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Homotopy equivalence of normalized and unnormalized complexes, revisited

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ABSTRACT. We consider the unnormalized and normalized complexes of a simplicial or a cosimplicial object coming from the Dold–Kan correspondence for an idempotent complete additive category (kernels and cokernels are not required). The normalized complex is defined as the image of certain idempotent in the unnormalized complex. We prove that this idempotent is homotopic to identity via homotopy which is expressed via faces and degeneracies. Hence, the normalized and unnormalized complex are homotopy isomorphic to each other. We provide explicit formulae for the homotopy.

Introduction

The study of simplicial modules began with pioneer works of Eilenberg and Mac Lane [EM53], Dold [Dol58] and Kan [Kan58]. The equivalence between the category of simplicial modules and the category of non-negatively graded complexes of modules is afterwards called the Dold–Kan correspondence. The unnormalized complex corresponds to a simplicial module in an obvious way, the normalized complex was introduced by Eilenberg and Mac Lane [EM53, Section 4] as a quotient of unnormalized complex. They prove in [EM53, Theorem 4.1] (see also Mac Lane's book [Mac63, Theorem VIII.6.1]) that the canonical projection of the unnormalized complex onto the normalized one is a homotopy equivalence.

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As noticed in [Lyu21] the normalized complex is not only the quotient but even a direct summand of the unnormalized one. The relevant idempotent has a simple expression via faces and degeneracies. This allows to work in an additive category with split idempotents in place of category of modules. A question arose whether there is a homotopy between this idempotent and the identity map of the unnormalized complex which also has a simple expression via faces and degeneracies. The present article answers this question affirmatively.

First we prove that a homotopy of the sought form exists using a spectral sequence (Section 3) associated with a double complex (Section 2). Secondly, we provide explicit formulae for this homotopy (Section 4). All that we do for simplicial and cosimplicial objects in an additive category with split idempotents.

Notation

We borrow some notation from the study [Lyu21]. The simplex category Δ has objects $[n] = \{0 < 1 < \cdots < n\}$, $n \in \mathbb{N} = \mathbb{Z}_{\geqslant 0}$, morphisms are non-decreasing maps. The cofaces are denoted as $\partial^i = \partial^i_n : [n-1] \hookrightarrow [n] \in \Delta$, $0 \leqslant i \leqslant n$, $i \notin \operatorname{Im} \partial^i$. The codegeneracies are denoted as $\sigma^j = \sigma^j_n$: $[n+1] \twoheadrightarrow [n] \in \Delta$, $0 \leqslant j \leqslant n$, $\sigma^j(j) = \sigma^j(j+1) = j$. For $0 \leqslant j < n$

$$P^j = \big([n] \xrightarrow{\sigma^j} [n-1] \xrightarrow{\partial^j} [n]\big)$$

is an idempotent, split in Δ and, *a forteriori*, in $\mathbb{Z}\Delta$. Define a morphism $\pi^k \in \mathbb{Z}\Delta([n],[n])$, $-1 \leqslant k < n$, via formula

$$\pi^k \overset{\text{def}}{=} (1 - P^k) \cdot \dots \cdot (1 - P^1) \cdot (1 - P^0) : [n] \to [n]$$

with the convention $\pi^{-2} = \pi^{-1} = 1_{[n]}$. It is an idempotent in $\mathbb{Z}\Delta$ by [Lyu21, Exercise 1.8(e)].

Denote by ∂ the sum $\sum_{i=0}^{n} (-1)^{n-i} \partial^i \in \mathbb{Z}\Delta([n-1],[n])$, $n \geqslant 1$. We have $\partial \cdot \partial = 0$, so there is a complex $\dots \xrightarrow{\partial} [n-1] \xrightarrow{\partial} [n] \xrightarrow{\partial} \dots$ in $\mathbb{Z}\Delta$.

Let $\mathcal{A}$ be an additive category. We use the unnormalized cochain complex functor from cosimplicial objects in $\mathcal{A}$ to non-negatively graded cochain complexes in $\mathcal{A}$

$$^{\circ}C : \cos \mathcal{A} \to \mathrm{Ch}^{\geqslant 0}(\mathcal{A}), \qquad ^{\circ}CB = \big((B^n), d = B(\partial) : B^{n-1} \to B^n\big)$$

and the unnormalized chain complex functor from simplicial objects in $\mathcal{A}$ to non-negatively graded chain complexes in $\mathcal{A}$

$$C : s\mathcal{A} \to \mathrm{Ch}_{\geqslant 0}(\mathcal{A}), \qquad CA = \big((A_n), d = A(\partial^{\mathrm{op}}) : A_n \to A_{n-1}\big).$$

In the following $\mathcal{A}$ denotes an additive category with split idempotents.

1. Generalities

Consider the family of idempotents of $\mathbb{Z}\Delta$

$$\big(\Pi^k|_{[n]} : [n] \to [n]\big) = \begin{cases} \pi^{n-1} & \text{if} \quad n \leqslant k+1, \\ \pi^k & \text{if} \quad n > k, \end{cases}$$

indexed by $k \geqslant -1$. In particular, $\Pi^{-1}|_{[n]} = 1$. Define $\Pi_k|_{[n]} = (\Pi^k|_{[n]})^{\mathrm{op}} \in \mathbb{Z}\Delta^{\mathrm{op}}([n],[n])$. Consider also $\Pi^\infty|_{[n]} = \pi^{n-1} : [n] \to [n]$ and $\Pi_\infty|_{[n]} = (\Pi^\infty|_{[n]})^{\mathrm{op}} \in \mathbb{Z}\Delta^{\mathrm{op}}([n],[n])$.

Proposition 1. *For any $k \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ the idempotents $\Pi^k|_{[n]} : [n] \to [n]$ form a chain map $\Pi^k : (([n])_n, \partial) \to (([n])_n, \partial)$ in $\mathrm{Ch}_{\geqslant 0}(\mathbb{Z}\Delta)$.*

Proof. Let us prove an identity in $\mathbb{Z}\Delta$

$$\Pi^k|_{[n-1]} \cdot \sum_{j=0}^{n} (-1)^{n-j} \partial^j = \sum_{j=0}^{n} (-1)^{n-j} \partial^j \cdot \Pi^k|_{[n]} : [n-1] \to [n]. \tag{1.1}$$

For $n \leqslant k+1$ the equality

$$\pi^{n-2} \cdot \sum_{j=0}^{n} (-1)^{n-j} \partial^j = \pi^{n-2} \cdot \partial^n \cdot \pi^{n-1} = \partial^n \cdot \pi^{n-1}$$
$$= \sum_{j=0}^{n} (-1)^{n-j} \partial^j \cdot \pi^{n-1} : [n-1] \to [n]$$

follows from [Lyu21, Exercise 1.8(e) and (1.16)]. For $n \geqslant k+2$ (1.1) reads

$$\pi^k \cdot \sum_{j=0}^{n} (-1)^{n-j} \partial^j = \sum_{j=0}^{n} (-1)^{n-j} \partial^j \cdot \pi^k \in \mathbb{Z}\Delta([n-1],[n]) \tag{1.2}$$

which we are going to prove. Since π^k commutes with ∂^j for $j \geqslant k+2$ and due to [Lyu21, Exercise 1.8(c)] the equation reduces to

$$\pi^k \cdot \sum_{j=1}^{k+1} (-1)^{k+1-j} \partial^j = \partial^{k+1} \cdot \pi^k \in \mathbb{Z}\Delta([n-1],[n]).$$

It follows from [Lyu21, (1.16)] that the right-hand side equals

$$\sum_{j=0}^{k+1} (-1)^{k+1-j} \pi^{j-2} \cdot \partial^j.$$

As a corollary,

$$\begin{aligned}\partial^{k+1}\cdot\pi^k &= \partial^{k+1}\cdot\pi^{k-1}\cdot\pi^k = \pi^{k-1}\cdot\partial^{k+1}\cdot\pi^k \\ &= \pi^{k-1}\cdot\sum_{j=0}^{k+1}(-1)^{k+1-j}\pi^{j-2}\cdot\partial^j = \pi^{k-1}\cdot\sum_{j=0}^{k+1}(-1)^{k+1-j}\partial^j,\end{aligned}$$

hence,

$$(1-P^k)\cdot\partial^{k+1}\cdot\pi^k = \pi^k\cdot\sum_{j=0}^{k+1}(-1)^{k+1-j}\partial^j.$$

It remains to notice that

$$\begin{aligned}P^k\cdot\partial^{k+1}\cdot\pi^k &= \sigma^k\cdot\partial^k\cdot\partial^{k+1}\cdot\pi^k = \sigma^k\cdot\partial^k\cdot\partial^{k+1}\cdot(1-\sigma^k\cdot\partial^k)\cdot\pi^{k-1} \\ &= \sigma^k\cdot\partial^k\cdot(\partial^{k+1}-\partial^k)\cdot\pi^{k-1} = 0.\end{aligned}$$

□

Corollary 1. *For any $k\in\mathbb{Z}_{\geqslant 0}\cup\{\infty\}$ and any cosimplicial object $B:\mathbb{Z}\Delta\to\mathcal{A}$, any simplicial object $A:\mathbb{Z}\Delta^{\mathrm{op}}\to\mathcal{A}$ the idempotents $B(\Pi^k):{}^\circ CB\to{}^\circ CB$, $A(\Pi_k):CA\to CA$ are cochain maps and chain maps, respectively.*

Proposition 2. *Let $\mathcal{A}$ be an additive category with split idempotents. Then the category $\mathrm{Ch}^\bullet(\mathcal{A})$ of cochain complexes in $\mathcal{A}$ has split idempotents.*

Proof. Let $e:M\to M\in\mathrm{Ch}^\bullet(\mathcal{A})$ be an idempotent, $(e)^2=e$. Then for each $n\in\mathbb{Z}$, the endomorphism $e^n:M^n\to M^n$ is an idempotent in $\mathcal{A}$. Therefore, it admits a splitting $e^n=\big(M^n\xrightarrow{p^n}L^n\xhookrightarrow{i^n}M^n\big)$, $i^n\cdot p^n=1_{L^n}$. The exterior of the following diagram commutes:

$$\begin{tikzcd} M^{n-1} \arrow[r, "p^{n-1}"] \arrow[d, "d"'] & L^{n-1} \arrow[r, hook, "i^{n-1}"] \arrow[d, "d'"'] & M^{n-1} \arrow[d, "d"] \\ M^n \arrow[r, "p^n"] & L^n \arrow[r, hook, "i^n"] & M^n \end{tikzcd} \tag{1.3}$$

There exists a unique morphism $d':L^{n-1}\to L^n$ which makes the both above squares commutative, namely, $d'=i^{n-1}\cdot d\cdot p^n$. In a similar diagram with rows indexed by $n-2$ and n uniqueness implies that $(d')^2=0$. We have constructed a splitting $e=\big((M,d)\xrightarrow{p}(L,d)\xhookrightarrow{i}(M,d)\big)$ in $\mathrm{Ch}^\bullet(\mathcal{A})$. □

Corollary 2. *Let $\mathcal{A}$ be an additive category with split idempotents. Then the categories $\mathrm{Ch}_\bullet(\mathcal{A})$ of chain complexes in $\mathcal{A}$, $\mathrm{Ch}^{\geqslant 0}(\mathcal{A})$, $\mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ have split idempotents.*

This corollary and a version of diagram (1.3) show that for any $k \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ the natural transformations $\text{-}(\Pi^k) : {}^\circ C \to {}^\circ C$, $\text{-}(\Pi_k) : C \to C$ split as follows:

$$\text{-}(\Pi^k) = \big({}^\circ C \xrightarrow{{}^\circ p} \operatorname{Im}(\text{-}(\Pi^k)) \xhookrightarrow{{}^\circ i} {}^\circ C\big), \quad {}^\circ i \cdot {}^\circ p = 1,$$
$$\text{-}(\Pi_k) = \big(C \xrightarrow{p} \operatorname{Im}(\text{-}(\Pi_k)) \xhookrightarrow{i} C\big), \qquad i \cdot p = 1,$$

where $\operatorname{Im}(\text{-}(\Pi^k)) : \cos \mathcal{A} \to \mathrm{Ch}^{\geqslant 0}(\mathcal{A})$, $\operatorname{Im}(\text{-}(\Pi_k)) : \mathrm{s}\,\mathcal{A} \to \mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ are functors and ${}^\circ p, {}^\circ i, p, i$ are natural transformations. In particular, $\operatorname{Im}(\text{-}(\Pi^k))$, $\operatorname{Im}(\text{-}(\Pi_k))$ are the normalised cochain complex functor ${}^\circ N$ and the normalised chain complex functor N, respectively (in the form of [Lyu21, Corollary 1.12]).

2. Double complex

Let us add to Δ initial object $[-1] = \varnothing$. The obtained category is denoted Δ^+. This is the category of all finite ordinals and non-decreasing maps. We view Δ as a full subcategory of Δ^+. Denote by ∂ the sum $\sum_{i=0}^n (-1)^{n-i} \partial^i \in \mathbb{Z}\Delta^+([n-1],[n])$, $n \geqslant 0$. In particular, the only map $\partial^0 : [-1] \to [0]$ is also denoted by ∂. We have $\partial \cdot \partial = 0$, so ∂ can be viewed as a differential.

We have a double complex D^+ (whose all squares anticommute)

$$\begin{tikzcd}
 & {} & {} & {} \\
{} \arrow[r, "\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([1],[3]) \arrow[u] \arrow[r, "-\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([0],[3]) \arrow[u] \arrow[r, "\mathbb{Z}\Delta^+(\partial,1)"] & \mathbb{Z}\Delta^+([-1],[3]) \arrow[u] \\
{} \arrow[r, "-\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([1],[2]) \arrow[u, "\mathbb{Z}\Delta(1,\partial)"'] \arrow[r, "\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([0],[2]) \arrow[u, "\mathbb{Z}\Delta(1,\partial)"'] \arrow[r, "-\mathbb{Z}\Delta^+(\partial,1)"] & \mathbb{Z}\Delta^+([-1],[2]) \arrow[u, "\mathbb{Z}\Delta^+(1,\partial)"'] \\
{} \arrow[r, "\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([1],[1]) \arrow[u, "\mathbb{Z}\Delta(1,\partial)"'] \arrow[r, "-\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([0],[1]) \arrow[u, "\mathbb{Z}\Delta(1,\partial)"'] \arrow[r, "\mathbb{Z}\Delta^+(\partial,1)"] & \mathbb{Z}\Delta^+([-1],[1]) \arrow[u, "\mathbb{Z}\Delta^+(1,\partial)"'] \\
{} \arrow[r, "-\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([1],[0]) \arrow[u, "\mathbb{Z}\Delta(1,\partial)"'] \arrow[r, "\mathbb{Z}\Delta(\partial,1)"] & \mathbb{Z}\Delta([0],[0]) \arrow[u, "\mathbb{Z}\Delta(1,\partial)"'] \arrow[r, "-\mathbb{Z}\Delta^+(\partial,1)"] & \mathbb{Z}\Delta^+([-1],[0]) \arrow[u, "\mathbb{Z}\Delta^+(1,\partial)"']
\end{tikzcd} \tag{2.1}$$

The first horizontal (homological) degree of $\mathbb{Z}\Delta^+([m],[n])$ is m, the second vertical (cohomological) degree is n. So we view this double complex as chain in horizontal direction and cochain in vertical direction. Otherwise, we use the conventions and notations of [Wei94]. The total chain complex $\operatorname{Tot}^\Pi D^+$ associated with double complex D^+ is

$$\cdots \xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta([m+2],[m])}_{2} \xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta([m+1],[m])}_{1}$$
$$\xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta([m],[m])}_{0} \xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=-1}^{\infty} \mathbb{Z}\Delta^{+}([m],[m+1])}_{-1} \xrightarrow{[\text{-},\partial]} \cdots .$$

So the degree of $\prod \mathbb{Z}\Delta^+([m+k],[m])$ is k. The differential $d^h + d^v$ is $[\text{-},\partial]$ since we use the right operators. Thus, $[f,\partial] = f\cdot\partial - (-1)^{\deg f}\partial\cdot f$, where $\deg f = m-n$ for $f \in \mathbb{Z}\Delta^+([m],[n])$.

Removing from (2.1) the rightmost column (indexed by -1) and the map from 0th column to -1st column, we get a doulde complex D. The associated total complex $\operatorname{Tot}^{\prod} D$ is

$$\cdots \xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta([m+2],[m])}_{2} \xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta([m+1],[m])}_{1}$$
$$\xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta([m],[m])}_{0} \xrightarrow{[\text{-},\partial]} \underbrace{\prod_{m=0}^{\infty} \mathbb{Z}\Delta^{+}([m],[m+1])}_{-1} \xrightarrow{[\text{-},\partial]} \cdots .$$

3. Spectral sequence

Besides the double complex D let us consider double complex $\bar{D}$ of the same shape as D which is $\mathbb{Z}$ concentrated in bidegree $(0,0)$. Denote by $p_0^0 : D \to \bar{D}$, $\psi_0 \mapsto 1$, the projection chain map. Associate with D a differential abelian group $G = \bigoplus_{t\in\mathbb{Z}} \prod_{n\geqslant\max\{0,-t\}} D_{t+n}^n$, $D_m^n = \mathbb{Z}\Delta([m],[n])$, graded by summands $G_t = \prod_{n\geqslant\max\{0,-t\}} D_{t+n}^n = \operatorname{Tot}_t^{\prod} D$. Notice that $d(G_t) \subset G_{t-1}$. Thus, $G = \oplus_{t\in\mathbb{Z}} \operatorname{Tot}_t^{\prod} D$. Consider the decreasing filtration of G

$$F^pG = \bigoplus_{t\in\mathbb{Z}} \prod_{n\geqslant\max\{p,-t\}} D_{t+n}^n, \qquad p \in \mathbb{N},$$

formed by pth and above rows of D. Since $G = F^0G$, this filtration is exhaustive. Since $\cap_{p\in\mathbb{N}} F^pG = 0$, this filtration is weakly convergent [McC01, §3.1, p. 62].

Let us compute few initial terms of the associated spectral sequence:

$$E_{0;q}^pG = F^pG_{q-p}/F^{p+1}G_{q-p} = D_q^p, \qquad d_0 = d_h;$$
$$E_{1;q}^pG = H_{q-p}(F^pG/F^{p+1}G) = H_q(D_*^p, d_h), \quad d_1 = d_v.$$

Since the topological simplex Δ^p_{top} is contractible, the cell complex $C_\bullet(\Delta^p_{\mathrm{top}}) = N(\mathbb{Z}\Delta^p)$ is homotopy isomorphic to $\mathbb{Z}$, concentrated in degree 0. By Dold-Kan correspondence the simplicial abelian group $\mathbb{Z}\Delta^p$ is homotopy isomorphic to the constant presheaf $\mathbb{Z}$, therefore, the complex $C(\mathbb{Z}\Delta^p)$ is homotopy isomorphic to $\mathbb{Z}$, and the former is isomorphic to the pth row of D. We conclude that

$$E^p_{1;q}G = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

More precisely, $E^p_{1;0}G \to \mathbb{Z}\Delta^+([-1],[p])$ is an isomorphism and the differential d_v induces

$$\big(\mathbb{Z}\Delta^+([-1],\partial) : \mathbb{Z}\Delta^+([-1],[p]) \to \mathbb{Z}\Delta^+([-1],[p+1])\big) = \begin{cases} 0 & \text{for } p \text{ even}, \\ 1_{\mathbb{Z}} & \text{for } p \text{ odd}. \end{cases}$$

The complex $E^\bullet_{1;0}$ is homotopy isomorphic to $\mathbb{Z}$, hence,

$$E^p_{2;q}G = \begin{cases} \mathbb{Z} & \text{for } p = q = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \quad d_2 = 0.$$

Thus, $E^p_{2;q}\phi : E^p_{2;q}G \to E^p_{2;q}\bar{G}$ is an isomorphism for all $p, q \in \mathbb{N}$.

The induced filtration of $H(G, d)$ is defined by

$$F^pH(G,d) = \mathrm{Im}\big(H(\hookrightarrow) : H(F^pG, d) \to H(G,d)\big).$$

It follows from Acyclic Assembly Lemma 2.7.3.1 [Wei94] that $\mathrm{Tot}^{\prod} D^+$ is acyclic. In fact, pth row of D^+ is isomorphic to $\wedge^{[\bullet]}\mathbb{Z}^{[p]} = \wedge^{1+\bullet}\mathbb{Z}^{1+p}$, $\bullet \geqslant -1$, with the differential d, given by right derivation, whose restriction to $\mathbb{Z}^{[p]} = \mathbb{Z}^{1+p}$ is determined by $(e_i)d = 1$, $0 \leqslant i \leqslant p$. A contracting homotopy for $\wedge^{[\bullet]}\mathbb{Z}^{[p]}$ is given e.g. by - $\wedge\, e_p$, therefore, pth row of D^+ is acyclic. Conventions for double complexes in [Wei94] differ from ours by reflection with respect to line $x + y = 0$. In particular, the fourth quadrant is the fourth quadrant in both conventions, however, horizontal and vertical directions exchange roles. Therefore, Acyclic Assembly Lemma 2.7.3.1 [Wei94] is applicable and we conclude that $\mathrm{Tot}^{\prod} D^+$ is acyclic. In few lines, for $p \geqslant \max\{0, -k-1\}$ an element $\alpha \in Z_kF^p \,\mathrm{Tot}^{\prod} D^+ \subset \prod_{m=p}^{\infty} \mathbb{Z}\Delta^+([m+k],[m])$ has among its coordinates a d_h-cycle $\alpha_p \in \mathbb{Z}\Delta^+([p+k],[p])$. Necessarily $\alpha_p = (\beta_p)d_h$ for some $\beta_p \in \mathbb{Z}\Delta([p+k+1],[p])$. Then $\alpha - [\beta_p, \partial] \in Z_kF^{p+1}\,\mathrm{Tot}^{\prod} D^+$ and we find β_{p+1}, and so on. Clearly, $\alpha = [\beta, \partial]$.

These reasonings show also that $F^p H_k(G,d) = 0$ for $p \geqslant \max\{0, 1-k\}$, hence, the filtered abelian group $(H(a,d), F)$ is *complete*, that is, the mapping

$$H(G) \to \lim_p H(G)/F^p H(G)$$

is an isomorphism. Hence, by definition, filtered differential module (G, F, d) is *strongly convergent*.

Similarity one produces a filtered differential abelian group $\bar{G}$ from $\bar{D}$. The filtration $(\bar{G}, F, d = 0)$ is exhaustive, weakly convergent and complete. In fact, $F^p \bar{G} = 0$ for $p > 0$. The map $p_0^0 : D \to \bar{D}$ induces a filtration preserving chain map $\phi : G \to \bar{G}$.

Theorem 3.9 of [McC01] implies

Proposition 3. *The map ϕ induces an isomorphism on homology, $H(\phi) : H(G,d) \to H(\bar{G}, d) = \bar{G}$.*

Briefly, cohomology of $\operatorname{Tot}^{\prod} D$ reduces to cohomology of the rightmost column of Δ^+, which is $\mathbb{Z}$ concentrated in degree 0.

Let $\mathcal{A}$ be an additive category, $B \in \cos \mathcal{A}$, $A \in s\mathcal{A}$ be cosimplicial and simplicial objects, respectively. Consider a cycle $\varphi \in Z_0 \operatorname{Tot}^{\prod} D$. Its image $B(\varphi) : {}^\circ C^\bullet B \to {}^\circ C^\bullet B$ is a cochain endomorphism of the cochain complex ${}^\circ C^\bullet B = ((B_n), B\partial)$ associated with B. The image $A(\varphi^{\mathrm{op}}) : C_\bullet A \to C_\bullet A$ is a chain endomorphism of the chain complex $C_\bullet A = ((A_n), d = A(\partial^{\mathrm{op}}))$ associated with A. Suppose now that $\varphi \in B_0 \operatorname{Tot}^{\prod} D$ is the boundary $\varphi = [\eta, \partial]$, $\eta \in \operatorname{Tot}_1^{\prod} D$. Then $B(\varphi) = B\eta \cdot B\partial + B\partial \cdot B\eta$ and $A(\varphi^{\mathrm{op}}) = A(\eta^{\mathrm{op}}) \circ d + d \circ A(\eta^{\mathrm{op}})$ are null-homotopic such that $B(\varphi)_0 = 0 : B_0 \to B_0$ and $A(\varphi^{\mathrm{op}})_0 = 0 : A_0 \to A_0$.

Summing up, Proposition 3 implies

Corollary 3. *Let $\mathcal{A}$ be an additive category with split idempotents. Let $\varphi \in Z_0 \operatorname{Tot}^{\prod} D$, $\varphi_n \in \mathbb{Z}\Delta([n],[n])$ be a cycle such that $\psi_0 = 0$. Then $B(\varphi) : {}^\circ C^\bullet B \to {}^\circ C^\bullet B$ and $A(\varphi^{\mathrm{op}}) : C_\bullet A \to C_\bullet A$ are null-homotopic. The contracting homotopy is given by the sequence $B(\eta_n)$, $A(\eta_n^{\mathrm{op}})$ respectively, where $\varphi = [\eta, \partial]$, $\eta_n \in \mathbb{Z}\Delta([n+1],[n])$, $n \geqslant 0$.*

Applying this corollary to the cycle $\varphi = 1 - \Pi^\infty$ we conclude that the morphisms in the decompositions of $B(\Pi^\infty) = \big({}^\circ CB \xrightarrow{{}^\circ p} {}^\circ NB \xhookrightarrow{{}^\circ i} {}^\circ CB\big)$ and $A(\Pi_\infty) = \big(CA \xrightarrow{p} NA \xhookrightarrow{i} CA\big)$ are homotopy inverse to each other.

4. Homotopies between projections

Projections Π^k and Π^{k+1} are homotopic for $k \geqslant -1$ by Corollary 3. Let us find the homotopy between them explicitly.

Proposition 4. *Define $t_k^n \in \mathbb{Z}\Delta([n+1],[n])$ for $n \geqslant 0$ as*

$$t_k^n = \begin{cases} 0 & \text{if } n \leqslant k+1, \\ \sum_{j=k+2}^{n} (-1)^{n+1-j} \sigma^j \cdot \pi^{k-1} & \text{if } n \geqslant k+2. \end{cases}$$

Then t_k is the homotopy $\Pi^k \sim \Pi^{k+1}$.

Proof. We have to prove for $n \geqslant 0$ that

$$\Pi^{k+1} - \Pi^k = \partial \cdot t_k^n + t_k^{n-1} \cdot \partial : [n] \to [n]. \tag{4.1}$$

where ∂ denotes the sum $\sum_{i=0}^{n} (-1)^{n-i} \partial^i \in \mathbb{Z}\Delta([n-1],[n])$, $n > 0$ and $t_k^{-1} = 0$.

Assume that $k \geqslant 0$. Notice that, for $k = 0$ the following holds as well taking into account that $\pi^{-1} = 1$ by convention. Consider empty sums to be 0.

For $n \leqslant k+1$ the sought equation (4.1) is obvious.

For $n \geqslant k+2$ notice that

$$\begin{aligned} \Pi^{k+1} - \Pi^k = \pi^{k+1} - \pi^k &= -P^{k+1} \cdot \left(1 - P^k\right) \cdot \pi^{k-1} \\ &= \left(\partial^k \cdot \sigma^{k+2} - \partial^{k+1} \cdot \sigma^{k+2}\right) \cdot \pi^{k-1}. \end{aligned}$$

Then for $n = k+2$ we have

$$\begin{aligned} &\sum_{i=0}^{k+3} (-1)^{k+3-i} \partial^i \cdot t_k^{k+2} + 0 \\ &\quad = \sum_{i=0}^{k+3} (-1)^{k+3-i} \partial^i \cdot (-\sigma^{k+2} \cdot \pi^{k-1}) = \sum_{i=0}^{k+3} (-1)^{k-i} \partial^i \cdot \sigma^{k+2} \cdot \pi^{k-1} \\ &\quad = \Pi^{k+1} - \Pi^k + \sum_{i=0}^{k-1} (-1)^{k-i} \sigma^{k+1} \cdot \partial^i \cdot \pi^{k-1} = \Pi^{k+1} - \Pi^k. \end{aligned}$$

Let $n \geqslant k+3$. Then

$$\sum_{i=0}^{n+1} (-1)^{n+1-i} \partial^i \cdot t_k^n + t_k^{n-1} \cdot \sum_{i=0}^{n} (-1)^{n-i} \partial^i$$

$$
\begin{aligned}
&= \sum_{i=0}^{n+1}(-1)^{n+1-i}\partial^i \cdot \sum_{j=k+2}^{n}(-1)^{n+1-j}\sigma^j \cdot \pi^{k-1} \\
&\qquad + \sum_{j=k+2}^{n-1}(-1)^{n-j}\sigma^j \cdot \pi^{k-1} \cdot \sum_{i=0}^{n}(-1)^{n-i}\partial^i. \qquad (4.2)
\end{aligned}
$$

Rewrite the first sum in the right-hand side as

$$
\begin{aligned}
&\sum_{j=k+2}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} \\
&= \sum_{j=k+2}^{n}\sum_{i=0}^{k-1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} + \sum_{j=k+2}^{n}\sum_{i=k}^{n+1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} \\
&= \sum_{j=k+2}^{n}\sum_{i=0}^{k-1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} + \sum_{j=k+3}^{n}\sum_{i=k}^{n+1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} \\
&\qquad + \left(\partial^k \cdot \sigma^{k+2} - \partial^{k+1} \cdot \sigma^{k+2}\right) \cdot \pi^{k-1} + \sum_{i=k+2}^{n+1}(-1)^{k+i}\partial^i \cdot \sigma^{k+2} \cdot \pi^{k-1} \\
&= \sum_{j=k+2}^{n}\sum_{i=0}^{k-1}(-1)^{i+j}\sigma^{j-1} \cdot \partial^i \cdot \pi^{k-1} + \sum_{j=k+3}^{n}\sum_{i=k}^{j-1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} \\
&\qquad + \sum_{j=k+3}^{n}\sum_{i=j}^{n+1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} + \Pi^{k+1} - \Pi^k \\
&\qquad + \sum_{i=k+4}^{n+1}(-1)^{k+i}\partial^i \cdot \sigma^{k+2} \cdot \pi^{k-1} \\
&\qquad + \left(-\partial^{k+3} \cdot \sigma^{k+2} + \partial^{k+2} \cdot \sigma^{k+2}\right) \cdot \pi^{k-1} \\
&= \sum_{j=k+3}^{n}\sum_{i=k}^{j-1}(-1)^{i+j}\sigma^{j-1} \cdot \partial^i \cdot \pi^{k-1} + \sum_{j=k+3}^{n}\sum_{i=j+2}^{n+1}(-1)^{i+j}\partial^i \cdot \sigma^j \cdot \pi^{k-1} \\
&\qquad - \sum_{j=k+3}^{n}\partial^{j+1} \cdot \sigma^j \cdot \pi^{k-1} + \sum_{j=k+3}^{n}\partial^j \cdot \sigma^j \cdot \pi^{k-1} + \Pi^{k+1} - \Pi^k \\
&\qquad + \sum_{i=k+4}^{n+1}(-1)^{k+i}\sigma^{k+2} \cdot \partial^{i-1} \cdot \pi^{k-1}
\end{aligned}
$$

$$= \sum_{j=k+3}^{n} \sum_{i=k}^{j-1} (-1)^{i+j} \sigma^{j-1} \cdot \partial^i \cdot \pi^{k-1} + \sum_{j=k+3}^{n} \sum_{i=j+2}^{n+1} (-1)^{i+j} \partial^i \cdot \sigma^j \cdot \pi^{k-1}$$
$$+ \sum_{i=k+4}^{n+1} (-1)^{k+i} \sigma^{k+2} \cdot \partial^{i-1} \cdot \pi^{k-1} + \Pi^{k+1} - \Pi^k.$$

Due to (1.2) we may rewrite the second sum in the right-hand side of (4.2) as

$$\sum_{j=k+2}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \sigma^j \cdot \partial^i \cdot \pi^{k-1} = \sum_{j=k+2}^{n-1} \sum_{i=0}^{k-1} (-1)^{i+j} \sigma^j \cdot \partial^i \cdot \pi^{k-1}$$
$$+ \sum_{j=k+2}^{n-1} \sum_{i=k}^{n} (-1)^{i+j} \sigma^j \cdot \partial^i \cdot \pi^{k-1} = \sum_{j=k+2}^{n-1} \sum_{i=k}^{j} (-1)^{i+j} \sigma^j \cdot \partial^i \cdot \pi^{k-1}$$
$$+ \sum_{j=k+2}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} \sigma^j \cdot \partial^i \cdot \pi^{k-1} = \sum_{j=k+2}^{n-1} \sum_{i=k}^{j} (-1)^{i+j} \sigma^j \cdot \partial^i \cdot \pi^{k-1}$$
$$+ \sum_{j=k+3}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} \partial^{i+1} \cdot \sigma^j \cdot \pi^{k-1} + \sum_{i=k+3}^{n} (-1)^{k+i} \sigma^{k+2} \cdot \partial^i \cdot \pi^{k-1}.$$

The firsts, the seconds, and the thirds summands cancel each out, and (4.2) reduces to $\Pi^{k+1} - \Pi^k$.

For $k = -1$ the general proof holds, defining $\pi^{-2} = 1$, $\partial^{-1} = 0$ and $\sigma^{-1} = 0$ as we show below. For $n = 0$ equation (4.1) obviously holds.

For $n \geqslant 1$

$$\Pi^0 - \Pi^{-1} = 1 - P^0 - 1 = -\sigma^0 \cdot \partial^0 = -\partial^0 \cdot \sigma^1$$

Then for $n = 1$ equation (4.1) follows from the identity

$$(\partial^2 - \partial^1 + \partial^0) \cdot (-\sigma^1) = -\partial^0 \cdot \sigma^1$$

And for $n \geqslant 2$ the right-hand side of (4.1) is

$$\sum_{i=0}^{n+1} (-1)^{n+1-i} \partial^i \cdot t_{-1}^n + t_{-1}^{n-1} \cdot \sum_{i=0}^{n} (-1)^{n-i} \partial^i \tag{4.3}$$
$$= \sum_{i=0}^{n+1} (-1)^{n+1-i} \partial^i \cdot \sum_{j=1}^{n} (-1)^{n+1-j} \sigma^j + \sum_{j=1}^{n-1} (-1)^{n-j} \sigma^j \cdot \sum_{i=0}^{n} (-1)^{n-i} \partial^i.$$

Rewrite the first sum as

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}\partial^i\cdot\sigma^j \\
&\quad= \sum_{j=2}^{n}\sum_{i=0}^{n+1}(-1)^{i+j}\partial^i\cdot\sigma^j - \partial^0\cdot\sigma^1 + \sum_{i=1}^{n+1}(-1)^{1+i}\partial^i\cdot\sigma^1 \\
&\quad= \sum_{j=2}^{n}\sum_{i=0}^{j-1}(-1)^{i+j}\partial^i\cdot\sigma^j + \sum_{j=2}^{n}\sum_{i=j}^{n+1}(-1)^{i+j}\partial^i\cdot\sigma^j \\
&\qquad\quad + \Pi^0 - \Pi^{-1} + \sum_{i=3}^{n+1}(-1)^{1+i}\partial^i\cdot\sigma^1 + (-\partial^2\cdot\sigma^1 + \partial^1\cdot\sigma^1) \\
&\quad= \sum_{j=2}^{n}\sum_{i=0}^{j-1}(-1)^{i+j}\sigma^{j-1}\cdot\partial^i + \sum_{j=2}^{n}\sum_{i=j+2}^{n+1}(-1)^{i+j}\partial^i\cdot\sigma^j - \sum_{j=2}^{n}\partial^{j+1}\cdot\sigma^j \\
&\qquad\quad + \sum_{j=2}^{n}\partial^j\cdot\sigma^j + \Pi^0 - 1 + \sum_{i=3}^{n+1}(-1)^{1+i}\sigma^1\cdot\partial^{i-1} \\
&\quad= \sum_{j=2}^{n}\sum_{i=0}^{j-1}(-1)^{i+j}\sigma^{j-1}\cdot\partial^i + \sum_{j=2}^{n}\sum_{i=j+2}^{n+1}(-1)^{i+j}\partial^i\cdot\sigma^j \\
&\qquad\quad + \sum_{i=3}^{n+1}(-1)^{1+i}\sigma^1\cdot\partial^{i-1} + \Pi^0 - 1.
\end{aligned}$$

The second sum in the right-hand side of (4.3) is

$$\begin{aligned}
\sum_{j=1}^{n-1}\sum_{i=0}^{n}(-1)^{i+j}\sigma^j\cdot\partial^i &= \sum_{j=1}^{n-1}\sum_{i=0}^{j}(-1)^{i+j}\sigma^j\cdot\partial^i + \sum_{j=1}^{n-1}\sum_{i=j+1}^{n}(-1)^{i+j}\sigma^j\cdot\partial^i \\
= \sum_{j=1}^{n-1}\sum_{i=0}^{j}(-1)^{i+j}\sigma^j\cdot\partial^i &+ \sum_{j=2}^{n-1}\sum_{i=j+1}^{n}(-1)^{i+j}\partial^{i+1}\cdot\sigma^j + \sum_{i=2}^{n}(-1)^{1+i}\sigma^1\cdot\partial^i.
\end{aligned}$$

The firsts, the seconds, and the thirds summands cancel each out, and (4.3) reduces to $\Pi^0 - \Pi^{-1}$. □

Let $P, Q : C \to C : s\mathcal{A} \to \mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ be chain idempotents, $P\cdot Q = P = Q\cdot P$. Then $Q - P$ is an idempotent and there exist $i : \operatorname{Im} P \hookrightarrow \operatorname{Im} Q$, $p : \operatorname{Im} Q \longrightarrow \operatorname{Im} P$ representing $\operatorname{Im} Q$ as a direct sum $\operatorname{Im} P \oplus \operatorname{Im}(Q - P)$. In particular, $i\cdot p = 1_{\operatorname{Im} P}$, $p\cdot i = P|_{\operatorname{Im} Q}$.

Let chain maps $P, Q : CA \to CA$ be homotopic: there exist $h_n : A_n \to A_{n+1} \in \mathcal{A}$, $n \geqslant -1$, $h_{-1} = 0$, such that $Q_n - P_n = d \cdot h_{n-1} + h_n \cdot d$ for $n \geqslant 0$. Then i, p are homotopy inverse to each other, the subcomplex $\operatorname{Im} P \subset \operatorname{Im} Q$ is homotopy isomorphic to $\operatorname{Im} Q$, $\operatorname{Im}(Q - P)$ is contractible in CA and in $\operatorname{Im} Q$. In fact, h can be replaced with $h'_n = Q_n \cdot h_n \cdot Q_{n+1} : \operatorname{Im} Q_n \to \operatorname{Im} Q_{n+1}$, since

$$d \cdot h'_{n-1} + h'_n \cdot d = Q_n \cdot (Q_n - P_n) \cdot Q_n = Q_n - P_n,$$
$$1_{\operatorname{Im} Q} - p \cdot i = Q - P = d \cdot h' + h' \cdot d.$$

Let $P, Q, R : C \to C : s\mathcal{A} \to \mathrm{Ch}_{\geqslant 0}(\mathcal{A})$ be chain idempotents, $P \cdot Q = P = Q \cdot P$, $Q \cdot R = Q = R \cdot Q$. Assume that $P \sim Q$, $Q \sim R$, that is,

$$Q - P = d \cdot h + h \cdot d, \qquad R - Q = d \cdot k + k \cdot d.$$

Then $P \sim R$ with the homotopy $l = h + k$,

$$R - P = d \cdot (h + k) + (h + k) \cdot d.$$

If $Q_n \cdot h_n \cdot Q_{n+1} = h_n$, $R_n \cdot k_n \cdot R_{n+1} = k_n$, then $R_n \cdot l_n \cdot R_{n+1} = l_n$ as well.

Summing the homotopies obtained in Proposition 4 we get a homotopy between $\mathrm{id} : \big(([n])_n, \partial\big) \to \big(([n])_n, \partial\big)$ and $\Pi^\infty : \big(([n])_n, \partial\big) \to \big(([n])_n, \partial\big)$ in $\mathrm{Ch}_{\geqslant 0}(\mathbb{Z}\Delta)$:

$$t^n = \sum_{k=-1}^{n-2} \sum_{j=k+2}^{n} (-1)^{n+1-j} \sigma^j \cdot \pi^{k-1} \in \mathbb{Z}\Delta([n+1], [n]), \qquad n \geqslant 0.$$

Therefore, the natural transformations ${}^\circ p : {}^\circ C \to {}^\circ N$ and ${}^\circ i : {}^\circ N \to {}^\circ C$ are homotopy inverse to each other. In fact, for any $B \in \operatorname{Ob} \cos \mathcal{A}$ the morphisms $1_{{}^\circ CB}$ and $B(\Pi^\infty) : {}^\circ CB \to {}^\circ CB$ are naturally homotopic via $B(t)$. Similarly, the natural transformations $p : C \to N$ and $i : N \to C$ are homotopy inverse to each other.

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