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On the kernel of the surgery map

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Abstract

A Jacobi diagram gives a clasper in the trivial homology cylinder, and then one obtains another homology cylinder by surgery along the clasper. This procedure defines a homomorphism $\mathfrak{s}_n\colon \mathcal{A}_n^c \to Y_n\mathcal{IC}_{g,1}/Y_{n+1}$ between abelian groups. Sato, Suzuki, and the author [15, 16] constructed a homomorphism on $Y_n\mathcal{IC}_{g,1}/Y_{n+1}$, and gave an application to the study of the surgery map $\mathfrak{s}_n$. The purpose of this article is to review the results in [16] and introduce related works on the surgery map.

1 Introduction

Let $\Sigma_{g,1}$ denote a connected oriented compact surface with one boundary component. For an oriented compact 3-manifold M and an orientation-preserving homeomorphism $m\colon \partial(\Sigma_{g,1} \times [-1,1]) \to \partial M$, we call (M,m) a *cobordism*. Here, (M,m) and (M',m') are identified if there exists a homeomorphism $f\colon M \to M'$ satisfying $f \circ m = m'$. In this article, we focus on cobordisms with a certain homological condition. A cobordism (M,m) is called a *homology cylinder* over $\Sigma_{g,1}$ if the restrictions $m_\pm$ of m to $\Sigma_{g,1} \times \{\pm 1\}$ induce the same isomorphism $(m_\pm)_*\colon H_*(\Sigma_{g,1};\mathbb{Z}) \to H_*(M;\mathbb{Z})$. The set $\mathcal{IC} = \mathcal{IC}_{g,1}$ of homology cylinders over $\Sigma_{g,1}$ has a monoid structure defined by stacking $M \circ M' = (M \cup_{m_+ = m'_-} M', m_- \cup m'_+)$. Our motivations for studying the monoid $\mathcal{IC}$ is as follows:

(1) Let $\mathcal{I} = \mathcal{I}_{g,1}$ denote the Torelli subgroup of the mapping class group of the surface $\Sigma_{g,1}$, and then $\mathcal{IC}$ is regarded as an extension of $\mathcal{I}$ to 3-dimensional topology. In fact, for $M = \Sigma_{g,1} \times [-1,1]$ and the map m defined by $m_+ = f \in \mathcal{I}$ and $m_- = \mathrm{id}_{\Sigma_{g,1}}$, the cobordism (M,m) is a homology cylinder. This construction gives a monoid homomorphism $\mathfrak{c}\colon \mathcal{I} \to \mathcal{IC}$, which is known to be injective.

(2) Also, $\mathcal{IC}$ is naturally defined in terms of clasper surgery (see Section 3.1). Indeed, the set of homology cylinders obtained from the trivial one $\mathfrak{c}(\mathrm{id}_{\Sigma_{g,1}})$ by clasper surgery coincides with $\mathcal{IC}$.

(3) The monoid $\mathcal{IC}$ is closely related to the homology cobordism group $\mathcal{IH} = \mathcal{IH}_{g,1}$ of homology cylinders (see Section 3.3). In particular, $\mathcal{IH}_{0,1}$ is isomorphic to the homology cobordism group $\Theta^3_{\mathbb{Z}}$ of oriented integral homology 3-spheres, and thus $\mathcal{IH}_{g,1}$ is regarded as an extension of $\Theta^3_{\mathbb{Z}}$.

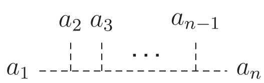
The topics (1)–(3) are respectively related to 2-, 3- and 4-dimensional topology, and $\mathcal{IC}$ attracts considerable attention in low-dimensional topology.

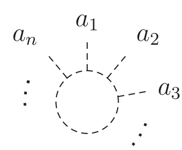
In general, studying $\mathcal{IC}_{g,1}$ is hard since it is not commutative when $g \geq 1$. Following Goussarov [5] and Habiro [6], we define a certain equivalence relation $\sim_{Y_n}$ among homology cylinders. We then consider a submonoid

$$Y_n\mathcal{IC} = \{(M,m) \in \mathcal{IC} \mid (M,m) \sim_{Y_n} \mathfrak{c}(\mathrm{id}_{\Sigma_{g,1}})\}$$

and a descending series $\mathcal{IC} = Y_1\mathcal{IC} \supset Y_2\mathcal{IC} \supset \cdots$ of submonoids. It is known that, for $n \geq 1$, the quotient $Y_n\mathcal{IC}/Y_{n+1}$ by $\sim_{Y_{n+1}}$ is a finitely generated abelian group. Since the group $Y_n\mathcal{IC}/Y_{n+1}$ measures the gap between consecutive terms $Y_n\mathcal{IC}$ and $Y_{n+1}\mathcal{IC}$, the study of these groups can be the first step toward understanding the monoid $\mathcal{IC}$.

Let us introduce a Jacobi diagram, which enables us to describe $Y_n\mathcal{IC}/Y_{n+1}$ combinatorially. A *Jacobi diagram* is a uni-trivalent graph such that each univalent vertex is colored by an element of the set $\{1^+,\ldots,g^+,1^-,\ldots,g^-\}$ and each trivalent vertex has a cyclic order. Throughout this article, cyclic orders are assumed to be counter-clockwise and graphs are drawn by dashed lines. Also, we use the following notation:

$T(a_1,a_2,\ldots,a_n) =$ 

, $O(a_1,a_2,\ldots,a_n) =$ 

.

Let $\mathcal{A}_n^c$ denote the $\mathbb{Z}$-module generated by connected Jacobi diagrams with n trivalent vertices subject to the AS, IHX, and self-loop relations. The module $\mathcal{A}_n^c$ naturally decomposes into the direct sum $\bigoplus_{l\geq 0}\mathcal{A}_{n,l}^c$ with respect to the first Betti number. For instance, $\mathcal{A}_2^c = \mathcal{A}_{2,0}^c \oplus \mathcal{A}_{2,1}^c \oplus \mathcal{A}_{2,2}^c$ is generated by $T(a_1,a_2,a_3,a_4)$'s, $O(b_1,b_2)$'s, and the θ-graph.

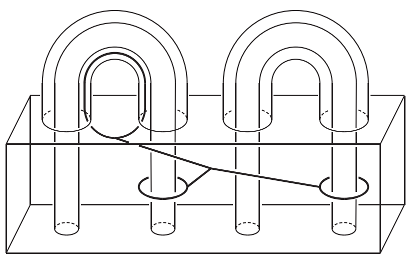


Figure 1: Clasper in $\Sigma_{2,1} \times [-1,1]$.

Let $J \in \mathcal{A}_n^c$ be a Jacobi diagram. We obtain a graph clasper G of degree n in $\Sigma_{g,1} \times [-1,1]$ from J, which defines an equivalence class $\mathfrak{s}_n(J) \in Y_n\mathcal{IC}/Y_{n+1}$ by surgery along G. See Figure 1 for the case $J = T(1^-,1^+,2^-)$. Roughly speaking, this procedure gives a homomorphism $\mathfrak{s}_n\colon \mathcal{A}_n^c \to Y_n\mathcal{IC}/Y_{n+1}$, which is called the *surgery map* and known to be surjective for $n \geq 2$. Therefore, the key to understanding the group $Y_n\mathcal{IC}/Y_{n+1}$ is to investigate the kernel $\mathrm{Ker}\,\mathfrak{s}_n$. In fact, for $n = 1,2$, Massuyeau and Meilhan [11, 12] described $\mathrm{Ker}\,\mathfrak{s}_n$ and determined the group structures of $Y_n\mathcal{IC}/Y_{n+1}$. In [15], Sato, Suzuki, and the author determined $Y_n\mathcal{IC}/Y_{n+1}$ when $n = 3$. Here note that it is shown in [1] that $\mathfrak{s}_n \otimes \mathrm{id}_{\mathbb{Q}}\colon \mathcal{A}_n^c \otimes \mathbb{Q} \to (Y_n\mathcal{IC}/Y_{n+1}) \otimes \mathbb{Q}$ is an isomorphism. This means that $Y_n\mathcal{IC}/Y_{n+1}$ is completely described by $\mathcal{A}_n^c$ if we ignore torsion elements. Therefore, the essential contribution of our paper [15] is investigations of the torsion subgroup $\operatorname{tor}\mathcal{A}_n^c$ and of the restriction map $\mathfrak{s}_n|_{\operatorname{tor}\mathcal{A}_n^c}$.

Let us review a related topic by Conant, Schneiderman, and Teichner [4]. They were interested in the homology cobordism group $\mathcal{IH}$ and determined the surjective map

$$\mathcal{A}_{n,0}^c \xrightarrow{\mathfrak{s}_{n,0}} Y_n\mathcal{IC}/Y_{n+1} \xrightarrow{q} Y_n\mathcal{IH}/Y_{n+1}$$

when $n \not\equiv 1 \bmod 4$, where $\mathfrak{s}_{n,0}$ is the restriction of $\mathfrak{s}_n$ and q is the quotient map. We obtain information about $\operatorname{Ker}\mathfrak{s}_{n,0}$ via $\operatorname{Ker}(q \circ \mathfrak{s}_{n,0})$. However, one has to solve at least two problems (Problems 3.2 and 3.4) for going in this direction. On the other hand, the study of $\operatorname{Ker}\mathfrak{s}_{n,1}$ is important as the next step after $\operatorname{Ker}\mathfrak{s}_{n,0}$.

Problem 1.1. Determine the module structure of the kernel $\operatorname{Ker}\mathfrak{s}_{n,1} \subset \mathcal{A}_{n,1}^c$.

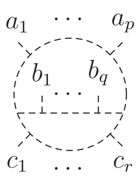
In [16], Sato, Suzuki, and the author made progress in Problem 1.1. We explain our results in Section 2 and review previous researches about claspers and the group $Y_n\mathcal{IC}/Y_{n+1}$ in Section 3. Finally, Section 4 is devoted to introducing a refinement of the surgery map and giving a sketch of the proof for the main theorem. Here, one of the keys to attacking Problem 1.1 is a homomorphism

$$\bar{z}_{n+1}\colon Y_n\mathcal{IC}_{g,1}/Y_{n+1} \to \mathcal{A}_{n+1}^c \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$$

introduced in [15]. This homomorphism is defined via the LMO functor constructed by Cheptea, Habiro, and Massuyeau [1]. These have been greatly developed in quantum topology (see Ohtsuki [17]).

2 Main results

Let us consider Problem 1.1. First note that $\operatorname{tor}\mathcal{A}_{n,1}^c = \{0\}$ ([15, Proposition 5.2]) implies $\operatorname{Ker}\mathfrak{s}_{n,1} = \{0\}$ if n is even. In [16], they gave an upper bound of $\operatorname{Ker}\mathfrak{s}_{n,1}$ when $n = 2m-1$ ($m \geq 2$). More precisely, let $\langle \Theta_{2m-1}^{\geq 1,s} \rangle$ denote the submodule of $\mathcal{A}_{2m-1}^c$ generated by symmetric 2-loop Jacobi diagrams



$$\left(\begin{array}{l} a_i, b_i, c_i \in \{1^\pm, \ldots, g^\pm\}, \\ a_i = a_{p-i+1},\ b_i = b_{q-i+1},\ c_i = c_{r-i+1}, \\ p, q, r \geq 1,\ p+q+r+2 = 2m-1 \end{array} \right) \tag{2.1}$$

and let $\pi\colon Y_{2m-1}\mathcal{IC}/Y_{2m} \to (Y_{2m-1}\mathcal{IC}/Y_{2m})/\mathfrak{s}(\langle \Theta_{2m-1}^{\geq 1,s} \rangle)$ be the quotient map. Then, they proved the next theorem.

Theorem 2.1 ([16])**.** *The kernel* $\operatorname{Ker}(\pi \circ \mathfrak{s}_{2m-1,1})$ *is generated by elements*

$$O(a_1, \ldots, a_{m-1}, a_m, a_{m-1}, \ldots, a_1) + O(a_m, \ldots, a_2, a_1, a_2, \ldots, a_m)$$

and is free $\mathbb{Z}/2\mathbb{Z}$*-module of rank* $\frac{1}{2}((2g)^m - (2g)^{\lceil m/2 \rceil})$*.*

Remark 2.2. The inclusion $\operatorname{Ker}\mathfrak{s}_{2m-1,1} \subset \operatorname{Ker}(\pi \circ \mathfrak{s}_{2m-1,1})$ holds if $m = 2, 3$. On the other hand, the author recently found an element in $\operatorname{Ker}(\pi \circ \mathfrak{s}_{7,1})$ but not in $\operatorname{Ker}\mathfrak{s}_{7,1}$. This observation implies that there is a non-trivial relation among graph claspers with different first Betti numbers.

The key of the proof is a homomorphism $\bar{z}_{2m,l}\colon Y_{2m-1}\mathcal{IC}/Y_{2m} \to \mathcal{A}^c_{2m,l} \otimes \mathbb{Q}/\mathbb{Z}$ introduced in [15], which gives an upper bound of $\operatorname{Ker}(\pi \circ \mathfrak{s}_{2m-1,1})$. Also, we use a homomorphism $\mathfrak{bu}\colon \mathcal{A}^c_{n,l} \to \mathcal{A}^c_{n+2,l+1}$ defined by blowing up a trivalent vertex of a Jacobi diagram, and obtain the following result.

Theorem 2.3 ([16])**.** *Let $n \geq 3$. Then, $\mathfrak{bu}\colon \mathcal{A}^c_{n-2,1} \to \mathcal{A}^c_{n,2}/\langle\Theta^{\geq 1}_n\rangle$ is an isomorphism.*

Here, $\langle\Theta^{\geq 1}_n\rangle \subset \mathcal{A}^c_{n,2}$ denotes the submodule of $\mathcal{A}^c_{2m-1}$ generated by 2-loop Jacobi diagrams as in (2.1) which is not necessarily symmetric. On the other hand, to give a lower bound, we introduce a refinement $\tilde{\mathfrak{s}}\colon \mathbb{Z}\widetilde{\mathcal{J}}^c_n \to Y_n\mathcal{IC}/Y_{n+2}$ of the surgery map and deduce relations among claspers (see Section 4).

As a consequence of these results, for $n = 4$, we can determine the group structure of $Y_n\mathcal{IC}/Y_{n+1}$ and solve the Goussarov-Habiro conjecture about the Y_{n+1}-equivalence and finite-type invariants of degree n ([16, Corollary 4.9]). This conjecture is known to be true for $n = 1, 2, 3$ ([12], [15]), and the case $n \geq 5$ is a challenging problem. Moreover, Sato, Suzuki and the author revealed the structures of interesting groups which were not being much studied.

Theorem 2.4 ([16])**.** *The abelian groups $Y_3\mathcal{IC}_{g,1}/Y_5$ and $Y_3\mathcal{IH}_{g,1}/Y_5$ are torsion-free.*

It follows from Theorem 2.4 that, in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_4\mathcal{IC}_{g,1}/Y_5 & \longrightarrow & Y_3\mathcal{IC}_{g,1}/Y_5 & \longrightarrow & Y_3\mathcal{IC}_{g,1}/Y_4 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y_4\mathcal{IH}_{g,1}/Y_5 & \longrightarrow & Y_3\mathcal{IH}_{g,1}/Y_5 & \longrightarrow & Y_3\mathcal{IH}_{g,1}/Y_4 & \longrightarrow & 0 \end{array}$$

consisting of two short exact sequences, the four groups on the left and in the middle are free abelian. On the other hand, the two groups on the right have torsion elements ([15]). In particular, the two exact sequences do not split.

3 Claspers and related groups

We briefly review terminologies appearing in Sections 1 and 2. Also, we summarize facts about the group $Y_n\mathcal{IC}/Y_{n+1}$.

3.1 Graph claspers and the Y-filtration

Clasper surgery[1] was independently initiated by Goussarov [5] and Habiro [6], which is indispensable for the study of the monoid $\mathcal{IC}$. First, a *graph clasper* is an embedded surface in a 3-manifold such that a decomposition into disks, bands, and annuli is specified

[1] Recently, Watanabe [19, 20] introduced clasper surgery for 4-manifolds and gave significant applications.

as in Figure 2. In general, a clasper has "boxes" which are used for zip constructions (see [6], [17] for details). Let G be a graph clasper. We write L_G for the framed link obtained from G by replacing each disk with the Borromean rings as in Figure 2. Let M_G denote the 3-manifold obtained by Dehn surgery along L_G.

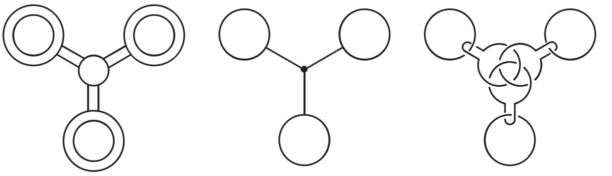


Figure 2: A graph clasper, its brief notation, and the corresponding framed link.

The number of disks in G is called the *degree* of G. Two homology cylinders $M, M' \in \mathcal{IC}$ are said to be *Y_n-equivalent*[2], $M \sim_{Y_n} M'$, if there exist graph claspers $G_1, \ldots, G_r$ of degree n such that $M_{G_1 \sqcup \cdots \sqcup G_r} = M'$. Define $Y_n\mathcal{IC} = \{M \in \mathcal{IC} \mid M \sim_{Y_n} \mathfrak{c}(\mathrm{id})\}$. Then we have a descending series $\mathcal{IC} = Y_1\mathcal{IC} \supset Y_2\mathcal{IC} \supset \cdots$ of submonoids, which is called the *Y-filtration* on $\mathcal{IC}$. We see the group $Y_n\mathcal{IC}/Y_{n+1}$ in Section 1. More generally, the quotients $Y_n\mathcal{IC}/Y_{n+k}$ $(1 \le k \le n)$ are also finitely generated abelian groups and are related to each other via the exact sequence

$$0 \to Y_{n+1}\mathcal{IC}/Y_{n+k} \to Y_n\mathcal{IC}/Y_{n+k} \to Y_n\mathcal{IC}/Y_{n+1} \to 0.$$

3.2 The structures of $\mathcal{A}^c_{n,l}$ and $Y_n\mathcal{IC}/Y_{n+1}$ for small n

Let $H = H_1(\Sigma_{g,1};\mathbb{Z})$ and $H_{(2)} = H \otimes \mathbb{Z}/2\mathbb{Z}$. When $n = 1$, we have $\mathcal{A}^c_1 \cong \left(\bigwedge^3 H\right) \oplus H_{(2)}^{\otimes 2}$. It follows from [11] that

$$0 \to \langle T(a,b,a) + T(b,a,b) \mid a \neq b\rangle \to \mathcal{A}^c_1 \xrightarrow{\mathfrak{s}_1} \mathcal{IC}/Y_2 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact and $\mathcal{IC}/Y_2 \cong \left(\bigwedge^3 H\right) \oplus \left(\bigwedge^2 H_{(2)}\right) \oplus H_{(2)} \oplus \mathbb{Z}/2\mathbb{Z}$.

When $n = 2$, we have $\mathcal{A}^c_2 = \mathcal{A}^c_{2,0} \oplus \mathcal{A}^c_{2,1} \oplus \mathcal{A}^c_{2,2}$ and the direct summands are expressed as in Table 1. Concerning $\mathcal{A}^c_{2,0}$, an exact sequence

$$0 \to \bigwedge^4 H \xrightarrow{\iota} S^2\left(\bigwedge^2 H\right) \to \mathcal{A}^c_{2,0} \to 0$$

is given in [12, Section 3.1], where ι is defined by

$$\iota(a \wedge b \wedge c \wedge d) = (a \wedge b)(c \wedge d) - (a \wedge c)(b \wedge d) + (a \wedge d)(b \wedge c)$$

and its image vanishes in $\mathcal{A}^c_{2,0}$ by the IHX relation. In particular, $\mathcal{A}^c_2$ is torsion-free and the surgery map $\mathfrak{s}_2\colon \mathcal{A}^c_2 \to Y_2\mathcal{IC}/Y_3$ is an isomorphism.

Next, in the case $n = 3$, we have $\mathcal{A}^c_3 = \mathcal{A}^c_{3,0} \oplus \mathcal{A}^c_{3,1}$. In Table 1, $\mathcal{A}^c_{3,0} \cong D'_3 = \mathrm{Ker}(H \otimes L'_4 \xrightarrow{[-,-]} L'_5)$, where L'_n denotes the degree n part of the free quasi-Lie algebra[3]

[2]Similarly, tree claspers define the C_{n+1}-equivalence among links.

[3]The L'_n is isomorphic, over $\mathbb{Q}$, to the degree n part of the free Lie algebra L_n. Also, $D'_n \otimes \mathbb{Q}$ is isomorphic to $\mathfrak{h}_{g,1}(n)$ appearing in Morita, Sakasai, and Suzuki [14, 13]

$n\setminus l$	0	1	2	3
1	$(\bigwedge^3 H)\oplus H_{(2)}^{\otimes 2}$	0	0	0
2	$S^2(\bigwedge^2 H)/\bigwedge^4 H$	$S^2(H)$	$\mathbb{Z}$	0
3	D_3'	$\mathcal{A}^c_{1,0}$	0	0
4	D_4'	$(H^{\otimes 4})_{\mathfrak{D}_8}$	$\mathcal{A}^c_{2,1}$	$\mathbb{Z}$

Table 1: The structures of the modules $\mathcal{A}^c_{n,l}$.

generated by H. Here, by Levine [9, Corollary 2.3], one has $\operatorname{tor}\mathcal{A}^c_{3,0}\cong(H\otimes L_2)\otimes\mathbb{Z}/2\mathbb{Z}\cong H^2_{(2)}\oplus\bigwedge^2 H_{(2)}$ and its free part is computed by Witt's formula [10, Theorem 5.11] as follows: $\operatorname{rank}_{\mathbb{Z}}\mathcal{A}^c_{3,0}=\frac{2}{5}(4g^5-5g^3+g)$. Furthermore, $Y_3\mathcal{IC}/Y_4$ is given in the next theorem.

Theorem 3.1 ([15])**.** $0\to\left(\bigwedge^3 H\oplus\bigwedge^2 H\right)\otimes\mathbb{Z}/2\mathbb{Z}\xrightarrow{j}\mathcal{A}^c_3\xrightarrow{\mathfrak{s}}Y_3\mathcal{IC}/Y_4\to 0$ *is exact, where* j *is the homomorphism defined by*

$$j(a\wedge b\wedge c)=T(a,b,c,b,a)+T(b,c,a,c,b)+T(c,a,b,a,c)=\Delta_{1,0}(T(a,b,c)),\qquad(3.1)$$
$$j(a\wedge b)=O(a,b,a)+O(b,a,b).\qquad(3.2)$$

Finally, the case $n=4$ is also listed in Table 1, where $\mathcal{A}^c_{4,1}$ is expressed as the coinvariant quotient with respect to the action of the dihedral group $\mathfrak{D}_8$ of order 8 ([15, Proposition 5.1]). In particular, $\mathcal{A}^c_4$ is torsion-free and $\mathfrak{s}_4\colon\mathcal{A}^c_4\to Y_4\mathcal{IC}/Y_5$ is an isomorphism. On the other hand, to go further, we need to attack the following problem.

Problem 3.2. Does $\Delta_{n,0}(T(a_1,a_2,\dots,a_{n+2}))\in\operatorname{Ker}\mathfrak{s}_{2n+1,0}$ hold?

Here, roughly speaking, the map $\Delta_{n,l}\colon\mathcal{A}^c_{n,l}\to A^c_{2n+1,2l}$ is defined to be the sum of the "doubles" with respect to univalent vertices of a Jacobi diagram (see [15, Definition 3.4] or [4, Definition 40] for the precise definition). The case $n=1$ in Theorem 3.1 is nothing but (3.1) and the case $n=2$ is also true[4], but the cases $n\geq 3$ remain open. Here, our invariant vanishes for these elements.

Theorem 3.3 ([15])**.** *Let* $l\geq 0$*. Then,*

(1) $\operatorname{Ker}(\bar{z}_{2n+2}\circ\mathfrak{s}\colon\ \operatorname{tor}\mathcal{A}^c_{2n+1,2l}\to\mathcal{A}^c_{2n+2}\otimes\mathbb{Q}/\mathbb{Z})\supset\operatorname{Im}\Delta_{n,l}$,

(2) $\operatorname{Ker}(\bar{z}_{2n+2}\circ\mathfrak{s}\colon\ \operatorname{tor}\mathcal{A}^c_{2n+1,0}\to\mathcal{A}^c_{2n+2}\otimes\mathbb{Q}/\mathbb{Z})=\operatorname{Im}\Delta_{n,0}$.

In the proof, Sato, Suzuki, and the author used deep results on the homology cobordism group and higher-order Sato-Levine invariants due to Conant, Schneiderman, and Teichner [2, 3, 4].

3.3 The homology cobordism group of homology cylinders

Two homology cylinders (M_1,m_1), $(M_2,m_2)\in\mathcal{IC}$ are said to be *homology cobordant*, $M_1\sim_H M_2$, if there exists a smooth 4-manifold W such that $\partial W=M_1\cup_{m_1=m_2}(-M_2)$ and the induced maps $H_*(M_j;\mathbb{Z})\to H_*(W;\mathbb{Z})$ $(j=1,2)$ are isomorphisms. The quotient $\mathcal{IH}=\mathcal{IC}/{\sim_H}$ is called the *homology cobordism group* of homology cylinders, and

[4]The structure of the abelian group $Y_3\mathcal{IC}/Y_6$ is determined via the case $n=2$.

the canonical projection $q\colon \mathcal{IC} \twoheadrightarrow \mathcal{IH}$ induces a homomorphism $\operatorname{Gr} q\colon Y_n\mathcal{IC}/Y_{n+1} \twoheadrightarrow Y_n\mathcal{IH}/Y_{n+1}$.

Here, it is natural to expect that the group $Y_n\mathcal{IH}/Y_{n+1}$ can be also described in terms of Jacobi diagrams. By Levine [8, Theorem 2], the composite map $\operatorname{Gr} q \circ \mathfrak{s}_n\colon \mathcal{A}^c_{n,\geq 1} \to Y_n\mathcal{IH}/Y_{n+1}$ is trivial, and thus, for $n \geq 2$, $\operatorname{Gr} q \circ \mathfrak{s}_{n,0}\colon \mathcal{A}^c_{n,0} \to Y_n\mathcal{IH}/Y_{n+1}$ is surjective. Furthermore, it gives an isomorphism over $\mathbb{Q}$ ([8, Theorem 3]). When $n = 1$, $\operatorname{Gr} q\colon \mathcal{IC}/Y_2 \twoheadrightarrow \mathcal{IH}/Y_2$ is an isomorphism ([7, Proposition 7.5], [4, p. 326]), and hence $\operatorname{Gr} q\circ\mathfrak{s}_{1,0}$ is neither surjective nor injective (see Section 3.2). On the other hand, $\operatorname{Gr} q\circ\mathfrak{s}_{2m,0}$ is an isomorphism ([2, Corollary 1.2]) and, for $n = 2m + 1$, the kernel is non-trivial:

$$\Delta_{m,0}(T(a_1, a_2, \ldots, a_{m+2})) \in \operatorname{Ker}(\operatorname{Gr} q \circ \mathfrak{s}_{2m+1,0}\colon \mathcal{A}^c_{2m+1,0} \to Y_{2m+1}\mathcal{IH}/Y_{2m+2}).$$

Moreover, $\operatorname{Ker}(\operatorname{Gr} q \circ \mathfrak{s}_{4m-1,0}) = \operatorname{Im} \Delta_{2m-1,0}$ holds ([4, Corollary 51]), while this equality is still open for $n = 4m + 1$ ([4, Section 4.1]). For instance, when $m = 1$, we need to attack the following problem.

Problem 3.4 ([18, Conjecture 2.25])**.** For $a \neq b$, does $T(a, b, a, b, a, b, a)+T(b, a, b, a, b, a, b) \notin \operatorname{Ker}(q \circ \mathfrak{s}_5)$ hold?

This element can be regarded as "$\frac{1}{2}\Delta_{2,0}(T(a, b, a, b))$", and its non-triviality is a subtle problem. On the other hand, the non-triviality in $Y_5\mathcal{IC}/Y_6$ is detected by our homomorphism $\bar{z}_{6,1}$:

$$\bar{z}_{6,1} \circ \mathfrak{s}_5(T(a, b, a, b, a, b, a) + T(b, a, b, a, b, a, b)) = O(a, b, a, a, b, a) + O(b, a, b, b, a, b) \neq 0.$$

4 The refined surgery map and its applications

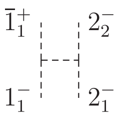
We need complicated clasper calculus to prove Theorem 2.1. To do it systematically, it is better to develop clasper calculus not in $Y_n\mathcal{IC}/Y_{n+1}$ but $Y_n\mathcal{IC}/Y_{n+2}$. We here introduce a module $\mathbb{Z}\widetilde{\mathcal{J}}^c_n$ and a homomorphism $\tilde{\mathfrak{s}}_n$ which commutes the diagram

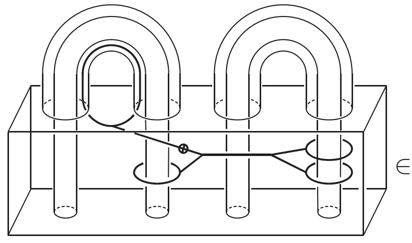
$$\begin{array}{ccc} \mathbb{Z}\widetilde{\mathcal{J}}^c_n & \xrightarrow{\tilde{\mathfrak{s}}_n} & Y_n\mathcal{IC}/Y_{n+2} \\ \downarrow & & \downarrow \\ \mathcal{A}^c_n & \xrightarrow{\mathfrak{s}_n} & Y_n\mathcal{IC}/Y_{n+1}. \end{array}$$

See [16, Section 3] for the precise definitions.

4.1 Refinements of the surgery map, AS relation, and STU relation

Let $\mathbb{Z}\widetilde{\mathcal{J}}^c_n$ denote the $\mathbb{Z}$-module generated by connected Jacobi diagrams with n trivalent vertices. Here, each univalent vertex is colored by an element of $\{1^\pm, \ldots, g^\pm\}$ with additional information, for example, 4^-_1, $\bar{4}^+_2$. The map $\tilde{\mathfrak{s}}_n$ is defined in much the same way as the ordinary surgery map $\mathfrak{s}_n$, while we take into account additional information of labels. The subscript of a label indicates the relative position of the corresponding annulus, and the overline stands for inserting a positive half-twist "⊕" to the corresponding band. For instance,

$\tilde{\mathfrak{s}}_2\Bigg($ 

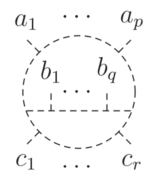
$\Bigg) =$  $\in Y_2\mathcal{IC}_{2,1}/Y_4.$

One can check that a homomorphism $\tilde{\mathfrak{s}}_n\colon \mathbb{Z}\widetilde{\mathcal{J}}_n^c \to Y_n\mathcal{IC}/Y_{n+2}$ is well-defined, which is called the *refined surgery map*. Furthermore, the AS and STU relations in $Y_n\mathcal{IC}/Y_{n+1}$ are lifted to $Y_n\mathcal{IC}/Y_{n+2}$. The following corollary is a consequence of the refined AS and STU relations.

Corollary 4.1 ([16])**.** *For $m \geq 2$,*

$$\begin{aligned} &O(a_1,\dots,a_{m-1},a_m,a_{m-1},\dots,a_1) + O(a_m,\dots,a_2,a_1,a_2,\dots,a_m) \\ &\quad + \sum_{i=2}^{m-1} \theta(a_{i-1},\dots,a_1,\dots,a_{i-1};a_i;a_{i+1},\dots,a_m,\dots,a_{i+1}) \in \operatorname{Ker}\mathfrak{s}_{2m-1} \end{aligned}$$

holds, where

$\theta(a_1,\dots,a_p;b_1,\dots,b_q;c_1,\dots,c_r) =$ 

.

4.2 Sketch of the proof of Theorem 2.1

First, it follows from Corollary 4.1 that

$$O(a_1,\dots,a_{m-1},a_m,a_{m-1},\dots,a_1) + O(a_m,\dots,a_2,a_1,a_2,\dots,a_m) \in \operatorname{Ker}(\pi\circ\mathfrak{s}_{2m-1}).$$

Next, to show that the images of the rest of the elements under $\pi\circ\mathfrak{s}_{2m-1}$ are non-trivial, we use our homomorphism $\bar{z}_{2m,l}$:

$$\begin{array}{ccccc} \mathcal{A}^c_{2m-1,1} & \xrightarrow{\pi\circ\mathfrak{s}_{2m-1,1}} & \pi(Y_{2m-1}\mathcal{IC}/Y_{2m}) & \xrightarrow{\bar{z}_{2m,1}} & \mathcal{A}^c_{2m,1}\otimes\mathbb{Q}/\mathbb{Z} \\ & & \downarrow{\scriptstyle \bar{z}_{2m,2}} & & \\ & & (\mathcal{A}^c_{2m,2}/\langle\Theta^{\geq 1}_{2m}\rangle)\otimes\mathbb{Q}/\mathbb{Z} & \xleftarrow[\cong]{\mathfrak{bu}} & \mathcal{A}^c_{2m-2,1}\otimes\mathbb{Q}/\mathbb{Z}. \end{array}$$

Since the two modules on the right are 1-loop parts whose structures are well studied ([15, Proposition 5.2]), it allows us to detect the non-triviality of the elements. Finally, we enumerate certain necklaces corresponding to Jacobi diagrams and compute the rank of the module $\operatorname{Ker}(\pi\circ\mathfrak{s}_{2m-1})$.

4.3 Applications to $Y_n\mathcal{IC}/Y_{n+k}$ and $Y_n\mathcal{IH}/Y_{n+k}$ (Theorem 2.4)

When $1 \leq k \leq n$, not only $Y_n\mathcal{IC}/Y_{n+1}$ but also $Y_n\mathcal{IC}/Y_{n+k}$ is a finitely generated abelian group. Also, for $1 \leq k' < k$, we have an exact sequence

$$0 \to Y_{n+k'}\mathcal{IC}/Y_{n+k} \to Y_n\mathcal{IC}/Y_{n+k} \to Y_n\mathcal{IC}/Y_{n+k'} \to 0.$$

We first consider the easiest case $Y_2\mathcal{IC}/Y_4$. Since the above exact sequence splits when $n = k = 2$ and $k' = 1$ (see Section 3.2), we know the structure of the group $Y_2\mathcal{IC}/Y_4$. We next observe $Y_3\mathcal{IC}/Y_5$. In this case, the exact sequence does not split and the relation between $Y_4\mathcal{IC}/Y_5$ and $Y_3\mathcal{IC}/Y_4$ is complicated and interesting. To see it, let focus on $J = O(a,b,a) \in \operatorname{tor}\mathcal{A}^c_{3,1}$ and a lift $\tilde{J} = O(a_1,b,a_2) \in \mathbb{Z}\widetilde{\mathcal{J}}_{3,1}$ of J. It follows from the exactness that there is $x \in \mathcal{A}^c_4$ satisfying $\mathfrak{s}_4(x) \mapsto 2\tilde{\mathfrak{s}}_3(\tilde{J}) \in Y_3\mathcal{IC}/Y_5$. It is difficult to compute x explicitly, though the refined relations enable us to do it. In fact, we have

$$x = -2O(a,a,a,b) - O(a,a,b,b) - \theta(a;;b).$$

By such a technique, one can show that $Y_3\mathcal{IC}/Y_5$ is free abelian (Theorem 2.4). Here, its rank is computed as follows. By Table 1, we have $\operatorname{rank}\mathcal{A}^c_3 = \frac{2}{5}(4g^5 - 5g^3 + g) + \binom{2g}{3}$ and

$$\operatorname{rank}\mathcal{A}^c_4 = \frac{1}{15}(32g^6 + 20g^3 - 2g^2 - 5g) + \frac{1}{2}(4g^4 + 4g^3 + 3g^2 + g) + \binom{2g+1}{2} + 1,$$

and hence

$$\operatorname{rank}_{\mathbb{Z}}(Y_3\mathcal{IC}/Y_5) = \frac{1}{30}(g+1)(2g+1)\left(32g^4 - 24g^3 + 50g^2 - 23g + 30\right).$$

Also, in a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_5\mathcal{IC}/Y_6 & \longrightarrow & Y_4\mathcal{IC}/Y_6 & \longrightarrow & Y_4\mathcal{IC}/Y_5 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_5\mathcal{IC}/Y_6 & \longrightarrow & Y_3\mathcal{IC}/Y_6 & \longrightarrow & Y_3\mathcal{IC}/Y_5 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y_3\mathcal{IC}/Y_4 & = & Y_3\mathcal{IC}/Y_4 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

consisting of four exact sequences, since $Y_4\mathcal{IC}/Y_5$ and $Y_3\mathcal{IC}/Y_5$ are free abelian, we conclude that $\operatorname{tor}(Y_5\mathcal{IC}/Y_6) = \operatorname{tor}(Y_4\mathcal{IC}/Y_6) = \operatorname{tor}(Y_3\mathcal{IC}/Y_6)$. Therefore, any element of $\operatorname{tor}(Y_3\mathcal{IC}/Y_4) \neq \{0\}$ does not lift to $\operatorname{tor}(Y_3\mathcal{IC}/Y_6) \neq \{0\}$. In particular, the two columns do not split.

One can define abelian groups $Y_n\mathcal{IH}/Y_{n+k}$ for homology cobordism group $\mathcal{IH}$ as well. However, the exactness in the middle of

$$0 \to Y_{n+1}\mathcal{IH}/Y_{n+2} \to Y_n\mathcal{IH}/Y_{n+2} \to Y_n\mathcal{IH}/Y_{n+1} \to 0 \tag{4.1}$$

is not necessarily satisfied. When $n = 2$, there is a homomorphism

$$\iota\colon Y_2\mathcal{IH}/Y_3 \cong \mathcal{A}^c_{2,0} \hookrightarrow \mathcal{A}^c_2 \cong Y_2\mathcal{IC}/Y_3$$

fitting into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y_3\mathcal{IC}/Y_4 & \longrightarrow & Y_2\mathcal{IC}/Y_4 & \longrightarrow & Y_2\mathcal{IC}/Y_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \uparrow \iota & & \\
0 & \longrightarrow & Y_3\mathcal{IH}/Y_4 & \longrightarrow & Y_2\mathcal{IH}/Y_4 & \longrightarrow & Y_2\mathcal{IH}/Y_3 & \longrightarrow & 0.
\end{array}$$

Since the top row is exact, so is the bottom row. Moreover, since the bottom row splits, the group $Y_2\mathcal{IH}/Y_4$ is determined.

When $n = 3$, the sequence (4.1) is exact and $Y_3\mathcal{IH}/Y_5$ is determined as well (Theorem 2.4). Note that the exact sequence does not split, and thus one needs to know the group $Y_3\mathcal{IC}/Y_5$ well. In general, it is a problem whether the inclusion $\operatorname{Ker}\mathfrak{s}_{n,0} \subset \operatorname{Ker}(\operatorname{Gr} q \circ \mathfrak{s}_{n,0})$ is an equality (see Problems 3.2 and 3.4).

5 Future perspectives

Recall that the main theme of this article is Problem 1.1, and Theorem 2.1 is a partial answer for it. The next step is to give a complete answer for Problem 1.1 without the projection π. More generally, it is important to determine the structures of the groups $Y_n\mathcal{IC}/Y_{n+k}$ $(1 \le k \le n)$. We are concretely interested in when $Y_n\mathcal{IC}/Y_{n+k}$ has torsion elements and whether there exist elements of order greater than 2. It might be possible to solve the Goussarov-Habiro conjecture via these investigations. Furthermore, the groups $Y_n\mathcal{IH}/Y_{n+k}$ are also important. Combining Sato, Suzuki, and the author [15, 16] with Conant, Schneiderman, and Teichner [2, 3, 4], we might obtain much fruitful results on $Y_n\mathcal{IH}/Y_{n+k}$.

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