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Algebraizable Weak Logics*

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Abstract

We extend the standard framework of abstract algebraic logic to the setting of logics which are not closed under uniform substitution. We introduce the notion of weak logics as consequence relations closed under limited forms of substitutions and we give a modified definition of algebraizability that preserves the uniqueness of the equivalent algebraic semantics of algebraizable logics. We provide several results for this novel framework, in particular a connection between the algebraizability of a weak logic and the standard algebraizability of its schematic fragment. We apply this framework to the context of logics defined over team semantics and we show that the classical version of inquisitive and dependence logic is algebraizable, while their intuitionistic versions are not.

Introduction

The algebraic approach to logic dates back to the beginning of mathematical logic itself, with the seminal works of the logical algebraists, e.g. Augustus De Morgan, George Boole, Arthur Cayley, William Stanley Jevons and Charles Sanders Peirce [19]. Amidst the subsequent developments of mathematical logic as a foundational science, the original approach of the logical algebraists was to be at least partially forgotten. It was especially in Poland that algebraic logic flourished again, with the work of Jan Łukasiewicz, Adolf Lindenbaum and Alfred Tarski concerning abstract consequence relations. Between the sixties and seventies, Helena Rasiowa made further advancements towards an abstract theory of logical systems and their relations with classes of algebras. In particular, she developed in [31] a general theory of algebraization for implicative logics. Finally, this approach was put in its contemporary formulation by Blok and Pigozzi, who introduced in [3] the notion of algebraizable logics and started developing what is nowadays known as *abstract algebraic logic*. We refer the reader to [15] for a general introduction to the subject.

In the setting of abstract algebraic logic, Blok and Pigozzi defined logics as consequence relations which are additionally closed under uniform substitutions. This is not a merely technical constraint, but also expresses a key property of *logicality*, which can be tracked back to Bolzano and that was clearly formulated by Tarski. In [23], Kennedy and Väänänen present Tarski's notion of logic and its relation to Felix Klein's 1872 *Erlanger Programm* as follows:

> In 1968 in a (posthumously published) lecture called "What are logical notions?" [34] Tarski proposed a definition of "logical notion" or alternatively of "logical constant", modelled on the Erlanger Program due to Felix Klein. The core observation is the

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> following: for a given subject area, the number of concepts classified as invariant are inversely related to the number of transformations – the more transformations there are, the fewer invariant notions there are. If one thinks of logic as the most general of all the mathematical sciences, why not then declare "logical" notions to be the limiting cases? Thus, a notion is to be thought of as logical if it is invariant under all permutations of the relevant domain. [23]

Although Tarski's notion of logicality was not immune from criticism, his idea of logical notions as being invariant under all transformations informed most of the abstract approaches to logic, and is still very much alive in philosophy of logic, e.g. in the reformulation of his criterion by Gila Sher [33].

In this article, we shall consider a generalization of Blok and Pigozzi's idea of algebraizable logics and introduce a suitable notion of algebraizability for logical systems which are not closed under uniform substitution, but that are nonetheless invariant under permutations of the variables in the language. The main motivation for our work is provided by the recognition of the multitude of systems which do possess a logical nature but still fail to be closed under uniform substitution in Tarski's original strong sense. The field of modal logic is particularly rich of such examples, as many systems which have been studied in recent years manifest such a failure of the law of uniform substitution: Buss' pure provability logic [5], public announcement logic [22, 21] and other epistemic logics are all examples of this behaviour. Furthermore, logics based on team semantics, such as inquisitive [8, 9] and dependence logic [35, 36], also do not satisfy the principle of uniform substitution.

The behaviour of these logical systems prevented so far a uniform and general study of these logics, and did not allow for immediately applying the facts and results from abstract algebraic logic, which hold only for logical systems in Tarski's strong sense. For example, algebraic semantics for several versions of inquisitive and dependence logics were introduced in [1, 2, 27, 30], but it remained an open question whether these semantics are in any sense unique. In fact, it is a seminal result from abstract algebraic logic that *the equivalent*[1] *algebraic semantics of a logic is unique*, e.g. Boolean algebras are the unique equivalent algebraic semantics of classical propositional logic and Heyting algebras are the unique equivalent algebraic semantics of intuitionistic propositional logic. However, as inquisitive and dependence logics do not make for logics in the standard sense, it was still an open question whether their algebraic semantics is unique.

In this article we introduce the notion of weak logic, generalizing previous definitions of Ciardelli [8] and Punčochář [28], and we show how to develop a theory of algebraizable logics for this framework. In Section 1 we define weak logics and introduce expanded algebras as their algebraic dual notion. In Section 2 we introduce a suitable notion of algebraizability and we prove a version of Maltsev's Theorem for our framework, to show that the equivalent algebraic semantics of a weak logic is unique. In Section 3.1 we investigate the relation between a weak logic and its schematic variant, and we provide a characterization of the algebraizability of weak logics in terms of the algebraizability of their schematic variant. In Section 3.2 we develop on these results and prove a version of Blok and Pigozzi's (theory-)isomorphism theorem for the framework of weak logics. Our results are then put to test in Section 4, where we apply them to inquisitive (dependence) logic – in particular we show that the classical version of inquisitive (dependence) logic is algebraizable, and has thereby a unique equivalent algebraic semantics, while the intuitionistic version of such logics is not algebraizable. Finally, in Section 5 we

[1] Logical systems in abstract algebraic logic are traditionally represented as relations over term algebras. Thus *equivalent* in the context refers to the equivalence between these relations and the algebraic semantics via suitable translation maps, see Definition 18 for a detailed exposition.

briefly explain how to adapt the usual matrix semantics to our setting and we make explicit some connection with model theory. We conclude the article with some remarks on possible directions of further study.

1 Weak Logics and Expanded Algebras

In this section we introduce *weak logics* as a generalization of propositional logical systems and we provide several examples of them. Alongside, we define *expanded algebras* and *core semantics* to provide an algebraic interpretation to these logical systems. We start by recalling some standard notation.

On notation We stick to the following conventions in a bid to improve the reading experience: the variables $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ stand for algebras or first order structures; classes of algebras are typeset in boldface – both for arbitrary collections ($\mathbf{Q}, \mathbf{K}, \dots$) and designated ones ($\mathbf{HA}, \mathbf{BA}$); blackboard bold font is used for operators on these classes $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{Q}, \dots$. Valuation maps are written as homomorphisms from the term algebra $\mathcal{F}m$ and labelled as $f, g, h \dots \in \mathtt{Hom}(\mathcal{F}m, \mathcal{A})$. We write $\vec{x}$ as a shorthand for a sequence of variables $(x_0, \dots, x_n)$. Equations are written as $\epsilon \approx \delta$ or $\epsilon(\vec{x}) \approx \delta(\vec{x})$ to emphasize on the free variables $\vec{x}$. We write Eq for the set of equations over some background signature $\mathcal{L}$. We also recall that a quasi-equation is a formula of the form $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta$. If $\mathcal{A}$ is an algebra and $h \in \mathtt{Hom}(\mathcal{F}m, \mathcal{A})$, we write $\mathcal{A} \vDash_h \epsilon \approx \delta$ if $h(\epsilon) = h(\delta)$. We will introduce further notation when needed.

1.1 Standard and Weak Logics

We recall the basic definition of logic as a consequence relation in the tradition of abstract algebraic logic. Let $\mathtt{Var}$ be a denumerable set of variables and $\mathcal{L}$ an algebraic (i.e. purely functional) signature. We denote by $\mathcal{F}m_{\mathcal{L}}$ both the set of formulae over $\mathtt{Var}$ in the signature $\mathcal{L}$ and the $\mathcal{L}$-term algebra over $\mathtt{Var}$. We omit the index $\mathcal{L}$ when it is clear from the context what signature we are working in.

Definition 1. A *consequence relation* is a relation $\vdash \subseteq \wp(\mathcal{F}m) \times \mathcal{F}m$ such that for all $\Gamma \subseteq \mathcal{F}m$:

1. For all $\phi \in \Gamma$, $\Gamma \vdash \phi$;
2. If $\Gamma \vdash \phi$, for all $\phi \in \Delta$, and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.

Moreover, we generally also assume that consequence relations are *finitary (compact)*, i.e:

$$\Gamma \vdash \phi \Rightarrow \text{there is } \Delta \subseteq \Gamma \text{ such that } |\Delta| < \omega \text{ and } \Delta \vdash \phi.$$

A *substitution* is an endomorphism $\sigma : \mathcal{F}m \to \mathcal{F}m$ on the term algebra. We denote by $\mathsf{Subst}(\mathcal{L})$ the set of all substitution of the $\mathcal{L}$-term algebra and by $\mathsf{At}(\mathcal{L})$ the set of all *atomic substitutions*, i.e. substitutions σ such that $\sigma[\mathtt{Var}] \subseteq \mathtt{Var}$. Given an atom p, we let $\Gamma[\phi/p]$ stand for substituting ϕ for all occurrences of p in the formulas in Γ. A consequence relation $\vdash$ is closed under *uniform substitution* ($\mathtt{US}$) if for any substitution σ, $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$. A *(standard) logic* is a consequence relation $\vdash$ which is closed under $\mathtt{US}$. Classical propositional logic, $\mathtt{CPC}$, and intuitionistic propositional logic, $\mathtt{IPC}$, are examples of logics in this sense.

In this work, we shall investigate logical systems which are essentially weaker than standard logics — while standard logics are consequence relations satisfying uniform substitutions, we define weak logics to be consequence relations closed only under suitable subsets of substitutions.

Definition 2 (Weak Logic)**.** A *weak logic* is a finitary consequence relation $\vdash$ such that for all atomic substitutions $\sigma \in \mathsf{At}(\mathcal{L})$, $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$.

A weak logic is thus a consequence relation $\vdash$ which is closed under the principle of atomic substitution `AS`. Intuitively, this principle reifies the least prerequisite a consequence relation must satisfy in order to be characterizable as a logic: the validity of the consequences in a weak logic can depend on the logical complexity of its formulae, but not on the specific variables that occur in them.

Clearly, standard logics are weak logics, as they are closed under atomic substitutions. More poignantly, there are many interesting examples of weak logics which are not standard logics and that have been extensively studied in literature. We briefly mention here some of them.

Example 3. Public Announcement Logic (`PAL`) [16, 21] is an example of a modal logic that is not closed under uniform substitution [22]. However, it can be shown that `PAL` is closed under atomic substitution [22, §2.1] and it is therefore a weak logic.

Introducing the proper syntax and semantics of `PAL` is out of scope for this paper. We consider the following example from [22] to provide the reader with some intuition why uniform substitution fails. Given a set of agents $\mathcal{A}$, the language of `PAL` extends the basic modal language with operators K_i, for $i \in \mathcal{A}$, and $\langle\phi\rangle$ for any formula ϕ. The sentence $K_i\phi$ should be read as "agent i knows that ϕ" and $\langle\phi\rangle\psi$ as "after the truthful announcement of ϕ to all agents, ψ holds". Let the atoms of the language stand for *facts* – that is, sentences that can be truly uttered at any time.

We can now turn our attention to the following example, taken from [22, §1.2]. Consider the principle:

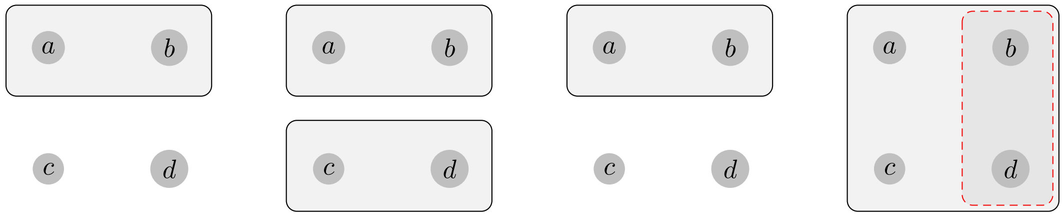
$$p \to \langle p\rangle p \quad \text{(if } p \text{ is true, } p \text{ remains true after a truthful announcement)} \qquad (\star)$$

The schema ($\star$) is valid for *facts*, but in general does not hold if we substitute p with a sentence talking about the epistemic state of an agent. Let L be the sentence *"Ljubljana became the capital of an independent Slovenia in 1991, and agent j does not know this."* with translation $c \wedge \neg K_i c$. Now substituting L for p in ($\star$) gives us a Moorean sentence – after truthfully announcing L, agent j learns that *"Ljubljana became the capital of an independent Slovenia in 1991"*, and thus the conclusion $\langle L\rangle L$ is no longer truthful.

Example 4. Logics based on team semantics [20], such as inquisitive and dependence logics [10, 11, 36, 37], offer a rich supply of examples. In Section 4 we will focus particularly on `InqB` and `InqI`, namely the classical and the intuitionistic version of inquisitive logic respectively.

However, for now we can provide a conceptual motivation why `InqB` is not closed under uniform substitution. One of the main goals of `InqB` is to serve as a basis for a uniform treatment of both truth-conditional statements and questions in natural language. To that end, the intended semantics of `InqB` must establish when a piece of information *supports* a statement or *settles* a question rather than their truth conditions. We call the evidence an *information state* and represent it as a set of possible worlds.

Let p be an arbitrary statement without inquisitive content, e.g. *"It is raining in Amsterdam."*. Assume that p holds in the possible worlds a and b, i.e. the information state $\{a, b\}$ supports p (see Figure 1). We form the polar question $?p$ – *"Is it raining in Amsterdam?"*, and model it as the set of alternatives $\{a, b\}$ and $\{c, d\}$. Let's check the validity of *Double Negation Elimination* (`DNE`) – $\neg\neg q \to q$; we interpret negation as the complement of the union of alternatives. Thus any information state supporting $\neg\neg p$ will support the statement p as well (Figure 1(c)), but this is not the case for questions – e.g. the state $\{b, d\}$ supports $\neg\neg ?p$, but does not settle $?p$ as the possible worlds b and d do not agree on a same answer. Hence we can conclude that the schema DNE is valid only for statements without inquisitive content, i.e. for propositional atoms.



(a) p (b) $?p$ (c) $\neg\neg p$ (d) $\neg\neg ?p$

Figure 1: Double-negation elimination for statements and polar questions

InqB is actually a concrete example of a wider class of weak logics – a *double negation atoms logic* or DNA-*logic*. A DNA-*logic* (or *negative variant* of an intermediate logic [8, 26]) is a set of formulae $\mathtt{L}^{\neg} = \{\phi[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n] : \phi \in \mathtt{L}\}$, where L is an intermediate logic, namely a logic comprised between IPC and CPC. It can be proved (see e.g. [8, Prop. 3.2.15]) that DNA-logics are closed under atomic substitutions. However, for any DNA-logic $\mathtt{L} \neq \mathtt{CPC}$ it is the case that $\neg\neg p \to p \in \mathtt{L}^{\neg}$, but $(\neg\neg(p \vee \neg p) \to p \vee \neg p) \notin \mathtt{L}^{\neg}$, showing that DNA-logics are not standard logics. DNA-logics can be further generalised to χ-logics, defined in [30], which offer another non-trivial example of weak logics.

If $\vdash$ is a weak logic, then we know that $\vdash$ is closed at least under atomic substitutions $\sigma \in \mathsf{At}(\mathcal{L})$, but in principle there might be more substitutions for which the logic $\vdash$ is closed. We call such substitutions *admissible*.

Definition 5 (Admissible Substitutions)**.** Let $\vdash$ be a weak logic. The set of *admissible substitutions* $\mathsf{AS}(\vdash)$ is the set of all substitutions σ such that, for all sets of formulae $\Gamma \cup \{\phi\} \subseteq \mathcal{F}m$, $\Gamma \vdash \phi \Longrightarrow \sigma[\Gamma] \vdash \sigma(\phi)$. In particular, $\mathsf{At}(\mathcal{L}) \subseteq \mathsf{AS}(\vdash)$.

In contrast with the set of atomic substitutions, determining the set of admissible substitutions is much harder. One example of such a characterization can be given for the case of inquisitive logic InqB: $\sigma \in \mathsf{AS}(\mathtt{InqB})$ if and only if σ is a classical substitution, namely if for all $p \in \mathtt{Var}$, $\sigma(p) \equiv_{\mathtt{InqB}} \psi$ where ψ is a disjunction-free formula.

Although in weak logics we cannot freely substitute formulae in place of variables, we often want to consider the subset of formulae for which this is possible. We refer to this subset as the core of a logic.

Definition 6 (Core of a Logic)**.** The *core* of a weak logic $\vdash$ is the set $\mathsf{core}(\vdash) \subseteq \mathcal{F}m$ of all formulae ψ such that for all sets of formulae $\Gamma \cup \{\phi\}$ we have that:

$$\Gamma \vdash \phi \implies \Gamma[\psi/p] \vdash \phi[\psi/p].$$

where p is any atomic variable. In particular, we always have $\mathtt{Var} \subseteq \mathsf{core}(\vdash)$.

One can verify that ψ is a core formula of $\vdash$ if and only if for all $p \in \mathtt{Var}$ the substitution σ such that $\sigma \restriction \mathtt{Var} \setminus \{p\} = \mathsf{id}$ and $\sigma(p) = \psi$ is admissible.

1.2 Expanded Algebras

In order to make sense of weak logics from an algebraic perspective, we need to adapt the usual algebraic framework to handle the failure in uniform substitution. To this end, we introduce *expanded algebras* as the expansion of standard algebras by an extra predicate symbol. We use calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$ to denote first-order structures, and we write $\mathsf{dom}(\mathcal{A})$ to refer to the underlying universe of $\mathcal{A}$. However, when it is not confusing, we use the same notation for a structure and its underlying universe. For all functional symbols $f \in \mathcal{L}$ and all relational symbols $R \in \mathcal{L}$, we write $f^{\mathcal{A}}$ and $R^{\mathcal{A}}$ for their interpretation in $\mathcal{A}$. We use the same notations for symbols and their interpretation when it does not cause confusion. If $X \subseteq \mathcal{A}$, we write $\langle X \rangle$ for the substructure of $\mathcal{A}$ generated by X. Finally, we also use $\vDash$ as the standard satisfaction symbol of first-order logic. We refer the reader to [4, 7, 24] for an introduction to universal algebra and the standard model-theoretic techniques.

Definition 7 (Expanded Algebra)**.** Let $\mathcal{A}$ be an $\mathcal{L}$-algebra and P a unary predicate. An *expanded algebra* is a structure in the vocabulary $\mathcal{L} \cup \{P\}$. We denote the interpretation $P^{\mathcal{A}}$ also by $\mathsf{core}(\mathcal{A})$.

A *strong homomorphism* between expanded algebras $h : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{L}$-algebra homomorphism that also preserves the core, i.e. $h[\mathsf{core}(\mathcal{A})] \subseteq \mathsf{core}(\mathcal{B})$. A *strict homomorphism* between expanded algebras $h : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{L}$-algebra homomorphism such that for all $a \in \mathcal{A}$ $a \in \mathsf{core}(\mathcal{A}) \iff h(a) \in \mathsf{core}(\mathcal{B})$. A *strong embedding* between expanded algebras is an injective strong homomorphism and a *strict embedding* is an injective strict homomorphism. We write $\mathcal{A} \preceq \mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$, i.e. if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\mathsf{core}(\mathcal{A}) = \mathsf{core}(\mathcal{B}) \cap \mathsf{dom}(\mathcal{A})$. In other words, $\mathcal{A} \preceq \mathcal{B}$ if the identity map $id : \mathcal{A} \to \mathcal{B}$ is a strict embedding.

We recall that a class of algebras $\mathbf{K}$ is a quasi-variety if it closed under the operators $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U$, i.e. if it is closed under isomorphic copies, subalgebras, products and ultraproducts. To extend the notion of quasi-variety to the setting of expanded algebras we first explain how to extend the usual class operators. If $\mathbf{K}$ is a class of expanded algebras, then let $\mathbb{I}(\mathbf{K}), \mathbb{S}(\mathbf{K}), \mathbb{P}(\mathbf{K}), \mathbb{P}_U(\mathbf{K})$ be defined as follows.

$$\mathcal{A} \in \mathbb{I}(\mathbf{K}) \iff \mathcal{A} \cong \mathcal{B} \text{ for some } \mathcal{B} \in \mathbf{K};$$
$$\mathcal{A} \in \mathbb{S}(\mathbf{K}) \iff \mathcal{A} \preceq \mathcal{B} \text{ for some } \mathcal{B} \in \mathbf{K};$$
$$\mathcal{A} \in \mathbb{P}(\mathbf{K}) \iff \mathcal{A} = \prod_{\alpha<\kappa} \mathcal{B}_\alpha,\ \mathsf{core}(\prod_{\alpha<\kappa} \mathcal{B}_\alpha) = \prod_{\alpha<\kappa} \mathsf{core}(\mathcal{B}_\alpha) \text{ and } \mathcal{B}_\alpha \in \mathbf{K} \text{ for all } \alpha < \kappa;$$
$$\mathcal{A} \in \mathbb{P}_U(\mathbf{K}) \iff \mathcal{A} = \prod_{\alpha<\kappa} \mathcal{B}_\alpha/\mathcal{U},\ \mathsf{core}(\prod_{\alpha<\kappa} \mathcal{B}_\alpha/\mathcal{U}) = \prod_{\alpha<\kappa} \mathsf{core}(\mathcal{B}_\alpha)/\mathcal{U} \text{ and } \mathcal{B}_\alpha \in \mathbf{K} \text{ for all } \alpha < \kappa.$$

Strict homomorphism enable us to extend also the notion of variety to the context of expanded algebras. We recall that a class of algebras $\mathbf{K}$ is a variety if it is closed under $\mathbb{H}, \mathbb{S}, \mathbb{P}$, i.e. if it is closed under homomorphic images, subalgebras and products. If $\mathbf{K}$ is a class of expanded algebras, then we let:

$$\mathcal{A} \in \mathbb{H}(\mathbf{K}) \iff \mathcal{A} = h[\mathcal{B}], \text{ where } h \text{ is a strict homomorphism and } \mathcal{B} \in \mathbf{K}.$$

In the light of the previous definitions, we can apply the operators $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{H}$ to expanded algebras. We then say that a class of expanded algebras $\mathbf{K}$ is a *quasi-variety* if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U$, and that it is a *variety* if it is closed under $\mathbb{H}, \mathbb{S}, \mathbb{P}$ — notice that closure under isomorphic copies follows immediately from closure under (strict) homomorphic images. Finally, if $\mathbf{K}$ is a class of expanded algebras, we let $\mathbb{Q}(\mathbf{K})$ be the least quasi-variety containing

$\mathbf{K}$ and we let $\mathbb{V}(\mathbf{K})$ be the least variety containing $\mathbf{K}$. Notice that the class $\mathbf{K}$ can be seen as a Horn class (e.g. see Gorbunov [17]). We will expand on the connection in Section 5.

As we have seen before, any class $\mathbf{K}$ of $\mathcal{L}$-algebras induces a semantic consequence relation $\vDash_{\mathbf{K}}$ by letting $\Theta \vDash_{\mathbf{K}} \epsilon \approx \delta$ if and only if for all $\mathcal{A} \in \mathbf{K}$, and for all $h \in \mathtt{Hom}(\mathcal{F}m, \mathcal{A})$, if $h(x) = h(y)$ for all $x \approx y \in \Theta$, then $h(\epsilon) = h(\delta)$. Expanded algebras allow us to define a more fine grained consequence relations, by restring the scope of the previous definition only to valuations of atomic variables onto the core elements of the algebra. We let $Fm_{\mathcal{L}}$ be the *expanded term algebra* $Fm_{\mathcal{L}} = (Fm, \mathtt{Var})$ and we write $\mathtt{Hom}^c(\mathcal{F}m, \mathcal{A})$ for the set of all *core assignments* from the term algebra $\mathcal{F}m$ into $\mathcal{A}$, i.e. the set of all assignments $h : \mathcal{F}m \to \mathcal{A}$ such that $h(p) \in \mathsf{core}(\mathcal{A})$ for all $p \in \mathtt{Var}$. It is easy to verify that a core assignment $h \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{A})$ is a strong homomorphism between the term algebra and $\mathcal{A}$. We define the core consequence relation over a class of expanded algebras as follows.

Definition 8 (Core Semantics)**.** Let $\mathbf{K}$ be a class of expanded algebras and $\Theta \cup \{\epsilon \approx \delta\}$ a set of equations, then we let:

$$\begin{aligned}\Theta \vDash^c_{\mathbf{K}} \epsilon \approx \delta \iff & \text{for all } \mathcal{A} \in \mathbf{K}, h \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{A}),\\ & \text{if } h(x) = h(y) \text{ for all } x \approx y \in \Theta, \text{ then } h(\epsilon) = h(\delta).\end{aligned}$$

If $\bigwedge_{i\leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta$ is a quasi-equation, we write $\vDash^c_{\mathbf{K}} \bigwedge_{i\leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta$ or $\mathbf{K} \vDash^c \bigwedge_{i\leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta$ if $\bigwedge_{i\leq n} \epsilon_i \approx \delta_i \vDash^c_{\mathbf{K}} \epsilon \approx \delta$. Also, we often write $\mathcal{A} \vDash^c \epsilon \approx \delta$ in place of $\vDash^c_{\{\mathcal{A}\}} \epsilon \approx \delta$. The related notion for equations are defined analogously. Notice that $\mathcal{A} \vDash^c \epsilon(\vec{x}) \approx \delta(\vec{x})$ if and only if $\mathcal{A} \vDash \forall x_0, \dots, \forall x_n(\epsilon(\vec{x}) \approx \delta(\vec{x}) \wedge \bigwedge_{i\leq n} \mathsf{core}(x_i))$, where $\vec{x} := (x_0, \dots, x_n)$ are the variables occurring in $\epsilon \approx \delta$ and $\vDash$ is the standard first-order satisfaction relation. We will come back later in Section 5 to the relationship between core semantics and first-order semantics.

In core semantics, atomic variables are thus assigned to core elements and, as a result of this feature, arbitrary formulae are always interpreted inside the subalgebra generated by the core elements. This motivates the interest in core-generated structures and in quasi-varieties which are generated by these structures.

Definition 9 (Core Generated Structures)**.** An expanded algebra $\mathcal{A}$ is *core-generated* if $\mathcal{A} = \langle \mathsf{core}(\mathcal{A}) \rangle$. A quasi-variety $\mathbf{Q}$ of expanded algebras is *core-generated* if $\mathbf{Q} = \mathbb{Q}(\mathbf{K})$, where $\mathbf{K}$ is a class of core-generated algebras. A variety $\mathbf{V}$ of expanded algebras is *core-generated* if $\mathbf{V} = \mathbb{V}(\mathbf{K})$, where $\mathbf{K}$ is a class of core-generated algebras.

The focus on core-generated structures motivates the introduction of a further class operator. We say that $\mathcal{B}$ is a *core superalgebra* of $\mathcal{A}$ if $\mathcal{A} \preceq \mathcal{B}$ and $\mathsf{core}(\mathcal{A}) = \mathsf{core}(\mathcal{B})$. If $\mathbf{K}$ is class of algebras, we write $\mathbb{C}(\mathbf{K})$ for the class of core superalgebras of elements of $\mathbf{K}$.

We can now see that unrestricted substitutions interacts nicely with core-generated quasi-varieties.

Lemma 10. If $\mathbf{Q}$ is a core-generated quasi-variety of expanded algebras, then for all $\sigma \in \mathsf{Subst}$, $\sigma(\Theta) \vDash^c_{\mathbf{Q}} \sigma(\epsilon \approx \delta) \iff \Theta \vDash_{\mathbf{Q}} \epsilon \approx \delta$.

Proof. ($\Longrightarrow$) Suppose that $\Theta \nvDash_{\mathbf{Q}} \epsilon \approx \delta$. Since $\mathbf{Q}$ is core-generated, there is a core-generated expanded algebra $\mathcal{A} \in \mathbf{Q}$ and some $h \in \mathtt{Hom}(Fm, \mathcal{A})$ such that $\mathcal{A} \vDash_h \Theta$ and $\mathcal{A} \nvDash_h \epsilon \approx \delta$. Since $\mathcal{A}$ is core-generated, we have that for all variables $p \in Var(\Theta \cup \{\epsilon \approx \delta\})$ there is a polynomial β_p such that $h(p) = \beta_p(x_0, \dots, x_n)$ with $x_i \in \mathsf{core}(\mathcal{A})$ for $i \leq n$. Now we construct a new assignment $s \in \mathtt{Hom}^c(Fm, \mathcal{A})$ by letting $s(q_i) = x_i$ for $0 \leq i \leq n$ which means that for all formula $\psi(q_0, \dots, q_n)$ with $q_i \in Var(\Theta \cup \{\epsilon \approx \delta\})$ we have $s(\psi(q_0, \dots q_n)) = \psi[\beta_p(x_0, \dots, x_n)/p]$. By construction s is a core assignment. Now, let σ be a substitution such that $\sigma(p) = \beta_p(x_0, \dots x_n)$ for all $p \in Var(\Theta \cup \{\epsilon \approx \delta\})$. Then $\mathcal{A} \vDash^c_s \sigma[\Theta]$, but $\mathcal{A} \nvDash^c_s \sigma(\epsilon \approx \delta)$.

($\Longleftarrow$) Suppose that $\sigma(\Theta) \nvDash^c_{\mathbf{Q}} \sigma(\epsilon \approx \delta)$, thus we can find an assignment $h \in \mathtt{Hom}^c(Fm, \mathcal{A})$ where $\mathcal{A} \in \mathbf{Q}$ is a core-generated algebra, such that $\mathcal{A} \vDash^c_h \sigma[\Theta]$, but $\mathcal{A} \nvDash^c_h \sigma(\epsilon \approx \delta)$. Then we have $h \circ \sigma \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{A})$, $\mathcal{A} \vDash^c_{h\circ\sigma} \Theta$ and $\mathcal{A} \nvDash^c_{h\circ\sigma} \epsilon \approx \sigma$. □

Expanded algebras are standard algebras augmented with an interpretation for a unary predicate. It is then natural to consider several ways in which this predicate could be defined. For the purpose of algebraizability, we are particularly interested in the following notion of equational definability.

Definition 11 (Equational Definability)**.** An expanded algebra $\mathcal{A}$ is *equationally definable* if there is some finite set of equations $\Sigma = \{\epsilon_i(x) \approx \delta_i(x) : i \leq n\}$ such that $\mathsf{core}(\mathcal{A}) = \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon_i(x) \approx \delta_i(x) \text{ for all } i \leq n\}$. A class of expanded algebras $\mathbf{K}$ is *(uniformly) equationally definable* if there is some finite set of equations Σ such that for all $\mathcal{A} \in \mathbf{K}$, $\mathsf{core}(\mathcal{A}) = \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon_i(x) \approx \delta_i(x) \text{ for all } i \leq n\}$.

If $\mathcal{A}$ is an algebra and Σ a set of equations, we let $\Sigma(x, \mathcal{A}) = \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon(x) \approx \delta(x) \text{ for all } \epsilon \approx \delta \in \Sigma\}$. If $\mathbf{K}$ is a core-generated class of expanded algebras such that $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$ for all $\mathcal{A} \in \mathbf{K}$, then we also say that $\mathbf{Q}$ is Σ-*generated*. In particular, we say that $\mathbf{Q}$ is a Σ-generated quasi-variety to mean that it is a quasi-variety of expanded algebras such that $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$ for all $\mathcal{A} \in \mathbf{Q}$. We say that $\mathbf{K}$ is Σ-defined if $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$ for all $\mathcal{A} \in \mathbf{K}$. Clearly, for any set of equations Σ, there is a unique way to expand some class of algebras $\mathbf{K}$ into a Σ-defined class of expanded algebras. We will show later in Proposition 16(i) that if $\mathbf{K}$ is Σ-defined, then the quasi-variety of expanded algebra generated by it is also Σ-defined.

Moreover, when we deal with equationally definable core predicates, we shall often be interested in a restricted notion of core superalgebras, i.e. superalgebras of a given structure whose core is defined by a given set of equations Σ. For any class of expanded algebras $\mathbf{K}$ with core defined by Σ, we let $\mathbb{C}^\Sigma(\mathbf{K})$ be the operator defined as follow:

$$\mathcal{A} \in \mathbb{C}^\Sigma(\mathbf{K}) \iff \mathcal{B} \preceq \mathcal{A} \text{ and } \mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A}) = \Sigma(x, \mathcal{B}) \text{ for some } \mathcal{B} \in \mathbf{K}.$$

We give some examples of cores over algebraic structures.

Example 12. Let $\mathcal{M}$ be a monoid in $\mathcal{L} = (\cdot, e)$ and define $\mathsf{core}(\mathcal{M}) = \{x \in \mathcal{M} : \mathcal{M} \vDash x^n \approx e\}$, then $\mathcal{M}$ is an expanded algebra with core defined by $\Sigma = \{x^n \approx e\}$. If $\mathcal{G} \in \mathbf{Grp}$ is a group this is equivalent to let $\Sigma = \{x^{n+1} \approx x\}$. In the latter case, we also say that the core is *polynomially definable*, for every equation in Σ is of the form $x \approx \delta(x)$ for a unary polynomial – i.e. a term δ.

Example 13. Let $\mathbf{HA}$ be the variety of Heyting algebras and for all $\mathcal{A} \in \mathbf{HA}$ let $\mathsf{core}(\mathcal{A}) = \mathcal{A}_\neg = \{x \in \mathcal{A} : \mathcal{A} \vDash x \approx \neg\neg x\}$, i.e. the core of $\mathcal{A}$ is its subset of regular elements. Clearly $\mathcal{A}_\neg$ is polynomially definable. Let $\mathbf{ML}$ be the variety of all Medvedev algebras, then $\mathbf{ML}$ is generated by its subclass of core-generated Heyting algebras $\mathcal{A}$ with core $\mathcal{A}_\neg$, see [1, 2] and Theorem 39 later.

Example 14. Let $\mathbf{BA}^*$ be any expansion of the class of Boolean algebras with a core $\mathsf{core}(\mathcal{B})$ for all $\mathcal{B} \in \mathbf{BA}$. Then, since the Boolean algebra 2 is always core-generated, as $2 = \langle \mathsf{core}(2) \rangle$ only contains the interpretations of the constants $0, 1$, and since $\mathbb{V}(2) = \mathbf{BA}$, it follows that $\mathbf{BA}^*$ is core generated, i.e. any expansion of the class of Boolean algebras is a core-generated variety.

The following propositions demonstrate how equational definability leads to classes of algebras with better behaviour. Firstly, the notions of regular algebra homomorphisms and strong homomorphisms of expanded algebras now coincide.

Proposition 15. *Let* $\mathbf{Q}$ *be a* Σ*-defined class of expanded algebras, then every homomorphism* $h : \mathcal{A} \to \mathcal{B}$ *for* $\mathcal{A}, \mathcal{B} \in \mathbf{Q}$ *is a strong homomorphism.*

Proof. Suppose $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism and let $a \in \mathsf{core}(\mathcal{A})$. Then for all equations $\epsilon \approx \delta \in \Sigma$, we have $\mathcal{A} \vDash \epsilon(a) \approx \delta(a)$ and thus $\mathcal{B} \vDash \epsilon(h(a)) \approx \delta(h(a))$, since h is a homomorphism. Hence, it follows that $h(a) \in \mathsf{core}(\mathcal{B})$, which yields that h is strong. □

By the following Proposition 16, the class-operators $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{C}^{\Sigma}$ are well-defined over expanded algebras with an equationally definable core, and moreover the core validity of quasi-equations is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{C}^{\Sigma}$. Notice that in Proposition 16 Item 2 follows from [17, §1.2.2].

Proposition 16.

1. *Let* $\mathbf{K}$ *be a class of expanded algebras such that for all* $\mathcal{A} \in \mathbf{K}$ *we have* $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$*, then for all* $\mathcal{B} \in \mathbb{O}(K)$ *we have* $\mathsf{core}(\mathcal{B}) = \Sigma(x, \mathcal{B})$*, for* $\mathbb{O} \in \{\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{C}^{\Sigma}\}$*.*

2. *Let* $\mathbf{K}$ *be a class of algebras with a core defined by a finite set of equations* Σ*, and let* $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \alpha \approx \beta$ *be a quasi-equation. For all* $\mathbb{O} \in \{\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{C}^{\Sigma}\}$ *we have that* $\vDash^c_{\mathbf{K}} \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \alpha \approx \beta$ *entails* $\vDash^c_{\mathbb{O}(\mathbf{K})} \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \alpha \approx \beta$*.*

Proof. We prove (1) and (2) together by considering each operator $\mathbb{O} \in \{\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U, \mathbb{C}^{\Sigma}\}$.

1. *Isomorphism*: Immediate.

2. *Subalgebras*: Suppose $\mathcal{A} \preceq \mathcal{B}$, then we have:

$$\mathsf{core}(\mathcal{A}) = \mathsf{core}(\mathcal{B}) \cap \mathsf{dom}(\mathcal{A}) = \Sigma(x, \mathcal{B}) \cap \mathsf{dom}(\mathcal{A}) = \Sigma(x, \mathcal{A}).$$

   Now, if $\mathcal{A} \nvDash^c \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \eta \approx \delta$, then there exists an assignment $h \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{A})$, such that $h(\epsilon_i) = h(\delta_i)$ for all $i \leq n$ and $h(\eta) \neq h(\delta)$. Now, since $\mathcal{A} \subseteq \mathcal{B}$, we have that $h \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{B})$ and thus $\mathcal{B} \nvDash^c \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \eta \approx \delta$.

3. *Products*: Firstly, notice that, for any term γ, we have: $\gamma^{\prod_{\alpha<\kappa} \mathcal{A}_\alpha} = (\gamma^{\mathcal{A}_\alpha})_{\alpha<\kappa}$. Let $\mathcal{A}_\alpha \in \mathbf{K}$ for all $\alpha < \kappa$, then:

$$\mathsf{core}(\prod_{\alpha<\kappa} \mathcal{A}_\alpha) = \prod_{\alpha<\kappa} \mathsf{core}(\mathcal{A}_\alpha) = \prod_{\alpha<\kappa} \Sigma(x, \mathcal{A}_\alpha) = \Sigma(x, \prod_{\alpha<\kappa} \mathcal{A}_\alpha).$$

   Now, suppose for some $h \in \mathtt{Hom}^c(\mathcal{F}m, \prod_{\alpha<\kappa} \mathcal{A}_\alpha)$ we have $\prod_{\alpha<\kappa} \mathcal{A}_\alpha \vDash_h \epsilon_i \approx \delta_i$ and also $\prod_{\alpha<\kappa} \mathcal{A}_\alpha \nvDash_h \eta \approx \delta$.

   Let $h_\alpha = \pi_\alpha \circ V$, where π_α is the α-th projection of $\prod_{\alpha<\kappa} \mathcal{A}_\alpha$. Since h is a core assignment, we have, for all $\epsilon \approx \delta \in \Sigma$ and for all $p \in \mathtt{Var}$ that $\epsilon(h(p)) = \delta(h(p))$ and therefore $\epsilon(h_\alpha(p)) = \delta(h_\alpha(p))$ for all $p \in \mathtt{Var}$ and $\alpha < \kappa$, meaning that every h_α is also a core valuation. Now, since $h(\epsilon) \neq h(\delta)$, there is some $\alpha < \kappa$ such that $h_\alpha(\epsilon) \neq h_\alpha(\delta)$ and, since $h(\epsilon_i) = h(\delta_i)$ for all $i \leq n$, also $h_\alpha(\epsilon_i) = h_\alpha(\delta_i)$ for all $i \leq n$. Finally, this shows that $\mathcal{A}_\alpha \nvDash^c \bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta$.

4. *Ultraproducts*: Firstly, notice that, for any term γ, we have: $\gamma^{\prod_{\alpha<\kappa}\mathcal{A}_\alpha/\mathcal{U}} = (\gamma^{\mathcal{A}_\alpha})_{\alpha<\kappa}/\mathcal{U}$. Let $\mathcal{A}_\alpha \in \mathbf{K}$ for all $\alpha < \kappa$, then:

$$\mathsf{core}(\prod_{\alpha<\kappa}\mathcal{A}_\alpha/\mathcal{U}) = \prod_{\alpha<\kappa}\mathsf{core}(\mathcal{A}_\alpha)/\mathcal{U} = \prod_{\alpha<\kappa}\Sigma(x,\mathcal{A}_\alpha)/\mathcal{U} = \Sigma(x,\prod_{\alpha<\kappa}\mathcal{A}_\alpha/\mathcal{U}).$$

Now, suppose for some core assignment $h \in \mathtt{Hom}^c(Fm, \prod_{\alpha<\kappa}\mathcal{A}_\alpha/\mathcal{U})$ we have $h(\epsilon_i) \approx h(\delta_i)$ for all $i \leq n$ and $h(\epsilon) \neq h(\delta)$. Let $x^0, \dots, x^m$ be the free variables in $\bigwedge_{i\leq n}\epsilon_i \approx \delta_i \to \epsilon \approx \delta$ and let $h(x^i) = \{a^i_\alpha\}_{\alpha<\kappa}/\mathcal{U}$ for all $i \leq m$, we then have:

$$\prod_{\alpha<\kappa}\mathcal{A}_\alpha/\mathcal{U} \vDash \bigwedge_{i\leq n}\epsilon_i(h(x^0),\dots h(x^m)) \approx \delta_i(h(x^0),\dots h(x^m));$$
$$\prod_{\alpha<\kappa}\mathcal{A}_\alpha/\mathcal{U} \nvDash \epsilon(h(x^0),\dots h(x^m)) \approx \delta(h(x^0),\dots h(x^m)).$$

Then, by Łoś Theorem:

$$\{\alpha<\kappa : \mathcal{A}_\alpha \vDash \bigwedge_{i\leq n}\epsilon_i(a^0_\alpha,\dots a^m_\alpha) \approx \delta_i(a^0_\alpha,\dots a^m_\alpha)\} \in \mathcal{U};$$
$$\{\alpha<\kappa : \mathcal{A}_\alpha \vDash \epsilon(a^0_\alpha,\dots a^m_\alpha) \approx \delta(a^0_\alpha,\dots a^m_\alpha)\} \notin \mathcal{U}.$$

Moreover, for all a^i_α with $\alpha < \kappa$, $i \leq m$, the following holds by Łoś Theorem together with h being a core assignment:

$$\{\alpha<\kappa : \mathcal{A}_\alpha \vDash \bigwedge_{\eta\approx\xi\in\Sigma}\bigwedge_{i\leq m}\eta(a^i_\alpha) \approx \xi(a^i_\alpha)\} \in \mathcal{U}.$$

Thus, we can find some $\alpha < \kappa$ such that $\mathcal{A}_\alpha \vDash \eta(a^i_\alpha) \approx \xi(a^i_\alpha)$ for all $\eta \approx \xi \in \Sigma$, $\mathcal{A}_\alpha \vDash \bigwedge_{i\leq n}\epsilon_i(a^0_\alpha,\dots a^m_\alpha) \approx \delta_i(a^0_\alpha,\dots x^m_\alpha)$ and $\mathcal{A}_\alpha \nvDash \alpha(a^0_\alpha,\dots a^m_\alpha) \approx \beta(a^0_\alpha,\dots a^m_\alpha)$. Then, by considering $h_\alpha : \mathcal{F}m \to \mathcal{A}_\alpha$ such that $h_\alpha(x^i) = a^i_\alpha$, it follows that h_α is a core assignment and $\mathcal{A}_\alpha \nvDash_{h_\alpha} \bigwedge_{i\leq n}\epsilon_i \approx \delta_i \to \alpha \approx \beta$.

5. *Core Superalgebras*: Firstly, we remark that by definition Σ-core superalgebra we have: $\Sigma(x,\mathcal{A}) = \mathsf{core}(\mathcal{A}) = \mathsf{core}(\mathcal{B}) = \Sigma(x,\mathcal{B})$. Then, suppose $\mathcal{A} \preceq \mathcal{B}$, $\Sigma(x,\mathcal{A}) = \Sigma(x,\mathcal{B})$ and that there is a core assignment $h : \mathcal{F}m \to \mathcal{B}$ such that $h(\epsilon_i) = h(\delta_i)$ for all $i \leq n$, $h(\epsilon) = h(\delta)$. Since $\mathsf{core}(\mathcal{A}) = \mathsf{core}(\mathcal{B})$, and $\mathcal{A} \preceq \mathcal{B}$, we can also consider h as a core assignment over $\mathcal{A}$, which in turn shows that $\mathcal{A} \nvDash \bigwedge_{i\leq n}\epsilon_i \approx \delta_i \to \alpha \approx \beta$. □

The previous result provides an important characterization of validity under the core consequence relation. Using that, we show that the core consequence relation is compact, if restricted it to quasi-varieties of expanded algebras. The following theorem is a straightforward adaptation of the usual proof of compactness via ultraproducts for first-order logic.

Theorem 17 (Compactness)**.** *Let* $\mathbf{Q}$ *be a quasi-variety whose underlying core is defined by* Σ*, then the induced consequence relation* $\vDash^c_{\mathbf{Q}}$ *is compact.*

Proof. It suffices to show that if T is a set of quasi-equations T such that for all finite $\Gamma \subseteq T$ there is some expanded algebra $\mathcal{A} \in \mathbf{Q}$ such that $\mathcal{A} \vDash^c \Gamma$, then there is $\mathcal{B} \in \mathbf{Q}$ such that $\mathbf{Q} \vDash^c T$.

We reason by induction on the cardinality of T. Let $\kappa = |T|$, by induction hypothesis every $\Gamma \subseteq T$ with $|\Gamma| < \kappa$ is satisfiable. Since the set $F = \{k \setminus \alpha : \alpha < \kappa\}$ has the finite intersection property, we can extend it to an ultrafilter $\mathcal{U}$. By induction hypothesis, for all $\alpha < \kappa$, there is an expanded algebra $\mathcal{A}_\alpha \in \mathbf{Q}$ such that $\mathcal{A}_\alpha \vDash^c \epsilon_\beta \approx \delta_\beta$ for all $\beta < \alpha$.

Consider the ultraproduct $\prod_{\alpha<\kappa} \mathcal{A}_\alpha/\mathcal{U}$, then since for every $\beta < \gamma < \kappa$, $\mathcal{A}_\gamma \vDash^c \epsilon_\beta \approx \delta_\beta$ and $\kappa \setminus \beta + 1 \in \mathcal{U}$, it follows by Proposition 16(ii) that $\prod_{\alpha<\kappa} \mathcal{A}_\alpha \vDash^c \epsilon_\beta \approx \delta_\beta$ for all $\beta < \kappa$. Finally, by Proposition 16(i) we also have that $\prod_{\alpha<\kappa} \mathcal{A}_\alpha/\mathcal{U} \in \mathbf{Q}$, proving our claim. □

2 Algebraizability of Weak Logics

In this section we employ the notions we have introduced so far to provide a suitable definition of algebraizability for the setting of weak logics. We then adapt a number of classical results, e.g. we prove a version of Maltsev's Theorem and use it to show that the equivalent algebraic semantics of a weak logic is unique up to some notion of equivalence.

Firstly, we briefly recall algebraizability for the setting of standard logics. Let $\mathcal{F}m$ and Eq be respectively the set of formulae and equations in some signature $\mathcal{L}$. We shall refer to two maps, called *transformers* [15], that allow us to translate formulae into equations and *vice versa*. We let $\tau : \mathcal{F}m \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathcal{F}m)$. For any substitution σ, we let $\sigma(x, y) = (\sigma(x), \sigma(y))$. We say that τ and Δ are *structural* if for all substitutions $\sigma \in \mathsf{Subst}(\mathcal{L})$, $\tau(\sigma(\phi)) = \sigma(\tau(\phi))$ and $\sigma(\Delta(x, y)) = \Delta(\sigma(x, y))$. For any set of formulae Γ we let $\tau(\Gamma) = \bigcup_{\phi\in\Gamma} \tau(\phi)$ and for all set of equations Θ we let $\Delta(\Theta) = \bigcup_{\epsilon\approx\delta\in\Theta} \Delta(\epsilon, \delta)$. Algebraizability for standard logics was introduced by Blok and Pigozzi in their seminal article [3].

Definition 18 (Algebraizability)**.** A (standard) logic $\vdash$ is *algebraizable* if there are a quasi-variety $\mathbf{Q}$ and structural transformers $\tau : \mathcal{F}m \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathcal{F}m)$ such that:

$$\Gamma \vdash \phi \iff \tau[\Gamma] \vDash_{\mathbf{Q}} \tau(\phi) \quad \text{(Alg1)}$$

$$\Delta[\Theta] \vdash \Delta(\eta, \delta) \iff \Theta \vDash_{\mathbf{Q}} \eta \approx \delta \quad \text{(Alg2)}$$

$$\phi \dashv\vdash \Delta[\tau(\phi)] \quad \text{(Alg3)}$$

$$\eta \approx \delta \equiv_{\mathbf{Q}} \tau[\Delta(\eta, \delta)]. \quad \text{(Alg4)}$$

We then say that $\mathbf{Q}$ is an *equivalent algebraic semantics* of $\vdash$.

Notice that the condition that a logic is algebraizable by a quasi-variety of algebras follows from the fact that in our context we are exclusively considering finitary systems. See [15, §3] for generalisations of this definition.

We say that the transformers τ, Δ are finitary if for all $x \in \mathcal{F}m$ and $\epsilon \approx \delta \in \mathsf{Eq}$, $|\tau(x)| < \omega$ and $|\Delta(\epsilon, \delta)| < \omega$. We recall the following two useful facts about algebraizability. We refer the reader to [15, Prop. 3.12, Thm. 3.37] for a proof of these results.

Lemma 19.

1. Let $\vdash$ be a standard logic, then to show that $\vdash$ is algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$ it suffices to check either Alg1 and Alg4 or Alg2 and Alg3.

2. If $\vdash$ is an algebraizable standard logic, then it is algebraized by finitary maps τ, Δ.

Moreover, it is a key property of algebraizability that the equivalent algebraic semantics of a standard logic is unique. See [15] for a proof of this result.

Proposition 20. *If the tuples $(\mathbf{Q}_0, \tau_0, \Delta_0)$ and $(\mathbf{Q}_1, \tau_1, \Delta_1)$ both witness the algebraizability of a standard logic $\vdash$, then:*

1. $\mathbf{Q}_0 = \mathbf{Q}_1$*;*

2. $\Delta_0(x, y) \dashv\vdash \Delta_1(x, y)$*;*

3. $\tau_0(\phi) \equiv_{\mathbf{Q}_i} \tau_1(\phi)$ *with* $i \in \{0, 1\}$.

By using the core consequence relation $\vDash^c$ in place of the standard one $\vDash$, we can adapt the notion of algebraizability to the setting of weak logics. In particular, we make use of expanded algebras with an equationally definable core.

Definition 21 (Algebraizability of Weak Logics)**.** A weak logic $\vdash$ is *algebraizable* if there is a core-generated quasi-variety $\mathbf{Q}$, a finite set of equations Σ defining its core, and structural transformers $\tau : \mathcal{F}m \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathcal{F}m)$ such that:

$$\Gamma \vdash \phi \Longleftrightarrow \tau[\Gamma] \vDash^c_{\mathbf{Q}} \tau(\phi) \quad \text{(Weak-Alg1)}$$

$$\Delta[\Theta] \vdash \Delta(\eta, \delta) \Longleftrightarrow \Theta \vDash^c_{\mathbf{Q}} \eta \approx \delta \quad \text{(Weak-Alg2)}$$

$$\phi \dashv\vdash \Delta[\tau(\phi)] \quad \text{(Weak-Alg3)}$$

$$\eta \approx \delta \equiv^c_{\mathbf{Q}} \tau[\Delta(\eta, \delta)]. \quad \text{(Weak-Alg4)}$$

We then say that $\mathbf{Q}$ is an *equivalent algebraic semantics* of $\vdash$.

Moreover, if $\mathbf{Q}$ is a quasi-variety and $\vdash$ a weak logic, we also say that $\mathbf{Q}$ is the equivalent algebraic semantics of $\vdash$ if there is a finite set $\Sigma \subseteq \mathsf{Eq}$ such that the expansion of $\mathbf{Q}$ obtained by letting $\mathsf{core}(\mathcal{A}) = \Sigma[\mathcal{A}]$ for all $\mathcal{A} \in \mathbf{Q}$, is core-generated and it algebraizes $\vdash$.

To show that our previous definition provides a workable framework, we prove a version of Maltsev's Theorem for our setting. We firstly recall that a *local subgraph* of an $\mathcal{L}$-algebra $\mathcal{A}$ is a tuple $(X, f_0{\restriction}X, \dots, f_n{\restriction}X)$ where $X \subseteq dom(\mathcal{A})$ is finite and $f_i \in \mathcal{L}$ for all $i \leq n$. A local subgraph $(X, f_0{\restriction}X, \dots, f_n{\restriction}X)$ of an expanded algebra $\mathcal{A}$ is a local subgraph of its algebraic reduct which in addition interprets the core predicate as $\mathsf{core}(X) = \mathsf{core}(\mathcal{A}) \cap X$. If $(X, f_0{\restriction}X, \dots, f_n{\restriction}X)$ and $(Y, f_0{\restriction}Y, \dots, f_m{\restriction}Y)$ are local subgraphs, then their intersection is the local subgraph $(X \cap Y, f_{i_0}{\restriction}X \cap Y, \dots, f_{i_k}{\restriction}X \cap Y)$ where $f_{i_0}, ... f_{i_k} = \mathcal{L}{\restriction}X \cap Y$. We say that $h : \mathcal{A} \to \mathcal{B}$ is a *strong embedding* if it is an injective strong homomorphism. The proof of the following lemma is the standard one that can be found e.g. in [4, Theorem 5.2.14].

Lemma 22. Let $\mathbf{K}$ be a class of expanded algebras, $\mathcal{A}$ a core-generated expanded algebra. If all local subgraphs of $\mathcal{A}$ strongly embed into some element of $\mathbf{K}$, then $\mathcal{A} \in \mathbb{ISP}_U(\mathbf{K})$.

Proof. Let $\mathcal{A}$ be a core-generated expanded algebra and fix some enumeration $\{X_\alpha\}_{\alpha<\kappa}$ of its local subgraphs; for all $\alpha, \beta < \kappa$, let $X_\alpha \cup X_\beta = X_{\gamma(\alpha,\beta)}$. By assumption, for every $\alpha < \kappa$ there is some expanded algebra $\mathcal{D}_\alpha \in \mathbf{K}$ and a strong embedding $h_\alpha : X_\alpha \to \mathcal{D}_\alpha$. For every $\alpha < \kappa$, we let $I_\alpha = \{\gamma : X_\alpha \preceq X_\gamma\}$. Then, the set $F = \{I_\alpha : \alpha < \kappa\}$ has the finite intersection property, as $I_\alpha \cap I_\beta = I_{\gamma(\alpha,\beta)}$, and we can thus extend it to an ultrafilter $\mathcal{U}$.

Now, consider the ultraproduct $\prod_{\alpha<\kappa} \mathcal{D}_\alpha/\mathcal{U}$ and for every $a \in \mathcal{A}$ let $\pi_a \in \prod_{\alpha<\kappa} \mathcal{D}_\alpha$ be the map $\pi_a(\alpha) = h_\alpha(a)$ if $a \in \mathsf{dom}(h_\alpha)$, and otherwise assign an arbitrary value to it. Then, it can be verified that the map $a \mapsto \pi_a/\mathcal{U}$ is a strong embedding of $\mathcal{A}$ into $\prod_{\alpha<\kappa} \mathcal{D}_\alpha/\mathcal{U}$, which means that $\mathcal{A} \in \mathbb{ISP}_U(\mathbf{K})$. □

For any set T of quasi-equations, we let $Mod^c(T)$ be the class of expanded algebras $\mathcal{A}$ such that $\mathcal{A} \vDash^c T$ and $Mod^c_{CG}(T)$ for its subclass of core-generated model. We let $Mod^\Sigma(T)$ be the class of expanded algebras $\mathcal{A}$ such that $\mathcal{A} \vDash^c T$ and $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$, and $Mod^\Sigma_{CG}(T)$ be its subclass of core-generated algebras. If $\mathbf{K}$ is a class of expanded algebras, we denote by $Log^c(\mathbf{K})$

the set of all quasi-equation true in $\mathbf{K}$ under core-semantics. The following theorem follows by adapting the proof of (the algebraic version of) Maltsev's Theorem to our context. It can also be seen as an immediate corollary of a more general (relational) version of Maltsev's Theorem presented in [17, §2.3.3].

Theorem 23. *Let* $\mathbf{Q}$ *be a quasi-variety of expanded algebras and* $\mathcal{A}$ *a core generated expanded algebra, then* $\mathcal{A} \in \mathbf{Q}_{CG}$ *if and only if* $\mathcal{A} \vDash^c Log^c(\mathbf{Q})$*.*

Proof. ($\Rightarrow$) Clearly, if $\mathcal{A} \in \mathbf{Q}_{CG}$ then $\mathcal{A} \vDash^c Log^c(\mathbf{Q})$, by the definition of $Log^c(\mathbf{Q})$.
($\Leftarrow$) Let $\mathcal{A}$ be core-generated algebra such that $\mathcal{A} \vDash^c Log^c(\mathbf{Q})$. We firstly show that there exists an algebra $\mathcal{D} \in \mathbf{Q}$, such that every local subgraph of $\mathcal{A}$ embeds into $\mathcal{D}$.

Let $B = \{b_0, \dots, b_n\} \subseteq \mathcal{A}$ be a local subgraph of $\mathcal{A}$. Since $\mathcal{A}$ is core generated, $B \subseteq \langle \mathsf{core}(\mathcal{A}) \rangle$ and in particular there is a finite set $C = \{c_0, \dots, c_p\} \subseteq \mathsf{core}(\mathcal{A})$ such that $\alpha_i(c_0, \dots, c_p) = b_i$ for some polynomial $\alpha_i(x_0, \dots, x_p)$. We let $\vec{x} = (x_0, \dots, x_p)$ and $\vec{c} = (c_0, \dots, c_p)$, then we define the two following set of equations:

$$D^+(X) := \{f(\alpha_{k_1}(\vec{x}), \dots, \alpha_{k_l}(\vec{x})) \approx \alpha_j(\vec{x}) : f(\alpha_{k_1}(\vec{c}), \dots, \alpha_{k_l}(\vec{c})) \approx \alpha_j(\vec{c}), f \in \mathcal{L}{\restriction}B\}$$
$$D^-(X) := \{\alpha_i(\vec{x}) \approx \alpha_j(\vec{x}) : \alpha_i(\vec{c}) \not\approx \alpha_j(\vec{c})\}.$$

Since $D^-(X)$ is finite we have an enumeration $D^-(X) = \{\epsilon_0 \approx \delta_0, \dots, \epsilon_l \approx \delta_l\}$, and for every $i \leq l$ we let ϕ_i be the quasi-equation $\phi_i := \bigwedge D^+(X) \to \epsilon_i \approx \delta_i$.

Now, fix a core-assignment $h : \mathcal{F}m \to \mathcal{A}$ by letting $h(x_i) = c_i$ for all $i \leq p$, then it follows by our construction that $\mathcal{A} \vDash^c_h f(\alpha_{k_1}(\vec{x}), \dots, \alpha_{k_l}(\vec{x})) \approx \alpha_j(\vec{x})$ for all $b_i = \alpha_i(\vec{c}) \in B$, $f \in \mathcal{L}{\restriction}B$ and $\mathcal{A} \nvDash^c_h \epsilon_i \approx \delta_i$ for all $i \leq l$. It then follows that $\mathcal{A} \nvDash^c \phi_i$ for all $i \leq l$, and since by assumption $\mathcal{A} \vDash^c Log^c(\mathbf{Q})$, we obtain that $\phi_i \notin Log^c(\mathbf{Q})$.

It follows that for every $i \leq l$ there is an expanded algebra $\mathcal{D}_i \in \mathbf{Q}$ such that $\mathcal{D}_i \nvDash^c \phi_i$, hence there is $h_i \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{D}_i)$ such that $\mathcal{D}_i \vDash^c_h f(\alpha_{k_1}(\vec{x}), \dots, \alpha_{k_l}(\vec{x})) \approx \alpha_j(\vec{x})$ for all $b_i = \alpha_i(\vec{c}) \in B$, $f \in \mathcal{L}{\restriction}B$ and $\mathcal{D}_i \nvDash^c_h \epsilon_i \approx \delta_i$. Let $d^i_j = h_i(x_j)$ for every $j \leq p$.

Consider now the product $\prod_{i \leq l} \mathcal{D}_i$ and let $h \in \mathtt{Hom}^c(\mathcal{F}m, \prod_{i \leq l} \mathcal{D}_i)$ be such that $h(x_j) = (d^0_j, \dots, d^l_j)$ for all $j \leq l$ – this is a core assignment by definition of the core in product algebras. Let $\vec{d} = (d^0_j, \dots, d^l_j)$, then since for all $i \leq p$ we have $\prod_{i \leq l} \mathcal{D}_i \nvDash^c \phi_i$, the map $g : B \to \prod_{i \leq l} \mathcal{D}_i$ such that $g(b_j) = (\alpha^0_j(\vec{d}), \dots, \alpha^l_j(\vec{d}))$ is an embedding. By Lemma 22 $\mathcal{A} \in \mathbb{ISP}_U(\prod_{i \leq l} \mathcal{D}_i) \subseteq \mathbb{ISP}_U(\mathbf{Q}) = \mathbf{Q}$, thus since $\mathcal{A}$ is core-generated, we obtain $\mathcal{A} \in \mathbf{Q}_{CG}$. □

The previous result provides us with a criterion to determine whether core-generated expanded algebras belong to a quasi-variety. We now use this characterization to adapt the proof of the uniqueness of the equivalent algebraic semantic of standard logics [15, Thm. 3.17] to our setting.

Proposition 24. *If both the tuples* $(\mathbf{Q}_0, \Sigma_0, \tau_0, \Delta_0)$ *and* $(\mathbf{Q}_1, \Sigma_1, \tau_1, \Delta_1)$ *witness the algebraizability of* $\vdash$*, then:*

1. $\mathbf{Q}_0 = \mathbf{Q}_1$;
2. $\Sigma_0 \equiv_{\mathbf{Q}_i} \Sigma_1$ *with* $i \in \{0, 1\}$;
3. $\Delta_0(x, y) \dashv\vdash \Delta_1(x, y)$;
4. $\tau_0(\phi) \equiv^c_{\mathbf{Q}_i} \tau_1(\phi)$ *with* $i \in \{0, 1\}$.

Proof. Firstly, notice that the two witnesses of algebraizability that we are assuming give rise to two different consequence relations, which we shall denote by $\vDash^0_{\mathbf{Q}_0}$ and $\vDash^1_{\mathbf{Q}_1}$.

(3) We prove $\Delta_0(x,y) \vdash \Delta_1(x,y)$. Let $\phi \in \Delta_1(x,y)$, then we clearly have that:

$$\tau_0(\phi(x,x)), \phi(x,x) \approx \phi(x,y) \vDash^0_{\mathbf{Q}_0} \tau_0(\phi(x,y))$$

By Weak-Alg2 it follows:

$$\Delta_0(\tau_0(\phi(x,x))), \Delta_0(\phi(x,x), \phi(x,y)) \vdash \Delta_0(\tau_0(\phi(x,y))),$$

hence, by Weak-Alg3:

$$\phi(x,x), \Delta_0(\phi(x,x), \phi(x,y)) \vdash \phi(x,y). \tag{1}$$

Now, we also have that $\varnothing \vDash^1_{\mathbf{Q}_1} x \approx x$, hence by $\phi \in \Delta_1(x,x)$ and Weak-Alg2, we obtain:

$$\varnothing \vdash \phi(x,x). \tag{2}$$

Moreover, it also follows that $x \approx y \vDash^0_{\mathbf{Q}_0} \phi(x,x) \approx \phi(x,y)$, hence by Weak-Alg2:

$$\Delta_0(x,y) \vdash \Delta_0(\phi(x,x), \phi(x,y)). \tag{3}$$

Finally, by (1), (2) and (3), it follows that $\Delta_0(x,y) \vdash \phi(x,y)$, hence $\Delta_0(x,y) \vdash \Delta_1(x,y)$. The converse direction is proven analogously.

(1) We first prove that $\mathbf{Q}_0$ and $\mathbf{Q}_1$ satisfy the same quasi-equations under core-semantics. We show only that $Log^c(\mathbf{Q}_0) \subseteq Log^c(\mathbf{Q}_1)$, as the other direction follows analogously. Let $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta \in Log^c(\mathbf{Q}_0)$, then it follows that $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \vDash^0_{\mathbf{Q}_0} \epsilon \approx \delta$ and this yields that $\bigcup_{i \leq n} \Delta_0(\epsilon_i, \delta_i) \vdash \Delta_0(\epsilon, \delta)$ by Weak-Alg2 . By point (3) above, it follows that $\bigcup_{i \leq n} \Delta_1(\epsilon_i, \delta_i) \vdash \Delta_1(\epsilon, \delta)$, hence by Weak-Alg2 we get $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \vDash^1_{\mathbf{Q}_1} \epsilon \approx \delta$. The latter finally entails $\bigwedge_{i \leq n} \epsilon_i \approx \delta_i \to \epsilon \approx \delta \in Log^c(\mathbf{Q}_1)$ and thus $Log^c(\mathbf{Q}_0) \subseteq Log^c(\mathbf{Q}_1)$. By reasoning analogously we obtain that $Log^c(\mathbf{Q}_1) \subseteq Log^c(\mathbf{Q}_0)$, hence $Log^c(\mathbf{Q}_0) = Log^c(\mathbf{Q}_1)$. It then follows by Theorem 23 above that $(\mathbf{Q}_0)_{CG} = (\mathbf{Q}_1)_{CG}$ and since both $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are core-generated $\mathbf{Q}_0 = \mathbf{Q}_1$.

(2) By (1) we have $\mathbf{Q}_0 = \mathbf{Q}_1$, so let $\mathbf{Q} = \mathbf{Q}_i$, $i \in \{0,1\}$. Let $\alpha_0 \approx \beta_0 \in \Sigma_0$, then $\vDash^0_{\mathbf{Q}} \alpha_0 \approx \beta_0$ hence by Weak-Alg2 we obtain that $\varnothing \vdash \Delta_0(\alpha_0 \approx \beta_0)$ and thus $\vDash^1_{\mathbf{Q}} \alpha_0 \approx \beta_0$, meaning that $\mathcal{A} \vDash^1 \alpha_0 \approx \beta_0$ for all $\alpha_0 \approx \beta_0 \in \Sigma_0$. It follows that $\mathsf{core}_1(\mathcal{A}) \subseteq \{x \in \mathcal{A} : \alpha_0(x) \approx \beta_0(x)\} = \mathsf{core}_0(\mathcal{A})$. The other direction is proven analogously.

(4) By (1), it suffices to prove $\tau_0(\phi) \equiv^0_{\mathbf{Q}_0} \tau_1(\phi)$. By Weak-Alg3, we have $\Delta_0(\tau_0(x)) \dashv\vdash \Delta_1(\tau_1(x))$ and by point (3) this is equivalent to $\Delta_0(\tau_0(x)) \dashv\vdash \Delta_0(\tau_1(x))$. It then follows by Weak-Alg2 that $\tau_1(x) \equiv^0_{\mathbf{Q}_0} \tau_2(x)$. □

We have thus established that every algebraizable weak logic has a unique equivalent algebraic semantics, up to equivalence under the core consequence relation. Finally, we conclude this section by proving an analogue of Lemma 19 for the setting of weak logics.

Lemma 25.

1. Let $\vdash$ be a weak logic, then to show that $\vdash$ is algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$ it suffices to check either Weak-Alg1 and Weak-Alg4 or Weak-Alg2 and Weak-Alg3.

2. If $\vdash$ is an algebraizable weak logic, then it is algebraized by finitary maps τ, Δ.

Proof. 1. Suppose $\vdash$ is a weak logic and let $(\mathbf{Q}, \Sigma, \tau, \Delta)$ satisfy Weak-Alg1 and Weak-Alg4. We verify that Weak-Alg2 and Weak-Alg3 hold as well.

By Weak-Alg4 $\Theta \vDash^c_{\mathbf{Q}} \epsilon \approx \delta$ is equivalent to $\tau[\Delta(\Theta)] \vDash^c_{\mathbf{Q}} \tau(\Delta(\epsilon, \delta))$, which by Weak-Alg1 is equivalent to $\Delta(\Theta) \vdash \Delta(\epsilon, \delta)$, proving Weak-Alg2.

Also, for any formula ϕ, we have that $\tau(\phi) \equiv^c_{\mathbf{Q}} \tau(\phi)$, hence by Weak-Alg4 $\tau[\Delta(\tau(\phi))] \equiv^c_{\mathbf{Q}} \tau(\phi)$ and by Weak-Alg1 $\Delta(\tau(\phi)) \dashv\vdash \phi$, proving Weak-Alg3.

If $(\mathbf{Q}, \Sigma, \tau, \Delta)$ satisfies Weak-Alg2 and Weak-Alg3, then we proceed analogously.

2. For any $\phi \in \mathcal{F}m$, we have by Weak-Alg3 that $\phi \dashv\vdash \Delta[\tau(\phi)]$, thus by $\vdash$ being finitary there is some $\tau_0(\phi) \subseteq \tau(\phi)$ such that $|\tau_0(\phi)| < \omega$ and $\phi \dashv\vdash \Delta[\tau_0(\phi)]$.

Moreover, for any equation $\epsilon \approx \delta \in \mathsf{Eq}$, we have by Weak-Alg4 that $\epsilon \approx \delta \equiv^c_{\mathbf{Q}} \tau_0[\Delta(\epsilon, \delta)]$ we obtain by Theorem 17 a finite subset $\Delta_0(\epsilon, \delta) \subseteq \Delta(\epsilon, \delta)$ such that $\epsilon \approx \delta \equiv^c_{\mathbf{Q}} \tau_0[\Delta_0(\epsilon, \delta)]$.

Finally, it follows by the choice of τ_0, Δ_0 that $\Delta[\tau(\phi)] \dashv\vdash \Delta(\tau_0(\phi))$, hence $\tau(\phi) \equiv^c_{\mathbf{Q}} \tau_0(\phi)$. Similarly, from $\tau_0[\Delta(\epsilon, \delta)] \equiv^c_{\mathbf{Q}} \tau_0[\Delta_0(\epsilon, \delta)]$ we obtain $\Delta_0(\epsilon, \delta) \dashv\vdash \Delta(\epsilon, \delta)$. Thus τ_0, Δ_0 together with $\mathbf{Q}$ and Σ witness the algebraizability of $\vdash$. □

3 Characterisations of Algebraizability

In the previous section we have established the basic properties of algebraizability and adapted some classical results from abstract algebraic logic to the setting of weak logics. In this section we shall provide two important characterizations of algebraizability. Firstly, we investigate more closely the relationship between standard algebraizability and algebraizability for weak logics and we establish a bridge between these two notions. Then, we will use this bridge theorem to provide a version of the (theory-)isomorphism theorem for algebraizable logics in the context of weak logics.

3.1 The Schematic Fragment of a Weak Logic

Given a weak logic, a natural question to ask is whether we can associate it to some standard logical system. The study of negative variants of intermediate logics has lead to the notion of schematic fragment, which was defined in [8, 26] and further investigated in [2, 29]. Here we generalize it to arbitrary weak logics.

Definition 26 (Schematic Fragment)**.** Let $\vdash$ be a weak logic, we define its *schematic fragment* $\mathsf{Schm}(\vdash)$ as follows:

$$\mathsf{Schm}(\vdash) := \{(\Gamma, \phi) : \forall \sigma \in \mathsf{Subst}(\mathcal{L}), \sigma[\Gamma] \vdash \sigma(\phi)\}.$$

For a weak logic $\vdash$, we write $\Gamma \vdash_S \phi$ if $(\Gamma, \phi) \in \mathsf{Schm}(\vdash)$. It is clear from the definition that $\mathsf{Schm}(\vdash)$ is a standard logic. Moreover, for any standard logic $L \subseteq \vdash$ we have that $L \subseteq \mathsf{Schm}(\vdash)$, i.e. $\mathsf{Schm}(\vdash)$ is the greatest standard logic contained in $\vdash$. For any set of formulae Γ, we let $\mathsf{At}[\Gamma]$ be the closure of Γ under atomic substitutions, i.e. $\mathsf{At}[\Gamma] = \{\sigma(\phi) : \phi \in \Gamma, \sigma \in \mathsf{At}\}$. The atomic closure of sets of equations is defined analogously. We next define the following notions.

Definition 27. We say that a weak logic $\vdash$ is *representable* if there is a set of formulae Λ such that for all $\Gamma \cup \{\phi\} \subseteq \mathcal{F}m$:

$$\Gamma \vdash \phi \iff \Gamma \cup \mathsf{At}[\Lambda] \vdash_S \phi.$$

We say that $\vdash$ is *finitely representable* if the condition above holds for some finite set Λ.

We proceed to obtain a characterization of algebraizable weak logics in terms of finite representability and algebraizability of the underlying schematic fragment.

Theorem 28. *For a weak logic $\vdash$, the following are equivalent:*

1. *$\vdash$ is algebraizable;*
2. *$\mathsf{Schm}(\vdash)$ is algebraizable and $\vdash$ is finitely representable.*

Proof. ($\Rightarrow$) Suppose that $\vdash$ is algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$. We claim that $(\mathbf{Q}, \tau, \Delta)$ algebraizes $\mathsf{Schm}(\vdash)$. Recall that $\Gamma \vdash_S \phi$ stands for $(\Gamma, \phi) \in \mathsf{Schm}(\vdash)$. Firstly, we check that $(\mathbf{Q}, \Sigma, \tau, \Delta)$ verifies Weak-Alg1:

$$\begin{aligned}
\Gamma \vdash_S \phi &\iff \forall \sigma \in \mathsf{Subst}.\ \sigma[\Gamma] \vdash \sigma(\phi) && \text{(by definition)}\\
&\iff \forall \sigma \in \mathsf{Subst}.\ \tau(\sigma[\Gamma]) \vDash^c_{\mathbf{Q}} \tau(\sigma(\phi)) && \text{(by Weak-Alg2)}\\
&\iff \forall \sigma \in \mathsf{Subst}.\ \sigma(\tau[\Gamma]) \vDash^c_{\mathbf{Q}} \sigma(\tau(\phi)) && \text{(by structurality)}\\
&\iff \tau[\Gamma] \vDash_{\mathbf{Q}} \tau(\phi). && \text{(by Lemma 10)}
\end{aligned}$$

Now, it suffices to check that $(\mathbf{Q}, \Sigma, \tau, \Delta)$ verifies Weak-Alg4. Suppose that $\tau(\Delta(x, y)) \nvDash_{\mathbf{Q}} x \approx y$, then by Lemma 10 and the fact that $\mathbf{Q}$ is core-generated, we get that there exists a substitution σ such that $\sigma(\tau(\Delta(x, y))) \nvDash^c_{\mathbf{Q}} \sigma(x \approx y)$ and by structurality $\tau(\Delta(\sigma(x), \sigma(y))) \nvDash^c_{\mathbf{Q}} \sigma(x) \approx \sigma(y)$, contradicting the algebraizability of $\vdash$.

Finally, we also check that $\vdash$ is finitely representable. For a set Σ of equations we let $\mathsf{At}[\Sigma] := \{\sigma(\eta) \approx \sigma(\delta) : \eta \approx \delta \in \Sigma, \sigma \in \mathsf{At}\}$. We proceed as follows:

$$\begin{aligned}
\Gamma \vdash \phi &\iff \tau[\Gamma] \vDash^c_{\mathbf{Q}} \tau(\phi) && \text{(by Weak-Alg1)}\\
&\iff \tau[\Gamma] \cup \mathsf{At}[\Sigma] \vDash_{\mathbf{Q}} \tau(\phi)\\
&\iff \Gamma \cup \Delta[\mathsf{At}[\Sigma]] \vdash_S \phi && \text{(by algebraizability of } \mathsf{Schm}(\vdash))\\
&\iff \Gamma \cup \mathsf{At}[\Delta[\Sigma]] \vdash_S \phi. && \text{(by structurality)}
\end{aligned}$$

($\Leftarrow$) Suppose $\mathsf{Schm}(\vdash)$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$ and that $\vdash$ is finitely represented by Λ.

$$\begin{aligned}
\Gamma \vdash \phi &\iff \Gamma \cup \mathsf{At}[\Lambda] \vdash_S \phi && \text{(by definition)}\\
&\iff \tau[\Gamma] \cup \tau[\mathsf{At}[\Lambda]] \vDash_{\mathbf{Q}} \tau(\phi) && \text{(by assumption)}\\
&\iff \tau[\Gamma] \vDash^c_{\mathbf{Q}} \tau(\phi).
\end{aligned}$$

The last step is justified by letting $\mathsf{core}(\mathcal{A}) := \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon(x) \approx \delta(x),\ \epsilon \approx \delta \in \tau[\Lambda]\}$, which is already stable under substitutions, so we can only restrict our attention at $\tau[\Lambda]$. Consider $\mathbf{Q}' := \mathbb{Q}(\langle \mathsf{core}(\mathcal{A}) \rangle_{\mathcal{A} \in \mathbf{Q}})$. Then $(\mathbf{Q}', \tau[\Lambda], \tau, \Delta)$ algebraizes $\vdash$, as Weak-Alg1 follows by the reasoning above and Weak-Alg4 follows similarly to the converse direction (since $\mathbf{Q}' \subseteq \mathbf{Q}$). □

The following corollary follows immediately from the proof of the previous theorem:

Corollary 29. Let $\vdash$ be finitely represented by Λ, then $(\mathbf{Q}, \tau, \Delta)$ algebraizes $\mathsf{Schm}(\vdash)$ if and only if $(\mathbf{Q}, \tau[\Lambda], \tau, \Delta)$ algebraizes $\vdash$.

3.2 The Isomorphism Theorem

Blok and Pigozzi's Isomorphism Theorem [15, §3.5] provides an important characterization of algebraizability, as it shows that the algebraizability of a logic is witnessed by the existence of a suitable isomorphism between the lattice of deductive filters of the logic and the lattice of congruences of the corresponding class of algebras. We shall now prove a partial analogue of this result for the setting of weak logics.

The Standard Isomorphism Theorem In the previous section we have provided a criterion for algebraizability of weak logics in terms of their underlying schematic variant, i.e. a weak logic $\vdash$ is algebraizable if and only if $\mathsf{Schm}(\vdash)$ is algebraizable and $\vdash$ is finitely representable in it. Building on this result, we formulate a version of Blok and Pigozzi's Isomorphism Theorem for our setting. Firstly, let us recall some preliminary definitions.

Definition 30. For any algebra $\mathcal{A}$ and standard logic $\vdash$, we say that $F \subseteq \mathcal{A}$ is a *deductive filter* over $\mathcal{A}$ with respect to $\vdash$ if:

$$\Gamma \vdash \phi \implies \forall h \in \mathtt{Hom}(\mathcal{F}m, \mathcal{A}),\ h[\Gamma] \subseteq F \text{ entails } h(\phi) \in F.$$

We let $\mathsf{Fi}_{\vdash}(\mathcal{A})$ be the set of all deductive filters of $\vdash$ over $\mathcal{A}$, then $\mathsf{Fi}_{\vdash}(\mathcal{A})$ forms a lattice under the subset ordering. It is possible to verify that if F is a deductive filter and σ an endomorphism over $\mathcal{A}$, then $\sigma^{-1}(F)$ is also a deductive filter. With some abuse of notation, we then let $\mathsf{Fi}_{\vdash}(\mathcal{A})$ also refer to this lattice expansion, i.e. $\mathsf{Fi}_{\vdash}(\mathcal{A}) = (\mathsf{Fi}_{\vdash}, \subseteq, \{\sigma^{-1} : \sigma \in \mathtt{End}(\mathcal{L})\})$.

Similarly, for any algebra $\mathcal{A}$ we let $\mathsf{Con}(\mathcal{A})$ be the set of all congruences over $\mathcal{A}$ and by $\mathsf{Con}_{\mathbf{Q}}(\mathcal{A})$ the subset of all congruences θ over $\mathcal{A}$ such that $\mathcal{A}/\theta \in \mathbf{Q}$. Analogously to deductive filters, it is possible to verify that $\mathsf{Con}_{\mathbf{Q}}(\mathcal{A})$ forms a lattice under the subset ordering and that it is closed under inverse endomorphisms of $\mathcal{A}$. We thus let $\mathsf{Con}_{\mathbf{Q}}(\mathcal{A})$ also refer to this lattice expansion, i.e. $\mathsf{Con}_{\mathbf{Q}}(\mathcal{A}) = (\mathsf{Con}_{\mathbf{Q}}(\mathcal{A}), \subseteq, \{\sigma^{-1} : \sigma \in \mathtt{End}(\mathcal{A})\})$.

Finally, if $\vdash$ is a standard logic, we denote by $\mathsf{Th}(\vdash)$ the set of all *(syntactic) theories* over $\vdash$, i.e. all sets $\Gamma \subseteq \mathcal{F}m$ such that if $\Gamma \vdash \phi$ then $\phi \in \Gamma$. It is possible to verify that $\mathsf{Th}(\vdash)$ forms a lattice under the subset relation, and that it is additionally closed under inverse substitutions. With a slight abuse of notation, we refer by $\mathsf{Th}(\vdash)$ also to this lattice expansion, namely we let $\mathsf{Th}(\vdash) = (\mathsf{Th}(\vdash), \subseteq, \{\sigma^{-1} : \sigma \in \mathsf{Subst}(\mathcal{L})\})$. Similarly, if $\mathbf{Q}$ is a quasi-variety, then $\mathsf{Th}(\vDash_{\mathbf{Q}})$ denotes the set of *(semantical) theories* over $\mathbf{Q}$, i.e. sets of equations $\Theta \subseteq \mathsf{Eq}$ such that $\Theta \vDash_{\mathbf{Q}} \alpha \approx \beta$ entails $\alpha \approx \beta \in \Theta$. It is possible to verify that $\mathsf{Th}(\vDash_{\mathbf{Q}})$ forms a lattice under the subset relation, and it is additionally closed under inverse substitutions. We let $\mathsf{Th}(\vDash_{\mathbf{Q}}) = (\mathsf{Th}(\vDash_{\mathbf{Q}}), \subseteq, \{\sigma^{-1} : \sigma \in \mathsf{Subst}(\mathcal{L})\})$, again with slight abuse of notation. It is easily verified that syntactic theories over $\vdash$ are filters over $\mathcal{F}m$ and semantic theories over $\vDash_{\mathbf{Q}}$ are $\mathbf{Q}$-congruences over $\mathcal{F}m$.

The isomorphism theorem for standard logics provides us with a criterion to determine if a logic is algebraizable based on these lattices of filters and congruences. We refer the reader to [15, §3.5] for a proof of this result.

Theorem 31 (Isomorphism Theorem)**.** *Let $\vdash$ be a standard logic and $\mathbf{Q}$ a quasi-variety, then the following are equivalent:*

1. *$\vdash$ is algebraizable with equivalent algebraic semantics $\mathbf{Q}$;*
2. *$\mathsf{Fi}_{\vdash}(\mathcal{A}) \cong \mathsf{Con}_{\mathbf{Q}}(\mathcal{A})$, for any algebra $\mathcal{A}$;*
3. *$\mathsf{Th}(\vdash) \cong \mathsf{Th}(\vDash_{\mathbf{Q}})$.*

In order to obtain a version of the isomorphism theorem for weak logics, we firstly notice that from Blok and Pigozzi's result together with Theorem 28 we obtain the following characterization of algebraizability of a weak logic in terms of filters and congruences of its schematic variant.

Proposition 32. *Let $\vdash$ be a weak logic and $\mathbf{Q}$ a Σ-generated quasi-variety, then the following are equivalent:*

1. *$\vdash$ is algebraizable with equivalent algebraic semantics $\mathbf{Q}$;*
2. *$\vdash$ is finitely representable and $\mathsf{Fi}_{\mathsf{Schm}(\vdash)}(\mathcal{A}) \cong \mathsf{Con}_{\mathbf{Q}}(\mathcal{A})$, for any algebra $\mathcal{A}$;*
3. *$\vdash$ is finitely representable and $\mathsf{Th}(\mathsf{Schm}(\vdash)) \cong \mathsf{Th}(\vDash_{\mathbf{Q}})$.*

Proof. ($1 \Rightarrow 2$) If (1) holds, then by Corollary 29 it follows that $\mathsf{Schm}(\vdash)$ is algebraized by $\mathbf{Q}$ and $\vdash$ is finitely representable. By Theorem 31 above it follows that $\mathsf{Fi}_{\mathsf{Schm}(\vdash)}(\mathcal{A}) \cong \mathsf{Con}_{\mathbf{Q}}(\mathcal{A})$.

($2 \Rightarrow 3$) Immediate by Theorem 31.

($3 \Rightarrow 1$) If (3) holds, then by Theorem 31 $\mathsf{Schm}(\vdash)$ is algebraizable with equivalent algebraic semantics $\mathbf{Q}$. Moreover, since $\vdash$ is finitely representable, it follows by Corollary 29 that $\vdash$ is algebraized by $\mathbf{Q}$ as well. □

The Isomorphism Theorem for Theories in Weak Logics The former proposition gives us a first criterion of algebraizability in terms of the lattices of filters and congruences. However, it is simply a direct translation of the corresponding criterion of algebraizability of the underlying schematic variant of a weak logic, together with the additional condition of finite representability. Here we provide a slight improvement of this result by showing a version of the isomorphism theorem limited to lattices of theories over weak logics.

If $\vdash$ is a weak logic, we denote by $\mathsf{Th}(\vdash)$ the set of all *(syntactic) theories* over $\vdash$, i.e. all sets $\Gamma \subseteq \mathcal{F}m$ such that if $\Gamma \vdash \phi$ then $\phi \in \Gamma$. Similarly, if $\mathbf{Q}$ is a quasi-variety of expanded algebras, then $\mathsf{Th}(\vDash^c_{\mathbf{Q}})$ denotes the set of *(semantical) theories* over $\mathbf{Q}$, i.e. sets of equations $\Theta \subseteq \mathsf{Eq}$ such that $\Theta \vDash^c_{\mathbf{Q}} \alpha \approx \beta$ entails $\alpha \approx \beta \in \Theta$. It is straightforward to check that both $\mathsf{Th}(\vdash)$ and $\mathsf{Th}(\vDash^c_{\mathbf{Q}})$ form lattices under the subset ordering and are additionally closed under inverse atomic substitutions.

To obtain a version of the isomorphism theorem for our context, one way is to relate syntactic and semantic theories over weak logics and expanded algebras to special instances of standard theories. For any set of formulae $\Lambda \subseteq \mathcal{F}m$, we let $\mathsf{Th}^{\Lambda}(\vdash)$ be the set of all (syntactical) theories Γ over $\vdash$ such that $\mathsf{At}[\Lambda] \subseteq \Gamma$ and for any set of equations $\Sigma \subseteq \mathsf{Eq}$. we let $\mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$ denote the (semantical) theories Θ such that $\mathsf{At}[\Sigma] \subseteq \Theta$. It can be checked by a routine argument that both $\mathsf{Th}^{\Lambda}(\vdash)$ and $\mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$ are lattices under the subset ordering and they are additionally closed under inverse substitutions of the term algebra.

The following lemma provides us with an important connection between theories in weak logics and expanded algebras on the one side, and standard theories over their schematic fragment and algebraic reducts on the other side. On the syntactic side, we can now express the finite representability of a weak logic in terms of the lattice of core syntactical theories. Similarly, on the semantical side, the lemma allows us to determine when a quasi-variety of expanded algebras has an equationally definable core by reference to core semantical theories. We recall that, for any standard or weak logic $\vdash$, the syntactical consequence operator $\mathsf{Cn}_{\vdash}$ is defined by letting $\mathsf{Cn}_{\vdash}(\Gamma) = \{\phi : \Gamma \vdash \phi\}$ for any set of formulae Γ. The semantical consequence operators $\mathsf{Cn}_{\vDash_{\mathbf{Q}}}$ and $\mathsf{Cn}_{\vDash^c_{\mathbf{Q}}}$ are defined analogously.

Proposition 33. *Let* $\mathbf{Q}$ *be a quasi-variety of expanded algebras with an equationally defined core,* Λ *a finite set of formulae and* Σ *a finite set of equations, the following facts hold:*

1. $\mathbf{Q}$ *is* Σ*-defined if and only if* $\mathsf{Th}(\vDash^c_{\mathbf{Q}}) = \mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$*;*

2. $\vdash$ *is finitely represented by* Λ *if and only if* $\mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash)) = \mathsf{Th}(\vdash)$*.*

Proof.

1. ($\Rightarrow$) Suppose for all $\mathcal{A} \in \mathbf{Q}$, $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$. (i) Suppose $\Theta \in \mathsf{Th}(\vDash^c_{\mathbf{Q}})$, then since for all $\mathcal{A} \in \mathbf{Q}$ $\mathcal{A} \vDash^c \Sigma$, we have $\mathsf{At}[\Sigma] \subseteq \Theta$. Now suppose $\Theta \vDash_{\mathbf{Q}} \alpha \approx \beta$, then by monotonicity $\Theta \cup \mathsf{At}[\Sigma] \vDash_{\mathbf{Q}} \alpha \approx \beta$. Now, this is equivalent to $\Theta \vDash^c_{\mathbf{Q}} \alpha \approx \beta$, hence $\alpha \approx \beta \in \Theta$. (ii) Suppose $\Theta \in \mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$, thus $\mathsf{At}[\Sigma] \subseteq \Theta$. Now, if $\Theta \vDash^c_{\mathbf{Q}} \alpha \approx \beta$, then since for all $\mathcal{A} \in \mathbf{Q}$, $\mathsf{core}(\mathcal{A}) = \Sigma(x, \mathcal{A})$, we have $\Theta \cup \mathsf{At}[\Sigma] \vDash_{\mathbf{Q}} \alpha \approx \beta$ and therefore $\alpha \approx \beta \in \Theta$.

   ($\Leftarrow$) Suppose $\mathsf{Th}(\vDash^c_{\mathbf{Q}}) = \mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$, then for any Θ we have $\mathsf{Cn}_{\vDash_{\mathbf{Q}}}(\Theta \cup \mathsf{At}[\Sigma]) = \mathsf{Cn}_{\vDash^c_{\mathbf{Q}}}(\Theta)$. Hence:

$$\begin{aligned} \Theta \vDash^c_{\mathbf{Q}} \alpha \approx \beta &\iff \mathsf{Cn}_{\vDash^c_{\mathbf{Q}}}(\alpha \approx \beta) \subseteq \mathsf{Cn}_{\vDash^c_{\mathbf{Q}}}(\Theta) \\ &\iff \mathsf{Cn}_{\vDash_{\mathbf{Q}}}(\{\alpha \approx \beta\} \cup \mathsf{At}[\Sigma]) \subseteq \mathsf{Cn}_{\vDash_{\mathbf{Q}}}(\Theta \cup \mathsf{At}[\Sigma]) \\ &\iff \Theta \cup \mathsf{At}[\Sigma] \vDash_{\mathbf{Q}} \alpha \approx \beta. \end{aligned}$$

   Since $\mathbf{Q}$ has an equationally definable core, we have for all $\mathcal{A} \in \mathbf{Q}$ that $\mathsf{core}(\mathcal{A}) = \Omega[x, \mathcal{A}]$ for some finite $\Omega \subseteq \mathsf{Eq}$. Then, we have that $\Theta \vDash^c_{\mathbf{Q}} \alpha \approx \beta$ if and only if $\Theta \cup \mathsf{At}[\Omega] \vDash_{\mathbf{Q}} \alpha \approx \beta$. By the display above, it follows that $\Omega \equiv_{\mathbf{Q}} \Sigma$.

2. ($\Rightarrow$) Suppose $\vdash$ is finitely represented by Λ. (i) If $\Gamma \in \mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash))$ and $\Gamma \vdash \phi$. By finite representability $\Gamma \cup \mathsf{At}[\Lambda] \vdash_S \phi$ and by $\Gamma \in \mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash))$ also $\mathsf{At}[\Lambda] \subseteq \Gamma$. Since Γ is a theory over $\mathsf{Schm}(\vdash)$, it follows that $\phi \in \Gamma$. (ii) Similarly, if $\Gamma \in \mathsf{Th}(\vdash)$, then since $\mathsf{At}[\Lambda] \vdash \mathsf{At}[\Lambda]$, it follows by Theorem 28 that $\varnothing \vdash_S \mathsf{At}[\Lambda]$ which means $\mathsf{At}[\Lambda] \subseteq \Gamma$. Now, if $\Gamma \vdash_S \phi$, then $\Gamma \cup \mathsf{At}[\Lambda] \vdash_S \phi$ hence by Theorem 28 $\Gamma \vdash \phi$ and $\phi \in \Gamma$.

   ($\Leftarrow$) Assume $\mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash)) = \mathsf{Th}(\vdash)$. Then for any Γ we have $\mathsf{Cn}_{\mathsf{Schm}(\vdash)}(\Gamma \cup \mathsf{At}[\Lambda]) = \mathsf{Cn}_{\vdash}(\Gamma)$. Whence:

$$\begin{aligned} \Gamma \vdash \phi &\iff \mathsf{Cn}_{\vdash}(\phi) \subseteq \mathsf{Cn}_{\vdash}(\Gamma) \\ &\iff \mathsf{Cn}_{\mathsf{Schm}(\vdash)}(\phi \cup \mathsf{At}[\Lambda]) \subseteq \mathsf{Cn}_{\mathsf{Schm}(\vdash)}(\Gamma \cup \mathsf{At}[\Lambda]) \\ &\iff \Gamma \cup \mathsf{At}[\Lambda] \vdash_S \phi. \end{aligned}$$

   which gives us finite representability via Λ. □

Finally, we use the previous proposition to obtain a version of the theory isomorphism theorem for weak logics.

Theorem 34. *Let* $\vdash$ *be a weak logic and* $\mathbf{Q}$ *a* Σ*-generated quasi-variety of expanded algebras. The following are equivalent:*

1. $\vdash$ *is algebraized by* $(\mathbf{Q}, \Sigma, \tau, \Delta)$

2. $\mathsf{Th}(\mathsf{Schm}(\vdash)) \cong \mathsf{Th}(\vDash_{\mathbf{Q}})$ *and there are finite* $\Lambda \subseteq \mathcal{F}m$*,* $\Sigma \subseteq \mathsf{Eq}$ *such that* $\mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash)) = \mathsf{Th}(\vdash)$ *and* $\mathsf{Th}(\vDash^c_{\mathbf{Q}}) = \mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$*.*

Proof.

- (1 ⇒ 2) It immediately follows by Proposition 33 that $\mathsf{Th}(\vDash^c_{\mathbf{Q}}) = \mathsf{Th}^{\Sigma}(\vDash_{\mathbf{Q}})$. Moreover, by Theorem 28 it follows that $\mathsf{Schm}(\vdash)$ is algebraizable, hence by the standard isomorphism theorem $\mathsf{Th}(\mathsf{Schm}(\vdash)) \cong \mathsf{Th}(\vDash_{\mathbf{Q}})$. Finally, again by Theorem 28, $\vdash$ is finitely representable, hence by Proposition 33 for some Λ we have that $\mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash)) = \mathsf{Th}(\vdash)$.

- (2 ⇒ 1) Since $\mathsf{Th}(\mathsf{Schm}(\vdash)) \cong \mathsf{Th}(\vDash_{\mathbf{Q}})$, it follows by the standard Isomorphism Theorem that $\mathsf{Schm}(\vdash)$ is algebraized by $(\mathbf{Q}, \tau, \Delta)$ for some structural transformers τ, Δ. Then, by Proposition 33, $\mathsf{Th}^{\Lambda}(\mathsf{Schm}(\vdash)) = \mathsf{Th}(\vdash)$ entails that $\vdash$ is finitely represented, hence by Theorem 28 it is algebraized by $\mathbf{Q}$. Finally, by uniqueness of the equivalent algebraic semantics, $\vdash$ is algebraized by $(\mathbf{Q}, \Sigma, \tau, \Delta)$. □

The result above shows that the algebraizability of a weak logic $\vdash$ by a quasi-variety $\mathbf{Q}$ of equationally definable algebras is equivalent to the algebraizability of $\mathsf{Schm}(\vdash)$ by $\mathbf{Q}$ together with the fact that the lattices of core syntactic and semantical theories coincide with suitable sublattices of the lattices of syntactic and semantical theories. This provides us with an alternative method to determine whether a weak logic is algebraizable or not. We leave it to future research whether such isomorphism theorem could be improved by extending it to suitable core filters and core congruences.

4 Applications to Inquisitive and Dependence Logics

Inquisitive logic and dependence logic are two related propositional systems, which are usually both defined in terms of so-called *team semantics*, originally introduced by Hodges in [20]. Here we introduce both of them in syntactic terms, analogously as in [30], and we investigate whether they are algebraizable or not in the sense provided by this article. We refer the reader to [8, 36] for the standard presentation of inquisitive and dependence logic by their team semantics.

An algebraic semantics for the classical version of inquisitive logic `InqB` was introduced in [1, 2, 29]. Such semantics was generalized in [30] to provide a sound and complete algebraic semantics to `InqI`, $\mathtt{InqB}^{\otimes}$, $\mathtt{InqI}^{\otimes}$ and other intermediate versions of inquisitive and dependence logic as well. Since these logical systems do not satisfy the rule of uniform substitution, it has so far been an open question whether such semantics are in any sense unique. The notion of algebraizability of weak logics that we have introduced in this article provides us with a framework to make sense of this question.

Interestingly, these logics behave quite differently from one another with respect to our notion of algebraizability. We shall prove in this section that the classical versions of inquisitive and dependence logic `InqB` and $\mathtt{InqB}^{\otimes}$ are algebraizable, while the intuitionistic versions `InqI` and $\mathtt{InqI}^{\otimes}$ are not.

4.1 Inquisitive and Dependence Logic

We provide an axiomatic presentation of inquisitive and dependence logic. The following Hilbert-style presentation of inquisitive and dependence logics adapts the natural-deduction characterizations given in [10].

Let $\mathcal{L}_{\mathtt{IPC}} = \{\wedge, \vee, \rightarrow, \bot\}$ be both the usual intuitionistic signature of `IPC` and the set of formulae in this signature obtained from a set `Var` of atomic variables. Negation is treated as a defined operation and can be introduced by letting $\neg\phi := \phi \rightarrow \bot$. A formula of $\mathcal{L}_{\mathtt{IPC}}$ is said to be *standard* if it is $\vee$-free. We write $\mathcal{L}_{\mathtt{CL}}$ for the set of all standard formulae and also for the signature $\mathcal{L}_{\texttt{CL}} = \{\wedge, \rightarrow, \bot\}$. We use Greek letters $\phi, \psi, \dots$ to denote arbitrary inquisitive formulae and $\alpha, \beta, \dots$ to denote standard inquisitive formulae.

Definition 35 (Intuitionistic Inquisitive Logic)**.** The system `InqI` of *intuitionistic inquisitive logic* is the smallest set of formulae of $\mathcal{L}_{\texttt{IPC}}$ such that, for all $\phi, \psi, \chi \in \mathcal{L}_{\texttt{IPC}}$ and for all $\alpha \in \mathcal{L}_{\texttt{CL}}$, `InqI` contains the following formulae:

(A1) $\phi \rightarrow (\psi \rightarrow \phi)$

(A2) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)$

(A3) $\phi \wedge \psi \rightarrow \phi$

(A4) $\phi \wedge \psi \rightarrow \psi$

(A5) $\phi \rightarrow (\psi \rightarrow \phi \wedge \psi)$

(A6) $\phi \rightarrow \phi \vee \psi$

(A7) $\psi \rightarrow \phi \vee \psi$

(A8) $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \vee \psi \rightarrow \chi))$

(A9) $\bot \rightarrow \phi$

(A10) $(\alpha \rightarrow (\phi \vee \psi)) \rightarrow ((\alpha \rightarrow \phi) \vee (\alpha \rightarrow \psi))$

and in addition it is closed under the rule of *modus ponens* (`MP`).

The classical version of inquisitive logic is the extension of `InqI` by the axioms $\neg\neg\alpha \rightarrow \alpha$, where $\alpha \in \mathcal{L}_{\texttt{CL}}$.

Definition 36. The system `InqB` of classical inquisitive logic is defined as the closure under `MP` of the set $\texttt{InqI} \cup \{\neg\neg\alpha \rightarrow \alpha\}_{\alpha \in \mathcal{L}_{\texttt{CL}}}$.

Dependence logics can be seen as an extension of inquisitive logic in a larger signature. Let $\mathcal{L}^{\otimes}_{\texttt{IPC}} = \{\wedge, \vee, \rightarrow, \otimes, \bot\}$, where $\otimes$ is the so-called *tensor disjunction*, and with slight abuse of notation let $\mathcal{L}^{\otimes}_{\texttt{IPC}}$ also be the set of formulae in this signature. We say that a formula in $\mathcal{L}^{\otimes}_{\texttt{IPC}}$ is *standard* if it does not contain the $\vee$ operator and we write $\mathcal{L}^{\otimes}_{\texttt{CL}}$ both for the set of standard dependence formulae and for the restricted signature $\mathcal{L}^{\otimes}_{\texttt{CL}} = \{\wedge, \rightarrow, \otimes, \bot\}$. We define negation again by $\neg\phi := \phi \rightarrow \bot$.

Definition 37 (Intuitionistic Dependence Logic)**.** The system $\texttt{InqI}^{\otimes}$ of *intuitionistic dependence logic* is the smallest set of formulae of $\mathcal{L}^{\otimes}_{\texttt{IPC}}$ such that, for all $\phi, \psi, \chi, \tau \in \mathcal{L}^{\otimes}_{\texttt{IPC}}$ and for all $\alpha, \beta, \gamma \in \mathcal{L}^{\otimes}_{\texttt{CL}}$, $\texttt{InqI}^{\otimes}$ contains the formulae (A1)–(A10) of Definition 35 and the following:

(A11) $\alpha \rightarrow (\alpha \otimes \beta)$

(A12) $(\alpha \otimes \beta) \rightarrow (\beta \otimes \alpha)$

(A13) $\phi \otimes (\psi \vee \chi) \rightarrow (\phi \otimes \psi) \vee (\phi \otimes \chi)$

(A14) $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \tau) \rightarrow (\phi \otimes \psi \rightarrow \chi \otimes \tau))$

(A15) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \otimes \beta \rightarrow \gamma))$

and in addition it is closed under *modus ponens*.

And, similarly as we did before for classical inquisitive logic, we define the classical version of dependence logic $\texttt{InqB}^{\otimes}$ as follows.

Definition 38. The system $\texttt{InqB}^{\otimes}$ of classical dependence logic is defined as the closure under `MP` of the set $\texttt{InqI}^{\otimes} + \{\neg\neg\alpha \rightarrow \alpha\}_{\alpha \in \mathcal{L}^{\otimes}_{\texttt{CL}}}$.

To see that the systems that we have defined so far are weak logics, it is convenient to present them via their induced consequence relations. Fix $L \in \{\texttt{InqB}, \texttt{InqB}^{\otimes}, \texttt{InqI}, \texttt{InqI}^{\otimes}\}$, then we let:

$$\Gamma \vdash_L \phi \Longleftrightarrow \bigwedge_{\psi \in \Gamma_0} \psi \rightarrow \phi \in L \text{ for some finite } \Gamma_0 \subseteq \Gamma.$$

It is then easy to verify that $\vdash_L$ is in fact a consequence relation and moreover that it is also closed under atomic substitutions. As a matter of fact, it is not hard to see that $\mathtt{InqI}$, $\mathtt{InqI}^{\otimes}$, $\mathtt{InqB}$, $\mathtt{InqB}^{\otimes}$ are closed under $\vee$-free substitutions, namely substitutions $\sigma \in \mathsf{Subst}(\mathcal{L})$ such that $\sigma(\alpha)$ is standard whenever α is standard. It then follows that $\mathtt{InqB}$, $\mathtt{InqB}^{\otimes}$, $\mathtt{InqI}$, $\mathtt{InqI}^{\otimes}$ are all weak logics.

Inquisitive and dependence logics also make for proper weak logics, i.e. for systems which are not closed under uniform substitution. That this is the case follows from Example 4 before. Another failure of $\mathtt{US}$ is the following: for any $L \in \{\mathtt{InqB}, \mathtt{InqB}^{\otimes}, \mathtt{InqI}, \mathtt{InqI}^{\otimes}\}$ we have by the axiomatisation above:

$$(p \to (q \vee r)) \to ((p \to q) \vee (p \to r)) \in L.$$

However, the result of the substitution $p \mapsto q \vee r$ is not a validity of these logics:

$$((q \vee r) \to (q \vee r)) \to (((q \vee r) \to q) \vee ((q \vee r) \to r)) \notin L.$$

We refer the reader to [8] and [10] for a lengthier discussion of the failure of uniform substitution in propositional inquisitive and dependence logic.

4.2 Algebraizability of InqB and InqB$^{\otimes}$

We prove in this section the algebraizability of the classical versions of inquisitive and dependence logic. We firstly recall some important facts about inquisitive logic and its algebraic semantics.

We recall that a *Heyting algebra* $\mathcal{H}$ is a bounded distributive lattice augmented by an operation $\to$ such that for all $a, b, c \in \mathcal{H}$:

$$a \wedge b \leq c \iff a \leq b \to c.$$

Negation is defined by letting $\neg x := x \to \bot$. An element $x \in \mathcal{H}$ is *regular* if $x = \neg\neg x$. We write $\mathcal{H}_{\neg}$ for the subset of regular elements of $\mathcal{H}$, and we say that a Heyting algebra $\mathcal{H}$ is *regularly generated*, or simply *regular*, if $\mathcal{H} = \langle \mathcal{H}_{\neg} \rangle$. Similarly, we say that a class of Heyting algebras is *regularly generated* if it is core-generated for $\Sigma = \{x \approx \neg\neg x\}$.

We also recall that an *intermediate logic* is a (standard) logic $\mathtt{L}$ such that $\mathtt{IPC} \subseteq \mathtt{L} \subseteq \mathtt{CPC}$. The logic $\mathtt{ML}$, namely Medvedev's logic of finite problems, is the logic of all Kripke frames of the form $(\wp^{+}(s), \supseteq)$, where $|s| < \omega$ and $\wp^{+}(s) = \wp(s) \setminus \{\emptyset\}$. As in Example 4, we define a $\mathtt{DNA}$-*logic*, or *negative variant* of an intermediate logic, as a set of formulae $\mathtt{L}^{\neg} = \{\phi[\neg p_0, \dots, \neg p_n / p_0, \dots, p_n] : \phi \in \mathtt{L}\}$, where $\mathtt{L}$ is an intermediate logic. Medvedev's logic was firstly introduced in [25], while negative variants were originally considered in [26].

The next theorem collects together some previous result by Ciardelli in [8] on the schematic variant of $\mathtt{InqB}$ and the characterization of regularly generated varieties from [2].

Theorem 39.

1. $\mathtt{ML}^{\neg} = \mathtt{InqB}$ *and* $\mathsf{Schm}(\mathtt{InqB}) = \mathtt{ML}$*;*
2. $Var(\mathtt{ML})$ *is regularly generated;*
3. *For all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}_{\mathtt{IPC}}$, $\Gamma \vdash_{\mathtt{InqB}} \phi \iff \Gamma \vDash^{c}_{Var(\mathtt{ML})} \phi$.

Proof. (1) both facts were proved by Ciardelli in [8]. (2) follows from (1) together with Proposition 4.17 from [2]. (3) From Theorem 3.32 in [2].[2] □

[2]Notice that in [2] it was proven $\Gamma \vdash_{\mathtt{InqB}} \phi \iff \Gamma \vDash^{c}_{Var(\mathtt{ND})} \phi$. The present statement follows by the fact that the varieties $\mathtt{ND}$ and $\mathtt{ML}$ contain the same regularly generated algebras.

The algebraizability of $\mathtt{InqB}$ is then immediate to prove.

Theorem 40. $\mathtt{InqB}$ *is algebraizable.*

Proof. Let $\mathbf{Q} = Var(\mathtt{ML})$, $\tau(\phi) = \phi \approx 1$, $\Delta(x, y) = x \leftrightarrow y$ and $\Sigma = \{x \approx \neg\neg x\}$. We prove that $(\mathbf{Q}, \Sigma, \tau, \Delta)$ algebraizes $\mathtt{InqB}$. Firstly, by Theorem 39 above we have that $Var(\mathtt{ML})$ is core-generated by Σ.

Now, by Lemma 25, it suffices to show that $(Var(\mathtt{ML}), \phi \approx 1, x \leftrightarrow y, x \approx \neg\neg x)$ satisfies Weak-Alg1 and Weak-Alg4. By Theorem 39(3), Weak-Alg1 immediately follows. Moreover, for all $\mathcal{H} \in Var(\mathtt{ML})$ and $x, y \in \mathcal{H}$, we have that $x = y$ if and only if $x \leq y$ and $y \leq x$. By the properties of Heyting implication, this is equivalent to $\mathcal{H} \vDash x \to y \approx 1$ and $\mathcal{H} \vDash y \to x \approx 1$. It follows that $x \approx y \equiv^c_{\mathbf{Q}} \{x \to y \approx 1, y \to x \approx 1\}$, showing Weak-Alg4 holds. It follows that $\mathtt{InqB}$ is algebraizable. □

To extend this result to dependence logic, we firstly need to introduce a suitable notion of dependence algebras.

Definition 41. A $\mathtt{InqB}^{\otimes}$-algebra $\mathcal{A}$ is a structure in the signature $\mathcal{L}^{\otimes}_{\mathtt{IPC}}$ such that: $\mathcal{A}\restriction\{\vee, \wedge, \to, \bot\} \in Var(\mathtt{ML})$, $\mathcal{A}_{\neg}\restriction\{\otimes, \wedge, \to, \bot\} \in \mathbf{BA}$ and, additionally:

$$(Dist) \quad \mathcal{A} \vDash x \otimes (y \vee z) \approx (x \otimes y) \vee (x \otimes z);$$
$$(Mon) \quad \mathcal{A} \vDash (x \to z) \to (y \to k) \approx (x \otimes y) \to (z \otimes k).$$

Our definition is similar to the definition of $\mathtt{InqB}^{\otimes}$-algebras from [30, 2.2], with the difference that here we assume the equations to hold for the full algebra and not only with respect to the subalgebra generated by the core.

By expanding the previous definition, one can see that it amounts to an equational definition of a class of algebras. We thus let $\mathsf{InqBAlg}^{\otimes}$ be the variety of all $\mathtt{InqB}^{\otimes}$-algebras and $\mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}}$ be the subclass of all finite, regular, subdirectly irreducible $\mathtt{InqB}^{\otimes}$-algebras. The following fact follows from [30, 2.15, 3.20].

Theorem 42. *For all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}^{\otimes}_{\mathtt{IPC}}$, $\Gamma \vdash_{\mathtt{InqB}^{\otimes}} \phi \iff \Gamma \vDash^c_{\mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}}} \phi$.

Now, we say that $\mathcal{A}$ is a *dependence algebra* if it belongs to the subvariety generated by all finite, regular, subdirectly irreducible $\mathtt{InqB}$-algebras. We let $\mathbf{DA} := \mathbb{V}(\mathsf{InqBAlg}^{\otimes}_{\mathsf{FRSI}})$ be the variety of dependence algebras. It immediately follows that $\mathbf{DA}$ witnesses the algebraizability of $\mathtt{InqB}^{\otimes}$.

Theorem 43. $\mathtt{InqB}^{\otimes}$ *is algebraizable.*

Proof. It follows from Theorem 42 analogously to Theorem 40 by letting $\mathbf{Q} = \mathbf{DA}$, $\tau(\phi) := \phi \approx 1$, $\Delta(x, y) = x \leftrightarrow y$ and $\Sigma = \{x \approx \neg\neg x\}$. □

The question whether $\mathbf{DA} = \mathsf{InqBAlg}^{\otimes}$ is open and should be subject of future investigation.

4.3 Failure of Algebraizability for $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$

We prove the failure of algebraizability of $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$. Before showing our main result about intuitionistic inquisitive and dependence logic, we recall some fact from [10].

Proposition 44 (Disjunctive Normal Form)**.**

- *Let* $\phi \in \mathcal{L}_{\mathtt{IPC}}$, *there are standard inquisitive formulas* $\{\alpha_i\}_{i \leq n}$ *such that* $\phi \equiv_{\mathtt{InqI}} \bigvee_{i \leq n} \alpha_i$.

- *Let* $\phi \in \mathcal{L}^{\otimes}_{\mathtt{IPC}}$*, there are standard dependence formulas* $\{\alpha_i\}_{i\leq n}$ *such that* $\phi \equiv_{\mathtt{InqI}^{\otimes}} \bigvee_{i\leq n} \alpha_i$*.*

Proposition 45 (Disjunction Property)**.**

- *Let* $\phi_0, \dots, \phi_n \in \mathcal{L}_{\mathtt{IPC}}$*, then* $\mathtt{InqI} \vdash \bigvee_{i\leq n} \phi_i$ *if and only if* $\mathtt{InqI} \vdash \phi_i$ *for some* $i \leq n$*.*
- *Let* $\phi_0, \dots, \phi_n \in \mathcal{L}^{\otimes}_{\mathtt{IPC}}$*, then* $\mathtt{InqI}^{\otimes} \vdash \bigvee_{i\leq n} \phi_i$ *if and only if* $\mathtt{InqI}^{\otimes} \vdash \phi_i$ *for some* $i \leq n$*.*

The next theorem shows that, contrary to $\mathtt{InqB}$ and $\mathtt{InqB}^{\otimes}$, $\mathtt{InqI}$ and $\mathtt{InqI}^{\otimes}$ are not algebraizable.

Theorem 46. $\mathtt{InqI}$ *and* $\mathtt{InqI}^{\otimes}$ *are not algebraizable.*

Proof. (i) We first consider the case of $\mathtt{InqI}$. Suppose by *reductio* that $\mathtt{InqI}$ is algebraized by a suitable tuple $(\mathbf{Q}, \Sigma, \tau, \Delta)$. By Theorem 28 it follows that $Log^{\tau}_{\Delta}(\mathbf{Q}) = \mathsf{Schm}(\mathtt{InqI}) \supseteq \mathtt{IPC}$, which means that the standard logic of $\mathbf{Q}$ is either an intermediate logic or the inconsistent logic $\mathcal{L}_{\mathtt{IPC}}$ containing all formulae in the signature of intuitionistic logic. However, since $\mathsf{Schm}(\mathtt{InqI}) \neq \mathcal{L}_{\mathtt{IPC}}$, we have that $Log^{\tau}_{\Delta}(\mathbf{Q})$ is an intermediate logic. Now, since $Log^{\tau}_{\Delta}(\mathbf{Q})$ is an intermediate logic, it is algebraized by $(\mathbf{Q}, \phi \approx 1, x \leftrightarrow y)$ (see e.g. [6]). Thus, by the uniqueness of the equivalent algebraic semantics of standard logics, it follows that $\mathbf{Q}$ is a subvariety of Heyting algebras and without loss of generality that $\tau(\phi) = \phi \approx 1$, $\Delta(x, y) = x \leftrightarrow y$.

Without loss of generality, we assume $|\Sigma| = 1$. Let $\Sigma = \{\epsilon(x) \approx \delta(x)\}$, then by algebraizability it follows that :

$$\epsilon(x) \approx \delta(x) \equiv_{\mathbf{Q}} \tau(\Delta(\epsilon(x), \delta(x))) = (\epsilon(x) \leftrightarrow \delta(x)) \approx 1.$$

Thus, without loss of generality, $\Sigma = \{\sigma(x) \approx 1\}$, where $\sigma \in \mathcal{L}_{\mathtt{IPC}}$. By the disjunctive normal form of $\mathtt{InqI}$ (Proposition 44), it follows $\sigma \equiv_{\mathbf{Q}} \bigvee_{j<l} \rho_j$, where each ρ_j is standard. Thus $\Sigma = \{\bigvee_{j<l} \rho_j(x) \approx 1\}$.

Then, for all $\mathcal{A} \in \mathbf{Q}$ with $\mathsf{core}(\mathcal{A}) = \Sigma[x, \mathcal{A}]$, it follows that $\mathcal{A} \vDash^c \bigvee_{j<l} \rho_j(x) \approx 1$. Therefore, we obtain that $\bigvee_{j<l} \rho_j(x) \leftrightarrow 1 \in \mathtt{InqI}$ and thus $\bigvee_{j<l} \rho_j(x) \in \mathtt{InqI}$. By the disjunction property of $\mathtt{InqI}$ (Proposition 45) it follows that $\rho_j(x) \in \mathtt{InqI}$ for some $j < l$.

Now, since $\mathtt{InqI}\restriction\{\wedge, \to, \bot\}$ contains the $\vee$-free fragment of $\mathtt{IPC}$ and $\rho_j(x)$ is univariate, it must be equivalent to one of the following formulae (See [32]):

$$\top,\ \neg\neg x,\ \neg\neg x \to x,\ x,\ \neg x,\ \bot.$$

However, it can be checked using the semantics from [10] or [30] that $\neg\neg x \notin \mathtt{InqI}$, $\neg\neg x \to x \notin \mathtt{InqI}$, $x \notin \mathtt{InqI}$, $\neg x \notin \mathtt{InqI}$, $\bot \notin \mathtt{InqI}$. Otherwise, if $\rho_j(x) = \top$, it follows that $\mathsf{core}(\mathcal{A}) = \mathcal{A}$ for all $\mathcal{A} \in \mathbf{Q}$, contradicting the fact that $\mathtt{InqI}$ is not closed under uniform substitution.

(ii) The non-algebraizability of $\mathtt{InqI}^{\otimes}$ follows analogously. By the same argument of above, it follows that $\rho(x) \in \mathtt{InqI}^{\otimes}$, where $\rho \in \mathcal{L}^{\otimes}_{\mathtt{IPC}}$ is some $\vee$-free formula. Notice that, for all $\mathcal{A} \in \mathbf{Q}$ and $x \in \mathsf{core}(\mathcal{A})$, we must have $\rho(x) = x$, which means that for every $n < \omega$, $x \dashv\vdash_{\mathtt{InqI}^{\otimes}} \rho^n(x)$, where $\rho^1(x) := \rho(x)$ and $\rho^{n+1}(x) = \rho(\rho^n(x))$.

Since ρ is $\vee$-free, it follows it is an $\mathtt{IPC}$-formula, for $\mathtt{IPC}$ expressed in the language $\{\wedge, \otimes, \to, \bot\}$. By the proof of Ruitenburg's Theorem there are only five such fixpoints in one variable in $\mathtt{IPC}$:

$$\top,\ \neg\neg x,\ x \otimes \neg x,\ x,\ \bot.$$

See e.g. [18] for a proof of this fact. In particular there is some $n < \omega$ such that $\rho^n(x) \dashv\vdash_{\mathtt{IPC}} \chi$ where $\chi \in \{\top, \neg\neg x, x \otimes \neg x, x,\ \bot\}$. Thus since $\mathtt{IPC} \subseteq \mathtt{InqI}^{\otimes}$ and $x \dashv\vdash_{\mathtt{InqI}^{\otimes}} \rho^n(x)$, it follows that $x \dashv\vdash_{\mathtt{InqI}^{\otimes}} \chi$ for $\chi \in \{\top, \neg\neg x, x \otimes \neg x, x, \bot\}$.

However, as it can be verified using the team semantics of dependence logic, the only one of these formulas χ such that $\chi(x) \leftrightarrow x \in \texttt{InqI}^{\otimes}$ is $\chi(x) = x$. It follows that $\rho(x) = \chi(x) = x$ and thus $\mathsf{core}(\mathcal{A}) = \mathcal{A}$ for all $\mathcal{A} \in \mathbf{Q}$, contradicting the fact that $\texttt{InqI}$ is not closed under uniform substitution. □

Corollary 47. $\texttt{InqI}$ is not finitely representable.

Proof. Since $\mathsf{Schm}(\texttt{InqI}) \supseteq \texttt{IPC}$ it follows immediately that $\mathsf{Schm}(\texttt{InqI})$ is an intermediate logic, thus it is algebraizable (see e.g. [6]). It then follows from Theorem 28 that $\texttt{InqI}$ is not finitely representable. □

It should be investigated in the future whether the schematic fragment of $\texttt{InqI}^{\otimes}$ can be algebraized and thus whether also $\texttt{InqI}^{\otimes}$ is not finitely representable.

Finally, we stress the fact that, by the results of this section, our notion of algebraizability is non-trivial and offers a proper dividing line among weak logics.

5 Matrix Semantics

In this final section we briefly explore matrices as an alternative, non-algebraic semantics for weak logics, and we study the relationship between core semantics and the standard first-order semantics. In the standard setting, it can be shown that every logic is complete with respect to a class of matrices. Furthermore, Dellunde and Jansana [14] have provided a characterization of the class of matrices of a (possibly infinitary) logic in terms of some model-theoretic results for first order logic without equality. Our goal is to show that these results still hold in the present setting, and to that end we shall adapt the matrix semantics to weak logics.

5.1 Completeness of Matrix Semantics

We first briefly recall matrix semantics for standard logics.

Definition 48 (Logical matrix)**.** A *(logical) matrix* of type $\mathcal{L}$ is a pair $(\mathcal{A}, \mathsf{truth}(\mathcal{A}))$ where $\mathcal{A}$ is a $\mathcal{L}$-algebra and $\mathsf{truth}(\mathcal{A}) \subseteq \mathsf{dom}(\mathcal{A})$.

Intuitively, $\mathsf{truth}(\mathcal{A})$ is the "truth set" of the algebra $\mathcal{A}$. Matrices induce a consequence relation over formulae – let $\mathbf{K}$ be a class of matrices and let $\Gamma \cup \{\phi\}$ be a set of propositional formulae – i.e. first-order terms in $\mathcal{L}$, then:

$$\begin{aligned}\Gamma \vDash_{\mathbf{K}} \phi \iff & \text{for all } \mathcal{A} \in \mathbf{K}, h \in \texttt{Hom}(\mathcal{F}m, \mathcal{A}),\\ & \text{if } h[\Gamma] \subseteq \mathsf{truth}(\mathcal{A}), \text{ then } h(\phi) \in \mathsf{truth}(\mathcal{A}).\end{aligned}$$

Given a logic L, we say that $(\mathcal{A}, \mathsf{truth}(\mathcal{A}))$ is a *model* of L and write $(\mathcal{A}, \mathsf{truth}(\mathcal{A})) \vDash L$, if for every $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, if $\Gamma \vdash_L \phi$ then $\Gamma \vDash_{\mathbf{K}} \phi$. We refer the reader to [15, §4] for a detailed study of matrix semantics in the context of standard propositional logics.

We can extend matrix semantics to the setting of weak logics by introducing a further predicate.

Definition 49 (Logical bimatrix)**.** The tuple $(\mathcal{A}, \mathsf{truth}(\mathcal{A}), \mathsf{core}(\mathcal{A}))$ is a *(logical) bimatrix* of type $\mathcal{L}$ if $\mathcal{A}$ is a $\mathcal{L}$-algebra and $\mathsf{truth}(\mathcal{A}), \mathsf{core}(\mathcal{A}) \subseteq \mathsf{dom}(\mathcal{A})$.

Bimatrices induce a consequence relation analogous to that of expanded algebras by restricting attention to assignments over core elements. We write $\mathtt{Hom}^c(\mathcal{F}m, \mathcal{A})$ for the set of all assignments $h : \mathcal{F}m \to \mathcal{A}$ such that $h[\mathtt{Var}] \subseteq \mathsf{core}(\mathcal{A})$. Then, for a class $\mathbf{K}$ of bimatrices and a set of propositional formulae $\Gamma \cup \{\phi\}$, let:

$$\Gamma \vDash^c_{\mathbf{K}} \phi \Longleftrightarrow \text{for all } \mathcal{A} \in \mathbf{K}, h \in \mathtt{Hom}^c(\mathcal{F}m, \mathcal{A}), \\ \text{if } h[\Gamma] \subseteq \mathsf{truth}(\mathcal{A}), \text{ then } h(\phi) \in \mathsf{truth}(\mathcal{A}).$$

Given a weak logic L, we say that $(\mathcal{A}, \mathsf{truth}(\mathcal{A}), \mathsf{core}(\mathcal{A}))$ is a *model* of L, which we write as $(\mathcal{A}, \mathsf{truth}(\mathcal{A}), \mathsf{core}(\mathcal{A})) \vDash L$, if for every $\Gamma \cup \{\phi\} \subseteq \mathcal{F}m$, if $\Gamma \vdash_L \phi$ then $\Gamma \vDash^c_{\mathbf{K}} \phi$. Thus, the main intuition behind bimatrices is the same of expanded algebras: we add a new predicate specifying the core of the matrix in order to preserve only the assignments sending atomic formulae to elements of the core.

Let $\mathbf{K}$ be a class of bimatrices in language $\mathcal{L}$, then we let:

$$\mathsf{Log}(\mathbf{K}) := \{(\Gamma, \phi) \in \mathcal{L} : \Gamma \vDash^c_{\mathbf{K}} \phi\};$$

where $\Gamma \cup \{\phi\}$ is a set of formulae in the given signature $\mathcal{L}$ and $\vDash^c_{\mathbf{K}}$ is the consequence relation defined above. It turns out that the set $\mathsf{Log}(\mathbf{K})$ is a weak logic — bimatrices are thus a natural source of several kinds of weak logics.

Proposition 50. *For any class of bimatrices* $\mathbf{K}$*, the set* $\mathsf{Log}(\mathbf{K})$ *is a weak logic.*

Proof. For all $\phi \in \Gamma$ we have that $\Gamma \vDash^c_{\mathbf{K}} \phi$ and that if $\Gamma \vDash^c_{\mathbf{K}} \phi$, for all $\phi \in \Delta$ and $\Delta \vDash^c_{\mathbf{K}} \psi$, then $\Gamma \vDash^c_{\mathbf{K}} \psi$. Hence, $\mathsf{Log}(\mathbf{K})$ is a consequence relation.

We now show that $\mathsf{Log}(\mathbf{K})$ is closed under atomic substitutions. Let $\mathcal{M} \in \mathbf{K}$ and suppose $\Gamma \vDash^c_{\mathcal{M}} \phi$. Let $\{q_i\}_{i \le n}$ enumerate the variables in $\Gamma \cup \{\phi\}$, we need to show that $\Gamma[\vec{p_i}/\vec{q_i}] \vDash^c_{\mathcal{M}} \phi[\vec{p_i}/\vec{q_i}]$. Suppose this is not the case, then for some valuation $h : \mathcal{F}m \to \mathcal{M}$ such that $h[\mathtt{Var}] \subseteq \mathsf{core}(\mathcal{M})$, we have $h[\Gamma[\vec{p_i}/\vec{q_i}]] \subseteq \mathsf{truth}(\mathcal{M})$ and $h[\phi[\vec{p_i}/\vec{q_i}]] \notin \mathsf{truth}(\mathcal{M})$. Define $k : \mathcal{F}m \to \mathcal{M}$ by letting $k(q_i) = h(p_i)$ for all $i \le n$ and $k(x) = h(x)$ otherwise. We then have that $k[\mathtt{Var}] = h[\mathtt{Var}] \subseteq \mathsf{core}(\mathcal{M})$ and also $k[\Gamma] = h[\Gamma[\vec{p_i}/\vec{q_i}]] \subseteq \mathsf{truth}(\mathcal{M})$ and $k(\phi) = h[\phi[\vec{p_i}/\vec{q_i}]] \notin \mathsf{truth}(\mathcal{M})$, contradicting $\Gamma \vDash^c_{\mathcal{M}} \phi$. □

By the previous proposition, every class of bimatrices determines a weak logic. More interestingly, we can also show the converse and prove that every weak logic is complete with respect to a class of suitable bimatrices, i.e. every weak logic is the logic of a class of bimatrices.

For every weak logic $\vdash$, we let $\mathcal{M}(\vdash, \Gamma)$ be the bimatrix with domain $\mathsf{dom}(\mathcal{M}(\vdash, \Gamma)) = \mathcal{F}m$ and predicates $\mathsf{truth}(\mathcal{M}(\vdash, \Gamma)) = Cl_{\vdash}(\Gamma)$ and $\mathsf{core}(\mathcal{M}(\vdash, \Gamma)) = \mathsf{core}(\vdash)$. Let $\mathcal{M}(\vdash)$ be the set of all matrices $\mathcal{M}(\vdash, \Gamma)$ for $\Gamma \subseteq \mathcal{F}m$. We can then prove the following result for weak logics, corresponding to the "completeness" theorem of standard matrices [15, Thm. 4.16].

Theorem 51. *Every weak logic* $\vdash$ *is complete with respect to the class of bimatrices* $\mathcal{M}(\vdash)$*.*

Proof. ($\Rightarrow$) Suppose $\Gamma \vdash \phi$ and $\Gamma \nvDash_{\mathcal{M}(\vdash)} \phi$. Then there is a bimatrix $\mathcal{M}(\vdash, \Delta)$ and a homomorphism $h : \mathcal{F}m \to \mathcal{M}(\vdash, \Delta)$ such that $h[\mathtt{Var}] \subseteq \mathsf{core}(\vdash)$, $h[\Gamma] \subseteq \mathsf{truth}(\mathcal{M}(\vdash, \Delta))$ and $h(\phi) \notin \mathsf{truth}(\mathcal{M}(\vdash, \Delta))$. Let $x_0, \dots, x_n$ list all variables in $\Gamma \cup \{\phi\}$ and let $\psi_i = h(x_i)$ for all $i \le n$. Since $\mathsf{dom}(\mathcal{M}(\vdash, \Gamma)) = \mathcal{F}m$ we then have that $h[\Gamma] = \Gamma[\psi_i/x_i]$, $h(\phi) = \phi[\psi_i/x_i]$ and consequently, $\Delta \vdash h[\Gamma] = \Gamma[\psi_i/x_i]$. Since $\Gamma \vdash \phi$ and ψ_i is core formula for all $i \le n$, it follows that $\Gamma[\psi_i/x_i] \vdash \phi[\psi_i/x_i]$ and thus by transitivity $\Delta \vdash \phi[\psi_i/x_i] = h(\phi)$, which contradicts $h(\phi) \notin \mathsf{truth}(\vdash, \Gamma)$.

($\Leftarrow$) Suppose $\Gamma \vDash_{\mathcal{M}(\vdash)} \phi$ and let $h : \mathcal{F}m \to \mathcal{M}(\vdash, \Gamma)$ be the identity map $h : x \mapsto x$. Then clearly $h[\mathtt{Var}] \subseteq \mathsf{core}(\vdash)$ and $h[\Gamma] \subseteq \mathsf{truth}(\mathcal{M}(\vdash, \Gamma))$. We thus obtain $h(\phi) \in \mathsf{truth}(\mathcal{M}(\vdash, \Gamma))$, hence $\phi \in Cl_{\vdash}(\Gamma)$ and $\Gamma \vdash \phi$. □

So far, we have established that every class of bimatrices defines a weak logic, and also that every weak logic is complete with respect to a suitable class of bimatrices. It is then natural to wonder what is exactly the class of bimatrices defined by a weak logic L, i.e. the bimatrices $\mathcal{M}$ such that $\mathcal{M} \vDash^c \Gamma$ entails $\mathcal{M} \vDash^c \phi$ whenever $(\Gamma, \phi) \in L$. We shall consider this problem in the following section by firstly providing a suitable translation of logics into Horn theories without equality.

5.2 Connections to Model Theory without Equality

In [14] Dellunde and Jansana provided a study of standard propositional logics by relating them to Horn Theories in first-order logic without equality. In particular, by developing some model theory for this fragment of first-order (and infinitary) logics, Dellunde and Jansana provided a novel proof of previous results by Czelakowski [13], which characterized the class of matrices complete with respect to a logic. We show in this section that their framework is expressive enough to capture also weak propositional logics and it allows us to extend Czelakowski's result to the context of weak logics.

Firstly, we remark that the consequence relations that we have defined for matrices and bimatrices can be easily translated in terms of the standard first-order satisfaction relation $\vDash$. The following proposition makes it clear the correspondence between these two notions.

Proposition 52.

(i) Let $\mathbf{K}$ *be a class of matrices,* $\Gamma \cup \{\phi\} \subseteq \mathcal{F}m$ *with* $|\Gamma \cup \{\phi\}| < \omega$*, then:*

$$\Gamma \vDash_{\mathbf{K}} \phi \Longleftrightarrow \mathbf{K} \vDash \forall x_0, \ldots, \forall x_n \Big(\bigwedge_{\gamma_i \in \Gamma} \mathsf{truth}(\gamma_i(\vec{x})) \to \mathsf{truth}(\phi(\vec{x})) \Big).$$

(ii) Let $\mathbf{K}$ *be a class of bimatrices,* $\Gamma \cup \{\phi\} \subseteq \mathcal{F}m$ *with* $|\Gamma \cup \{\phi\}| < \omega$*, then:*

$$\Gamma \vDash^c_{\mathbf{K}} \phi \Longleftrightarrow \mathbf{K} \vDash \forall x_0, \ldots, \forall x_n \Big(\bigwedge_{\gamma_i \in \Gamma} \mathsf{truth}(\gamma_i(\vec{x})) \wedge \bigwedge_{j \leq n} \mathsf{core}(x_j) \to \mathsf{truth}(\phi(\vec{x})) \Big).$$

Proof. Immediate from the definition of $\vdash_{\mathbf{K}}$ and $\vDash^c_{\mathbf{K}}$. □

We recall that a *basic Horn formula* is a formula of the form $\bigvee_{i<l} \psi_i$, where every ψ_i is a literal and at most one of them is atomic. A Horn formula $\bigvee_{i<l} \psi_i$ is *strict* if exactly one ψ_i is atomic. A *universal Horn formula* is then a formula $\forall x_0, \ldots, \forall x_n \bigwedge_{j \leq k} \psi_j(\vec{x})$, where every ψ_j is a basic Horn formula. A universal Horn formula $\forall x_0, \ldots, \forall x_n \bigwedge_{j \leq k} \psi_j(\vec{x})$ is *strict* if every ψ_j is strict. A first order theory T is a *Horn theory* if it is axiomatized by universal Horn formulae. (We refer the reader also to [17] for a study of Horn theories). It is straightforward to see that Proposition 52 provides us with a translation of pairs (Γ, ϕ) into universal Horn formulae. This observation motivates the translation of finitary logics into (strict) Horn theories as described in [14]. We note that Dellunde and Jansana's results are more general and cover both finitary and infinitary logics, but we will be concerned only with the finitary case.

In the standard case, a (finitary) propositional logic $\vdash$ can be translated into a (strict) Horn theory without equality via the following translation ρ:

$$\rho : (\Gamma, \phi) \in \vdash \longmapsto \forall x_0, \ldots, \forall x_n \Big(\bigwedge_{\gamma_i \in \Gamma} \mathsf{truth}(\gamma_i(\vec{x})) \to \mathsf{truth}(\phi(\vec{x})) \Big).$$

Similarly, a weak logic $\vdash$ can be translated into a (strict) Horn theory without equality via the following translation:

$$\rho : (\Gamma, \phi) \in \vdash \longmapsto \forall x_0, \ldots, \forall x_n \Big(\bigwedge_{\gamma_i \in \Gamma} \mathsf{truth}(\gamma_i(\vec{x})) \wedge \bigwedge_{j \leq n} \mathsf{core}(x_j) \to \mathsf{truth}(\phi(\vec{x})) \Big).$$

The only difference being that in the translation of weak logics we also make explicit the fact that the interpretation of variables ranges only over the core elements of the underlying models.

We associate every weak logic $\vdash$ to a corresponding strict universal Horn theory without equality $\mathsf{Horn}(\vdash)$, defined by letting $\rho(\Gamma, \phi) \in \mathsf{Horn}(\vdash)$ whenever $\Gamma \vdash \phi$. It then follows by Proposition 52 and by the finitarity of $\vdash$ that a bimatrix $\mathcal{M}$ satisfies $\mathsf{Horn}(\vdash)$ if and only if $\mathcal{M} \vDash^c \Gamma$ entails $\mathcal{M} \vDash^c \phi$ whenever $(\Gamma, \phi) \in L$. We say that $\mathsf{Mod}(\mathsf{Horn}(\vdash))$ is the class of bimatrices defined by a weak logic L and we also write $\mathsf{Mod}(\vdash)$ for it.

Now, whilst Czelakowski's original approach in [12] was specifically tailored to logical matrices, Dellunde and Jansana considered arbitrary model classes axiomatized by Horn theories without equality, thus making it possible to apply their results to the setting of bimatrices and expanded algebras. We denote by $\mathcal{L}^-$ the language and the set of formulae in some signature $\mathcal{L}$ which do not include equality symbols. We say that a function $f : \mathcal{A} \to \mathcal{B}$ is a *strict homomorphism* if it is a homomorphism and for all n-ary relation symbols $R \in \mathcal{L}$ and tuples $a \in \mathcal{A}^n$, $a \in R^{\mathcal{A}}$ if and only if $f(a) \in R^{\mathcal{B}}$ — this generalizes the notion of strict homomorphism between expanded algebras that we introduced in Section 1. Let $\mathbb{H}$ be the closure operator associating a class of models to the class of its strict homomorphic images and $\mathbb{H}^{-1}$ be the operator associating a class of models to the class of its strict homomorphic pre-images. The operators $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U$ are the usual closure operators under isomorphism, submodels, products and ultraproducts, with predicate symbols treated as we made explicit in Section 1 for the restricted case of the core predicate. We recall only the finitary version of [14, Thm. 9]:

Theorem 53 (Dellunde, Jansana)**.** *Let* $\mathbf{C}$ *be a class of* $\mathcal{L}^-$*-structures, then the following are equivalent:*

1. $\mathbf{C}$ *is axiomatised by strict universal Horn formulae in* $\mathcal{L}^-$*;*
2. $\mathbf{C}$ *is closed under the operators* $\mathbb{H}^{-1}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U$ *and contains a trivial structure;*
3. $\mathbf{C} = \mathbb{H}^{-1}\mathbb{H}\mathbb{S}\mathbb{P}\mathbb{P}_U(\mathbf{K})$ *for some class* $\mathbf{K}$ *of* $\mathcal{L}$*-structures containing a trivial structure.*

It follows from the previous theorem that the validity of Horn formulae is closed under the operators $\mathbb{H}^{-1}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_U$ (See [14]). Moreover, under the translation we have given above, every weak logic $\vdash$ induces a corresponding universal Horn theory without equality $\mathsf{Horn}(\vdash)$. We can then obtain a characterization of the class of models $\mathsf{Mod}(\vdash)$.

Corollary 54. Let $\mathbf{C}$ be a class of bimatrices and let $T = \mathsf{Horn}(\mathsf{Log}(\mathbf{C}))$, then:

$$\mathsf{Mod}(T) = \mathbb{H}^{-1}\mathbb{H}\mathbb{S}\mathbb{P}\mathbb{P}_U(\mathbf{C}').$$

where $\mathbf{C}'$ is $\mathbf{C}$ together with some trivial bimatrices.

Recall that by Theorem 51 every weak logic is complete with respect to a suitable class of bimatrices and so the previous theorem applies to all weak logics $\vdash$ and provides us with a characterization of the class of all bimatrices defined by $\vdash$.

However, one can see that such classes will contain many pathological examples. For example, let $\mathbf{C}$ be the class of all `InqB`-algebras, and consider them as bimatrices by letting, for all

$\mathcal{A} \in \mathbf{C}$, $\mathsf{truth}(\mathcal{A}) = \{x \in \mathcal{A} : x = 1\}$ and $\mathsf{core}(\mathcal{A}) = \{x \in \mathcal{A} : x = \neg\neg x\}$. Then, we have by the previous corollary that $\mathsf{Mod}(\mathtt{InqB}) = \mathbb{H}^{-1}\mathbb{HSPP}_U(\mathbf{C})$, giving us a class of bimatrices which is strictly larger that the class of expanded algebras which forms the equivalent algebraic semantics of $\mathtt{InqB}$. The reason is that the strict homomorphic preimage of an inquisitive algebra is not necessarily an inquisitive algebra.

To obtain a characterization of the non-pathological models of a (strict) universal Horn theory without equality, and thus of propositional logics, Dellunde and Jansana introduce so-called *reduced structures* and *reduced matrices*. Given the generality of their approach, it is again straightforward to adapt their results to our current setting of bimatrices. Let $\mathcal{M}$ be a first-order structure and $\mathcal{D} \subseteq \mathcal{M}$, we let the *type without equality of a over $\mathcal{D}$ in $\mathcal{M}$* be the following set of equality-free formulae:

$$\mathrm{tp}^{-}_{\mathcal{M}}(a/D) = \{\phi(x) \in \mathcal{L}^{-}(X) : (\mathcal{M}, d)_{d\in D} \vDash \phi(a)\}$$

where $(\mathcal{M}, d)_{d\in D}$ denotes the expansion of $\mathcal{M}$ with constants for all elements of $\mathcal{D}$. We then define the so-called *Leibniz Equality* $\sim_*$ on $\mathcal{A}$ by letting, for $a, b \in \mathcal{M}$:

$$a \sim_* b \iff \mathrm{tp}^{-}_{\mathcal{M}}(a/\mathcal{M}) = \mathrm{tp}^{-}_{\mathcal{M}}(b/\mathcal{M}).$$

We can then verify that $\sim_*$ is the largest non-trivial consequence relation $\mathcal{M}$, meaning that it is the largest congruence θ over its algebraic reduct such that, for any relation symbol $R \in \mathcal{L}$ and $a, b \in \mathcal{M}$ such that $a\theta b$, $a \in R^{\mathcal{M}}$ if and only if $b \in R^{\mathcal{M}}$. This induces a natural projection $\pi : \mathcal{M} \to \mathcal{M}/\sim_*$ by $\pi : a \mapsto a/\sim_*$. We let $\mathcal{M}^* := \mathcal{M}/\sim_*$ and for any class operator $\mathbb{O}$ we let $\mathbb{O}^*(C) := \{\mathcal{A}^* : \mathcal{A} \in \mathbb{O}(C)\}$. We say that a model or a bimatrix $\mathcal{M}$ is *reduced* if $\mathcal{M} = \mathcal{M}/\sim_*$.

Theorem 53 then entails the following result [14, Thm. 18]. Notice that our formulation differs from the original one as Dellunde and Jansana assume that the closure operators are already closed under isomorphic copies.

Theorem 55 (Dellunde, Jansana)**.** *Let $\mathbf{C}$ be a class of reduced $\mathcal{L}^{-}$-structures, then the following are equivalent:*

1. *$\mathbf{C}$ is the class of reduced structures of a class axiomatised by $\mathcal{L}^{-}$-universal Horn formulae;*
2. *$\mathbf{C}$ is closed under the operators $\mathbb{I}^*, \mathbb{S}^*, \mathbb{P}^*, \mathbb{P}^*_U$;*
3. *$\mathbf{C} = \mathbb{I}^*\mathbb{S}^*\mathbb{P}^*\mathbb{P}^*_U(\mathbf{K})$ for some class $\mathbf{K}$ of $\mathcal{L}^{-}$-structures.*

From the closure of the validity of Horn formulae under $\mathbb{I}^*$, $\mathbb{S}^*$, $\mathbb{P}^*$, $\mathbb{P}^*_U$ we obtain a corresponding corollary for the special case of bimatrices:

Corollary 56. Let $\mathbf{C}$ be a class of reduced bimatrices and let $T = \mathsf{Horn}(\mathsf{Log}(\mathbf{C}))$, then:

$$\mathsf{Mod}^*(T) = \mathbb{I}^*\mathbb{S}^*\mathbb{P}^*\mathbb{P}^*_U(\mathbf{C}).$$

Thus, since every weak logic is complete with respect to some class of bimatrices, we obtain a characterization of the class of reduced bimatrices defined by any weak logic $\vdash$.

Finally, consider a weak logic $\vdash$ which is algebraized by some suitable tuple $(\mathbf{Q}, \Sigma, \tau, \Delta)$, then every expanded algebra $\mathcal{A}$ can be regarded as logical matrix with $\mathsf{truth}(\mathcal{A}) = \{x \in \mathcal{A} : \mathcal{A} \vDash^c \tau(x)\}$ and $\mathsf{core}(\mathcal{A}) = \Sigma[x, \mathcal{A}]$. The next corollary shows that in this case we obtain $\mathsf{Mod}^*(\vdash) = \mathbf{Q}$.

Corollary 57. Let $\vdash$ be an algebraizable weak logic with equivalent algebraic semantics given by $(\mathbf{Q}, \Sigma, \tau, \Delta)$, then $\mathbf{Q} = \mathsf{Mod}^*(\vdash)$.

Proof. Firstly, we show that every $\mathcal{A} \in \mathbf{Q}_{CG}$ is reduced. Let $a, b \in \mathcal{A}$ such that $a \neq b$, since $\mathcal{A}$ is core generated, we have without loss of generality that $a = \alpha(\vec{c})$, $b = \beta(\vec{c})$ for $c_i \in \mathsf{core}(\mathcal{M})$ for all $i \leq n$. By algebraizability, we have that $\vDash^c_{\mathbf{Q}} \alpha(\vec{c}) \approx \alpha(\vec{c})$, hence $\vDash^c_{\mathbf{Q}} \tau(\Delta(\alpha(\vec{c}), \alpha(\vec{c})))$ and so $\Delta(\alpha(\vec{c}), \alpha(\vec{c})) \in \mathsf{truth}(\mathcal{A})$. Similarly, since $\nvDash^c_{\mathbf{Q}} \alpha(\vec{c}) \approx \beta(\vec{c})$, then $\nvDash^c_{\mathbf{Q}} \tau(\Delta(\alpha(\vec{c}), \beta(\vec{c})))$ and so $\Delta(\alpha(\vec{c}), \beta(\vec{c})) \notin \mathsf{truth}(\mathcal{A})$, which shows $\mathrm{tp}^-_{\mathcal{M}}(a/\mathcal{M}) \neq \mathrm{tp}^-_{\mathcal{M}}(b/\mathcal{M})$. We then obtain that $\vdash = \mathsf{Log}(\mathbf{Q}_{CG})$, whence $\mathsf{Mod}^*(\vdash) = \mathbb{I}^*\mathbb{S}^*\mathbb{P}^*\mathbb{P}^*_U(\mathbf{Q}_{CG}) = \mathbf{Q}$. □

Finally, this shows that, in the case of an algebraizable weak logic $\vdash$, the class of reduced matrices defined by $\vdash$ coincide with the class of expanded algebras of the equivalent algebraic semantics.

6 Conclusions and Open Problems

In this article, we introduced weak logics as consequence relations closed under atomic substitutions and we showed how to adapt the standard notion of algebraizability from Blok and Pigozzi to this setting. In Section 2 we proved that the equivalent algebraic semantics of a weak logic is unique and in Section 3 we gave two characterization of the algebraizability of weak logics, one in terms of its schematic variant and one in terms of its lattices of theories. In Section 5 we also briefly touched upon the issue of matrix semantics for weak logics.

The results of this article provide us with a useful extension of the framework of abstract algebraic logic to the setting of logics which do not satisfy the rule of uniform substitution, but which are nonetheless closed under atomic substitutions. The fact that this is a working extension was explored and elaborated upon in Section 4, where we applied our setting to the context of inquisitive and dependence logics. In particular, we showed that the algebraic semantics for classical inquisitive and dependence logic investigated in [1, 2, 30] is unique in the strong sense of Proposition 24.

On the other hand, we have seen in Theorem 46 that the intuitionistic versions of inquisitive and dependence logics are not algebraizable. This fact hints at a first potential generalization and extension of our current work. In fact, even if they are not algebraizable, intuitionistic inquisitive and dependence logic have a workable algebraic semantics [27, 30]. This calls for possible generalization of our framework to account for other classes of weak logics, e.g. pseudo-algebraizable logics. As a matter of fact, it seems that the whole spectrum of refinements of a logic within abstract algebraic logic should have counterparts in the domain of weak logics. In particular, one should consider the case of weak logics whose algebraic semantics consists of expanded algebras whose core is first-order definable, but not definable by a finite set of polynomials Σ, as it seems to be the case of `InqI` [30].

Secondly, one possible object of further research would be to lift the restriction to finitary systems that we considered in this article. On the one hand, as we already remarked, it seems straightforward to extend the present setting to that of weak logics which are truly infinitary. To this end, it might suffice to consider generalized quasi-varieties in place of regular ones. On the other hand, it might be more complex to allow for classes of expanded algebras whose core is defined by a infinite set of polynomials Σ.

A further direction for future research is that of expanding the theory of algebraizable weak logics, in particular by providing a suitable version of the bridge theorems which are usually studied in abstract algebraic logic. For example, can we prove for our setting a version of Czelakowski's result relating logics which satisfy the deduction detachment theorem to quasi-varieties satisfying the relative congruence extension property? In a similar spirit, we can ask whether it is possible to improve Theorem 34 and obtain a version of Blok and Pigozzi's isomorphism theorem for suitable core filters and core congruences.

Also, we have remarked in the proof of the non-algebraizability of $\mathtt{InqI}^{\otimes}$ the connection between polynomials defining the core of an algebra and the fixed points of the corresponding logic. This connection seems fruitful and should be pushed further. For example, can we classify all logics with a given schematic fragments in terms of a corresponding fixed-point formula?

Finally, we believe it is important to the test the applicability of the framework of weak logics by finding other examples of algebraizable (and not algebraizable) weak logics. Can we show that epistemic logics, such as PAL, or provability logics, such as Buss' logic, are algebraizable? We leave these and further problems to future research.

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