_{np}\}$ which effectively converges to $\{x_n\}$.

Proof There is a recursive double sequence of rational numbers, say $\{r_{nk}\}$, which converges to $\{x_n\}$ with a recursive modulus of convergence $\beta(n,p)$. Applying Proposition 3.1 to $\{r_{nk}\}$, we obtain a computable triple irrational sequence, say $\{z_{nkl}\}$, which converges to $\{r_{nk}\}$ with a recursive modulus of convergence $\delta(n,k,p)$. Put $w_{np} = z_{n\beta(n,p+1)\delta(n,\beta(n,p+1),p+1)}$. $\{w_{np}\}$ will do, with a recursive modulus of convergence $\gamma(n,p) = p$.

Proposition 3.3 A sequence of integers is recursive if and only if it can be effectively approximated by a computable irrational sequence.

Proof Sufficiency is obvious from Proposition 3.1. Neccesity follows from the general properties that computability is closed under effective convergence and that a computable sequence of integers is recursive.

With those preparations, we can present an alternative definition of computability of real sequences based on computable irrational sequences.

Definition 3.3 (CF-computability) (i) An *irrational sequence* is called *CF-computable* if its continued fraction representation is recursive.

This case is especially called *initially CF-computable.*

(ii) *Any real sequence* is called *CF-computable* if it is effectively approximated by a double *initially CF-computable irrational sequence.*

Proposition 3.4 (1) (ii) in Definition 3.3 does not expand CF-computable irrational sequences.

(2) An irrational sequence is CF-computable if and only if it is computable.

Proof (1) Apply Proposition 2.5 twice and refer to (i) of Definition 3.3.
(2) By virtue of (1), an irrational sequence is CF-computable exactly when its continued fraction representation is recursive. Thus follows (2) from Proposition 2.5.

Summing up, the following are all equivalent for an *irrational sequence* $\mathbf{a} = \{\alpha_m\}$. (Apply (1) of Proposition 3.4 and Proposition 2.5.)

(a) $\mathbf{a}$ is CF-computable.
(b) $\mathbf{a}$ is initially CF-computable.
(c) The continued fraction corresponding to $\mathbf{a}$ is recursive.
(d) $\mathbf{a}$ is computable.

Using (a)$\sim$(b), we obtain the following.

Proposition 3.5 For any sequence of real numbers, the notion of computability and that of CF-computability coincide. As a consequence, CF-computability is closed under effective convergence.

4 Liouville's theorem

In this and the following sections, two examples will be presented in order to show the utility of recursive continued fraction representations of computable irrational sequences. The classical results are taken from [2]. Effective procedures to obtain some objects are mostly effectivizations of the proofs in [2].

Classical Liouville's Theorem (Theorem 27, Section 9 of [2]) For any irrational algebraic real number α of degree n, there is a number $C > 0$ such that, for any natural numbers p and q ($q > 0$), holds:

$$|\alpha - \frac{p}{q}| > \frac{C}{q^n}.$$

In order to state the effective version of this theorem, we employ the expression that, with a computable sequence of irrational algebraic numbers $\{\alpha_m\}$, a recursive sequence of integral polynomials $\{f_m(x)\}$ is associated. Precisely, $\{f_m(x)\}$ is a sequence of defining polynomials of $\{\alpha_m\}$ and the sequence of degrees of $\{f_m(x)\}$ as well as the sequence of finite sequences of coefficients of $\{f_m(x)\}$ are recursive.

Theorem 1 (Effective version of Liouville's theorem) Let $\{\alpha_m\}$ be a computable sequence of real irrational algebraic numbers, and suppose the sequence of degrees of its associated recursive sequence of integral polynomials $\{f_m(x)\}$ be $\{n_m\}$. Then there is a computable sequence of positive numbers $\{C_m\}$ so that, for any natural numbers p and q ($q > 0$),

$$|\alpha_m - \frac{p}{q}| > \frac{C_m}{q^{n_m}}.$$

Proof Let us first deal with the case of a single α and show that C can be effectively constructed from the information concerning α.

We consult the classical proof of Theorem 27 in [2], and refine it in order to effectivize the process. Let α be a computable irrational algebraic number of degree n with the associated integral polynomial $f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$, where $b_n > 0$. Since $f(x)$ is an integral polynomial, it is a computable real function.

Put $f(x) = (x - \alpha)g_1(x)$. $g_1(x)$ is also a computable real function, since the coefficients of $g_1(x)$ can be effectively determined. It holds that $g_1(\alpha) \neq 0$, and hence either $g_1(\alpha) > 0$ or $g_1(\alpha) < 0$. It can be effectively determined which is the case. We will deal with the case $g_1(\alpha) > 0$. The other case can be treated in exactly the same way.

1° Since $g_1(x)$ is effectively continuous and α is computable, one can effectively find a rational number $\delta > 0$ so that $g_1(x) > 0$ if $x \in [\alpha - \delta, \alpha + \delta]$. δ does not depend on p and q.

Each step of the foregoing process is effective, and hence it can be extended to the case of a computable sequence $\{\alpha_m\}$. 2° below and the rest of the proof can be extended to a sequence likewise.

2° $|\alpha - \frac{p}{q}| > \delta$ or $|\alpha - \frac{p}{q}| < \delta$, and it can be effectively detemined which is the case.

This is so since δ is rational, while $|\alpha - \frac{p}{q}|$ is irrational and computable.

Using 2°, we proceed as follows.

Case (a) $|\alpha - \frac{p}{q}| < \delta$. Then $\frac{p}{q} \in [\alpha - \delta, \alpha + \delta]$, and hence $g_1(\frac{p}{q}) > 0$ by 1°. In this case holds $\frac{p}{q} - \alpha \neq 0$. Put $M = \max_{x \in [\alpha-\delta, \alpha+\delta]} g_1(x)$. M is computable and effectively obtained, independent of p and q, and satisfies $0 < g_1(\frac{p}{q}) \leq M$. So, $|\frac{p}{q} - \alpha| \geq \frac{1}{q^n M}$.

Case (b) $|\alpha - \frac{p}{q}| > \delta$. Then, since $q \geq 1$, $|\alpha - \frac{p}{q}| > \frac{\delta}{q^n}$.

Now put $C = \frac{\min(\delta, \frac{1}{M})}{2}$. C is computable and is effectively determined from f and α, independent of p and q, and satisfies $0 < C < \delta, \frac{1}{M}$. C then satisfies, for both cases, $|\alpha - \frac{p}{q}| > \frac{C}{q^n}$.

[**Corollary** to Theorem 1: Constructing computable transcendental numbers] The classical construction in Section 9 of [2] itself produces computable transcendental numbers.

5 Quadratic irrational numbers

Classical Theorem (Quadratic numbers: Theorem 28 of [2]) Let α be an irrational number. α is quadratic if and only if its continued fraction is periodic.

Theorem 2 (Effective version of quadratic number theorem) Let $\{\alpha_m\}$ be a computable irrational sequence.

$\{\alpha_m\}$ is effectively quadratic if and only if its continued fraction representation is effectively periodic.

Precisely, the sufficient condition can be stated as follows: there is a recursive sequence of quadratic integral polynomials (the sequence of coefficients are recursive) which have $\{\alpha_m\}$ as simple roots.

The precise form of the necessary condition is the following: there are recursive sequences of positive integers $\{\kappa_m\}$ and $\{\lambda_m\}$, and a recursive (double) sequence of integer tuples $\{(a_{mi})_{i \leq (\kappa_m + \lambda_m - 1)}\}_m$, where a_{mi} is positive except for $i = 0$, so that α_m is represented by

$$A_m = [a_{m0}; a_{m1}, \cdots, a_{m(\kappa_m - 1)}, \overline{a_{m\kappa_m}, \cdots, a_{m(\kappa_m + \lambda_m - 1)}}],$$

where $\overline{a_{m\kappa_m}, \cdots, a_{m(\kappa_m + \lambda_m - 1)}}$ represents the block that iterates in A_m.

Proof The desired recursive objects are inherent in the classical proof in Section 10 of [2]. We thus merely pont out where effectivity matters in the classical proof.

'If' part: We will deal with one α, which is irrational and computable, hence its representation is recursive (Proposition 2.5) and effectively periodic. A quadratic polynomial associated with α will be determined effectively. Let α be represnted by the following A.

$$A \equiv [a_0; a_1, \cdots, a_{(\kappa-1)}, \overline{a_\kappa, \cdots, a_{(\kappa+\lambda-1)}}].$$

Let r_k be the k-th remainder of A. Since A is recursive and periodic, so is $\{r_k\}$: $r_k = r_{k+\lambda}$ for $k \geq \kappa$, and it is a computable and irrational sequence.

The corresponding $\{p_k\}$ and $\{q_k\}$ are also recursive. From the classical result, $\{r_k\}$ is a sequence of solutions of integral quadratic equations for $k \geq \kappa$. The construction in [2] is in fact effective, and hence is not reproduced here.

Due to effectivity of obtaining the coefficients of the desired quadratic polynomial, the argument above can easily be extended to a sequence $\{\alpha_m\}$.

'Only if' part: The classical proof in [2] is furnished with a method to obtain, for an irrational number α, three sequences of integers $\{A_n\}, \{B_n\}, \{C_n\}$ which have uniform bounds and satisfy

$$A_n r_n^2 + B_n r_n + C_n = 0. \tag{1}$$

This fact implies that there can be only finitely many values of $\{r_n\}$. Classically, one can then claim that $\{r_n\}$ is periodic. We will effectivize this process.

Let α be a computable, quadratic irrational number, with which are associated an integral equation $ax^2 + bx + c = 0$, so that

$$a\alpha^2 + b\alpha + c = 0, \tag{2}$$

and a recursive continued fraction representation $A \equiv [a_0; a_1, a_2, \cdots, a_n, \cdots]$. Put

$$X_n \equiv \frac{p_{n-1} r_n + p_{n-2}}{q_{n-1} r_n + q_{n-2}} = \alpha.$$

Substituting X_n for α in (2), we obtain $aX_n^2 + bX_n + c = 0$, from which follows

$$A_n r_n^2 + B_n r_n + C_n = 0,$$

where A_n, B_n, C_n are integers obtained effectively from the given data as below.

$$A_n = ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2;$$

$$B_n = 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + 2cq_{n-1}q_{n-2};$$

$$C_n = ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2.$$

This implies that r_n is a root of the equation

$$A_n x^2 + B_n x + C_n = 0. \tag{3}$$

By simple calculations, it can be derived that

$$C_n = A_{n-1}; \quad B_n^2 - 4A_nC_n = b^2 - 4ac. \tag{4}$$

Following the classical argument, we can find a positive integer E, uniformly for all n, such that

$$|A_n|, |B_n|, |C_n| < E. \tag{5}$$

E can be effectively acquired as an application of (4). From the definition and the computability of $2|a\alpha| + |a| + |b|$, one can effectively find a positive integer D such that, for all n,

$$|A_n| < 2|a\alpha| + |a| + |b| < D.$$

Then, $|C_n| < D$ also follows, hence $|B_n| < \sqrt{|b^2 - 4ac + 4D^2|} < D'$ for an integer D', effectively evaluated. Now put $E = \max(D, D')$. There can be less than $F = 8E^3$ many triples $G_n = (A_n, B_n, C_n)$.

Having all this information, we can find effectively, numbers κ and λ such that $\kappa \leq 2F + 1$ and $\lambda \leq 2F + 1 - \kappa$ so that

$$\forall k \geq \kappa . a_k = a_{k+\lambda}. \tag{6}$$

This assures us that $\{a_n\}$ is periodic from κ onward with period λ. To find such κ and λ effectively, we proceed as follows.

Among G_n for $n \leq 2F+1$, there occur at least three distinct numbers n_i, $i = 1, 2, 3$, such that $G_{n_1} = G_{n_2} = G_{n_3}$. Let $G \equiv (A, B, C)$ be the common triple of these three. The equation (3) with the coefficients A, B, C have at most two roots, which can be effectively calculated, hence are computable. Let them be s_1 and s_2. Since the discriminant of the equation is an integer, it can be determined whether they are equal or not. Corresponding to $G_{n_1}, G_{n_2}, G_{n_3}$, there are remainders, say, r^1, r^2, r^3, and at least two of them must be equal. $ax^2 + bx + c$ is the quadratic polynomial for α, and so $b^2 - 4ac > 0$. Since $B_n^2 - 4A_nC_n = b^2 - 4ac > 0$ by the equation (4), $s_1 \neq s_2$. Compare each of r^1, r^2, r^3 with each of s_1, s_2. At least two of the former must be inequal with one of the latter, say s_1 for simplicity. Then they must be equal to the other, which means that the two are equal. All this can be effectively determined, since the inequalities involved here are decidable, and the equality is not directly judged. Suppose r_κ is the one with the least subscript, and $r_{(\kappa+\lambda)}$ be the second. From the definition of $\{r_n\}$, follows that $r_\kappa = r_{\kappa+\lambda}$ implies $a_\kappa = a_{\kappa+\lambda}$ and, for any i, $a_{\kappa+i} = a_{\kappa+\lambda+i}$, entailing (6).

The argument above can be extended effectively to a sequence $\{\alpha_m\}$.

6 Recursively defined function sequence

One of the significant topics in [2] is Gauss's problems (Section 15 of Chapter III in [2]). On the way to it, 'Kuz'min's result' is proved at length in Section 7 of [2]. In order to effectivize it, a meta-lemma (Theorem 3) will be proved in this section.

[**General assumption**] Throughout the rest of this article, the domain of functions is $[0, 1]$, a compact and computable interval.

Kuz'min considers the following function sequence, which is recursively defined from an initial function f_0.

Definition 6.1 (Kuz'min's function sequence: cf. Theorem 33 in [2]) $\{f_n(x) : n = 0, 1, 2, \dots\}$ is a sequence of real functions defined by the equation

$$f_{n+1}(x) = \Sigma_{k=1}^{\infty} \frac{1}{(k+x)^2} f_n(\frac{1}{k+x}), \quad (n \geq 0). \tag{7}$$

It is assumed that $f_0(x)$ is a differentiable function, and that $0 < f_0(x) < M$ and $|f_0^{'}(x)| < \mu$ for some $M > 0$ and $\mu > 0$.

Remark 2 (1) Since f_0 is continuous on a compact interval, $M = \max\{f_0(x) : 0 \leq x \leq 1\} + 1$ will do for the above M. If $f_0(x)$ is computable, then M is a computable number.

(2) It is classically true that the infinite summation of (7) converges and that continuity of $f_n(x)$ for every n is proved by induction on n.

The following lemma is placed here for later use. It is a consequence of Lemma 3 in Section 15 of [2], and no effectivity problem is involved.

Lemma 1 Of the functions defined by (7), the assumption $0 < f_0(x) < M$ implies $0 < f_n(x) < G = 2M$ for all $n = 0, 1, 2, \dots$.

Theorem 3 (Computable case of Kuz'min's function sequence) Let $f_0(x)$ be a positive computable function. Then the function sequence $\{f_n(x)\}$ defined by the formula (7) is computable.

Proof [3] First, notice that $0 < f_0(x) < M$, M being the computable number as in (1) of Remark 2, and Lemma 1 is valid.

Define some functions and numbers.

$$s_{n,l}(x) = \Sigma_{k=1}^{l} \frac{1}{(k+x)^2} f_n(\frac{1}{k+x});$$

$$t_l(x) = \Sigma_{k=l+1}^{\infty} \frac{1}{(k+x)^2};$$

$$w_l = \Sigma_{k=1}^{l} \frac{1}{k^2};$$

$$\delta(p) = \min_l [G(\frac{\pi^2}{6} - w_l) < \frac{1}{2^p}],$$

where $G = 2M$. Then, $\{t_l(x)\}$ is a computable function sequence, $\{w_l\}$ is a computable sequnce of numbers, and $\delta(p)$ is a recursive function. Note that those are independent of n.

The theorem will be established by showing the following **I** and **II** simultaneously.

I The function sequence $\{f_n(x)\}$ is computable.

II The double sequence of functions $\{s_{n,l}(x)\}$ is computable.

What are to be worked out are the following.

(i) (Sequential computability) Given a computable sequence of numbers $\{x_m\}$. $\{f_n(x_m)\}$ is a computable double sequence of real numbers and $\{s_{n,l}(x_m)\}$ is a computable triple sequence of real numbers.

(ii) (Effective and uniform continuity) There are recursive functions $d(n,p)$ and $\gamma(n,p)$ such that

$$\forall p, n, l \forall x, y. |x - y| < \frac{1}{2^{d(n,p)}} \Rightarrow |s_{n,l}(x) - s_{n,l}(y)| < \frac{1}{2^p};$$

$$\forall p, n \forall x, y. |x - y| < \frac{1}{2^{\gamma(n,p)}} \Rightarrow |f_n(x) - f_n(y)| < \frac{1}{2^p}.$$

Notice that d depends only on n, p and is independent of l.

For $n = 0$, computability of f_0 is assumed, and hence hold (i) and (ii) for f_0. $s_{n,l}$ is not irrelivant here.

By sheer computation, we obtain the following.

$$t_l(x) \le \Sigma_{k=l+1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - \Sigma_{k=1}^{l} \frac{1}{k^2} = \frac{\pi^2}{6} - w_l.$$

By Lemma 1,

$$f_{n+1}(x) - s_{nl}(x) = \Sigma_{k=l+1}^{\infty} \frac{1}{(k+x)^2} f_n(\frac{1}{k+x}) \le G \Sigma_{k=l+1}^{\infty} \frac{1}{(k+x)^2} = G t_l(x).$$

[3] The facts $\Sigma_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and $1 < \Sigma_{k=1}^{\infty} \frac{1}{k^3} = 1.202 \cdots < 2$ will be utilized in the proof.

From those follows

$$0 < f_{n+1}(x) - s_{nl}(x) \leq G(\frac{\pi^2}{6} - w_l).$$

With the function δ above holds

$$\forall p \forall l \geq \delta(p) \forall x. |f_{n+1}(x) - s_{nl}(x)| < \frac{1}{2^p}. \tag{8}$$

Summing up, the facts below are at our disposal.

(a) $0 < f_n(x) < G = 2M$ for all n: by Lemma 1.

(b) There is a recursive function $\delta(p)$ such that (8) holds.

Proof of (i) We take the advantage of Definition 5a in Section 2, Chapter 0, of [3], an alterntive definition of computable real sequences.

Definition 5a,[3]: A real sequence $\{x_n\}$ is computable if there is a recursive sequence of rationals, $\{r_{nk}\}$ satisfying

$$\forall n \forall k. |x_n - r_{nk}| < \frac{1}{2^k}. \tag{9}$$

When (9) holds, $\{r_{nk}\}$ is said to effectively converge to $\{x_n\}$. This definition of computability is equivalent to the original one.

Given a computable sequence of numbers, say $\{x_m\}$. In order to claim computability of the double sequence $\{f_n(x_m)\}_{nm}$, it suffices, in view of Definition 5a above, to construct a recursive triple sequence of rational numbers $\{r_{nmp}\}$ such that

$$\forall n \forall m \forall p. |f_n(x_m) - r_{nmp}| < \frac{1}{2^p}. \tag{10}$$

f_0 is assumed to be computable, hence there is a sequence $\{r_{0mp}\}$ effectively converging to $\{f_0(x_m)\}_{0m}$.

Suppose such a sequence has been defined effectively for $i \leq n$ (for all m and p), $\{r_{imp}\}_{i \leq n}$, so that (10) holds for $i \leq n$. The construction of a recursive rational sequence converging to $\{s_{nl}(x_m)\}$ will be assumed here. It is constructed effectively from $\{r_{nmp}\}_{m,p}$. Let it be $\{q_{nlmp}\}$, [4] so that

$$\forall l \forall p. |s_{nl}(x_m) - q_{nlmp}| < \frac{1}{2^p}. \tag{11}$$

From (8) follows

$$\forall p \forall l \geq \delta(p) \forall m. |f_{n+1}(x_m) - s_{nl}(x_m)| < \frac{1}{2^p}. \tag{12}$$

[4] The construction of $\{q_{nlmp}\}$ is omitted here, since it is a mere technicality, though laborious.

By (11) and (12),

$$|f_{n+1}(x_m) - q_{n\delta(p)mp}| < \frac{1}{2^{p-1}}.$$

Now put $r_{(n+1)mp} = q_{n\delta(p+1)m(p+1)}$. Then (10) holds for $n+1$.

$\{r_{nmp}\}$ is a recursive triple sequence of rational numbers satisfying (10).

Proof of (ii) Effective and uniform continuity is proved as follows.

First, define recursive functions ν, d, γ. (δ above will also be used.) Since f_0 is computable, there is a recursive modulus of uniform continuity for it, say ζ.

$$\nu(p) = \min_m . \frac{1}{2^m} < \frac{2 - \frac{\pi^2}{6}}{2^{p+3}G};$$

$$d(0,p) = \max(\zeta(p), \nu(p)); \quad d(n,p) = \max(d(n-1, p+3), \nu(p+3)) \quad (n \geq 1);$$

$$\gamma(0,p) = \zeta(p); \quad \gamma(n,p) = d(n-1, p+3) \quad (n \geq 1).$$

What is to be proved is the following: for all n, p.

$$|x - y| < \frac{1}{2^{\gamma(n,p)}} \Rightarrow |f_n(x) - f_n(y)| < \frac{1}{2^p}. \tag{13}$$

The proof is carried out by induction on n. Namely, assume (13) for n and for all p, and then establish (13) for $n+1$:

$$|x - y| < \frac{1}{2^{\gamma(n+1,p)}} \Rightarrow |f_{n+1}(x) - f_{n+1}(y)| < \frac{1}{2^p}. \tag{14}$$

For the proof of (14), we will first show the following.

$$|x - y| < \frac{1}{2^{\gamma(n+1,p)}} = \frac{1}{2^{d(n,p+3)}} \Rightarrow |s_{n,l}(x) - s_{n,l}(y)| < \frac{1}{2^{p+2}}. \tag{15}$$

Notice that this holds uniformly in l.

For the proof of (15), assume

$$|x - y| < \frac{1}{2^{\gamma(n+1,p)}}.$$

By the definitions of functions, for $n \geq 1$, [5]

$$\gamma(n+1, p) = d(n, p+3) = \max(d(n-1, p+6), \nu(p+3))$$

$$= \max(\gamma(n, p+3), \nu(p+3)) \geq \gamma(n, p+3), \nu(p+3),$$

hence from the assumption follows

$$|x - y| < \frac{1}{2^{\gamma(n,p+3)}}; \quad |x - y| < \frac{1}{2^{\nu(p+3)}}. \tag{16}$$

[5] For $n = 0$: $\gamma(1,p) = d(0,p+3) = \max(\zeta(p+3), \nu(p+3)) = \max(\gamma(0,p+3), \nu(p+3)) \geq \gamma(0, p+3), \nu(p+3)$.

$$|s_{n,l}(x) - s_{n,l}(y)| \leq \Sigma_{k=1}^{l}(I_k + J_k),$$

where

$$I_k = |\frac{1}{(k+x)^2} f_n(\frac{1}{k+x}) - \frac{1}{(k+x)^2} f_n(\frac{1}{k+y})| = \frac{1}{(k+x)^2}|f_n(\frac{1}{k+x}) - f_n(\frac{1}{k+y})|.$$

$$J_k = |\frac{1}{(k+x)^2} f_n(\frac{1}{k+y}) - \frac{1}{(k+y)^2} f_n(\frac{1}{k+y})| = f_n(\frac{1}{k+y})|\frac{1}{(k+x)^2} - \frac{1}{(k+y)^2}|.$$

As for the arguments of f_n, using (16),

$$|\frac{1}{k+x} - \frac{1}{k+y}| \leq |y - x| < \frac{1}{2^{\gamma(n,p+3)}}.$$

Then by the induction hypothesis, the conclusion of (13) holds for $p+3$. We have thus

$$I_k \leq \frac{1}{(k+x)^2}\frac{1}{2^{p+3}} \leq \frac{1}{k^2}\frac{1}{2^{p+3}} \tag{17}$$

Next consider J_k. By (a), (16) and the definition of ν,

$$J_k < \frac{4G|x-y|}{k^3} < \frac{4G}{k^3}\frac{1}{2^{\nu(p+3)}} < \frac{2 - \frac{\pi^2}{6}}{k^3 2^{p+4}}. \tag{18}$$

By (17) and (18),

$$\Sigma_{k=1}^{l}(I_k + J_k) = \Sigma_{k=1}^{l} I_k + \Sigma_{k=1}^{l} J_k \leq \frac{1}{2^{p+3}}\Sigma_{k=1}^{l}\frac{1}{k^2} + \frac{2 - \frac{\pi^2}{6}}{2^{p+4}}\Sigma_{k=1}^{l}\frac{1}{k^3}$$

$$\leq \frac{1}{2^{p+3}}\Sigma_{k=1}^{\infty}\frac{1}{k^2} + \frac{2 - \frac{\pi^2}{6}}{2^{p+4}}\Sigma_{k=1}^{\infty}\frac{1}{k^3} \leq \frac{1}{2^{p+3}}\frac{\pi^2}{6} + \frac{2 - \frac{\pi^2}{6}}{2^{p+4}}2 = \frac{1}{2^{p+2}}$$

Summing up, under the assumption $|x - y| < \frac{1}{2^{\gamma(n+1,p)}}$, we have the conclusion of (15): $|s_{n,l}(x) - s_{n,l}(y)| < \frac{1}{2^{p+2}}$.

Now the proof of (14). Assume $|x - y| < \frac{1}{2^{\gamma(n+1,p)}}$, and put $l_p = \delta(p+2)$. By (8) and (15), we obtain

$$|f_{n+1}(x) - f_{n+1}(y)|$$

$$\leq |f_{n+1}(x) - s_{n,l_p}(x)| + |s_{n,l_p}(x) - s_{n,l_p}(y)| + |s_{n,l_p}(y) - f_{n+1}(y)|$$

$$< \frac{1}{2^{p+2}} + \frac{1}{2^{p+2}} + \frac{1}{2^{p+2}} < \frac{1}{2^p}.$$

This completes the proof.

7 Effectivization of Kuz'min's result

Theorem 4 (Effective version of Kuz'min's basic result: cf. Theorem 33 of [2]) Consider a sequence of functions $\{f_n\}$ on the compact and computable interval $[0,1]$, which satisfies the conditions below.
(K1) (Condition on f_0)
f_0 is computable, differentiable, and possessed of the following properties:

$$0 < f_0(x) < M, \quad |f_0'(x)| < \mu, \quad x \in [0,1].$$

where M and μ are positive computable numbers.
(K2) (Recursive definition of function sequence) The functions in the sequence are successively defined as in (7).

From the conditions (K1) and (K2) follow the facts below.
(R1) (Computability)
$\{f_n\}$ is a computable sequence of functions.
(R2) (Effective representation)
There are computable numbers $a_0, A = A(M,\mu), \lambda$, and a computable sequence of functions $\{\theta_n(x)\}$ with which $\{f_n\}$ is represented:

$$f_n(x) = \frac{a_0}{1+x} + \theta_n(x)Ae^{-\lambda\sqrt{n}}, \tag{19}$$

where $a_0 = \frac{1}{\log 2}\int_0^1 f_0(z)dz$, $|\theta_n(x)| < 1$ and $\lambda > 0$. a_0 and λ are absolute constants, and $0 < A = A(M,\mu)$ is obtained from M and μ effectively.

Remark 3 The computable constants A and λ are to be constructed, independent of n, x. As for $\theta_n(x)$, the existence is classically secured, and its computability is forced by (R1) and the formula (19).

Proof (R1) is Theorem 3 in Section 6.

(R2) Those constants can be obtained by faithfully tracing the proof of Theorem 33 in [2] (cf. pp.74-81). The proof in [2] is not reproduced here. Only relevant lines will be quoted, by pointing out where effectivization is necessary. There are four lemmas to Kuz'min's theorem. Those are used for mathematical proofs and no effectivity is questioned thereby. Computable (sequences of) numbers will be successively gained.

1^* Let $m > 0$ be the minimum value of $f_0(x)$, which is computable, and put $g = \frac{m}{2}$, $G = 2M$. Then

$$\frac{g}{(1+x)} < f_0(x) < \frac{G}{(1+x)}. \tag{20}$$

2^* For any (temporarily) fixed n and for any $l = 1, 2, \ldots, n$, one can define numbers $g_{n\cdot l}, G_{n\cdot l}$ so that $0 < g < g_{n\cdot(l-1)} < g_{n\cdot l} < G_{n\cdot l} < G_{n\cdot(l-1)} < G$, and $\frac{g_{n\cdot l}}{1+x} < f_{n\cdot l}(x) < \frac{G_{n\cdot l}}{1+x}$. In particular,

$$\frac{g_{n^2}}{1+x} < f_{n^2}(x) < \frac{G_{n^2}}{1+x}.$$

($n \cdot l$ is meant here the multiple of n and l.)

3^* Define $\mu_{nl} = \frac{\mu}{2^{n\cdot l-3}} + 4M$. This is a computable double sequence of positive numbers (for $l \le n$), depending also on μ, M. Then, by Lemma 2 in Section 15 of [2], [6]

$$|f'_{n\cdot(l-1)}(x)| < \mu_{n(l-1)}.$$

4^* There are several points at which a formula is valid 'for sufficiently large n'. They are the following three.

(i) Put $a = \frac{1}{2}\int_0^1 (f_0(x) - \frac{g}{1+x})dx$. For sufficiently large n, $a - \frac{\mu+g}{2^{n-1}} > 0$.
(ii) Put $a' = \frac{1}{2}\int_0^1 (\frac{G}{1+1} - f_0(x))dx$. For sufficiently large n, $\frac{\mu+G}{2^{n-1}} - a' < 0$.
(iii) $\mu_{nl} < 5M$, or $\frac{\mu}{2^{n\cdot l-3}} < M$ holds for sufficiently large n.

In any of (i)-(iii), all the terms are computable, and hence a relevant n can be effectively found for each case from M, μ and computability of f_0. Let $n_0 = n_0(M, \mu)$ denote the greatest among them. We will call an n sufficiently large if $n \ge n_0$. Then for all sufficiently large n, the inequalities in (i)-(iii) hold.

5^* For sufficiently large n,

$$G_{n^2} - g_{n^2} < (G-g)\delta^n + 2^{-n+2}[(\mu+2M)\delta^{n-1} + 7M\Sigma_{i=n-2}^{i=0}\delta^i](\equiv H_m), \quad (21)$$

where $\delta = 1 - \frac{\log 2}{2}$, hence $0 < \delta < 1$. Put $L = \max\{\mu + 2M, 7M\}$.

$$(\mu + 2M)\delta^{n-1} + 7M\Sigma_{i=n-2}^{i=0}\delta^i \le L\Sigma_{i=n-1}^{0}\delta^i = L\frac{1-\delta^n}{1-\delta} < \frac{L}{1-\delta}.$$

From this and (21), follows

$$0 < G_{n^2} - g_{n^2} < (G-g)\delta^n + 2^{-n}4\frac{L}{1-\delta} \le G\delta^n + 2^{-n}4\frac{L}{1-\delta}(\equiv H). \quad (22)$$

6^* Define λ and B by:

$$\lambda = \min(-\log\delta, \log 2)(= -\log\delta); \quad B = G + 4\frac{L}{1-\delta}.$$

[6] Differentiability of f_n for all n is classically ensured.

Then , $\lambda, B > 0$, and $\delta^n \leq e^{-\lambda n}$ and $2^{-n} \leq e^{-\lambda n}$, implying $H < Be^{-\lambda n}$.

7* By virtue of (22), $G_{n^2} - g_{n^2}$ converges effectively to 0, or $\lim G_{n^2} = \lim g_{n^2}$, whose value a_0 is, in fact, $a_0 = \frac{1}{\log 2}\int_0^1 f_0(z)dz$. [7].

8* Put $A = 2Be^{\lambda}$.

9* For an arbitrary $N = 0, 1, 2, \ldots$, let $\nu(N)$ denote the n satisfying $n^2 \leq N < (n+1)^2$. ν is a recursive function. Put $N_0 = n_0^2$ for n_0 in 4*. Then $N \geq N_0$ implies $\nu(N) = n \geq n_0$.

10* For $N \geq N_0$ holds

$$|f_N(x) - \frac{a_0}{1+x}| < Ae^{-\lambda\sqrt{N}}. \tag{23}$$

a_0, B, A, λ are the constants defined in 7* and 6*.

11* For each $N = 0, 1, 2, \ldots, N_0 - 1$, an $A = A(N)$ can be selected so that (23) holds.

12* Put $A^* = \max\{A(N)(N = 0, 1, 2, \ldots, N_0 - 1), A\}$.

13* Due to 10*, 11*, and 12*, the formula in (23) with A^* in the place of A holds for all N:

$$|f_N(x) - \frac{a_0}{1+x}| < A^* e^{-\lambda\sqrt{N}}. \tag{24}$$

14* Define $\theta_N(x)$ on $[0, 1]$ by

$$\theta_N(x) = \frac{e^{\lambda\sqrt{N}}}{A^*}(f_N(x) - \frac{a_0}{1+x}).$$

$\{\theta_N(x)\}$ is a computable sequence of functions, since $\{f_N(x)\}$ is (Theorem 3). $|\theta_N(x)| < 1$, and we have

$$f_N(x) = \frac{a_0}{1+x} + \theta_N(x)A^* e^{-\lambda\sqrt{N}}.$$

[**Application**] Kuz'min's result is a prelude to a Gauss's problem (cf. Section 15 of [2]). There is nothing to add to it here. We only lightly reference that a function sequence relevant to it, $\{m_n(x)\}$, is in fact computable.

$$\omega = [0; a_1, a_2, \ldots, a_n \ldots]; \quad z_n(\omega) = [0; a_{n+1}, a_{n+2}, \ldots].$$

$$S_n(x) = \{\omega : z_n(\omega) < x\}; \; m_n(x) = \mathcal{M}(S_n(x)),$$

[7] Effectivity of convergence of $G_{n^2} - g_{n^2}$ is established without any mention of computability of $\{G_n\}$ and $\{g_n\}$

where $\mathcal{M}$ denotes Lebesgue measure.

The function sequence $\{m_n(x)\}$ satisfies

$$m_{n+1}(x) = \Sigma_{k=1}^{\infty}(m_n(\frac{1}{k}) - m_n(\frac{1}{k+x})), x \in [0,1], n \geq 0. \tag{25}$$

The formal differentiation of equation (25),

$$m'_{n+1}(x) = \Sigma_{k=1}^{\infty}\frac{1}{(k+x)^2}m'_n(\frac{1}{k+x}), \tag{26}$$

deternines a function.

In order to apply Theorem 4, put $f_n(x) := m'_n(x)$. Since $m_0(x) = x$, $f_0(x)$ and $f_0^{'}(x)$ are computable. So, the conditions (K1) and (K2) hold for $\{f_n(x)\}$ with $M = 2$ and $\mu = 1$, and hence the results (R1) and (R2) are valid. Computability of $\{m'_n(x)\}$, hence also that of $\{m_n(x)\}$, now follows.

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