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Perfect and Quasi-Perfect Codes for the Bosonic Classical-Quantum Channel

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ABSTRACT In this article, we explore perfect and quasi-perfect codes for the Bosonic channel, where information is generated by a laser and conveyed in the form of coherent states. In particular, we consider the phase-modulation codebook for coherent states in a Bosonic channel. We show that these phase-modulation codes are quasi-perfect as long as the cardinality of the code is the same as the dimension of the coherent states. These codes feature the smallest error probability among all codes of the same cardinality and the same dimension of the channel Hilbert space. We study the performance of these codes in terms of error probability, incorporating the degradation caused by a depolarizing or an erasure quantum channel.

INDEX TERMS Classical-quantum channel, coherent state, finite blocklength analysis, perfect code, quantum meta-converse, quasi-perfect code.

I. INTRODUCTION

We study the limits of communication systems in the finite-blocklength regime. In these cases, nonasymptotic bounds on the error probability provide a description of the system performance. In order to derive these bounds, it is common to use hypothesis testing concepts. Hypothesis testing was used by Shannon et al. in [1] in order to derive the sphere-packing exponent, and also by Blahut in [2]. More recently, hypothesis testing has been used by Nagaoka in order to derive strong converse bounds in classical-quantum channels [3], and by Hayashi [4, Sec. 4.6] in order to derive the converse part of the channel-coding theorem. Polyanskiy et al. [5, Th. 27] introduced a hypothesis testing finite-blocklength bound for classical channels. Matthews and Wehner [6] obtained finite-length bounds for general quantum channels. In classical channels, perfect and quasi-perfect codes were generalized beyond binary alphabets in [7]. These codes attain the meta-converse bound and so they are optimum. In quantum channels, quasi-perfects codes were introduced in [8] and the Bell codebook was provided as an example.

This work follows as a continuation of [8] and focuses on quasi-perfect codes for coherent states. Information is conveyed by the electromagnetic field generated by a laser. The received signal is represented by a coherent state that describes the photon statistics of the field. The coherent state was first introduced by Klauder (see [9]) and later on formally defined by Glauber (see [10]). The channel used to transmit coherent states is the Bosonic channel. This channel can be modeled using an infinite dimensional Hilbert space, but for practical purposes, it is necessary to consider an equivalent channel with a reduced dimension. We consider a truncation of the Bosonic channel as in [11], where the coherent states has instead a finite dimension (with the corresponding normalization).

The line of work in this article and the one in [8] are focused on solving problems related to state discrimination. Helstrom [23] and [24] developed the theory of quantum detection and quantum hypothesis testing. Our work studies the optimality of a codebook used to transmit classical information over a quantum channel rather than the optimality of the detector. The classical information is recovered directly by means of an optimum measurement operation. A different line of work in the literature considers error correction codes, which are codes that use redundancy in order to correct errors that may be caused by the channel or other effects. In quantum systems, Shor showed in [12] that errors can be corrected by encoding the state of the system in a quantum code and perform measurements on the redundant parts of the code to detect errors. Detected errors can be corrected by simply applying unitary transformations to the state. Later quantum stabilizer codes were introduced by Calderbank [13], [14] and Gottesman [15]. Error correction codes include, for example, surface codes (see [16], [17]) and LDPC quantum codes (see [18], [19], [20]). The results obtained in this work could also be applied in an error correction setting by considering a classical-quantum channel where the error correction procedure is included in the channel model. Here, we only focus on the state discrimination problem for the Bosonic channel.

The rest of this article is organized as follows. In Section II, we formalize the problems of binary and multiple hypothesis testing and establish a connection between them. We also define quasi-perfect codes and show that they attain the meta-converse bound. In Section III, we define a quasi-perfect code for the Bosonic channel based on phase-modulation and study its performance in terms of error probability. We also consider the inclusion of a depolarizing and an erasure channel, since they are symmetric and it is possible to show results for both of them. Finally, Section IV concludes this article.

A. NOTATION

Let $\mathcal{D}(\mathcal{H})$ denote the space of density operators acting on a Hilbert space $\mathcal{H}$. In the general case, a quantum state is described by a density operator $\rho \in \mathcal{D}(\mathcal{H})$. Density operators are self-adjoint, positive semidefinite, and have unit trace. A measurement on a quantum system is a mapping from the state of the system ρ to a classical outcome $m \in \{1, \dots, M\}$. A measurement is represented by a collection of positive self-adjoint operators $\{\Pi_1, \dots, \Pi_M\}$ such that $\sum \Pi_m = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. These operators form a *positive operator-valued measure* (POVM). A POVM measurement $\{\Pi_1, \dots, \Pi_M\}$ applied to ρ has outcome m with probability $\mathrm{Tr}(\rho\Pi_m)$ where Tr is the trace operator.

For self-adjoint operators A, B, the notation $A \geq B$ means that $A - B$ is positive semidefinite. Similarly $A \leq B$, $A > B$, and $A < B$ means that $A - B$ is negative semidefinite, positive definite, and negative definite, respectively.

For a self-adjoint operator A with spectral decomposition $A = \sum_i \lambda_i E_i$, where $\{\lambda_i\}$ are the eigenvalues and $\{E_i\}$ are the orthogonal projections onto the corresponding eigenspaces, we define

$$\{A > 0\} \triangleq \sum_{i:\lambda_i > 0} E_i. \tag{1}$$

This corresponds to the projector associated to the positive eigenspace of A. We shall also use $\{A \geq 0\} \triangleq \sum_{i:\lambda_i \geq 0} E_i$ and $\{A = 0\} \triangleq \sum_{i:\lambda_i = 0} E_i$.

II. QUASI-PERFECT CODES

A. BINARY HYPOTHESIS TESTING

Consider a binary hypothesis test discriminating among two quantum states ρ_0 and ρ_1 acting on $\mathcal{H}$. We define a test measurement $\{T, \bar{T}\}$ such that T and $\bar{T}$ are positive semidefinite, self-adjoint operators, and $T + \bar{T} = \mathbb{1}$. We define the probability of false alarm $\epsilon_{1|0}$ and the probability of miss-detection $\epsilon_{0|1}$ as follows:

$$\epsilon_{1|0}(T) \triangleq 1 - \mathrm{Tr}(\rho_0 T) = \mathrm{Tr}\left(\rho_0 \bar{T}\right) \tag{2}$$

$$\epsilon_{0|1}(T) \triangleq \mathrm{Tr}(\rho_1 T). \tag{3}$$

We define $\alpha_\beta(\rho_0\|\rho_1)$ as the minimum probability of false alarm $\epsilon_{1|0}$ among all tests with $\epsilon_{0|1} \leq \beta$

$$\alpha_\beta(\rho_0\|\rho_1) \triangleq \inf_{T:\epsilon_{0|1}(T)\leq\beta} \epsilon_{1|0}(T). \tag{4}$$

B. MULTIPLE HYPOTHESIS TESTING

Consider a multiple hypothesis testing problem where we want to discriminate between M quantum states acting on $\mathcal{H}$. In particular, we consider that the quantum states $\tau_1, \dots, \tau_M$ are associated with classical probabilities $p_1, \dots, p_M$, respectively. We define an M-ary test as a POVM $\mathcal{P} \triangleq \{\Pi_1, \Pi_2, \dots, \Pi_M\}$ satisfying $\sum \Pi_i = \mathbb{1}$ and $\Pi_m \geq 0$. This test will decide τ_j when the true hypothesis is τ_i with a probability of $\mathrm{Tr}\left(\tau_i \Pi_j\right)$. The average error probability is

$$\epsilon(\mathcal{P}) \triangleq 1 - \sum_{i=1}^{M} p_i \mathrm{Tr}(\tau_i \Pi_i). \tag{5}$$

We are interested in the minimum average error probability among all test $\mathcal{P}$

$$\epsilon^\star \triangleq \min_{\mathcal{P}} \epsilon(\mathcal{P}). \tag{6}$$

The following lemma states that a test is optimum under certain conditions.

Lemma 1 (Holevo–Yuen–Kennedy–Lax Conditions): A test $\mathcal{P}^\star = \{\Pi_1^\star, \dots, \Pi_M^\star\}$ minimizes (6) if and only if, for each $m = 1, \dots, M$

$$\left(\Lambda(\mathcal{P}^\star) - p_m\tau_m\right)\Pi_m^\star = \Pi_m^\star\left(\Lambda(\mathcal{P}^\star) - p_m\tau_m\right) = 0 \tag{7}$$

$$\Lambda(\mathcal{P}^\star) - p_m\tau_m \geq 0 \tag{8}$$

where

$$\Lambda(\mathcal{P}^\star) \triangleq \sum_{i=1}^{M} p_i\tau_i\Pi_i^\star = \sum_{i=1}^{M} p_i\Pi_i^\star\tau_i \tag{9}$$

is required to be self-adjoint.

Proof: This result follows from [21, Th. 4.1, Eq. (4.8)] or [22, Th. I] after simplifying the resulting optimality conditions. ■

Define matrices $\mathcal{T}$ and $\mathcal{D}$ as follows:

$$\mathcal{T} \triangleq \mathrm{diag}\left(p_1\tau_1, \dots, p_M\tau_M\right) \tag{10}$$

$$\mathcal{D}(\mu_0) \triangleq \mathrm{diag}\left(\tfrac{1}{M}\mu_0, \dots, \tfrac{1}{M}\mu_0\right) \tag{11}$$

where $\mathrm{diag}(\rho_1, \dots, \rho_A)$ is a diagonal matrix with diagonal blocks $\rho_1, \dots, \rho_A$ and μ_0 is an arbitrary density operator acting on $\mathcal{H}$.

Lemma 2: The minimum error probability of a Bayesian M-ary test discriminating among states $\{\tau_1, \dots, \tau_M\}$ with prior probabilities $\{p_1, \ldots, p_M\}$ satisfies

$$\epsilon^\star = \max_{\mu_0} \alpha_{\frac{1}{M}}\big(\mathcal{T} \,\|\, \mathcal{D}(\mu_0)\big). \tag{12}$$

Proof: The proof can be found in [8]. ■

C. META-CONVERSE BOUND

We consider the channel coding problem of transmitting M equiprobable messages over a one-shot classical-quantum channel $x \to W_x$, with $x \in \mathcal{X}$ and $W_x \in \mathcal{D}(\mathcal{H})$. A channel code is defined as a mapping from the message set $\{1, \ldots, M\}$ into a set of M codewords $\mathcal{C} = \{x_1, \ldots, x_M\}$. For a source message m, the decoder receives the associated density operator W_{x_m} and must decide on the transmitted message.

With some abuse of notation, for a fixed code, sometimes we shall write $W_m \triangleq W_{x_m}$. The minimum error probability for a code $\mathcal{C}$ is then given by

$$\mathsf{P}_{\mathrm{e}}(\mathcal{C}) \triangleq \min_{\{\Pi_1,\ldots,\Pi_M\}} \left\{1 - \frac{1}{M}\sum_{m=1}^{M} \mathrm{Tr}\big(W_m \Pi_m\big)\right\}. \tag{13}$$

Lemma 3 (Classical-Quantum Meta-Converse Bound): Let $\mathcal{C}$ be any codebook of cardinality M for a channel $W_x \in \mathcal{D}(\mathcal{H})$. Define the following:

$$P_{\mathcal{C}} W = \frac{1}{M}\sum_{x\in\mathcal{C}} \big(|x\rangle\langle x| \otimes W_x\big) \tag{14}$$

$$P_{\mathcal{C}} \otimes \mu = \left(\frac{1}{M}\sum_{x\in\mathcal{C}} |x\rangle\langle x|\right) \otimes \mu \tag{15}$$

$$PW \triangleq \sum_{x\in\mathcal{X}} P(x)\Big(|x\rangle\langle x| \otimes W_x\Big) \tag{16}$$

$$P \otimes \mu \triangleq \left(\sum_{x\in\mathcal{X}} P(x)\,|x\rangle\langle x|\right) \otimes \mu. \tag{17}$$

Then

$$\mathsf{P}_{\mathrm{e}}(\mathcal{C}) = \sup_{\mu}\left\{\alpha_{\frac{1}{M}}\big(P_{\mathcal{C}} W \,\|\, P_{\mathcal{C}} \otimes \mu\big)\right\} \tag{18}$$

$$\geq \inf_{P} \sup_{\mu}\left\{\alpha_{\frac{1}{M}}\big(PW \,\|\, P \otimes \mu\big)\right\} \tag{19}$$

where the maximization is over auxiliary states $\mu \in \mathcal{D}(\mathcal{H})$, and the minimization is over (classical) input distributions P.

Proof: The proof can be found in [8]. ■

D. PERFECT AND QUASI-PERFECT CODES

For any density operator μ in $\mathcal{D}(\mathcal{H})$, and $t \in \mathbb{R}$ we define

$$\mathcal{E}_x(t,\mu) \triangleq \big\{W_x - t\mu \geq 0\big\} \tag{20}$$

$$\mathcal{E}^\bullet_x(t,\mu) \triangleq \big\{W_x - t\mu > 0\big\} \tag{21}$$

$$F_x(t,\mu) \triangleq \mathrm{Tr}\big(W_x \mathcal{E}_x(t,\mu)\big) \tag{22}$$

$$F^\bullet_x(t,\mu) \triangleq \mathrm{Tr}\big(W_x \mathcal{E}^\bullet_x(t,\mu)\big) \tag{23}$$

$$G_x(t,\mu) \triangleq \mathrm{Tr}\big(\mu\, \mathcal{E}_x(t,\mu)\big) \tag{24}$$

$$G^\bullet_x(t,\mu) \triangleq \mathrm{Tr}\big(\mu\, \mathcal{E}^\bullet_x(t,\mu)\big). \tag{25}$$

Definition 1: We say that a classical-quantum channel $\{W_x\}$, $x \in \mathcal{X}$, is *symmetric* if

$$W_x = U_x \bar{W} U_x^\dagger \tag{26}$$

for every $x \in \mathcal{X}$, where $\bar{W} \in \mathcal{D}(\mathcal{H})$ does not depend on x and U_x is a unitary linear operator acting on $\mathcal{H}$ and parametrized by x. Equivalently, define

$$\mathcal{U}_W \triangleq \big\{\mu \in \mathcal{D}(\mathcal{H}) \mid U_x\mu = \mu U_x\big\}. \tag{27}$$

For any symmetric channel and $\mu \in \mathcal{U}_W$, $F_x(t,\mu)$, and $G_x(t,\mu)$ do not depend on x as shown in [8].

We denote by $\bar{t}$ the smallest value of t such that $\big\{\mathcal{E}^\bullet_x(t,\mu)\big\}_{x\in\mathcal{C}}$ are orthogonal to each other for a certain code $\mathcal{C}$. Define

$$\bar{\mathcal{E}}(\bar{t},\mu) \triangleq \big\{\bar{W} - \bar{t}\mu \geq 0\big\} \tag{28}$$

and also

$$\bar{\mathcal{E}}^\bullet(\bar{t},\mu) \triangleq \big\{\bar{W} - \bar{t}\mu > 0\big\} \tag{29}$$

$$\bar{\mathcal{E}}^\circ(\bar{t},\mu) \triangleq \big\{\bar{W} - \bar{t}\mu = 0\big\}. \tag{30}$$

We define the orthogonal basis $\{\bar{E}(i)\}$ associated to the eigenspace of $\big\{\bar{W} - \bar{t}\mu \geq 0\big\}$ such that

$$\bar{\mathcal{E}}^\bullet(\bar{t},\mu) = \sum_{i\in\mathcal{I}^\bullet} \bar{E}(i) \tag{31}$$

$$\mathcal{E}^\bullet_x(\bar{t},\mu) = U_x \bar{\mathcal{E}}^\bullet(\bar{t},\mu) U_x^\dagger = \sum_{i\in\mathcal{I}^\bullet} U_x \bar{E}(i) U_x^\dagger = \sum_{i\in\mathcal{I}^\bullet} E_x(i) \tag{32}$$

where $\mathcal{I}^\bullet$ denotes the set of basis indexes associated to the strictly positive eigenvalues. We also write

$$\bar{\mathcal{E}}^\circ(\bar{t},\mu) = \sum_{i\in\mathcal{I}^\circ} \bar{E}(i) \tag{33}$$

$$\mathcal{E}^\circ_x(\bar{t},\mu) = U_x \bar{\mathcal{E}}^\circ(\bar{t},\mu) U_x^\dagger = \sum_{i\in\mathcal{I}^\circ} U_x \bar{E}(i) U_x^\dagger = \sum_{i\in\mathcal{I}^\circ} E_x(i) \tag{34}$$

where $\mathcal{I}^\circ$ denote the set of basis indexes associated to the zero eigenvalues.

Definition 2 (See [8]): A code $\mathcal{C}$ is *perfect* for a classical-quantum channel $x \to W_x$, if there exist a scalar t and a state $\mu \in \mathcal{D}(\mathcal{H})$ such that the projectors $\big\{\mathcal{E}_x(t,\mu)\big\}_{x\in\mathcal{C}}$ are orthogonal to each other and $\sum_{x\in\mathcal{C}} \mathcal{E}_x(t,\mu) = \mathbb{1}$. More generally, a code is *quasi-perfect* if there exist t and $\mu \in \mathcal{D}(\mathcal{H})$ such that the projectors $\big\{\mathcal{E}^\bullet_x(t,\mu)\big\}_{x\in\mathcal{C}}$ are orthogonal to each other, and for $I_\bullet \triangleq \sum_{x\in\mathcal{C}} \mathcal{E}^\bullet_x(t,\mu)$, $I_\circ \triangleq \mathbb{1} - I_\bullet$, it holds that $\sum_{x\in\mathcal{C}} \mathcal{E}^\circ_x(t,\mu) = cI_\circ$ where $c \in \mathbb{R}$, $c > 0$ is a normalizing constant that depends on the code $\mathcal{C}$.

Lemma 4 (Quasi-Perfect Codes Attain the Meta-Converse Bound): Let the channel $x \to W_x$ be symmetric and let $\mathcal{C}$ be perfect or quasi-perfect with parameters t and $\mu \in \mathcal{U}_W$. Then, for $M = |\mathcal{C}|$

$$\mathsf{P}_e(\mathcal{C}) = \inf_P \sup_{\mu'} \alpha_{\frac{1}{M}}\left(PW \,\|\, P \otimes \mu'\right). \tag{35}$$

Proof: See [8] for the proof. ∎

Remark 1: A special channel to consider is the erasure channel that takes a quantum state on Hilbert space $\mathcal{H}_A$ and outputs a quantum state on Hilbert space $\mathcal{H}_B$, where systems A and B have dimensions d_A and d_B, respectively. The erasure channel is defined by

$$\mathcal{N}^E_{A\to B}(\rho_A) = (1-\epsilon)\mathcal{I}_{A\to B}(\rho_A) + \epsilon|e\rangle\langle e|_B \tag{36}$$

where ρ_A is the input quantum state and the Isometric channel $\mathcal{I}_{A\to B}(\rho_A) = I_{A\to B}\rho_A I^{\dagger}_{A\to B}$ is defined using the isometry

$$I_{A\to B} = \begin{bmatrix} & \mathbb{1}_A & \\ 0 & \ldots & 0 \end{bmatrix} \tag{37}$$

as unique Kraus operator and where $\{|0\rangle, \ldots, |M-1\rangle, |e\rangle\}$ form an orthonormal basis on $\mathcal{H}_B$. In this case we express the output state $W_x = \mathcal{N}^E_{A\to B}(\rho_A)$ as $W_x = W_x\mathbb{1}_B = W_x(I_{A\to B}I^{\dagger}_{A\to B} + |e\rangle\langle e|_B) = W_x I_{A\to B}I^{\dagger}_{A\to B} + \epsilon|e\rangle\langle e|_B$. The eigenspace of $\left\{W_x - \bar{t}\mu = 0\right\}$ consist of the eigenspace of $\left\{\mathcal{I}_{A\to B}(\rho_A) - \bar{t}\mu = 0\right\}$ plus the eigenvector $|e\rangle\langle e|_B$. Equivalently, we can express $\mathcal{E}^{\circ}_x(\bar{t},\mu)$ as $\mathcal{E}^{\circ}_x(\bar{t},\mu) = \sum_{i\in\mathcal{I}^{\circ}} E_x(i) = \mathcal{E}'^{\circ}_x(\bar{t},\mu) + |e\rangle\langle e|_B$, where $\mathcal{E}'^{\circ}_x(\bar{t},\mu)$ is the eigenspace of $\left\{\mathcal{I}_{A\to B}(\rho_A) - \bar{t}\mu = 0\right\}$ and $|e\rangle\langle e|_B$ does not depend on x (i.e., all codewords share the same eigenvector $|e\rangle\langle e|_B$). The input state has no effect on the term $\epsilon|e\rangle\langle e|_B$, so for this case, we introduce the following generalized definition of quasi-perfect codes, which can accommodate the different input and output dimensions of the erasure channel.

Definition 3: A code $\mathcal{C}$ is generalized quasi-perfect if there exists t and $\mu \in \mathcal{D}(\mathcal{H})$ such that the projectors $\left\{\mathcal{E}^{\bullet}_x(t,\mu)\right\}_{x\in\mathcal{C}}$ are orthogonal to each other, fulfilling $\sum_{x\in\mathcal{C}} \mathcal{E}^{\bullet}_x(t,\mu) = I_{\bullet}$. Moreover, we also require that $\sum_{x\in\mathcal{C}} \mathcal{E}'^{\circ}_x(\bar{t},\mu) = cI_{\circ A}$, where $I_{\circ A} = I_{A\to B}I^{\dagger}_{A\to B} - I_{\bullet}$, $\mathcal{E}'^{\circ}_x(\bar{t},\mu) = \mathcal{E}^{\circ}_x(\bar{t},\mu) - |e\rangle\langle e|_B$ and $c \in \mathbb{R}$, $c > 0$ is a normalizing constant that depends on the code $\mathcal{C}$.

The following lemma shows that generalized quasi-perfect codes are optimum among all codes of the same dimension of the channel Hilbert space and cardinality.

Lemma 5 (Generalized Quasi-Perfect Codes Attain the Meta-Converse Bound): Let the channel $x \to W_x$ be symmetric and let $\mathcal{C}$ be generalized quasi-perfect with parameters t and $\mu \in \mathcal{U}_W$. Then, for $M = |\mathcal{C}|$

$$\mathsf{P}_e(\mathcal{C}) = \inf_P \sup_{\mu'} \alpha_{\frac{1}{M}}\left(PW \,\|\, P \otimes \mu'\right). \tag{38}$$

Proof: We define the POVM $\mathcal{T} = \{\Pi_1, \ldots, \Pi_M\}$ as follows:

$$\Pi_m = \mathcal{E}^{\bullet}_{x_m}(\bar{t},\mu) + \frac{1}{c}\mathcal{E}^{\circ}_{x_m}(\bar{t},\mu) \tag{39}$$

$$= U_{x_m}\bar{\mathcal{E}}^{\bullet}(\bar{t},\mu)U^{\dagger}_{x_m} + \frac{1}{c}U_{x_m}\bar{\mathcal{E}}^{\circ}(\bar{t},\mu)U^{\dagger}_{x_m} \tag{40}$$

$$= U_{x_m}\bar{\Pi}U^{\dagger}_{x_m}, \quad m = 1, \ldots, M \tag{41}$$

with

$$\bar{\Pi} = \bar{\mathcal{E}}^{\bullet}(\bar{t},\mu) + \frac{1}{c}\bar{\mathcal{E}}'^{\circ}(\bar{t},\mu) + \frac{1}{M}|e\rangle\langle e|_B \tag{42}$$

and $\bar{\mathcal{E}}'^{\circ}(\bar{t},\mu) = \bar{\mathcal{E}}^{\circ}(\bar{t},\mu) - |e\rangle\langle e|_B$, $c = \frac{M|\mathcal{I}'^{\circ}|}{d_{\circ A}}$ where $|\mathcal{I}'^{\circ}|$ is the cardinality of the set $\mathcal{I}^{\circ}$ of the input state and $d_{\circ A}$ is the dimension of $I_{\circ A}$. Following similar steps as in [8, the proof of Theorem 3], it is possible to show that the optimality conditions from Lemma 1 are satisfied. We have

$$\Lambda(\mathcal{T}) = \frac{1}{M}\sum_{\ell=1}^{M} U_l\bar{W}\bar{\mathcal{E}}^{\bullet}U^{\dagger}_l + \frac{1}{Mc}\sum_{\ell=1}^{M} U_l\bar{W}\bar{\mathcal{E}}'^{\circ}(\bar{t},\mu)U^{\dagger}_l$$
$$+ \frac{1}{M}W|e\rangle\langle e|_B = \Lambda(\mathcal{T})' + \frac{1}{M}W|e\rangle\langle e|_B \tag{43}$$

and so

$$\left(\Lambda(\mathcal{T}) - \frac{1}{M}W_m\right)\Pi_m$$
$$= \left(\Lambda(\mathcal{T})' - \frac{1}{M}W_m I_{A\to B}I^{\dagger}_{A\to B}\right)\Pi_m$$
$$+ \left(\frac{1}{M}W|e\rangle\langle e|_B - \frac{1}{M}W|e\rangle\langle e|_B\right)\Pi_m = 0 \tag{44}$$

where we used that $(\Lambda(\mathcal{T})' - \frac{1}{M}W_m I_{A\to B}I^{\dagger}_{A\to B})\Pi_m = 0$ as shown in [8]. Similarly $\Pi_m(\Lambda(\mathcal{T}) - \frac{1}{M}W_m) = 0$, showing that (7) holds. Also, $\Lambda(\mathcal{T}) - \frac{1}{M}W_m$ is semidefinite positive because $\Lambda(\mathcal{T}) - \frac{1}{M}W_m = \Lambda(\mathcal{T})' - \frac{1}{M}W_m I_{A\to B}I^{\dagger}_{A\to B} + \frac{1}{M}W|e\rangle\langle e|_B - \frac{1}{M}W|e\rangle\langle e|_B = \Lambda(\mathcal{T})' - \frac{1}{M}W_m I_{A\to B}I^{\dagger}_{A\to B}$. We conclude that the conditions from Lemma 1 are also satisfied in this case. The rest of the proof follows the same steps as in [8, Theorems 3 and 4]. ∎

III. QUASI-PERFECT AND GENERALIZED QUASI-PERFECT CODES FOR THE BOSONIC CLASSICAL-QUANTUM CHANNEL

This section introduces quasi-perfect codes for the Bosonic channel, where coherent states are transmitted. A coherent state is represented by the following expression:

$$|\alpha\rangle \triangleq e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{45}$$

where α is a complex amplitude, $|\alpha|^2$ is the average number of photons associated with state $|\alpha\rangle$ and $|n\rangle$ is the Fock, or photon number, state. The Bosonic classical-quantum channel is the mapping of a classical variable x to the quantum state W_x defined as $W_x = |\alpha_x\rangle\langle\alpha_x|$, $x \to W_x$, with $\alpha_x \triangleq ae^{i\theta_x}$, $\theta_x \in [0, 2\pi)$ and $a = |\alpha_x|$.

Here, we consider a finite-dimensional quantum receiver implementing collective measurements in the form of a POVM defined in a Hilbert space of dimension N, $\mathcal{H}_N$, i.e., restricting the coherent state measurements to Fock states $|n\rangle$ for $n \in \{0, \ldots, N-1\}$.

Consider the Nth-order *approximation* $|\alpha\rangle_N \in \mathcal{H}_N$ to coherent state $|\alpha\rangle$

$$|\alpha\rangle_N \triangleq \frac{1}{\sqrt{C_N}} \sum_{n=0}^{N-1} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{46}$$

$$C_N = \sum_{n=0}^{N-1} \frac{|\alpha|^{2n}}{n!}. \tag{47}$$

Similar to the Bosonic classical-quantum channel, the truncated Bosonic classical-quantum channel is the mapping of a classical variable x to the quantum state W_x defined as $W_x = |\alpha_x\rangle\langle\alpha_x|_N$, $x \to W_x$, with $\alpha_x \triangleq ae^{i\theta_x}$, $\theta_x \in [0, 2\pi)$, and $a = |\alpha_x|$. In order to assess the closeness of the approximation to the original state we will use the concept of *pure-state fidelity*, which can be easily computed as a function of the order of approximation N

$$F(N) = |\langle\alpha|\alpha\rangle_N|^2 = \frac{C_N}{e^{|\alpha|^2}} = \frac{1}{1 + C_N^{-1} \sum_{n=N}^{\infty} \frac{|\alpha|^{2n}}{n!}} = \frac{1}{1+\epsilon_N} \tag{48}$$

for $\epsilon_N = C_N^{-1} \sum_{n=N}^{\infty} \frac{|\alpha|^{2n}}{n!}$. Note that $\lim_{N\to\infty} \epsilon_N = 0$ so that if N is big enough, the Fidelity between the original and the approximated state is close to one. The fidelity and the trace distance $|||\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|_N||_1$ for pure states can be related as follows:

$$\frac{1}{2}|||\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|_N||_1 = \sqrt{1-F(N)} = \sqrt{\frac{\epsilon_N}{1+\epsilon_N}} \approx \sqrt{\epsilon_N}. \tag{49}$$

Now, since $\lim_{N\to\infty} \epsilon_N = 0$, for sufficiently big values of N, we know that any measurement using an arbitrary operator Π on the approximated state $|\alpha\rangle_N$ succeeds with high probability, it also does so if applied to the original state since

$$\frac{1}{2}|||\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|_N||_1 = \max_{0\le\Delta\le I}\{\mathrm{Tr}\{\Delta(|\alpha\rangle\langle\alpha| - |\alpha\rangle\langle\alpha|_N)\}\} \geq |\langle\alpha|\Pi|\alpha\rangle - \langle\alpha|_N\Pi|\alpha\rangle_N|. \tag{50}$$

We consider the channel coding problem of transmitting M equiprobable messages. Messages are modeled by the classical random variable x, over a one-shot approximated coherent quantum channel $x \to |\alpha_x\rangle\langle\alpha_x|_N$, with $\alpha_x \triangleq ae^{i\theta_x}$, $\theta_x \in [0, 2\pi)$. A channel code is defined as a mapping from the message set $\{1, \ldots, M\}$ into a set of M codewords $\mathcal{C} = \{x_1, \ldots, x_M\}$. The decoder is operating in finite-dimensional Hilbert space of dimension N, $\mathcal{H}_N$. For a source message m, the decoder must decide on the transmitted message. We define $\mathcal{C} = \{x_1, \ldots, x_M\}$, with $\alpha_{x_m} \triangleq a\delta_{x_m} \triangleq ae^{i\theta_{x_m}}$, $\theta_{x_m} = \frac{2\pi(m-1)}{M}$ for a code with cardinality M, and for each message $m = 1, \ldots, M$, i.e., as in a classical PSK modulation. Note that we have defined $a = |\alpha_{x_m}|$.

We consider the properties of this channel codes for the particular case of defining the dimension of the collective measurement's Hilbert space N equal to the cardinality of the message set M, i.e., $N = M$.

Let ρ_A be the density matrix as observed by the M-dimensional decoder. For $M \geq 2$ it follows that

$$\rho_A = \frac{1}{M}\sum_{m=1}^{M} |\alpha_{x_m}\rangle\langle\alpha_{x_m}|_A = \frac{1}{C_M}\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & a^2 & 0 & \ldots & 0 \\ 0 & 0 & \frac{a^4}{2} & \ldots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \ldots & \frac{a^{2(M-1)}}{(M-1)!} \end{bmatrix} \tag{51}$$

where

$$|\alpha_{x_m}\rangle_A \triangleq \frac{1}{\sqrt{C_M}} \sum_{n=0}^{M-1} \frac{\alpha_{x_m}^n}{\sqrt{n!}} |n\rangle. \tag{52}$$

Notice that $|n\rangle$ is the photon number state, so the density matrix in (51) is in the photon basis. Let us define the state density when message x_m is transmitted as W_m, i.e., $W_m \triangleq |\alpha_{x_m}\rangle\langle\alpha_{x_m}|_A$. Also, in order to simplify notation, let $\alpha_m \triangleq \alpha_{x_m}$ and $\delta_m \triangleq \delta_{x_m}$. We consider the decoder $\mathcal{P} = \{\Pi_1, \ldots, \Pi_M\}$ where

$$\Pi_m = \frac{1}{M}\rho_A^{-\frac{1}{2}} W_m \rho_A^{-\frac{1}{2}} = \frac{1}{M}\begin{bmatrix} 1 & \delta_m^* & \delta_m^{*2} & \ldots & \delta_m^{*M-1} \\ \delta_m & 1 & \delta_m^* & \ldots & \delta_m^{*M-2} \\ \delta_m^2 & \delta_m & 1 & \ldots & \delta_m^{*M-3} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \delta_m^{M-1} & \delta_m^{M-2} & \delta_m^{M-3} & \ldots & 1 \end{bmatrix}. \tag{53}$$

One can check that $\Pi_m \geq 0$ and that $\sum_{m=1}^{M} \Pi_m = \mathbb{1}_A$. Moreover

$$\Lambda(\mathcal{P}) \triangleq \frac{1}{M}\sum_{m=1}^{M} W_m \Pi_m = \frac{1}{M}\frac{B_M}{C_M}\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & a & 0 & \ldots & 0 \\ 0 & 0 & \frac{a^2}{\sqrt{2}} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{a^{M-1}}{\sqrt{(M-1)!}} \end{bmatrix} \tag{54}$$

with $C_M = \sum_{n=0}^{M-1} \frac{a^{2n}}{n!}$, and $B_M = \sum_{n=0}^{M-1} \frac{a^n}{\sqrt{n!}}$. Then, one can check that

$$\Lambda(\mathcal{P})\Pi_m = \frac{1}{M} W_m \Pi_m. \tag{55}$$

Also, for any unit vector $|v\rangle$

$$\frac{\langle v|\Lambda(\mathcal{P})|v\rangle}{\frac{1}{M}\langle v|W_m|v\rangle} = \frac{\langle v|\Lambda(\mathcal{P})^{\frac{1}{2}}\Lambda(\mathcal{P})^{\frac{1}{2}}|v\rangle}{\frac{1}{M}|\langle v|\alpha_m\rangle|^2} = \frac{M\langle u|u\rangle}{|\langle u|\Lambda(\mathcal{P})^{\frac{1}{2}}|\alpha_m\rangle|^2} \geq \frac{M}{\langle\alpha_m|\Lambda(\mathcal{P})^{-1}|\alpha_m\rangle} = 1 \quad (56)$$

which proves that $\Lambda(\mathcal{P}) - \frac{1}{M}W_m \geq 0$.

We check the symmetry of the one-shot coherent channel $x \to |\alpha_x\rangle\langle\alpha_x|_A$, with $|\alpha_x| = a$. In our channel we have $\alpha_y = \alpha_x e^{i(\theta_y - \theta_x)}$, i.e., $|\alpha_y\rangle = \Theta|\alpha_x\rangle$, where Θ is a diagonal matrix which elements incorporate the corresponding phase shifts. Note that $\Theta^H\Theta = I$. Since (26) holds, we conclude that the channel is symmetric.

Next we show that $\mathcal{C}$ is quasi-perfect for $\mu = \mu_0$ and $t = t_0$, which will be defined as follows. Recall that a code is quasi-perfect with respect to μ_0 and t_0 if it satisfies that $\{\mathcal{E}_x^\bullet(t_0, \mu_0)\}$ for $x \in \mathcal{C}$ are orthogonal to each other and also that $\sum_{x\in\mathcal{C}} \mathcal{E}_x^\circ(t, \mu) = cI_\circ$. From the optimality condition of the decoder (56), we can see that $\Lambda(\mathcal{P}) - \frac{1}{M}W_x \geq 0$, which implies that $\mathcal{E}_x(t_0, \mu_0) \triangleq \{W_x - t_0\mu_0 \geq 0\} = \{W_x - t_0\mu_0 = 0\} = \mathcal{E}_x^\circ(t_0, \mu_0)$ is the null eigenspace of $\frac{1}{M}W_x - \Lambda(\mathcal{P})$. This also implies that $\mathcal{E}_x^\bullet(t, \mu_0) = 0$, hence $\{\mathcal{E}_x^\bullet(t_0, \mu)\}$ for $x \in \mathcal{C}$ are orthogonal to each other. Also, $d_\circ = n = M$ because $d_\bullet = 0$, which implies that $c = 1$.

Recall that

$$\mathcal{E}_x(t, \mu_0) = \{|\alpha_x\rangle\langle\alpha_x|_A - t\mu_0 \geq 0\}. \quad (57)$$

Let $\mu_0 = \frac{MC_M}{(B_M)^2}\Lambda(\mathcal{P})$, we prove that

$$\mathcal{E}_x(t_0, \mu_0) = |v_{x,t_0}\rangle\langle v_{x,t_0}|_A. \quad (58)$$

We obtain the eigenvector associated to the largest eigenvalue of $|\alpha_x\rangle\langle\alpha_x| - t\mu_0$. To this end, we consider an arbitrary unit-norm vector $|v\rangle$. The largest eigenvalue of $|\alpha_m\rangle\langle\alpha_m| - t\mu_0$ is given by

$$\max_{|v\rangle:\langle v|v\rangle=1} \langle v|\big(|\alpha_x\rangle\langle\alpha_x| - t\mu_0\big)|v\rangle = \max_{|v\rangle:\langle v|v\rangle=1}\Big\{\langle v|\alpha_x\rangle\langle\alpha_x|v\rangle - t\langle v|\mu_0|v\rangle\Big\}. \quad (59)$$

We can observe that $t = t_0$ corresponds to the case for which the maximum eigenvalue of $|\alpha_m\rangle\langle\alpha_m| - t\mu_0$ is equal to zero, which implies

$$|\alpha_x\rangle\langle\alpha_x|v\rangle = t_0\mu_0|v\rangle. \quad (60)$$

Note that (60) implies

$$|v_{x,t_0}\rangle = \frac{1}{\sqrt{M}}\begin{bmatrix} 1 \\ \delta_x \\ \delta_x^2 \\ \vdots \\ \delta_x^{M-1}\end{bmatrix}. \quad (61)$$

Multiplying by $\langle\alpha_x|\mu_0^{-1}$ at both sides of (60) we obtain

$$\langle\alpha_x|\mu_0^{-1}|\alpha_x\rangle\langle\alpha_x|v\rangle = \frac{(B_M)^2}{C_M}\langle\alpha_x|v\rangle = t_0\langle\alpha_x|v\rangle \quad (62)$$

from which we obtain $t_0 = \frac{(B_M)^2}{C_M}$.

Now, we can see that using (61)

$$\sum_{x\in\mathcal{C}} \mathcal{E}_x(t_0, \mu) = \sum_{x\in\mathcal{C}} |v_{x,t_0}\rangle\langle v_{x,t_0}| \quad (63)$$

$$= \frac{1}{M}\sum_{m=1}^{M}\begin{bmatrix} 1 & \delta_m^* & \delta_m^{*2} & \dots & \delta_m^{*M-1} \\ \delta_m & 1 & \delta_m^* & \dots & \delta_m^{*M-2} \\ \delta_m^2 & \delta_m & 1 & \dots & \delta_m^{*M-3} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \delta_m^{M-1} & \delta_m^{M-2} & \delta_m^{M-3} & \dots & 1 \end{bmatrix} = \mathbb{1}_A. \quad (64)$$

So, we conclude that the code is quasi-perfect.

We can also easily find the error probability of this code, which is the minimum error probability among all possible codes of cardinality M for this channel. Using the optimal decoder $\mathcal{P}$, we obtain that the probability of error is

$$\mathsf{P}_e(\mathcal{C}) = 1 - \frac{1}{M}\sum_{m=1}^{M}\mathrm{Tr}\big(W_i\Pi_i\big) \quad (65)$$

$$= 1 - \frac{1}{M}\frac{B_M^2}{C_M}. \quad (66)$$

Notice that the code is quasi-perfect only for the truncated Bosonic channel. However, due to (50), for sufficiently large N, the error probability in (66) is close to the optimal error probability for the generic Bosonic channel.

A. QUASI-PERFECT CODES FOR THE BOSONIC CLASSICAL-QUANTUM CHANNEL INCORPORATING A DEPOLARIZING CHANNEL

Consider the Nth-order approximation of the Bosonic classical-quantum channel observed after a quantum depolarizing channel, defined as

$$\mathcal{N}_{A\to B}^{D}(\rho_A) = p\frac{1}{M}\mathbb{1}_M + (1-p)\rho_A. \quad (67)$$

The combined classical-quantum channel is, thus, $W_x = \mathcal{N}_{A\to B}^{D}\big(|\alpha_x\rangle\langle\alpha_x|_A\big)$. Using the codebook defined in (52), the channel output is given by $W_m = \mathcal{N}_{A\to B}^{D}\big(|\alpha_{x_m}\rangle\langle\alpha_{x_m}|_A\big)$, $m = 1, \dots, M$. For $N = M$, the combined channel W_x is symmetric and the code $\mathcal{C}$ is quasi-perfect for this channel for $\mu_0 = \frac{1}{c_0}\frac{1}{M}\sum_{m=1}^{M} W_m\Pi_m$, where c_0 is a normalizing constant.

Moreover

$$\mathsf{P}_e(\mathcal{C}) = \alpha_{\frac{1}{M}}\big(W_x \,\|\, \mu_0\big) = 1 - \frac{(1-p)B_M^2}{MC_M} - \frac{p}{M} \quad (68)$$

which is obtained using decoder $\mathcal{P} = \{\Pi_1, \dots, \Pi_M\}$ with

$$\Pi_m = |\pi_m\rangle\langle\pi_m| \quad (69)$$

$$\pi_m = \big[1, e^{j\theta_m}, e^{2j\theta_m}, e^{3j\theta_m}, \dots, e^{(M-1)j\theta_m}\big]^T. \quad (70)$$

Proof: The proof follows similar steps as in the case of the Nth-order approximation Bosonic channel and is omitted. ■

B. GENERALIZED QUASI-PERFECT CODES FOR THE BOSONIC CLASSICAL-QUANTUM CHANNEL INCORPORATING AN ERASURE CHANNEL

Consider the Nth-order approximation of the Bosonic classical-quantum channel observed after a quantum erasure channel, defined as

$$\mathcal{N}^E_{A\to B}(\rho_A) = (1-\epsilon)\mathcal{I}_{A\to B}(\rho_A) + \epsilon|e\rangle\langle e|_B$$

where the isometric channel $\mathcal{I}_{A\to B}(\rho_A) = I_{A\to B}\rho_A I^\dagger_{A\to B}$ is defined using the isometry

$$I_{A\to B} = \begin{bmatrix} & \mathbb{1}_A & \\ 0 & \dots & 0 \end{bmatrix} \tag{71}$$

as unique Kraus operator and where $\{|0\rangle, \dots, |M-1\rangle, |e\rangle\}$ form an orthonormal basis in $\mathcal{H}_B$. The combined classical-quantum channel is then $W_x = \mathcal{N}^E_{A\to B}\big(|\alpha_x\rangle\langle\alpha_x|_A\big)$. Using the codebook defined in (52), the channel output is given by $W_m = \mathcal{N}^E_{A\to B}\big(|\alpha_{x_m}\rangle\langle\alpha_{x_m}|_A\big)$, $m = 1, \dots, M$. The combined channel W_x is symmetric and the code $\mathcal{C}$ is generalized quasi-perfect for this channel for $\mu_0 = \frac{1}{c_0}\frac{1}{M}\sum_{m=1}^M W_m\Pi_m$, where c_0 is a normalizing constant.

Moreover

$$\mathsf{P}_\mathrm{e}(\mathcal{C}) = \alpha_{\frac{1}{M}}\big(W_x \,\|\, \mu_0\big) = 1 - \frac{(1-\epsilon)B_M^2}{MC_M} - \frac{\epsilon}{M} \tag{72}$$

which is obtained using decoder $\mathcal{P} = \{\Pi_1, \dots, \Pi_M\}$ with

$$\Pi_m = \frac{1}{M}\begin{bmatrix} 1 & \delta_m^* & \delta_m^{*2} & \dots & \delta_m^{*M-1} & 0 \\ \delta_m & 1 & \delta_m^* & \dots & \delta_m^{*M-2} & 0 \\ \delta_m^2 & \delta_m & 1 & \dots & \delta_m^{*M-3} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \delta_m^{M-1} & \delta_m^{M-2} & \delta_m^{M-3} & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{73}$$

Proof: The proof follows similar steps as in the case of the Nth-order approximation Bosonic channel and is omitted. ■

IV. CONCLUSION

This work can be seen as a continuation to [8], which explored perfect and quasi-perfect codes in classical-quantum channels. First, we summarized the results in [8] and we introduced a generalized definition of quasi-perfect codes in order to include quasi-perfect codes for the erasure channel. Lemma 5 shows that the error probability of generalized quasi-perfect codes is equal to the meta-converse bound and so is optimum among codes of the same cardinality and dimension of the channel Hilbert space.

In the second part of this work, we defined a family of quasi-perfect codes for the Bosonic channel. To do that, we consider the Nth-order approximation to coherent states. We show that phase-modulated coherent states can be used as codewords constituting a quasi-perfect code for the truncated Bosonic channel. The error introduced by the approximation for the coherent states is negligible for a sufficiently large dimension of the Hilbert space N. With this consideration, we conclude that these codes are quasi-perfect for the original Bosonic channel as well.

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