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On certain semigroups of contraction mappings of a finite chain

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Abstract. Let $[n] = \{1, 2, \ldots, n\}$ be a finite chain and let $\mathcal{P}_n$ (resp., $\mathcal{T}_n$) be the semigroup of partial transformations on $[n]$ (resp., full transformations on $[n]$). Let $\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n :$ (for all $x, y \in \operatorname{Dom}\alpha$) $|x\alpha - y\alpha| \leqslant |x - y|\}$ (resp., $\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n :$ (for all $x, y \in [n]$) $|x\alpha - y\alpha| \leqslant |x - y|\}$) be the subsemigroup of partial contraction mappings on $[n]$ (resp., subsemigroup of full contraction mappings on $[n]$). We characterize all the starred Green's relations on $\mathcal{CP}_n$ and it subsemigroup of order preserving and/or order reversing and subsemigroup of order preserving partial contractions on $[n]$, respectively. We show that the semigroups $\mathcal{CP}_n$ and $\mathcal{CT}_n$, and some of their subsemigroups are left abundant semigroups for all n but not right abundant for $n \geqslant 4$. We further show that the set of regular elements of the semigroup $\mathcal{CT}_n$ and its subsemigroup of order preserving or order reversing full contractions on $[n]$, each forms a regular subsemigroup and an orthodox semigroup, respectively.

1. Introduction and preliminaries

Let $[n] = \{1, 2, \ldots, n\}$ be a finite chain, a map α which has domain and image both subsets of $[n]$ is said to be a *transformation.* A transformation

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α whose domain is a subset of $[n]$ (i.e., $\operatorname{Dom}\alpha \subseteq [n]$) is said to be *partial*, and it is said to be *full* if its domain is $[n]$. The collection of all partial transformations of $[n]$ (resp., full transformations of $[n]$) is known as the *partial transformation semigroup* (resp., *full transformation semigroup*), usually denoted by $\mathcal{P}_n$ (resp., $\mathcal{T}_n$). A map $\alpha \in \mathcal{P}_n$ is said to be *order preserving* (resp., *order reversing*) if (for all $x, y \in \operatorname{Dom}\alpha$) $x \leqslant y$ implies $x\alpha \leqslant y\alpha$ (resp., $x\alpha \geqslant y\alpha$); is *order decreasing* if (for all $x \in \operatorname{Dom}\alpha$) $x\alpha \leqslant x$; an *isometry* (i.e., *distance preserving*) if (for all $x, y \in \operatorname{Dom}\alpha$) $|x\alpha - y\alpha| = |x - y|$; a *contraction* if (for all $x, y \in \operatorname{Dom}\alpha$) $|x\alpha - y\alpha| \leqslant |x - y|$.

The full transformation semigroup $\mathcal{T}_n$, is known to be a regular semigroup (see [[3], p.33. Ex.1]). The idempotents in $\mathcal{T}_n$ do not form a subsemigroup for $n \geqslant 2$, however the semigroup generated by idempotents in $\mathcal{T}_n$ was investigated by Howie [22] in 1996.

Let

$$\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \operatorname{Dom}\alpha)\ |x\alpha - y\alpha| \leqslant |x - y|\}$$

and

$$\mathcal{OCP}_n = \{\alpha \in \mathcal{CP}_n : (\text{for all } x, y \in \operatorname{Dom}\alpha)\ x \leqslant y \text{ implies } x\alpha \leqslant y\alpha\}$$

be the subsemigroups of *partial contractions* and of *order preserving partial contractions* of $[n]$, respectively.

Further, notice that the collection of all order preserving or order reversing partial contractions denoted by $\mathcal{ORCP}_n$ is a subsemigroup of $\mathcal{ORP}_n$ (where $\mathcal{ORP}_n$ denotes the semigroup of order preserving or order reversing partial transformations of $[n]$).

Moreover, let

$$\mathcal{CT}_n = \{\alpha \in \mathcal{T}_n : (\text{for all } x, y \in [n])\ |x\alpha - y\alpha| \leqslant |x - y|\}, \tag{1}$$

$$\mathcal{OCT}_n = \{\alpha \in \mathcal{CT}_n : (\text{for all } x, y \in [n])\ x \leqslant y \text{ implies } x\alpha \leqslant y\alpha\}, \tag{2}$$

and

$$\mathcal{ORCT}_n = \mathcal{OCT}_n \cup \{\alpha \in \mathcal{CT}_n : (\text{for all } x, y \in [n])\ x \leqslant y \text{ implies } x\alpha \geqslant y\alpha\} \tag{3}$$

be the subsemigroups of *full contractions*, of *order preserving full contractions* and of *order preserving or reversing full contractions* on $[n]$, respectively. A general study of these semigroups was proposed in a 2013 research proposal by Umar and Alkharousi [9] supported by a grant from

The Research Council of Oman (TRC). In the proposal [9], notations for the semigroups and their subsemigroups were given, as such we maintain the same notations in this paper. The earliest reference to contraction mappings in the context of algebraic semigroup theory is in [28]. Later in 2012, Zhao and Yang [31] characterized the Green's relations on the subsemigroup $\mathcal{OCP}_n$. Recently, Adeshola and Umar [11] studied combinatorial properties of certain subsemigroups of $\mathcal{CT}_n$ while Ali *et al.* [12,13] obtained a necessary and sufficient condition for an element in $\mathcal{CP}_n$ and $\mathcal{CT}_n$ to be regular and also described all their Green's equivalences. Most of the results concerning regularity and Green's relations for some subsemigroups of $\mathcal{CP}_n$ can be deduced from the results obtained in Ali *et al.* [12,13]. Zhao and Yang [31] have shown that the semigroup $\mathcal{OCP}_n$ is nonregular for $n > 2$. Similarly, Ali *et al.* [12,13] have shown that the semigroups $\mathcal{CP}_n$ and $\mathcal{ORCP}_n$ are nonregular for $n > 2$. Thus, there is a need to identify the class of semigroups to which they belong, for example, whether they are *abundant* semigroups [18] and/or *idempotent-generated* semigroups [22]. Therefore, this paper is a natural sequel to Ali *et al.* [12,13].

This section includes a brief introduction giving some basic definitions and introducing some new concepts. In section 2, we characterize all the starred Green's relations on the semigroups $\mathcal{CP}_n$, $\mathcal{ORCP}_n$ and $\mathcal{OCP}_n$ and show that $\mathcal{D}^* = \mathcal{J}^*$ (see [18]). We also show that the semigroups $\mathcal{CP}_n$, $\mathcal{ORCP}_n$ and $\mathcal{OCP}_n$ (resp., the semigroups $\mathcal{CT}_n$, $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$), are left abundant for all n but not right abundant for $n \geqslant 4$. In section 3, we give a characterization of idempotent elements in $\mathcal{CT}_n$ and show that product of two idempotents in $\mathcal{CT}_n$ is not necessarily an idempotent. Moreover, we show that the regular elements in $\mathcal{CT}_n$ forms a subsemigroup. In section 4, we show that the semigroups $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$ are left quasi-adequate. In section 5, we explore some *orthodox* subsemigroups of $\mathcal{CT}_n$ and its *R*ees factor semigroups. For standard concepts in semigroup theory, we refer the reader to Howie [19], Higgins [27] and Ganyushkin and Mazorchuk [26].

Let α be an element of $\mathcal{CP}_n$ and let Dom α, $\operatorname{Im}\alpha$, $h(\alpha)$ and $F(\alpha)$ denote, the *domain of* α, *image of* α, $|\operatorname{Im}\alpha|$ and $\{x \in \operatorname{Dom}\alpha : x\alpha = x\}$ (i.e., the *set of fixed points of* α), respectively. For $\alpha, \beta \in \mathcal{CP}_n$, the composition of α and β is defined as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for any x in $\operatorname{Dom}\alpha\beta = (\operatorname{Im}\alpha \cap \operatorname{Dom}\beta)\alpha^{-1}$. Without ambiguity, we shall be using the notation $\alpha\beta$ to denote $\alpha \circ \beta$.

Next, given any transformation α in $\mathcal{ORCP}_n$, the domain of α is partitioned into $p-$ *blocks* by the relation $\ker\alpha = \{(x, y) \in \operatorname{Dom}\alpha \times$

$\mathrm{Dom}\,\alpha : x\alpha = y\alpha\}$ and so as in [21], α can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ x_1 & x_2 & \dots & x_p \end{pmatrix} \qquad (1 \leqslant p \leqslant n), \tag{4}$$

where, $1 \leqslant x_1 < x_2 < \ldots < x_p \leqslant n$ or $1 \geqslant x_1 > x_2 > \ldots > x_p \geqslant n$, and A_i $(1 \leqslant i \leqslant p)$ are equivalence classes under the relation $\ker\alpha$, i.e., $A_i = x_i\alpha^{-1}$ $(1 \leqslant i \leqslant p)$. Thus for the rest of the content of the paper we shall consider α to be as expressed in equation (4) unless otherwise specified. Next, let $\ker\alpha = \{A_1, A_2, \ldots, A_p\}$ be the partition of $\mathrm{Dom}\,\alpha$. Observe that in this case, $\ker\alpha$ is ordered under the usual ordering, i.e., $A_i < A_j$ if and only if $i < j$ for all $i, j \in \{1, \ldots, n\}$. As such $A_1 < A_2 < \ldots < A_p$. A subset T_α of $\mathrm{Dom}\,\alpha$ is said to be a *transversal* of the partition $\ker\alpha$ if $|A_i \cap T_\alpha| = 1$ $(1 \leqslant i \leqslant p)$. A transversal T_α is said to be *convex* (resp., *relatively convex*) if for all $x, y \in T_\alpha$ with $x \leqslant y$ and if $x \leqslant z \leqslant y$ $(z \in [n])$(resp., $z \in \mathrm{Dom}\,\alpha$), then $z \in T_\alpha$. Notice that every convex transversal is necessarily relatively convex but not *vice-versa*. A transversal T_α is said to be *admissible* if and only if the map $A_i \mapsto t_i$ $(\{t_i\} = A_i \cap T_\alpha,\ i \in \{1, 2, \ldots, p\})$ is a contraction, see [12]. Notice that every (relatively) convex transversal is admissible but not *vice-versa*.

An element a in a semigroup S is said to be an *idempotent* if and only if $a^2 = a$. It is well known that an element $\alpha \in \mathcal{P}_n$ is an idempotent if and only if $\mathrm{Im}\,\alpha = F(\alpha) = \{x \in \mathrm{Dom}\,\alpha : x\alpha = x\}$. Equivalently, α is an idempotent if and only if $x_i \in A_i$ for $1 \leqslant i \leqslant p$, that is to say the *blocks* A_i are *stationary* [21]. As usual $E(S)$ denotes the set of all idempotents in S. An element $a \in S$ is said to be *regular* if there exists $b \in S$ such that $a = aba$.

The following lemmas from [11–13] would be useful in what follows:

Lemma 1 ([12], Lemma 1.3 and [13], Lemma 1.4). *For $n \geqslant 4$, let $\alpha \in \mathcal{CP}_n$ be such that there exists $k \in \{2, \ldots, p-1\}$ $(3 \leqslant p \leqslant n)$ and $|A_k| \geqslant 2$. Suppose $A_i < A_j$ if and only if $i < j$ for all $i, j \in \{1, 2, \ldots, p\}$. Then the partition $\ker\alpha = \{A_1, A_2, \ldots, A_p\}$ of $\mathrm{Dom}\,\alpha$ has no relatively convex transversal and hence has no admissible transversal.*

Lemma 2 ([12], Lemma 1.8 and [13], Lemma 1.5). *Let $\alpha \in \mathcal{CP}_n$ and let A be a convex subset of $\mathrm{Dom}\,\alpha$. Then $A\alpha$ is convex.*

Lemma 3 ([11], Lemma 1.2). *Let $\alpha \in \mathcal{CT}_n$. Then $\mathrm{Im}\,\alpha$ is convex.*

Lemma 4 ([10], Corollary 1.15 and [13], Corollary 5.17). *Let $\alpha \in \mathcal{ORCT}_n$. Then α is regular if and only if $\min A_p - x_p = \max A_1 - x_1 = d$ and $A_i = \{x_i + d\}$ or $\min A_p - x_1 = \max A_1 - x_p = d$ and $A_i = \{x_{p-i+1} + d\}$, for $i = 2, \ldots, p-1$.*

2. Starred Green's relations

Let S be a semigroup. The relation $\mathcal{L}^*$ defined as (for all $a, b \in S$) $a\mathcal{L}^*b$ *if and only if* a, b are related by the Green's $\mathcal{L}$ relation in some oversemigroup of S, is known as the starred Green's $\mathcal{L}$ relation. The relation $\mathcal{R}^*$ is defined dually, while the relation $\mathcal{D}^*$ is defined as the join of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. The intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$. A semigroup S is said to be *left abundant* (resp., *right abundant*) if each $\mathcal{L}^*-$class (resp., $\mathcal{R}^*-$class) contains an idempotent, and it is called *abundant* if each $\mathcal{L}^*-$class and each $\mathcal{R}^*-$class of S contains an idempotent. An abundant semigroup S in which the set $E(S)$, of its idempotents is a subsemigroup of S is called *quasi-adequate* and if $E(S)$ is commutative then S is called *adequate* [2, 17, 18]. In the 1980s and 1990s, Fountain and his students introduced and studied various classes of abundant and adequate semigroups, for example see [1, 2, 24, 25, 33, 34, 37].

The class of abundant semigroups include in particular the class of cancellative monoids, and any subsemigroup of a regular semigroup S that contains $E(S)$ is abundant [27]. The starred Green's relations play a role in the theory of abundant semigroups analogue to that of Green's relations in the theory of regular semigroups.

Many nonregular classes of transformation semigroups were shown to be either abundant or adequate, for example see [4–7, 16, 23, 30, 32]. Recently, AlKharousi *et al.* have shown that the semigroup $\mathcal{OCI}_n$, of all order preserving one to one contraction maps of a finite chain is adequate [14]. In this section we are going to show that the semigroups $\mathcal{CP}_n, \mathcal{OCP}_n, \mathcal{CT}_n$ and $\mathcal{OCT}_n$ are all left abundant (for all n) but not right abundant for $n \geqslant 4$.

We shall use the following notation from ([19], Chapter 2). If U is a subsemigroup of a semigroup S then $a\mathcal{L}^U b$ means that there exist $u, v \in U^1$ such that $ua = b$ and $vb = a$, while $a\mathcal{L}^S b$ means that there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$. Similarly, for the relation $\mathcal{R}$. Furthermore, We shall write 1_A to denote a *partial identity* mapping defined on $A \subseteq [n]$.

Some of the earlier results concerning starred Green's relations on a transformation semigroup were obtained by Umar [4–7], where he described all the starred relations on the semigroups of order decreasing full and of order decreasing partial one-one transformations of a chain, these papers marked the beginning of the study of these relations on a transformation semigroup. Recently, Garba *et al.* characterized these relations on the semigroup of full contraction maps and of order preserving full contraction maps of a finite chain: $\mathcal{CT}_n$ and $\mathcal{OCT}_n$, respectively [15]. In this section, we characterize these relations on the more general semigroup of partial contractions $\mathcal{CP}_n$ and its subsemigroups of order preserving or order reversing partial contraction maps of a finite chain $\mathcal{ORCP}_n$, and of order preserving partial contraction maps of a finite chain $\mathcal{OCP}_n$, respectively. We equally show that the relations $\mathcal{D}^*$ and $\mathcal{J}^*$ coincide on these semigroups.

To begin our investigation let us start with the following. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ have the following characterizations as described in ([19], Exercise 2.6.7-9) or as described in [17].

$$\mathcal{L}^* = \{(a,b) : (\text{for all } x, y \in S^1)\ ax = ay \Leftrightarrow bx = by\} \tag{5}$$

and

$$\mathcal{R}^* = \{(a,b) : (\text{for all } x, y \in S^1)\ xa = ya \Leftrightarrow xb = yb\} \tag{6}$$

It is worth noting that the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ do not commute, in general. Also, for regular semigroups, $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$.

Denote

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ x_1 & x_2 & \dots & x_p \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ y_1 & y_2 & \dots & y_p \end{pmatrix} (1 \leqslant p \leqslant n), \tag{7}$$

We next give the characterizations of these relations on the semigroups $\mathcal{CP}_n$, $\mathcal{ORCP}_n$ and $\mathcal{OCP}_n$ as follows: Let S be a semigroup in $\{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$.

Theorem 1. *Let $\alpha, \beta \in S$ be as expressed in equation (7). Then*
(i) *$\alpha\mathcal{L}^*\beta$ if and only if $\operatorname{Im}\alpha = \operatorname{Im}\beta$.*
(ii) *$\alpha\mathcal{R}^*\beta$ if and only if $\ker\alpha = \ker\beta$.*
(iii) *$\alpha\mathcal{H}^*\beta$ if and only if $\operatorname{Im}\alpha = \operatorname{Im}\beta$ and $\ker\alpha = \ker\beta$.*
(iv) *$\alpha\mathcal{D}^*\beta$ if and only if $|\operatorname{Im}\alpha| = |\operatorname{Im}\beta|$.*

Proof. (i) Let α, β be elements in $S \in \{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$ such that $\alpha\mathcal{L}^*\beta$ and $\operatorname{Im}\alpha = \{x_1, x_2, \dots, x_p\}$. Further, let $\gamma = \begin{pmatrix} x_1 & x_2 & \dots & x_p \\ x_1 & x_2 & \dots & x_p \end{pmatrix}$.

Then clearly $\gamma \in S$ and

$$\alpha \circ \begin{pmatrix} x_1 & x_2 & \dots & x_p \\ x_1 & x_2 & \dots & x_p \end{pmatrix} = \alpha \circ 1_{[n]}$$

$$\Leftrightarrow \beta \circ \begin{pmatrix} x_1 & x_2 & \dots & x_p \\ x_1 & x_2 & \dots & x_p \end{pmatrix} = \beta \circ 1_{[n]} \quad (\text{by equation } (5))$$

which implies that $\operatorname{Im}\beta \subseteq \{x_1, x_2, \ldots, x_p\} = \operatorname{Im}\alpha$ or $\operatorname{Im}\beta = \varnothing \subseteq \{x_1, x_2, \ldots, x_p\} = \operatorname{Im}\alpha$. Similarly, in the same manner we can show that $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$. Thus, $\operatorname{Im}\alpha = \operatorname{Im}\beta$.

Conversely, suppose that $\operatorname{Im}\alpha = \operatorname{Im}\beta$. Then by ([19], Exercise 2.6.17) $\alpha\mathcal{L}^{\mathcal{P}_n}\beta$, and it follows from definition that $\alpha\mathcal{L}^*\beta$. Thus, the result follows.

(ii) Suppose that $\alpha, \beta \in S$ and $\alpha\mathcal{R}^*\beta$. Now if $(x, y) \in \ker\alpha$ then there are 3 instances. i.e., either $x, y \in \operatorname{Dom}\beta$ or $x, y \notin \operatorname{Dom}\beta$ or $x \in \operatorname{Dom}\beta$ and $y \notin \operatorname{Dom}\beta$. If $x, y \in \operatorname{Dom}\beta$. Then $(x, y) \in \ker\alpha$ if and only if

$$\begin{aligned}
&\binom{\operatorname{Dom}\alpha}{x} \circ \alpha = \binom{\operatorname{Dom}\alpha}{y} \circ \alpha \\
&\Leftrightarrow \binom{\operatorname{Dom}\alpha}{x} \circ \beta = \binom{\operatorname{Dom}\alpha}{y} \circ \beta \quad \text{(by equation (6))}. \\
&\Leftrightarrow x\beta = y\beta \\
&\Leftrightarrow (x, y) \in \ker\beta.
\end{aligned}$$

If $x, y \notin \operatorname{Dom}\beta$. Then

$$\begin{aligned}
&\binom{\operatorname{Dom}\alpha}{x} \circ \beta = \varnothing = \binom{\operatorname{Dom}\alpha}{y} \circ \beta \\
&\Leftrightarrow \binom{\operatorname{Dom}\alpha}{x} \circ \alpha = \varnothing = \binom{\operatorname{Dom}\alpha}{y} \circ \alpha \quad \text{(by equation (6))}. \\
&\Leftrightarrow x, y \notin \operatorname{Dom}\alpha.
\end{aligned}$$

Finally if $x \in \operatorname{Dom}\beta$ and $y \notin \operatorname{Dom}\beta$. Then $(x, y) \notin \ker\beta$ if and only if

$$\begin{aligned}
&\binom{\operatorname{Dom}\alpha}{x} \circ \beta = \binom{\operatorname{Dom}\alpha}{x} \neq \varnothing = \binom{\operatorname{Dom}\alpha}{y} \circ \beta \\
&\Leftrightarrow \binom{\operatorname{Dom}\alpha}{x} \circ \alpha = \binom{\operatorname{Dom}\alpha}{x} \neq \varnothing = \binom{\operatorname{Dom}\alpha}{y} \circ \alpha \quad \text{(by equation (6))}. \\
&\Leftrightarrow x \in \operatorname{Dom}\alpha \text{ and } y \notin \operatorname{Dom}\alpha. \\
&\Leftrightarrow (x, y) \notin \ker\alpha.
\end{aligned}$$

Hence $\ker\alpha = \ker\beta$.

Conversely, suppose that $\ker\alpha = \ker\beta$. Then by ([19], Exercise 2.6.17) $\alpha\mathcal{R}^{\mathcal{P}_n}\beta$, and it follows from definition that $\alpha\mathcal{R}^*\beta$.

(iii) This follows directly from (i) and (ii) above.

(iv) Suppose that $\alpha\mathcal{D}^*\beta$. Then by ([19], Proposition 1.5.11) there exist elements $\gamma_1, \gamma_2, \ldots, \gamma_{2n-1} \in S$ such that $\alpha\mathcal{L}^*\gamma_1$, $\gamma_1\mathcal{R}^*\gamma_2$, $\gamma_2\mathcal{L}^*\gamma_3, \ldots,$

$\gamma_{2n-1}\mathcal{R}^*\beta$ for some $n \in \mathbb{N}$. Thus, by (i) and (ii) we have $\operatorname{Im}\alpha = \operatorname{Im}\gamma_1$, $\ker\gamma_1 = \ker\gamma_2$, $\operatorname{Im}\gamma_2 = \operatorname{Im}\gamma_3\ldots$, $\ker\gamma_{2n-1} = \ker\beta$. This implies that $|\operatorname{Im}\alpha| = |\operatorname{Im}\gamma_1| = |\operatorname{Dom}\gamma_1/\ker\gamma_1| = |\operatorname{Dom}\gamma_2/\ker\gamma_2| = \ldots = |\operatorname{Dom}\gamma_{2n-1}/\ker\gamma_{2n-1}| = |\operatorname{Dom}\beta/\ker\beta| = |\operatorname{Im}\beta|$.

Conversely, suppose $|\operatorname{Im}\alpha| = |\operatorname{Im}\beta|$ where

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_p \\ x_1 & x_2 & \ldots & x_p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & B_2 & \ldots & B_p \\ y_1 & y_2 & \ldots & y_p \end{pmatrix} \quad (p \leqslant n) \quad (8)$$

where we may without loss of generality assume that $1 \leqslant x_1 < x_2 < \ldots < x_p \leqslant n$ and $1 \leqslant y_1 < y_2 < \ldots < y_p \leqslant n$. Now let $\{x+1, x+2, \ldots, x+p\}$ be an arbitrary convex subset of $[n]$. Notice that $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$ are ordered. Now consider

$$\gamma_1 = \begin{pmatrix} A_1 & \ldots & A_p \\ x+1 & \ldots & x+p \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} B_1 & \ldots & B_p \\ x+1 & \ldots & x+p \end{pmatrix} \in \mathcal{CP}_n.$$

Then by Theorem 1 (i) and (ii), it follows that $\alpha\mathcal{R}^*\gamma_1\mathcal{L}^*\gamma_2\mathcal{R}^*\beta$ which imply $\alpha\mathcal{R}^*o\mathcal{L}^*o\mathcal{R}^*\beta$. On the other hand suppose $\alpha\mathcal{R}^*o\mathcal{L}^*o\mathcal{R}^*\beta$. This means there exist $\gamma_1, \gamma_2 \in \mathcal{CP}_n$ such that $\alpha\mathcal{R}^*\gamma_1\mathcal{L}^*\gamma_2\mathcal{R}^*\beta$. It follows that $|\operatorname{Im}\alpha| = |\operatorname{Dom}\alpha/\ker\alpha| = |\operatorname{Dom}\gamma_1/\ker\gamma_1| = |\operatorname{Im}\gamma_1| = |\operatorname{Im}\gamma_2| = |\operatorname{Dom}\gamma_2/\ker\gamma_2| = |\operatorname{Dom}\beta/\ker\beta| = |\operatorname{Im}\beta|$. This means by ([19], Proposition 1.5.11) that $\alpha\mathcal{D}^*\beta$, as required. □

In the last paragraph of the proof above, we have proved the following lemma.

Lemma 5. *Let S be a semigroup in $\{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$. Then $\mathcal{D}^* = \mathcal{R}^*o\mathcal{L}^*o\mathcal{R}^*$.*

The following theorem gives characterizations of starred Green's relations for a semigroup $S \in \{\mathcal{CT}_n, \mathcal{OCT}_n\}$ from [15].

Theorem 2 ([15], Theorem 4.1)**.** *Let α, β be elements in $S \in \{\mathcal{CT}_n, \mathcal{OCT}_n\}$ be as expressed in (7). Then we have the following:*

(i) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{Im}\alpha = \operatorname{Im}\beta$*;*
(ii) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\ker\alpha = \ker\beta$*;*
(iii) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{Im}\alpha = \operatorname{Im}\beta$ *and* $\ker\alpha = \ker\beta$*;*
(iv) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{Im}\alpha| = |\operatorname{Im}\beta|$.

Remark 1. (i) The statements of Theorems 1 and 2 are the same. However the proofs are different since $S \in \{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$ contains partial maps. In our proof of Theorem 2.1(i), we have to consider where $\operatorname{Im}\beta$

could be empty, and in (ii) we have to consider the cases where either $x, y \in \operatorname{Dom}\beta$ or $x, y \notin \operatorname{Dom}\beta$ or $x \in \operatorname{Dom}\beta$ and $y \notin \operatorname{Dom}\beta$.

(ii) The starred Green's relations characterizations in Theorem 2 also hold in $\mathcal{ORCT}_n$.

Proposition 1. *Let S be a semigroup in $\{\mathcal{CP}_n, \mathcal{ORCP}_n\}$. Then for $n \geqslant 5$, $\mathcal{D}^* = \mathcal{R}^* o \mathcal{L}^* o \mathcal{R}^* \neq \mathcal{L}^* o \mathcal{R}^* o \mathcal{L}^*$.*

Proof. Let $\alpha = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 2 & 4 & 5 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 \end{pmatrix}$ be elements in S. Define

$$\gamma_1 = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Then it is clear that $\alpha \mathcal{R}^* \gamma_1 \mathcal{L}^* \gamma_2 \mathcal{R}^* \beta$. However, if $\alpha \mathcal{L}^* \gamma_1 \mathcal{R}^* \gamma_2 \mathcal{L}^* \beta$ then $\operatorname{Im}\gamma_1 = \operatorname{Im}\alpha = \{1, 2, 4, 5\}$ and $\operatorname{Im}\gamma_2 = \operatorname{Im}\beta = \{1, 3, 4, 5\}$, but it is impossible to find $\operatorname{Dom}\gamma_1 = \operatorname{Dom}\gamma_2 \subseteq \{1, 2, 3, 4, 5\}$ that will admit the two possible image sets and, for γ_1 and γ_2 to be contractions. Hence $\mathcal{D}^* \neq \mathcal{L}^* o \mathcal{R}^* o \mathcal{L}^*$. □

Proposition 2. *On the semigroup $\mathcal{OCP}_n$ $(n \geqslant 4)$, $\mathcal{D}^* = \mathcal{R}^* o \mathcal{L}^* o \mathcal{R}^* \neq \mathcal{L}^* o \mathcal{R}^* o \mathcal{L}^*$.*

Proof. Let $\alpha = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix}$. Define

$$\gamma_1 = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then it is clear that $\alpha \mathcal{R}^* \gamma_1 \mathcal{L}^* \gamma_2 \mathcal{R}^* \beta$. However, if $\alpha \mathcal{L}^* \gamma_1 \mathcal{R}^* \gamma_2 \mathcal{L}^* \beta$ then $\operatorname{Im}\gamma_1 = \operatorname{Im}\alpha = \{1, 2, 4\}$ and $\operatorname{Im}\gamma_2 = \operatorname{Im}\beta = \{1, 3, 4\}$, but it is impossible to find $\operatorname{Dom}\gamma_1 = \operatorname{Dom}\gamma_2 \subseteq \{1, 2, 3, 4\}$ that will admit the two possible image sets and, for γ_1 and γ_2 to be order preserving contractions. In fact for γ_1 and γ_2 to be contractions, they must be $\begin{pmatrix} 1 & 3 & 4 \\ 4 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix}$, respectively or $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \end{pmatrix}$, respectively. Notice that in the former γ_1 is a contraction but not order preserving, i.e., $\gamma_1 \notin \mathcal{OCP}_n$ and also in the latter γ_2 is a contraction but not order preserving, i.e., $\gamma_2 \notin \mathcal{OCP}_n$. Hence $\mathcal{D}^* \neq \mathcal{L}^* o \mathcal{R}^* o \mathcal{L}^*$. □

Fountain [18] introduced the notion of $*-$*ideal* to study the starred Green's relation J^*. A *left* (resp., *right*) $*-$*ideal* of a semigroup S is defined as the *left* (resp., *right*) ideal of S for which $L_a^* \subseteq I$ (resp., $R_a^* \subseteq I$) for all $a \in I$. A subset I of a semigroup S is a $*-$ideal if it is both left and right $*-$ideal of S. The principal $*-$ideal, $J^*(a)$, generated by $a \in S$ is the intersection of all $*-$ideal of S containing a, where the relation $\mathcal{J}^*$ is defined as: $a\mathcal{J}^*b$ if and only if $J^*(a) = J^*(b)$ for all $a, b \in S$. We now recognize the following lemma from Fountain [18].

Lemma 6 ([18], Lemma 1.7(3))**.** *Let a, b be elements of a semigroup S. Then $b \in J^*(a)$ if and only if there are elements $a_0, a_1, \ldots, a_n \in S$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}^*$ for $i = 1, 2, \ldots, n$.*

As in [5], we immediately have:

Lemma 7. *Let S be in $\{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$. Then for $\alpha, \beta \in S$, $\alpha \in J^*(\beta)$ implies $|\operatorname{Im}\alpha| \leqslant |\operatorname{Im}\beta|$.*

Proof. Let $\alpha \in J^*(\beta)$. Then by Lemma 6, there exist $\eta_0, \eta_1 \ldots, \eta_n \in S$, $\rho_1, \ldots, \rho_n, \tau_1, \ldots, \tau_n \in S^1$ such that $\beta = \eta_0, \alpha = \eta_n$ and $(\eta_i, \rho_i \eta_{i-1} \tau_i) \in \mathcal{D}^*$ for $i = 1, 2, \ldots, n$. Thus, by Theorem 1(iv), it implies that

$$|\operatorname{Im}\eta_i| = |\operatorname{Im}\rho_i\eta_{i-1}\tau_i| \leqslant |\operatorname{Im}\eta_i| \text{ for } i = 1, 2, \ldots, n,$$

which implies that $|\operatorname{Im}\alpha| \leqslant |\operatorname{Im}\beta|$. □

Notice that, $\mathcal{D}^* \subseteq \mathcal{J}^*$ and together with Lemma 7 we have:

Corollary 1. *On the semigroups $\mathcal{CP}_n$, $\mathcal{ORCP}_n$ or $\mathcal{OCP}_n$ we have $\mathcal{D}^* = \mathcal{J}^*$.*

We now are going to show in the next lemma that if $S \in \{\mathcal{CP}_n, \mathcal{OCP}_n, \mathcal{ORCP}_n\}$ then S is left abundant.

Lemma 8. *Let $S \in \{\mathcal{CP}_n, \mathcal{OCP}_n, \mathcal{ORCP}_n\}$. Then S is left abundant.*

Proof. Let $\alpha \in S$ and L_α^* be an $\mathcal{L}^* -$ *class* of α in S, where $\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_p \\ x_1 & x_2 & \ldots & x_p \end{pmatrix}$ $(1 \leqslant p \leqslant n)$. Define $\gamma = \begin{pmatrix} x_1 & x_2 & \ldots & x_p \\ x_1 & x_2 & \ldots & x_p \end{pmatrix}$. Clearly $\gamma^2 = \gamma \in S$ and $\operatorname{Im}\alpha = \operatorname{Im}\gamma$, therefore by Theorem 1(i), $\alpha\mathcal{L}^*\gamma$, which means that $\gamma \in L_\alpha^*$. Thus, S is left abundant, as required. □

Theorem 3. *Let $S \in \{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$. Then for $n \geqslant 4$, S is not right abundant.*

Proof. Let $n = 4$ and consider $\alpha = \begin{pmatrix} 1 & \{2,3\} & 4 \\ 1 & 2 & 3 \end{pmatrix}$. It is clear that α is in S and

$$R^*_\alpha \subseteq \left\{ \begin{pmatrix} 1 & \{2,3\} & 4 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & \{2,3\} & 4 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \{2,3\} & 4 \\ 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & \{2,3\} & 4 \\ 4 & 3 & 2 \end{pmatrix} \right\},$$

which has no idempotent element. □

Remark 2. Let $S \in \{\mathcal{CP}_n, \mathcal{ORCP}_n, \mathcal{OCP}_n\}$. Then for $1 \leqslant n \leqslant 3$, S is right abundant.

The starred Green's relations for the semigroups $\mathcal{CT}_n$ and $\mathcal{OCT}_n$ were characterized by Garba *et al.* [15] and curiously they did not show whether they are abundant or not. We are now going to show that the semigroup $\mathcal{CT}_n$ and its subsemigroups $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$ are left abundant but not right abundant, in general.

One of the essential differences between the usual Green's relations and their starred analogues is that $\mathcal{L}^*$ and $\mathcal{R}^*$ may not commute in an arbitrary semigroup. However, in the case of $\mathcal{CT}_n$, $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$, they do commute as shown in the proposition below.

Proposition 3. *Let S be a semigroup in $\{\mathcal{CT}_n, \mathcal{ORCT}_n, \mathcal{OCT}_n\}$. Then $\mathcal{D}^* = \mathcal{L}^* o \mathcal{R}^* = \mathcal{R}^* o \mathcal{L}^*$.*

Proof. Suppose $\alpha \mathcal{D}^* \beta$. Then by Theorem 2 (iv), $|\operatorname{Im}\alpha| = |\operatorname{Im}\beta|$. Notice that $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$ are convex by Lemma 3. Thus we can write α and β as:

$$\alpha = \begin{pmatrix} A_1 & \dots & A_p \\ x+1 & \dots & x+p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & \dots & B_p \\ y+1 & \dots & y+p \end{pmatrix} \quad (p \leqslant n) \tag{9}$$

where we may without loss of generality assume that $1 \leqslant x+1 < x+2 < \dots < x+p \leqslant n$ and $1 \leqslant y+1 < y+2 < \dots < y+p \leqslant n$. It is now easy to see that the map defined as $\begin{pmatrix} x+1 & x+2 & \dots & x+p \\ y+1 & y+2 & \dots & y+p \end{pmatrix}$ is an isometry. Therefore the maps defined as

$$\gamma_1 = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ y+1 & y+2 & \dots & y+p \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} B_1 & B_2 & \dots & B_p \\ x+1 & x+2 & \dots & x+p \end{pmatrix}$$

are also contractions in S. Thus by Theorem 2 (i) and (ii), it follows that $\alpha \mathcal{L}^* \gamma_2$ and $\gamma_2 \mathcal{R}^* \beta$ which imply $\alpha \mathcal{L}^* o \mathcal{R}^* \beta$. Similarly by Theorem 2 (i) and (ii), it is easy to see that $\alpha\mathcal{R}^*\gamma_1$ and $\gamma_1\mathcal{L}^*\beta$, which imply $\alpha\mathcal{R}^*o\mathcal{L}^*\beta$. It follows that:

$$\mathcal{D}^* \subseteq \mathcal{L}^*o\mathcal{R}^* \subseteq \mathcal{D}^* \text{ and } \mathcal{D}^* \subseteq \mathcal{R}^*o\mathcal{L}^* \subseteq \mathcal{D}^*.$$

The result now follows. □

The following result is the version of a well known result about regular semigroups that applies to the semigroup $\mathcal{CT}_n$ and some of its subsemigroups. A similar version of this result about a nonregular semigroup was recorded in ([4], Lemma 3.1).

Lemma 9. *Let $\alpha, \beta \in S$, where $S \in \{\mathcal{CT}_n, \mathcal{OCT}_n, \mathcal{ORCT}_n\}$. If $(\alpha, \beta) \in \mathcal{D}^*$ and $(\alpha, \alpha\beta) \in \mathcal{D}^*$, then $(\alpha, \alpha\beta) \in \mathcal{R}^*$ and $(\alpha\beta, \beta) \in \mathcal{L}^*$.*

Proof. Let $(\alpha, \beta) \in \mathcal{D}^*$ and $(\alpha, \alpha\beta) \in \mathcal{D}^*$. Then by Theorem 2 (iv)

$$|\operatorname{Im}\alpha| = |\operatorname{Im}\beta| = |\operatorname{Im}\alpha\beta|.$$

It follows that $\operatorname{Im}\alpha\beta = \operatorname{Im}\beta$ and $\ker\alpha\beta = \ker\alpha$, which respectively implies $(\alpha\beta, \beta) \in \mathcal{L}^*$ and $(\alpha, \alpha\beta) \in \mathcal{R}^*$ by Theorem 2 (i) and (ii), respectively. □

We now prove the following theorem:

Theorem 4. *Let $S \in \{\mathcal{CT}_n, \mathcal{OCT}_n, \mathcal{ORCT}_n\}$. Then S is left abundant.*

Proof. Let L^*_α be an $\mathcal{L}^* - class$ of α in S. First notice that by Lemma 3 an arbitrary $\alpha \in S$ can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ x+1 & x+2 & \dots & x+p \end{pmatrix} \quad (1 \leqslant p \leqslant n).$$

Now consider

$$\gamma = \begin{pmatrix} \{1, 2, \dots, x{+}1\} & x{+}2 & \dots & x{+}p{-}1 & \{x{+}p, x{+}p{+}1, \dots, n\} \\ x+1 & x+2 & \dots & x+p-1 & x+p \end{pmatrix} \in S.$$

It is clear that γ is an idempotent with $\operatorname{Im}\alpha = \operatorname{Im}\gamma$ so that $\gamma \in L^*_\alpha$ by Theorem 1(i). This completes the proof. □

Now similarly, as in Theorem 3 we deduce the following remark.

Remark 3. Let $S \in \{\mathcal{CT}_n, \mathcal{ORCT}_n, \mathcal{OCT}_n\}$. Then
(i) for $n \geqslant 4$, S is not right abundant;
(ii) for $1 \leqslant n \leqslant 3$, S is right abundant.

3. Regular elements of $\mathcal{CT}_n$

A regular semigroup S is said to be *orthodox* if $E(S)$ is a subsemigroup of S. An orthodox semigroup is said to be $\mathcal{R}$-*unipotent* (resp., $\mathcal{L}$-*unipotent*) if every $\mathcal{R}$-class (resp., every $\mathcal{L}$-class) has a unique idempotent (see [27], Exercise 1.2.19). If an orthodox semigroup S is both $\mathcal{R}$ and $\mathcal{L}$-unipotent then S is an inverse semigroup. For detailed account on regular semigroups, we refer the reader to [20, 29, 36]. Regular elements in $\mathcal{CT}_n$ were characterized in [12] and [13]. Furthermore, we denote by $\mathrm{Reg}(S)$ to be the collection of all regular elements of S. If A is a subset of S then $\langle A\rangle$ denotes the semigroup generated by A. Moreover, $\langle A\rangle = A$ if and only if A is a subsemigroup of S and if $\langle A\rangle = S$ then A is said to generate S. Recall from section one that an element $\alpha \in \mathcal{CT}_n$ is an idempotent if and only if $x_i \in A_i$ for $1 \leqslant i \leqslant p$, that is to say the *blocks* A_i are *stationary* [21]. We begin by recalling the following known characterization of regular elements in $\mathcal{CT}_n$ from [10].

Theorem 5 ([10], Corollary 1.13 and [13], Theorem 5.15)**.** *Let $\alpha \in \mathcal{CT}_n$. Then α is regular if and only if* $\ker\alpha$ *has a convex transversal, T_α.*

We now have the following lemma.

Lemma 10. *Let $\alpha \in \mathcal{CT}_n$ be as expressed in equation (4). Then α is an idempotent if and only if* $\mathrm{Im}\,\alpha = T_\alpha$*, where $T_\alpha = \{t+1, \ldots, t+p\}$ $(t+i \in A_i,\ 1 \leqslant i \leqslant p)$ is a convex transversal of* $\ker\alpha$*.*

Proof. The result follows from the definition of an idempotent and the fact that $\mathrm{Im}\,\alpha$ and the transversal $\{t+1, \ldots, t+p\}$ are necessarily convex by Lemma 3 and Theorem 5, respectively. □

Remark 4. It is worth noting that product of two idempotents in $\mathcal{CT}_n$ is not necessarily an idempotent. For example, consider the idempotents

$$\epsilon_1 = \begin{pmatrix} 1 & 2 & \{3,4\} \\ 1 & 2 & 3 \end{pmatrix} \text{ and } \epsilon_2 = \begin{pmatrix} \{1,2,4\} & 3 \\ 4 & 3 \end{pmatrix}$$

in $\mathcal{CT}_4$. The product $\epsilon_1\epsilon_2 = \begin{pmatrix} \{1,2\} & \{3,4\} \\ 4 & 3 \end{pmatrix}$ is not an idempotent.

Moreover, the non-regular element $\alpha = \begin{pmatrix} 1 & \{2,3\} & 4 \\ 1 & 2 & 3 \end{pmatrix} \in \mathcal{CT}_4$ cannot be expressed as a product of idempotents in $\mathcal{CT}_4$. Notice that the only idempotent in $\mathcal{CT}_4$ with rank greater than 3 is the identity map which is not useful in product of idempotents. Furthermore the collection of all idempotents in $\mathcal{CT}_4$ of rank 3 is

$$\left\{\begin{pmatrix}1 & 2 & \{3,4\}\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}\{1,3\} & 2 & 4\\ 3 & 2 & 4\end{pmatrix}, \begin{pmatrix}\{1,2\} & 3 & 4\\ 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & \{2,4\} & 3\\ 1 & 2 & 3\end{pmatrix}\right\}.$$

It is easy to verify that no products of idempotents from this collection gives α. This shows that $\mathcal{CT}_n$ $(n \geqslant 4)$ is not generated by idempotents.

We thus conclude from the above remark that $\langle E(\mathcal{CT}_n)\rangle \neq E(\mathcal{CT}_n)$ and $\langle E(\mathcal{CT}_n)\rangle \neq \mathcal{CT}_n$ $(n \geqslant 4)$. The next result is from [35].

Proposition 4 ([35], Proposition 1)**.** *Let S be an arbitrary semigroup. Then the following are equivalent:*

(i) *For all idempotents e and f of S, the element ef is regular;*
(ii) $\text{Reg}(S)$ *is a regular subsemigroup;*
(iii) $\langle E(S)\rangle$ *is a regular semigroup.*

Then we have the following lemma.

Lemma 11. *Let $\epsilon, \tau \in E(\mathcal{CT}_n)$. Then $\epsilon\tau$ is regular.*

Proof. Let $\epsilon, \tau \in E(\mathcal{CT}_n)$. Then by Lemma 10 we may denote

$$\epsilon = \begin{pmatrix}A_1 & A_2 & \dots & A_p\\ t+1 & t+2 & \dots & t+p\end{pmatrix} \text{ and } \tau = \begin{pmatrix}B_1 & B_2 & \dots & B_s\\ t'+1 & t'+2 & \dots & t'+s\end{pmatrix}$$

for some $p, s \in [n]$, where $T_\epsilon = \{t+1, \dots, t+p\}$ $(t+i \in A_i,\ 1 \leqslant i \leqslant p)$ and $T_\tau = \{t'+1, \dots, t'+s\}$ $(t'+j \in B_j,\ 1 \leqslant j \leqslant s)$ are convex transversals of **Ker** ϵ and **Ker** τ, respectively. Now since $T_\epsilon = \{t+1, t+2, \dots, t+p\}$ is convex, $T_\epsilon\epsilon\tau = \{(t+1)\epsilon\tau, (t+2)\epsilon\tau, \dots, (t+p)\epsilon\tau\}$ whose elements are not necessarily distinct but is nevertheless convex. Moreover, it is not difficult to see that it is a convex transversal of **Ker** $\epsilon\tau$. Hence by Theorem 5 $\epsilon\tau$ is regular. □

As a consequence of Proposition 4 and Lemma 11 we have the following:

Corollary 2. *Let $\mathcal{CT}_n$ be as defined in equation (1). Then we have*

(i) $\text{Reg}(\mathcal{CT}_n)$ *is a regular subsemigroup of $\mathcal{CT}_n$;*
(ii) $\langle E(\mathcal{CT}_n)\rangle$ *is a regular subsemigroup of $\mathcal{CT}_n$.*

Remark 5. It is worth noting that $\text{Reg}(\mathcal{CT}_n) \neq \langle E(\mathcal{CT}_n)\rangle$ $(n \geqslant 2)$. To see this, consider $\alpha = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} \in \text{Reg}(\mathcal{CT}_2)$. Notice that $E(\mathcal{CT}_2) = \left\{\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\right\}$. However, it is easy to see that for all $\epsilon_1, \epsilon_2 \in E(\mathcal{CT}_2)$, $\epsilon_1\epsilon_2 \neq \alpha$.

4. Semigroup of order preserving or order reversing full contractions: $\mathcal{ORCT}_n$

A semigroup S is said to be *left quasi-adequate* (resp., *right quasi-adequate*) when it is left abundant (resp., right abundant) and its set of idempotents forms a subsemigroup, and it is said to be *quasi-adequate* when it is both left and right quasi-adequate. For a detailed account of the structure theory and examples of quasi-adequate semigroups, we refer the reader to [2] and [8], respectively. Now let $\mathrm{Reg}(\mathcal{ORCT}_n)$ denote the set of all regular elements in $\mathcal{ORCP}_n$. Then we have the following lemma:

Lemma 12. *Let $\alpha \in \mathcal{ORCT}_n$ be as expressed in equation (4). Then α is regular if and only if α is of the form*

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+1 & x+2 & \ldots & x+p-1 & x+p \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+p & x+p-1 & \ldots & x+2 & x+1 \end{pmatrix}.$$

Proof. Let $\alpha \in \mathcal{ORCT}_n$ be as expressed in equation (4). Now suppose α is a regular element in $\mathcal{ORCT}_n$. Then by the contrapositive of Lemma 1 we see that $\ker\alpha = \{A_1, \{a+2\}, \ldots, \{a+p-1\}, A_p\}$ (where $A_1 < \{a+2\} < \cdots < \{a+p-1\} < A_p$). Now since α is regular, $T_\alpha = \{\max A_1, \{a+2\}, \ldots, \{a+p-1\}, \min A_p\}$ is convex. Therefore $\max A_1 = a+1$ and $\min A_p = a+p$. The fact that α is a full map and $\ker\alpha$ is ordered implies $A_1 = \{1,\ldots,a+1\}$ and $A_p = \{a+p,\ldots,n\}$. Moreover, by Lemma 3, $T_\alpha\alpha = \mathrm{Im}\,\alpha$ is convex say $\mathrm{Im}\,\alpha = \{x+1, x+2, \ldots, x+p\}$ and hence since α is order preserving or reversing, we have:

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+1 & x+2 & \ldots & x+p-1 & x+p \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+p & x+p-1 & \ldots & x+2 & x+1 \end{pmatrix},$$

as required.

Conversely, if

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & a+3 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+1 & x+2 & x+3 & \ldots & x+p-1 & x+p \end{pmatrix}$$

or

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & a+3 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+p & x+p-1 & x+p-2 & \ldots & x+2 & x+1 \end{pmatrix}.$$

Notice that in either case, $a+1 = \max\{1,\ldots,a+1\}$ and $a+p = \min\{a+p,\ldots,n\}$ and also both $T_\alpha = \{a+1, a+2, \ldots, a+p\}$ and $\operatorname{Im}\alpha = \{x+1, x+2, \ldots, x+p\}$ are convex. It is easy to see from Lemma 4 that α is regular in either of the cases. □

Theorem 6. $\operatorname{Reg}(\mathcal{ORCT}_n)$ *is a regular subsemigroup of* $\mathcal{ORCT}_n$.

Proof. The proof follows from Corollary 2(i) coupled with the fact that $\operatorname{Reg}(\mathcal{ORCT}_n) = \operatorname{Reg}(\mathcal{CT}_n) \cap \mathcal{ORCT}_n$. □

Now, we have the following lemma.

Lemma 13. *Let* ϵ *be an idempotent element in* $\operatorname{Reg}(\mathcal{ORCT}_n)$. *Then* ϵ *can be expressed as*

$$\begin{pmatrix} \{1,\ldots,a+1\} & a+2 & a+3 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ a+1 & a+2 & a+3 & \ldots & a+p-1 & a+p \end{pmatrix}.$$

Proof. Let $\epsilon \in \operatorname{Reg}(\mathcal{ORCT}_n)$ be of height p. Then by Lemma 12 ϵ can be expressed as

$$\epsilon = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & a+3 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+1 & x+2 & x+3 & \ldots & x+p-1 & x+p \end{pmatrix}.$$

However, since ϵ is an idempotent, the blocks of $\ker\epsilon$ are stationary i.e., $x+1 \in \{1,\ldots,a+1\}$, $x+p \in \{a+p,\ldots,n\}$, and $x+i = a+i$ $(i = 2,\ldots,p-1)$. Notice also that $T_\epsilon = \{a+1,\ldots,a+p\}$ and $\operatorname{Im}\epsilon = \{x+1,\ldots,x+p\}$ are both convex, this means that $a+1 = x+1$ and $a+p = x+p$. Thus, $x+i = a+i$ $(i = 1,\ldots,p)$, which implies $x = a$, as required. □

Theorem 7. *Let* $\mathcal{ORCT}_n$ *be as defined in equation* (3). *Then* $\operatorname{Reg}(\mathcal{ORCT}_n)$ *is orthodox.*

Proof. Let $\epsilon, \tau \in E(\operatorname{Reg}(\mathcal{ORCT}_n))$. Thus by Lemma 13 we may suppose

$$\epsilon = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ a+1 & a+2 & \ldots & a+p-1 & a+p \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} \{1,\dots,b+1\} & b+2 & \dots & b+s-1 & \{b+s,\dots,n\} \\ b+1 & b+2 & \dots & b+s-1 & b+s \end{pmatrix}$$

with $p, s \in \{1, 2, \dots, n\}$.

Let $c = \max\{a+1, b+1\}$ and $d = \min\{a+p, b+s\}$ and let the blocks of the product $\epsilon\tau$ be $D_1, D_2, \dots, D_k$, where $k \leqslant \min\{p, s\}$. It is worth noting that $F(\epsilon) \cap F(\tau) \neq \varnothing$, since ϵ and τ are full maps, and therefore $c \leqslant d$ and so $F(\epsilon\tau) = \{c, \dots, d\}$. We shall consider four subcases:

(i) If $a+1 = c$ and $a+p = d$ then $b+1 \leqslant a+1$ and $a+p \leqslant b+s$. Using convexity, it is now not difficult to see that $D_1 = \{1, \dots, a+1\}$, $D_i = \{a+i\}$ $(i = 2, \dots, k-1)$ and $D_k = \{a+p, \dots, n\}$. Moreover, $D_1\epsilon\tau = a+1 = \max D_1$ and $D_k\epsilon\tau = a+p = \min D_k$. Hence $\epsilon\tau$ is an idempotent;

(ii) If $a+1 = c$ and $b+s = d$. Using convexity, it is now not difficult to see that $D_1 = \{1, \dots, a+1\}$, $D_i = \{a+i\}$ $(i = 2, \dots, k-1)$ and $D_k \subseteq [b+s, a+p] \cup \{a+p, \dots, n\}$. Moreover, $D_1\epsilon\tau = a+1 = \max D_1$ and $D_k\epsilon\tau = b+s = \min D_k$. Hence $\epsilon\tau$ is an idempotent;

(iii) If $b+1 = c$ and $a+p = d$. If $a+1 = c$ and $b+s = d$. Using convexity, it is now not difficult to see that $D_1 \subseteq \{1, \dots, a+1\} \cup [a+p, b+1]$, $D_i = \{a+i\}$ $(i = 2, \dots, k-1)$ and $D_k = \{a+p, \dots, n\}$. Moreover, $D_1\epsilon\tau = b+1 = \max D_1$ and $D_k\epsilon\tau = a+p = \min D_k$. Hence $\epsilon\tau$ is an idempotent;

(iv) If $b+1 = c$ and $b+s = d$. Using convexity, it is now not difficult to see that $D_1 \subseteq \{1, \dots, a+1\} \cup [a+p, b+1]$, $D_i = \{a+i\}$ $(i = 2, \dots, k-1)$ and $D_k \subseteq [b+s, a+p] \cup \{a+p, \dots, n\}$. Moreover, $D_1\epsilon\tau = b+1 = \max D_1$ and $D_k\epsilon\tau = b+s = \min D_k$. Hence $\epsilon\tau$ is an idempotent.

Therefore, in either eventuality $\epsilon\tau \in E(\mathrm{Reg}(\mathcal{ORCT}_n))$.

Notice that $\mathrm{Reg}(\mathcal{ORCT}_n)$ is a subsemigroup of $\mathcal{ORCT}_n$ by Theorem 6. Thus $\mathrm{Reg}(\mathcal{ORCT}_n)$ is orthodox, as required. □

We verify case 1 of the above proof with the following example:

Example 1. Choose $n = 8$, $a = 3$, $b = 1$ so that $c = 4$. Let $p = 4$ and $s = 7$. Then $a+p \leqslant b+s$. Now

$$\epsilon = \begin{pmatrix} \{1,2,3,4\} & 5 & 6 & \{7,8\} \\ 4 & 5 & 6 & 7 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} \{1,2\} & 3 & 4 & 5 & \{6,7,8\} \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Thus $D_1 = \{1,2,3,4\}$, $D_2 = \{5\}$ and $D_3 = \{6,7,8\}$. Furthermore, $D_1 = \{1,2,3,4\}\epsilon\tau = 4$, $D_2 = \{5\}\epsilon\tau = 5$ and $D_3 = \{6,7,8\}\epsilon\tau = 6$. Hence the product

$$\epsilon\tau = \begin{pmatrix} \{1,2,3,4\} & 5 & \{6,7,8\} \\ 4 & 5 & 6 \end{pmatrix}$$

is an idempotent and it is easy to verify that $k \leqslant \min\{4,5\}$ and also $F(\epsilon\tau) = \{4,5,6\}$.

Next, we are now going to show that $\text{Reg}(\mathcal{ORCT}_n)$ is indeed a special orthodox semigroup. However, first we establish the following lemma:

Lemma 14. $\text{Reg}(\mathcal{ORCT}_n)$ *is an* $\mathcal{L}$*-unipotent semigroup.*

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & a+2 & \dots & a+p-1 & A_p \\ x+1 & x+2 & \dots & x+p-1 & x+p \end{pmatrix}$$

be an arbitrary element of $\text{Reg}(\mathcal{ORCT}_n)$. It is enough to show that every L_α contains a unique idempotent. However, the map

$$\epsilon = \begin{pmatrix} \{1,\dots,x+1\} & x+2 & \dots & x+p-1 & \{x+p,\dots,n\} \\ x+1 & x+2 & \dots & x+p-1 & x+p \end{pmatrix}$$

is obviously the unique idempotent in L_α, as required. □

We now have as a consequence of Theorems 4 and 7 the following result:

Theorem 8. *Let S be a semigroup in $\{\mathcal{ORCT}_n, \mathcal{OCT}_n\}$. Then S is left quasi-adequate.*

Notice that $\text{Reg}(\mathcal{ORCT}_n)$ is not an $\mathcal{R}$-unipotent semigroup for $n \geqslant 2$. To see this, consider $\alpha = \begin{pmatrix} [n] \\ x \end{pmatrix}$ and $\alpha' = \begin{pmatrix} [n] \\ x' \end{pmatrix}$ where $x', x \in [n]$ $(n \geqslant 2)$ and $x \neq x'$ are distinct idempotents in R_α.

Remark 6. (i) The results proved in this section for the semigroup $\mathcal{ORCT}_n$ hold when $\mathcal{ORCT}_n$ is replaced with $\mathcal{OCT}_n$.

(ii) The results proved in this section for the semigroup $\mathcal{ORCT}_n$ do not necessarily hold when $\mathcal{ORCT}_n$ is replaced with $\mathcal{ORCP}_n$. This is due to the fact that product of regular elements in $\mathcal{ORCP}_n$ is not necessarily regular. For example, consider the regular elements

$$\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \{1,2\} & 3 & 4 \\ 4 & 3 & 2 \end{pmatrix}$$

in $\mathcal{ORCP}_4$. Their product $\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \{1,2\} & 3 & 4 \\ 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}$ is not regular.

5. Rees quotients of $\mathrm{Reg}(\mathcal{ORCT}_n)$

In this section we construct a Rees quotient semigroup from $\mathrm{Reg}(\mathcal{ORCT}_n)$ and show that it is an inverse semigroup. For $n \geqslant p \geqslant 2$, let

$$K(n,p) = \{\alpha \in \mathrm{Reg}(\mathcal{ORCT}_n) : |\operatorname{Im}\alpha| \leqslant p\} \tag{10}$$

be the two-sided ideal of $\mathrm{Reg}(\mathcal{ORCT}_n)$ consisting of all elements of height less than or equal to p. Further, let

$$Q_p(n) = K(n,p)/K(n,p-1) \tag{11}$$

be the Rees factor or quotient semigroup on the two-sided ideal $K(n,p)$. The product of two elements in $Q_p(n)$ is zero if its height is less than p, otherwise it is as in $\mathrm{Reg}(\mathcal{ORCT}_n)$.

Immediately, we have the following lemma.

Theorem 9. *The semigroup $Q_p(n)$ is an inverse semigroup.*

Proof. It is clear from Lemma 14 that $Q_p(n)$ is $\mathcal{L}$-unipotent. To show it is $\mathcal{R}$-unipotent, let $\alpha \in Q_p(n)$, where

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ x+1 & x+2 & \ldots & x+p-1 & x+p \end{pmatrix} \quad (p \geqslant 2),$$

and consider $R_\alpha-$class. Notice that the map defined as

$$\epsilon = \begin{pmatrix} \{1,\ldots,a+1\} & a+2 & \ldots & a+p-1 & \{a+p,\ldots,n\} \\ a+1 & a+2 & \ldots & a+p-1 & a+p \end{pmatrix}$$

is in $Q_p(n)$ and clearly $\ker\alpha = \ker\epsilon$, thus $\epsilon \in R_\alpha$ by ([13], Corollary 5.3(ii)). Furthermore, notice that the blocks of ϵ are stationary, i.e., ϵ is an idempotent and obviously unique in R_α. Hence $Q_p(n)$ is $\mathcal{R}$-unipotent and hence $Q_p(n)$ is an inverse semigroup, as required. □

Remark 7. The results proved in this section for the semigroup $\mathcal{ORCT}_n$ hold when $\mathcal{ORCT}_n$ is replaced with $\mathcal{OCT}_n$.

We conclude the paper with the following questions suggested by the referee:

(i) When is the product of two regular elements in $\mathcal{ORCP}_n$ regular?

(ii) Is it possible to describe or characterize the idempotent generated subsemigroup of $\mathcal{CT}_n$?

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