Gromov hyperbolicity of fine curve graphs for nonorientable surfaces

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1 Introduction

Let $N = N_{g,n}$ be a connected finite type nonorientable surface of genus g with n punctures and $S = S_{g,n}$ be a connected finite type orientable surface of genus g with n punctures. We will abbreviate $N_{g,0}$ as N_g (we use the same notation for orientable surfaces). If we are not interested in whether a given surface is orientable or not, we write F for the surface. In [2], Bowden, Hensel, and Webb introduced a new curve graph $C^\dagger(S)$ called the *fine curve graph* in order to study the diffeomorphism group $\mathrm{Diff}_0(S)$ on a surface S. They proved that the graph $C^\dagger(S)$ is Gromov hyperbolic and the action of the group $\mathrm{Diff}_0(S)$ on $C^\dagger(S)$ satisfies the condition of Bestvina–Fujiwara [1]. The Gromov hyperbolicity of metric spaces is defined as follows:

Definition 1.1. Let (X, d) be a metric space. For points x, y, w of (X, d), the *Gromov product* is defined to be

$$\langle x, y\rangle_w \coloneqq \frac{1}{2}(d(w, x) + d(w, y) - d(x, y)).$$

A metric space X is δ-*hyperbolic* if for all $w, x, y, z \in X$ we have

$$\langle x, z\rangle_w \geq \min\{\langle x, y\rangle_w, \langle y, z\rangle_w\} - \delta.$$

Morover, the definition of fine curve graphs for surfaces by Bowden, Hensel, and Webb [2] is the following:

Definition 1.2. Let $F = F_g$ be a closed surface of genus $g \geq 3$ if F is nonorientable and $g \geq 2$ if F is orientable. A *fine curve graph* $\mathcal{C}^\dagger(N)$ of N is a graph whose vertices are the essential simple closed curves on N, and two vertices form an edge if the corresponding curves are disjoint.

We remark that for low genera,that is, $g = 0, 1$ if F is orientable and $g = 1, 2$ if F is nonorientable, we modify the definition of the fine cuve graph $\mathcal{C}^\dagger(F)$ of F so that two vertices form an edge if the corresponding curves intersect at most once.

In this article, we will give the outline of the proof of the uniform hyperbolicity of the fine curve graphs $\mathcal{C}^\dagger(N)$ of nonorientable surfaces. Here curve graphs are *uniformly* hyperbolic if we can choose the hyperbolicity constant δ which is independent of the topological types of the surfaces:

Theorem 1.3. *There exists $\delta > 0$ such that for any closed nonorientable surface $N = N_g$ of genus $g \geq 2$, $\mathcal{C}^\dagger(N)$ is δ-hyperbolic.*

2 Preliminaries

A simple closed curve c on F is *essential* if c does not bound a disk, a disk with one marked point, or a Möbius band. The *curve graph* $\mathcal{C}(F)$ of F is the simplicial graph whose vertex set consists of the isotopy classes of all essential simple closed curves on F and whose edge set consists of all non-ordered pairs of isotopy classes of essential simple closed curves which can be represented disjointly. We define a *nonseparating curve graph* $\mathcal{NC}(F)$ of F as the full subgraph of $\mathcal{C}(F)$ consists of the nonseparating curves on F. Two curves c_1 and c_2 in F are in *minimal position* if the number of intersections of c_1 and c_2 is minimal in the isotopy classes of c_1 and c_2. Note that two essential simple closed curves are in minimal position in F if and only if they do not bound a bigon on F (see [4, Proposition 2.1] for nonorientable surfaces). Slightly abusing the notation, we consider vertices of curve graphs as the (essential) simple closed curves on N which are in minimal position. Furthermore, so long as it does not cause confusion, we might say curves as essential simple closed curves on F. We define the distances $d_{\mathcal{C}(F)}(\cdot,\cdot)$ on $\mathcal{C}(F)$ by the minimal lengths of edge-paths connecting two vertices. Thus, we consider $\mathcal{C}(F)$, as a geodesic space.

3 Proof of uniform hyperbolicity of fine curve graphs for nonorientable surfaces

In this section, we prove Theorem 1.3 for $g \geq 3$, that is, the fine curve graph $\mathcal{C}^\dagger(N)$ of a closed nonorientable surface N of genus $g \geq 3$ is uniformly hyperbolic. First, we define some notations. Let $N = N_{g,n}$ be a nonorientable surface of genus $g \geq 3$ with $n \geq 0$ punctures.

Definition 3.1. The *surviving curve graph* $\mathcal{C}^s(N)$ is a full subgraph of the original curve graph $\mathcal{C}(N)$ whose vertices correspond to the isotopy classes of curves on N which are essential even after filling in the punctures.

Note that each nonseparating curve is surviving, so the *nonseparating curve graph* $\mathcal{NC}(N)$ is a full subgraph of $\mathcal{C}^s(N)$. For a nonorientable surface N, a curve c is said to be *one-sided* if the regular neighborhood of c is a Möbius band. Moreover, c is said to be *two-sided* if the regular neighborhood of c is an annulus. Then, we define the *two-sided curve graph* $\mathcal{C}_{\text{two}}(N)$ of N, which is the subgraph of $\mathcal{C}(N)$ induced by the isotopy classes of all two-sided curves on N. We also denote by $\mathcal{C}^\pm(N)$ the curve graph whose vertices are the usual vertices and the isotopy classes of curves bounding a Möbius band, and two vertices form an edge if the corresponding curves can be realized disjointly. We call $\mathcal{C}^{\pm}(N)$ the *extended curve graph* of N. We use the same notation for $\mathcal{NC}(N)$, $\mathcal{C}^s(N)$, and $\mathcal{C}^{\dagger}(N)$. We may even use the notations at the same time; for instance, $\mathcal{C}^{\pm\dagger}_{\text{two}}(N)$ is the extended two-sided fine curve graph of N whose vertices are the two-sided essential curves and curves bounding a Möbius band.

Definition 3.2. Let (X, d) and (X', d') be two metric spaces. Let φ be a map from X to X'. The map φ is a *quasi-isometric embedding* if there exists a constant $\lambda \geq 1$ such that, for all $x, y \in X$, the following inequality is satisfied:

$$\frac{1}{\lambda}d(x,y) - \lambda \leq d(\varphi(x), \varphi(y)) \leq \lambda d(x,y) + \lambda.$$

The map φ is *quasi-dense* if there exists a constant $\lambda \geq 0$ such that, for any point $y \in X'$, there exists some $x \in X$ such that $d(\varphi(x), y) \leq \lambda$. Finally, The map φ is a *quasi-isometry* if it is both a quasi-isometric embedding and is quasi-dense, and we call X is quasi-isometric to X'. We will use the symbol $X \underset{\text{q.i.}}{\sim} X'$ to mean that two metric spaces X and X' are quasi-isometric.

We use the following two lemmas to prove Theorem 1.3, and we omit the proofs of the two lemmas in this article.

Lemma 3.3. *Let $N = N_{g,p}$ be a nonorientable surface of genus $g \geq 3$ with $p \geq 0$ punctures. Then, $\mathcal{C}^s(N)$, $\mathcal{C}^{\pm s}(N)$, and $\mathcal{C}^{\pm s}_{\text{two}}(N)$ are path-connected, and*

$$\mathcal{NC}(N) \underset{q.i.}{\sim} \mathcal{C}^s(N) \underset{q.i.}{\sim} \mathcal{C}^{\pm s}(N) \underset{q.i.}{\sim} \mathcal{C}^{\pm s}_{\text{two}}(N).$$

Lemma 3.4. *Let $N = N_g$ be a closed nonorientable surface of genus $g \geq 3$. Then, $\mathcal{C}^{\dagger}(N)$, $\mathcal{C}^{\pm\dagger}(N)$, and $\mathcal{C}^{\pm\dagger}_{\text{two}}(N)$ are path-connected, and*

$$\mathcal{C}^{\dagger}(N) \underset{q.i.}{\sim} \mathcal{C}^{\pm\dagger}(N) \underset{q.i.}{\sim} \mathcal{C}^{\pm\dagger}_{\text{two}}(N).$$

We know that the nonseparating curve graphs $\mathcal{NC}(N)$ of the finite type nonorientable surfaces N are uniformly hyperbolic (see [3]):

Theorem 3.5. ([3]) *Let N be any finite type nonorientable surface of genus $g \geq 3$. Then, there exists a constant $\delta'' > 0$ not depending on N such that the nonseparating curve graph $\mathcal{NC}(N)$ is connected and δ''-hyperbolic.*

By combining Lemma 3.3 and Theorem 3.5, we obtain the following corollary:

Corollary 3.6. *There exists $\delta' > 0$ such that the extended two-sided surviving curve graphs $\mathcal{C}^{\pm s}_{\text{two}}(N)$ are δ'-hyperbolic.*

We are now ready to prove Theorem 1.3. Due to Lemma 3.4, it is enough to show the following proposition to prove Theorem 1.3.

Proposition 3.7. *The extended two-sided fine curve graphs $\mathcal{C}^{\pm\dagger}_{\text{two}}(N)$ are uniform hyperbolic.*

We need the following three lemmas to prove Theorem 3.7. The lemmas come from Bowden, Hensel, and Webb [2, Lemmas 3.4, 3.5, 3.6] but with the assumption of orientable surfaces changed to nonorientable ones. Instead of our giving the proofs of Lemmas 3.8, 3.9, and 3.10, readers may refer to the proofs of [2, Lemmas 3.4, 3.5, 3.6].

Let $P \subset N$ be a finite set. In the following, for a curve a in N, let $[a]_{N-P}$ denote the isotopy class of a in $N - P$. We also write $d_{\mathcal{C}^{\pm s}_{\text{two}}(N-P)}(\cdot, \cdot)$, for instance, for the metric of $\mathcal{C}^{\pm s}_{\text{two}}(N)$.

Lemma 3.8. (cf. *[2, Lemma 3.4]*) *Suppose that vertices a and b in $\mathcal{C}^{\pm\dagger}_{\text{two}}(N)$ are transverse, and are in minimal position in $N \setminus P$, where $P \subset N$ is finite and disjoint from a and b. Then,*

$$d_{\mathcal{C}^{\pm s}_{\text{two}}(N-P)}([a]_{N-P}, [b]_{N-P}) = d_{\mathcal{C}^{\pm\dagger}_{\text{two}}(N)}(a, b).$$

Lemma 3.9. (cf. *[2, Lemma 3.5]*) *Suppose that $a_1, \cdots, a_n$ are two-sided curves including curves bounding a Möbius band that are pairwise in minimal position in $N - P$. Let $b_1, \cdots, b_m$ be two-sided curves including curves bounding a Möbius band that are disjoint from P. Then, the b_i can be isotoped in $N - P$ such that $a_1, \cdots, a_n, b_1, \cdots, b_m$ are pairwise in minimal position in $N - P$.*

Lemma 3.10. (cf. *[2, Lemma 3.6]*) *Let $a, b \in \mathcal{C}^{\pm\dagger}_{\text{two}}(N)$ and $P \subset N$ be a finite set. Then, we can find a geodesic $a = \nu_0, \cdots, \nu_k = b$ such that $\nu_i \cap P = \emptyset$ for all $0 < i < k$.*

Now we can prove Theorem 3.7.

Proof of Theorem 3.7. In particular, we will prove that, for all $u, a, b, c \in \mathcal{C}^{\pm\dagger}_{\text{two}}(N)$,

$$\langle a, c\rangle_u \geq \min\{\langle a, b\rangle_u, \langle b, c\rangle_u\} - \delta' - 4, \tag{3.1}$$

where δ' is the uniform constant of $\mathcal{C}^{\pm s}_{\text{two}}(N)$ in Corollary 3.6.

We relate the vertices $u, a, b, c \in \mathcal{C}^{\pm\dagger}_{\text{two}}(N)$ with the vertices $u', a', b', c' \in \mathcal{C}^{\pm s}_{\text{two}}(N - P)$ satisfying the assumptions of Lemma 3.8, that is, the two properties ,

(i) $d_{\mathcal{C}^{\pm\dagger}_{\text{two}}(N)}(a, a') \leq 1$, $d_{\mathcal{C}^{\pm\dagger}_{\text{two}}(N)}(b, b') \leq 1$, and $d_{\mathcal{C}^{\pm\dagger}_{\text{two}}(N)}(c, c') \leq 1$,

(ii) the vertices u', a', b', c' are transversal.

Set $u' = u$. Find a'' that is disjoint from and isotopic to a (note that the two-sidedness of curves is needed in this step); then, find a small enough perturbation a' of a'' satisfying (ii) (do this for b and c as well).

Now, choose a finite set $P \subset N$ so that any bigon between a pair from u', a', b', c' contains a point of P. Then, by the bigon criterion, this ensures that u', a', b', are c' are pairwise in minimal position in $N - P$.

By Lemma 3.8, for any pair $d, e \in \{a, b, c\}$ we have that

$$d_{\mathcal{C}^{\pm\dagger}_{\text{two}}(N)}(d, e) = d_{\mathcal{C}^{\pm s}_{\text{two}}(N-P)}(d', e'), \tag{3.2}$$

and

$$|\langle d', e'\rangle_{u'} - \langle d, e\rangle_u| \leq 2. \tag{3.3}$$

Finally, we obtain formula (3.1) from (3.2), (3.3) above, and Corollary 3.6. ∎

Acknowledgments

The second author wishes to express her great appreciation to the organizers for giving her a chance to talk at Intelligence of Low-dimensional Topology, and supporting her a lot. The first author was supported by JSPS KAKENHI Grant Number 20H00114. The secound author was supported by Foundation of Kinoshita Memorial Enterprise, by JSPS KAKENHI Grant-in-Aid for Early-Career Scientists, Grant Number 21K13791, and by JST, ACT-X, Grant Number JPMJAX200D.

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