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Geometric aspects of reduction for dynamical systems with symmetry

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Introduction

Reduction theory has played a prominent role in the study of dynamical systems in both mathematics and physics, and more specifically in the domain of what nowadays is commonly described as "geometric mechanics". The range of subjects which have an impact or which have benefited from the study of reduction of dynamical systems is too broad to allow for an exhaustive enumeration: symplectic geometry, Lie group theory, integrable systems and stability theory are only a few of the most celebrated examples. The growth in number of papers which relate to this technique in the last decades is the best evidence of the increasing interest of the mathematics and physics community in this topic. We have decided not to include any references in this introductory part, but the reader will be referred to the relevant literature in the course of the manuscript.

Presumably, the strategy of reduction of a dynamical system by means of a group of symmetries is not entirely new to reader. Roughly speaking, one attempts to make use of a symmetry of the system to split the dynamical equations into a set of *reduced equations*, on the one hand, and a set of *reconstruction equations*, on the other hand. The reduced system is typically defined on a space with a lower number of degrees of freedom and, therefore, is expected to be easier to solve. Once a solution of the reduced system is known, a complete solution is obtained by solving the reconstruction equations which encode the relation between the solutions of the reduced and the original system.

In geometric mechanics there are mainly two general reduction theories, corresponding to the two classical formulations of mechanics: Hamiltonian and Lagrangian. In a way, one could say that Hamiltonian reduction aims at reducing the geometric structure underlying the dynamics, whereas Lagrangian reduction mainly focuses on reducing the variational principle. But, as usual, the terminology is debatable. For instance, though the classical procedure of Routh reduction for Lagrangian systems with symmetry may be realized by means of a symplectic reduction on a tangent space, few people will dispute that it is a genuine form of Lagrangian reduction.

The Rigid body. Here is a simple but telling example of how reduction essentially works. In classical treatments of the dynamics of a rigid body spinning freely about a fixed point, one eventually arrives at the following "Lagrangian"

$$l(\omega) = \frac{1}{2}\langle \omega, \mathbb{I}\omega \rangle \,,$$

where $\mathbb{I}$ is the moment of inertia tensor w.r.t. the fixed point, and ω is the angular velocity of the body. The Lagrangian l is not defined in a tangent bundle, and certainly one does not get to the Euler equations for the rigid body by simply writing down the Euler-Lagrange equations in the variables $\omega = (\omega_1, \omega_2, \omega_3)$. There is an elegant reduction technique, the so-called *Euler-Poincaré reduction*, which allows both to correctly interpret l as a Lagrangian function and to derive the associated equations of motion.

Let us briefly describe the Euler-Poincaré reduction method. Let $L : TG \to \mathbb{R}$ be a Lagrangian defined on the tangent bundle of a Lie group G, and assume that L is invariant under the tangent lift of the left translation of G on itself. Denote by $l : \mathfrak{g} \to \mathbb{R}$ the restriction of L to the Lie algebra $\mathfrak{g}$ of G. Then we have the following result, already known to Poincaré in the case of $SO(3)$:

Theorem (Euler-Poincaré reduction). *A curve $g(t)$ in G satisfies the Euler-Lagrange equations for L if, and only if, the curve $\xi(t) = g^{-1}(t)\dot{g}(t)$ satisfies the Euler-Poincaré equations for l:*

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) = ad^*_\xi \frac{\partial l}{\partial \xi} \,.$$

For example, in the case of the rigid body discussed above, the reduced Lagrangian l corresponds to the restriction of $L : TSO(3) \to \mathbb{R}$ to the Lie algebra $(\mathfrak{so}(3), [\cdot,\cdot])$, which in turn is isomorphic to $(\mathbb{R}^3, \wedge)$, where $\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the standard vector product. It is now an easy exercise to show that the Euler-Poincaré equations of l are precisely the Euler equations of the free rigid body with a fixed point $\mathbb{I}\dot{\omega} = \mathbb{I}\omega \wedge \omega$.

The Euler-Poincaré equations obey a reduced variational principle. Indeed, one can show that the variational principle for l reads

$$\delta \int l(\xi)\, dt = 0 \,,$$

with variations $\delta\xi$ of $\xi \in \mathfrak{g}$ of the form the form $\delta\xi = \dot{\zeta} + [\xi, \zeta]$, where $\zeta(t)$ is an arbitrary curve in $\mathfrak{g}$ with vanishing endpoints. These variations can be interpreted as the variations on $\mathfrak{g}$ induced by the arbitrary variations δg of curves in G.

There is a similar story going on in the Hamiltonian picture. One usually obtains the Hamiltonian $h : \mathbb{R}^3 \to \mathbb{R}$ of the rigid body from the Lagrangian l by means of the Legendre transformation. Denoting by $\Pi := \mathbb{I}\,\omega \in \mathbb{R}^3$, one has

$$h(\Pi) = \frac{1}{2}\langle \mathbb{I}^{-1}\Pi, \Pi \rangle \,.$$

Again, one cannot derive the equations of motion corresponding to the Hamiltonian h in the standard way. This is apparent for many reasons, starting with the fact that h is not a function defined in a cotangent bundle. Once more, the reduction theory provides a solution in the framework of the *Lie-Poisson reduction.*

The setting is analogous to the one for the Euler-Poincaré case discussed above. One considers an invariant Hamiltonian on T^*G (where now invariance is understood w.r.t. the cotangent lift of the left translation) and denotes by $h : \mathfrak{g}^* \to \mathbb{R}$ the induced Hamiltonian in the dual of the Lie algebra. Under the well-known identification $T^*G/G \cong \mathfrak{g}^*$, we have the following theorem:

Theorem (Lie-Poisson reduction). *Let X_H be the Hamiltonian vector field on T^*G. Then X_H projects onto the Hamiltonian vector field X_h on $\mathfrak{g}^*$, determined by the following Poisson bracket:*

$$\{f, g\}(\mu) = \left\langle \mu, \left[\frac{\partial f}{\partial \mu}, \frac{\partial g}{\partial \mu}\right]\right\rangle ,$$

where $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$.

The usual way to prove the previous theorem is to invoke the general results on reduction of Poisson manifolds which guarantee that if a Lie group G acts freely and properly on a Poisson manifold $(M, \{\cdot, \cdot\})$ and preserves the Poisson bracket, then the quotient M/G is a Poisson manifold in a natural way: functions on the quotient may be regarded as G-invariant functions on M, and this naturally induces a bracket on the quotient.

In the case of the rigid body, the bracket reads

$$\{f, g\}(\Pi) = \Pi \cdot (\nabla f(\Pi) \wedge \nabla g(\Pi)) ,$$

and this leads to the Euler equations for the rigid body in momentum representation: $\dot{\Pi} = \Pi \wedge \mathbb{I}^{-1}\Pi$. This reflects the general fact that Hamiltonian reduction is typically obtained by reduction of the geometric structure responsible for the dynamics; in the case of the rigid body, this structure is the canonical Poisson structure in $T^*SO(3)$.

Momentum maps and Routh reduction

There is a large class of reduction theories that take into account the conserved quantities associated with the symmetry group G. In this case, one first restricts the attention to the submanifold given by level set of the conserved quantities, and only then quotients by the symmetry group of that submanifold which is, in general, a proper subgroup of G.

The fact that the action of a symmetry group in the case of a Lagrangian or a Hamiltonian system relates to the existence of first integrals of the dynamics is a deep result which has proven to be very fruitful in the modern theories of reduction. One refers to this correspondence between symmetries and conserved quantities as Noether's theorem, although for historical reasons this terminology is usually reserved for the Lagrangian formalism only. In the case where (P, Ω) is a symplectic manifold, one may collect the first integrals of Hamilton equations in a *momentum map* $J : P \to \mathfrak{g}^*$ which is equivariant w.r.t. the coadjoint action of G on $\mathfrak{g}^*$. If we fix a regular value $\mu \in \mathfrak{g}^*$ of J, the submanifold $J^{-1}(\mu)$ is invariant under a certain subgroup G_μ of G, and it is an important result that the space of orbits $J^{-1}(\mu)/G_\mu$ can be endowed with a symplectic structure[a].

The symplectic reduction theorem applies directly to the case of Hamiltonian dynamics, where the symplectic space is the cotangent bundle of the configuration space Q with its canonical symplectic form, namely (T^*Q, Ω_Q). This case encompasses many of the early theories of reduction, such as the reduction by a family of integrals in involution due to Liouville and Jacobi. Surprisingly, although the main results regarding the symplectic reduction of a cotangent bundle have been known in the literature for a few decades, the Lagrangian counterpart has received much less attention, and it is only recently that a genuine interest in tangent bundle reduction has resulted in a number of papers dealing, in one way or another, with the so-called *Routh Reduction*.

The method of Routh is applicable to Lagrangians L with cyclic variables, a situation that we identify today with an Abelian group of symmetries. Routh himself was able to successfully reduce the number of unknowns by making use of the conserved momenta, and derived a criteria for the stability of steady motions (or, in the modern terminology, *relative equilibria)*. This method can be generalized to non-Abelian group actions, and it is precisely within this context that the first part of this thesis is to be situated, where we aim at constructing a geometric framework for this reduction technique.

While doing this, we come across a generalization of the standard Lagrangian systems that we call *magnetic Lagrangian systems*. This class of systems allows for a systematic treatment of the different aspects of Routh reduction by means of what we call *compatible transformations* between two magnetic Lagrangian systems and, in particular, we will show how Routh reduction can be understood as a reduction technique in the "class" of magnetic Lagrangian systems. Although the emphasis of our exposition is put on the Lagrangian formalism, we will also briefly study the Hamiltonian counterpart, that of *magnetic Hamiltonian systems*. As as illustration of the interest of our framework, we will discuss the case of Routh semidirect product reduction in a simple situation.

We would like to point out that our approach is symplectic rather than variational in nature. Of course, the reduction of the variational principle in the context of momentum constraints is an interesting topic in its own right.

Hamilton-Poincaré and Lagrange-Poincaré reduction

The Euler-Poincaré and Lie-Poisson reduction theories, reviewed at the beginning of this introduction, are particular instances of more sophisticated reduction techniques applicable to systems whose phase space is an arbitrary manifold Q, and not necessarily a Lie group G. These reduction theories, known as Hamilton-Poincaré reduction (in

[a]Apparently, many of the ingredients of this modern construction of the symplectic reduced space were known to Lie, including the symplectic nature of the coadjoint orbits. See [MW01] for an overview on the history of symplectic reduction.

the Hamiltonian case) and Lagrange-Poincaré reduction (in the Lagrangian case), do not take into account the possible existence of first integrals, but rather use the full group of symmetries to reduce the cotangent bundle T^*Q and the tangent bundle TQ, respectively.

Nowadays, the standard formulation of these reduction theories is based on the geometry of a Lie algebroid (in the Lagrangian case) or its dual (in the Hamiltonian case). In either case, and unlike in the situation encountered after symplectic reduction, one no longer stays in the category of symplectic manifolds.

In the second part of this thesis we intend to show that it is possible to derive these reduction theories from a symplectic framework. More specifically, we will obtain the reduced equations by "reduction of the Tulcyjew triple". The idea of reducing the Tulcyjew triple is not new in the literature, and we shall briefly comment on these approaches later in Chapter 4. The main novelty in our description lies in the fact that we combine the symplectic reduction theorem with the description of the dynamics in terms of Lagrangian submanifolds in such a way that the reduced triple consists of symplectic manifolds.

Outline of the dissertation

In the **first chapter** we provide the basic mathematical background that will be used throughout this manuscript. Besides recalling the basic definitions of symplectic and Poisson manifolds and introducing the terminology and notations that will be used in the text, this chapter discusses three main topics:

1) *Lie group actions* (Section 1.2): We present the main results concerning the action of a Lie group on a manifold and we recall the standard assumptions that guarantee that the quotient space of a manifold by a Lie group action is again a smooth manifold. We introduce the notion of invariance of a differential form under a Lie group action and, in particular, we define what a *canonical action* of a Lie group on a symplectic manifold is.

2) *Bundles and connections* (Section 1.3): After recalling the basic definition of a connection and of its curvature on a general fibre bundle, we will focus on the case of a principal fibre bundle $P \to P/G$. We introduce the connection 1-form associated to a principal connection on $P \to P/G$, define its curvature and relate them by means of the *Cartan structure equations*. We also review the definition of the adjoint bundle that plays an important role in the Lagrange-Poincaré equations.

3) *Reduction* (Section 1.4): The final section of the first chapter consist of a short overview on the symplectic and Poisson reduction theories. An important part of this thesis builds to a large extent on the results reviewed here. Therefore, we will also include a reasonably self-contained proof of the two main reduction theorems: the Marsden-Weinstein reduction theorem and the Poisson reduction theorem.

The main goal of the **second chapter** is to present a unified version of Routh reduction following the symplectic reduction approach. Section 2.1 reviews the classical method of Routh in classical mechanics and discusses its limitations. Section 2.2 aims at giving an account of the modern theory of Routh reduction. To that end, we begin by proving the following basic results in the theory of cotangent bundle reduction and tangent bundle reduction:

1) Cotangent bundle reduction: Starting from a (free and proper) Lie group action ϕ of G on a manifold Q, one may lift ϕ to a canonical action of G on the symplectic manifold (T^*Q, Ω_Q), where T^*Q denotes the cotangent bundle of Q and Ω_Q is the canonical symplectic form. This action admits an equivariant momentum map $J : T^*Q \to \mathfrak{g}^*$ and it is possible to show that, for any fixed value $\mu \in \mathfrak{g}^*$, there exists a symplectomorphism
$$((T^*Q)_\mu, (\Omega_Q)_\mu) \cong \left(T^*(Q/G) \times_{Q/G} Q/G_\mu, \pi_1^*\Omega_{Q/G} + \pi_2^*\mathcal{B}_\mu\right) ,$$
where we have used the usual notation for symplectic reduced spaces. In the previous expression, $\mathcal{B}_\mu$ denotes the so-called magnetic term (induced by the μ-component of the exterior differential of a chosen connection 1-form) and the maps π_1 and π_2 are the projections given in the following diagram:
$$\begin{tikzcd} T^*(Q/G) \times_{Q/G} Q/G_\mu \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Q/G_\mu \\ T^*(Q/G) & \end{tikzcd}$$

2) Tangent bundle reduction: We rely on the standard symplectic description of a regular Lagrangian system with Lagrangian L on a tangent bundle TQ, given by the symplectic manifold (TQ, Ω_Q^L). Here $\Omega_Q^L = \mathbb{F}L^*\Omega_Q$, $\mathbb{F}L : TQ \to T^*Q$ being the fibre derivative of the Lagrangian L. If there is a G-action action ϕ on Q such that its tangent lift ϕ^{TQ} leaves the Lagrangian L invariant, then the action is Hamiltonian and there is a symplectomorphism:
$$\left((TQ)_\mu, (\Omega_Q^L)_\mu\right) \cong ((T^*Q)_\mu, (\Omega_Q)_\mu) \; .$$

It should be noted, however, that in general the manifold $(TQ)_\mu$ cannot be realized as a fibred product, precisely due to the fact that the momentum map in the tangent bundle involves the fibre derivative of the Lagrangian L. To overcome this difficulty, we review the definition of G-regularity of a Lagrangian L, which guarantees the existence of the desired identification
$$(TQ)_\mu \cong T(Q/G) \times_{Q/G} (Q/G_\mu) \, . \tag{$\star$}$$
It turns out that there is a way to interpret this reduced (Hamiltonian) dynamics as being Lagrangian. This requires a generalization of the standard definition of Lagrangian

systems (on a tangent bundle) to include systems which are defined on a fibred products such as the space $(TQ)_\mu$ in ($\star$), and which possibly include terms of gyroscopic type. The role of the reduced Lagrangian function is played by the *Routhian*, whose definition appears at the end of Section 2.2.

The new contributions of this dissertation begin in Section 2.3, where we describe Routh reduction in the case where the configuration space is a direct product $Q = S \times G$, with S an arbitrary manifold and where G acts on Q by translation on the G factor. We obtain explicit expressions for the reduced dynamics. These expressions are subsequently applied in Section 2.4 to the case of a rigid body with rotors and shown to agree with the known expressions in the literature.

The starting point of the **third chapter** is the concept of *magnetic Lagrangian system*. A magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ consists of the following data: (1) a fibre bundle $\epsilon : E \to Q$, (2) a smooth function L (the Lagrangian) on the fibred product $TQ \times_Q E$, and (3) a closed 2-form $\mathcal{B}$ on E which we will refer to as the *magnetic form* of the system. Within the framework of magnetic Lagrangian systems, we give generalizations to the usual definitions of fibre derivative and *energy* associated with the Lagrangian.

Let us denote by (q^i) a set of coordinates in Q, and by (q^i, r^a) adapted coordinates on E. The dynamics of a magnetic Lagrangian system is given by the following set of *Euler-Lagrange* equations:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} &= \quad \mathcal{B}_{ij}\dot{q}^j + \mathcal{B}_{ia}\dot{r}^a \,, \\ -\frac{\partial L}{\partial r^b} &= -\mathcal{B}_{ib}\dot{q}^i + \mathcal{B}_{ab}\dot{r}^a \,, \end{aligned} \tag{$\star\star$}$$

where $\mathcal{B}_{ij}$, $\mathcal{B}_{ia}$ and $\mathcal{B}_{ab}$ are the components of the magnetic form $\mathcal{B}$. Under some regularity conditions on the Lagrangian L and on the magnetic form $\mathcal{B}$ that are discussed in Section 3.1, the Euler-Lagrange equations ($\star\star$) can be shown to be symplectic with respect to a specific symplectic form $\Omega^{L,\mathcal{B}}$ on $T_EQ := TQ \times_Q E$, where the Hamiltonian function is the energy E_L of the Lagrangian L. A large number of dynamical systems fit into the category of magnetic Lagrangian systems and, in particular, the systems arising from Routh reduction of a Lagrangian system are magnetic Lagrangian systems. There is also a natural notion of *magnetic Hamiltonian system* which will be discussed at the end Section 3.1.

Section 3.2 is devoted to the study of a class of mappings between fibred products that we call *compatible transformations*, and that are used to construct "morphisms" ψ between two given magnetic Lagrangian systems. To construct a compatible transformation $\psi : T_{E_1}Q_1 \to T_{E_2}Q_2$, one starts with a couple of submersions (F, f) making the diagram

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\epsilon^{(1)}"'] & E_2 \arrow[d, "\epsilon^{(2)}"] \\ Q_1 & Q_2 \arrow[l, "f"] \end{tikzcd}$$

commutative, and such that all fibrations in the previous diagram become bundles. We then say that a transformation $\psi : T_{E_1}Q_1 \to T_{E_2}Q_2$ is compatible (w.r.t. the submersions F and f) if it respects the previous scheme. Although the definition of compatible transformation may seem rather restrictive, it becomes apparent later on that the definition is flexible enough to describe many practical situations arising in symmetry reduction of Lagrangian systems with symmetry.

After introducing compatible transformations, we turn to the case where the fibred products correspond to the configuration space of a magnetic Lagrangian system, and study transformations

$$(\epsilon^{(1)} : E_1 \to Q_1, L_1, \mathcal{B}_1) \xrightarrow{\psi} (\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$$

which relate the dynamics of the corresponding magnetic Lagrangian systems. There are many good reasons to explore the properties of these maps in view of possible applications in geometric mechanics. A first example, in the context of Routh reduction, appears when studying the reduction of a Lagrangian system which is invariant by two symmetry groups G and H. The situation is summarized in the following diagram

$$\begin{tikzcd} & TQ \arrow[dl, bend right, "H\text{-red}"'] \arrow[dr, bend left, "G\text{-red}"] & \\ T(Q/H) \times_{Q/H} Q/H_\mu \arrow[rr, dashed, "\psi"'] & & T(Q/G) \times_{Q/G} Q/G_\mu \end{tikzcd}$$

and the natural question is whether one can relate the dynamics on the reduced spaces by means of a compatible transformation ψ. In Section 3.3 we will encounter the previous scheme in the context of semidirect product reduction.

We introduce a class of compatible transformations $\psi_{L_2,\beta}$ depending on the Lagrangian L_2 and on a chosen map $\beta : E_1 \to V^*f$ (where $Vf \subset TQ_2$ is the vertical space w.r.t. the fibration f) such that, if the Lagrangian L_1 and the magnetic form $\mathcal{B}_1$ satisfy a certain relation (depending on $\psi_{L_2,\beta}$, L_2 and $\mathcal{B}_2$), then $\psi_{L_2,\beta}$ preserves the (pre)symplectic structures and the Hamiltonian functions. We combine the previous result with the well known presymplectic constraint algorithm to derive conditions under which the transformation

$\psi_{L_2,\beta}$ relates the solutions of the Euler-Lagrange equations for $(\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$ and $(\epsilon^{(1)} : E_1 \to Q_1, L_1, \mathcal{B}_1)$. At the end of Section 3.2 we give an overview of the corresponding transformations in the framework of magnetic Hamiltonian systems.

One of the most interesting cases corresponds to the case when $\psi_{L_2,\beta}$ is a diffeomorphism. We can then induce a magnetic Lagrangian system on $\epsilon^{(1)} : E_1 \to Q_1$ which is symplectomorphic to the original one. Section 3.3 exhibits a first situation where this occurs in the context of semidirect product reduction by stages. We study the case of a Lagrangian system defined on a product $S \times GV$, where GV denotes the semidirect product of the Lie group G and the vector space V.

In Section 3.4 we use the class $\psi_{L_2,\beta}$ to carry out Routh reduction of a magnetic Lagrangian system defined on a general fibred product. In particular, we show that choosing a suitable β the transformation $\psi_{L_2,\beta}$ may be regarded as a "restriction to the level set of the momentum map". Once the system is restricted to the level set of the momentum map, the reduction by the isotropy group of symmetries is immediate. The resulting *magnetic Routh reduction* is in agreement with the literature, where it appears in the context of Routh reduction by stages.

The **fourth chapter** is mostly concerned with the Hamilton-Poincaré and Lagrange-Poincaré reduction theories. It intends to show that Tulczyjew's symplectic formulation of dynamics on the one hand, and symplectic reduction on the other hand, can be combined into a model for Lagrange-Poincaré reduction and Hamilton-Poincaré reduction within the framework of a reduced Tulczyjew triple.

In Section 4.1 we review the main ingredients of the Tulcyjew triple, for which the notations in the following diagram are used:

$$\begin{tikzcd}
T^*TQ & & TT^*Q \arrow[ll, "\alpha_Q"'] \arrow[rr, "\beta_Q"] \arrow[dl, "T\pi_Q"] \arrow[dr, "\tau_{T^*Q}"'] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"] \\
& TQ \arrow[ul, bend left, "dL"] \arrow[rr, "\mathbb{F}L"'] & & T^*Q \arrow[ur, bend right, "dH"'] &
\end{tikzcd}$$

with $T^*TQ \xrightarrow{\pi_{TQ}} TQ$.

We discuss how Lagrangian submanifolds behave under reduction and, roughly speaking, it is precisely through the reduction of the Lagrangian submanifolds S_H and S_L given by

$$S_H = \beta_Q^{-1}\left(dH(T^*Q)\right), \qquad S_L = \alpha_Q^{-1}\left(dL(TQ)\right),$$

that we obtain, respectively, the Hamilton-Poincaré equations and Lagrange-Poincaré equations. This is the subject of Section 4.2, where we also clarify the equivalence of both descriptions in the regular case. The general theory is illustrated with a concrete example: the Lie-Poisson dynamics.

References. Part of the work presented in this thesis has already been published or accepted for publication. Many of the new results concerning compatible transformations and their applications in the second and third chapter can be found in either [LGTAC12] or [GTALC14]. The intrinsic derivation of the Hamilton-Poincaré and Lagrange-Poincaré equations within the context of the Tulczyjew triple has appeared in [GTAGMM14].

Chapter 1

Preliminaries

The aim of this chapter is to collect, for later use, some standard results concerning Lie groups, Lie group actions on symplectic manifolds, principal bundles, connections and reduction theory. In this way we hope to provide a concise reference for later use and to fix the notations. We will skip the technical aspects, for which reference will be made to the literature.

The content is organized as follows. Section 1.1 recalls some basic definitions and fixes the sign conventions that will be used in this manuscript. Section 1.2 presents some basic facts about actions on manifolds with an eye towards reduction. Section 1.3 reviews some basic results about principal bundles, connections and curvature. Finally, Section 1.4 is a brief exposition of some of the reduction theories which are of special interest in mechanics. In this last section we will pay special attention to the case of symplectic reduction, where the important notion of momentum maps will be introduced.

1.1 Geometric structures and conventions

Before we go on, we need to introduce some standard notations and make some basic assumptions.

Notations on linear spaces. Consider a finite dimensional vector space V. The annihilator of a vector subspace $W \subset V$ is denoted by W^0, and it is defined as:

$$W^0 = \{\alpha \in V^* \mid \langle \alpha, w \rangle = 0, \quad \text{for all } w \in W\} .$$

If $B : V \times V \to \mathbb{R}$ is a bilinear form on V, we write $B^\flat = \flat_B$ for the following map:

$$\begin{aligned} B^\flat : V &\to V^* , \\ v &\mapsto B^\flat(v) = B(v, \cdot) . \end{aligned}$$

We say that B is nondegenerate if $B^\flat : V \to V^*$ is an isomorphism.

A bilinear form Ω which is nondegenerate and skew is called a *symplectic form*, and the pair (V, Ω) is a *symplectic vector space*. In this situation, the *symplectic orthogonal* of a subspace $W \subset V$ is denoted by W^Ω. Its definition is:

$$W^\Omega = \{v \in V \mid \Omega(v, w) = 0, \quad \text{for all } w \in W\} .$$

Geometric structures on manifolds. We will always consider manifolds which are finite dimensional, Hausdorff and second countable.

Definition 1.1. *Let P be a manifold. A* symplectic form Ω *on P is a closed nondegenerate 2-form on P. The pair (P, Ω) is a* symplectic manifold.

Nondegeneracy means that Ω_p is a symplectic form on the vector space T_pP for each $p \in P$.

The celebrated *Darboux Theorem* states that, around each $p \in P$, it is possible to find a chart such that Ω is constant. In particular, using the canonical form for symplectic forms on vector spaces, it follows that around each point $p \in P$ there exist local coordinates $(q^1, \ldots, q^n, p_1, \ldots p_n)$ such that

$$\Omega = dp_i \wedge dq^i \, , \tag{1.1}$$

where the summation over repeated indexes is understood (this convention will be used overall in this thesis).

Given a vector field X on (P, Ω), we say that X is *Hamiltonian* if there is a function f on P such that

$$i_X \Omega = -df \, . \tag{1.2}$$

In this case, we write $X = X_f$. One usually refers to (1.2), or to the associated evolution equation

$$\dot{p} = X_f(p) \, ,$$

as the *Hamilton equations*. The previous convention for the sign of Ω and for the sign in Hamilton equations agrees with [LM87], but differs from some other references that we shall often cite, such as [AM78] or [MR99].

Definition 1.2. *A* Poisson structure *on a manifold P is bilinear map $\{\cdot, \cdot\}$ on the algebra of smooth functions $\mathcal{C}^\infty(P)$, such that:*

1) It is skew symmetric.

2) It obeys Leibniz's rule: $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

3) It satisfies the Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$.

We call $\{\cdot,\cdot\}$ the *Poisson bracket* and the pair $(P,\{\cdot,\cdot\})$ is said to be a *Poisson manifold*. By the derivation property of the bracket, for each $f \in \mathcal{C}^\infty(P)$ there exists a *Hamiltonian vector field* X_f such that

$$X_f(\cdot) = \{f,\cdot\}\,.$$

Any symplectic manifold (P,Ω) admits a Poisson bracket defined by $\{f,g\} = \Omega(X_f, X_g)$, and the Hamiltonian vector field w.r.t. this bracket coincides with the one obtained from (1.2).

From the definition of the Poisson bracket $\{\cdot,\cdot\}$, one can show that there exists a bivector $\Lambda \in \Lambda^2(TM)$ such that

$$\{f,g\} = \Lambda(df,dg)\,.$$

We will denote by $\Lambda^\sharp : T^*M \to TM$ (or sometimes $\sharp_\Lambda$) the associated vector bundle morphism defined as by the following relation:

$$\langle \beta, \Lambda^\sharp(\alpha)\rangle = \Lambda\left(\alpha,\beta\right)\,,$$

for each $\alpha,\beta \in T^*P$. The *characteristic distribution* $\mathcal{S}(P)$ of the Poisson manifold $(P,\{\cdot,\cdot\})$ is the generalized distribution defined by $\mathcal{S}(P) = \Lambda^\sharp(T^*M)$. A very well known result states the integrability (in the sense of Sussmann and Stefan) of this generalized distribution. Each leaf of the resulting foliation has a unique symplectic structure such that its canonical immersion in P is a Poisson map (see Definition 1.3 below). We refer to [Vai94] for details and references.

Definition 1.3. *A smooth map $f : P_1 \to P_2$ between two Poisson manifolds $(P_1,\{\cdot,\cdot\}_{P_1})$ and $(P_2,\{\cdot,\cdot\}_{P_2})$ is a* Poisson map *if it preserves Poisson brackets, namely:*

$$f^*\{g,h\}_{P_2} = \{f^*g, f^*h\}_{P_1}\,,$$

for all $g,h \in \mathcal{C}^\infty(P_2)$.

Lie groups. A *Lie group* is a manifold G with a group structure such that the group multiplication $(g,h) \mapsto g\cdot h = gh$ and inversion $g \mapsto g^{-1}$ are smooth maps. The *left translation* is the map

$$\begin{aligned} L_g : G &\to G\,,\\ h &\mapsto L_g(h) = gh\,. \end{aligned}$$

Similarly we define the *right translation* R_g. We use the standard shorthand notation for the tangent maps of L_g and R_g: if $v_h \in T_hG$, we write $gv_h = T_hL_g(v_h)$ and $v_hg = T_hR_g(v_h)$.

We identify the Lie algebra $\mathfrak{g}$ of G with the set of left invariant vector fields on G. The bracket on G is denoted by $[\cdot,\cdot]_{\mathfrak{g}}$, or more often $[\cdot,\cdot]$ when there is no risk of confusion. Finally, we write $\exp : \mathfrak{g} \to G$ of the exponential map of the Lie algebra.

1.2 Lie group actions

A left action of a Lie group G on a manifold M is a (smooth) mapping $\phi : G \times M \to M$ which satisfies:

i) $\phi(e, m) = m$.

ii) $\phi(g, \phi(h, m)) = \phi(gh, m), \quad$ for all $g, h \in G$.

Intuitively, an action is a group of transformations in a manifold which is compatible with the group structure of G. Actions will be an important ingredient in the forthcoming chapters because they provide a formal way to deal with the notion of a continuous symmetry of a system (such as a rotation).

From the conditions above one can show that $\phi_g = \phi(g, \cdot) : M \to M$ is a diffeomorphism for each $g \in G$, whose inverse is $\phi_{g^{-1}}$. A right action ψ is defined in a similar way as a map $\psi : M \times G \to M$ which satisfies $\psi(\psi(m, h), g) = \psi(m, hg)$. In what follows, and without loss of generality, we will always work with left actions: if ϕ was a right action, one can associate with it the left action $g \mapsto \phi_{g^{-1}}$ (and vice-versa). A convenient notation for a (left) Lie group action is $gm = g \cdot m := \phi(g, m)$. This notation has many advantages and brings no risk of confusion: if $g, h \in G$, we can write an expression such as $g \cdot h \cdot m$ without ambiguity. A manifold M with an action ϕ is sometimes called a *G-space* and denoted by (M, ϕ).

The *infinitesimal generator* of the action ϕ corresponding to an element $\xi \in \mathfrak{g}$ is the vector field on M defined as

$$\xi_M(m) = \left.\frac{d}{dt}\right|_{t=0} \phi\left(\exp(t\xi), m\right) = \left.\frac{d}{dt}\right|_{t=0} \left(\exp(t\xi) \cdot m\right) .$$

One usually refers to the association $\mathfrak{g} \mapsto \xi_M$ as the *infinitesimal action.* It is a Lie algebra antihomomorphism from $\mathfrak{g}$ to $\mathfrak{X}(M)$, i.e., for all $\xi, \eta \in \mathfrak{g}$ we have

$$[\xi, \eta]_M = -\left[\xi_M, \eta_M\right],$$

and it defines a Lie algebra action in the usual sense (cf. [OR04]), for which we will write

$$\begin{aligned} \sigma_m : \mathfrak{g} \times M &\to TM\,, \\ (\xi, m) &\mapsto \xi_M(m)\,, \end{aligned} \tag{1.3}$$

which is the so-called *infinitesimal generator map.*

The *stabilizer* or *isotropy subgroup* of $m \in M$ is the subgroup

$$G_m = \{g \in G \mid gm = m\}\,,$$

which is closed and, therefore, a Lie subgroup of G. Its Lie algebra is often referred to as the *isotropy algebra* or the *symmetry algebra* of m, and it is denoted by $\mathfrak{g}_m$.

Definition 1.4. *An action on M is said to be:*

1) Free *if $G_m = \{e\}$ for each $m \in M$.*

2) Proper *if the map*

$$\begin{aligned} G \times M &\to M \times M\,,\\ (g,m) &\mapsto (m, gm)\,, \end{aligned}$$

is proper (i.e., the preimage of every compact set is compact).

Note that a proper action is not defined to be an action ϕ such that $\phi : G \times M \to M$ is proper. The usual characterization of proper actions is as follows: if the sequences $\{m_n\}$ and $\{g_n m_n\}$ converge in M, then $\{g_n\}$ has a convergent subsequence. In particular, this condition is satisfied if G is compact.

An action ϕ on a symplectic manifold (M, Ω) is said to be *canonical* or *symplectic* if Ω is invariant under the action, namely: if for all g we have $\phi_g^* \Omega = \Omega$. We will later address the case of canonical actions in more detail when we discuss reduction (see Section 1.4).

The adjoint action. The *inner automorphism* of G is the map $I_g : G \to G$ given by $I_g(h) = ghg^{-1}$ for $h \in G$. Differentiating at the identity we obtain the *adjoint action*:

$$\begin{aligned} Ad_g : \mathfrak{g} &\to \mathfrak{g}\,,\\ \xi &\mapsto Ad_g \xi = g\xi g^{-1}\,. \end{aligned}$$

Note that $(gh)\xi(gh)^{-1} = g(h\xi h^{-1})g^{-1}$ or, in other words $Ad_{gh}(\cdot) = Ad_g\left(Ad_h(\cdot)\right)$, and therefore Ad is a left action on $\mathfrak{g}$. Related to the adjoint action one defines the *coadjoint action* of G on $\mathfrak{g}^*$ as the inverse dual of the adjoint action, namely

$$\begin{aligned} Coad_g : \mathfrak{g}^* &\to \mathfrak{g}^*\,,\\ \mu &\mapsto Coad_g \mu = Ad^*_{g^{-1}} \mu\,. \end{aligned}$$

It is easily checked to be a left action (inverse and dual change left to right and vice-versa). Note that Ad_g and $Coad_g$ are the tangent and cotangent lift of the inner automorphism I_g at the identity and, due to the linearity of these lifts, they are representations in the usual sense. $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ is called the *adjoint representation* (of the group G on the vector space $\mathfrak{g}$) and $Coad : G \times \mathfrak{g}^* \to \mathfrak{g}^*$, referred to as the *coadjoint representation*, is its dual representation (in the sense of representation theory).

One can compute the infinitesimal generators corresponding to the adjoint and coadjoint action which, in agreement with the notations before, are denoted by $\xi_{\mathfrak{g}}$ and $\xi_{\mathfrak{g}^*}$, respectively. For an element $\eta \in \mathfrak{g}$, some computation leads to (see [HSS09])

$$\xi_{\mathfrak{g}}(\eta) = \left.\frac{d}{dt}\right|_{t=0} \left(Ad_{\exp(t\xi)}\eta\right) = [\xi, \eta]\,. \tag{1.4}$$

It is common to denote the infinitesimal generator map of the adjoint action as ad_ξ, and therefore the previous relation (1.4) simply reads $ad_\xi(\cdot) = [\xi, \cdot]$. Using this notation and writing $\xi_{\mathfrak{g}^*}$ for the generator of the coadjoint action, an easy computation shows that $\xi_{\mathfrak{g}^*} = -ad_\xi^*$, i.e., for each $\eta \in \mathfrak{g}$ we have $\langle \xi_{\mathfrak{g}^*}(\cdot), \eta \rangle = -\langle \cdot, [\xi, \eta] \rangle$.

Another property of the infinitesimal generators that we will use later is the following:

Proposition 1.5. $(Ad_g\xi)_M = (\phi_{g^{-1}})^*\xi_M$.

The proof of the previous proposition is based on the identity $\exp\left(Ad_g\xi\right) = g\left(\exp(\xi)\right)g^{-1}$.

Invariance and the case of connected Lie groups. An interesting case is that of actions by connected Lie groups. This situation is often encountered in reduction theories in mechanics, where one typically restricts the attention to the connected component of the identity $G^0 \subset G$. As far as computations are concerned, we will see below that in case of an action of a connected Lie group, invariance of a differential form (or a tensor field) under the Lie derivative with respect to the infinitesimal generators of the action suffices to conclude invariance under the group action. In particular, this applies to symplectic forms.

Lemma 1.6. *Any open subgroup of a Lie group is closed.*

Proof. Let $H \subset G$ be an open subset. It is clear that gH is open for each $g \in G$. If $k \in G \setminus H$, it follows that the open set kH satisfies $kH \subset G \setminus H$, hence

$$G \setminus H = \bigcup_{k \in G \setminus H} (kH),$$

which is open. □

Lemma 1.7. *Let G be a connected Lie group and $U \subset G$ an open neighborhood of the identity. Then G is finitely generated by U (each element $g \in G$ is written as a finite product of elements in U).*

Proof. We need to show that $G = \bigcup_{n=1}^{\infty} U^n$, where $U^n = \{g_1 \cdot \ldots \cdot g_n \mid g_i \in U\}$. Define the (open) set $V = \bigcup_{n=1}^{\infty} U^n$, which is easily seen to be a subgroup of G. Applying Lemma 1.6 the result follows. □

The previous two results hold for any topological group. In the case of Lie groups, considering an open $V \subset \mathfrak{g}$ such that $\exp : V \to \exp(V) \subset G$ becomes a diffeomorphism it follows that any element $g \in G^0$ is a finite product of exponentials, and we write

$$g = \prod_{i=1}^{n} (\exp(\xi_i)),$$

for some $\xi_i \in \mathfrak{g}$. This property characterizes canonical actions in terms of Lie derivatives:

Proposition 1.8. *Let ϕ be an action of a connected Lie group G on the symplectic manifold (M, Ω). Then ϕ is canonical if, and only if, $\pounds_{\xi_M}\Omega = 0$ for all $\xi \in \mathfrak{g}$.*

Proof. $\boxed{\rightarrow}$ Using that the flow $\{\varphi_t\}$ of ξ_M around $m \in M$ is $\exp(t\xi)m$, it follows

$$\pounds_{\xi_M}\Omega = \frac{d}{dt}\bigg|_0 (\varphi_t^*\Omega) = \frac{d}{dt}\bigg|_0 (\phi^*_{\exp(t\xi)}\Omega) = \frac{d}{dt}\bigg|_0 \Omega = 0\,.$$

$\boxed{\leftarrow}$ From $\pounds_{\xi_M}\Omega = 0$ we observe that the flow of ξ_M satisfies $\varphi_t^*\Omega = \phi^*_{\exp(t\xi)}\Omega = \Omega$. Take an arbitrary element $g \in G$. In view of the assumed connectedness of G, we obtain from the previous lemma that $g = \prod_{i=1}^n (\exp(\xi_i))$ for some $\xi_i \in \mathfrak{g}$. Then it follows that:

$$\begin{aligned}\phi_g^*\Omega &= \phi^*_{(\exp(\xi_1))\ldots(\exp(\xi_n))}\Omega = \phi^*_{(\exp(\xi_n))}\cdots\phi^*_{(\exp(\xi_2))}\phi^*_{(\exp(\xi_1))}\Omega \\ &= \phi^*_{(\exp(\xi_n)}\cdots\phi^*_{(\exp(\xi_2)}\Omega = \cdots = \Omega\,.\end{aligned}$$

□

Similar statements hold for forms and vector fields. For example a function f on M is invariant if, and only if, $\pounds_{\xi_M} f = 0$ for all $\xi \in \mathfrak{g}$, and a vector field $X \in \mathfrak{X}(M)$ is invariant if, and only if, the bracket with all generators vanishes: $[\xi_M, X] = 0$ for all $\xi \in \mathfrak{g}$. The latter condition is often used to construct vector fields that are invariant under the action.

A map f between two G-spaces (M, ϕ^M) and (N, ϕ^N) is said to be *equivariant* if it satisfies $f(gm) = gf(m)$, namely if the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi_g^M"'] & N \arrow[d, "\phi_g^N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

It is clear that an equivariant map $f : M \to N$ sends infinitesimal generators to infinitesimal generators,

$$Tf(\xi_M(x)) = \xi_N(f(x))\,.$$

Using the same argument as before, the converse is true if the Lie group is connected.

Orbits and quotients manifolds. If $m \in M$, the *orbit* of m under an action of G on M is defined as the set

$$\mathcal{O}_m = \{gm \mid g \in G\}\,,$$

which is an immersed submanifold. The set of orbits defines an equivalence relation $\sim_G$ on M as follows: $m, m' \in M$ are related if there exists an element $g \in G$ such that

$m = gm'$ (equivalently, if $\mathcal{O}_m = \mathcal{O}_n$). We write M/G for the quotient space (i.e. the set of orbits), whose elements are denoted by $[m]_G$, or simply by $[m]$ when there is no risk of confusion. In other words

$$M/G = \{[m] \mid m \in M\}\,.$$

The orbit $\mathcal{O}_m$ is the image of the map $\theta_m : G \to M$ which sends $g \in G$ to $gm \in M$. Clearly, this map is not injective unless $G_m = \{e\}$ and, in particular, the orbit of a point may have self-intersections. Under the assumption of properness of the action, however, the orbits are regular submanifolds.

Proposition 1.9. *Consider the following map:*

$$\begin{aligned} \bar{\theta} : G/G_m &\to \mathcal{O}_m \subset M\,, \\ [g]_{G_m} &\mapsto gm\,. \end{aligned}$$

Then:

1) *The map $\bar{\theta}_m$ is an injective immersion (in particular, $\mathcal{O}_m$ is an immersed submanifold).*

2) *If the action is proper, then $\bar{\theta}_m$ is a diffeomorphism ($\mathcal{O}_m$ is then an embedded submanifold).*

For a proof, see e.g. [AM78, OR04]. From now on, we consider the orbit $\mathcal{O}_m$ endowed with the manifold structure such that $\bar{\theta}_m$ becomes a diffeomorphism. For a proof of the next result, see [OR04].

Proposition 1.10. *The tangent space at m' to the orbit $\mathcal{O}_m$ is:*

$$T_{m'}\mathcal{O}_m = \{\xi_M(m') \mid \xi \in \mathfrak{g}\}.$$

Given a point $m \in M$, considering curves of the form $\exp(t\xi)m$ with $\xi \in \mathfrak{g}_m$ and differentiating the relation $\exp(t\xi)m = m$, one may show that $\mathfrak{g}_m$ is given by:

$$\mathfrak{g}_m = \{\xi \in \mathfrak{g} \mid \xi_M(m) = 0\}. \tag{1.5}$$

The main property concerning the orbit space M/G is the following:

Theorem 1.11. *If the action is free and proper, then M/G admits a unique smooth structure such that $\pi : M \to M/G$ is a submersion.*

There are many proofs of the previous result in the literature: a detailed one can be found in [Lee13].

1.3 Fibre bundles and connections

We will assume that the reader is familiar with the definition of a fibre bundle $\pi : P \to M$ with typical fibre F (see e.g. [KMS93] for a detailed exposition). Given two bundles fibred over the same base manifold $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to M$, *the fibred product* is the bundle with base manifold M and with total space

$$P_1 \times_M P_2 := \{(p_1, p_2) \in P_1 \times P_2 \mid \pi_1(p_1) = \pi_2(p_2)\} \ .$$

Principal bundles. A *principal bundle* is a fibre bundle with typical fibre a Lie group, and such that the local trivializations respect the group operation. More precisely we have:

Definition 1.12. *Let M be a manifold and G a Lie group. A* principal fibre bundle *over M with group G also called* principal G-bundle *consists of a G-space P and a (smooth) projection $\pi : P \to M$ satisfying the following conditions:*

1) G acts freely on P.

2) $\pi(G \cdot p) = \pi(p)$ for all $p \in P$.

3) P is locally trivial. This means that around each $m \in M$ there exists a neighborhood U and a diffeomorphism $\varphi : \pi^{-1}(U) \to G \times U$ which is equivariant, where G acts on $G \times U$ by left translations on the first factor. In other words, Diagram 1.1 commutes.

$$\begin{tikzcd}
P \supset \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & G \times U \arrow[dl, "pr_2"] \\
& U &
\end{tikzcd}$$

Diagram 1.1: Local triviality on a principal bundle

Note that the fibres $\pi^{-1}(m)$, where $m \in M$, are the orbits of the action and M is the orbit space of the G-space P, that we denote by $M = P/G$. Definition 1.12 is a classical one (see e.g. [KN96]), but we remark that local triviality actually holds once the action is assumed to be free. Therefore, we may equivalently define a principal bundle by simply requiring the action of G on P to be free and M to be the orbit space (such that the fibres of π coincide with the orbits).

Connections and Curvature. If $\pi : P \to M$ is a fibre bundle, then the fibration π determines the vertical tangent space V_p at a point $p \in P$ in the following sense: $V_p = \ker T_p\pi$. The union of these spaces defines the vertical subbundle $VP \subset TP$, which is often denoted by $V\pi$ to emphasize the fibration π. Recall that an *(Ehresmann) connection* on P is a vector valued 1-form $\mathcal{A}$ with values in VP which is a projection: in other words, $\mathcal{A} \in \Omega^1(P, VP)$ satisfies $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$. This determines a *horizontal* subbundle $HP = \ker \mathcal{A}$ such that $TP = VP \oplus HP$, and for any tangent vector v_p we define its vertical and horizontal components as

$$\mathrm{Ver}(v_p) = \mathcal{A}(v_p)\,, \qquad \mathrm{Hor}(v_p) = v_p - \mathcal{A}(v_p)\,.$$

The connection also determines the *horizontal lift* of tangent vectors to M (see e.g. [KMS93]), that we write as follows: $(v_m)^h_p$ is the horizontal lift of $v_m \in T_m M$ to the point $p \in \pi^{-1}(m)$.

The curvature of a connection $\mathcal{A}$ is defined as the following vertical vector valued 2-form:

$$\mathcal{B}(v_p, w_p) = d\mathcal{A}(\mathrm{Hor}(v_p), \mathrm{Hor}(w_p))\,, \tag{1.6}$$

where d is the exterior derivative (of a vector valued form). Here we adopt the usual convention for the sign of the curvature in geometric mechanics.

Taking into account the definition of the exterior derivative for 1-forms, the action of $\mathcal{B}$ on two vector fields X, Y on P is

$$\mathcal{B}(X, Y) = -\mathcal{A}\left([\mathrm{Hor}(X), \mathrm{Hor}(Y)]\right)\,,$$

with $\mathrm{Hor}(X)(p) = \mathrm{Hor}(X_p)$, $\mathrm{Hor}(Y)(p) = \mathrm{Hor}(Y_p)$, and where the bracket is Lie bracket of vector fields. This provides a nice geometric interpretation: the curvature measures lack of integrability of the horizontal distribution induced by the connection.

We now want to specialize the concepts above to the case of principal bundles. Before giving the definition, we point out that it is natural to demand some compatibility of the connection with the group structure. We also observe that the vertical subbundle of a principal bundle is spanned by infinitesimal generators of the group action, and as a result the vertical space at a point may be identified with $\mathfrak{g}$.

Definition 1.13. *A principal connection $\mathfrak{A}$ on the principal fibre bundle $\pi : P \to M$ is a Lie algebra valued 1-form $\mathfrak{A} : TP \to \mathfrak{g}$ with the following properties:*

1) $\mathfrak{A}(\xi_P) = \xi$.

2) $\phi_g^* \mathfrak{A}(\cdot) = Ad_g\,(\mathfrak{A}(\cdot))$.

To retrieve the Ehresmann connection $\mathcal{A}$ associated to the form $\mathfrak{A}$ one simply defines

$$\mathcal{A}(v_p) = \left(\mathfrak{A}(v_p)\right)_P(p)\,.$$

We see that the horizontal subbundle is equivariant under the group action in the sense that $gH_p = H_{gp}$ (actually, it is easy to check that an Ehresmann connection on a principal bundle is a principal connection if, and only if, this condition holds).

Definition 1.14. *Given a principal connection $\mathfrak{A}$ on the principal bundle $\pi : P \to M$, the* curvature *is the Lie algebra valued 2-form on P given by*

$$\mathfrak{B}(v_p, w_p) = d\mathfrak{A}(\mathrm{Hor}(v_p), \mathrm{Hor}(w_p))\,. \tag{1.7}$$

Sometimes the relation (1.7) is also written as $\mathcal{B} = d^{\mathfrak{A}}\mathfrak{A}$, where $d^{\mathfrak{A}}$ is the exterior covariant derivative w.r.t. $\mathfrak{A}$ (see e.g. [MMO+07]). For a arbitrary arbitrary Lie algebra valued n-form α, $d^{\mathfrak{A}}\alpha$ is defined by its action on tangent vectors $v_1, v_2, \dots v_{n+1} \in T_pP$ as follows:

$$d^{\mathfrak{A}}\alpha(v_1, v_2, \dots, v_{n+1}) = d\alpha(\mathrm{Hor}(v_1), \mathrm{Hor}(v_2), \dots, \mathrm{Hor}(v_{n+1}))\,.$$

A classic result to which we shall refer later is the so-called *Cartan structure equation* whose proof, with our conventions, may be found in e.g. [MMO+07].

Theorem 1.15 (Cartan structure equations). *For any pair of vector fields X, Y on P we have*

$$\mathfrak{B}(X, Y) = d\mathfrak{A}(X, Y) - [\mathfrak{A}(X), \mathfrak{A}(Y)]_{\mathfrak{g}}\,, \tag{1.8}$$

where the bracket is the Lie algebra bracket on $\mathfrak{g}$.

In short, $\mathfrak{B} = d\mathfrak{A} - [\mathfrak{A}, \mathfrak{A}]$. This structure equation is sometimes written with a factor $1/2$ in front of the bracket (see e.g. [KN96]), namely $\mathfrak{B} = d\mathfrak{A} - 1/2[\![\mathfrak{A}, \mathfrak{A}]\!]$. It should be noted that the bracket $[\![\cdot, \cdot]\!]$ is understood as the "wedge", namely for two Lie algebra valued 1-forms A and A' one has $[\![A, A']\!](X, Y) = [A(X), A'(Y)]_{\mathfrak{g}} - [A(Y), A'(X)]_{\mathfrak{g}}$.

If α is a Lie-algebra valued n-form on P, $\alpha \in \Omega^n(P, \mathfrak{g})$, and $\mu \in \mathfrak{g}^*$, then we denote by α_μ or also $\alpha^*(\mu)$ the following n-form on P:

$$\alpha_\mu(\cdot) := \langle \alpha(\cdot), \mu \rangle \in \Omega^n(P)\,. \tag{1.9}$$

Remark 1.16. *The way we have introduced curvature is suitable for the computations in this manuscript. For a rigorous treatment of the curvature associated to a connection, see e.g. [KN96] or [KMS93].*

Coordinate expressions. For later use, we will now provide local expressions of a connection on a principal fibre bundle and of its curvature. The reader may want to take a look at [Blo03] for a detailed exposition of this standard topic.

Consider a connection $\mathcal{A}$ on a fibre bundle $P \to M$, and take adapted coordinates $(x, y) = (x^i, y^a)$ on P, where $\pi(x, y) = x$. The connection 1-form $\mathcal{A}$ reads

$$\mathcal{A} = \omega^a \frac{\partial}{\partial y^a}, \quad \text{where} \quad \omega^a = dy^a + \mathcal{A}_i^a dx^i\,,$$

where the coefficients $\mathcal{A}_i^a = \mathcal{A}_i^a(x, y)$ are the *connection coefficients.* For a tangent vector $v_p = (\dot{x}, \dot{y})$, the decomposition into horizontal and vertical parts is

$$\left(\dot{x}^i, \dot{y}^a\right) = \left(\dot{x}^i, -\mathcal{A}_i^a \dot{x}^i\right) + \left(0, \dot{y}^a + \mathcal{A}_i^a \dot{x}^i\right).$$

If we let the curvature components be denoted by $\mathcal{B}^a_{\alpha\beta}$, i.e.

$$\mathcal{B} = \mathcal{B}^a_{ij}\frac{\partial}{\partial y^a},$$

then a standard computation leads to the following expression:

$$\mathcal{B}^a_{ij} = \left(\frac{\partial \mathcal{A}^a_j}{\partial x^i} - \frac{\partial \mathcal{A}^a_i}{\partial x^j} + \mathcal{A}^b_j\frac{\partial \mathcal{A}^a_i}{\partial y^b} - \mathcal{A}^b_i\frac{\partial \mathcal{A}^a_j}{\partial y^b}\right).$$

The adjoint bundle. One of the bundles that plays an essential role in the theory of reduction is the so-called *adjoint bundle*. We will now briefly recall its construction, which uses the notion of *associated bundle* (see [KN96]).

Besides a principal G-bundle $P \to P/G$, we consider a left representation $\Psi_g : W \to W$ of the group G in the vector space W. Denote the quotient space by $P \times_G W := (P \times W)/G$, where G acts on the left on $P \times W$ in the following way:

$$g \cdot (p, w) = (g \cdot p, \Psi_g(w)),$$

with $p \in P$ and $w \in W$. The associated bundle is then the bundle $P \times_G W \to P/G$ with fibre W, and this is a vector bundle as we explain next. Let $[p, w]_G \in P \times_G W$ represent a typical element in the fibre over $[p]_G \in P/G$; then, the vector space operations in that fibre are:

$$\lambda \cdot [p, w]_G = [p, \lambda w]_G,$$
$$[p, w_1]_G + [p, w_2]_G = [p, w_1 + w_2]_G.$$

If we choose $W = \mathfrak{g}$ and $\Psi_g = Ad_g$ in the construction above, we obtain the *adjoint bundle* denoted by $\tilde{\mathfrak{g}}$. It carries a Lie algebra structure on each fibre whose bracket is:

$$[[p, \xi]_G, [p, \eta]_G] = [p, [\xi, \eta]]_G.$$

This bundle is related to the geometry underlying the Lagrange-Poincaré equations. Given a principal connection on $P \to P/G$, there is a natural way to induce a (affine) connection on the associated bundle $P \times_G W$. The importance of this connection in the case of the adjoint bundle $\tilde{\mathfrak{g}}$ is highlighted in [CMR01], where it shows up as one of the ingredients needed to obtain an intrinsic formulation of the Lagrange-Poincaré equations.

If $P \to P/G$ is principal bundle, it is possible to define an action of G on TP by tangent lifts. Consider the map $[\tau]_G : TP/G \to P/G$ given by $[\tau]_G([v_p]_G) = [p]_G$, which defines a bundle structure. The bundle $[\tau]_G : TP/G \to P/G$ admits a vector bundle structure with operations:

$$\lambda \cdot [v_p]_G = [\lambda v_p]_G,$$
$$[v_q]_G + [w_q]_G = [v_q + w_q]_G.$$

Note that the fibre dimension remains the same, as $(TP/G)_x$ is isomorphic to T_pP for each p such that $x = [p]_g$, and therefore the dimensional reduction of TP/G to P/G only takes place in the base space P. Choosing a connection $\mathfrak{A}$ in $P \to P/G$ gives an identification between the bundles TP/G and $T(P/G) \oplus \tilde{\mathfrak{g}}$, and this ultimately allows to write down the (global) Lagrange-Poincaré equations (see [CMR01]). This identification is obtained via the following map:

$$\begin{aligned} \alpha_{\mathfrak{A}} : TP/G &\to T(P/G) \oplus \tilde{\mathfrak{g}}\,, \\ ([v_p]_G) &\mapsto \alpha_{\mathfrak{A}}([v_p]_G) = T\pi(v_p) \oplus [p, (\mathfrak{A}(v_p))_G]\,. \end{aligned} \tag{1.10}$$

Lemma 1.17. *The map $\alpha_{\mathfrak{A}}$ is an isomorphism.*

Proof. First note that the map is well defined in view of the equivariance of $\mathfrak{A}$. We now construct the inverse $\alpha_{\mathfrak{A}}^{-1}$. Consider a tangent vector $v_x \in T(P/G)$ and define

$$\alpha_{\mathfrak{A}}^{-1}\,(v_x \oplus [p,\xi]_G) = [(v_x)^h_p + \xi_P(p)]_G\,.$$

Note that by equivariance of the connection, $g(v_x)^h_p = (v_x)^h_{gp}$. Together with Proposition 1.5 this implies that the inverse is well defined. Indeed, if $[gp, Ad_g\xi]_G$ is another representative it follows:

$$\begin{aligned} \alpha_{\mathfrak{A}}^{-1}\,(v_x \oplus [gp, Ad_g\xi]_G) &= [(v_x)^h_{gp} + (Ad_g\xi)_P(gp)]_G \\ &= [g(v_x)^h_p + g\xi_P(p)]_G \\ &= [(v_x)^h_p + \xi_P(p)]_G\,. \end{aligned}$$

□

1.4 Reduction

The subject of reduction of dynamical systems is too broad to allow of reviewing all of its aspects here, so our attention will be directed to the specific results on which this work relies upon. To give a flavor on the nature of the reduction results in the context of geometric mechanics, consider the following proposition:

Proposition 1.18. *Let (P,Ω), $(M,\tilde{\Omega})$ be presymplectic manifolds where Ω and $\tilde{\Omega}$ have constant rank, and let $\pi : P \to M$ be a submersion. Let H and $\tilde{H}$ be Hamiltonian functions on P and M respectively. Assume that the following conditions hold:*

1) $\pi^*\tilde{\Omega} = \Omega$,

2) $\pi^*\tilde{H} = H$.

Then, any solution $\gamma(t)$ of the Hamilton equations on P associated with H projects onto a solution of the Hamilton equations in M associated with $\tilde{H}$. Conversely, consider a solution $\tilde{\gamma}(t)$ of the Hamilton equations on M associated with $\tilde{H}$. Any curve $\gamma(t)$ in P satisfying $\pi(\gamma(t)) = \tilde{\gamma}(t)$ is then a solution of the Hamilton equations on P associated with H.

Proof. By definition of the Hamilton equations and the conditions on π we have

$$\imath_{\dot\gamma(t)}(\pi^*\tilde\Omega) = -d(\pi^*\tilde H) = \pi^*(-d\tilde H)\,.$$

Writing $\tilde\gamma = \pi(\gamma)$ for the projected curve, and considering an arbitrary vector $X \in T_{\gamma(t)}P$ yields

$$\tilde\Omega(\dot{\tilde\gamma}(t), T\pi(X)) = -d\tilde H(T\pi(X)).$$

Since π is a submersion, $T\pi(X)$ is an arbitrary element of $T_{\tilde\gamma(t)}M$, and hence $\imath_{\dot{\tilde\gamma}(t)}(\pi^*\tilde\Omega) = -d\tilde H$. The converse follows reversing the argument: consider γ such that $\pi(\gamma) = \tilde\gamma$ and an arbitrary $X \in T_{\gamma(t)}P$. Then:

$$\imath_{\dot\gamma(t)}\Omega(X) = \tilde\Omega(T\pi(\dot\gamma(t)), T\pi(X))) = -d\tilde H(T\pi(X)) = -dH(X)\,.$$

□

The previous proposition contains the three main elements of reduction (see Figure 1.2):

1) The presymplectic structures on P and M are related, or in other words, the geometric data on P *reduce* to those on M;

2) The Hamiltonian functions on P and M are related, i.e. H *reduces* to $\tilde H$;

3) There is a relation between the solutions of the Hamilton equations on P and M.

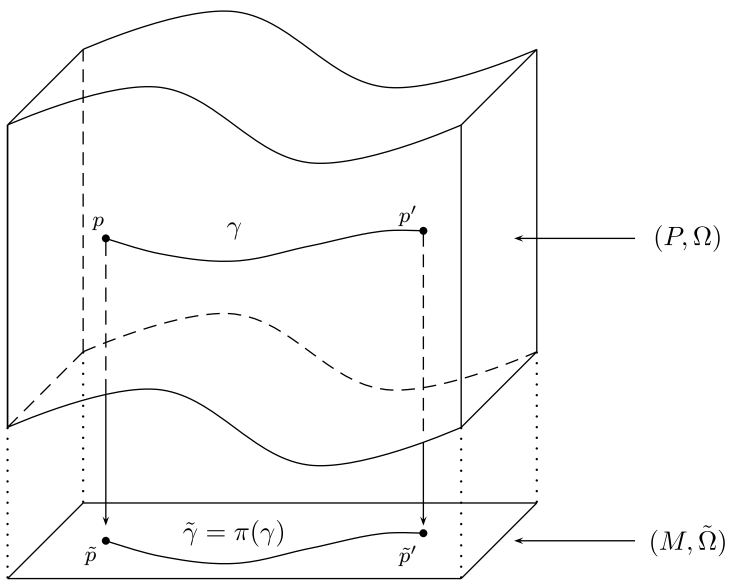

FIGURE 1.2: Reduction scheme

The reduction is often achieved by making use of symmetries. In the Poisson case, for instance, one assumes that the manifold is endowed with a Lie group action which leaves invariant both the Poisson bracket and the Hamiltonian function. One may then reduce the Poisson bracket and Hamiltonian function to P/G, and relate the solutions of the Hamilton equations for both systems. In the symplectic case the scheme is somehow more involved, but essentially the use of the so-called *momentum maps* guarantees that the reduced space will again be symplectic.

The procedure to retrieve solutions of the original system from a solution of the reduced system is usually referred to as *reconstruction*. Among the many references for this topic, we would like to draw the attention of the reader to [MMR90], where the notion of *geometric phases* in the reconstruction method is nicely discussed with some illustrative examples.

1.4.1 Reduction of a symplectic manifold

We will now discuss in some detail the case of reduction of a symplectic manifold. The results below (Theorem 1.20 and Theorem 1.21) will later be used to give a version of the Marsden-Weinstein Theorem (see [MW74]). Our approach follows [LM87] and [MMO$^+$07].

Recall that if $M \subset P$ is a submanifold of the symplectic manifold (P, Ω), we denote by $(TM)^\Omega$ the symplectic orthogonal of TM in TP, namely

$$(T_mM)^\Omega = \{v \in T_mP \mid \Omega(v, w) = 0 \quad \text{for all } w \in T_mM\}\,.$$

Lemma 1.19. *Let (P, Ω) be a symplectic manifold and $M \subset P$ a submanifold such that $\Omega_M = i_M^*\Omega$ has constant rank. Then $TM \cap (TM)^\Omega$ is an integrable vector subbundle of TM.*

Proof. Note that $\ker \Omega_M = TM \cap (TM)^\Omega$ is of constant dimension and therefore defines a subbundle of TN. The corresponding distribution is integrable since $\ker \Omega_M$ is checked to be involutive: for two vector fields X and Y with values in $TM \cap (TM)^\Omega$, it follows easily $\imath_{[X,Y]}\Omega_M = 0$. □

The maximal connected integral submanifolds of an integrable distribution are the leaves of the foliation. Recall that a foliation $\mathcal{F}$ on P is *regular* or *simple* if the set of leaves of the foliation, $P/\mathcal{F}$, is a manifold such that the canonical projection $\pi : P \to P/\mathcal{F}$ is a submersion. Necessary and sufficient conditions for a foliation to be regular may be found, for instance, in [AMR88].

Theorem 1.20. *In the situation above (Lemma 1.19), assume that the foliation $\mathcal{F}$ defined by $\ker \Omega_M$ is regular. Then there exists a unique symplectic form $\tilde{\Omega}$ on $P/\mathcal{F}$ such that*

$$\pi^*\tilde{\Omega} = \Omega_M = i_M^*\Omega\,.$$

Proof. The form $\tilde{\Omega}$ is defined as

$$\tilde{\Omega}([x])([v],[w]) = \Omega(x)(v,w)\,,$$

where $x \in \pi^{-1}([x])$ and $v, w \in T_xP$ are such that $T\pi(v) = [v]$, $T\pi(w) = [w]$. One checks that it is well defined (this is not immediate) and that it is symplectic:

i) $\tilde{\Omega}$ IS CLOSED. This follows easily from $\pi^*\tilde{\Omega} = \Omega_M$ and the fact that π is a submersion.

ii) $\tilde{\Omega}$ IS NONDEGENERATE. If $\tilde{\Omega}([x])([v],[w]) = 0$ for all $[w] \in T_{[x]}P/\mathcal{F}$, we would have $\Omega_M(x)(v,w) = 0$ for all $w \in T_xM$ and therefore $v \in \ker \Omega_M$. But this implies $[v] = T\pi(v) = [0]$.

□

The following result is not hard to prove.

Theorem 1.21. *In the situation above, assume H is a Hamiltonian on P such that $X_H(x) \in T_xM$ for all $x \in M$ (the Hamiltonian field is tangent to M). There exists a unique function $\tilde{H}$ on $P/\mathcal{F}$ such that*

$$H_{|M} = \pi^*\tilde{H}\,.$$

Moreover, the restriction of X_H to M projects onto $X_{\tilde{H}}$.

Momentum maps. Momentum maps are the basic construction to work with conserved quantities of a Hamiltonian system with symmetry.

In what follows we will assume that the manifold P is connected. Recall that a vector field X on the symplectic manifold (P,Ω) is said to be:

i) *locally Hamiltonian* if $\pounds_X\Omega = 0$ or, equivalently, if the flow $\{\varphi_t\}$ of X consists of symplectic transformations;

ii) *Hamiltonian* (or, sometimes, *globally Hamiltonian*) if there exists a function $f \in \mathcal{C}^\infty(P)$ such that $\imath_X\Omega = -df$. The function f is said to be a *Hamiltonian function* associated to the vector field X.

If G acts canonically on (P,Ω), from the relation $\phi_g^*\Omega = \Omega$ it is immediate to check that the infinitesimal generators are locally Hamiltonian. However, in general, the generators ξ_P need not be (globally) Hamiltonian.

Definition 1.22. *A symplectic action ϕ is* Hamiltonian *if, for each $\xi \in \mathfrak{g}$, the infinitesimal generator ξ_P is Hamiltonian.*

In this situation, we write $J_\xi \in \mathcal{C}^\infty(P)$ for the Hamiltonian function associated to the generator ξ_P, namely

$$\imath_{\xi_P}\Omega = -dJ_\xi\,,$$

and therefore we write $X_{J_\xi} = \xi_P$. The linear map

$$\begin{aligned} \mathfrak{g} &\to \mathcal{C}^\infty(P)\,, \\ \xi &\mapsto J_\xi\,, \end{aligned}$$

is often referred to as generalized Hamiltonian of the action ϕ.

Definition 1.23. *With the notations above, a map $J : P \to \mathfrak{g}^*$ is a* momentum map *(for the action ϕ) if $J_\xi = \langle J, \xi\rangle$. In other words, we have*

$$\imath_{\xi_P}\Omega = -d\langle J, \xi\rangle = -dJ_\xi\,,$$

for the function $J_\xi \in \mathcal{C}^\infty(P)$ defined as $J_\xi(p) = \langle J(p), \xi\rangle$.

The existence of a momentum map is guaranteed if the action is Hamiltonian. Indeed given a base $\{\xi_i\}_i$ of $\mathfrak{g}$, we can choose Hamiltonian functions J_{ξ_i} associated to each ξ_i, and then we extend by linearity. If J, J' are two momentum maps (for the same action) we have $d(J_\xi - J'_\xi) = 0$ and therefore the momentum map is defined up to a constant $\mu \in \mathfrak{g}^*$.

The basic observation about momentum maps is the following: if $H : P \to \mathbb{R}$ is G-invariant, then J is a first integral of X_H. This is an easy verification.

A momentum map J is said to be *equivariant* if it is equivariant with respect to the G-action ϕ on P and the coadjoint action on $\mathfrak{g}^*$

$$J(g \cdot p) = Ad^*_{g^{-1}} J(p)\,, \tag{1.11}$$

namely the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "J"] \arrow[d, "\phi_g"'] & \mathfrak{g} \arrow[d, "Ad^*_{g^{-1}}"] \\ P \arrow[r, "J"'] & \mathfrak{g} \end{tikzcd}$$

Differentiating (1.11) yields the following *infinitesimal equivariance* property of the momentum map: $T_pJ(\xi_P(p)) = -ad^*_\xi J(p)$.

For the time being, and for the sake of simplicity, we will focus on the case of equivariant momentum maps. Later we shall see that the results obtained can be generalized to non-equivariant momentum maps when one considers an affine action on $\mathfrak{g}$. The following useful lemma about the geometry of the level sets of the momentum may be used in the proof of Marsden-Weinstein Theorem.

Lemma 1.24. *Let J be an equivariant momentum map and $\mu \in \mathfrak{g}^*$ a regular value. Then:*

1) $G_\mu \cdot p = (G \cdot p) \cap J^{-1}(\mu)$.

2) $J^{-1}(\mu)$ *intersects* $G \cdot p$ *cleanly,*

$$T_p(G_\mu \cdot p) = T_p(G \cdot p) \cap T_p\left(J^{-1}(\mu)\right) .$$

3) $T_p\left(J^{-1}(\mu)\right) = (T_p(G \cdot p))^\Omega$.

Proof. See [MMO+07]. □

Recall that if $f : M \to N$ is a smooth map between manifolds, we say that $n \in N$ is a *regular value* of f if, for each $m \in f^{-1}(n)$, $T_m f$ is surjective. More generally n is a *weakly regular value* if $f^{-1}(n)$ is a submanifold and for each $m \in f^{-1}(n)$ we have $T_m(f^{-1}(n)) = \ker(T_m f)$.

The following proposition shows that regular points of the momentum map are precisely those whose symmetry algebra is trivial.

Proposition 1.25. *μ is regular value of J if, and only if, $\mathfrak{g}_p = \{0\}$ for all $p \in J^{-1}(\mu)$.*

Proof. Take a tangent vector v_p in p. From the definition of momentum map it follows

$$-\langle T_p J(v_p), \xi\rangle = \Omega_p(\xi_P(p), v_p) .$$

μ being regular means that $T_p J$ is surjective for all $p \in J^{-1}(\mu)$, or equivalently:

$$\{0\} = \{\xi \in \mathfrak{g} \mid \Omega_p(\xi_P(p), v_p) = 0 \quad \text{for all } v_p .$$

In view of (1.5), we conclude that $\mathfrak{g}_p = 0$. □

The previous proposition has an important consequence:

Corollary 1.26. *If the action is free, then any element $\mu \in \mathfrak{g}^*$ is a regular value of J.*

Marsden-Weinstein reduction. Building on the results above, we are now ready to discuss the Marsden-Weinstein reduction Theorem (see [MW74]).

Given a weakly regular value $\mu \in \mathfrak{g}^*$, consider the submanifold $J^{-1}(\mu)$ and denote its canonical inclusion by $i_\mu : J^{-1}(\mu) \hookrightarrow P$. Due to the equivariance of J, the action restricts to an action of the isotropy group that we denote ϕ_μ:

$$\phi_\mu \colon G_\mu \times J^{-1}(\mu) \to J^{-1}(\mu) .$$

We now compute the kernel of the induced symplectic form $i^*\Omega$ using Lemma 1.24:

$$\begin{aligned}(\ker i^*\Omega)_p &= T_p\left(J^{-1}(\mu)\right) \cap T_p\left(J^{-1}(\mu)\right)^{\Omega} \\ &= T_p\left(J^{-1}(\mu)\right) \cap T_p\left(G \cdot p\right) \\ &= T_p\left(G_\mu \cdot p\right) .\end{aligned}$$

From this it follows that $\ker i^*\Omega$ at p has dimension $\dim G_\mu - \dim G_p$. One may check that this is indeed constant because $\dim G_p$ does not depend on the point $p \in J^{-1}(\mu)$ (see [LM87]). We conclude that the rank of $i^*\Omega$ is constant and an integrable subbundle of $T\left(J^{-1}(\mu)\right)$ (see Lemma 1.19), and the leaves of the foliation of $J^{-1}(\mu)$ defined by $\ker(i^*\Omega)$ are the connected components of the orbits of ϕ_μ. If the foliation is simple and H is an invariant Hamiltonian, we may apply Theorem 1.20 and Theorem 1.21 and show that there exist reduced (symplectic) dynamics which are related to the original (unreduced) dynamics.

The above essentially gives the content of the Marsden-Weinstein reduction Theorem. One usually starts with a free and proper action which guarantees that the quotient $J^{-1}(\mu)/G_\mu$ is a manifold, with

$$\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$$

being a submersion.

Theorem 1.27 (Marsden-Weinstein reduction). *Let ϕ be a free and proper Hamiltonian action of G on (P, Ω) with equivariant momentum map J. Then the space $P_\mu = J^{-1}(\mu)/G_\mu$ has a unique symplectic form Ω_μ characterized by*

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega \, .$$

Figure 1.3 illustrates the various spaces and maps in Marsden-Weinstein reduction. A point p on the level set $J^{-1}(\mu)$ still carries some symmetry (due to the equivariance of the momentum) which, roughly speaking, amounts to the degeneracy $i_\mu^*\Omega$. Taking the quotient by G_μ removes the degeneracy and induces a symplectic form Ω_μ on P_μ.

The Marsden-Weinstein Theorem 1.27 also holds with the following (weaker assumptions): (1) $\mu \in \mathfrak{g}^*$ is weakly regular, and (2) ϕ_μ is free and proper. However, unless otherwise stated, we will assume to be in the setting of Theorem 1.27 (free and proper action ϕ which implies that $\mu \in \mathfrak{g}^*$ is a regular value, see Corollary 1.26). Exceptionally, when discussing examples, we will encounter non-free actions. In these cases we will check that the quotients are manifolds, and therefore the results about reduction contained in this section will apply with no change.

We now focus on the dynamical consequences of Theorem 1.27. Note that if H is an invariant Hamiltonian on P, the flow $\{\varphi_t\}$ of X_H preserves $J^{-1}(\mu)$ and commutes with the action, so it induces a reduced flow $\{\tilde{\varphi}_t\}$. The next Theorem states that the reduced flow is precisely the (Hamiltonian) flow corresponding to the reduced Hamiltonian.

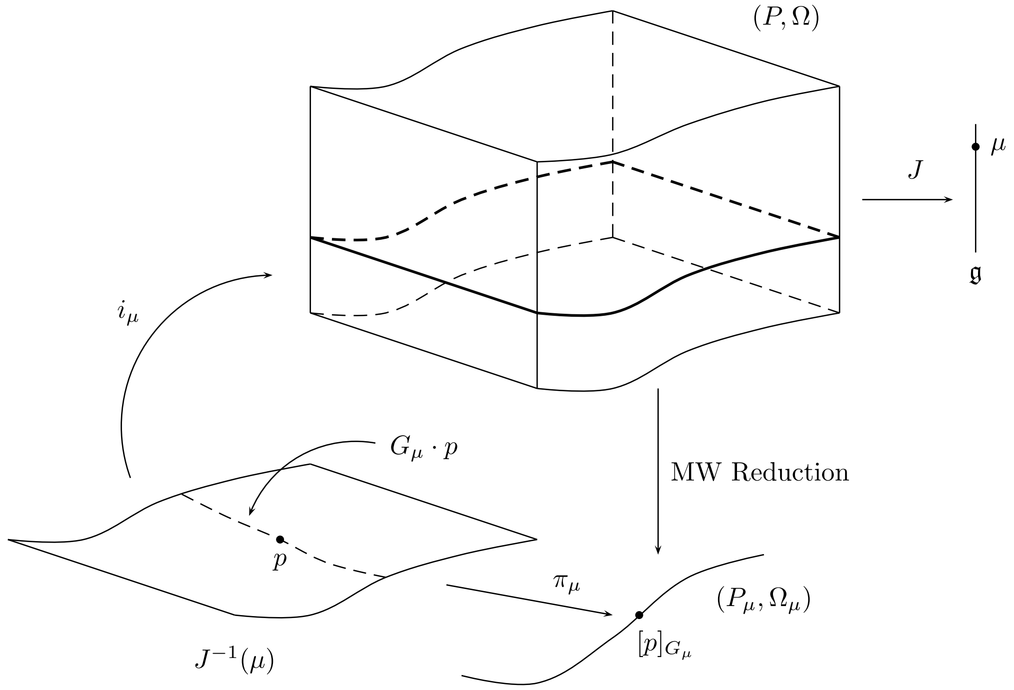

FIGURE 1.3: Marsden-Weinstein reduction

Theorem 1.28. *Let H be an invariant Hamiltonian on P and denote by $\tilde{H}$ the induced Hamiltonian on P_μ. The flows $\{\varphi_t\}$ and $\{\tilde{\varphi}_t\}$ corresponding to X_H and $X_{\tilde{H}}$ satisfy $\pi_\mu \circ \varphi_t = \tilde{\varphi}_t \circ \pi_\mu$.*

Proof. This may be proved directly with a similar argument to the one used in the proof of Proposition 1.18. See also [AM78]. □

Remark 1.29. *The Poisson bracket on P_μ can be characterized as follows. Consider $f, g \in \mathcal{C}^\infty(P_\mu)$ and let $F, G \in \mathcal{C}^\infty(P)$ be G-invariant extensions of $f \circ \pi_\mu$ and $g \circ \pi_\mu$, then*

$$\{F, G\}_P \circ i_\mu = \{f, g\}_{P_\mu} \circ \pi_\mu \,.$$

Consider two symplectic manifolds (P_1, Ω_1) and (P_2, Ω_2), both equipped with a G-action, such that:

i) There is an equivariant symplectomorphism $f : (P_1, \Omega_1) \to (P_2, \Omega_2)$;

ii) (P_2, Ω_2) is Marsden-Weinstein reducible.

In this situation, if $J_2 : P_2 \to \mathfrak{g}^*$ denotes the momentum map for the G-action on (P_2, Ω_2), G acts canonically on (P_1, Ω_1) with (equivariant) momentum map $J_1 = J_2 \circ f$. Note that f restricts to a G_μ-equivariant diffeomorphism $f_\mu : J_1^{-1}(\mu) \to J_2^{-1}(\mu)$. In particular, when dropped to the reduced spaces, this map provides a symplectomorphism between the Marsden-Weinstein reduced spaces. For later use, we formalize this observation:

Proposition 1.30. *In the situation above, the map* $[f_\mu]$ *defined as*

$$\begin{aligned}[f_\mu] : (P_1, (\Omega_1)_\mu) &\to (P_2, (\Omega_2)_\mu) \ , \\ [p]_{G_\mu} &\mapsto [f(p)]_{G_\mu} \ ,\end{aligned}$$

is a symplectomorphism.

Proof. This follows from the characterization of the reduced symplectic forms $(\Omega_1)_\mu$ and $(\Omega_2)_\mu$. See [LCV10] for more details. □

We close this overview of symplectic reduction discussing the case of non-equivariant momentum maps and the case of presymplectic forms (of constant rank). We describe how the results above apply with small changes to these situations.

Affine actions. Consider the case of a momentum map which is not necessarily equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, and assume as before that P is connected. We will now explain how to redefine the action on $\mathfrak{g}^*$ in such a way that J becomes equivariant.

Define the following map (see e.g. [AM78]):

$$\begin{aligned}\sigma : G &\to \mathfrak{g}^* \ , \\ g &\mapsto \sigma(g) = J(g \cdot p) - Ad^*_{g^{-1}} J(p) \, ,\end{aligned}$$

where $p \in P$ is arbitrary (σ doesn't depend on p by connectedness of P). It is not hard to show that σ satisfies the cocycle identity

$$\sigma(gh) = \sigma(g) + Ad^*_{g^{-1}} \sigma(h) \, , \tag{1.12}$$

and therefore we call σ the *cocycle* of the momentum map. Clearly $\sigma = 0$ if, and only if, J is equivariant. Using (1.12), it is immediate to prove the following result:

Proposition 1.31. *In the situation above, consider the map*

$$\begin{aligned}\rho \colon G \times \mathfrak{g}^* &\to \mathfrak{g}^*, \\ (g, \mu) &\mapsto \rho_g(\mu) = Ad_{g^{-1}}(\mu) + \sigma(g) \, .\end{aligned}$$

Then:

1) ρ *is a (left) action on* $\mathfrak{g}^*$.

2) J *is equivariant with respect to the action* ρ *on* $\mathfrak{g}^*$.

To sum up, given a momentum map, it becomes equivariant with respect to the affine action ρ on $\mathfrak{g}^*$. Thus, the Marsden-Weinstein Theorem 1.27 holds with obvious modifications.

Presymplectic reduction. If (P, Ω) is a presymplectic manifold (where Ω has constant rank), one can generalize in an obvious way the notions of canonical action, momentum map, Hamiltonian action, etc. It is possible to extend Marsden-Weinstein Theorem to this setting and construct a presymplectic form Ω_μ on the quotient space $P_\mu = J^{-1}(\mu)/G_\mu$.

Note that in the case of a presymplectic form Ω, the induced presymplectic form $i_\mu^*\Omega$ has the following kernel (this follows from Lemma 1.24):

$$\ker \left(i_\mu^*\Omega\right)_p = T_p \left(G_\mu \cdot p\right) + \ker \Omega_p \,.$$

Therefore, with the same reasoning as in the symplectic case, the quotient will be symplectic if, and only if, $\ker \left(i_\mu^*\Omega\right)_p = T_p \left(G_\mu \cdot p\right)$. The next result from [EEMLRR99] summarizes these observations:

Theorem 1.32. *There is a closed* 2*-form* Ω_μ *on* $J^{-1}(\mu)/G_\mu$ *such that* $\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$ *and:*

1) Ω_μ *is symplectic if, and only if,* $\ker(\Omega_\mu)_p = T_p(G_\mu \cdot p)$.

2) Otherwise Ω_μ *is a presymplectic (degenerate) form.*

1.4.2 Reduction of a Poisson manifold

We will now describe an easy version of Poisson reduction which will be used later on. For generalizations, see [MR86].

We assume we have a (free and proper) canonical action ϕ on the Poisson manifold (P, Ω). Let H be an invariant Hamiltonian function, and write $\{\varphi_t\}$ for the corresponding Hamiltonian flow.

Theorem 1.33. *In the situation above:*

1) There is a unique Poisson bracket $\{\cdot,\cdot\}_{P/G}$ *on* P/G *such that* $\pi : P \to P/G$ *is a Poisson map.*

2) Define the reduced Hamiltonian $\tilde{H}$ *by* $H = \tilde{H} \circ \pi$ *and write* $\{\tilde{\varphi}_t\}$ *for the corresponding Hamiltonian flow. Then* $\pi \circ \varphi_t = \tilde{\varphi}_t \circ \pi$.

The situation is illustrated in in the following diagram:

$$\begin{tikzcd}
 & P \arrow[r, "\pi"] \arrow[dl, "H"'] \arrow[dd, "\varphi_t"] & P/G \arrow[dr, "\tilde{H}"] \arrow[dd, "\tilde{\varphi}_t"] & \\
\mathbb{R} & & & \mathbb{R} \\
 & P \arrow[r, "\pi"'] \arrow[ul, "H"] & P/G \arrow[ur, "\tilde{H}"'] &
\end{tikzcd}$$

Proof. 1) If $f, g \in \mathcal{C}^\infty(P/G)$, by invariance of the bracket there exists $h \in \mathcal{C}^\infty(P/G)$ such that $\{f \circ \pi, g \circ \pi\} = h \circ \pi$. We define the reduced bracket as $\{f, g\}_{P/G} = h$. The properties of the bracket follow easily.

2) For each $f \in \mathcal{C}^\infty(P/G)$, writing the definition of the flow

$$\frac{d}{dt}(f \circ \pi \circ \varphi_t) = \{f \circ \pi, H\} \circ \varphi_t$$

and comparing this expression with the definition of $\{\tilde{\varphi}_t\}$ (using $H = \tilde{H} \circ \pi$) gives the relation $\pi \circ \varphi_t = \tilde{\varphi}_t \circ \pi$.

□

Remark 1.34. *Let (P, Ω) be a symplectic manifold regard P as a Poisson manifold with respect to the Poisson bracket induced by Ω (see Section 1.1). Then the leaves of the symplectic foliation of the reduced Poisson space P/G are precisely P_μ. The details can be found in [KKS78].*

Chapter 2

Tangent bundle reduction

The goal of this chapter is to describe the dynamical system that arises after symplectic reduction of a Lagrangian system with symmetry. In particular, we will discuss in which sense this new system can be regarded as being "Lagrangian" and how this relates to the classical procedure of elimination of ignorable variables due to Edward Routh.

The structure of the chapter is as follows. In Section 2.1 we recall the classical construction of the Routhian for a Lagrangian with cyclic coordinates. The modern geometric definition of the Routhian will be presented in Section 2.2, where we study the symplectic reduction of a tangent bundle equipped with the Poincaré-Cartan form associated to a Lagrangian. In Section 2.3 we specialize Routh reduction to the case where the configuration space is a product. Finally, in Section 2.4 we discuss some illustrative examples.

2.1 Introduction: Routh method

There exists a large class of systems for which the Lagrangian does not depend on some of the generalized coordinates. Such coordinates are called *cyclic* or *ignorable*, and it is well known that the generalized momenta corresponding to the cyclic coordinates are constants of motion. The Routh reduction procedure is a classical reduction technique which takes advantage of this conservation law to define a reduced Lagrangian function, so-called the *Routhian*, such that the solutions of the Euler-Lagrange equations for the Routhian are in correspondence with the solutions of the original Lagrangian when the conservation of momenta is taken into account.

Routh's procedure, as described in classical textbooks (cf. [Par65], [LL76] or [Gol80]), consists of the following steps:

1) Let $L(x^i, \dot{x}^i, \dot{\theta})$ be a Lagrangian with cyclic coordinate θ, and denote by p_θ the generalized momentum corresponding to θ.

2) Fix a value of the momentum $\mu = p_\theta$, and consider the function

$$R_c^\mu(x^i, \dot{x}^i) = \left(L - \dot{\theta} p_\theta\right)_{\{p_\theta = \mu\}} ,$$

where the notation means that we have used the relation $\mu = p_\theta$ to replace all the appearances of $\dot{\theta}$ in terms of $(x^i, \dot{x}^i)$ and the parameter μ. R_c^μ is the *(classical) Routhian.*

3) If we regard R_c^μ as a new Lagrangian in the variables $(x^i, \dot{x}^i)$, then the solutions of the Euler-Lagrange equations for R_c^μ are in correspondence with those of L when one takes into account the relation $p_\theta = \mu$. More precisely:

> Any solution $(x^i(t), \theta(t))$ of the Euler-Lagrange equations for L with momentum $p_\theta = \mu$ projects onto a solutions $x^i(t)$ of the Euler-Lagrange equations for R_c^μ.
>
> Conversely, any solution of the Euler Lagrange equations for R_c^μ can be lifted to a solution of the Euler-Lagrange equations for L with momentum $p_\theta = \mu$.

The method is best understood by means of an example.

Elroy's beanie. Consider the system in Figure 2.1, often referred to as Elroy's beanie (see e.g. [MMR90]). It consists of two parallel planar rigid bodies, joined by a pin at their center of mass O. For the sake of simplicity, we will assume that O is fixed, so that each body rotates around it. The moments of inertia of the bodies w.r.t. an axis through O and perpendicular to the plane are denoted by I_1 and I_2. The system is subjected to a potential V depending on the relative angle between the bodies.

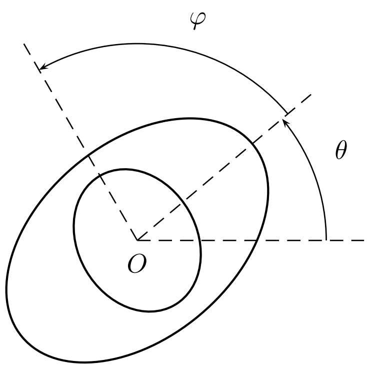

FIGURE 2.1: Elroy's Beanie

The configuration space of this system is $Q = S^1 \times S^1$ with the coordinates indicated on the figure: we write θ for the angle of the first body w.r.t. the x-axis and φ for the relative angle of the second body w.r.t. the first body. The Lagrangian of the system is

$$L(\varphi, \dot{\varphi}, \theta, \dot{\theta}) = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \big(\dot{\theta} + \dot{\varphi}\big)^2 - V(\varphi) .$$

The coordinate θ is a cyclic coordinate of L, and the corresponding generalized momentum equals $p_\theta = (I_1 + I_2)\dot{\theta} + I_2\dot{\varphi}$. In particular, for a fixed value μ of p_θ, we have

$$\dot{\theta} = \frac{-I_2\dot{\varphi} + \mu}{I_1 + I_2}. \tag{2.1}$$

A computation shows that

$$R_c^\mu(\varphi, \dot{\varphi}) = \frac{I_1 I_2}{2(I_1 + I_2)}\dot{\varphi}^2 + \frac{I_2\mu}{I_1 + I_2}\dot{\varphi} - V(\varphi) - \frac{\mu^2}{I_1 + I_2},$$

and that the Euler-Lagrange equation for R_c^μ is

$$\ddot{\varphi} = -\left(\frac{I_1 + I_2}{I_1 I_2}\right)\frac{dV}{d\varphi}. \tag{2.2}$$

First, one solves (2.2) to obtain $\varphi(t)$. Once $\varphi(t)$ is known, one can retrieve $\theta(t)$ out of the momentum equation (2.1) and the initial condition θ_0.

Spinning top. Let us discuss now the applicability of the Routh method in the case of a symmetric spinning top (see e.g. [Par65] or [Gol80]). The symmetric top is a rigid body of mass m with axial symmetry, namely, the principal moments of inertia $\{I_1, I_2, I_3\}$ satisfy the relation $I_1 = I_2$. One of the points in the axis, called the *peg*, remains fixed and the body is free to spin about it under the gravitational force. The configuration space is $SO(3)$.

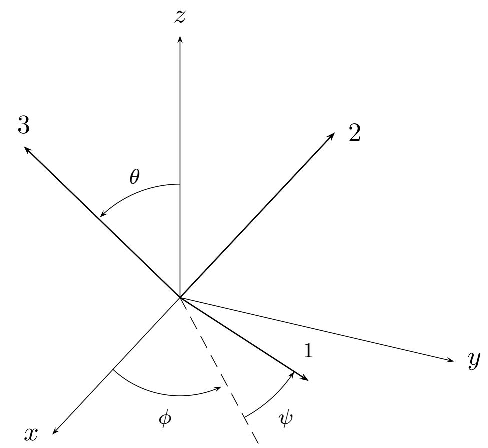

FIGURE 2.2: Euler angles on the top

Figure 2.2 shows our convention for the Euler angles, where O denotes the peg and the axis labeled “3” coincides with the axis of symmetry of the top. In this chart, the Lagrangian reads (see e.g. [Gol80]):

$$L(\theta, \dot{\theta}, \phi, \dot{\phi}, \psi, \dot{\psi}) = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - V(\theta),$$

where $V(\theta) = mgl\cos(\theta)$, with l the distance of the center of mass of the top from the (fixed) point O. L has two cyclic coordinates: ϕ and ψ. For simplicity in the exposition, let us consider the Routh method with respect to only one of these cyclic variables, say ψ. Let μ be a fixed value of the momentum p_ψ. A computation shows that the classical Routhian for the symmetric top is given by:

$$\begin{aligned}
R_c^\mu(\theta,\dot\theta,\phi,\dot\phi) &= \Big(L(\theta,\dot\theta,\phi,\dot\phi,\psi,\dot\psi) - \dot\psi p_\psi\Big)_{\{p_\psi=\mu\}} \\
&= \frac{1}{2}\left(I_1\dot\theta^2 + (2\mu\cos\theta)\dot\phi + (I_1\sin^2\theta)\dot\phi^2\right) - \frac{\mu^2}{2I_3} - V(\theta)\,.
\end{aligned}$$

Note, however, that the elimination of the velocity coordinate $\dot\psi$ is not intrinsic. Intuitively, the removal of the cyclic coordinate ψ corresponds to the projection $(\theta,\phi,\psi)\mapsto(\theta,\phi)$, and $\dot\psi$ should then be interpreted as the vertical part of the velocity w.r.t. this projection. We can overcome this difficulty by making an intrinsic assignment of the vertical part of the velocity, namely, by choosing a connection.

A global Routhian. We consider an arbitrary principal connection on the principal S^1-bundle

$$\pi : SO(3)\to S^2\,,$$

where S^1 acts on $SO(3)$ by rotations[a]. The connection 1-form $\mathfrak{A}$ reads

$$\mathfrak{A} = d\psi + A_\theta d\theta + A_\phi d\phi\,,$$

with $A_\theta = A_\theta(\theta,\phi)$ and $A_\phi = A_\phi(\theta,\phi)$, because of the infinitesimal equivariance of the 1-form $\mathfrak{A}$ (see Definition 1.13). The Routhian R^μ associated to the connection $\mathfrak{A}$ is:

$$\begin{aligned}
R^\mu &= \Big(L(\theta,\dot\theta,\phi,\dot\phi,\psi,\dot\psi) - \langle\mathfrak{A}(\theta,\dot\theta,\phi,\dot\phi,\psi,\dot\psi),\mu\rangle\Big)_{\{p_\psi=\mu\}} \\
&= R_c^\mu(\theta,\dot\theta,\phi,\dot\phi) - \mu A_\theta(\theta,\phi)\dot\theta - \mu A_\phi(\theta,\phi)\dot\phi\,.
\end{aligned}$$

As we have indicated above while discussing the classical procedure of Routh, the Euler-Lagrange equations for R_c^μ give the desired (local) correspondence with the solutions of the Euler-Lagrange equations for L. Therefore, if we want to preserve the correspondence between solutions of L and R^μ, we must take into account the Euler-Lagrange equations of the terms $\mu A_\theta\dot\theta$ and $\mu A_\phi\dot\phi$. This leads to the following the equations for R^μ:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial R^\mu}{\partial\dot\theta}\right) - \left(\frac{\partial R^\mu}{\partial\theta}\right) &= -\mu\dot\phi\left(\frac{\partial A_\theta}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right)\,, \\
\frac{d}{dt}\left(\frac{\partial R^\mu}{\partial\dot\phi}\right) - \left(\frac{\partial R^\mu}{\partial\phi}\right) &= -\mu\dot\theta\left(\frac{\partial A_\phi}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\,.
\end{aligned}$$

From the Cartan structure equation (1.8), we identify the extra terms in the right hand side as the curvature components of the connection $\mathfrak{A}$ contracted with the value $\mu\in\mathbb{R}$ (note that S^1 is Abelian and thus the bracket vanishes).

[a] In the chart under consideration, $\tau\in S^1$ maps an element (θ,ϕ,ψ) into $(\theta,\phi,\psi+\tau)$.

Gyroscopic forces. The argument above shows that, in general, one needs to introduce terms of gyroscopic type in the Euler-Lagrange equations for the Routhian R^μ.

In the case of the spinning top, these terms arise from the curvature of a chosen connection on the bundle $\pi : SO(3) \to S^2$. Hence, the Routhian R^μ will satisfy the standard Euler-Lagrange equations if we can choose a flat connection. The following proposition from [KN96] shows that such a choice is not possible in the case of the spinning top:

Proposition 2.1. *Assume that $\mathfrak{A}$ is a flat principal connection on the principal G-bundle $\pi : P \to M$. If M is simply connected, then P is isomorphic to the trivial bundle $M \times G$.*

The obstruction to the existence of ordinary Euler-Lagrange equations for the reduced dynamics cannot, in general, be solved by adding a velocity dependant potential. In the case of the spinning top this will only be possible if the reduction of the curvature form of the chosen connection $\mathfrak{A}$ is exact on the base space. As an illustration, let us consider the following connection form $\mathfrak{A}_M$ on $\pi : SO(3) \to S^2$:

$$\mathfrak{A}_M = d\psi + \cos(\theta) d\phi \,.$$

This is actually a natural connection for the spinning top, called the *mechanical connection.* The contraction of the curvature $d\mathfrak{A}_M$ of $\mathfrak{A}_M$ and the value of the momentum, μ, drops to a 2-form on S^2 that we denote by $\mathcal{B}_\mu$. With a suitable orientation of S^2, we find

$$\int_{S^2} \mathcal{B}_\mu = 4\pi\mu \,.$$

From Stoke's theorem it follows that, for $\mu \neq 0$, $\mathcal{B}_\mu$ is not an exact 2-form on S^2. As a matter of fact, this is not a special feature of the connection $\mathfrak{A}_M$. One can prove (cf. [Mor01]) that all the terms $\mathcal{B}_\mu$ arising from the curvature of different principal connections on the bundle $\pi : SO(3) \to S^2$ define the same de Rham cohomology class in $H^2_{DR}(S^2)$.

2.2 Geometric Routh reduction

The observations in Section 2.1 can be used to formulate a *global* Abelian version of Routh reduction which includes gyroscopic forces, as in [AKN88]. Some interesting remarks on the topological obstructions to the existence of a classical Routh function (in the sense we have just discussed in Section 2.1) are described in [Har77]. The generalization of Routh reduction to the case of non-Abelian symmetry groups was initiated in [MS93a] and [MS93b] and, since then, a number of works related to different aspects of Routhian reduction have appeared in the literature.

In this section we describe in detail the construction of the Routhian and its precise relation with Routh reduction, mostly following the symplectic approach in [LCV10]. As we shall see below, Routh reduction can be thought of as a particular case of the Marsden-Weinstein reduction theory when applied to the tangent bundle of the configuration space endowed with the Poincaré-Cartan form induced by the Lagrangian. This procedure (at least when certain regularity conditions on the Lagrangian are assumed) closely resembles the well known case of cotangent bundle reduction. We will devote special attention to the similarities between tangent and cotangent bundle reduction, and as a matter of fact this analogy will be a recurring theme throughout the next chapters. To make the exposition as self contained as possible, we start with a brief review of this theory.

2.2.1 Cotangent bundle reduction and tangent bundle reduction

There is an immense amount of literature dealing with the symplectic reduction of a cotangent bundle. A very detailed exposition, including different approaches and a good account of references, can be found in [MMO+07]. We will content ourselves with a description of the so-called *embedding picture* which is enough for our purposes.

Cotangent bundle reduction. Consider a free and proper action ϕ of a Lie group G on a manifold Q, and write as usual $\pi : Q \to Q/G$. The cotangent lift action ϕ^{T^*Q} (obtained for each g by cotangent lift of the diffeomorphism ϕ_g) defines a symplectic action on (T^*Q, Ω_Q), where Ω_Q is the canonical 2-form on Q. This action admits a momentum map $J : T^*Q \to \mathfrak{g}^*$ which is characterized by:

$$\langle J(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle \,, \tag{2.3}$$

for each $\xi \in \mathfrak{g}$. Using Proposition (1.5), it follows that J is equivariant. In particular, $G \cdot J^{-1}(0) \subset J^{-1}(0)$.

Lemma 2.2. *The spaces $J^{-1}(0)$ and $T^*(Q/G) \times_{Q/G} Q$ are diffeomorphic.*

Proof. The identification is obtained through the map:

$$\begin{aligned} J^{-1}(0) &\to T^*(Q/G) \times_{Q/G} Q \,, \\ \alpha_q &\mapsto (\tilde{\alpha}_{[q]}, q) \,, \end{aligned} \tag{2.4}$$

where $\tilde{\alpha}_{[q]}$ is defined by $\langle \tilde{\alpha}_{[q]}, \tilde{v}_{[q]} \rangle = \langle \alpha_q, v_q \rangle$ for any v_q such that $T\pi(v_q) = \tilde{v}_{[q]}$. Note that this is well defined because by the definition of J, $J^{-1}(0) = (V\pi)^0$ (the annihilator of the vertical subbundle). □

From the proof above it follows that, under the previous identification, G acts on the bundle $T^*(Q/G) \times_{Q/G} Q$ in the following manner:

$$g \cdot (\alpha_{[q]}, q) = (\alpha_{[q]}, g \cdot q) \,.$$

Fix a principal connection $\mathfrak{A}$ on $Q \to Q/G$, fix an element $\mu \in \mathfrak{g}^*$ and define the shift map S^μ as follows:

$$\begin{aligned} S^\mu : J^{-1}(\mu) &\to J^{-1}(0) \,, \\ \alpha_q &\mapsto S^\mu(\alpha_q) = \alpha_q - \mathfrak{A}_\mu(q) \,, \end{aligned} \tag{2.5}$$

where $\mathfrak{A}_\mu$ is defined as in (1.9). As usual, let us denote by G_μ the isotropy subgroup of μ. One checks that S^μ is a G_μ-equivariant diffeomorphism and therefore it drops to a diffeomorphism between the reduced spaces that we denote by $[S^\mu] : J^{-1}(\mu)/G_\mu \to J^{-1}(0)/G_\mu$.

Lemma 2.3. *The form $d\mathfrak{A}_\mu$ is projectable on Q/G_μ.*

Proof. $\mathfrak{A}_\mu$ is clearly G_μ-invariant. It remains to check that $d\mathfrak{A}_\mu$ annihilates vertical vectors to the fibration $Q \to Q/G_\mu$, but this follows easily. Indeed, for each $\xi \in \mathfrak{g}_\mu$ we have

$$\imath_{\xi_Q} d\mathfrak{A}_\mu = \pounds_{\xi_Q}\mathfrak{A}_\mu - d\imath_{\xi_Q}\mathfrak{A}_\mu = -d\langle \mu, \xi\rangle = 0\,.$$

□

It is customary to denote the projection of the 2-form $d\mathfrak{A}_\mu$ on Q/G_μ by $\mathcal{B}_\mu$ and call it the *magnetic term*. We make an important observation regarding this notation in the following remark.

Remark 2.4. *The notation $\mathcal{B}_\mu$ might be confusing in the following sense. If one picks a principal connection $\mathfrak{A}$, the Cartan structure equations* (1.8) *contracted with an element $\mu \in \mathfrak{g}^*$ read:*

$$\mathfrak{B}_\mu = d\mathfrak{A}_\mu - [\mathfrak{A}, \mathfrak{A}]_\mu\,.$$

In general, unless the group G is Abelian, the form $\mathcal{B}_\mu$ defined on Q/G_μ as above does not come solely *from the μ-component of the curvature $\mathfrak{B}_\mu$ of the chosen connection. The reason why we have decided to keep this notation is to match as much as possible the notation in the main references we cite.*

There is a way to realize the magnetic term as the μ-component of the curvature of a connection on the bundle $p_\mu : Q \to Q/G_\mu$, but this requires a slightly different approach to describe cotangent bundle reduction.

Define the following maps:

i) π_1 denotes the projection $\pi_1 : T^*(Q/G) \times_{Q/G} Q/G_\mu \to T^*(Q/G)$;

ii) π_2 denotes the projection $\pi_2 : T^*(Q/G) \times_{Q/G} Q/G_\mu \to Q/G_\mu$;

iii) π_μ is the projection $\pi_\mu : J^{-1}(\mu) \to P_\mu = J^{-1}(\mu)/G_\mu$;

iv) π_0 is the projection $\pi_0 : J^{-1}(0) \to T^*(Q/G) \times_{Q/G} Q/G_\mu$.

For a proof of the following result, see e.g. [MMO+07].

Theorem 2.5 (Cotangent bundle reduction). *In the situation above, $[S^\mu]$ induces a symplectomorphism*

$$((T^*Q)_\mu, (\Omega_Q)_\mu) \cong \left(T^*(Q/G) \times_{Q/G} Q/G_\mu, \pi_1^*\Omega_{Q/G} + \pi_2^*\mathcal{B}_\mu\right)\,.$$

$$\begin{tikzcd}
J^{-1}(\mu) \arrow[r, "S^\mu"] \arrow[d, "\pi_\mu"] & J^{-1}(0) \arrow[d, "\pi_0"] & \\
(T^*Q)_\mu \arrow[r, "{[S^\mu]}"] & T^*(Q/G) \times_{Q/G} Q/G_\mu \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Q/G_\mu \\
 & T^*(Q/G) &
\end{tikzcd}$$

DIAGRAM 2.3: Cotangent bundle reduction

The situation is illustrated in Diagram 2.3. The case of reduction at $\mu = 0$ is of special interest because it leads to a reduced space which is again a cotangent bundle. This next proposition formalizes this observation:

Proposition 2.6. *Under the conditions of Theorem 2.5, we have*

$$((T^*Q)_0, (\Omega_Q)_0) \cong (T^*(Q/G), \Omega_{Q/G}) \, .$$

The symplectomorphism Ψ_0 which realizes the previous identification is characterized as follows:

$$\langle \Psi_0(p_0(\alpha_q)), T_q\pi(v_q) \rangle = \langle \alpha_q, v_q \rangle \, , \tag{2.6}$$

where $p_0 : J^{-1}(0) \to (T^*Q)_0$ denotes the projection, $\alpha_q \in J^{-1}(0)$ and $v_q \in T_qQ$.

Again, for a proof, see e.g. [MMO+07].

Tangent bundle reduction. The basic setup here is a Lagrangian function L on the tangent bundle TQ of the configuration space Q, i.e. a *Lagrangian system* (Q, L). We begin by recalling the main ingredients of the symplectic picture of Lagrangian mechanics:

i) The Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ is defined as

$$\langle \mathbb{F}L(v_q), w_q \rangle = \left.\frac{d}{dt}\right|_{t=0} L(v_q + tw_q) \, .$$

If $\mathbb{F}L$ is a (global) diffeomorphism, we say that L is *(hyper-)regular*.

ii) We denote by $\Omega_Q^L = \mathbb{F}L^*\Omega_Q$ the presymplectic form induced by L on TQ. When the Lagrangian is regular, Ω_Q^L is symplectic.

iii) The *energy* E_L is the following function on TM:

$$E_L(v_q) = \langle \mathbb{F}L(v_q), v_q \rangle - L(v_q) \, .$$

It is a well known result that the Euler-Lagrange equations can be cast into a presymplectic formulation on (TQ, Ω_Q^L) with Hamiltonian function E_L. For simplicity, we will assume that our Lagrangian is hyperregular.

Assume that there is a (free and proper) G-action ϕ on the configuration space Q, and consider the lifted action ϕ^{TQ} of G on TQ. It is defined, for each $g \in G$, as $\phi_g^{TQ} = T(\phi_g)$. The Lagrangian L is said to be invariant if it is invariant with respect to ϕ^{TQ}, i.e. if $L(g \cdot v_q) = L(v_q)$. It is then immediate to check that $\mathbb{F}L$ is an equivariant diffeomorphism between TQ and T^*Q w.r.t. the actions ϕ^{TQ} and ϕ^{T^*Q} respectively. In particular, from Proposition 1.30 it follows that the map $J_L : TQ \to \mathfrak{g}^*$ defined as

$$\langle J_L(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q \rangle \,, \tag{2.7}$$

for each $\xi \in \mathfrak{g}$, is an equivariant momentum map, and that there exists a symplectomorphism between the corresponding Marsden-Weinstein reduced spaces

$$\left(J_L^{-1}(\mu)/G_\mu, (\Omega_Q^L)_\mu\right) \cong \left(J^{-1}(\mu)/G_\mu, (\Omega_Q)_\mu\right) \,.$$

According to the notation in Proposition 1.30, the previous symplectomorphism if denoted by $[\mathbb{F}L_\mu]$ and is obtained by G_μ equivariance of the map $\mathbb{F}L_{|J^{-1}(\mu)}$ (see Diagram 2.4).

$$\begin{tikzcd}
(TQ, \Omega_Q^L) \arrow[r, "\mathbb{F}L"] \arrow[d] & (T^*Q, \Omega_Q^L) \arrow[d] \\
\left(J_L^{-1}(\mu)/G_\mu, (\Omega_Q^L)_\mu\right) \arrow[r, "{[\mathbb{F}L_\mu]}"] & \left(J^{-1}(\mu)/G_\mu, (\Omega_Q)_\mu\right)
\end{tikzcd}$$

Diagram 2.4: Tangent bundle reduction

2.2.2 G-regularity

Just like in the case of cotangent bundle reduction, it is convenient to have a somehow more explicit description of the reduced space $J_L^{-1}(\mu)/G_\mu$ as a fibred product. However, because the definition of the Lagrangian momentum map $J_L = J \circ \mathbb{F}L$ involves the fibre derivative $\mathbb{F}L$, this would only be possible if an additional regularity condition is imposed on the Lagrangian.

Definition 2.7. *Let L be an invariant Lagrangian on TQ and $v_q \in T_qQ$ an arbitrary vector.*

1) The map $\mathcal{J}_L^{v_q} : \mathfrak{g} \to \mathfrak{g}^*$ is defined as follows:

$$\begin{aligned} \mathcal{J}_L^{v_q} : \mathfrak{g} &\to \mathfrak{g}^* \,, \\ \xi &\mapsto J_L\left(v_q + \xi_Q(q)\right) \,. \end{aligned}$$

2) L is locally G-regular *if $\mathcal{J}_L^{v_q}$ is a local diffeomorphism at 0 for every $v_q \in TQ$.*

3) L is G-regular *if $\mathcal{J}_L^{v_q}$ is a diffeomorphism for every $v_q \in TQ$.*

It is obvious that G-regularity implies local G-regularity. The main consequence of this regularity condition is the following identification:

Lemma 2.8. *Let L be G-regular (invariant) Lagrangian and $\mu \in \mathfrak{g}^*$. Then there exists a diffeomorphism:*

$$J_L^{-1}(\mu)/G_\mu \cong T(Q/G) \times_{Q/G} Q/G_\mu \,.$$

Proof. Consider the following map:

$$\begin{aligned} \Pi_\mu : J_L^{-1}(\mu) &\to T(Q/G) \times_{Q/G} Q \,, \\ v_q &\mapsto (T\pi(v_q), q) \;, \end{aligned} \tag{2.8}$$

where $\pi : Q \to Q/G$. It is clearly smooth. We will now define an inverse Δ_μ for Π_μ.

Given a point $(v_{[q]}, q) \in T(Q/G) \times_{Q/G} Q$, choose $v_q \in TQ$ such that $T\pi(v_q) = v_{[q]}$. By G-regularity, there exists a unique $\xi \in \mathfrak{g}$ such that $J_L(v_q + \xi_Q(q)) = \mu$. In this way we define the (smooth) map $\Delta_\mu(v_{[q]}, q) = v_q + \xi_Q(q)$, which is easily checked to be the inverse of Π_μ. Note that Δ_μ is well defined. Indeed, if $v'_q \in TQ$ is another vector such that $T\pi(v'_q) = v_{[q]}$, it follows $v'_q = v_q + \eta_Q(q)$ for some $\eta \in \mathfrak{g}$, and therefore

$$\mu = J_L(v_q + \xi_Q(q)) = J_L\left(v'_q - \eta_Q(q) + (\xi - \eta)_Q(q)\right) \,.$$

On the other hand, Π_μ is clearly G_μ equivariant, with respect to the following action of G_μ on $T(Q/G) \times_{Q/G} Q$:

$$g \cdot (v_{[q]}, q) = \left(v_{[q]}, g \cdot q\right) \,.$$

Thus, the reduced map

$$[\Pi_\mu] : J_L^{-1}(\mu)/G_\mu \to T(Q/G) \times_{Q/G} Q/G_\mu$$

is a diffeomorphism. □

The previous result has a local version: if L is locally G-regular, then Π_μ defines a local diffeomorphism. The proof is as follows. Choose a *small* neighborhood U of v_q in $J_L^{-1}(\mu)$. By local G-regularity, it is possible to obtain a local inverse for the map $(\Pi_\mu)_{|U}$: given $(w_{[q]}, q) \in \Pi_\mu(U)$, we choose a tangent vector $w_q \in U$ such that $T\pi(w_q) = w_{[q]}$. If U is small, by local G-regularity there exists a unique $\xi \in \mathfrak{g}$ such that $J_L(w_q + \xi_Q(q)) = \mu$, and $\Delta_\mu(w_{[q]}, q) = w_q + \xi_Q(q)$ defines a smooth inverse.

From the definition, it follows easily that local G-regularity ensures the existence of adapted charts on the level set $J_L^{-1}(\mu)$ in the following sense. Take coordinates $(x, g) = (x^i, g^a)$ on Q which are adapted to the fibration $\pi : Q \to Q/G$. Then, the implicit function theorem[b] guarantees that (x^i, v^i, g^a) is a coordinate chart on $J_L^{-1}(\mu)$, where (x^i, v^i) are standard lifted coordinates on $T(Q/G)$. This is an important observation and actually, it is the ultimate reason why Routh reduction works just fine under the assumption of local G-regularity, see [CM08]. Anyhow, for the sake of simplicity and to make some identifications global, we will henceforth assume that we are working with a G-regular Lagrangian.

Remark 2.9. *If L is a mechanical Lagrangian (kinetic energy minus potential), then L is G-regular: In this case $\mathbb{F}L$ is linear, and therefore*

$$\mathcal{J}_L^{v_q}(\xi) = J_L(v_q) + J_L(\xi_Q) = J_L(v_q) + \mathbb{I}_q(\xi)\,,$$

where $\mathbb{I}_q$ is the locked inertia tensor (see [Mar92] for an explanation of the terminology). Note that in this case J_L is an affine map modeled on $\mathbb{I}_q$.

More on G-regularity. There is an alternative, useful characterization of G-regularity in terms of the reduced Lagrangian on TQ/G. Recall from Section 1.3 that there exists a diffeomorphism

$$\alpha_{\mathfrak{A}} : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$$

depending on an arbitrary principal connection $\mathfrak{A}$ on the bundle $Q \to Q/G$, defined as follows (see also (1.10) for more details):

$$[v_q]_G \mapsto \ \alpha_{\mathfrak{A}}([v_q]_G) = T\pi(v_q) \oplus [q, (\mathfrak{A}(v_q))_G] = v_x \oplus [q,\xi]_G\,.$$

To ease the notation, let us write $\tilde{\xi} = [q,\xi]_G$. Using the identification $\alpha_{\mathfrak{A}}$, any invariant Lagrangian L determines a reduced Lagrangian l on $T(Q/G) \oplus \tilde{\mathfrak{g}}$. We define the map

$$\mathbb{F}_{\mathfrak{g}} l : T(Q/G) \oplus \tilde{\mathfrak{g}} \to T(Q/G) \oplus \tilde{\mathfrak{g}}^*\,,$$

fibred over the identity on $T(Q/G)$, by

$$\langle \mathbb{F}_{\mathfrak{g}} l(v_x, \tilde{\xi}), (v_x, \tilde{\eta})\rangle = \left.\frac{d}{dt}\right|_{t=0} l(v_x, \tilde{\xi} + t\tilde{\eta})\,,$$

where the pairing is between the elements in $\tilde{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}}$.

Proposition 2.10. *With the notations above, an invariant Lagrangian L is G-regular if, and only if, $\mathbb{F}_{\mathfrak{g}} l$ is a diffeomorphism.*

Proof. Consider an element $v_q \in TQ$ such that $\alpha_{\mathfrak{A}}([v_q]_G) = (v_x, \tilde{\xi})$. Then:

$$\begin{aligned}\langle J_L(v_q), \zeta\rangle = \langle \mathbb{F}L(v_q), \zeta_Q(q)\rangle &= \left.\frac{d}{dt}\right|_{t=0} L(v_q + t\zeta_Q(q)) = \left.\frac{d}{dt}\right|_{t=0} l(v_x, \tilde{\xi} + t\tilde{\zeta}) \\ &= \langle \mathbb{F}_{\mathfrak{g}} l(v_x, \tilde{\xi}), (v_x, \tilde{\zeta})\rangle\,.\end{aligned}$$

From this equality, the result follows easily. ∎

[b] Note that in coordinates, local G-regularity is equivalent to L being regular with respect to the *group* variables.

We will return later to the notion of G-regularity in the more general context of "magnetic Lagrangian systems" in Section 3.1. For a more detailed discussion of G-regularity, we refer the interested reader to [LL10] and [LCV10].

2.2.3 The Routhian $\mathcal{R}^\mu$

Let us take a moment to summarize the situation so far. We have started with a hyperregular G-invariant Lagrangian L, whose dynamics is governed by Hamilton equations with symplectic form Ω_Q^L and Hamiltonian function E_L. E_L is an invariant function, and thus we can define the reduced Hamiltonian system on $(J_L^{-1}(\mu)/G_\mu)$ whose symplectic form is given by symplectic reduction and whose Hamiltonian is obtained by G_μ-invariance of $i_\mu^* E_L$ (where $i_\mu : J_L^{-1}(\mu) \to TQ$ denotes the inclusion).

Moreover, under the additional assumption of G-regularity on L, we have constructed a symplectomorphism (see Diagram 2.5):

$$[S^\mu] \circ [\mathbb{F}L_\mu] \circ [\Delta_\mu] : T(Q/G) \times_{Q/G} Q/G_\mu \to T^*(Q/G) \times_{Q/G} Q/G_\mu \,.$$

$$\begin{tikzcd}
T(Q/G) \times_{Q/G} Q \arrow[r, "\Delta_\mu"] \arrow[d] & J_L^{-1}(\mu) \arrow[r, "\mathbb{F}L_\mu"] \arrow[d, "\pi_\mu^L"] & J^{-1}(\mu) \arrow[r, "S^\mu"] \arrow[d, "\pi_\mu"] & J^{-1}(0) \arrow[d, "\pi_0"] \\
T(Q/G) \times_{Q/G} Q/G_\mu \arrow[r, "{[\Delta_\mu]}"] \arrow[rrr, bend right, "{[S^\mu]\circ[\mathbb{F}L_\mu]\circ[\Delta_\mu]}"'] & J_L^{-1}(\mu)/G_\mu \arrow[r, "{[\mathbb{F}L_\mu]}"] & J^{-1}(\mu)/G_\mu \arrow[r, "{[S^\mu]}"] & T^*(Q/G) \times_{Q/G} Q/G_\mu
\end{tikzcd}$$

DIAGRAM 2.5: Relations among the different maps

We would like to reinterpret this dynamics as the one associated to a *reduced Lagrangian* function that we call the Routhian $\mathcal{R}^\mu$. In order for the analogy to be faithful, it should satisfy the following properties:

I) The map $[S^\mu] \circ [\mathbb{F}L_\mu] \circ [\Delta_\mu]$ should be the fibre derivative of the function $\mathcal{R}^\mu$ on the reduced space $T(Q/G) \times_{Q/G} Q/G_\mu$.

II) The reduced Hamiltonian should be expressed as the energy of $\mathcal{R}^\mu$.

Before we can give a precise meaning to this, we need to adapt a few definitions to the context of reduced Lagrangian systems:

i) There is a natural pairing between elements of the reduced spaces, defined as follows. Consider elements $(\alpha_x, y) \in T^*(Q/G) \times_{Q/G} Q/G_\mu$ and $(v_x, y) \in T(Q/G) \times_{Q/G} Q/G_\mu$, then:

$$\langle (\alpha_x, y), (v_x, y) \rangle = \langle \alpha_x, v_x \rangle \,.$$

ii) Let $f \in C^\infty(T(Q/G) \times_{Q/G} Q/G_\mu)$. The fibre derivative of f

$$\mathbb{F}f : (T(Q/G) \times_{Q/G} Q/G_\mu) \to (T^*(Q/G) \times_{Q/G} Q/G_\mu),$$

is defined by

$$\langle \mathbb{F}f(v_x, y), (w_x, y) \rangle = \left.\frac{d}{dt}\right|_{t=0} f(v_x + tw_x, y).$$

iii) Let $f \in C^\infty(T(Q/G) \times_{Q/G} Q/G_\mu)$. The energy E_f of f is the function

$$E_f(v_x, y) = \langle \mathbb{F}f(v_x, y), (v_x, y) \rangle - f(v_x, y).$$

Remark 2.11. *The previous definitions apply also to the case of fibred products of the type $TQ \times_Q E$ and $T^*Q \times_Q E$ where Q is an arbitrary manifold and $E \to Q$ is a fibre bundle. We will see more details later in Section 3.1.*

Define the function R^μ by

$$R^\mu = L - \mathfrak{A}_\mu, \tag{2.9}$$

where $\mathfrak{A}$ is the connection on $Q \to Q/G$ used to construct the map S^μ. It is manifestly G_μ-invariant, so its restriction to $J_L^{-1}(\mu)$ defines a reduced function $[R^\mu]$ on $J_L^{-1}(\mu)/G_\mu$. Let the corresponding function on $T(Q/G) \times_{Q/G} Q/G_\mu$ be denoted $\mathcal{R}^\mu$:

$$\mathcal{R}^\mu = [\Delta_\mu]^*[R^\mu] \in C^\infty(T(Q/G) \times_{Q/G} Q/G_\mu). \tag{2.10}$$

We call $\mathcal{R}^\mu$ the *Routhian* or the *Routh function*.

Proposition 2.12. *Under the above conditions we have:*

1) $[S^\mu] \circ [\mathbb{F}L_\mu] \circ [\Delta_\mu] = \mathbb{F}\mathcal{R}^\mu$.

2) The reduced Hamiltonian function $[i_\mu^* E_L]$ is precisely $E_{\mathcal{R}^\mu}$, i.e.

$$(\Pi_\mu \circ \pi_\mu^L)^* E_{\mathcal{R}^\mu} = i_\mu^* E_L.$$

Proof. The reader is invited to take a look back at Diagram 2.5. Our proof follows [LCV10].

1) Consider an arbitrary element $(v_x, q) \in T(Q/G) \times_{Q/G} Q$ and denote by $v_q \in J_L^{-1}(\mu)$ the vector which projects onto (v_x, q) (i.e. $\Pi_\mu(v_q) = ((v_x, q))$, see (2.8)). By the definition of the maps involved we have

$$(S^\mu \circ \mathbb{F}L \circ \Delta_\mu)(v_x, q) = \mathbb{F}L(v_q) - \mathfrak{A}_\mu(q) \in J^{-1}(0),$$

or in other words, using the identification (2.4) in Lemma 2.2,

$$\langle (S^\mu \circ \mathbb{F}L \circ \Delta_\mu)(v_x, q), (w_x, q) \rangle = \langle \mathbb{F}L(v_q) - \mathfrak{A}_\mu(q), w_q \rangle. \tag{2.11}$$

On the other hand, consider a curve $(v_x + tw_x, q) \subset T(Q/G) \times_{Q/G} Q$. Its image under the map Δ_μ defines a curve

$$\gamma(t) = \Delta_\mu((v_x + tw_x, q)) \subset J_L^{-1}(\mu)$$

which satisfies:

(a) $\gamma(0) = v_q$, with $T\pi(v_q) = v_x$.

(b) $\gamma'(0)$ is the vertical lift of some w_q with $T\pi(w_q) = w_x$. This follows directly from the definition of Δ_μ. (See also [LCV10] for an argument involving coordinates.)

Denote by $\bar{\mathcal{R}}^\mu = \Delta_\mu^* R^\mu$. Then

$$\left.\frac{d}{dt}\right|_0 \bar{\mathcal{R}}^\mu(v_x + tw_x, q) = \left.\frac{d}{dt}\right|_0 (L(\gamma(t)) - \mathfrak{A}_\mu(\gamma(t))) = \langle \mathbb{F}L(v_q) - \mathfrak{A}_\mu(q), w_q \rangle .$$

In the usual form, this reads:

$$\langle \mathbb{F}\bar{\mathcal{R}}^\mu(v_x, q), (w_x, q) \rangle = \langle \mathbb{F}L(v_q) - \mathfrak{A}_\mu(q), w_q \rangle . \tag{2.12}$$

Comparison between (2.11) and (2.12) yields $\mathbb{F}\bar{\mathcal{R}}^\mu = (S^\mu \circ \mathbb{F}L \circ \Delta_\mu)$. The result now follows easily from equivariance of the maps and the definition of the Routhian (2.10).

In short, if (v_x, y) and (w_x, y) are arbitrary elements of $T(Q/G) \times_{Q/G} Q/G_\mu$, we have:

$$\langle \mathbb{F}\mathcal{R}^\mu(v_x, y), (w_x, y) \rangle = \langle ([S^\mu] \circ [\mathbb{F}L_\mu] \circ [\Delta_\mu])\, (v_x, y), (w_x, y) \rangle . \tag{2.13}$$

2) Using the previous relation (2.13), the result now follows by "diagram chasing". Indeed, consider a vector $v_q \in J_L^{-1}(\mu)$ with $(\Pi_\mu \circ \pi_\mu^L) = (v_x, y)$. Then, from the definition of the Routhian it follows:

$$\begin{aligned} i_\mu^* E_L(v_q) &= \langle \mathbb{F}L(v_q), v_q \rangle - L(v_q) = \langle S^\mu \circ \mathbb{F}L(v_q), v_q \rangle - (L(v_q) - \mathfrak{A}_\mu(v_q)) \\ &= \langle ([S^\mu] \circ [\mathbb{F}L_\mu] \circ [\Delta_\mu])(v_x, y), (v_x, y) \rangle - \mathcal{R}^\mu(v_x, y) \\ &= \langle \mathbb{F}\mathcal{R}^\mu(v_x, y), (v_x, y) \rangle - \mathcal{R}^\mu(v_x, y) \\ &= E_{\mathcal{R}^\mu}(v_x, y) . \end{aligned}$$

□

Collecting all the results above, we can make a precise sense of the reduced dynamics as a Lagrangian system.

Theorem 2.13 (Routh reduction). *Let L be an hyperregular G-invariant, G-regular Lagrangian on the configuration space Q. Then the Marsden-Weinstein reduction of (TQ, Ω_Q^L) with momentum value μ is the symplectic manifold*

$$\left(T(Q/G) \times_{Q/G} Q/G_\mu, (\mathbb{F}\mathcal{R}^\mu)^* \left(\pi_1^* \Omega_{Q/G} + \pi_2^* \mathcal{B}_\mu\right)\right) .$$

The reduced Hamiltonian corresponding to E_L is $E_{\mathcal{R}^\mu}$.

We denote this reduced system by $(Q/G_\mu \to Q/G, \mathcal{R}^\mu, \mathcal{B}_\mu)$, and refer to it as a *magnetic Lagrangian system.* This terminology and notation will be clarified later in Section 3.1 of the following chapter.

2.3 Routh reduction on product manifolds

In this section we describe Routh reduction for Lagrangian systems whose configuration manifold is of the form $Q = S \times G$, where S is an arbitrary manifold, and the Lagrangian L is defined on $TQ = TS \times TG$. Our exposition follows [LGTAC12].

There is a left action of G on Q given by $\phi^Q_{g'}(s, g) = (s, L_{g'} g) = (s, g'g)$, with $L_{g'}$ left multiplication on G by g'. The lifted action ϕ^{TQ} on TQ has the form $\phi^{TQ} : G \times TQ; (g', (v_s, v_g)) \mapsto (v_s, g'v_g)$, where $g'v_g$ is a shorthand notation for $TL_{g'}(v_g)$. (Similarly, we will write $v_g g'$ for $TR_{g'}(v_g)$, with $R_{g'}$ right translation.)

The left identification. We use the left identification of TG with $G\times\mathfrak{g}$, i.e. $v_g \mapsto (g, \xi)$ with $\xi = g^{-1}v_g \in \mathfrak{g}$. The tangent bundle $TQ = TS \times TG$ is then isomorphic with $TS\times G\times\mathfrak{g}$, and we write $L(v_s, g\xi)$ accordingly. The lifted action ϕ^{TQ} on TQ corresponds to left multiplication in the middle factor of $TS \times G \times \mathfrak{g}$: if $(v_s, v_g) \mapsto (v_s, g, \xi)$ then $\phi^{TQ}_{g'}(v_s, v_g) \mapsto (v_s, g'g, \xi)$. With this left identification in mind, it is easy to check that the fundamental vector field ξ_Q corresponding to $\xi \in \mathfrak{g}$ takes the form:

$$\xi_Q(s, g) = (0_s, g, Ad_{g^{-1}}\xi) \in TS \times G \times \mathfrak{g}\,.$$

If the given Lagrangian L is invariant with respect to the lifted action ϕ^{TQ}, the corresponding expression for L becomes independent of G, i.e. $L(v_s, g\xi) = L(v_s, g'\xi)$ for any $g, g' \in G$. After identifying TQ with $TS \times G \times \mathfrak{g}$, an invariant function L determines a function ℓ on $TS \times \mathfrak{g}$:

$$\ell(v_s, \xi) := L(v_s, g\xi)\,.$$

The purpose now is to express Routh reduction of the G-invariant Lagrangian system $(Q = S \times G, L)$ in terms of the function ℓ. It turns out that in this case we can write down an explicit form for the reduced equations of motion.

The momentum map. We recall the definition of the momentum map J_L, evaluated at $(v_s, g\xi) \in TQ = TS \times TG$, and we substitute $\ell(v_s, \xi)$ for $L(v_s, g\xi)$:

$$\begin{aligned}\langle J_L(v_s, g\xi), \eta\rangle &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L\big(v_s, g(\xi + \epsilon Ad_{g^{-1}}\eta)\big) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(v_s, \xi + \epsilon Ad_{g^{-1}}\eta)\\ &= \langle \mathbb{F}_2\ell(v_s, \xi), Ad_{g^{-1}}\eta\rangle\,,\end{aligned}$$

for all $\eta \in \mathfrak{g}$, and where $\mathbb{F}_2\ell : TS \times \mathfrak{g} \to \mathfrak{g}^*$ is precisely by the relation

$$\langle \mathbb{F}_2\ell(v_s, \xi), \tau\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(v_s, \xi + \epsilon\tau), \quad \text{for all } \tau \in \mathfrak{g}\,.$$

Consequently, we conclude from the above that

$$J_L(v_s, g\xi) = Ad^*_{g^{-1}}\mathbb{F}_2\ell(v_s, \xi)\,. \tag{2.14}$$

Therefore, for every $\mu \in \mathfrak{g}^*$ the equation $J_L(v_s, g\xi) = \mu$ can be equivalently written as $\mathbb{F}_2\ell(v_s, \xi) = Ad^*_g\mu$.

G-regularity. Next, we investigate the G-regularity of L. Recall that this is in fact a condition on J_L: for every $g \in G$, $\xi \in \mathfrak{g}$, $v_s \in TS$ and $\nu \in \mathfrak{g}^*$, there exists a unique $\eta \in \mathfrak{g}$ such that

$$J_L(v_s, g\xi + g\eta) = \nu\,.$$

Using the foregoing result this is equivalent to $\mathbb{F}_2\ell(v_s, \xi + \eta) = Ad_g^*\nu$. In particular, it follows that G-regularity of L is equivalent here to the condition that the map $\mathbb{F}_2\ell(v_s, \cdot) : \mathfrak{g} \to \mathfrak{g}^*$ is invertible. Denote the inverse map by $\chi^{(v_s)} : \mathfrak{g}^* \to \mathfrak{g}$, where $\chi^{(v_s)}$ depends smoothly on the 'parameter' $v_s \in TS$. For later use we define $\chi_\mu^{(v_s)}$ to be the restriction of $\chi^{(v_s)}$ to the coadjoint orbit $\mathcal{O}_\mu$ of some $\mu \in \mathfrak{g}^*$.

The connection 1-form. The standard principal connection on $Q = S \times G$ regarded as a G-bundle over S is $\mathfrak{A}(v_s, v_g) = v_g g^{-1} \in \mathfrak{g}$. This is in fact the trivial extension to $S \times G$ of the canonical connection on G associated to the left multiplication. The corresponding map from $TS \times G \times \mathfrak{g} \to \mathfrak{g}$ is $(v_s, g\xi) \mapsto \mathfrak{A}(v_s, g\xi) = Ad_g\xi$. We now check that $\mathfrak{A}$ satisfies the two conditions to be a connection 1-form (see Definition 1.13):

(1) $\mathfrak{A}(\xi_Q(s,g)) = \xi$, for $\xi \in \mathfrak{g}$ arbitrary. Indeed, from $\xi_Q(s,g) = (0_s, gAd_{g^{-1}}\xi)$, it follows that $\mathfrak{A}(0_s, gAd_{g^{-1}}\xi) = Ad_g(Ad_{g^{-1}}\xi) = \xi$.

(2) Equivariance: $\mathfrak{A}(v_s, g'g\xi) = Ad_{g'}\mathfrak{A}(g\xi)$ for all $g', g \in G$ and $\xi \in \mathfrak{g}$. The left-hand side of this equality becomes $\mathfrak{A}(v_s, g'g\xi) = Ad_{g'g}\xi = Ad_{g'}Ad_g\xi$, which is precisely equal to the right-hand side.

The horizontal distribution for this connection is the subbundle $TS \times 0_G$ of TQ.

Next, for $\mu \in \mathfrak{g}^*$ we compute the 1-form $\mathfrak{A}_\mu$ on $S \times G$. Note that the connection $\mathfrak{A}$ is flat (the horizontal subbundle is obviously integrable) and, therefore, using Cartan structure equation (1.8) it follows:

$$\begin{aligned} d\mathfrak{A}_\mu(s,g)\big((v_s, g\xi), (w_s, g\xi')\big) &= \langle \mu, [\mathfrak{A}(v_s, g\xi), \mathfrak{A}(w_s, g\xi')]\rangle = \langle \mu, [Ad_g\xi, Ad_g\xi']\rangle \\ &= \langle \mu, Ad_g[\xi, \xi']\rangle = \langle Ad_g^*\mu, [\xi, \xi']\rangle\,. \end{aligned}$$

Computation of the 2-form $\mathcal{B}_\mu$. We now reduce the 2-form $d\mathfrak{A}_\mu$ to $Q/G_\mu = S \times G/G_\mu$. Recall that G/G_μ is diffeomorphic to $\mathcal{O}_\mu$, with diffeomorphism defined by $G/G_\mu \to \mathcal{O}_\mu, [g]_{G_\mu} = G_\mu g \to Ad_g^*\mu$. The tangent space to $\mathcal{O}_\mu$ at some $\nu \in \mathcal{O}_\mu$ is given by (cf. [MR99]):

$$T_\nu\mathcal{O}_\mu = \{ad_\xi^*\nu \mid \xi \in \mathfrak{g}\}\,.$$

Recall that $\mathcal{B}_\mu$ is defined as the reduction of $d\mathfrak{A}_\mu$. In particular in this case $\mathcal{B}_\mu$ will be a 2-form on $S \times \mathcal{O}_\mu$.

Lemma 2.14. *The form $\mathcal{B}_\mu$ is given by:*

$$\mathcal{B}_\mu(s,\nu)\left((v_s, ad_\xi^*\nu), (w_s, ad_{\xi'}^*\nu)\right) := \langle \nu, [\xi, \xi']\rangle\,.$$

Proof. Let $g\xi$ be a tangent vector at $g \in G$ and consider the curve $g(t) = g \cdot exp(t\xi)$. Under the map $G \to \mathcal{O}_\mu$; $g \mapsto Ad^*_g\mu$, $g\xi$ projects onto the tangent vector:

$$\left.\frac{d}{dt}\right|_0 Ad^*_{g(t)}\mu = ad^*_\xi Ad^*_g\mu\,.$$

Assume that g satisfies $Ad^*_g\mu = \nu$. Then from the definition of $d\mathfrak{A}_\mu$, it follows

$$\begin{aligned}\mathcal{B}_\mu(s,\nu)\big((v_s, ad^*_\xi\nu),(w_s, ad^*_{\xi'}\nu)\big) &= d\mathfrak{A}_\mu(s,g)\big((v_s,g\xi),(w_s,g\xi')\big)\\ &= \langle \nu, [\xi,\xi']\rangle\,.\end{aligned}$$

□

One can verify that this is a closed 2-form. In fact, $\mathcal{B}_\mu$ is the trivial extension to $S \times \mathcal{O}_\mu$ of the standard Kostant-Kirillov-Souriau symplectic form ω^+ on $\mathcal{O}_\mu$ (see e.g. [MR99]).

Routh reduction. The expression for R^μ (given by (2.9)) reads

$$R^\mu(v_s, g\xi) = L(v_s, g\xi) - \mathfrak{A}_\mu(v_s, g\xi) = \ell(v_s,\xi) - \langle Ad^*_g\mu, \xi\rangle\,.$$

Since in the case under consideration we have that $J_L^{-1}(\mu)/G_\mu \cong TS \times \mathcal{O}_\mu$ and, hence, the map Π_μ defined in (2.8) becomes the identity, we find the Routhian $\mathcal{R}^\mu$ by taking the restriction of $\ell - \mathfrak{A}_\mu$ to $J_L^{-1}(\mu)$ and projecting it onto $J_L^{-1}(\mu)/G_\mu$. The momentum equation $J_L = \mu$ is equivalent to $\mathbb{F}_2\ell(v_s, \xi+\eta) = Ad^*_g\mu$. Thus, if we denote $\nu = Ad^*_g\mu \in \mathcal{O}_\mu$, we have:

$$\mathcal{R}^\mu(v_s,\nu) = \left(\ell(v_s,\xi) - \langle\nu,\xi\rangle\right)\big|_{\xi=\chi_\mu^{(v_s)}(\nu)}\,, \tag{2.15}$$

where $\nu = Ad^*_g\mu \in \mathcal{O}_\mu$.

The local equations of motion. In view of the observations above, the reduced symplectic manifold is

$$\big(TS \times \mathcal{O}_\mu, (\mathbb{F}\mathcal{R}^\mu)^*(\pi_S^*\Omega_S + \pi_{\mathcal{O}}^*\omega^+)\big)\,.$$

where $\pi_S : T^*S \times \mathcal{O}_\mu \to S$ and $\pi_{\mathcal{O}} : T^*S \times \mathcal{O}_\mu \to \mathcal{O}_\mu$ are the canonical projections. The Euler-Lagrange equations can be split in two parts that we describe next.

i) The component in TS has the structure of standard Euler-Lagrange equations: if $x = (x^i)$ is a coordinate system on S, then the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial\mathcal{R}^\mu}{\partial\dot x}(x,\dot x,\nu)\right) - \frac{\partial\mathcal{R}^\mu}{\partial x}(x,\dot x,\nu) = 0\,.$$

ii) The evolution equation on the coadjoint orbit is computed as follows. Let e^a denote a basis for $\mathfrak{g}^*$ and let $\dot\nu = \dot\nu_a e^a$ and $\dot\nu' = \dot\nu'_a e^a$ be arbitrary tangent vectors to $\mathcal{O}_\mu$ at ν. Hamilton equations, when contracted with a tangent vector of the form $(0_{v_s}, \dot\nu')$ at a point (v_s, ν) are:

$$\omega^+_\nu\left(\dot\nu, \dot\nu'\right) = \frac{\partial \mathcal{R}^\mu}{\partial \nu_a}(v_s, \nu)\dot\nu'_a\,.$$

The right hand side is easily computed from (2.15):

$$\begin{aligned}\frac{\partial \mathcal{R}^\mu}{\partial \nu_a}(v_s, \nu)\dot\nu'_a &= \left\langle \mathbb{F}_2\ell(v_s, \chi^{(v_s)}(\nu)), \frac{\partial \chi^{(v_s)}}{\partial \nu_a}(\nu)\dot\nu'_a \right\rangle - \dot\nu'_a\big(\chi^{(v_s)}(\nu)\big)^a - \left\langle \nu, \frac{\partial \chi^{(v_s)}}{\partial \nu_a}(\nu)\dot\nu'_a \right\rangle\\ &= -\langle \dot\nu', \chi^{(v_s)}(\nu)\rangle\,.\end{aligned}$$

Therefore, the reduced equation of motion is $\omega^+_\nu(\dot\nu, \dot\nu') = -\langle \dot\nu', \chi^{(v_s)}_\mu(\nu)\rangle$, with $\dot\nu'$ arbitrary in $T_\nu\mathcal{O}_\mu$. We conclude that one component of the Euler-Lagrange equation is precisely

$$\dot\nu = ad^*_{\chi^{(v_s)}_\mu(\nu)}\nu\,.$$

Summarizing, we have proved the following result:

Theorem 2.15. *Let ℓ denote the restriction to $TS \times \mathfrak{g}$ of a left G-invariant Lagrangian L on $T(S \times G)$ and let $\mathbb{F}_2\ell : TS \times \mathfrak{g} \to \mathfrak{g}^*$ denote the fibre derivative w.r.t. the second argument. Fix an element μ in $\mathfrak{g}^*$ and assume that there exists a map $\chi^{(v_s)} : \mathfrak{g}^* \to \mathfrak{g}$ which smoothly depends on $v_s \in TS$, such that $\mathbb{F}_2\ell(v_s, \chi^{(v_s)}(\nu)) \equiv \nu$ for arbitrary $(v_s, \nu) \in TS \times \mathfrak{g}^*$. Then, the reduced system is the magnetic Lagrangian system $(S \times \mathcal{O}_\mu \to S, \mathcal{R}^\mu, \mathcal{B}_\mu)$ where the 2-form $\mathcal{B}_\mu$ on $S \times \mathcal{O}_\mu$ and the Routhian $\mathcal{R}^\mu$ on $TS \times \mathcal{O}_\mu$ are given by, respectively,*

$$\mathcal{B}_\mu(s, \nu)\left((v_s, ad^*_\xi\nu), (w_s, ad^*_{\xi'}\nu)\right) = \langle \nu, [\xi, \xi']\rangle\,,$$

and

$$\mathcal{R}^\mu(v_s, \nu) = \big(\ell(v_s, \xi) - \langle \nu, \xi\rangle\big)_{\xi = \chi^{(v_s)}_\mu(\nu)}\,.$$

(Here, $\chi^{(v_s)}_\mu$ is the restriction of $\chi^{(v_s)}$ to the coadjoint orbit $\mathcal{O}_\mu$). In a local coordinate chart $x = (x^i)$ on S, the equations of motion for the reduced system are a system of coupled first and second order differential equations:

$$\left\{\begin{aligned} &\dot\nu = ad^*_{\chi^{(v_s)}_\mu(\nu)}\nu\,,\\ &\frac{d}{dt}\left(\frac{\partial \mathcal{R}^\mu}{\partial \dot x}(x, \dot x, \nu)\right) - \frac{\partial \mathcal{R}^\mu}{\partial x}(x, \dot x, \nu) = 0\,.\end{aligned}\right. \tag{2.16}$$

A similar result holds in case we are dealing with a Lagrangian on $Q = S \times G$ which is right invariant, i.e. which is invariant under the lifted action of $\Psi^Q : G \times Q \to Q$, $(g', (s, g)) \mapsto (s, gg')$. Given the appropriate function ℓ, the reduced equation of motion in this case will slightly differ from those obtained above: the component along $\mathcal{O}_\mu$ becomes $\dot\nu = -ad^*_{\chi^{(v_s)}_\mu(\nu)}\nu$.

2.4 Examples

The "3-beanie"

This example is taken from [GTALC14]. Consider three planar rigid bodies with a common fixed point O, so that each body is free to rotate about the axis through O, orthogonal to the plane. The configuration space is $S^1 \times S^1 \times S^1$, with coordinates (θ, φ, ψ) where θ is the angle which the first body makes with a fixed direction in the plane, φ is the relative angle of the second rigid body w.r.t. the first and finally ψ denotes the relative angle of the third rigid body w.r.t. the second (see Figure 2.6).

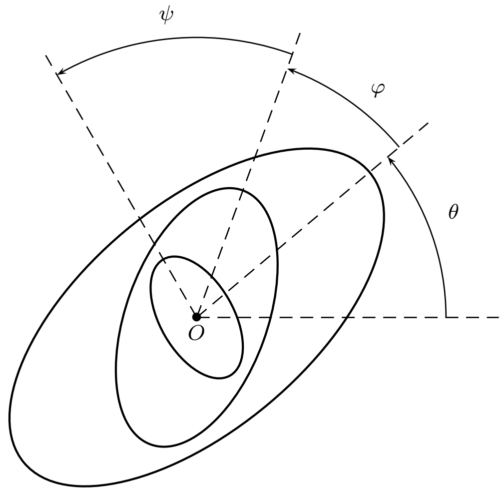

FIGURE 2.6: Coordinates for the 3-beanie

The system moves under the influence of a potential of the form $V(\varphi, \psi)$. Then the S^1-invariant Lagrangian is

$$L = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 (\dot{\theta} + \dot{\varphi})^2 + \frac{1}{2} I_3 (\dot{\theta} + \dot{\varphi} + \dot{\psi})^2 - V(\varphi, \psi) \,,$$

and the associated Euler-Lagrange equations (in normal form) are:

$$\ddot{\theta} = \frac{1}{I_1} \frac{\partial V}{\partial \varphi} \,, \quad \ddot{\varphi} = -\left(\frac{I_1 + I_2}{I_2 I_1} \right) \frac{\partial V}{\partial \varphi} + \frac{1}{I_2} \frac{\partial V}{\partial \psi} \,, \quad \ddot{\psi} = -\left(\frac{I_2 + I_3}{I_3 I_2} \right) \frac{\partial V}{\partial \psi} + \frac{1}{I_2} \frac{\partial V}{\partial \varphi} \,.$$

The Routhian is a function on $T(S^1 \times S^1)$. Fix a value μ for the momentum $J = (I_1 + I_2 + I_3)\dot{\theta} + (I_2 + I_3)\dot{\varphi} + I_3 \dot{\psi}$. Then on the level set $\{J = \mu\}$ we have $\dot{\theta} = (\mu - (I_2 + I_3)\dot{\varphi} + I_3 \dot{\psi})/(I_1 + I_2 + I_3)$. We work out the Routhian for three different connections:

(1) The *classical Routhian* R_c^μ discussed in Section 2.1 corresponds to the connection

given by $\mathfrak{A}^0 = d\theta$. We have:

$$
\begin{aligned}
R_c^\mu :=& \frac{1}{2}\left[\frac{I_1\,(I_2+I_3)}{(I_1+I_2+I_3)}\right]\dot{\varphi}^2 + \frac{1}{2}\left[\frac{I_3\,(I_1+I_2)}{(I_1+I_2+I_3)}\right]\dot{\psi}^2 + \left[\frac{I_1 I_3}{(I_1+I_2+I_3)}\right]\dot{\varphi}\dot{\psi} \\
&+ \left[\frac{I_3\mu}{(I_1+I_2+I_3)}\right]\dot{\psi} + \left[\frac{(I_2+I_3)\mu}{(I_1+I_2+I_3)}\right]\dot{\varphi} - V(\varphi,\psi) - \frac{\mu^2}{2\,(I_1+I_2+I_3)}\,.
\end{aligned}
$$

(2) Consider the *mechanical connection* whose horizontal spaces are orthogonal to the G-orbits with respect to the metric given by the kinetic energy. It is easy to check that the corresponding connection 1-form has the following expression:

$$
\mathfrak{A}^M = d\theta + \frac{I_2+I_3}{I_1+I_2+I_3}d\varphi + \frac{I_3}{I_1+I_2+I_3}d\psi\,.
$$

The Routhian $\mathcal{R}_M^\mu(\varphi,\dot{\varphi},\psi,\dot{\psi}) = \big(L - \langle\mathfrak{A}^M,\mu\rangle\big)_{\{J=\mu\}}$ satisfies:

$$
\begin{aligned}
\mathcal{R}_M^\mu =& \frac{1}{2}\left[\frac{I_1\,(I_2+I_3)}{(I_1+I_2+I_3)}\right]\dot{\varphi}^2 + \frac{1}{2}\left[\frac{I_3\,(I_1+I_2)}{(I_1+I_2+I_3)}\right]\dot{\psi}^2 + \left[\frac{I_1 I_3}{(I_1+I_2+I_3)}\right]\dot{\varphi}\dot{\psi} \\
&- V(\varphi,\psi) - \frac{\mu^2}{2\,(I_1+I_2+I_3)}\,.
\end{aligned}
$$

Note that with this choice of the connection the Routhian is again of mechanical type.

(3) Take now the non-flat connection given by $\mathfrak{A}^t = d\theta + \cos(\psi)d\varphi$. The Routhian $\mathcal{R}_t^\mu(\varphi,\dot{\varphi},\psi,\dot{\psi}) = \big(L - \langle\mathfrak{A}^t,\mu\rangle\big)_{\{J=\mu\}}$ reads:

$$
\mathcal{R}_t^\mu = R_c^\mu - \mu\cos(\psi)\dot{\varphi}\,.
$$

The force term is $d\mathfrak{A}_\mu^t = \mu\sin(\psi)d\varphi\wedge d\psi$.

An easy computation shows that the Euler-Lagrange equations for any of the three Routhian functions above are equivalent to the Euler-Lagrange equations for the variables (φ,ψ) of L. Together with the momentum equation, they provide complete solutions of the original system.

This example illustrates an important fact about Routh reduction: the choice of the connection is arbitrary and always leads to the same Euler-Lagrange equations.

The rigid body with a rotor

This example is analized in [BKMSdA92]. Following [LGTAC12], we will apply the results on Section 2.3 to obtain the reduced equations of motion via Routh reduction.

We consider a rigid body with a single rotor along the third principal axis of the body. The configuration space of this system is $Q = S^1 \times SO(3)$, where $SO(3)$ is the configuration space of the rigid body and S^1 measures the angle of the rotor relative to the body frame which we denote by x. In the body frame of the principal inertia axes, the (reduced) Lagrangian $\ell : TS^1 \times so(3) \to \mathbb{R}$ has the following expression:

$$\ell(x, \dot{x}, \omega) = \frac{1}{2}\left(\omega \mathbb{I} \omega + (\omega + \alpha)\mathbb{J}(\omega + \alpha)\right),$$

where $\mathbb{I}$ and $\mathbb{J}$ are the inertia tensors corresponding to the rigid body and the rotor, respectively, $\omega = (\omega_1, \omega_2, \omega_3)$ denotes the angular velocity of the body and $\alpha := (0, 0, \dot{x})$ corresponds to the angular velocity of the rotor, both in the body frame. Introducing the quantities $\lambda_i = I_i + J_i$, $i = 1, 2, 3$, the Lagrangian becomes explicitly

$$\ell(x, \dot{x}, \omega) = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2 + J_3\dot{x}^2) + J_3\dot{x}\omega_3 .$$

The map $\mathbb{F}_2\ell(x, \dot{x}, \cdot) : \mathbb{R}^3(\cong so(3)) \to \mathbb{R}^3(\cong so^*(3))$ is given by

$$\mathbb{F}_2\ell(\dot{x}, \omega) = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3 + J_3\dot{x}),$$

and its inverse equals

$$\chi^{(x,\dot{x})}(m) = \left(\frac{1}{\lambda_1}m_1, \frac{1}{\lambda_2}m_2, \frac{1}{\lambda_3}(m_3 - J_3\dot{x})\right),$$

where, in the notations of Section 2.3, $m = (m_1, m_2, m_3) \in \mathbb{R}^3 \cong so^*(3)$ corresponds to ν and $(x, \dot{x})$ corresponds to v_s. For the actual computation of the Routhian, we use the property that for Lagrangians of mechanical type with potential energy $V(s)$, the Routhian can be computed from the following identity (see [Par65]):

$$2\big(\mathcal{R}^\mu(v_s, \nu) + V(s)\big) = \left(\langle \mathbb{F}_1\ell(v_s, \xi), v_s\rangle - \langle \mathbb{F}_2\ell(v_s, \xi), \xi\rangle\right)\big|_{\xi = \chi_\mu^{(v_s)}(v_s, \nu)} .$$

Using the above expression, the Routhian is obtained in a straightforward way:

$$\begin{aligned}
\mathcal{R}^{m_0}(x, \dot{x}, m) &= \frac{1}{2}\left((J_3\dot{x}^2 + J_3\dot{x}\omega_3) - \lambda_1\omega_1^2 - \lambda_2\omega_2^2 - \omega_3(\lambda_3\omega_3 + J_3\dot{x})\right)\big|_{\omega = \chi_{m_0}^{(x,\dot{x})}(m)} \\
&= \frac{1}{2}\left(J_3\dot{x}^2\left(1 - \frac{J_3}{\lambda_3}\right) - \frac{m_1^2}{\lambda_1} - \frac{m_2^2}{\lambda_2} - \frac{m_3^2}{\lambda_3}\right) + \frac{J_3}{\lambda_3}\dot{x}m_3 \\
&= \frac{1}{2}\left(\frac{J_3 I_3}{\lambda_3}\dot{x}^2 - \frac{m_1^2}{\lambda_1} - \frac{m_2^2}{\lambda_2} - \frac{m_3^2}{\lambda_3}\right) + \frac{J_3}{\lambda_3}\dot{x}m_3 .
\end{aligned}$$

Note that the difference with the Routhian obtained in [JM00], which was computed using the mechanical connection, is the appearance of the product term $\dot{x}m_3$. The reduced equations on $so^*(3)$ read $\dot{m} = ad^*_{\chi_{m_0}^{(x,\dot{x})}(m)} m = m \times \chi_{m_0}^{(x,\dot{x})}(m)$. Finally, the full reduced set of equations of motion corresponding to $\mathcal{R}^{m_0}$ are:

$$\begin{aligned}
\dot{m}_1 &= \left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right)m_2m_3 - \frac{m_2J_3}{\lambda_3}\dot{x}, &\qquad \dot{m}_2 &= \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right)m_1m_3 + \frac{m_1J_3}{\lambda_3}\dot{x}, \\
\dot{m}_3 &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)m_1m_2, & I_3\ddot{x} &= -\dot{m}_3 .
\end{aligned}$$

Remark 2.16. *In [BLM01] the previous example is also treated in the context of controlled Lagrangians (see also [BLM00] and [BCLM01]).*

Chapter 3

Lagrangian systems closed under reduction

In the previous chapter we have described the structure of the dynamical system that arises after symplectic reduction of a Lagrangian system (Q, L) by a Lie group of symmetries G. In particular we have shown that, under some regularity conditions, this dynamical system is represented by the Euler-Lagrange equations of a generalized Lagrangian system $(Q/G_\mu \to Q/G, \mathcal{R}^\mu, \mathcal{B}_\mu)$ that we have called of *magnetic type*.

The aim of this chapter is to describe the dynamics associated to a magnetic Lagrangian system and to introduce a class of transformations $\psi_{L,\beta}$ between magnetic Lagrangian systems which preserves the dynamics. Building on these transformations we will carry out Routh reduction of a magnetic Lagrangian system, and it will turn out that the class of magnetic Lagrangian systems is closed under Routh reduction.

The content is organized as follows. Section 3.1 formalizes the concept of magnetic Lagrangian system and presents its main properties. Section 3.2 defines the aforementioned class of transformations $\psi_{L,\beta}$ between magnetic Lagrangian systems. Section 3.3 discusses a first application of these transformations in the setting of reduction by a semi-direct product. Section 3.4 describes the more general procedure of Routh reduction for magnetic Lagrangian systems by making use of the class $\psi_{L,\beta}$.

3.1 Systems of magnetic type

A magnetic Lagrangian system is a Lagrangian system whose configuration space is the total space of a bundle $\epsilon : E \to Q$, and where the Lagrangian is independent of the velocities tangent to the fibres of ϵ. Additionally, the system may be subjected to a gyroscopic force term. More precisely, we have the following definition (see [LMV11]):

Definition 3.1. *A* magnetic Lagrangian system *(MLS) consists of a triple* $(\epsilon : E \to Q, L, \mathcal{B})$ *where* $\epsilon : E \to Q$ *is a fibre bundle,* L *is a smooth function on the fibred product* $TQ \times_Q E$ *and* $\mathcal{B}$ *is a closed 2-form on* E*. We say that* E *is the configuration manifold of the system,* L *is the Lagrangian and* $\mathcal{B}$ *is the magnetic 2-form.*

For simplicity, we will always assume that the magnetic form $\mathcal{B}$ is of constant rank. Note that the definition of a MLS includes the standard definition of a Lagrangian system: this case corresponds to $E = Q$, $\epsilon = id_Q$ and $\mathcal{B} = 0$. In this sense, the concept of a magnetic Lagrangian systems extends the standard concept of Lagrangian systems.

Points in Q and E are denoted by q and e, respectively. Assuming that $\dim Q = n$ and $\dim E = n+k$, local coordinates on Q will be denoted by (q^i) and adapted coordinates on E will be denoted by (q^i, r^a), with $i = 1, \ldots, n = \dim Q$, $a = 1, \ldots, k = \dim E - \dim Q$. The induced bundle coordinates on $TQ \times_Q E$ are then given by (q^i, v^i, r^a) where (q^i, v^i) are the coordinates of a point on TQ, and thus the Lagrangian L is locally expressed as a function of (q^i, v^i, r^a). In particular, we observe that L does not depend on the velocities in the fibre coordinates and, therefore, becomes singular when interpreted as a Lagrangian on the "full" tangent bundle TE. The 2-form $\mathcal{B}$ has the following coordinate expression:

$$\mathcal{B} = \frac{1}{2}\mathcal{B}_{ij} dq^i \wedge dq^j + \mathcal{B}_{ia} dq^i \wedge dr^a + \frac{1}{2}\mathcal{B}_{ab} dr^a \wedge dr^b\,. \tag{3.1}$$

Before we go on, we need some notations:

i) The fibred product $TQ \times_Q E$ will be abbreviated by $T_E Q$ and a point in $T_E Q$ will be denoted by (v_q, e), where $v_q \in T_q Q$ and $e \in E$ are such that $\epsilon(e) = q$. Similarly, $T^*_E Q$ denotes the fibred product $T^*Q \times_Q E$ and (p_q, e) represents an arbitrary point in $T^*_P Q$, with $p_q \in T^*_q Q$ and $\epsilon(e) = q$.

ii) $V\epsilon$ denotes the distribution on E of tangent vectors vertical with respect to ϵ.

iii) $\hat{\epsilon} : TE \to T_E Q$ is the projection fibred over E that maps $v_e \in TE$ onto $(T\epsilon(v_e), e) \in T_E Q$.

iv) $\tau_1 : T_E Q \to TQ$ is the projection that maps $(v_q, e) \in T_E Q$ onto $v_q \in TQ$.

v) $\tau_2 : T_E Q \to E$ is the projection that maps $(v_q, e) \in T_E Q$ onto $e \in E$.

vi) $\pi_1 : T^*_E Q \to T^*Q$ is the projection that maps $(p_q, e) \in T^*_E Q$ onto $p_q \in T^*Q$.

vii) $\pi_2 : T^*_E Q \to E$ is the projection that maps $(p_q, e) \in T^*_E Q$ onto $e \in E$.

In the same way, when working with several bundles $\epsilon^{(i)} : E_i \to Q_i$ we let $\tau_1^{(i)}$ and $\tau_2^{(i)}$ (respectively, $\pi_1^{(i)}$ and $\pi_2^{(i)}$) denote the corresponding projections maps of $T_{E_i} Q_i$ (resp. $T^*_{E_i} Q_i$).

Definition 3.2. *Assume a magnetic Lagrangian system* $(\epsilon : E \to Q, L, \mathcal{B})$ *is given.*

1) The Legendre transform corresponding to L is the map $\mathbb{F}L : T_EQ \to T_E^*Q$ sending $(v_q, e) \in T_EQ$ into $(p_q, e) \in T_E^*Q$, where $p_q \in T_q^*Q$ is uniquely determined by the relation

$$\langle p_q, w_q \rangle = \left.\frac{d}{du}\right|_{u=0} L(v_q + uw_q, e)\,,$$

for all $w_q \in T_qQ$.

2) The function on T_EQ defined by $E_L(v_q, e) = \langle \mathbb{F}L(v_q, e), (v_q, e)\rangle - L(v_q, e)$ is called the energy of the magnetic Lagrangian system. (Here, the contraction of an element $(p_q, e) \in T_E^*Q$ with $(v_q, e) \in T_EQ$ is defined naturally as $\langle (p_q, e), (v_q, e)\rangle := \langle p_g, v_q\rangle$.)

3) Let $\Omega_Q = d\theta_Q$ be the canonical symplectic form on T^*Q. By means of the Legendre transform, we can pull-back the closed 2-form $\pi_1^*\Omega_Q + \pi_2^*\mathcal{B}$ on T_E^*Q to a closed 2-form on T_EQ

$$\Omega^{L,\mathcal{B}} := \mathbb{F}L^*(\pi_1^*\Omega_Q + \pi_2^*\mathcal{B})\,. \tag{3.2}$$

For convenience of the reader, we summarize the main maps introduced before in the following diagram:

$$\begin{tikzcd}
 & TQ \arrow[dl, "\tau_Q"'] & T^*Q \\
Q & T_EQ \arrow[u, "\tau_1"] \arrow[r, "\mathbb{F}L"] \arrow[d, "\tau_2"'] & T_E^*Q \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & E \arrow[ul, "\epsilon"] \arrow[r, "id_E"'] & E
\end{tikzcd}$$

Let us now specify the dynamical system we associate with a magnetic Lagrangian system. A curve $\big(q^i(t), r^a(t)\big)$ in E is called a solution of the magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ if the induced curve $\gamma(t) = (\dot{q}^i(t), r^a(t)) \in T_EQ$ satisfies

$$i_{\dot{\gamma}(t)}\Omega^{L,\mathcal{B}}(\gamma(t)) = -dE_L(\gamma(t))\,.$$

The local expressions for the 2-form $\Omega^{L,\mathcal{B}}$ and the 1-form dE_L read

$$\Omega^{L,\mathcal{B}} = d\left(\frac{\partial L}{\partial v^i}\right) \wedge dq^i + \frac{1}{2}\mathcal{B}_{ij}dq^i \wedge dq^j + \mathcal{B}_{ia}dq^i \wedge dr^a + \frac{1}{2}\mathcal{B}_{ab}dr^a \wedge dr^b \tag{3.3}$$

and

$$dE_L = v^i d\left(\frac{\partial L}{\partial v^i}\right) + \frac{\partial L}{\partial v^i}dv^i - dL = v^i d\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i}dq^i - \frac{\partial L}{\partial r^a}dr^a\,, \tag{3.4}$$

respectively. Using (3.3) and (3.4) one can readily check that a curve $(q^i(t), r^a(t))$ in E is a solution of the magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ if, and only if, it satisfies the following set of mixed second and first order ordinary differential equations, referred to as the *Euler-Lagrange* equations of the MLS $(\epsilon : E \to Q, L, \mathcal{B})$:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} &= \mathcal{B}_{ij}\dot{q}^j + \mathcal{B}_{ia}\dot{r}^a \,, \\ -\frac{\partial L}{\partial r^b} &= -\mathcal{B}_{ib}\dot{q}^i + \mathcal{B}_{ab}\dot{r}^a \,, \end{aligned} \tag{3.5}$$

for $i = 1, \ldots, n$ and $a = 1, \ldots, k$. We remark that these equations are the standard Euler-Lagrange equations for the singular Lagrangian $\hat{\epsilon}^* L$ on TE subjected to a magnetic force term. It follows that a curve $\gamma(t)$ in E is a solution of the MLS $(\epsilon : E \to Q, L, \mathcal{B})$ if it is a critical curve of the (ordinary) Lagrangian system $(E, \hat{\epsilon}^* L)$ with gyroscopic force $\mathcal{B}$. In other words, γ satisfies the following variational principle (in the sense of D'Alembert):

$$\delta \int_a^b L(\gamma, \dot{\gamma})\, dt = \int_a^b \langle \iota_{\dot{\gamma}} \mathcal{B}, \delta\gamma \rangle \, dt \,,$$

for arbitrary variations with fixed endpoints.

Definition 3.3. *A magnetic Lagrangian system is called* regular *if the following two conditions are satisfied:*

1) The 2-form $\pi_1^* \Omega_Q + \pi_2^* \mathcal{B}$ is symplectic.

2) $\mathbb{F}L$ is a local diffeomorphism.

If, in addition, $\mathbb{F}L$ is a global diffeomorphism, the magnetic Lagrangian system is called hyperregular*.*

From the local expression for the magnetic 2-form (3.1), the first of these conditions is equivalent to $\det \mathcal{B}_{ab} \neq 0$ if $\dim E > \dim Q$. The following proposition is an immediate consequence of Definition 3.3.

Proposition 3.4. *If a magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ is hyperregular, the 2-form $\Omega^{L,\mathcal{B}} = \mathbb{F}L^*(\pi_1^* \Omega_Q + \pi_2^* \mathcal{B})$ determines a symplectic structure on $T_E Q$.*

We conclude that a hyperregular magnetic Lagrangian system induces a symplectic structure on $T_E Q$ and its dynamics is represented by the Hamiltonian vector field with respect to this symplectic structure and with the energy function as Hamiltonian:

$$i_{X_{E_L}} \Omega^{L,\mathcal{B}} = -dE_L \,.$$

Clearly, each integral curve of X_{E_L} projects onto a solution of the magnetic Lagrangian system.

A large supply of regular magnetic Lagrangians is provided by the kind of magnetic Lagrangians which are inspired upon mechanical systems.

Definition 3.5. *A magnetic Lagrangian system* $(\epsilon : E \to Q, L, \mathcal{B})$ *is said to be of mechanical type if* $L(v_q, e) = \frac{1}{2}\langle\langle (v_q, e), (v_q, e)\rangle\rangle_{\tau_2} - V(e)$*, where* $\langle\langle \cdot, \cdot \rangle\rangle_{\tau_2}$ *is a metric on the vector bundle* $\tau_2 : T_E Q \to E$ *and* V *is a function on* E*.*

The Hamiltonian picture of magnetic systems is obtained in an analogous way adapting the definitions from the ordinary Hamiltonian systems on a cotangent bundle (cf. [AM78]). We give the definitions with no further explanation.

Definition 3.6. *A magnetic Hamiltonian system (MHS) is a triple* $(\epsilon : E \to Q, H, \mathcal{B})$ *where* $\epsilon : E \to Q$ *is a fibre bundle,* H *is a smooth function on the fibred product* $T^*_E Q$ *and* $\mathcal{B}$ *is a closed 2-form on* P*.* H *is called the* Hamiltonian*.*

There is a presymplectic structure $\Omega^{\mathcal{B}}$ on $T^*_E Q$ given by $\Omega^{\mathcal{B}} := \pi_1^*\Omega_Q + \pi_2^*\mathcal{B}$. A curve $(q^i(t), p_i(t), r^a(t)) \subset T^*_E Q$ is a solution of the Hamilton equations for the presymplectic manifold $(T^*_E Q, \Omega^{\mathcal{B}})$ i, and only if, it satisfies the following system of differential equations:

$$\dot{q}^i = \frac{\partial H}{\partial p^i}\,, \qquad \dot{p}_i + \mathcal{B}_{ij}\dot{q}^j - \mathcal{B}_{ia}\dot{r}^a = -\frac{\partial H}{\partial q^i}\,, \qquad \mathcal{B}_{ib}\dot{q}^i + \mathcal{B}_{ab}\dot{r}^a = -\frac{\partial H}{\partial r^b}\,. \tag{3.6}$$

The fibre derivative $\mathbb{F}H : T^*_E Q \to T_E Q$ of the Hamiltonian H sends $(p_q, e) \in T^*_E Q$ into $(v_q, e) \in T_E Q$, where $(v_q, e) \in T_E Q$ is determined by the relation:

$$\langle \bar{p}_q, v_q \rangle = \left.\frac{d}{du}\right|_{u=0} H(p_q + u\bar{p}_q, e)\,,$$

for all $\bar{p}_q \in T^*_q Q$. A magnetic Hamiltonian system is *(hyper-)regular* if $\Omega^{\mathcal{B}}$ is symplectic and $\mathbb{F}H$ is a (global) diffeomorphism.

Hyperregular systems and equivalence. Just like in the case of standard Hamiltonian and Lagrangian systems, there exists a bijective correspondence between hyperregular magnetic Lagrangian systems and hyperregular magnetic Hamiltonian systems. Formally, the construction of a hyperregular MHS from a hyperregular MLS (or viceversa) is identical to the one for standard systems and, therefore, we shall not describe it here[a].

3.2 Transformations between magnetic Lagrangian systems

In this section we introduce a particular type of mappings relating two magnetic Lagrangian systems. This can be seen as a generalization of the concept of point transformations in Lagrangian mechanics and is partially inspired upon the techniques encountered in the theory of Routh reduction. First, we need to recall some generalities concerning the pull-back of a symplectic structure and investigate the relationship between Hamiltonian vector fields that are connected by such a pull-back operation.

[a]In the case of standard Lagrangian and Hamiltonian systems, this equivalence is detailed in e.g. [MR99].

3.2.1 Pull-back Hamiltonian systems

Consider the situation where we are given two manifolds N, M and a smooth map $f : N \to M$ of constant rank. Assume, in addition, that M is a symplectic manifold with symplectic form ω_M, and that we are given a Hamiltonian h_M on M. Let us denote by X_{h_M} the corresponding Hamiltonian vector field, which satisfies $i_{X_{h_M}}\omega_M = -dh_M$. Consider then the presymplectic form $\omega_N = f^*\omega_M$ and the Hamiltonian $h_N = f^*h_M$ induced on N. A Hamiltonian vector field on N with respect to ω_N, corresponding to h_N, is determined by the presymplectic equation

$$i_X\omega_N = -dh_N \,, \tag{3.7}$$

and we are interested in those cases where (some of) the integral curves of X_{h_M} can be retrieved from integral curves of a solution to (3.7). More precisely, we investigate when X_{h_M} is f-related to a solution X of (3.7). Recall that solutions to (3.7), if they exist, are determined up to elements in the kernel of ω_N, which we denote by TN^{ω_N}, and that $Tf\,(TN^{\omega_N}) = [Tf(TN)]^{\omega_M} \cap Tf(TN)$, where $[Tf(TN)]^{\omega_M}$ is the kernel of the restriction of ω_M to $TM_{|f(N)}$.

First note that any vector field Y on N which is f-related to X_{h_M}, solves (3.7): for any $x \in N$ and $Z_x \in T_xN$ it follows that

$$\begin{aligned}\omega_N(x)\big(Y_x, Z_x\big) &= \omega_M(f(x))\big(Tf(Y_x), Tf(Z_x)\big) = -dh_M(f(x))\big(Tf(Z_x)\big)\\ &= -dh_N(x)(Z_x)\,.\end{aligned}$$

A necessary condition for X_{h_M} to be f-related to a vector field on N is $X_{h_M|f(N)} \in Tf(TN)$, or equivalently

$$\langle dh_M, [Tf(TN)]^{\omega_M}\rangle_{|f(N)} = 0\,. \tag{3.8}$$

If X solves (3.7) and (3.8) holds, the vector $Tf(X_x) - X_{h_M}(f(x))$ is in $Tf(T_xN^{\omega_N})$ for all x in the domain of X, i.e. X can be gauged by an element in the kernel of ω_N so that it becomes f-related to X_{h_M}. To show that (3.8) is also a sufficient condition for the existence of an f-related solution of (3.7), we need to show that it implies the existence of a solution (3.7). For that purpose, we rely on the presymplectic constraint algorithm developed by M. Gotay, J.M. Nester and G. Hinds (see [GNH78, GN79]).

The starting point of the presymplectic constraint algorithm is the observation that (3.7) admits a solution at a point $x \in N$ if the following condition holds: $\langle Z_x, dh_N(x)\rangle = 0$ for all $Z \in TN^{\omega_N}$. The set of all these points is assumed to form a (immersed) submanifold of N, i.e.

$$N_2 = \{x \in N \,:\, \langle dh_N(x), T_xN^{\omega_N}\rangle = 0\}\,,$$

called the secondary constraint submanifold. The next step then consists in requiring that one should be able to find a vector field solution to (3.7) which is tangent to N_2. This possibly leads to new constraints defining a constraint submanifold $N_3 = \{x \in N_2 : \langle dh_N(x), T_xN_2^{\omega_N}\rangle = 0\}$, where $TN_2^{\omega_N} = \{X \in TN_{|N_2} \,:\, \omega_N(X,Y) = 0 \text{ for all } Y \in TN_2\}$.

Proceeding this way one generates a descending sequence of constraint submanifolds $\ldots \subset N_k \subset \ldots \subset N_2 \subset N := N_1$, where

$$N_k = \{x \in N_{k-1} \,:\, \langle dh_N(x), T_x N_{k-1}^{\omega_N} \rangle = 0\}$$

for $k = 2, \ldots$, with $TN_{k-1}^{\omega_N} = \{X \in TN_{|N_{k-1}} \,:\, \omega_N(X,Y) = 0 \text{ for all } Y \in TN_{k-1}\}$. If this sequence stabilizes at some finite step $K \in \mathbb{N}$, in the sense that $N_K \neq \emptyset$ and $N_{K+1} = N_K$, we say that N_K is the *final constraint (sub-)manifold*. In that case, equation (3.7) admits solutions on N_K, and we say that the presymplectic equation leads to a consistent dynamics on N_K.

Returning to the situation described above, we are now able to prove that (3.7) admits a consistent dynamics on N provided the Hamiltonian vector field X_{h_M} is everywhere tangent to $f(N)$. In fact we have:

Proposition 3.7. *There exists a solution X of* (3.7) *which is f-related to X_{h_M} if and only if $X_{h_M|f(N)} \in Tf(TN)$.*

Proof. It suffices to check the first step of the presymplectic constraint algorithm. Indeed, from $Tf\,(TN^{\omega_N}) \subset [Tf(TN)]^{\omega_M}$ and using equation (3.8) it follows that for all $x \in N$

$$\langle dh_N(x), T_x N^{\omega_N} \rangle = \langle dh_M(f(x)), T_x f\,(T_x N^{\omega_N}) \rangle = 0\,,$$

proving that N is the final constraint manifold for (3.7) which therefore admits a solution. Hence, according to a previous observation, there also exists a solution which is f-related to X_{h_M}. □

3.2.2 Compatible transformations

Our purpose now is to specialize the symplectic framework described above to the case of interest in the study of magnetic Lagrangian systems, namely fibred products with presymplectic structures of the form $\Omega^{L,\mathcal{B}}$. First, we define the notion of compatible transformation $\psi : T_{E_1}Q_1 \to T_{E_2}Q_2$ between two fibred products and characterize its coordinate expression. Next, in Subsection 3.2.3, we will define a class of compatible transformations which closely relates to Routh reduction. To make the exposition somehow more explicit, we will use the case of Routh reduction (for ordinary Lagrangian systems) as the main example.

Definition 3.8. *Let $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$ be two given fibre bundles. If $F : E_1 \to E_2$ and $f : Q_2 \to Q_1$ are two surjective submersions we say that the pair (F, f) forms a* transformation pair *between both bundles if the following equality holds:*

$$f \circ \epsilon^{(2)} \circ F = \epsilon^{(1)},$$

and all the arrows in Diagram 3.1 are fibre bundles.

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\epsilon^{(1)}"'] & E_2 \arrow[d, "\epsilon^{(2)}"] \\ Q_1 & Q_2 \arrow[l, "f"'] \end{tikzcd}$$

DIAGRAM 3.1: Transformation pair

We write $\dim Q_i = n_i$ and $\dim E_i = n_i + k_i$ for $i = 1, 2$. Because F and f are submersions, it follows that $n_1 + k_1 \geq n_2 + k_2$ and $n_1 \leq n_2$. This way we find the relation $k_1 \geq k_2$ between the dimensions of the fibers of the bundles $\epsilon^{(i)} : E_i \to Q_i$. A transformation pair induces a chain of bundle structures $E_1 \to E_2 \to Q_2 \to Q_1$. Choosing coordinates adapted to these fibrations, we let (q^i) denote coordinates on Q_1, $(q^i, \bar{q}^a)$ coordinates on Q_2, $(q^i, \bar{q}^a, \bar{r}^\alpha)$ on E_2 and finally $(q^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma)$ on E_1. We have then the following natural sets of coordinates: $(q^i, v^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma)$ on $T_{E_1}Q_1$ and $(q^i, \bar{q}^a, v^i, \bar{v}^a, \bar{r}^\alpha)$ on $T_{E_2}Q_2$.

Definition 3.9. *Let (F, f) be a transformation pair between $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$. Then:*

1) *Two points $(v_{q_i}, e_i) \in T_{E_i}Q_i$, $i = 1, 2$ are* (F, f)-compatible *if $F(e_1) = e_2$ and $Tf(v_{q_2}) = v_{q_1}$.*

2) *A smooth map $\psi : T_{E_1}Q_1 \to T_{E_2}Q_2$ is* compatible *with the transformation pair (F, f) if for every point $s_1 = (v_{q_1}, e_1) \in T_{E_1}Q_1$, the points s_1 and $\psi(s_1)$ are (F, f)-compatible.*

We simply say that ψ is a *compatible transformation* or *compatible map*. Compatibility for a map ψ is equivalently specified by the following two conditions:

i) $\tau_2^{(2)} \circ \psi = F \circ \tau_2^{(1)}$;

ii) $Tf \circ \tau_1^{(2)} \circ \psi = \tau_1^{(1)}$.

The situation is summarized in Diagram 3.2:

$$\begin{tikzcd} T_{E_1}Q_1 \arrow[r, "F\circ\tau_2^{(1)}"] \arrow[d, "\tau_1^{(1)}"'] \arrow[dr, "\psi"] & E_2 \\ TQ_1 & T_{E_2}Q_2 \arrow[l, "Tf\circ\tau_1^{(2)}"'] \arrow[u, "\tau_2^{(2)}"'] \end{tikzcd}$$

DIAGRAM 3.2: Commutative diagram for a compatible map ψ

We use coordinates adapted to the fibrations as introduced before to describe both a point and its image by ψ. It is then readily checked that compatible maps convey to the following coordinate expression:

$$\psi(q^i, v^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma) = (q^i, \bar{q}^a, v^i, \bar{v}^a = \psi^a(q^i, v^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma), \bar{r}^\alpha).$$

Note that the rank of a transformation ψ is determined by the rank of the matrix $(\partial\psi^a/\partial r^\gamma)_{a,\gamma}$ in the following way: $\operatorname{rank}\psi = \dim E_2 + \dim Q_1 + \operatorname{rank}(\partial\psi^a/\partial r^\gamma)_{a,\gamma}$. In particular, for ψ to be a diffeomorphism the dimension of the fibers corresponding to f and F must the same and $\det(\partial\psi^a/\partial r^\gamma)_{a,\gamma} \neq 0$.

The compatibility of points gives naturally a notion of compatibility of vectors by lifting the conditions to the tangent spaces:

Definition 3.10. *Let (F, f) be a transformation pair between $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$, and let $s_1 = (v_{q_1}, e_1) \in T_{E_1}Q_1$ and $s_2 = (v_{q_2}, e_2) \in T_{E_2}Q_2$ be arbitrary points. Given Y_{s_1} and X_{s_2} tangent vectors at s_1 and s_2 respectively, we say that Y_{s_1} and X_{s_2} are* (F, f)-compatible *if the following two conditions are satisfied:*

1) $T\big(F \circ \tau_2^{(1)}\big)(Y_{s_1}) = T\tau_2^{(2)}(X_{s_2})$;

2) $T\tau_1^{(1)}(Y_{s_1}) = T\big(Tf \circ \tau_1^{(2)}\big)(X_{s_2})$.

In particular, s_1 and s_2 need to be (F, f)-compatible points (see Diagram 3.3).

$$\begin{tikzcd}
T_{E_1}Q_1 \arrow[r, "F\circ\tau_2^{(1)}"] \arrow[d, "\tau_1^{(1)}"'] & E_2 \\
TQ_1 & T_{E_2}Q_2 \arrow[l, "Tf\circ\tau_1^{(2)}"'] \arrow[u, "\tau_2^{(2)}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T(T_{E_1}Q_1) \arrow[r, "T\left(F\circ\tau_2^{(1)}\right)"] \arrow[d, "T\tau_1^{(1)}"'] & TE_2 \\
T(TQ_1) & T(T_{E_2}Q_2) \arrow[l, "T\left(Tf\circ\tau_1^{(2)}\right)"'] \arrow[u, "T\tau_2^{(2)}"']
\end{tikzcd}$$

DIAGRAM 3.3: Compatible points and tangent vectors

Consider an arbitrary vector Y_{s_1} tangent to $T_{E_1}Q_1$ at the point s_1. Its coordinate expression is

$$Y_{s_1} = Y^i_{s_1}\frac{\partial}{\partial q^i}\bigg|_{s_1} + Y^a_{s_1}\frac{\partial}{\partial \bar{q}^a}\bigg|_{s_1} + Y^\alpha_{s_1}\frac{\partial}{\partial \bar{r}^\alpha}\bigg|_{s_1} + Y^\gamma_{s_1}\frac{\partial}{\partial r^\gamma}\bigg|_{s_1} + \hat{Y}^i_{s_1}\frac{\partial}{\partial v^i}\bigg|_{s_1}, \tag{3.9}$$

and reading the local expressions of the previous definition, a compatible tangent vector X_{s_2} at the compatible point s_2 assumes the following form:

$$X_{s_2} = Y^i_{s_1}\frac{\partial}{\partial q^i}\bigg|_{s_2} + Y^a_{s_1}\frac{\partial}{\partial \bar{q}^a}\bigg|_{s_2} + Y^\alpha_{s_1}\frac{\partial}{\partial \bar{r}^\alpha}\bigg|_{s_2} + \hat{Y}^i_{s_1}\frac{\partial}{\partial v^i}\bigg|_{s_2} + \hat{X}^a_{s_2}\frac{\partial}{\partial \bar{v}^a}\bigg|_{s_2}. \tag{3.10}$$

Given a compatible transformation ψ between $\epsilon^{(1)}: E_1 \to Q_1$ and $\epsilon^{(2)}: E_2 \to Q_2$, it is clear that Y_{s_1} and $X_{s_2} = T\psi(Y_{s_1})$ are compatible vectors for any $Y_{s_1} \in T_{E_1}Q_1$. In this particular case, from the coordinate expression of a compatible map, we find:

$$\hat{X}^a_{s_2} = Y^i_{s_1}\left.\frac{\partial \psi^a}{\partial q^i}\right|_{s_1} + \hat{Y}^i_{s_1}\left.\frac{\partial \psi^a}{\partial v^i}\right|_{s_1} + Y^a_{s_1}\left.\frac{\partial \psi^a}{\partial \bar{q}^a}\right|_{s_1} + Y^\alpha_{s_1}\left.\frac{\partial \psi^a}{\partial \bar{r}^\alpha}\right|_{s_1} + Y^\gamma_{s_1}\left.\frac{\partial \psi^a}{\partial r^\gamma}\right|_{s_1}.$$

Example (Routh reduction). Consider a hyperregular standard Lagrangian system $(Q \to Q, L, \mathcal{B} = 0)$ amenable to Routh reduction, i.e. there is a left G-action and L is G-invariant and G-regular. Consider the (trivial) bundles $\epsilon^{(2)} = \mathrm{id}_Q : E_2 = Q \to Q_2 = Q$ and $\epsilon^{(1)} = \pi : E_1 = Q \to Q_1 = Q/G$ (Diagram 3.4). The maps $F = \mathrm{id}_Q : E_1 \to E_2$ and $f = \pi : Q_2 = Q \to Q_1 = Q/G$ are a transformation pair between $\epsilon^{(1)}$ and $\epsilon^{(2)}$. Then $T_{E_1}Q_1 = T_Q(Q/G)$, $T_{E_2}Q_2 = TQ$ and it follows:

i) points $(v_{[q]_G}, q)$ and v_q in $T_Q(Q/G)$ and TQ respectively, are compatible if $v_{[q]_G} = T\pi(v_q)$;

ii) a map $\psi : T_Q(Q/G) \to TQ$ is compatible if it sends $(v_{[q]_G}, q)$ to a tangent vector in TQ projectable to $v_{[q]_G}$, i.e. the map is determined up to a gauge in $\mathfrak{g}$;

iii) tangent vectors $X \in T(TQ)$ and $Y = (Y^Q, Y^{T(Q/G)}) \in T(T_Q(Q/G)) \cong TQ \times_{T(Q/G)} T(T(Q/G))$ are compatible if $T\tau_Q(X) = Y^Q$ and $T(T\pi)(X) = Y^{T(Q/G)}$.

$$\begin{tikzcd}
Q \arrow[r, "id_Q"] \arrow[d] & Q \arrow[d] \\
Q/G & Q \arrow[l, "\pi"']
\end{tikzcd}$$

DIAGRAM 3.4: Routh reduction scheme in TQ

Remark 3.11. *We already pointed out that an ordinary Lagrangian system is a special instance of a magnetic Lagrangian system with $E \equiv Q$ and $T_EQ \equiv TQ$. Consider a point transformation between E_2 and E_1, i.e., a diffeomorphism $f : Q_2 \to Q_1$. Then the pair $(F = f^{-1}, f)$ is a transformation pair and the tangent lift of f^{-1}, i.e. $\psi = Tf^{-1}$, is a compatible transformation.*

3.2.3 A family of compatible transformations

We now proceed with the case where in addition to a transformation pair (F, f) between $\epsilon^{(1)}$ and $\epsilon^{(2)}$, a magnetic Lagrangian system $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$ is given. It is our purpose to construct a class of (F, f)-compatible transformations by means of the Lagrangian

L_2 such that, under suitable regularity conditions, pulls-back the Hamiltonian vector field $X_{E_{L_2}}$ on $T_{E_2}Q_2$ to a vector field on $T_{E_1}Q_1$ which is the Hamiltonian vector field associated to a new magnetic Lagrangian system on $\epsilon^{(1)}$.

Assume we are given a magnetic Lagrangian system $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$, together with a transformation pair (F, f) between $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$. We will now construct a family of compatible transformations $\psi_{L_2,\beta} : T_{E_1}Q_1 \to T_{E_2}Q_2$ between these spaces. As the notation suggests, this family depends on the Lagrangian L_2 and on an arbitrary map $\beta : E_1 \to V^*f$, where V^*f is the dual of the bundle Vf of tangent vectors vertical to the fibration f.

First we introduce the notion of f-regularity of the Lagrangian L_2. Consider the map $\alpha_{L_2} : T_{E_2}Q_2 \to V^*f$ which is defined as he composition of $\pi_1^{(2)} \circ \mathbb{F}L_2 : T_{E_2}Q_2 \to T^*Q_2$ with the projection of T^*Q_2 onto V^*f (see Diagram 3.5).

$$\begin{tikzcd}
T_{E_2}Q_2 \arrow[r, "\mathbb{F}L_2"] \arrow[ddr, "\alpha_{L_2}"'] & T^*_{E_2}Q_2 \arrow[d] \\
& T^*Q_2 \arrow[d] \\
& V^*f
\end{tikzcd}$$

DIAGRAM 3.5: Definition of α_{L_2}

Definition 3.12. *The Lagrangian L_2 is f-*regular *if for any given $s_2 = (v_{q_2}, e_2) \in T_{E_2}Q_2$ the map*

$$\begin{aligned}
\alpha_{L_2}^{s_2} : V_{q_2}f &\to V^*_{q_2}f \,, \\
w_{q_2} &\mapsto \alpha_{L_2}(v_{q_2} + w_{q_2}, e_2)\,,
\end{aligned}$$

is a diffeomorphism.

It is easily verified in coordinates that this condition is (locally) equivalent to the non-vanishing of the Hessian of L_2 with respect to the velocities, i.e.

$$\det\left(\frac{\partial^2 L_2}{\partial \bar{v}^a \partial \bar{v}^b}\right) \neq 0\,.$$

For f-regular Lagrangians, we are now ready to introduce a family of compatible maps $\psi_{L_2,\beta} : T_{E_1}Q_1 \to T_{E_2}Q_2$. For clarity, we describe the construction in three steps:

Step 1: Consider the map $\alpha_{L_2} : T_{E_2}Q_2 \to V^*f$, defined as above;

Step 2: Fix a map $\beta : E_1 \to V^*f$ such that $f \circ pr_{|V^*f} \circ \beta = \epsilon^{(1)}$, where $pr : T^*Q_2 \to Q_2$ denotes the standard projection on the cotangent bundle T^*Q_2 (see also Diagram 3.6);

$$\begin{tikzcd} E_1 \arrow[r, "\beta"] \arrow[d, "\epsilon^{(1)}"'] & V^*f \subset T^*Q_2 \arrow[d, "pr"] \\ Q_1 & Q_2 \arrow[l, "f"'] \end{tikzcd}$$

DIAGRAM 3.6: Commutative diagram for the map β

Step 3: Let $s_1 = (v_{q_1}, e_1)$ be an arbitrary point in $T_{E_1}Q_1$ and let $s_2 = (v_{q_2}, e_2) \in T_{E_2}Q_2$ be a compatible point (such a point always exists). Due to the f-regularity of L_2, there exists a unique tangent vector $w_{q_2} \in V_{q_2}f$ that satisfies $\alpha_{L_2}^{s_2}(w_{q_2}) = \beta(e_1)$, or alternatively

$$\pi_1^{(2)}\big(\mathbb{F}L_2(v_{q_2} + w_{q_2}, e_2)\big)_{|Vf} = \beta(e_1)\,.$$

We take the point $(v_{q_2} + w_{q_2}, e_2)$ as the image of $s_1 = (v_{q_1}, e_1)$ under $\psi_{L_2,\beta}$. The fact that L_2 is f-regular implies that the construction is independent of the choice of s_2.

By construction, the map $\psi_{L_2,\beta}$ is compatible and it satisfies $\alpha_{L_2} \circ \psi_{L_2,\beta} \equiv \beta \circ \tau_2^{(1)}$. In coordinates, writing $\beta = \beta_a d\bar{q}^a$, this relation takes the form:

$$\frac{\partial L_2}{\partial \bar{v}^a}\left(q^i, \bar{q}^a, \bar{r}^\alpha, v^i, \bar{v}^a = \psi^a(q^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma, v^i)\right) \equiv \beta_a(q^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma)\,.$$

Proposition 3.13. *$\psi_{L_2,\beta}$ is uniquely characterized by the following two conditions:*

1. *It is a compatible transformation;*
2. *It satisfies $\alpha_{L_2} \circ \psi_{L_2,\beta} \equiv \beta \circ \tau_2^{(1)}$.*

Proof. Take $s_1 \in T_{Q_1}E_1$ and let ψ be a compatible map that satisfies the relation above. This condition reads:

$$\frac{\partial L_2}{\partial \bar{v}^a}\,(\psi(s_1)) = \beta_a\big(\tau_2^{(1)}(s_1)\big)\,.$$

If we use regularity of α_{L_2} and apply the implicit function theorem, it follows that ψ is unique. $\square$

Example (Routh reduction). Recall the setup for a standard Lagrangian system on Q amenable to Routh reduction: $\epsilon^{(1)} = \pi : E_1 = Q \to Q_1 = Q/G$, $\epsilon^{(2)} = \mathrm{id}_Q : E_2 = Q \to Q_2 = Q$, $F = \mathrm{id}_Q$, $f = \pi : Q \to Q/G$ and $L_2 = L$, $\mathcal{B}_2 = 0$. The bundle Vf is the bundle of symmetry vectors $\{\xi_Q \,|\, \xi \in \mathfrak{g}\}$ and $\sigma^* \circ \alpha_L = J_L$. The map $\beta : Q \to V^*f \cong Q \times \mathfrak{g}^*$ is equivalent to a $\mathfrak{g}^*$-valued map on Q. Although we are running ahead of things, in the case of Routh reduction the map β is determined from a fixed value $\mu \in \mathfrak{g}^*$. Indeed $\beta : Q \to V^*\pi$ is characterized in the following way:

$$\langle \beta(q), \xi_Q(q) \rangle = \langle \mu, \xi \rangle\,,$$

for all $\xi \in \mathfrak{g}$. Thus, the second equation in Proposition 3.13 coincides with the momentum equation $J_{L_2} = \mu$. From the definition of the map α_L, the image of $(v_{[q]_G}, q)$ by the map $\psi_{L,\beta}$ is the element $(v_q + \eta_Q(q)) \in TQ$ with η is determined by the equality:

$$\langle \mathbb{F}L(v_q + \eta_Q(q)), \xi_Q(q) \rangle = \langle \beta(q), \xi_Q(q) \rangle = \langle \mu, \xi \rangle \,,$$

for all $\xi \in \mathfrak{g}$. From the definition of J_L it follows $\langle \mathbb{F}L(v_q + \eta_Q(q)), \xi_Q(q) \rangle = \langle J_L(v_q + \eta_Q(q)), \xi \rangle$ and hence, with the notations in the previous chapter, $\psi_{L,\beta} = \imath_\mu \circ \Pi_\mu^{-1}$.

Pull-back of a magnetic Lagrangian system under $\boldsymbol{\psi_{L_2,\beta}}$. In the next paragraphs we study the pull-back under $\psi_{L_2,\beta}$ of the (pre)symplectic system $\Omega^{L_2,\mathcal{B}_2}$ with energy (Hamiltonian) E_{L_2}. First, we show that the pull-back system is associated to a new magnetic Lagrangian system on $E_1 \to Q_1$. Afterwards, we find conditions on the map β such that the Euler-Lagrange equations of the pull-back system are related to the Euler-Lagrange equations of the initial magnetic Lagrangian system.

In order to define in an intrinsic way a Lagrangian on $E_1 \to Q_1$ whose associated 2-form equals $\psi^*_{L_2,\beta}\Omega^{L_2,\mathcal{B}_2}$, we choose a connection $\mathcal{A}$ on the bundle $f : Q_2 \to Q_1$. Recall from the introduction that $\mathcal{A}$ is a Vf-valued 1-form on Q_2, satisfying $\mathcal{A}(v_{q_2}) = v_{q_2}$, for all $v_{q_2} \in Vf$. Consider now the associated Vf-valued 1-form $\mathcal{A}_{E_1}$ on E_1 defined by $\mathcal{A}_{E_1}(v) = \mathcal{A}\big(T(\epsilon^{(2)} \circ F)(v)\big)$ for $v \in TE_1$. Contraction of β and $\mathcal{A}_{E_1}$ gives rise to the following 1-form on E_1:

$$\langle \beta, \mathcal{A}_{E_1} \rangle (e_1) = \langle \beta(e_1), \mathcal{A}_{E_1}(e_1) \rangle \in T^*_{e_1}E_1 \,.$$

If we denote the TQ_2-component of the transformation $\psi_{L_2,\beta} : T_{E_1}Q_1 \to T_{E_2}Q_2$ by $\psi^{TQ_2}_{L_2,\beta}$ (i.e. $\psi^{TQ_2}_{L_2,\beta} = \tau_1^{(2)}(\psi_{L_2,\beta})$), we have the following important result:

Theorem 3.14. *Let (F, f) be a transformation pair between $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$ and let $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$ be a magnetic Lagrangian systems such that L_2 is f-regular. Fix a connection $\mathcal{A}$ on the bundle $f : Q_2 \to Q_1$ and a map $\beta : E_1 \to V^*f$, and let $\psi_{L_2,\beta} : T_{E_1}Q_1 \to T_{E_2}Q_2$ be the (F, f)-compatible transformation constructed above. Consider the magnetic Lagrangian system $(\epsilon^{(1)}, L_1, \mathcal{B}_1)$ defined by*

i) $L_1(v_{q_1}, e_1) = \left(\psi^*_{L_2,\beta} L_2\right)(v_{q_1}, e_1) - \langle \beta(e_1), \mathcal{A}(\psi^{TQ_2}_{L_2,\beta}(v_{q_1}, e_1)) \rangle$;

ii) $\mathcal{B}_1 = F^*\mathcal{B}_2 + d\left(\langle \beta, \mathcal{A}_{E_1} \rangle\right)$.

Then $\psi_{L_2,\beta}$ satisfies:

1) $\psi^*_{L_2,\beta}\Omega^{L_2,\mathcal{B}_2} = \Omega^{L_1,\mathcal{B}_1}$;

2) $\psi^*_{L_2,\beta}E_{L_2} = E_{L_1}$.

Proof. The transformation pair (F, f) induces a chain of bundle structures:

$$E_1 \xrightarrow{F} E_2 \xrightarrow{\epsilon^{(2)}} Q_2 \xrightarrow{f} Q_1 \,.$$

As before, we choose coordinate charts that are adapted to these fibrations. Then the map $\psi_{L_2,\beta}$ has only nontrivial components in $\dot{\bar{q}}^a = \psi^a_{L_2,\beta}(q, \bar{q}, \dot{q}, \bar{r}, r)$ and the map β in coordinates reads $\beta(q, \bar{q}, \bar{r}, r) = \beta_a(q, \bar{q}, \bar{r}, r) d\bar{q}^a$.

We let $\Gamma^a_i(q, \bar{q})$ denote the connection coefficients of $\mathcal{A}$:

$$\mathcal{A} = (d\bar{q}^a + \Gamma^a_i dq^i) \otimes \frac{\partial}{\partial \bar{q}^a} \,,$$

The vertical component of a vector $v_{q_2} = (q^i, \bar{q}^a, \dot{q}^i, \dot{\bar{q}}^a)$ at $q_2 = (q^i, \bar{q}^a)$ is then expressed as $v^V_{q_2} = (q^i, \bar{q}^a, 0, \dot{\bar{q}}^a + \Gamma^a_j(q, \bar{q})\dot{q}^j)$. From the definition of $\psi_{L_2,\beta}$ we have the following identities:

$$\frac{\partial L_2}{\partial \dot{\bar{q}}^a}(q, \bar{q}, \dot{q}, \psi_{L_2,\beta}(q, \dot{q}, \bar{q}, \bar{r}, r), \bar{r}) = \beta_a(q, \bar{q}, \bar{r}, r) \,.$$

The Lagrangian L_1 and the magnetic form $\mathcal{B}_1$ are then written as:

$$\begin{aligned} L_1(q, \bar{q}, \dot{q}, \bar{r}, r) &= L_2(q, \bar{q}, \dot{q}, \psi_{L_2,\beta}(q, \bar{q}, \dot{q}, \bar{r}, r), \bar{r}) \\ &\qquad - \beta_a(q, \bar{q}, \bar{r}, r)\big(\psi^a_{L_2,\beta}(q, \bar{q}, \dot{q}, \bar{r}, r) + \Gamma^a_i(q, \bar{q})\dot{q}^i\big) \,, \\ \mathcal{B}_1 &= F^*\mathcal{B}_2 + d\Bigg(\beta_a(q, \bar{q}, \bar{r}, r)(d\bar{q}^a + \Gamma^a_i(q, \bar{q})dq^i)\Bigg) \,. \end{aligned}$$

The fact that $\psi_{L_2,\beta}$ relates the symplectic structures follows from a straightforward computation:

$$\begin{aligned} \psi^*_{L_2,\beta}(\Omega^{L_2,\mathcal{B}_2}) &= \psi^*_{L_2,\beta}\left(d\left(\frac{\partial L_2}{\partial \dot{q}^i}\right) \wedge dq^i + d\left(\frac{\partial L_2}{\partial \dot{\bar{q}}^a}\right) \wedge d\bar{q}^a + \mathcal{B}_2\right) \\ &= d\left(\frac{\partial L_1}{\partial \dot{q}^i}\right) \wedge dq^i + F^*\mathcal{B}_2 + d\Bigg(\beta_a(q, \bar{q}, \bar{r}, r)(d\bar{q}^a + \Gamma^a_i(q, \bar{q})dq^i)\Bigg) \,, \end{aligned}$$

i.e. $\psi^*_{L_2}(\Omega^{L_2,\mathcal{B}_2}) = \Omega^{L_1,\mathcal{B}_1}$. It now remains to check that $\psi^*_{L_2,\beta}E_{L_2} = E_{L_1}$:

$$\begin{aligned} \psi^*_{L_2,\beta}E_{L_2} &= \psi^*_{L_2,\beta}\left(\dot{q}^i \frac{\partial L_2}{\partial \dot{q}^i} + \dot{\bar{q}}^a \frac{\partial L_2}{\partial \dot{\bar{q}}^a}\right) - \psi^*_{L_2,\beta}L_2 \\ &= \dot{q}^i \frac{\partial L_1}{\partial \dot{q}^i} - \big(\psi^*_{L_2,\beta}L_2 - \beta_a\Gamma^a_i\dot{q}^i - \beta_a\psi^a_{L_2,\beta}\big) \,, \end{aligned}$$

and the last term on the right-hand side is precisely L_1. This completes the proof. □

Example (Routh reduction). The magnetic Lagrangian system on $\pi : Q \to Q/G$ has the following properties:

i) $L_1 = (i_\mu \circ \Pi^{-1}_\mu)^* L - \mathfrak{A}_\mu$, i.e. $L_1(T\pi(v_q), q) = L(v_q) - \langle \mu, \mathfrak{A}(v_q)\rangle$ for $v_q \in J^{-1}_L(\mu)$ arbitrary,

ii) $\mathcal{B}_1 = d\mathfrak{A}_\mu$.

The tangency condition. Under some restrictive conditions (to be discussed later) the map $\psi_{L_2,\beta}$ can be proved to be a diffeomorphism. In this situation, the two Hamiltonian vector fields $X_{E_{L_1}}$ and $X_{E_{L_2}}$ from Theorem 3.14 are $\psi_{L_2,\beta}$-related since $\psi_{L_2,\beta}$ satisfies $\psi^*_{L_2,\beta}(\Omega^{L_2,\mathcal{B}_2}) = \Omega^{L_1,\mathcal{B}_1}$ and $\psi^*_{L_2,\beta}E_{L_2} = E_{L_1}$.

However, in general, the magnetic Lagrangian system on $T_{E_1}Q_1$ is not regular even if the original system on $\epsilon^{(2)} : E_2 \to Q_2$ was. Theorem 3.14 states that the two Lagrangian systems are related, but this does not guarantee that the solution curves to the (pre)symplectic equations are related.

Here, we will use the results from Subsection 3.2.1 to get a sufficient condition for the solutions of the Euler-Lagrange equations of $(\epsilon^{(1)} : E_1 \to Q_1, L_1, \mathcal{B}_1)$ to be related to those of a regular system $(\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$. From Proposition 3.7, it is necessary and sufficient that $X_{E_{L_2}}$ is contained in the image of $T\psi_{L_2,\beta}$. The next proposition provides information on the image of $T\psi_{L_2,\beta}$.

Proposition 3.15. *Let X_{s_2} denote an arbitrary tangent vector to $M = T_{E_2}Q_2$ at $s_2 = (e_2, v_{q_2}) = \psi_{L_2,\beta}(s_1)$. Then $X_{s_2} = T\psi_{L_2,\beta}(Y_{s_1})$ for some Y_{s_1} tangent to $N = T_{E_1}Q_1$ at $s_1 = (e_1, v_{q_1})$ if, and only if, the following two conditions are satisfied:*

1) X_{s_2} and Y_{s_1} are compatible.

2)

$$X_{s_2}\left(\frac{\partial L_2}{\partial \bar{v}^a}\right) = \left(T\tau_2^{(1)}(Y_{s_1})\right)(\beta_a)\,.$$

Proof. We first show that the two conditions hold if $X_{s_2} = T\psi_{L_2,\beta}(Y_{s_1})$. Since $\psi_{L_2,\beta}$ is a compatible map, the pair X_{s_2}, Y_{s_1} is compatible. Deriving the left hand side of the equality $\alpha_{L_2}\circ\psi_{L_2,\beta} = \beta\circ\tau_2^{(1)}$, becomes $T\alpha_{L_2}\circ T\psi_{L_2,\beta}(Y_{s_1}) = T\alpha_{L_2}(X_{s_2})$. The right hand side equals $T\beta\big(T\tau_2^{(1)}(Y_{s_1})\big)$. In components, we have $T(\alpha_{L_2})_a(X_{s_2}) = T\beta_a(T\tau_2^{(1)}Y_{s_1})$ which is the second condition.

For the converse statement, let X_{s_2}, Y_{s_1} denote a pair of vectors satisfying 1) and 2). Note that the pair $T\psi_{L_2,\beta}(Y_{s_1}), Y_{s_1}$ also satisfies 1. and 2., and that the proof is concluded if we can show uniqueness, i.e. two pairs $\bar{X}_{s_2}, Y_{s_1}$ and $\bar{X}'_{s_2}, Y_{s_1}$ satisfying conditions 1) and 2), will necessarily be equal: $\bar{X}_{s_2} = \bar{X}'_{s_2}$.

From the second condition (and using the coordinate expressions for compatible vectors given before in (3.10)) it follows

$$(\bar{X}_{s_2} - \bar{X}'_{s_2})\left(\frac{\partial L_2}{\partial \bar{v}^a}\right) = 0\,,\ \text{or}\ (\hat{\bar{X}}^b_{s_2} - \hat{\bar{X}}'^b_{s_2})\frac{\partial^2 L_2}{\partial \bar{v}^a \partial \bar{v}^b} = 0\,.$$

f-regularity of L_2 implies uniqueness: $\bar{X}_{s_2} = \bar{X}'_{s_2}$. □

Denote as before by β_a the component of β along $d\bar{q}^a$ and observe that the coordinate expressions for α_{L_2} is simply

$$\alpha_{L_2} : (q^i, \bar{q}^a, v^i, \bar{v}^a, \bar{r}^\alpha) \mapsto \left(\frac{\partial L_2}{\partial \bar{v}^a}\right)\,.$$

Taking tangent vectors points $s_2 = \psi_{L_2,\beta}(s_1)$, and using coordinate expressions for X_{s_2} and Y_{s_1} as in Equation (3.10), one finds that the equation $(T\alpha_{L_2})_b(X_{s_2}) = T\beta_b\big(T\tau_2^{(1)}Y_{s_1}\big)$ reads:

$$\begin{aligned} Y^i_{s_1}\frac{\partial^2 L_2}{\partial q^i \partial \bar{v}^b} + Y^a_{s_1}\frac{\partial^2 L_2}{\partial \bar{q}^a \partial \bar{v}^b} + Y^\alpha_{s_1}\frac{\partial^2 L_2}{\partial \bar{r}^\alpha \partial \bar{v}^b} + \hat{Y}^i_{s_1}\frac{\partial^2 L_2}{\partial v^i \partial \bar{v}^b} + \hat{X}^a_{s_1}\frac{\partial^2 L_2}{\partial \bar{v}^a \partial \bar{v}^b} \\ = Y^i_{s_1}\frac{\partial \beta_b}{\partial q^i} + Y^a_{s_1}\frac{\partial \beta_b}{\partial \bar{q}^a} + Y^\alpha_{s_1}\frac{\partial \beta_b}{\partial \bar{r}^\alpha} + Y^\gamma_{s_1}\frac{\partial \beta_b}{\partial r^\gamma}\,. \end{aligned} \tag{3.11}$$

Example (Routh reduction). The tangency condition holds if β is defined by a constant chosen momentum μ. Given any vector $X_{s_2=v_q}$, then a compatible vector $Y_{s_1=(v_{[q]_G},q)}$ in $T(T_Q(Q/G))$ is completely determined from X_{v_q}. The relation (3.11) can be rewritten as:

$$X_{v_q}\left(\frac{\partial L_2}{\partial \bar{v}^a}\right) = \big(T\tau_Q(X_{v_q})\big)\,(\beta_a)\,, \tag{3.12}$$

with $\tau_Q : TQ \to Q$. If $X_{v_q} = X_{E_L}(v_q)$ the previous equality will hold if β is defined from a chosen fixed momentum μ: (3.12) then becomes $X_{E_L}(J_L - \mu) = 0$. It is well-known that this is satisfied: an invariant Hamiltonian vector field is tangent to the level set of a momentum map.

The diffeomorphic case. Arguably, the most interesting case of a compatible transformation $\psi_{L,\beta}$ arises precisely when this map is a diffeomorphism. In particular, one has an induced system which is symplectomorphic to the original one, and hence its dynamics faithfully represent that of the original system. In a way, in this specific case the transformation allows one to transform a magnetic Lagrangian system into a new magnetic Lagrangian system with an enlarged configuration space E, but with a greater number of "constraints" in order to compensate for the raise in degrees of freedom.

We will now prove here a useful condition for this to happen. Assume that the dimensions of $T_{E_i}Q_i$ agree, i.e. with the notations of Subsection 3.2.2 the following equality holds: $2n_1 + k_1 = 2n_2 + k_2$. Then we have $n_1 + k_1 - n_2 - k_2 = n_2 - n_1$, i.e. the dimensions of the fibers of F and f coincide, a necessary condition for $\psi_{L,\beta}$ to be a diffeomorphism. Note that in this case both the indices a and γ run from 1 to $n_2 - n_1$.

Proposition 3.16. *In the situation above, assume the following regularity condition holds: the map $\beta_{|F^{-1}(e_2)} : F^{-1}(e_2) \to V^*_{q_2}f$ is a diffeomorphism for each $e_2 \in E_2$, with $\epsilon^{(2)}(e_2) = q_2$. Then $\psi_{L,\beta}$ is a diffeomorphism.*

Proof. The fibre submanifold $F^{-1}(e_2)$ has coordinates r^γ, and in particular

$$\operatorname{rank}(\partial\beta_a/\partial r^\gamma)_{a,\gamma} = n_2 - n_1\,.$$

The rank of $\psi_{L,\beta}$ is maximal if, and only if, $\operatorname{rank}(\partial\psi^a/\partial r^\gamma)_{a,\gamma}$ is maximal, where ψ^a are the components of ψ, implicitly defined as:

$$\frac{\partial L}{\partial \bar{v}^b}\,(q,\bar{q},\dot{q},\psi^a(q,\dot{q},\bar{q},\bar{r},r),\bar{r}) = \beta_b\big(q,\bar{q},\bar{r},r\big)\,.$$

By f-regularity of L it follows $\mathrm{rank}(\partial\psi^a/\partial r^\gamma)_{a,\gamma} = \mathrm{rank}(\partial\beta_a/\partial r^\gamma)_{a,\gamma}$. Since $\psi_{L,\beta}$ is a bijection (this is easily checked using the condition on β) and has constant maximal rank, the result follows. □

Moreover, from the proof it is clear that the previous proposition fully characterizes the case where $\psi_{L,\beta}$ is a diffeomorphisms, i.e., the condition on β in Proposition 3.16 is also necessary.

The following proposition guarantees the regularity of the induced systems under the transformation $\psi_{L,\beta}$ in this situation.

Proposition 3.17. *Assume $\psi_{L,\beta}$ is a diffeomorphism and that $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$ is hyperregular. Then the induced magnetic Lagrangian system on $\epsilon^{(1)} : E_1 \to Q_1$ is hyperregular.*

Proof. It is clear that $\mathbb{F}L_1$ is a global diffeomorphism, because $\psi_{L,\beta}$ is a diffeomorphism. On the other hand, since $\psi_{L,\beta}$ is a symplectomorphism, it follows that the form $\left(\pi_1^{(1)}\right)^*\Omega_{Q_1} + \left(\pi_2^{(1)}\right)^*\mathcal{B}_1$ is symplectic. □

For later reference, we reformulate Theorem 3.14 in the setting of Proposition 3.16 and Proposition 3.17. This situation corresponds to the case studied in [LGTAC12].

Theorem 3.18. *Let (F, f) be a transformation pair between $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$ and let $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$ be a hyperregular magnetic Lagrangian systems such that L_2 is f-regular. Fix a map β as in Proposition 3.16 and let $\psi_{L_2,\beta} : T_{E_1}Q_1 \to T_{E_2}Q_2$ be the (F, f)-compatible diffeomorphism. Consider the hyperregular magnetic Lagrangian system $(\epsilon^{(1)}, L_1, \mathcal{B}_1)$ defined by*

1. $L_1(v_{q_1}, e_1) = \left(\psi^*_{L_2,\beta} L_2\right)(v_{q_1}, e_1) - \langle \beta(e_1), \mathcal{A}(\psi^{TQ_2}_{L_2,\beta}(v_{q_1}, e_1))\rangle$;
2. $\mathcal{B}_1 = F^*\mathcal{B}_2 + d\left(\langle \beta, \mathcal{A}_{E_1}\rangle\right)$.

Then $\psi_{L_2,\beta}$ is a symplectomorphism between the two symplectic structures associated with the two magnetic Lagrangian systems $(\epsilon^{(1)} : E_1 \to Q_1, L_1, \mathcal{B}_1)$ and $(\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$, and the corresponding Hamiltonian vector fields $X_{E_{L_1}}$ and $X_{E_{L_2}}$ are $\psi_{L_2,\beta}$-related.

Remark 3.19. *The fact that the two Hamiltonian vector fields are $\psi_{L_2,\beta}$-related, implies that every solution $e_1(t) \in E_1$ to the Euler-Lagrange equations for the system $(\epsilon^{(1)}, L_1, \mathcal{B}_1)$ projects under F to a solution $e_2(t) = F(e_1(t)) \in E_2$ of the Euler-Lagrange equations for the system $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$.*

3.2.4 The Hamiltonian picture

We end this section with a brief description of the Hamiltonian counterpart of the transformations studied in Subsection 3.2.3. The goal is now to define a class of transformations $\psi_{\mathcal{A},\beta} : T^*_{E_1}Q_1 \to T^*_{E_2}Q_2$ analogous to the class $\psi_{L,\beta}$.

Consider a transformation pair (F, f) for the bundles $\epsilon^{(1)} : E_1 \to Q_1$ and $\epsilon^{(2)} : E_2 \to Q_2$ inducing adapted coordinates $(q^i, \bar{q}^a, \bar{r}^\alpha)$ on E_2 and $(q^i, \bar{q}^a, \bar{r}^\alpha, r^\gamma)$ on E_1. The corresponding natural coordinates on $T^*_{E_1}Q_1$ and $T^*_{E_2}Q_2$ are denoted by $(q^i, p_i, \bar{q}^a, \bar{r}^\alpha, r^\gamma)$ and $(q^i, \bar{q}^a, p_i, \bar{p}_a, \bar{r}^\alpha)$ respectively.

To determine the analogue of the transformation $\psi_{L_2,\beta}$ one begins with the following observation. The coordinate expression for the Lagrangian in Theorem 3.14 induced by a transformation $\psi_{L_2,\beta}$ is

$$L_1(q, \dot{q}, \bar{q}, \bar{r}, r) = \psi^*_{L_2,\beta} L_2(q, \bar{q}, \dot{q}, \dot{\bar{q}}, \bar{r}) - \beta_a(q, \bar{q}, \bar{r}, r)\big(\psi^a_{L_2,\beta}(q, \bar{q}, \dot{q}, \bar{r}, r) + \Gamma^a_i(q, \bar{q})\dot{q}^i\big),$$

where Γ^a_i are the connection coefficients. A computation shows that the momenta $p = \partial L/\partial \dot{q}$ transform under $\psi_{L_2,\beta}$ as $p \mapsto p + \langle \beta, \mathcal{A}\rangle$. More precisely, using the definition of $\psi_{L_2,\beta}$ one finds:

$$\begin{aligned}
\psi^*_{L_2,\beta}\frac{\partial L_2}{\partial \dot{\bar{q}}^a} &= \beta_a\,,\\
\psi^*_{L_2,\beta}\frac{\partial L_2}{\partial \dot{q}^i} &= \frac{\partial L_1}{\partial \dot{q}^i} - \big(\psi^*_{L_2,\beta}\frac{\partial L_2}{\partial \dot{\bar{q}}^a}\big)\frac{\partial \psi^a}{\partial \dot{q}^i} + \beta_a\big(\frac{\partial \psi^a}{\partial \dot{q}^i} + \Gamma^a_i\big) = \frac{\partial L_1}{\partial \dot{q}^i} - \beta_a\Gamma^a_i\,.
\end{aligned}$$

Having the transformation law for the momenta, which depends on a chosen connection $\mathcal{A}$ and on the map β, one can naturally define a transformation $\psi_{\mathcal{A},\beta}$ for a magnetic Hamiltonian systems on $E_2 \to Q_2$ as the transformation which satisfies the aforementioned transformation law for the momenta and covers (F, f). The explicit expression is given by:

$$\psi_{\beta,\mathcal{A}}(p_{q_1}, e_1) = \big(T^*_{q_2}f(p_{q_1}) + \langle \beta, \mathcal{A}\rangle, F(e_1)\big)\,,$$

with $q_2 = \epsilon^{(2)}(F(e_1))$. If one then defines the magnetic form $\mathcal{B}_1$ as

$$\mathcal{B}_1 = F^*\mathcal{B}_2 + d\left(\langle \beta, \mathcal{A}_{E_1}\rangle\right)\,,$$

one has the following result:

Proposition 3.20. *In the situation above, $\psi^*_{\mathcal{A},\beta}\Omega^{\mathcal{B}_2} = \Omega^{\mathcal{B}_1}$.*

Proof. A point with coordinates $(q^i, p_i, \bar{q}^a, \bar{r}^\alpha, r^\gamma) \in T^*_{E_1}Q_1$ is mapped into the point $(q^i, \bar{q}^a, p_i + \beta_a\Gamma^a_i, \beta_a, \bar{r}^\alpha) \in T^*_{E_2}Q_2$ by $\psi_{\mathcal{A},\beta}$. Using $\Omega_2 = \big(dp_i \wedge dq^i + d\bar{p}_a \wedge d\bar{q}^a + \mathcal{B}_2\big)$, it follows easily

$$\begin{aligned}
(\psi_{\mathcal{A},\beta})^*\Omega^{\mathcal{B}_2} &= dp_i \wedge dq^i + d(\beta_a\Gamma^a_i dq^i + \beta_a d\bar{q}^a) + F^*\mathcal{B}_2\\
&= dp_i \wedge dq^i + \mathcal{B}_1 = \Omega^{\mathcal{B}_1}\,.
\end{aligned}$$

□

Starting from this result one defines the *induced* magnetic Hamiltonian system on $T^*_{E_1}Q_1$, denoted $(\epsilon^{(1)}, H_1, \mathcal{B}_1)$, whose Hamiltonian function is given by

$$H_1(p_{q_1}, p_1) = \psi^*_{\mathcal{A},\beta} H_2(p_{q_2}, p_2)\,.$$

Much like in the case of magnetic Lagrangian systems, one can then relate the dynamics of $(\epsilon^{(2)}, H_2, \mathcal{B}_2)$ to that of $(\epsilon^{(1)}, H_1, \mathcal{B}_1)$.

Example (Momentum shift in cotangent bundle reduction). Consider the following scheme, which is the same as in Routh reduction: $Q_1 = Q/G$, $Q_2 = Q$, $E_1 = Q$, $E_2 = Q$ and the transformation pair $(F, f) = (id_Q, \pi)$. The situation is the same as in Diagram 3.4. The map β is given by

$$\langle \beta(q), \xi_Q(q)\rangle = \langle \mu, \xi\rangle\,,$$

for all $\xi \in \mathfrak{g}$. It is easy to check that the map $\psi_{\mathcal{A},\beta} : T^*_Q(Q/G) \to T^*Q,\ p \mapsto \pi^*p + \mathfrak{A}_\mu$ equals $\imath_\mu \circ (S^\mu)^{-1}$, where $\imath_\mu : J^{-1}(\mu) \to T^*Q$ denotes the inclusion and S^μ is the shift map (2.5). $\psi_{\mathcal{A},\beta}$ induces the magnetic Hamiltonian system on $T^*_Q(Q/G)$ whose Hamiltonian function and magnetic term are $(\imath_\mu \circ S_\mu^{-1})^*H$ and $d\mathfrak{A}_\mu$ respectively. The situation is summarized in the Diagram 3.7.

$$\begin{tikzcd}
J^{-1}(0) \arrow[d, equal] & J^{-1}(\mu) \arrow[l, "S^\mu"'] \arrow[r, "\imath_\mu"] & T^*Q \\
T^*_Q(Q/G) \arrow[urr, bend right, "\psi_{\mathcal{A},\beta}"'] & &
\end{tikzcd}$$

DIAGRAM 3.7: Momentum shift

We have already mentioned in Section 3.1 that there exists an equivalence between hyperregular MLS and hyperregular MHS. It is an important observation that the transformations $\psi_{\mathcal{A},\beta}$ and $\psi_{L,\beta}$ respect this equivalence. More precisely, in the situation above, let $(\epsilon^{(2)}, L_2, \mathcal{B}_2)$ be a given MLS with L_2 f-regular, and let $(\epsilon^{(1)}, L_1, \mathcal{B}_1)$ denote the MLS induced on $T_{E_1}Q_1$ (given by Theorem 3.14), where $\mathcal{A}$ denotes the chosen connection. Then the following diagram commutes:

$$\begin{tikzcd}
T^*_{E_1}Q_1 \arrow[r, "\psi_{\mathcal{A},\beta}"] & T^*_{E_2}Q_2 \\
T_{E_1}Q_1 \arrow[u, "\mathbb{F}L_1"] \arrow[r, "\psi_{L_2,\beta}"'] & T_{E_2}Q_2 \arrow[u, "\mathbb{F}L_2"']
\end{tikzcd}$$

3.3 Semidirect product reduction

We will now study Routh reduction of a Lagrangian system whose configuration space is a product of a manifold S and a semi-direct product group $G \ltimes V$ of a Lie group G and a linear space V. In this case, there are two natural ways to apply Routh reduction: reducing with respect to the full symmetry group $G \ltimes V$ or reducing with respect to the Abelian subgroup V. If the dual action of G on V^* is free, it follows from Routh reduction by stages that there exists a symplectic diffeomorphism relating the symplectic structures of both reduced systems. This symplectic diffeomorphism belongs to the class of transformations $\psi_{L,\beta}$ between magnetic Lagrangian systems we have introduced in Subsection 3.2.3. We will discuss the case of Elroy's beanie as an illustrative example.

3.3.1 Lagrangian systems on semi-direct products

Semi-direct products. Consider a representation of a Lie group G on the vector space V on the left, and write as usual gv for the action of $g \in G$ on $v \in V$. The semidirect product $GV = G \ltimes V$ is the set $G \times V$ with the following group multiplication:

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2)\,,$$

where (g_1, v_1) and (g_2, v_2) are elements in GV. The inverse of (g, v) is $(g^{-1}, -g^{-1}v)$, and the identity element is $(e_G, 0)$, where e_G is the identity on G. The representation induces an infinitesimal action of $\mathfrak{g}$ on G obtained by taking the derivative of the map $g \mapsto gv$ at the identity. We also denote by concatenation this induced action.

The Lie algebra of $G \ltimes V$ is the semidirect product Lie algebra $\mathfrak{g}V = \mathfrak{g} \ltimes V$ whose bracket is given by the following expression:

$$[(\xi_1, u_1), (\xi_2, u_2)]_{\mathfrak{g}V} = ([\xi_1, \xi_2]_{\mathfrak{g}}, \xi_1 u_2 - \xi_2 u_1)\ , \tag{3.13}$$

for $(g, v) \in GV$ and $(\xi, u) \in \mathfrak{g}V$ arbitrary. The adjoint action of GV on its Lie algebra is

$$Ad_{(g,v)}(\xi, u) = (Ad_g \xi, gu - (Ad_g \xi)v)\,, \tag{3.14}$$

for $(g, v) \in GV$ and $(\xi, u) \in \mathfrak{g}V$.

The dual representation of G on V^* is defined by $G \times V^* \to V^*, (g, a) \mapsto g^* a$, where $\langle g^* a, v\rangle = \langle a, gv\rangle$, for arbitrary $v \in V$. Note that we do not take the inverse of g in the definition of the dual representation, and thus our expressions may differ from those in [MRW84]. The dual space of $\mathfrak{g}V$ is given by

$$(\mathfrak{g}V)^* = \{(\mu, a) \,|\, \mu \in \mathfrak{g}^*, a \in V^*\}\,.$$

The coadjoint action of $(g, v) \in GV$ on $(\mu, a) \in (\mathfrak{g}V)^*$ is

$$Ad^*_{(g,v)}(\mu, a) = (Ad^*_g(\mu - v^*(a)), g^* a) \tag{3.15}$$

where $v^* : V^* \to \mathfrak{g}^*$ is defined by $\langle v^*(a), \xi\rangle = \langle a, \xi v\rangle$. If we rewrite (3.13) as

$$ad_{(\xi_1,u_1)}(\xi_2, u_2) = (ad_{\xi_1}\xi_2, \xi_1 u_2 - \xi_2 u_1) \ ,$$

then we find that the coadjoint operator ad^* is given by:

$$ad^*_{(\xi_1,u_1)}(\mu, a) = \left(ad^*_{\xi_1}\mu - u_1^*(a), \xi_1^* a\right) \ . \tag{3.16}$$

The closed subgroup (e_G, V) of GV is normal and $GV/(e_G, V) = G$. Similarly, $\mathfrak{g}V/(0, V) = \mathfrak{g}$. From (3.15), we see that isotropy subgroup of $(\mu, a) \in (\mathfrak{g}V)^*$ w.r.t. the coadjoint action is

$$(GV)_{(\mu,a)} = \{(g, v) \in GV \mid g^* a = a \text{ and } Ad^*_g(\mu - v^*(a)) = \mu\} \,.$$

From the previous expression it follows that $(g, v) \in (GV)_{(\mu,a)}$ implies that $g \in G_a$, where G_a is the isotropy subgroup of $a \in V^*$ under the action of G. In what follows we will assume that the isotropy subgroup G_a is trivial, i.e. $G_a = \{e_G\}$. In this particular case any element in the isotropy group $(GV)_{(\mu,a)}$ is of the form (e_G, v) with $v^*(a) = 0$, and therefore $(GV)_{(\mu,a)}$ determines a subgroup of the Abelian group V.

We now turn to a Lagrangian system (Q, L) with configuration space $Q = S \times GV$ and such that L is invariant under the (lifted) action of GV onto the second factor. We will carry out Routh reduction of the Lagrangian system (Q, L) in two ways: (1) with respect to the full semi-direct product group GV, and (2) with respect to its Abelian subgroup V. Both reduced systems are Lagrangian magnetic systems, and will be equivalent in the sense of Theorem 3.18.

GV-regularity of the Lagrangian L. Following the notations and definitions in Section 2.3, the Lagrangian L determines a function $\ell : TS \times \mathfrak{g}V \to \mathbb{R}$. Fix an element $(\mu, a) \in (\mathfrak{g}V)^*$. From (2.14) and (3.15), the momentum relation $J_L\left(v_s, (g, v)(\xi, u)\right) = (\mu, a)$ is equivalent to the following two relations:

$$\begin{aligned}
\mathbb{F}_2\ell(v_s, \xi, u) &= Ad^*_g(\mu - v^* a) \,, \\
\mathbb{F}_3\ell(v_s, \xi, u) &= g^* a \,,
\end{aligned}$$

where $\mathbb{F}_2\ell$ and $\mathbb{F}_3\ell$ denote the fibre derivatives of ℓ with respect to the second and third argument, respectively. The Lagrangian is GV-regular if for each $(v_s, (g, v)(\xi, u)) \in TS \times GV \times \mathfrak{g}V$ the map $\mathfrak{g}V \to (\mathfrak{g}V)^*, (\eta, w) \mapsto J_L\left(v_s, (g, v)(\xi + \eta, u + w)\right)$ is a diffeomorphism. This translates into the existence, for each fixed $v_s \in TS$, of a mapping $(\chi_1, \chi_2) : TS \times (\mathfrak{g}V)^* \to \mathfrak{g}V$ such that, for arbitrary $(v_s, (\nu, b)) \in TS \times (\mathfrak{g}V)^*$,

$$\begin{aligned}
\mathbb{F}_2\ell\left(v_s, \chi_1(v_s, \nu, b), \chi_2(v_s, \nu, b)\right) &= \nu \,, \\
\mathbb{F}_3\ell\left(v_s, \chi_1(v_s, \nu, b), \chi_2(v_s, \nu, b)\right) &= b \,.
\end{aligned}$$

We will further assume that a map $\tau : TS \times \mathfrak{g} \times V^* \to V$ exists such that

$$\mathbb{F}_3\ell\left(v_s, \xi, \tau(v_s, \xi, b)\right) = b \,,$$

for arbitrary $(v_s, \xi, b) \in TS \times \mathfrak{g} \times V^*$. From the GV-regularity it then follows that the condition $\mathbb{F}_2\ell\,(v_s, \xi, \tau(v_s, \xi, b)) = \nu$ is equivalent to $\xi = \chi_1(v_s, \nu, b)$ and, additionally, $\tau(v_s, \chi_1(v_s, \nu, b), b) = \chi_2(v_s, \nu, b)$.

Remark 3.21. *One can easily show that the existence of the map τ is equivalent to the Lagrangian L being V-regular, where V is identified with the Abelian subgroup (e_G, V) of GV (see [LGTAC12] for more details). The assumption on the existence of τ amounts then to saying that we can do Routh reduction* by stages. *Note that in our case G_a is trivial and therefore we need not impose any further regularity conditions on L.*

Routh reduction w.r.t. GV. Fix a local coordinate chart (x^i) on S and choose a regular momentum value $(\mu, a) \in (\mathfrak{g}V)^*$. We will write $\hat{\chi}_1(x, \dot{x}, \cdot,)$ and $\hat{\chi}_2(x, \dot{x}, \cdot)$ for the restrictions of the maps $\chi_1(x, \dot{x}, \cdot)$ and $\chi_2(x, \dot{x}, \cdot)$, respectively, to the coadjoint orbit $\mathcal{O}_{(\mu,a)} \subset (\mathfrak{g}V)^*$ of (μ, a). According to the results in Section 2.3 and taking into account the expression of the ad^*-operator (3.16), the reduced Euler-Lagrange equations of motion become (see (2.16)

$$\begin{cases} \dot{\nu} = ad^*_{\hat{\chi}_1(x,\dot{x},\nu,b)}\nu - (\hat{\chi}_2(x, \dot{x}, \nu, b))^*\, b\,, \\ \dot{b} = (\hat{\chi}_1(x, \dot{x}, \nu, b))^*\, b\,, \\ \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{R}_1^{(\mu,a)}}{\partial \dot{x}^i}(x, \dot{x}, \nu, b)\right) - \dfrac{\partial \mathcal{R}_1^{(\mu,a)}}{\partial x^i}(x, \dot{x}, \nu, b) = 0\,. \end{cases} \tag{3.17}$$

where the Routhian $\mathcal{R}_1^{(\mu,a)}$ is the function on $TS \times \mathcal{O}_{(\mu,a)}$ given by

$$\begin{aligned} \mathcal{R}_1^{(\mu,a)}(x, \dot{x}, \nu, b) = \ell\,(v_s, \hat{\chi}_1(x, \dot{x}, \nu, b), \hat{\chi}_2(x, \dot{x}, \nu, b)) \\ - \langle \nu, \hat{\chi}_1(x, \dot{x}, \nu, b)\rangle - \langle b, \hat{\chi}_2(x, \dot{x}, \nu, b)\rangle\,. \end{aligned}$$

For later use, we will now compute the magnetic 2-form $\mathcal{B}_{(\mu,a)}$ explicitly. Recall from Section 2.3 that the connection 1-form is given by:

$$\mathfrak{A}\big(v_s, (g, v)(\xi, u)\big) = Ad_{(g,v)}(\xi, u) = (Ad_g\xi, gu - (Ad_g\xi)v)\,,$$

and hence

$$\begin{aligned} \mathfrak{A}_{(\mu,a)}\big(v_s, (g, v)(\xi, u)\big) &= \langle \mu, Ad_g\xi\rangle + \langle a, gu - (Ad_g\xi)v\rangle \\ &= \langle Ad^*_g(\mu - v^*(a)), \xi\rangle + \langle g^*a, u\rangle\,. \end{aligned} \tag{3.18}$$

Definition 3.22. *$\theta_{(\mu,a)}$ is the 1-form on $S \times \mathcal{O}_{(\mu,a)}$ that satisfies*

$$\theta_{(\mu,a)}(s, \nu, b)\big(v_s, \dot{\nu}, \dot{b}\big) = \langle \nu, \xi\rangle\,,$$

with $(\dot{\nu}, \dot{b} = \xi^*b) \in T_{(\nu,b=g^*a)}\mathcal{O}_{(\mu,a)} \subset (\mathfrak{g}V)^*$ arbitrary.

The form is well defined because, by assumption, G_a is trivial. Indeed, since the action of G on V^* is free, it is also infinitesimally free and therefore there is a unique $\xi \in \mathfrak{g}$ such that $\dot{b} = \xi^* b$. Then we can prove:

Lemma 3.23. $\mathcal{B}_{(\mu,a)} = d\theta_{(\mu,a)}$.

Proof. First note that the term $\langle g^*a, u\rangle$ on the right-hand side of (3.18) does not contribute to the computation of $\mathcal{B}_{(\mu,a)}$: it is the contraction of the fixed 'momentum' a with the tangent vector gu to the linear space V and therefore vanishes when taking the exterior derivative. It is then sufficient to show that $\theta_{(\mu,a)}$ is the reduction to $S \times \mathcal{O}_{(\mu,a)}$ of the 1-form $\mathfrak{A}_{(\mu,a)}$ with the term $\langle g^*a, u\rangle$ omitted.

The computation is similar to the one in the proof of Lemma 2.14. The tangent map of the projection $GV \to GV/GV_{(\mu,a)} \cong \mathcal{O}_{(\mu,a)}$ equals:

$$(g, v, g\xi, gu) \in T(GV) \mapsto (\nu = Ad^*_g(\mu - v^*(a)), b = g^*a, \dot{\nu} = ad^*_\xi \nu, \dot{b} = \xi^* b) \in T\mathcal{O}_{(\mu,a)}.$$

This shows that $\langle Ad^*_g(\mu - v^*(a)), \xi\rangle$ projects onto $\langle \nu, \xi\rangle$. $\square$

Routh reduction w.r.t. V. We consider the V-principal connection on $S \times GV$

$$\bar{\mathfrak{A}}(v_s, g\xi, gu) = gu \in V\,,$$

which is the pull-back to $S \times GV$ of the standard V-principal connection on the Abelian group $V = (e_G, V)$. One thinks of V as the Lie algebra of (e_G, V) in a natural way.

If a is a regular value of the momentum map, the associated Routhian is a function on $T(S \times G)$ and equals

$$\mathcal{R}^a_2(v_s, g\xi) = \ell(v_s, \xi, \tau(v_s, \xi, g^*a)) - \langle g^*a, \tau(v_s, \xi, g^*a)\rangle\,,$$

where we have used (2.15).

The magnetic 2-form $\mathcal{B}_a$ vanishes because V is Abelian and $\bar{\mathfrak{A}}$ is flat.

The compatible transformation. Diagram 3.8 represents the different maps involved.

$$\begin{tikzcd} E_1 = S \times \mathcal{O}_{(\mu,a)} \arrow[r, "F"] \arrow[d] & E_2 = S \times G \arrow[d, "\mathrm{id}"] \\ Q_1 = S & Q_2 = S \times G \arrow[l, "f"'] \end{tikzcd}$$

DIAGRAM 3.8: The compatible transformation for the semidirect product

The surjection F is determined from the projection $\mathcal{O}_{(\mu,a)} \to G,\ (\nu, b) \mapsto g$ where g is uniquely determined from $g^*a = b$. The map f is simply the projection onto the first factor and then $Vf = \ker Tf = 0_S \times TG \subset T(S \times G)$. It is not hard to check that the pair (F, f) is a transformation pair.

Theorem 3.24. *Assume that $G_a = \{e_G\}$ for $a \in V^*$ and that the map $\cdot^*a : V \to \mathfrak{g}^*$; $v \mapsto v^*a$ is onto. Then the two magnetic Lagrangian systems $(S \times \mathcal{O}_{(\mu,a)} \to S, \mathcal{R}_1^{(\mu,a)}, \mathcal{B}_{(\mu,a)})$ and $(S \times G \to S \times G, \mathcal{R}_2^a, 0)$ are equivalent in the sense of Theorem 3.18, i.e. there is a (F, f)-compatible diffeomorphism of the form $\psi_{\mathcal{R}_2^a,\beta}$ and a connection $\mathcal{A} : TQ_2 \to Vf$ such that the Lagrangians and the magnetic 2-forms satisfy*

1. $\mathcal{R}_1^{(\mu,a)}(v_{q_1}, e_1) = \left(\psi^*_{\mathcal{R}_2^a,\beta}\mathcal{R}_2^a\right)(v_{q_1}, e_1) - \langle \beta(e_1), \mathcal{A}(\psi^{TQ_2}_{\mathcal{R}_2^a,\beta}(v_{q_1}, e_1))\rangle$;
2. $\mathcal{B}_1 = F^*\mathcal{B}_2 + d\left(\langle \beta, \mathcal{A}_{E_1}\rangle\right)$.

The Hamiltonian vector fields $X_{E_{\mathcal{R}_1^{(\mu,a)}}}$ and $X_{E_{\mathcal{R}_2^a}}$ are $\psi_{\mathcal{R}_2^a,\beta}$-related.

Proof. We now introduce the remaining elements needed to apply Theorem 3.18, i.e. a map $\beta : E_1 \to V^*f$ and a connection $\mathcal{A}$ on $f : Q_2 \to Q_1$.

1) The map β is defined as follows: $\beta : E_1 \to V^*f,\ (s, \nu, b) \mapsto (s, g, 0_s, \nu \circ TL_{g^{-1}})$ where $g \in G$ is such that $g^*a = b$. Note that the conditions $G_a = \{e_G\}$ and $\mathrm{im}\cdot^* a = \mathfrak{g}^*$ imply that the fibres $F^{-1}(s, g) \cong V/\ker \cdot^*a$ and $V_s^*f \cong \mathfrak{g}^*$ are diffeomorphic. We show this by constructing pointwise an inverse for β. Consider an arbitrary element in $V^*_{(s,g)}f$ and let ν be the corresponding element in the dual of the Lie algebra $\mathfrak{g}$. Because $\cdot^*a$ is onto, a vector $v \in V$ exists such that $v^*a = \mu - Ad^*_{g^{-1}}\nu$. The element $(\nu, b = g^*a)$ then determines a point in $\mathcal{O}_{(\mu,a)}$, is unique and, by construction, it determines the inverse image for ν under $\beta|_{F^{-1}(s,g)}$.
2) The connection used to relate the dynamics is the pull-back to $S \times G$ of the standard zero-curvature connection with horizontal distribution $0_G \times TS \subset T(G \times S)$.

Note that the contraction of the β-map and the vertical part of the standard connection precisely equals the 1-form $\theta_{(\mu,a)}$ on $\mathcal{O}_{(\mu,a)}$: $(\nu, b, \dot\nu, \dot b) \mapsto \langle \nu, \xi\rangle$, with $\xi^*b = \dot b$. From Lemma 3.23, the exterior derivative of $\theta_{(\mu,a)}$ is precisely $\mathcal{B}_{(\mu,a)}$.

It now remains to show that the two Routhians $\mathcal{R}_1^{(\mu,a)}$ and $\mathcal{R}_2^a$ are transformed into each other by means of $\psi_{\mathcal{R}_2^a,\beta}$ and $\mathcal{A}$. For that purpose we derive an explicit formula for the second condition in Proposition 3.13 (with $L_2 = \mathcal{R}_2^a$). Let $(v_s, g\xi)$ be arbitrary in $T(S \times G)$. Fix an element $g\eta \in T_gG$. Then

$$\begin{aligned}
\langle \mathbb{F}_2\mathcal{R}_2^a(v_s, g\xi), g\eta\rangle &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{R}_2^a(v_s, g\xi + \epsilon g\eta) \\
&= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Big(\ell(v_s, \xi + \epsilon\eta, \tau(v_s, \xi + \epsilon\eta, g^*a)) - \langle g^*a, \tau(v_s, \xi + \epsilon\eta, g^*a)\rangle\Big) \\
&= \langle \mathbb{F}_2\ell(v_s, \xi, \tau(v_s, \xi, g^*a)), \eta\rangle\,.
\end{aligned}$$

Therefore, to construct the transformation $\psi_{\mathcal{R}_2,\beta}$ we have to solve the following equation for ξ:

$$\mathbb{F}_2\ell(v_s, \xi, \tau(v_s, \xi, g^*a)) = \beta(s, \nu, b) \circ TL_g = \nu.$$

By definition of τ, the solution ξ is precisely $\chi_1(v_s, \nu, b)$. From this, we necessarily have that the composition $\tau(v_s, \chi_1(v_s, \nu, b), b)$ equals $\chi_2(v_s, \nu, b)$. We now compute the transformation of $\mathcal{R}_2$ under $\psi_{\mathcal{R}_2,\beta}$ and $\mathcal{A}$:

$$\begin{aligned}\left(\mathcal{R}_2^a(v_s, g\xi) - \langle \nu, \xi\rangle\right)\Big|_{\mathbb{F}_2\ell(v_s,\xi,\tau(v_s,\xi,g^*a))=\nu} &= \ell(v_s, \chi_1(v_s, \nu, b), \chi_2(v_s, \nu, b)) \\ &\quad - \langle b, \chi_2(v_s, \nu, b)\rangle - \langle \nu, \chi_1(v_s, \nu, b)\rangle \,.\end{aligned}$$

This is precisely the Routhian $\mathcal{R}_1^{(\mu,a)}$ and, using Theorem 3.18, this concludes the proof. □

3.3.2 Example: Elroy's beanie

We will now revisit Elroy's beanie (see Section 2.1) to illustrate the results discussed above. To have a symmetry group which is a nontrivial semidirect product, we consider that the center of mass O moves on the plane (see Figure 3.9).

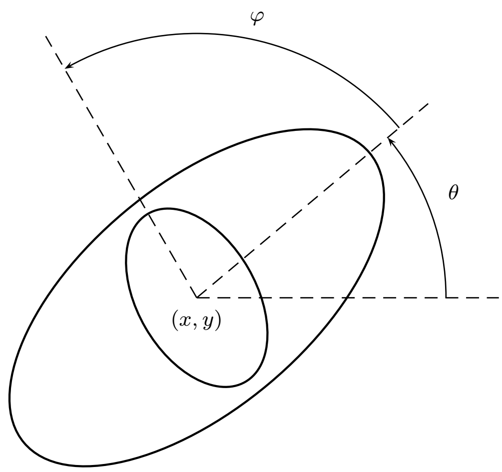

DIAGRAM 3.9: Elroy's beanie

The configuration space is then $S^1 \times SE(2) = S^1 \ltimes \mathbb{R}^2$, with coordinates (φ, θ, x, y) as in the figure. The kinetic energy of the system is $SE(2)$-invariant and we will also suppose that the potential is $SE(2)$-invariant. This actually implies that only the relative position of the two bodies matters for the dynamics of the system. The Lagrangian is:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\varphi})^2 - V(\varphi)\,.$$

The Euler-Lagrange equations of the system are, written in normal form,

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{\theta} = \frac{1}{I_1}\frac{dV}{d\varphi}, \quad \ddot{\varphi} = -\left(\frac{I_1 + I_2}{I_1 I_2}\right)\frac{dV}{d\varphi}\,.$$

The semi-direct product $SE(2)$**.** The special Euclidean group $SE(2)$ is the semi-direct product of the Lie group $G = S^1$ with $V = \mathbb{R}^2$, parametrized by (θ, x, y), where G acts on V by rotations in the plane. For convenience we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way: $(x, y) \mapsto z = x + iy$. Then the action of an element $\theta \in S^1$ on $z \in \mathbb{C}$ is by complex multiplication $e^{i\theta}z$. With this convention, the group multiplication is given by

$$(\theta_1, z_1) * (\theta_2, z_2) = (\theta_1 + \theta_2, e^{i\theta_1}z_2 + z_1)\,.$$

and the identity of $SE(2)$ corresponds to $(\theta = 0, z = 0)$. Elements of the Lie-algebra $se(2)$ of $SE(2)$ are denoted by $(\xi, w) \in \mathbb{R} \times \mathbb{C}$. The associated infinitesimal action of the Lie algebra $\mathbb{R}$ of S^1 on $\mathbb{C}$ then reads $\xi(z) = i\xi z$, with $\xi \in \mathbb{R}$ and $z \in \mathbb{C}$ arbitrary. Using (3.14), the adjoint action equals

$$Ad_{(\theta,z)}(\xi, w) = (\xi, e^{i\theta}w - i\xi z)\,.$$

If $(\theta, x, \dot{\theta}, \dot{z})$ is an element in $TSE(2)$ (with $\dot{z} = \dot{x} + i\dot{y}$), the corresponding element in the left identification with $SE(2) \times se(2)$ is $(\theta, z, \dot{\theta}, w)$, with $w = e^{-i\theta}\dot{z}$. Denote the real and complex part of w by u, v respectively, $w = u + iv$. This allows us to write down the Lagrangian ℓ on $TS^1 \times se(2)$ in the left identification as

$$\ell(\varphi, \dot{\varphi}, \dot{\theta}, w) = \frac{1}{2}m(u^2 + v^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\varphi})^2 - V(\varphi)\,.$$

Elements of the dual $se^*(2) \cong \mathbb{R} \times \mathbb{C}$ of $se(2)$ are written as (μ, a) and the contraction with an arbitrary element $(\xi, w) \in se(2)$ is $\mu\xi + \mathfrak{Re}(a\overline{w})$. The dual action of G on $\mathbb{C}^*$ is given by $g^*a = e^{-i\theta}a$, and the corresponding infinitesimal action of an element $\xi \in \mathbb{R}$ is $\xi^*a = -i\xi a$. Clearly the isotropy group of $a \in \mathbb{C}^*$ is trivial for any $a \neq 0$. Finally, for the element z^*a in the dual of the Lie algebra of S^1 we obtain:

$$\langle z^*a, \xi\rangle = \langle a, \xi(z)\rangle = \langle a, i\xi z\rangle = \mathfrak{Re}(-ia\bar{z})\xi\,,$$

i.e. $z^*a = \mathfrak{Re}(-ia\overline{z})$. It follows that the map $\cdot^*a : \mathbb{C} \to \mathbb{R}$ is onto for any $a \neq 0$.

Reduction with respect to $SE(2)$**.** The Lagrangian being of mechanical type we can compute the Routhian as follows (see [Par65]):

$$\begin{aligned}
2(\mathcal{R}_1^{(\mu,a)} + V)(\varphi, \dot{\varphi}, \nu, b) &= \left(\frac{\partial \ell}{\partial \dot{\varphi}}\dot{\varphi} - \frac{\partial \ell}{\partial \dot{\theta}}\dot{\theta} - \frac{\partial \ell}{\partial u}u - \frac{\partial \ell}{\partial v}v\right)\Bigg\{\begin{smallmatrix} \nu = (I_1+I_2)\dot{\theta} + I_2\dot{\varphi} \\ b = mw \end{smallmatrix} \\
&= \left(I_2\dot{\varphi}^2 - (I_1 + I_2)\dot{\theta}^2 - mu^2 - mv^2\right)\Bigg\{\begin{smallmatrix} \nu = (I_1+I_2)\dot{\theta} + I_2\dot{\varphi} \\ b = mw \end{smallmatrix}\,.
\end{aligned}$$

The momentum relations are regular, and with the notations used earlier we find:

$$\dot\theta = \frac{\nu - I_2\dot\varphi}{I_1+I_2} = \chi_1(\varphi,\dot\varphi,\nu,b)\,,$$
$$w = \frac{b}{m} = \chi_2(\varphi,\dot\varphi,\nu,b)\,.$$

Finally we obtain the Routhian after a straightforward computation:

$$\mathcal{R}_1^{(\mu,a)}(\varphi,\dot\varphi,\nu,b) = \frac{1}{2}\frac{I_1I_2}{I_1+I_2}\dot\varphi^2 + \frac{I_2}{I_1+I_2}\nu\dot\varphi - V(\varphi) + \frac{b\bar b}{2m} - \frac{1}{2}\frac{\nu^2}{I_1+I_2}\,.$$

The reduced equations of motion. The Routh reduced equations of motion for the beanie are obtained from (3.17). Note that, in the present case, the the first equation in (3.17) simplifies greatly because the term $ad^*_{\hat\chi_1}\nu$ vanishes.

$$\begin{aligned}
&\dot\nu = \mathfrak{Re}\left(-\frac{i}{m}\bar b b\right) = 0\,,\\
&\dot b = -i\left(\frac{\nu - I_2\dot\varphi}{I_1+I_2}\right)b\,,\\
&\frac{d}{dt}\left(\frac{\partial\mathcal{R}_1^{(\mu,a)}}{\partial\dot\varphi}(\nu,b,\varphi,\dot\varphi)\right) - \frac{\partial\mathcal{R}_1^{(\mu,a)}}{\partial\varphi}(\nu,b,\varphi,\dot\varphi) = \frac{I_1I_2}{I_1+I_2}\ddot\varphi + \frac{I_2}{I_1+I_2}\dot\nu + V'(\varphi) = 0\,.
\end{aligned}$$

The second equation of motion is clearly a rotation of the momentum b with angular velocity $(I_2\dot\varphi - \nu)/(I_1+I_2)$. The choice of the fixed momentum a is reflected in these equations as $b\bar b = a\bar a$.

Abelian reduction. We now perform Routh reduction w.r.t the Abelian symmetry group $V = \mathbb{R}^2$ of translations in the x and y direction. The conserved (complex) momentum for this action is $a = m\dot z$. We use the same momentum values as before: $b = e^{-i\theta}a$. The map τ is given by $\tau(\dot\theta,b,\varphi,\dot\varphi) = \frac{b}{m}$. The Routhian is obtained by computing

$$\begin{aligned}
2(\mathcal{R}_2^a + V)(\theta,\varphi,\dot\theta,\dot\varphi) &= \left(\frac{\partial\ell}{\partial\dot\varphi}\dot\varphi + \frac{\partial\ell}{\partial\dot\theta}\dot\theta - \frac{\partial\ell}{\partial u}u - \frac{\partial\ell}{\partial v}v\right)_{e^{-i\theta}a=mw}\\
&= \left(I_1\dot\theta^2 + I_2(\dot\theta+\dot\varphi)^2 - mw\overline{w}\right)_{e^{-i\theta}a=mw}\\
&= I_1\dot\theta^2 + I_2(\dot\theta+\dot\varphi)^2 - \frac{a\bar a}{m}\,.
\end{aligned}$$

Thus the Routh reduced system is a standard Lagrangian system on $S^1\times S^1$ with Lagrangian $\frac{1}{2}I_1\dot\theta^2 + \frac{1}{2}I_2(\dot\theta+\dot\varphi)^2 - V(\varphi)$ (we ignore irrelevant constant terms).

Equivalence. Using Theorem 3.18, both reduced systems are equivalent in the sense that a transformation $\psi_{\mathcal{R}_2^a,\beta}$ exists relating both magnetic Lagrangian systems. In this case, the β-map fixes the remaining momentum $\beta(\nu,b,\varphi) = \nu$. The diffeomorphism $\psi_{\mathcal{R}_2^a,\beta}$ then satisfies $\psi_{\mathcal{R}_2^a,\beta}(\varphi,\dot\varphi,\nu,b = e^{-i\theta}a) = (\theta,\varphi,\dot\varphi,\dot\theta = (\nu - I_2\dot\varphi)/(I_1+I_2))$.

3.4 Routh reduction of magnetic Lagrangian systems

Fibrewise reducible magnetic Lagrangian systems. In this paragraph we introduce a special type of magnetic Lagrangian systems where the dynamics is easily reducible. These systems posses symmetry along the fibers of $\epsilon : E \to Q$ which, roughly speaking, allows for a reduction of the base space space E of $T_E Q$ while leaving the tangent part TQ invariant. We will refer to these systems as *fiberwise reducible.*

Example (Routh reduction). In Section 3.2 we have shown that a general Lagrangian system, amenable to Routh reduction, can be transformed into a magnetic Lagrangian system $(Q \to Q/G, (i_\mu \circ \Pi_\mu^{-1})^* L - \mathfrak{A}_\mu, d\mathfrak{A}_\mu)$ in such a way that the solution to the Euler-Lagrange equations are mapped into solution of the original Euler-Lagrange equations with fixed momentum μ. There is still symmetry left in the transformed system: the fibers of $Q \to Q/G$ are diffeomorphic to G, and we will show below that the transformed magnetic Lagrangian system is reducible under the fiberwise action of the isotropy subgroup G_μ.

Let $\epsilon : E \to Q$ be a fibre bundle and let Φ^E denotes a G-action on E such that $\epsilon \circ \Phi^E = \epsilon$. Φ^E induces a G-action $\Phi^{T_E Q}$ on $T_E Q$ as follows:

$$\Phi_g^{T_E Q}(v_q, e) = (v_q, \Phi_g^E(e)) = (v_q, ge)\,.$$

Definition 3.25. *A magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ together with a G-action Φ^E on E is* fiberwise-reducible *if the following conditions hold:*

1) *The action of G on E is tangent to the fibers, i.e. $\epsilon(\Phi_g^E(e)) = \epsilon(e)$.*

2) *L is G-invariant with respect to the lift of Φ^E to $T_E Q$: $L(v_q, \Phi_g^E(e)) = L(v_q, e)$.*

3) *The 2-form $\mathcal{B}$ on E is reducible to E/G, i.e. $\mathcal{B}$ is G-invariant and satisfies $i_{\xi_E}\mathcal{B} = 0$ for all $\xi \in \mathfrak{g}$.*

We write $\bar{B}$ for the projection of B onto E/G and $\bar{L}$ for the projection of L onto $T_{E/G}Q$. The quotient manifold E/G fibers over Q with the submersion $\bar{\epsilon} : E/G \to Q$ given by $\bar{\epsilon} : [e] \mapsto \epsilon(e)$. Since Φ^E is assumed to be free, the fibration $\bar{\epsilon} : E/G \to Q$ is a principal G-bundle (that is, local triviality holds). Note that $\mathcal{B}$ projects to $\bar{\mathcal{B}}$, and that $\bar{\mathcal{B}}$ is closed.

Definition 3.26. *Let $(\epsilon : E \to Q, L, \mathcal{B}, G)$ be a fiberwise-reducible magnetic Lagrangian system. We call $(\bar{\epsilon} : E/G \to Q, \bar{L}, \bar{\mathcal{B}})$ the* (associated) reduced magnetic Lagrangian *system.*

We use the following notations, in agreement with the notations used before:

i) $\tau_G : T_E Q \to T_{E/G} Q$ is the projection that maps $(v_q, e) \in T_E Q$ onto $(v_q, [e]) \in T_{E/G} Q$.

ii) $p_G : T^*_E Q \to T^*_{E/G} Q$ is the projection that maps $(p_q, e) \in T^*_E Q$ onto $(p_q, [e]) \in T^*_{E/G} Q$.

iii) $\bar\pi_1 : T^*_{E/G} Q \to T^* Q$ is the projection that maps $(p_q, [e]) \in T^*_{E/G} Q$ onto $p_q \in T^* Q$.

iv) $\bar\pi_2 : T^*_{E/G} Q \to E/G$ is the projection that maps $(p_q, [e]) \in T^*_E Q$ onto $[e] \in E/G$.

We are interested in reducing the dynamics in a fiberwise-reducible Lagrangian system to the associated reduced magnetic Lagrangian system $(\bar\epsilon : E/G \to Q, \bar L, \bar{\mathcal{B}})$. To that end, we need the following two lemmas:

Lemma 3.27. *Let $(\epsilon : E \to Q, L, \mathcal{B}, G)$ be a fiberwise-reducible Lagrangian system and consider the reduced magnetic Lagrangian system $(\bar\epsilon : E/G \to Q, \bar L, \bar{\mathcal{B}})$. Then $p_G \circ \mathbb{F}L = \mathbb{F}\bar L \circ \tau_G$, i.e., Diagram 3.10 commutes.*

$$\begin{tikzcd} T_E Q \arrow[r, "\mathbb{F}L"] \arrow[d, "\tau_G"'] & T^*_E Q \arrow[d, "p_G"] \\ T_{E/G} Q \arrow[r, "\mathbb{F}\bar L"'] & T^*_{E/G} Q \end{tikzcd}$$

Diagram 3.10: Lemma 3.27

Proof. Obvious. □

Lemma 3.28. *The map $\tau_G : T_E Q \to T_{E/G} Q$ between the presymplectic manifolds $(T_E Q, \Omega^{L,\mathcal{B}})$ and $(T_{E/G} Q, \bar\Omega^{\bar L, \bar{\mathcal{B}}})$ satisfies $\tau_G^* \bar\Omega^{\bar L, \bar{\mathcal{B}}} = \Omega^{L,\mathcal{B}}$ and $\tau_G^* E_{\bar L} = E_L$.*

Proof. The first statement follows by diagram chasing:

$$\tau_G^* \mathbb{F}\bar L^* (\bar\pi_1^* \omega_Q + \bar\pi_2^* \bar{\mathcal{B}}) = \mathbb{F}L^* p_G^* (\bar\pi_1^* \omega_Q + \bar\pi_2^* \bar{\mathcal{B}}) = \mathbb{F}L^* (\pi_1^* \omega_Q + \pi_2^* \mathcal{B}) \,.$$

The second part is easily checked in coordinates:

$$\tau_G^* E_{\bar L} = \tau_G^* \left(\frac{\partial \bar L}{\partial v_i} v_i - \bar L \right) = \frac{\partial L}{\partial v_i} v_i - L = E_L \,.$$

□

Note that the map $T_E Q \to T_{E/G} Q$ is a submersion and therefore, in view of the previous lemma, we can apply the basic reduction result from Proposition 1.18 to the case of fiberwise reducible magnetic Lagrangians systems.

Proposition 3.29. *Consider a fiberwise reducible magnetic system $(\epsilon : E \to Q, L, \mathcal{B}, G)$. The associated reduced magnetic Lagrangian system $(\bar\epsilon : E/G \to Q, \bar L, \bar B)$ is such that any solution to the Euler-Lagrange equations $(\bar\epsilon : E/G \to Q, \bar L, \bar B)$ is the projection of a solution to the Euler-Lagrange equations of the reducible system $(\epsilon : E \to Q, L, \mathcal{B}, G)$.*

Example (Routh reduction). Clearly the Lagrangian $(i_\mu \circ \Pi_\mu^{-1})^* L - \mathfrak{A}_\mu$ on $T_Q(Q/G)$ and the magnetic force term $d\mathfrak{A}_\mu$ on Q of the transformed Lagrangian system are G_μ invariant and fiberwise reducible to a magnetic Lagrangian system on $Q/G_\mu \to Q$. The reduced Lagrangian and magnetic 2-form correspond to $\mathcal{R}^\mu$ and $\mathcal{B}_\mu$ from Theorem 2.13.

Throughout Section 3.2, we have used the specific case of a standard Lagrangian system amenable to Routh reduction to demonstrate and develop the general theory on transformations between magnetic Lagrangian systems. Our aim now is to show that Routh reduction itself can be cast into the framework of the compatible transformations and we also consider the more general framework of Routh reduction for magnetic Lagrangian systems (see [LMV11]). In both cases, Routh's reduction procedure for a G-invariant magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ is realized as the result of two steps:

Step 1 We construct an equivalent magnetic Lagrangian $(\epsilon_\mu : E \to Q/G, L_\mu, \mathcal{B}_\mu)$ system by means of a compatible transformation $\psi_{L,\beta}$ for a suitable β.

Step 2 We check that $(\epsilon_\mu : E \to Q/G, L_\mu, \mathcal{B}_\mu)$ is fiberwise reducible.

Reduction for invariant magnetic Lagrangian systems. Let G act on $E \to Q$ by bundle automorphisms, i.e. there's a G-action on both E and Q such that $\epsilon \circ \Phi_g^E = \Phi_g^Q \circ \epsilon$. The projections of the principal bundles are denoted by $\pi^E : E \to E/G$ and $\pi^Q : Q \to Q/G$. This action naturally lifts to an action $\Phi^{T_E Q}$ on $T_E Q$ in the following way:

$$\Phi_g^{T_E Q}(v_q, e) := (T\Phi_g^Q(v_q), \Phi_g^E(e)) .$$

Definition 3.30. *A magnetic Lagrangian system $(\epsilon : E \to Q, L, \mathcal{B})$ is G-invariant if $\mathcal{B}$ is invariant w.r.t. Φ^E and L is invariant w.r.t. $\Phi^{T_E Q}$.*

In this case, $\Phi^{T_E Q}$ is symplectic w.r.t. $\Omega^{L,\mathcal{B}}$. In order to obtain a momentum map for this action, we introduce the notion of $\mathcal{B}\mathfrak{g}$-potential (see e.g. [MMO+07]).

Definition 3.31. *Given an invariant closed 2-form $\mathcal{B}$ on E. Then a $\mathfrak{g}^*$-valued function δ on E is a* $\mathcal{B}\mathfrak{g}$-potential *if $i_{\xi_E}\mathcal{B} = d\langle \delta, \xi\rangle$ for any $\xi \in \mathfrak{g}$.*

From now on and to ease notation, given a $\mathfrak{g}^*$-valued function f, f_ξ for any $\xi \in \mathfrak{g}$ will be a shortcut for $\langle f, \xi\rangle$. For instance, the defining property of a $\mathcal{B}\mathfrak{g}$-potential $\delta \in \mathcal{C}^\infty(P, \mathfrak{g}^*)$ is $i_{\xi_E}\mathcal{B} = d\delta_\xi$ for any $\xi \in \mathfrak{g}$. If E is connected, we have

$$\begin{aligned} d\big[(\Phi_g^E)^*\delta_\xi\big] &= (\Phi_g^E)^* d\delta_\xi = (\Phi_g^E)^*(i_{\xi_E}\mathcal{B}) = i_{(\Phi_g^E)^*\xi_E}(\Phi_g^E)^*\mathcal{B} \\ &= i_{(\Phi_g^E)^*\xi_E}\mathcal{B} = i_{(Ad_{g^{-1}}\xi)_E}\mathcal{B} = d\delta_{(Ad_{g^{-1}}\xi)} . \end{aligned}$$

From $d\big((\Phi_g^E)^*\delta_\xi - \delta_{Ad_g\xi}\big) = 0$, it follows that the map $\sigma_\delta(g) = \delta \circ \Phi_g^E - Ad_{g^{-1}}^* \cdot \delta$ is a $\mathfrak{g}^*$-valued 1-cocycle on G. (This definition is independent of the point $p \in P$ because of connectedness.)

A momentum map $J_{L,\delta}$ for $\Phi^{T_E Q}$ is given by:

$$\langle J_{L,\delta}(v_q, e), \xi\rangle = \langle \mathbb{F}L(v_q, e), (\xi_Q(q), e)\rangle - \delta_\xi(e) .$$

This momentum map has non-equivariant cocycle $-\sigma_\delta$. We consider the affine action of G on $\mathfrak{g}^*$ that makes $J_{L,\delta}$ equivariant (see Section 1.4), and let G_μ denote the isotropy group of an element $\mu \in \mathfrak{g}^*$ w.r.t. this action. We will now prove that, under some regularity conditions, the level set of this momentum map may be identified with the subbundle $T_E(Q/G) \subset T_EQ$, and this identification will eventually allow us to define a suitable transformation scheme to describe Routh reduction on T_EQ.

Definition 3.32. *The Lagrangian L of a G-invariant magnetic Lagrangian system is called G-regular if the map $\mathcal{J}_{L,\delta}^{(v_q,e)} : \mathfrak{g} \to \mathfrak{g}^*; \xi \mapsto J_{L,\delta}(v_q + \xi_Q(q), e)$ is a diffeomorphism for all $(v_q, e) \in T_EQ$.*

Similar to the standard case, we consider the map

$$\Pi_{\delta,\mu} : J_{L,\delta}^{-1}(\mu) \to T_E(Q/G);\ (v_q, e) \mapsto (T\pi^Q(v_q), e)\,.$$

Lemma 3.33. *$\Pi_{\delta,\mu}$ is a diffeomorphism if the Lagrangian is G-regular.*

Proof. The construction of a map $\Delta_{\mu,\delta}$ inverse to $\Pi_{\delta,\mu}$ is similar to the construction of the map Δ_μ in the proof of Lemma 2.8. □

The compatible transformation. Analogous as the standard case, we consider the following transformation scheme in Diagram 3.4: $E_1 = E_2 = E$, $Q_1 = Q/G$, $Q_2 = Q$ and transformation the pair $(F = id_E, f = \pi^Q = \pi : Q \to Q/G)$. We have $T_{E_1}Q_1 = T_E(Q/G)$ and $T_{E_2}Q_2 = T_EQ$, and π-regularity of L is equivalent to G-regularity of L.

$$\begin{tikzcd}
E \arrow[r, "id_E"] \arrow[d] & E \arrow[d] \\
Q/G & Q \arrow[l, "\pi^Q"]
\end{tikzcd}$$

Diagram 3.11: Transformation scheme for T_EQ

W let coordinates on Q/G be denoted by (q^i), adapted coordinates on Q are then $(q^i, \bar{q}^a)$ and finally $(q^i, \bar{q}^a, \bar{r}^\alpha)$ represent coordinates on E (in particular, there are no components in r^γ).

The components of the infinitesimal generator of symmetries $\sigma : Q \times \mathfrak{g} \to TQ$ of Φ^Q are denoted by σ_b^a, i.e. $(q = (q^i, \bar{q}^a), \xi = \xi^b e_b) \mapsto \sigma_b^a(q^i, \bar{q}^a)\xi^b$, with $\{e_b\}_b$ a basis for $\mathfrak{g}$ (and e^b the dual basis). Since the action is free, σ_b^a is invertible, and we put $\Sigma_b^a := (\sigma^{-1})_b^a$.

Define $\beta : E \to V^*\pi$ in the following way:

$$\langle \beta(e), \xi_Q(q) \rangle = \langle \mu, \xi \rangle + \langle \delta(e), \xi \rangle\,.$$

In local coordinates, the map $\psi_{L,\beta}$ takes the form:

$$\psi_{L,\beta}(q^i, \bar{q}^a, v^i, \bar{r}^\alpha) = (q^i, \bar{q}^a, v^i, \bar{v}^a, \bar{r}^\alpha)\,,$$

with $\bar{v}^\alpha$ implicitly determined from

$$\frac{\partial L}{\partial \bar{v}^\alpha}(q^i, \bar{q}^a, v^i, \bar{v}^a, \bar{r}^\alpha) = \beta_a = \Sigma_a^b \mu_b + \Sigma_a^b \delta_b\,,$$

which is equivalent to the momentum equation and $\psi_{L,\beta}$, equals $\imath_\mu \circ \Pi^{-1}_{\delta,\mu}$.

Verifying the tangency condition. Because the momentum map is conserved along solution to the Euler-Lagrange equations, the tangency condition from Proposition 3.15 is fulfilled for any tangent vector that solves the presymplectic equation for the invariant Lagrangian system on $E \to Q$ at a point on the level set of the momentum map. Here we will check (3.11) for an arbitrary function β. More precisely, we study the tangency condition for a tangent vector X_s to $T_E Q$ in the case of a compatible transformation map $\psi_{L,\beta}$ with β arbitrary and where $X_{s=(v_q,e)}$ solves the Euler-Lagrange equation $i_{X_s}\Omega^{L,\mathcal{B}} = -dE_L$.

Because of the SODE nature Euler-Lagrange equation, the tangent vector X_s to $T_E Q$ is of the form:

$$X_s = v^i \frac{\partial}{\partial q^i} + \bar{v}^a \frac{\partial}{\partial \bar{q}^a} + \dot{r}^\alpha \frac{\partial}{\partial \bar{r}^\alpha} + \ddot{q}^i \frac{\partial}{\partial v^i} + \ddot{q}^a \frac{\partial}{\partial \bar{v}^a}\,,$$

where $(\ddot{q}, \dot{r}^\alpha)$ are implicitly determined from the Euler Lagrange equations. A tangent vector $Y_{\bar{s}=(T\pi(v_q),e)}$ to $T_E(Q/G)$ compatible to X_s completely determined by this condition and is of the form

$$Y = v^i \frac{\partial}{\partial q^i} + \bar{v}^a \frac{\partial}{\partial \bar{q}^a} + \dot{r}^\alpha \frac{\partial}{\partial \bar{r}^\alpha} + \ddot{q}^i \frac{\partial}{\partial v^i}\,.$$

From Proposition 3.15, X_s is in the image of $T\psi_{L,\beta}$ if

$$X_s\left(\frac{\partial L}{\partial \bar{v}^a}\right) = Y_{\bar{s}}(\beta_a).$$

Since β is a function on E, the right hand side can be written (with a slight abuse of notation) as $X_s(\beta_a)$, and the tangency condition becomes

$$X_s\left(\frac{\partial L}{\partial \bar{v}^a} - \beta_a\right) = 0.$$

For a G-invariant Lagrangian, only $\beta_a = \Sigma_a^b \mu_b + \Sigma_a^b \delta_b$ will provide a transformation that satisfies the tangency conditions.

The reduction step. Fix a principal connection $\mathcal{A}$ on the bundle $\pi : Q \to Q/G$ whose corresponding connection 1-form is denoted $\mathfrak{A}$. We apply the construction of Theorem 3.14 to induce a magnetic Lagrangian system on $T_E(Q/G)$. The resulting system has Lagrangian function

$$L_\mu = (\imath_\mu \circ \Pi_{\delta,\mu}^{-1})^* L - \langle \mu + \delta, \mathfrak{A}_E((\imath_\mu \circ \Pi_{\delta,\mu}^{-1})^{TQ}) \rangle \,,$$

and magnetic term $\mathcal{B}_\mu = \mathcal{B} + d\langle \mu + \delta, \mathfrak{A}_E \rangle$. Note that this new magnetic Lagrangian system $(E \to Q/G, L_\mu, \mathcal{B}_\mu)$ is G_μ-fiberwise reducible because:

i) $\Pi_{\delta,\mu}^{-1}$ is equivariant, L is invariant, and the term involving $\mathfrak{A}_E$ is G_μ-invariant.

ii) $\mathcal{B}_\mu$ is G_μ invariant and satisfies $\imath_{\xi_E} \mathcal{B}_\mu = 0$ for all $\xi \in \mathfrak{g}_\mu$.

The last assertion can be checked using Cartan's formula and the fact that the infinitesimal 2-cocycle corresponding to σ_δ equals $\Sigma_\delta(\xi, \zeta) = -\langle T_e \sigma_\delta(\xi), \zeta \rangle = -\xi_E(\delta_\zeta(e)) - \delta_{[\xi,\zeta]}(e)$. A detailed proof may be found in [MMO+07] (the sign convention differs though).

We conclude with a diagram (Diagram 3.12) that summarizes the equivalence of Routh reduction with the procedure described above: a transformation $\psi_{L,\beta}$ followed by a fiberwise reduction. The presymplectic structures on $J_{L,\delta}^{-1}(\mu)$ and $T_E(Q/G)$ (the former given by $\imath_\mu^* \Omega^{L,\mathcal{B}}$ and latter given by Theorem 3.14) are related by $\Delta_{\mu,\delta}$. Finally, $\Delta_{\mu,\delta}$ drops to a symplectomorphism $\bar{\Delta}_{\mu,\delta}$ on the quotient.

$$\begin{tikzcd}
J_{L,\delta}^{-1}(\mu)/G_\mu & J_{L,\delta}^{-1}(\mu) \arrow[l] \arrow[r, "\imath_\mu"'] & T_E Q \\
T_{E/G_\mu}(Q/G) \arrow[u, "\bar{\Delta}_{\mu,\delta}"'] & T_E(Q/G) \arrow[l] \arrow[u, "\Delta_{\mu,\delta}"'] \arrow[ur, bend right, "\psi_{L,\beta}"']
\end{tikzcd}$$

Diagram 3.12: Routh reduction of a magnetic Lagrangian system

Chapter 4

Reduced dynamics

In the previous chapters, our attention has been directed to reduction theories which take into account the conservation of the momentum map. We have seen that, when this conservation law is taken into account and some regularity conditions are satisfied, there exists a natural symplectic framework for the reduction of the dynamics in both the Hamiltonian and the Lagrangian descriptions. The reduced dynamics, being defined on a symplectic manifold, may again be interpreted by means of the Hamilton equations.

Nevertheless, it is sometimes convenient to reduce the dynamics *directly* by means of the Lie group action. In this case, one does not take into account the (possible) existence of conservation laws, but rather quotients the manifold where the dynamics is defined by the group action, and describes the induced dynamics on this reduced space. In the Lagrangian case, a suitable category to study the so-called *Lagrange-Poincaré* equations is that of Lie algebroids. This approach is based in the formulation of classical mechanics in terms of Lie algebroids due to E. Martínez (see [Mar01], and also [Wei96] for some background). Moreover, the dual of a Lie algebroid is endowed with a Poisson structure which is responsible for the dynamics in the Hamiltonian case, namely the *Hamilton-Poincaré* equations. In either case, one rapidly loses the symplectic description of the dynamics.

In this chapter, we wish to take an alternative view at these reduction theories, and derive them from a purely symplectic framework. More precisely, we will show how the reduced equations of Hamilton-Poincaré and Lagrange-Poincaré can be obtained by symplectic reduction of the Tulczyjew triple. Besides the theoretical interest of this derivation, this approach has also the advantage of treating singular Lagrangians on the same footing as the regular ones. There are, at least, three seemingly related approaches in the literature:

1) In [GUG06] the authors obtain a Tulczyjew triple in a Lie algebroid setting. If we apply these results to the case when the Lie algebroid is the Atiyah algebroid, we obtain rather Poisson answer than a symplectic one. For our convenience, and in

order to expedite the exposition, we will rely on some of the results in this approach: see also Subsection 4.2.4 for a more careful discussion.

2) In [dLMM05] one may find a different Tulczyjew triple for Lie algebroids. This triple consists of so-called prolongation bundles of Lie algebroids, which are all so-called *symplectic Lie algebroids*. It should be emphasized that a symplectic Lie algebroid is a generalization of a symplectic manifold to the level of a vector bundle, but not a genuine symplectic manifold in its own right.

3) A third approach, within the context of Dirac structures, is sketched in [YM06] (though the context in this work is different).

The material is organized as follows. In Section 4.1 we review the definition of the Tulcyjew triple and recall its main properties. Additionally, we also study how submanifolds behave under reduction and obtain conditions which guarantee that the reduction of a Lagrangian submanifold is again Lagrangian. The results are used in Section 4.2 to obtain the reduced dynamics corresponding to the Hamilton-Poincaré and Lagrange-Poincaré equations. We also state the equivalence of both reductions under the assumptions of regularity.

4.1 Tulczyjew triple and Lagrangian submanifolds

Let Q be the configuration space of a given system. The construction of the Tulczyjew triple relies on the existence of two canonical diffeomorphisms $\beta_Q : TT^*Q \to T^*T^*Q$ and $\alpha_Q : TT^*Q \to T^*TQ$ whose definition we recall below. Before doing that, we need to review the structure of the (double) vector bundle T^*TQ.

The manifold T^*TQ can be endowed with the structure of a vector bundle over T^*Q where the vector bundle projection $\mathsf{v}^* : T^*TQ \to T^*Q$ is the dual of the vertical lift [a], namely

$$\left\langle \mathsf{v}^*(\alpha_{v_q}), w_q \right\rangle = \left\langle \alpha_{v_q}, (w_q)^{\mathsf{v}}_{v_q} \right\rangle ,$$

for all $\alpha_{v_q} \in T^*TQ$ and $w_q \in TQ$.

There exists a vector bundle isomorphism $R : T^*TQ \to T^*T^*Q$ over the identity of T^*Q between the vector bundles $\mathsf{v}^* : T^*TQ \to T^*Q$ and $\pi_{T^*Q} : T^*T^*Q \to T^*Q$ which is completely determined by the condition:

$$\left\langle R(\alpha_{v_q}), W_{\mathsf{v}^*(\alpha_{v_q})} \right\rangle = -\left\langle \alpha_{v_q}, \bar{W}_{v_q} \right\rangle + \left\langle W_{\mathsf{v}^*(\alpha_{v_q})}, \bar{W}_{v_q} \right\rangle^T , \tag{4.2}$$

[a] As usual, we denote by $(\cdot)^{\mathsf{v}}_{v_q} : T_qQ \to T_{v_q}TQ$ the standard vertical lift:

$$(w_q)^{\mathsf{v}}_{v_q}(f) = \left.\frac{d}{ds}\right|_{s=0} f(v_q + sw_q) , \tag{4.1}$$

for each function f on TQ.

for all $\alpha_{v_q} \in T^*TQ$, $\bar{W}_{v_q} \in TTQ$ and $W_{\mathsf{v}^*(\alpha_{v_q})} \in TT^*Q$ satisfying

$$T\tau_Q\big(\bar{W}_{v_q}\big) = T\pi_Q\big(W_{\mathsf{v}^*(\alpha_{v_q})}\big) . \tag{4.3}$$

Here $\langle \cdot, \cdot \rangle^T : TT^*Q \times_{TQ} TTQ \to \mathbb{R}$ is the pairing defined by the tangent map of the usual pairing $\langle \cdot, \cdot \rangle : T^*Q \times_Q TQ \to \mathbb{R}$, and $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ are the standard projections.

We can now give a precise definition of the diffeomorphisms α_Q and β_Q:

1) β_Q is the contraction with respect to the symplectic form $(-\Omega_Q)$ on T^*Q. With the notations in Section 1.1, we have

$$\beta_Q = (-\Omega_Q^\flat)$$

2) α_Q is the composition

$$\alpha_Q = R^{-1} \circ \beta_Q .$$

The space TT^*Q, being the tangent bundle of the symplectic manifold (T^*Q, Ω_Q), is itself symplectic when endowed with the complete lift Ω_Q^c of Ω_Q (c.f. [YI73, GU95]). In particular, if we consider the manifolds T^*T^*Q and T^*TQ with their canonical symplectic structures, we have the following important result:

Theorem 4.1. *In the situation above:*

1) β_Q is an anti-symplectomorphism from (TT^*Q, Ω_Q^c) to (T^*T^*Q, Ω_{T^*Q}).

2) α_Q is a symplectomorphism from (TT^*Q, Ω_Q^c) to (T^*TQ, Ω_{TQ}).

Besides the (anti)symplectomorphisms α_Q and β_Q, the other main ingredients of the triple are a pair of Lagrangian submanifolds S_L and S_H describing, respectively, the Lagrangian and Hamiltonian dynamics. Before we can turn to their construction, we need to review some basic facts about Lagrangian submanifolds:

Definition 4.2. *Let (M, ω) be a symplectic manifold and let $i : L \to M$ be an immersion. We say that L is an immersed* Lagrangian submanifold (M, ω) *if* $\dim L = 1/2 \dim P$ *and* $i^*\omega = 0$.

A well known result (see e.g. [AM78]) states that if f is a function on a manifold M then the image of its differential $df(M) \subset T^*M$ is a Lagrangian submanifold of (T^*M, Ω_M). In particular, given a Hamiltonian H on T^*Q or a Lagrangian L on TQ, one may define the following Lagrangian submanifolds of (TT^*Q, Ω_Q^c):

$$S_H = \beta_Q^{-1}\left(dH(T^*Q)\right) , \qquad S_L = \alpha_Q^{-1}\left(dL(TQ)\right) .$$

The relation between the dynamics of a given mechanical system and the associated Lagrangian submanifold is the following (see [Tul76a, Tul76b]):

(1) Solutions of the Euler-Lagrange equations of L are in one-to-one correspondence with curves in S_L which are tangent lifts of curves in T^*Q.

(2) Solutions of the Hamilton equations of H are in one-to-one correspondence with curves in S_H which are tangent lifts of curves in T^*Q.

It is customary to depict the situation in a diagram known as Tulczyjew triple (see Diagram 4.1).

$$
\begin{tikzcd}
T^*TQ \arrow[dr, "\pi_{TQ}"] & & TT^*Q \arrow[ll, "\alpha_Q"'] \arrow[rr, "\beta_Q"] \arrow[dl, "T\pi_Q"'] \arrow[dr, "\tau_{T^*Q}"] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"'] \\
& TQ \arrow[rr, "\mathbb{F}L"] \arrow[dr, "\tau_Q"'] \arrow[ul, bend left, "dL"] & & T^*Q \arrow[dl, "\pi_Q"] \arrow[ur, bend right, "dH"'] & \\
& & Q & &
\end{tikzcd}
$$

DIAGRAM 4.1: Tulczyjew triple

Coordinate expressions. For a better understanding, we will provide coordinate expressions of the maps and symplectic forms introduced above. As usual, we denote the coordinates on TQ by $(q^i, \dot{q}^i)$ and the coordinates on T^*Q by (q^i, p_i). Coordinates on TT^*Q are then $(q^i, p_i, \dot{q}^i, \dot{p}_i)$.

Using the local expression $\Omega_Q = dp_i \wedge dq^i$, the map $\beta_Q = (dq^i \wedge dp_i)^\flat$ is readily checked to be:

$$
\begin{aligned}
\beta_Q : TT^*Q &\to T^*T^*Q, \\
(q^i, p_i, \dot{q}^i, \dot{p}_i) &\mapsto (q^i, p_i, -\dot{p}_i, \dot{q}^i) \,.
\end{aligned}
\tag{4.4}
$$

It takes some more work to derive the coordinate expression of α_Q. This can be done regarding α_Q as the dual of the canonical flip of the double tangent bundle (see [Mic08]). The result is the following local expression:

$$
\begin{aligned}
\alpha_Q : TT^*Q &\to T^*TQ, \\
(q^i, p_i, \dot{q}^i, \dot{p}_i) &\mapsto (q^i, \dot{q}^i, \dot{p}_i, p_i) \,.
\end{aligned}
\tag{4.5}
$$

Let M be a manifold and ω a closed 2-form on M. Denote by ω_{ij} the local components of ω, i.e. $\omega = {}^1\!/_2\, \omega_{ij} dx^i \wedge dx^j$, where x^i are local coordinates on M. Then the complete lift ω^c of ω has local expression (see [YI73])

$$
\omega^c = \frac{1}{2} \frac{\partial \omega_{ij}}{\partial x^k} \dot{x}^k dx^i \wedge dx^j + \omega_{ij} d\dot{x}^i \wedge dx^j \,.
$$

In particular, for the symplectic form Ω_Q on T^*Q we find

$$\Omega_Q^c = d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i \,. \tag{4.6}$$

Using (4.4), (4.5) and (4.6), it is easy to check that α_Q is a symplectomorphism and that β_Q is an anti-symplectomorphism (Theorem 4.1).

Let us denote by $(q^i, \dot{q}^i, \delta q^i, \delta \dot{q}^i)$ the coordinates on TTQ. Then from the expression of the usual pairing $\langle \cdot, \cdot \rangle : T^*Q \times_Q TQ \to \mathbb{R}$, it follows that its tangent map $\langle \cdot, \cdot \rangle^T$ has the following coordinate expression:

$$\big\langle (q^i, p_i, \delta q^i, \delta p_i), (q^i, \dot{q}^i, \delta q^i, \delta \dot{q}^i) \big\rangle = \delta p_i \dot{q}^i + p_i \delta \dot{q}^i \,.$$

Reduction of submanifolds. We now discuss how submanifolds behave under reduction. In particular, we show that in the presence of a Hamiltonian G action on a symplectic manifold (M, ω), a Lagrangian submanifold of M can be reduced to a submanifold on the symplectic reduced space. Moreover, we will find conditions under which this reduced submanifold is again Lagrangian.

Lemma 4.3. *Let ϕ be a (free and proper) action of a Lie group G on manifold M and S be a G-invariant embedded (respectively connected, closed) submanifold of M. Then the quotient manifold S/G is an embedded (respectively connected, closed) submanifold of M/G.*

Proof. The action restricts to a (free and proper) action $\phi_S : G \times S \to S$, and therefore S/G is a smooth manifold. We will denote by $p_M : M \to M/G$ and $p_S : S \to S/G$ the canonical projections, by $i : S \to M$ the canonical inclusion of S on M and by $\tilde{i} : S/G \to M/G$ the corresponding inclusion between the quotient manifolds.

Since S is embedded, S is diffeomorphic to its image $i(S)$ under the inclusion map, and from the quotient manifold structure $\tilde{i}$ is a diffeomorphism between S/G and $i(S)/G = \tilde{i}(S/G)$. It follows that $\tilde{i}$ is an embedding.

The statement about connectedness follows from the fact that connectedness is preserved by quotient maps.

Finally, assume that S is closed. Take a point $x \in M \setminus S$ and consider an open neighborhood V of x. Then the set

$$\bar{V} = G \cdot V = \bigcup_{g \in G} \{g \cdot V\} \,,$$

is open and invariant, and descends to an open neighborhood of $p_M(x) = [x]$ in the quotient. This proves that S/G is closed. $\square$

Using Lemma 4.3, we can now prove an important result concerning the reduction of Lagrangian submanifolds.

Theorem 4.4. *Let ϕ be a (free and proper) Hamiltonian G-action on a symplectic manifold (M, ω) with equivariant momentum map J. Assume that S is a Lagrangian submanifold of M which is embedded, closed and connected. Then:*

1) There exists a value $\mu \in \mathfrak{g}^*$ such that $S \subset J^{-1}(\mu)$.

2) The space of orbits $S_\mu = S/G_\mu$ is an isotropic submanifold of the reduced symplectic manifold (M_μ, ω_μ).

3) S_μ is Lagrangian if, and only if, $\mathfrak{g} = \mathfrak{g}_\mu$.

Proof. First we recall that S is Lagrangian if the following two conditions are satisfied: (1) $\dim S = 1/2 \dim M$, and (2) S is isotropic, i.e. $T_xS \subset (T_xS)^\omega$ for all $x \in S$.

1) We will check that the map $J_{|S} : S \to \mathfrak{g}^*$ is constant. This is the same as checking that for each $\xi \in \mathfrak{g}$, the real function $J_{\xi_{|S}} : S \to \mathbb{R}$ given by

$$J_{\xi_{|S}}(x) = \langle J(x), \xi \rangle$$

is constant.

From the invariance of the submanifold S, it follows that $\xi_M(x) \in T_xS$ for each $\xi \in \mathfrak{g}$. Together with the isotropy of S this implies that for each $v_x \in T_xS$ we have:

$$\langle d(J_\xi)_{|S}(x), v_x \rangle = \langle (dJ_\xi)(x), v_x \rangle = -\langle (i_{\xi_M(x)}\omega), v_x \rangle = -\omega(\xi_M(x), v_x) = 0\,.$$

Observing that S is connected by assumption, it follows that $J_{\xi_{|S}}$ is constant.

2) From Lemma 4.3 applied to $S \subset J^{-1}(\mu)$, it follows that $S_\mu = S/G_\mu$ is an embedded, closed and connected submanifold of $\big(M_\mu = J^{-1}(\mu)/G_\mu, \omega_\mu\big)$.

We will now show that S/G_μ is an isotropic submanifold of $\big(J^{-1}(\mu)/G_\mu, \omega_\mu\big)$. If we denote by $p_S : S \to S/G_\mu$, we need to prove that, for each $p_S(x) \in S/G_\mu$, we have:

$$T_{(p_S(x))}(S/G_\mu) \subseteq \big(T_{(p_S(x))}(S/G_\mu)\big)^{\omega_\mu}\,. \tag{4.7}$$

Let $u_x, v_x \in T_xS$. In view of the commutativity of the following diagram

$$\begin{tikzcd}
S \arrow[r, "i_S"] \arrow[d, "p_S"'] & J^{-1}(\mu) \arrow[d, "p_{J^{-1}(\mu)}"] \\
S/G_\mu \arrow[r, "i_{S_\mu}"'] & J^{-1}(\mu)/G_\mu
\end{tikzcd}$$

it follows that

$$\begin{aligned}\omega_\mu(p_S(x))\,(T_x p_S(u_x), T_x p_S(v_x)) &= ((p_{J^{-1}(\mu)})^*\omega_\mu)(x)\,(u_x, v_x)\\ &= (i^*\omega)(x)(u_x, v_x)\\ &= \omega(x)(u_x, v_x)\,.\end{aligned}$$

Since, by isotropy of S, we have $\omega(x)(u_x, v_x) = 0$, (4.7) holds and S/G_μ is isotropic.

3) It suffices to check that $\dim S_\mu = 1/2 \dim M_\mu$. By assumption, $\dim S = 1/2 \dim M$, and therefore:

$$\dim\,(S/G_\mu) = \dim\, S - \dim\, G_\mu = 1/2\, \dim M - \dim\, G_\mu\,.$$

On the other hand

$$\dim\,\left(J^{-1}(\mu)/G_\mu\right) = \dim\, M - \dim\, G - \dim\, G_\mu\,.$$

It follows that S/G_μ is Lagrangian if, and only if, $\dim\, G = \dim\, G_\mu$ or in other words: S/G_μ is Lagrangian if, and only if, $\mathfrak{g} = \mathfrak{g}_\mu$.

□

Example 4.5. Let ϕ be an action of a Lie group G on a connected manifold M and $H \in C^\infty(M)$ an invariant Hamiltonian. Then, the image of the differential of H, $dH(M)$, is a Lagrangian submanifold of the cotangent bundle (T^*M, Ω_M) which is invariant w.r.t. the cotangent lift action ϕ^{T^*M}. Indeed, for each $g \in G$ and $q \in M$ we find

$$\phi_g^{T^*M}(dH(q)) = d(H \circ \phi_{g^{-1}})(gq) = dH(gq)\,.$$

Applying Theorem 4.4 to the (closed, connected and embedded) Lagrangian submanifold $dH(M)$, there exists a value μ of the momentum map $J_{T^*M} : T^*M \to \mathfrak{g}$ such that $dH(M) \subseteq J^{-1}_{T^*M}(\mu)$. In fact, from (2.3)

$$\langle J_{T^*M}\,(dH(q))\,, \xi\rangle = \langle dH(q), \xi_M(q)\rangle = 0\,,$$

for all $\xi \in \mathfrak{g}$ and $q \in M$. This shows that $dH(M) \subseteq J^{-1}_{T^*M}(0)$, so in this particular case $\mu = 0$.

It follows from Theorem 4.4 that the reduced submanifold $dH(M)/G \subset (T^*M)_\mu$ is Lagrangian. We have already seen (c.f. Proposition 2.6) that $J^{-1}_{T^*M}(0)/G$ may be identified with $T^*(M/G)$. It is not hard to check that, under this identification, the Lagrangian submanifold $dH(M)/G$ coincides with $dh(M/G)$, where $h : M/G \to \mathbb{R}$ is the reduced Hamiltonian induced by H.

4.2 Reduced dynamics

Building on the previous results concerning the reduction of Lagrangian submanifolds, we will now discuss a symplectic approach to the Hamilton-Poincaré and Lagrange-Poincaré reduction theories based on the reduction of the Tulczyjew triple.

4.2.1 Hamilton-Poincaré reduction

In this subsection we will derive an intrinsic description of the solutions of the Hamilton-Poincaré equations.

Let ϕ be a G-action on the symplectic manifold (M, ω) which need not be symplectic. The cotangent lift of this action defines an action ϕ^{T^*M} on the symplectic manifold (T^*M, Ω_M) which is always Hamiltonian: this is a consequence of the fact that cotangent lifts preserve the Liouville 1-form. The tangent lift action ϕ^{TM} also defines an action on the symplectic manifold (TM, ω^c), but in this case the action is not, in general, Hamiltonian. We prove below in Proposition 4.6 that if ϕ is Hamiltonian, then also ϕ^{TM} is Hamiltonian.

We will make use the following result from [Tul76a]. Let ω be a closed two-form on a manifold M and consider the vector bundle morphism $\omega^\flat : TM \to T^*M$; $v_q \mapsto \imath_{v_q}\omega$. Then, using the coordinate expressions in Section 4.1, one can show that the canonical symplectic form Ω_M on T^*M and the complete lift ω^c of the closed two-form ω to TM are related by the morphism $\omega^\flat$ in the following way:

$$(\omega^\flat)^*(\Omega_M) = \omega^c\,. \tag{4.8}$$

The relation (4.8) may in fact be used as an alternative definition of the complete lift ω^c of the form ω. If ω is symplectic (i.e. nondegenerate) then ω^c is symplectic and $\omega^\flat$ is a symplectomorphism (and an isomorphism) between (TM, ω^c) and (T^*M, Ω_M).

Proposition 4.6. *Let (M, ω) be a symplectic manifold equipped with a Hamiltonian G-action ϕ with equivariant momentum J. Then:*

1) The vector bundle isomorphism $\omega^\flat : TM \to T^*M$ is G-equivariant with respect to ϕ^{TM} and ϕ^{T^*M}.

2) ϕ^{TM} is a Hamiltonian G-action on the symplectic manifold (TM, ω^c) whose associated equivariant momentum map $J_{TM} : TM \to \mathfrak{g}^*$ is given by

$$\langle J_{TM}(v_x), \xi\rangle = -v_x(J_\xi)\,.$$

Equivalently, J_{TM} satisfies $J_{TM} = J_{T^*M} \circ \omega^\flat$, where $J_{T^*M} : T^*M \to \mathfrak{g}^*$ is the momentum map associated with the symplectic action ϕ^{T^*M}.

Proof. 1) Let $x \in M$, $v_x \in T_xM$ and $w_{gx} \in T_{gx}M$. Then using that ϕ is symplectic we have:

$$\langle \omega^\flat(gv_x), w_{gx}\rangle = \omega\left(gv_x, w_{gx}\right) = \omega\left(v_x, g^{-1}w_{gx}\right) = \langle \omega^\flat(v_x), g^{-1}w_{gx}\rangle\,,$$

i.e. $\omega^\flat$ is equivariant.

2) Recall that the equivariant momentum map $J_{T^*M} : T^*M \to \mathfrak{g}^*$ associated to the cotangent action is given by (2.3):

$$\langle J_{T^*M}(\alpha_x), \xi \rangle = \langle \alpha_x, \xi_M(x) \rangle \,.$$

Define $J_{TM} : TM \to \mathfrak{g}^*$ by the equality $J_{T^*M} \circ \omega^\flat = J_{TM}$. Using that $\omega^\flat$ is an equivariant symplectomorphism, it follows that J_{TM} is an equivariant momentum map which satisfies

$$\langle J_{TM}(v_x), \xi \rangle = \langle \omega^\flat(v_x), \xi_M(x) \rangle = \omega\left(v_x, \xi_M(x)\right) = -v_x(J_\xi) \,.$$

□

In particular, when the previous theorem is applied to the cotangent bundle T^*Q endowed with the symplectic form $-\Omega_Q$, one finds that β_Q is an anti-symplectomorphism

$$\beta_Q : (TT^*Q, \Omega_Q^c) \to (T^*T^*Q, \Omega_{T^*Q}) \,,$$

which is equivariant w.r.t. the actions ϕ^{TT^*Q} and $\phi^{T^*T^*Q}$ obtained as the tangent and the cotangent lift of ϕ^{T^*Q}. Moreover β_Q preserves the momentum maps of these actions, namely $J_{T^*T^*Q} \circ \beta_Q = -J_{TT^*Q}$ where $J_{T^*T^*Q}$ and J_{TT^*Q} are defined as

$$\begin{aligned} \left\langle J_{T^*T^*Q}(\beta_{\alpha_q}), \xi \right\rangle &= \left\langle \beta_{\alpha_q}, \xi_{T^*Q}(\alpha_q) \right\rangle , \\ \left\langle J_{TT^*Q}(v_{\alpha_q}), \xi \right\rangle &= v_{\alpha_q}((J_{T^*Q})_\xi) \,, \end{aligned}$$

for all $\beta_{\alpha_q} \in T^*T^*Q$, $v_{\alpha_q} \in TT^*Q$ and $\xi \in \mathfrak{g}$. This is an important result that we summarize in the following theorem.

Theorem 4.7. *Under the above conditions, the Tulczyjew diffeomorphism β_Q is equivariant and satisfies*

$$J_{T^*T^*Q} \circ \beta_Q = -J_{TT^*Q} \,.$$

Using Proposition 1.30, the reduced spaces $J_{TT^*Q}^{-1}(0)/G$ and $J_{T^*T^*Q}^{-1}(0)/G$ are anti-symplectomorphic via the map

$$[(\beta_Q)_0] : J_{TT^*Q}^{-1}(0)/G \to J_{T^*T^*Q}^{-1}(0)/G$$

which is characterized by the condition

$$[(\beta_Q)_0] \circ p_{J_{TT^*Q}^{-1}(0)} = p_{J_{T^*T^*Q}^{-1}(0)} \circ (\beta_Q)_{|J_{TT^*Q}^{-1}(0)} \,. \tag{4.9}$$

The geometry of $[(\beta_Q)_0]$. The geometric description of the Hamilton-Poincaré equations is closely related to the quotient Poisson structure on T^*Q/G. The bracket on T^*Q/G is obtained by Poisson reduction (see Subsection 1.4.2), and it is the only bracket which makes the projection $p_{T^*Q} : T^*Q \to T^*Q/G$ a Poisson epimorphism, i.e.

$$\left\{\hat{f} \circ p_{T^*Q}, \hat{g} \circ p_{T^*Q}\right\}_{T^*Q} = \left\{\hat{f}, \hat{g}\right\}_{T^*Q/G} \circ p_{T^*Q} \,,$$

for all $\hat{f}, \hat{g} \in \mathcal{C}^\infty(T^*Q/G)$. With the notations of Section 1.1, we write

$$\sharp_{T^*Q/G} : T^*(T^*Q/G) \to T(T^*Q/G)$$

for the vector bundle morphism induced by the Poisson structure on T^*Q/G. It is defined as

$$\left(\sharp_{T^*Q/G}\right)(d\hat{f}) = X_{\hat{f}} \,, \tag{4.10}$$

where $X_{\hat{f}} \in \mathfrak{X}(T^*Q/G)$ is the Hamiltonian vector field given by

$$X_{\hat{f}}(\hat{g}) = \{\hat{f}, \hat{g}\}_{T^*Q/G} \,.$$

In the triple picture, the map $[(\beta_Q)_0]$ will define the reduced Hamiltonian dynamics. In order to relate $[(\beta_Q)_0]$ to the bundle map $\sharp_{T^*Q/G}$, we make the following observations:

1) The symplectic space $(J^{-1}_{T^*T^*Q}(0)/G, (\Omega_{T^*Q})_0)$ is obtained by cotangent reduction at $\mu = 0$. It is symplectomorphic to the canonical symplectic space $T^*(T^*Q/G)$, where the symplectomorphism

   $$\Psi_0 : \left(J^{-1}_{T^*T^*Q}(0)/G, (\Omega_{T^*Q})_0\right) \to \left(T^*(T^*Q/G), \Omega_{T^*Q/G}\right)$$

   is defined by (see (2.6)):

   $$\left\langle \Psi_0(p_{J^{-1}_{T^*T^*Q}(0)}(\alpha_{\beta_q})), T_{\beta_q} p_{T^*Q}(v_{\beta_q}) \right\rangle = \left\langle \alpha_{\beta_q}, v_{\beta_q} \right\rangle \,, \tag{4.11}$$

   for all $\beta_q \in T^*Q$, $\alpha_{\beta_q} \in J^{-1}_{T^*T^*Q}(0)$ and $v_{\beta_q} \in TT^*Q$.

2) The symplectic space $(J^{-1}_{TT^*Q}(0)/G, (\Omega^c_Q)_0)$ is not, in general, symplectomorphic to a tangent bundle. However, it is possible to define a vector bundle morphism

   $$\Xi : J^{-1}_{TT^*Q}(0)/G \to T(T^*Q/G)$$

   over the identity of T^*Q/G, which is characterized by the condition

   $$\Xi\left(p_{J^{-1}_{TT^*Q}(0)}(v_{\alpha_q})\right) = T_{\alpha_q} p_{T^*Q}(v_{\alpha_q}) \,, \tag{4.12}$$

   for all $v_{\alpha_q} \in J^{-1}_{TT^*Q}(0)$.

We now prove a key lemma which relates the reduced symplectic spaces with the spaces on which the Hamilton-Poincaré equations live.

Lemma 4.8. *The following diagram*

$$
\begin{tikzcd}
J^{-1}_{TT^*Q}(0)/G \arrow[d, "\Xi"'] & & J^{-1}_{T^*T^*Q}(0)/G \arrow[ll, "{[(\beta_Q)_0]^{-1}}"'] \\
T(T^*Q/G) \arrow[dr, "\tau_{T^*Q/G}"'] & & T^*(T^*Q/G) \arrow[ll, "\sharp_{T^*Q/G}"] \arrow[u, "(\Psi_0)^{-1}"'] \arrow[dl, "\pi_{T^*Q/G}"] \\
& T^*Q/G &
\end{tikzcd}
$$

is commutative.

Proof. It is sufficient to prove that

$$\sharp_{T^*Q/G}\big((d\hat{f})(p_{T^*Q}(\alpha_q))\big) = \big(\Xi \circ [(\beta_Q)_0]^{-1} \circ \Psi_0^{-1}\big)\big((d\hat{f})(p_{T^*Q}(\alpha_q))\big)\,,$$

for all $\hat{f} \in C^\infty(T^*Q/G)$ and $\alpha_q \in T^*Q$. For the sake of clarity, we divide the proof in steps:

I) Consider the function $\hat{f} \circ p_{T^*Q} \in C^\infty(T^*Q)$. Then $d(\hat{f} \circ p_{T^*Q})(\alpha_q) \in J^{-1}_{T^*T^*Q}(0)$. From the definition of Ψ_0 in (4.11) it follows that, for all $v_{\alpha_q} \in TT^*Q$,

$$\begin{aligned}\Big\langle \Psi_0\big(p_{J^{-1}_{T^*T^*Q}(0)}\big(d(\hat{f} \circ p_{T^*Q})(\alpha_q)\big)\big), T_{\alpha_q}p_{T^*Q}(v_{\alpha_q})\Big\rangle &= \big\langle d(\hat{f} \circ p_{T^*Q})(\alpha_q), v_{\alpha_q}\big\rangle \\ &= \big\langle d\hat{f}(p_{T^*Q}(\alpha_q)), T_{\alpha_q}p_{T^*Q}(v_{\alpha_q})\big\rangle\,,\end{aligned}$$

and this means that:

$$\Psi_0^{-1}((d\hat{f})(p_{T^*Q}(\alpha_q))) = p_{J^{-1}_{T^*T^*Q}(0)}(d(\hat{f} \circ p_{T^*Q})(\alpha_q))\,. \tag{4.13}$$

II) Similarly to the relation (4.9) defining $[(\beta_Q)_0]$, we have for its inverse $[(\beta_Q)_0]^{-1}$ the following equality:

$$\big[(\beta_Q)_0\big]^{-1} \circ p_{J^{-1}_{T^*TQ}(0)} = p_{J^{-1}_{TT^*Q}(0)} \circ (\beta_Q^{-1})_{|J^{-1}_{T^*TQ}(0)}\,. \tag{4.14}$$

Combining (4.14) with (4.13), we have the following:

$$\begin{aligned}[(\beta_Q)_0]^{-1}\big(\Psi_0^{-1}((d\hat{f})(p_{T^*Q}(\alpha_q)))\big) &= p_{J^{-1}_{TT^*Q}(0)}\Big((\beta_Q^{-1})_{|J^{-1}_{T^*T^*Q}(0)}(d(\hat{f} \circ p_{T^*Q})(\alpha_q)\Big) \\ &= p_{J^{-1}_{TT^*Q}(0)}\big(X_{(\hat{f}\circ p_{T^*Q})}(\alpha_q)\big)\,.\end{aligned}$$

In view of (4.12), we get:

$$(\Xi \circ [(\beta_Q)_0]^{-1} \circ \Psi_0^{-1})((d\hat{f})(p_{T^*Q}(\alpha_q))) = (T_{\alpha_q}p_{T^*Q})(X_{(\hat{f}\circ p_{T^*Q})}(\alpha_q))\,. \tag{4.15}$$

III) Since the Hamiltonian vector fields $X_{(\hat{f}\circ p_{T^*Q})} \in \mathfrak{X}(T^*Q)$ and $X_{\hat{f}} \in \mathfrak{X}(T^*Q/G)$ are p_{T^*Q}-related

$$Tp_{T^*Q} \circ X_{(\hat{f}\circ p_{T^*Q})} = X_{\hat{f}} \circ p_{T^*Q}\,,$$

we can rewrite (4.15) as

$$\begin{aligned}\big(\Xi \circ [(\beta_Q)_0]^{-1} \circ \Psi_0^{-1}\big)((d\hat{f})(p_{T^*Q}(\alpha_q))) &= X_{\hat{f}}(p_{T^*Q}(\alpha_q)) \\ &= \big(\sharp_{T^*Q/G}\big)((d\hat{f})(p_{T^*Q}(\alpha_q)))\,.\end{aligned}$$

□

Let $H : T^*Q \to \mathbb{R}$ be a G-invariant Hamiltonian and consider the invariant Lagrangian submanifold $dH(T^*Q) \subset J^{-1}_{T^*T^*Q}(0)$. Following Example 4.5, the reduced submanifold $dH(T^*Q)/G$ is again Lagrangian and can be mapped into a Lagrangian submanifold of $J^{-1}_{TT^*Q}(0)/G$ using the anti-symplectomorphism $[(\beta_Q)_0]$:

$$S_H/G = [(\beta_Q)_0]^{-1}(dH(T^*Q)/G) \subset J^{-1}_{TT^*Q}(0)/G\,. \tag{4.16}$$

Moreover, this Lagrangian submanifold coincides with the submanifold

$$S_h = ([(\beta_Q)_0]^{-1} \circ \Psi_0^{-1} \circ dh)(T^*Q/G) \subset J^{-1}_{TT^*Q}(0)/G\,,$$

obtained from the reduced Hamiltonian $h : T^*Q/G \to \mathbb{R}$. The results above imply the existence of a one-to-one correspondence between curves in T^*Q/G and curves in the Lagrangian submanifold S_h. The correspondence is naturally defined as follows:

(1) If γ is a curve in T^*Q/G, then the curve $\bar{\gamma}$ given by

$$\bar{\gamma}(t) = ([(\beta_Q)_0]^{-1} \circ \Psi_0^{-1} \circ dh)(\gamma(t))$$

is the corresponding curve in S_h.

(2) Conversely, let $\bar{\gamma}$ be a curve in S_h. Then it projects onto the following curve γ in T^*Q/G:

$$\gamma(t) = (\pi_{T^*Q/G} \circ \Psi_0 \circ [(\beta_Q)_0])(\bar{\gamma}(t)) = (\tau_{T^*Q/G} \circ \Xi)(\bar{\gamma}(t))\,.$$

Hamilton-Poincaré equations. Roughly speaking, the Hamilton-Poincaré equations equations follow from the symmetry reduction of the Hamilton equations. Here we will use the following well known result:

Lemma 4.9. *A curve γ in T^*Q/G is a solution of the Hamilton-Poincaré equations for H if, and only if, it is an integral curve of the Hamiltonian vector field $X_h \in \mathfrak{X}(T^*Q/G)$ with respect to the linear Poisson structure on T^*Q/G.*

Proof. See [dLMM05]. □

In other words, Lemma 4.9 gives the following characterization of the solutions γ of the Hamilton-Poincaré equations:

$$\sharp_{T^*Q/G}(dh(\gamma(t))) = X_h(\gamma(t)) = \frac{d}{dt}\gamma(t)\,. \tag{4.17}$$

Theorem 4.10. *Let $H : T^*Q \to \mathbb{R}$ be a G-invariant Hamiltonian. Then, in the one-to-one correspondence between curves in T^*Q/G and curves in S_h, the solutions of the Hamilton-Poincaré equations correspond to curves in S_h whose image by Ξ are tangents lifts of curves in T^*Q/G.*

Proof. Consider a solution $\gamma : I \to T^*Q/G$ of the Hamilton-Poincaré equations. From (4.17) and Lemma 4.8, it follows that

$$(\Xi \circ [(\beta_Q)_0]^{-1} \circ {\Psi_0}^{-1})(dh(\gamma(t))) = \sharp_{T^*Q/G}(dh(\gamma(t))) = \frac{d}{dt}\gamma(t)\,.$$

Thus, if we take the curve $\bar{\gamma} : I \to S_h$ defined as

$$\bar{\gamma}(t) = \big([(\beta_Q)_0]^{-1} \circ \Psi_0^{-1}\big)(dh(\gamma(t))\,,$$

then $\Xi \circ \bar{\gamma}$ is the tangent lift of γ.

Conversely, let $\bar{\gamma} : I \to S_h$ be a curve on S_h such that

$$(\Xi \circ \bar{\gamma})(t) = \frac{d}{dt}\gamma(t)\,,$$

where $\gamma : I \to T^*Q/G$ is a curve on T^*Q/G. Then,

$$(\tau_{T^*Q/G} \circ \Xi \circ \bar{\gamma})(t) = \gamma(t)\,,$$

which implies that

$$\bar{\gamma}(t) = ([\beta_Q)_0]^{-1} \circ \Psi_0^{-1})(dh(\gamma(t))\,.$$

As a consequence, γ is the corresponding curve in T^*Q/G associated with $\bar{\gamma}$ and

$$\frac{d}{dt}\gamma(t) = (\Xi \circ \bar{\gamma})(t) = \sharp_{T^*Q/G}(dh(\gamma(t)))\,.$$

We conclude that the curve γ on T^*Q/G solves the Hamilton-Poincaré equations for H. □

In particular, one obtains the intrinsic description of the Hamilton-Poincaré equations:

Corollary 4.11. *A curve $\gamma : I \to T^*Q/G$ is a solution of the Hamilton-Poincaré equations for H if, and only if, the image by Ξ of the corresponding curve in S_h,*

$$t \to \bar{\gamma}(t) = \big([(b_{\omega_Q})_0]^{-1} \circ \Psi_0^{-1} \circ dh\big)(\gamma(t))\,,$$

is the tangent lift of γ.

The situation is summarized in Diagram 4.2.

$$
\begin{tikzcd}
 & J^{-1}_{TT^*Q}(0)/G \arrow[dl, "\Xi"'] \arrow[dr] \arrow[rr, "\Psi_0\circ[(\beta_Q)_0]"] & & T^*(T^*Q/G) \arrow[dl, "\pi_{T^*Q/G}"'] \\
T(T^*Q/G) \arrow[rr, "\tau_{T^*Q/G}"] & & T^*Q/G \arrow[ur, bend right, "dh"'] & \\
 & & I \arrow[u, "\gamma"] \arrow[ull, "\frac{d}{dt}\gamma"] &
\end{tikzcd}
$$

DIAGRAM 4.2: Hamilton-Poincaré reduction

4.2.2 Example: Lie-Poisson equations

It is possible to give local expressions of the results in Subsection 4.2.1 in full generality. This would lead to the coordinate version of the so-called vertical and horizontal Hamilton-Poincaré equations which can be found in e.g. [CMPR03, dLMM05, Mes05, YM09]. However in view of the many technicalities involved with these local computations (such as invoking a principal connection and its curvature, choosing adapted coordinates, etc.) we will only treat here the special case where the configuration space is a Lie group G.

We consider the left action of G on itself by left translations, and identify $TG \cong G \times \mathfrak{g}$ and $T^*G \cong G \times \mathfrak{g}^*$ in the usual way:

$$
\begin{aligned}
v_g \in T_gG &\to (g, g^{-1}v_g) \in G \times \mathfrak{g}^*\,, \\
\alpha_g \in T_g^*G &\to (g, g^{-1}\alpha_g) \in G \times \mathfrak{g}^*\,.
\end{aligned}
$$

Applying these trivializations twice, we further identify

$$
TT^*G \cong (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)\,, \quad T^*T^*G \cong (G \times \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g})\,,
$$

whose elements will be denoted as follows:

$$
((g,\pi),(\omega,\dot{\pi})) \in (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)\,, \quad ((g,\pi)\,,(\tilde{\pi},\omega)) \in (G \times \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g})\,.
$$

Under the identifications above, G acts by left translations on the first factor, and therefore

$$
T^*G/G \cong \mathfrak{g}^*\,, \quad TT^*G/G \cong \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*)\,, \quad T^*T^*G/G \cong \mathfrak{g}^* \times (\mathfrak{g}^* \times \mathfrak{g})\,.
$$

The vector bundle isomorphism $\boldsymbol{\beta_Q}$. Using the definition of the Liouville 1-form θ on T^*G, it is easy to check that:

$$
\theta_G(g,\pi)((g,\pi),(\omega,\dot{\pi})) = \langle \pi, \omega \rangle\,.
$$

where $((g,\pi),(\omega,\dot{\pi})) \in (G\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*) \cong T(T^*G)$. In particular, this implies that the canonical 2-form Ω_G has the following expression:

$$\begin{aligned}\Omega_G(g,\pi)\left(((g,\pi),(\omega_1,\dot{\pi}_1)),((g,\pi),(\omega_2,\dot{\pi}_2))\right) &= \langle\dot{\pi}_1,\omega_2\rangle - \langle\dot{\pi}_2,\omega_1\rangle - \langle\pi,[\omega_1,\omega_2]\rangle \\ &= \langle\dot{\pi}_1,\omega_2\rangle - \langle\dot{\pi}_2,\omega_1\rangle - \langle\pi, ad_{\omega_1}\omega_2\rangle \ ,\end{aligned}$$

for all $((g,\pi),(\omega_1,\dot{\pi}_1)),((g,\pi),(\omega_2,\dot{\pi}_2)) \in (G\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*)$.

From the expression of the canonical symplectic form Ω_G it is straightforward to show that the vector bundle isomorphism

$$\beta_G : TT^*G \cong (G\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*) \to T^*T^*G \cong (G\times\mathfrak{g}^*)\times(\mathfrak{g}^*\times\mathfrak{g})$$

is given by

$$\beta_G((g,\pi),(\omega,\dot{\pi})) = ((g,\pi),(-\dot{\pi}+ad^*_\omega\pi,\omega))\,, \tag{4.18}$$

where $ad^* : \mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the adjoint operator.

Reduced spaces. Let $J_{T^*G} : T^*G \cong (G\times\mathfrak{g}^*) \to \mathfrak{g}^*$ be the momentum map on T^*G, defined as

$$\langle J_{T^*G}(g,\pi),\xi\rangle = \left\langle T_g^*L_{g^{-1}}(\pi),\xi_G(g)\right\rangle \ ,$$

for all $(g,\pi)\in(G\times\mathfrak{g}^*)$ and $\xi\in\mathfrak{g}$. Since the action on G is left translation, its infinitesimal generators are the right invariant vector fields and thus

$$\langle J_{T^*G}(g,\pi),\xi\rangle = \left\langle T_g^*L_{g^{-1}}(\pi),T_eR_g(\xi)\right\rangle = \left\langle \pi, Ad_{g^{-1}}\xi\right\rangle = \left\langle Ad^*_{g^{-1}}\pi,\xi\right\rangle,$$

or, in other words, $J_{T^*G}(g,\pi) = Ad^*_{g^{-1}}\pi$. With a similar computation we get the following expression for $J_{T^*T^*G} : T^*T^*G \cong (G\times\mathfrak{g}^*)\times(\mathfrak{g}^*\times\mathfrak{g}) \to \mathfrak{g}^*$:

$$J_{T^*T^*G}((g,\pi),(\pi',\omega)) = Ad^*_{g^{-1}}\pi' \, .$$

In view of the expression (4.18) for β_G and Theorem 4.4, we immediately obtain the expression for the trivialized momentum J_{TT^*G}:

$$J_{TT^*G}((g,\pi),(\omega,\dot{\pi})) = Ad^*_{g^{-1}}\left(\dot{\pi} - ad^*_\omega\pi\right) \, .$$

In particular, for the zero level sets of the momentum maps we then have:

$$\begin{aligned}J^{-1}_{T^*T^*G}(0) &= \{((g,\pi),(0,\omega)) \in (G\times\mathfrak{g}^*)\times(\mathfrak{g}^*\times\mathfrak{g})\} \cong (G\times\mathfrak{g}^*)\times\mathfrak{g}\,, \\ J^{-1}_{TT^*G}(0) &= \{((g,\pi),(\omega,ad^*_\omega\pi)) \in (G\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*)\} \cong (G\times\mathfrak{g}^*)\times\mathfrak{g}\,,\end{aligned}$$

and therefore the reduced spaces

$$\begin{aligned}J^{-1}_{T^*T^*G}(0)/G &= \{(\pi,(\pi',\omega)) \in \mathfrak{g}^*\times(\mathfrak{g}^*\times\mathfrak{g}) : \pi' = 0\} \cong \mathfrak{g}^*\times\mathfrak{g}\,, \\ J^{-1}_{TT^*G}(0)/G &= \{(\pi,(\omega,\dot{\pi})) \in \mathfrak{g}^*\times(\mathfrak{g}\times\mathfrak{g}^*) : \dot{\pi} = ad^*_\omega\pi\} \cong \mathfrak{g}^*\times\mathfrak{g}\,,\end{aligned}$$

can both be identified with $\mathfrak{g}^*\times\mathfrak{g}$.

The maps $[(\boldsymbol{\beta_G})_0]$, $\boldsymbol{\Psi_0}$ and $\boldsymbol{\Xi}$. In view of the above identifications and (4.18), the map $[(\beta_G)_0]$ is simply given by the identity

$$\begin{aligned}[(\beta_G)_0] : \mathfrak{g}^* \times \mathfrak{g} &\to \mathfrak{g}^* \times \mathfrak{g} \\ (\pi, \omega) &\mapsto [(\beta_G)_0](\pi, \omega) = (\pi, \omega) .\end{aligned}$$

We also have identifications $T^*(T^*G/G) \cong \mathfrak{g}^* \times \mathfrak{g}$ and $T(T^*G/G) \cong \mathfrak{g}^* \times \mathfrak{g}^*$, so we may as well work with trivialized expressions for the maps Ψ_0 and Ξ. One can check that these are given by:

$$\begin{aligned}\Psi_0 : \mathfrak{g}^* \times \mathfrak{g} &\to \mathfrak{g}^* \times \mathfrak{g} \\ (\pi, \omega) &\mapsto \Psi_0(\pi, \omega) = (\pi, \omega) , \\ \Xi : \mathfrak{g}^* \times \mathfrak{g} &\to \mathfrak{g}^* \times \mathfrak{g}^* \\ (\pi, \omega) &\mapsto \Xi(\pi, \omega) = (\pi, ad^*_\omega \pi) .\end{aligned}$$

The Lie-Poisson dynamics. Let $H : T^*G \cong G \times \mathfrak{g}^* \to \mathbb{R}$ be a G-invariant Hamiltonian and denote by $h : \mathfrak{g}^* \to \mathbb{R}$ the reduced Hamiltonian. Define the Lagrangian submanifold S_h by:

$$S_h = \{(\pi, dh(\pi)) \in \mathfrak{g}^* \times \mathfrak{g}\} \cong \mathfrak{g}^* .$$

Consider a curve $\bar{\gamma}(t) = (\bar{\pi}(t), \bar{\omega}(t)) \in J^{-1}_{TT^*G}(0)/G \cong \mathfrak{g}^* \times \mathfrak{g}$ with values in S_h and which is such that its image by Ξ is the tangent lift of a curve $t \mapsto \pi(t) \in T^*G/G \cong \mathfrak{g}^*$. Then, it is clear that

$$\bar{\pi}(t) = \pi(t) , \quad \bar{\omega}(t) = dh(\bar{\pi}(t)) , \quad ad^*_{\bar{\omega}(t)}\bar{\pi}(t) = \frac{d}{dt}\pi(t) .$$

Thus, it follows that

$$ad^*_{dh(\pi(t))}\pi(t) = \frac{d}{dt}\pi(t) .$$

Therefore, the curve $t \mapsto \pi(t)$ in $\mathfrak{g}^*$ solves the well known Lie-Poisson equations.

Conversely, assume that a curve in $\mathfrak{g}^*$, $t \mapsto \pi(t)$, is a solution of the Lie-Poisson equations for H and consider the following curve in S_h:

$$t \mapsto \bar{\gamma}(t) = [(\beta_G)_0]^{-1} \circ \Psi_0^{-1})(dh(\pi(t)) = (\pi(t), dh(\pi(t)) \in \mathfrak{g}^* \times \mathfrak{g} \cong J^{-1}_{TT^*G}(0)/G .$$

Its image by the map Ξ is the curve

$$t \mapsto \Big(\pi(t), ad^*_{dh(\pi(t))}\pi(t)\Big) \in \mathfrak{g}^* \times \mathfrak{g}^* \cong T(T^*G/G) .$$

Using that $t \to \pi(t)$ is a solution of the Lie-Poisson equations, it follows that

$$\frac{d}{dt}\pi(t) = ad^*_{dh(\pi(t))}\pi(t) ,$$

i.e., the curve $\Xi \circ \gamma$ is the tangent lift of the curve $t \to \pi(t) \in \mathfrak{g}^*$.

Dynamics on direct products. In the case where the configuration space Q is a direct product $G \times S$ of the Lie group G and a manifold S and the G-action is by left translations on the first factor, the Hamilton-Poincaré equations can be derived easily from the computations above. If we denote by $\pi_1 : Q \to G$ and $\pi_2 : Q \to S$ the corresponding projections, then $\Omega_Q = \pi_1^*\Omega_G + \pi_2^*\Omega_S$ and it follows:

$$\beta_Q = (\beta_G, \beta_S) : TT^*G \times TT^*S \to T^*T^*G \times T^*T^*S\,.$$

The momentum maps can be computed as before, in the case of a Lie group. In particular,

$$\langle J_{T^*Q}((g,\pi),\alpha_x),\omega\rangle = \langle J_{T^*G}(g,\pi),\omega\rangle = \left\langle \pi, Ad_{g^{-1}}\omega\right\rangle\,,$$

for all $((g,\pi),\alpha_x) \in (G\times\mathfrak{g}^*)\times T^*S$, with $x \in S$. As a consequence we obtain directly the expressions for J_{TT^*Q} and $J_{T^*T^*Q}$ from those in the previous subsection. For example, we find:

$$J_{TT^*Q}(((g,\pi),(\omega,\dot\pi)),X_{p_x}) = J_{TT^*G}((g,\pi),(\omega,\dot\pi)) = Ad^*_{g^{-1}}(\dot\pi - ad^*_\omega\pi)\,,$$

for all $(((g,\pi),(\omega,\dot\pi)),X_{p_x}) \in ((G\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*))\times TT^*S$, with $p_x \in T^*_xS$, and similarly for $J_{T^*T^*Q}$. Therefore, on the zero level sets we have the identifications:

$$J^{-1}_{TT^*Q}(0)/G \cong J^{-1}_{TT^*G}(0)/G \times TT^*S \cong (\mathfrak{g}^*\times\mathfrak{g})\times TT^*S\,,$$
$$J^{-1}_{T^*T^*Q}(0)/G \cong J^{-1}_{T^*T^*G}(0)/G \times T^*T^*S \cong (\mathfrak{g}^*\times\mathfrak{g})\times T^*T^*S\,.$$

Taking the previous expressions into account, we deduce that the reduced map $[(\beta_Q)_0]$ is of the form

$$[(\beta_Q)_0] = ([\beta_G]_0, \beta_S) : J^{-1}_{TT^*G}(0)/G \times TT^*S \to J^{-1}_{T^*T^*G}(0)/G\times T^*T^*S\,.$$

The map Ψ_0 is the identity and the map $\Xi : J^{-1}_{TT^*Q}(0)/G \simeq (\mathfrak{g}^*\times\mathfrak{g})\times TT^*S \to T(T^*Q/G) \simeq (\mathfrak{g}^*\times\mathfrak{g}^*)\times TT^*S$ is given by

$$\Xi((\pi,\omega),X_{p_x}) = ((\pi, ad^*_\omega\pi), X_{p_x})\,.$$

Now, suppose that $H : T^*Q \simeq G\times\mathfrak{g}^*\times T^*S \to \mathbb{R}$ is a G-invariant Hamiltonian function and $h : T^*Q/G \simeq \mathfrak{g}^*\times T^*S \to \mathbb{R}$ is the reduced Hamiltonian function. Then, the Lagrangian submanifold S_h in $J^{-1}_{TT^*Q}(0)/G \simeq (\mathfrak{g}^*\times\mathfrak{g})\times TT^*S$ is given by

$$S_h = \{(\pi, dh_{\alpha_x}(\pi), X_{h_\pi}(\alpha_x)) \mid \pi\in\mathfrak{g}^*, \alpha_x \in T^*S\}$$

where $h_{\alpha_x} : \mathfrak{g}^* \to \mathbb{R}$ (respectively, $h_\pi : T^*S \to \mathbb{R}$) is the real function on $\mathfrak{g}^*$ (respectively, T^*S) defined by

$$h_{\alpha_x}(\pi') = h(\pi',\alpha_x)\,,\quad \text{for } \pi'\in\mathfrak{g}^*$$

(respectively, $h_\pi(\alpha'_{x'}) = h(\pi,\alpha'_{x'})$, for $\alpha'_{x'} \in T^*S$), and X_{h_π} is the Hamiltonian vector field in T^*S of h_π.

Thus, a curve $t \mapsto (\pi_a(t), x^i(t), p_i(t))$ on $\mathfrak{g}^*\times T^*S \simeq T^*Q/G$ satisfies the conditions of Corollary 4.11 if, and only if,

$$\dot\pi = ad^*_{\frac{\partial h}{\partial\pi}}\pi\,,\quad \dot x^i = \frac{\partial h}{\partial p_i}\,,\quad \dot p_i = -\frac{\partial h}{\partial x^i}\,,$$

which are the Hamilton-Poincaré equations for H in this case.

4.2.3 Lagrange-Poincaré reduction

To get an intrinsic description of the reduced Lagrange-Poincaré equations, we will follow the same strategy as in the Hamiltonian case.

As before, we consider an action ϕ of G on Q and the corresponding G-actions ϕ^{TQ} on TQ, ϕ^{T^*Q} on T^*Q, ϕ^{TT^*Q} on TT^*Q and ϕ^{T^*TQ} on T^*TQ, obtained by tangent or cotangent lift of ϕ_g. The first thing to prove is that Tulczyjew's diffeomorphism α_Q is G-equivariant and preserves the momentum maps associated to the actions on TT^*Q and T^*TQ. In order to do that, we need two preparatory lemmas.

Lemma 4.12. *The projection* $\mathsf{v}^* : T^*TQ \to T^*Q$ *is equivariant.*

Proof. First, from the definition of the vertical lift $(\cdot)^{\mathsf{v}}_{v_q} : T_qQ \to T_{v_q}TQ$ in (4.1), it follows that it is equivariant: for each $w_q \in TQ$ we have $g(w_q)^{\mathsf{v}}_{v_q} = (gw_q)^{\mathsf{v}}_{gv_q}$. Then:

$$\begin{aligned}\left\langle \mathsf{v}^*(g\alpha_{v_q}), (gw_q)\right\rangle &= \left\langle g\alpha_{v_q}, (gw_q)^{\mathsf{v}}_{gv_q}\right\rangle = \left\langle g\alpha_{v_q}, g(w_q)^{\mathsf{v}}_{v_q}\right\rangle \\ &= \left\langle \alpha_{v_q}, (w_q)^{\mathsf{v}}_{v_q}\right\rangle = \left\langle \mathsf{v}^*(\alpha_{v_q}), (w_q)\right\rangle .\end{aligned}$$

This completes the proof. □

Lemma 4.13. *Consider the anti-symplectomorphism* $R : T^*TQ \to T^*T^*Q$*. Then is* R *equivariant and satisfies:*

$$J_{T^*T^*Q} \circ R = -J_{T^*TQ}\,.$$

Proof. Using the equivariance of the maps $T\tau_Q$ and $T\pi_Q$ and the invariance of the pairings, we find

$$\begin{aligned}\Big\langle R\big(g\alpha_{v_q}\big), g\big(W_{\mathsf{v}^*(\alpha_{v_q})}\big)\Big\rangle &= -\Big\langle g\alpha_{v_q}, g\bar{W}_{v_q}\Big\rangle + \Big\langle gW_{\mathsf{v}^*(\alpha_{v_q})}, g\bar{W}_{v_q}\Big\rangle^T \\ &= -\Big\langle \alpha_{v_q}, \bar{W}_{v_q}\Big\rangle + \Big\langle W_{\mathsf{v}^*(\alpha_{v_q})}, \bar{W}_{v_q}\Big\rangle^T .\end{aligned}$$

This implies $R\left(g\alpha_{v_q}\right) = gR\left(\alpha_{v_q}\right)$, i.e. R is equivariant.

From the definition of the momentum map $J_{T^*T^*Q}$ it follows that, for all $\alpha_{v_q} \in T^*TQ$ and $\xi \in \mathfrak{g}$,

$$\left\langle J_{T^*T^*Q}(R(\alpha_{v_q})), \xi\right\rangle = \left\langle R(\alpha_{v_q}), \xi_{T^*Q}(\mathsf{v}^*(\alpha_{v_q}))\right\rangle .$$

Recalling the definition (4.2) of R for $W = \xi_{T^*Q}(\mathsf{v}^*(\alpha_{v_q}))$ and $\bar{W} = \xi_{TQ}(v_q)$ (note that this choice satisfies (4.3)), we obtain the following:

$$\begin{aligned}\left\langle R(\alpha_{v_q}), \xi_{T^*Q}(\mathsf{v}^*(\alpha_{v_q}))\right\rangle &= -\left\langle \alpha_{v_q}, \xi_{TQ}(v_q)\right\rangle + \left\langle \xi_{T^*Q}(\mathsf{v}^*(\alpha_{v_q})), \xi_{TQ}(v_q)\right\rangle^T \\ &= -\left\langle J_{T^*TQ}(\alpha_{v_q}), \xi\right\rangle + \left\langle \xi_{T^*Q}(\mathsf{v}^*(\alpha_{v_q})), \xi_{TQ}(v_q)\right\rangle^T ,\end{aligned}$$

where $\langle\cdot,\cdot\rangle^T : TT^*Q \times_{TQ} TTQ \to \mathbb{R}$ is the pairing defined at the beginning of Section 4.1.

We will now prove that the second term vanishes. If we write φ_t for the flow of ξ_Q around $q \in Q$, i.e. $\varphi_t = \exp(t\xi)q$, then the flows of ξ_{TQ} and ξ_{T^*Q} are $T\varphi_t$ and $T^*\varphi_t$ respectively. If we take into account this observation, we conclude:

$$\begin{aligned}\left\langle \xi_{T^*Q}(\mathsf{v}^*(\alpha_{v_q})), \xi_{TQ}(v_q)\right\rangle^T &= \left.\frac{d}{dt}\right|_{t=0} \left\langle (T^*\varphi_t)(\mathsf{v}^*(\alpha_{v_q})), T\varphi_t(v_q)\right\rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \left\langle \mathsf{v}^*(\alpha_{v_q}), v_q\right\rangle = 0\,.\end{aligned}$$

Therefore,

$$\left\langle J_{T^*T^*Q}(R(\alpha_{v_q})), \xi\right\rangle = -\left\langle \alpha_{v_q}, \xi_{TQ}(v_q)\right\rangle = -\left\langle J_{T^*TQ}(\alpha_{v_q}), \xi\right\rangle\,.$$

□

We can now prove the analogous of Theorem 4.7 for the map α_Q:

Theorem 4.14. *Under the conditions above, the Tulczyjew diffeomorphism α_Q is equivariant and satisfies*

$$J_{TT^*Q} = J_{T^*TQ} \circ \alpha_Q\,.$$

Proof. In view of Lemma 4.13 and recalling the definition of α_Q as the composition $R^{-1} \circ \beta_Q$, it follows directly that α_Q is equivariant and that the momentum map satisfies

$$J_{T^*TQ} \circ \alpha_Q = -J_{T^*T^*Q} \circ \beta_Q = J_{TT^*Q}\,.$$

□

The results above imply the existence of the reduced maps

$$\begin{aligned}[R_0]\colon J^{-1}_{T^*TQ}(0)/G &\to J^{-1}_{T^*T^*Q}(0)/G\,,\\ [(\alpha_Q)_0]\colon J^{-1}_{TT^*Q}(0)/G &\to J^{-1}_{T^*TQ}(0)/G\,,\end{aligned}$$

defined by

$$[R_0] \circ p_{J^{-1}_{T^*TQ}(0)} = p_{J^{-1}_{T^*T^*Q}(0)} \circ R_{|J^{-1}_{T^*TQ}(0)} \tag{4.19}$$

and

$$[(\alpha_Q)_0] \circ p_{J^{-1}_{TT^*Q}(0)} = p_{J^{-1}_{T^*TQ}(0)} \circ \alpha_{Q|J^{-1}_{TT^*Q}(0)}\,,$$

respectively. Taking into account the definition of Tulczyjew's diffeomorphism, it is clear that

$$[(\alpha_Q)_0] = [R_0]^{-1} \circ [(\beta_Q)_0]\,. \tag{4.20}$$

One could in fact take the previous above expression as an alternative definition of the map $[(\alpha_Q)_0]$, which is then analogous to the definition of the Tulczyjew diffeomorphism. We shall refer to $[(\alpha_Q)_0]$ as the *reduced Tulczyjew diffeomorphism*.

The geometry of $[(\alpha_Q)_0]$. Before we enter the discussion about Lagrange-Poincaré reduction we need to introduce a few more mappings and fix some notations:

1) The space $(J^{-1}_{T^*TQ}(0)/G, (\Omega_{TQ})_0)$ is obtained after a cotangent bundle reduction at $\mu = 0$ and it hence it is identified with a cotangent bundle. More precisely, the exists a diffeomorphism
$$\varphi_0\colon \big(J^{-1}_{T^*TQ}(0)/G, (\Omega_{TQ})_0\big) \to \big(T^*(TQ/G), \Omega_{TQ/G}\big)$$
given by (see 2.6):
$$\big\langle \varphi_0\big(p_{J^{-1}_{T^*TQ}(0)}(\alpha_{v_q})\big), T_{v_q}p_{TQ}(u_{v_q})\big\rangle := \big\langle \alpha_{v_q}, u_{v_q}\big\rangle\,, \tag{4.21}$$
for all $\alpha_{v_q} \in J^{-1}_{T^*TQ}(0)$ and $u_{v_q} \in TTQ$.

2) The space $T^*(TQ/G)$ admits a vector bundle structure over T^*Q/G with projection $T^*(TQ/G) \to T^*Q/G$ that we define next. First, note that the space $T^*(TQ/G)$ is a vector subbundle (over TQ/G) of the vector bundle $[\pi_{TQ}]\colon T^*TQ/G \to TQ/G$, with inclusion
$$i : T^*(TQ/G) \to T^*TQ/G$$
defined by
$$i\big(\alpha_{p_{TQ}(v_q)}\big) = p_{T^*TQ}\big(T^*_{v_q}p_{TQ}(\alpha_{p_{TQ}(v_q)})\big)\,,$$
for $\alpha_{p_{TQ}(v_q)} \in T^*(TQ/G)$ and $v_q \in TQ$. On the other hand, T^*TQ/G is a vector bundle over T^*Q/G and with vector bundle projection $[\mathsf{v}^*]\colon T^*TQ/G \to T^*Q/G$ induced by v^*. Then we define
$$\tilde{\mathsf{v}}^* : T^*(TQ/G) \to T^*Q/G$$
as the composition
$$\tilde{\mathsf{v}}^* = [\mathsf{v}^*] \circ i\,.$$
Then $\tilde{\mathsf{v}}^* : T^*(TQ/G) \to T^*Q/G$ is

3) The vector bundle map $R_{Q/G}$ is defined similarly to the map R. Explicitly, the map $R_{Q/G} : T^*(TQ/G) \to T^*(T^*Q/G)$ is the isomorphism between the vector bundles $\tilde{\mathsf{v}}^* : T^*(TQ/G) \to T^*Q/G$ and $\pi_{T^*Q/G} : T^*(T^*Q/G)$ such that:
$$\begin{aligned}&\Big\langle R_{Q/G}\big(\alpha_{p_{TQ}(v_q)}\big), W_{\tilde{\mathsf{v}}^*\left(\alpha_{p_{TQ}(v_q)}\right)}\Big\rangle = \\ &\qquad -\Big\langle \alpha_{p_{TQ}(v_q)}, \bar{W}_{p_{TQ}(v_q)}\Big\rangle + \Big\langle W_{\tilde{\mathsf{v}}^*\left(\alpha_{p_{TQ}(v_q)}\right)}, \bar{W}_{p_{TQ}(v_q)}\Big\rangle^T,\end{aligned} \tag{4.22}$$
for all $\alpha_{p_{TQ}(v_q)} \in T^*(TQ/G)$ and $W_{\tilde{\mathsf{v}}^*(\alpha_{p_{TQ}(v_q)})} \in T(T^*Q/G)$, with $\bar{W}_{p_{TQ}(v_q)} \in T(TQ/G)$ satisfying
$$T[\tau_Q]\big(\bar{W}_{p_{TQ}(v_q)}\big) = T[\pi_Q]\Big(W_{\tilde{\mathsf{v}}^*(\alpha_{p_{TQ}(v_q)})}\Big)\,,$$
where $[\tau_Q] : TQ/G \to Q/G$ and $[\pi_Q] : T^*Q/G \to Q/G$ are the canonical projections and $\langle\cdot,\cdot\rangle^T : T(T^*Q/G) \times_{T(Q/G)} T(TQ/G) \to \mathbb{R}$ is the tangent map of the natural pairing $\langle\cdot,\cdot\rangle : T^*Q/G \times_{Q/G} TQ/G \to \mathbb{R}$.

4) There is a vector bundle morphism $\Lambda : T^*(TQ/G) \to T(T^*Q/G)$ (over the identity in T^*Q/G) given by
$$\Lambda = \sharp_{T^*Q/G} \circ R_{Q/G} \,. \tag{4.23}$$

The relation between the different maps introduced so far is summarized in the following lemma, which may be regarded as the Lagrangian analogue of Lemma 4.8.

Lemma 4.15. *The diagram*

$$\begin{tikzcd} J^{-1}_{TT^*Q}(0)/G \arrow[d, "\Xi"'] & J^{-1}_{T^*TQ}(0)/G \arrow[l, "{[(\alpha_Q)_0]^{-1}}"'] \\ T(T^*Q/G) & T^*(TQ/G) \arrow[l, "\Lambda"] \arrow[u, "(\varphi_0)^{-1}"'] \end{tikzcd}$$

is commutative.

Proof. We claim that it is sufficient to check the commutativity of the following diagram:

$$\begin{tikzcd} J^{-1}_{T^*TQ}(0)/G \arrow[r, "{[R_0]}"] & J^{-1}_{T^*T^*Q}(0)/G \arrow[d, "\Psi_0"] \\ T^*(TQ/G) \arrow[u, "(\varphi_0)^{-1}"] \arrow[r, "R_{Q/G}"'] & T^*(T^*Q/G) \end{tikzcd}$$

Indeed, Lemma 4.8 guarantees that the diagram

$$\begin{tikzcd} J^{-1}_{TT^*Q}(0)/G \arrow[d, "\Xi"'] & J^{-1}_{T^*T^*Q}(0)/G \arrow[l, "{[(\beta_Q)_0]^{-1}}"'] \\ T(T^*Q/G) & T^*(T^*Q/G) \arrow[l, "\sharp_{T^*Q/G}"] \arrow[u, "(\Psi_0)^{-1}"'] \end{tikzcd}$$

is commutative. Together with the definition (4.20) of the map $[(\alpha_Q)]_0$ and assuming the commutativity of the first diagram, the result follows directly by diagram chasing:

$$\Lambda = \sharp_{T^*Q/G} \circ R_{Q/G} = (\Xi \circ [(\beta_Q)_0]^{-1} \circ \Psi_0^{-1}) \circ (\Psi_0 \circ [R_0] \circ \varphi_0^{-1}) = \Xi \circ [(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1} \,.$$

The proof of the commutativity of the diagram is straightforward but involves a tedious computation. The details can be found in [GTAGMM14]. $\square$

Let $L : TQ \to \mathbb{R}$ be a G-invariant Lagrangian. Much like in the Hamiltonian case, we can show that $dL(TQ)/G$ is a Lagrangian submanifold of $J^{-1}_{T^*TQ}(0)/G$ which, by means of the reduced Tulczyjew's diffeomorphism, can be mapped into a Lagrangian submanifold of $J^{-1}_{TT^*Q}(0)/G$. More precisely, let S_L be the invariant Lagrangian submanifold of the symplectic manifold (TT^*Q, Ω^c_Q) defined by $S_L = (\alpha_Q)^{-1}(dL(TQ))$. The space of orbits S_L/G is Lagrangian submanifold of $J^{-1}_{TT^*Q}(0)/G$ which satisfies:

$$S_L/G = [(\alpha_Q)_0]^{-1}(dL(TQ)/G)\,. \tag{4.24}$$

This submanifold coincides with the submanifold S_l obtained from the reduced Lagrangian, i.e.

$$S_L/G = S_l := ([(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1} \circ dl)(TQ/G)\,,$$

where $l : TQ/G \to \mathbb{R}$ is the reduced Lagrangian, i.e. $L = l \circ p_{TQ}$.

This implies the existence of a one-to-one correspondence between curves in TQ/G and curves in the Lagrangian submanifold S_l defined as follows: if $\gamma : I \to TQ/G$ is a curve in TQ/G, then

$$t \to ([(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1} \circ dl)(\gamma(t))$$

is the corresponding curve in S_l.

Lagrange-Poincaré equations. There exist many different geometric frameworks for the theory of Lagrange-Poincaré reduction. The reader might be familiar with the following coordinate form of the Lagrange-Poincaré equations on the bundle $T(Q/G) \oplus \tilde{\mathfrak{g}}$:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial l}{\partial \bar{v}}\right) &= \frac{\partial l}{\partial \bar{v}^a}(C^a_{db}\bar{v}^d - C^a_{db}A^d_\alpha \dot{x}^\alpha)\,, \\ \frac{\partial l}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}^\alpha}\right) &= \frac{\partial l}{\partial \bar{v}^a}(B^a_{\beta\alpha}\dot{x}^\beta + C^a_{db}\bar{v}^d A^b_\alpha)\,, \end{aligned} \tag{4.25}$$

which can be found in e.g. [CMR01]. Here $(x, \dot{x}) \oplus \bar{v}$ denote coordinates on $T(Q/G) \oplus \tilde{\mathfrak{g}}$, A^d_α and $B^a_{\alpha\beta}$ denote the *local* components of the connection and the curvature forms, C^a_{db} are the structure constants of $\mathfrak{g}$, and l is the reduced Lagrangian[b].

We will use a somewhat indirect approach to the Lagrange-Poincaré equations. It is shown in [dLMM05] that the Lagrange-Poincaré equations can be thought of as Euler-Lagrange equations on a Lie algebroid, where the Lie algebroid is the Atiyah algebroid TQ/G. In [GUG06] the authors give a characterization of the set of solutions of the Euler-Lagrange equations on a Lie algebroid, which applied to the case of the Atiyah algebroid is as follows: a curve $\sigma : I \to TQ/G$ is a solution of the Lagrange–Poincaré equations if, and only if, it satisfies the equation

$$\frac{d}{dt}(\mathbb{F}l \circ \sigma)(t) = \Lambda(dl(\sigma(t)))\,, \tag{4.26}$$

[b] We refer the reader to the literature for the precise the meaning of all the terms in (4.25). Besides [CMR01], we also suggest [dLMM05] and [Mes05] for a careful and detailed derivation of the Lagrange-Poincaré equations.

where $\mathbb{F}l : TQ/G \to T^*Q/G$ is the Legendre transformation associated with l defined by

$$\mathbb{F}l(p_{TQ}(v_q)) = \tilde{\mathsf{v}}^*(dl(p_{TQ}(v_q)))\,, \tag{4.27}$$

for all $v_q \in TQ$.

Theorem 4.16. *Let $L : TQ \to \mathbb{R}$ be a G-invariant Lagrangian function. Then, in the one-to-one correspondence between curves in TQ/G and curves in S_l, the solutions of the Lagrange-Poincaré equations correspond with curves in S_l whose image by Ξ are tangent lifts of curves in T^*Q/G.*

Proof. Let us assume that a curve $\sigma : I \to TQ/G$ is a solution of the Lagrange-Poincaré equations for L. Then, using (4.26) and Lemma 4.15, it follows that

$$\frac{d}{dt}(\mathcal{F}_l(\sigma(t))) = \Lambda(dl(\sigma(t))) = (\Xi \circ [(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1})(dl(\sigma(t)))\,.$$

Thus, if we take the curve $\bar{\sigma} : I \to S_l$ in S_l associated with σ, namely

$$\bar{\sigma}(t) = ([(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1})(dl(\sigma(t)))\,,$$

we deduce that the curve $\Xi \circ \bar{\sigma}$ is the tangent lift of the curve $\mathbb{F}l \circ \sigma$.

Conversely, let $\bar{\sigma} : I \to S_l$ be a curve in S_l such that

$$(\Xi \circ \bar{\sigma})(t) = \frac{d}{dt}\gamma(t)\,, \tag{4.28}$$

where $\gamma : I \to T^*Q/G$ is a curve in T^*Q/G. Suppose that $\sigma : I \to TQ/G$ is the curve on TQ/G associated with $\bar{\sigma}$, that is,

$$\bar{\sigma}(t) = ([(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1})(dl(\sigma(t)))\,. \tag{4.29}$$

Then, using (4.28) and Lemma 4.15, it follows that

$$\gamma(t) = (\tau_{T^*Q/G} \circ \Xi \circ \bar{\sigma})(t) = (\tau_{T^*Q/G} \circ \Lambda)(dl(\sigma(t)))\,.$$

From (4.23), (4.19) and (4.27), we obtain that

$$\gamma(t) = \tilde{\mathsf{v}}^*(dl(\sigma(t))) = \mathbb{F}l(\sigma(t)).$$

Using (4.28) and (4.29) and Lemma 4.15, this proves that

$$\frac{d}{dt}(\mathbb{F}l \circ \sigma)(t) = \Lambda(dl(\sigma(t)))\,.$$

Therefore, σ is a solution of the Lagrange-Poincaré equations for L. $\square$

Using this theorem, we obtain an intrinsic description of the Lagrange-Poincaré equations.

Corollary 4.17. *Let $L : TQ \to \mathbb{R}$ be a G-invariant Lagrangian function, $l : TQ/G \to \mathbb{R}$ the reduced Lagrangian function and $\mathbb{F}l : TQ/G \to T^*Q/G$ the Legendre transformation associated with l. A curve $\sigma : I \to TQ/G$ is a solution of the Lagrange-Poincaré equations for L if, and only if, the image by Ξ of the corresponding curve in S_l,*

$$t \to ([(\alpha_Q)_0]^{-1} \circ \varphi_0^{-1})(dl(\sigma(t))),$$

is the tangent lift of the curve $\mathbb{F}l \circ \sigma$.

We summarize the situation in Diagram 4.3.

$$\begin{tikzcd}
T^*(TQ/G) \arrow[dr] & & J^{-1}_{TT^*Q}(0)/G \arrow[ll, "\varphi_0\circ[(\alpha_Q)_0]"'] \arrow[dl] \arrow[d] \arrow[dr, "\Xi"] & \\
& TQ/G \arrow[ul, bend left, "dl"] \arrow[r, "\mathbb{F}l"] & T^*Q/G & T(T^*Q/G) \arrow[l, "\tau_{T^*Q/G}"'] \\
& I \arrow[u, "\sigma"] \arrow[ur, "(\mathbb{F}l\circ\sigma)"] \arrow[urr, "\frac{d}{dt}(\mathbb{F}l\circ\sigma)"'] & &
\end{tikzcd}$$

DIAGRAM 4.3: Lagrange-Poincaré reduction

4.2.4 Equivalence

Let $L : TQ \to \mathbb{R}$ be an invariant hyperregular Lagrangian and consider its energy E_L, which is G-invariant. Then the corresponding Hamiltonian function $H = E_L \circ (\mathbb{F}L)^{-1}$ is G-invariant and it is not hard to see (c.f. [dLR89]) that in this situation the submanifolds describing the dynamics coincide:

$$S_L = S_H\,.$$

This equivalence is preserved under reduction, namely $S_H/G = S_L/G$. In view of the relations (4.16) and (4.24), this can be equivalently expressed as $S_h = S_l$. This is an important result.

Theorem 4.18. *In the situation above, let L be an hyperregular Lagrangian and consider the associated Hamiltonian H. Denote by $l : TQ/G \to \mathbb{R}$ and $h : T^*Q/G \to \mathbb{R}$ the reduced Lagrangian and Hamiltonian. Then $S_l = S_h$.*

The results in this chapter are depicted in the *reduced Tulczyjew triple*, Diagram 4.4. The map $[(T\pi_Q)_0]$ is the canonical projection induced by the vector bundle projection

$$(T\pi_Q)_0 = (T\pi_Q)_{|J^{-1}_{TT^*Q}(0)} : J^{-1}_{TT^*Q}(0) \to TQ\,,$$

and analogously for the map $[(\tau_{T^*Q})_0]$.

$$\begin{tikzcd}
T^*(TQ/G) \arrow[dr, "\pi_{TQ/G}"] & & J^{-1}_{TT^*Q}(0)/G \arrow[ll, "\varphi_0\circ[(\alpha_Q)_0]"'] \arrow[rr, "\Psi_0[\circ(\beta_Q)_0]"] \arrow[dl, "{[(T\pi_Q)_0]}"'] \arrow[dr, "{[(\tau_{T^*Q})_0]}"] & & T^*(T^*Q/G) \arrow[dl, "\pi_{T^*Q/G}"'] \\
& TQ/G \arrow[ul, bend left, "dl"] \arrow[rr, "\mathbb{F}l"] \arrow[dr, "{[\tau_Q]}"'] & & T^*Q/G \arrow[ur, bend right, "dh"'] \arrow[dl, "{[\pi_Q]}"] & \\
& & Q/G & &
\end{tikzcd}$$

Diagram 4.4: Reduced Tulczyjew triple

A final comment. During this section, we have extensively made use of the map Ξ defined in (4.12) to obtain the reduced dynamics. As a matter of fact, this map allows us to relate our triple with one that appeared in [GUG06], associated with an arbitrary Lie algebroid. More precisely, our triple relates to the construction in [GUG06] when applied to the Atiyah algebroid $[\tau_Q] : TQ/G \to Q/G$ associated with the principal G-bundle $Q \to Q/G$.

In our opinion, it seems natural to preserve the symplectic nature of the Tulczyjew triple after reduction. Since the morphism $\Xi\colon J^{-1}_{TT^*Q}(0)/G \to T(T^*Q/G)$ relates both Tulczyjew's triples, we have conveniently made use of it to relate the reduced dynamics, as described in [GUG06], with the reduced Lagrangian submanifolds. Again, we invite the interested reader to look at [GTAGMM14] for more details.

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Nederlandstalige samenvatting

De theorie omtrent reductietechnieken voor dynamische systemen met symmetrie, neemt een heel belangrijke plaats in binnen het domein dat algemeen omschreven wordt als "geometrische mechanica". Getuige daarvan zijn de vele publicaties die over dit onderwerp verschenen zijn. In algemene bewoordingen bestaat een reductietechniek erin om een bepaald dynamisch systeem met symmetrie te gaan opsplitsen in een stelsel van zogeheten "reconstructievergelijkingen", enerzijds, en een ontkoppeld systeem van "gereduceerde vergelijkingen", anderzijds. Het gereduceerde systeem is daarbij gedefinieerd op een ruimte met een kleiner aantal vrijheidsgraden en is in vele gevallen eenvoudiger te integreren dan het oorspronkelijk systeem. Bij de geometrische reductiemethodes wordt er daarbij naar gestreefd om zoveel mogelijk van de oorspronkelijke geometrische structuur van het gegeven systeem over te dragen naar het gereduceerde systeem. In het bijzonder werd de reductie van Hamiltoniaanse systemen met symmetrie op een symplectische variëteit of, meer algemeen, op een Poissonvariëteit, reeds druk bestudeerd in de literatuur. Over reductietechnieken voor Lagrangiaanse systemen met symmetrie is er daarentegen tot nu toe veel minder werk verricht. In deze verhandeling wordt het onderwerp reductie daarom hoofdzakelijk vanuit Lagrangiaans oogpunt bestudeerd.

De opbouw van deze scriptie is als volgt.

In het **eerste hoofdstuk** wordt de noodzakelijke wiskundige achtergrond geschetst waarop verder zal gesteund worden. Naast de herhaling van de basisdefinities omtrent symplectische en Poissonvariëteiten en de invoering van de terminologie en notaties waarvan gebruik zal gemaakt worden, bespreken we in dit hoofdstuk drie belangrijke onderwerpen:

1) *De actie van een Liegroep op een (differentieerbare) variëteit* (Section 1.2): We vermelden de voornaamste resultaten in verband met Liegroepacties op een variëteit en in het bijzonder herhalen we de standaardhypothesen die garanderen dat de quotiëntruimte van een variëteit onder een Liegroepactie terug een differentieerbare variëtiet is. We voeren de notie in van invariantie van een differentiaalvorm onder een actie van een Liegroep en in het bijzomder definiëren we wat een *canonische actie* van een Liegroep is op een symplectische variëteit.

2) *Bundels en connecties* (Section 1.3): Na de voornaamste definities herhaald te hebben van een connectie en de kromming van een connectie op een algemene vezelbundel, concentreren we ons vooral op het geval van een hoofdvezelbundel $P \to P/G$ (met P een differentieerbare variëteit en G een Liegroep die een actie definiëert op P die aan zekere voorwaarden moet voldoen). We voeren het begrip connectie 1-vorm in, geassocieerd met een hoofdconnectie op de bundel $P \to P/G$, alsook de corresponderende kromming, en leggen het verband tussen beide door middel van de *structuurvergelijking van Cartan*. We herhalen ook de definitie van de toegevoegde bundel welke een voorname rol speelt bij het opstellen van de zogenaamde Lagrange-Poincaré vergelijkingen.

3) *Reductie* (Section 1.4): De laatste sectie van het eerste hoofdstuk bestaat uit een kort overzicht van de theorie omtrent symplectische- en Poissonreductie. Een belangrijk deel van deze scriptie steunt voornamelijk op resultaten die hier besproken worden. Daarom zullen we ook een redelijk zelfbevattend bewijs geven van de twee voornaamste reductiestellingen: de Marsden-Weinstein reductiestelling en de Poisson reductiestelling.

Hoewel de basisresultaten omtrent de symplectische reductie van een co-raakbundel in de literatuur reeds verschillende tientallen jaren bekend zijn, heeft de Lagrangiaanse tegenhanger hiervan veel minder aandacht gekregen. Het is slechts vrij recent dat er belangstelling gegroeid is voor reductietheoriën in het kader van raakbundels. Dit weerspiegelt zich in een aantal artikels die op de één of andere wijze te maken hebben met zogenaamde Routh reductie van Lagrangiaanse systemen. Het voornaamste doel van het **tweede hoofdstuk** is dan ook om een ééngemaakte versie voor te stellen van Routh reductie in navolging van de symplectische reductiemethode. Sectie 2.1 geeft een overzicht van de klassieke methode van Routh, zoals we die aantreffen in de klassieke mechanica, en geeft er tevens de beperkingen van aan. Sectie 2.2 beoogt een behandeling van de moderne theorie over Routh reductie. Met het oog daarop beginnen we met het geven van de volgende basisresultaten uit de theorie omtrent co-raakbundel- en raakbundelreductie:

1) *Co-raakbundelreductie*: Vertekkend van een (vrije en eigenlijke) Liegroepactie ϕ van een Liegroep G op een variëteit Q, kan men deze actie liften naar een canonische actie van G op de symplectische variëteit (T^*Q, Ω_Q), waarbij T^*Q de co-raakbundel voorstelt van Q en Ω_Q de canonische symplectische vorm is op T^*Q. Deze actie heeft een equivariante momentumafbeelding $J : T^*Q \to \mathfrak{g}^*$ en een belangrijk resultaat is dat voor elke vaste $\mu \in \mathfrak{g}^*$ er een symplectisch diffeomorfisme

$$((T^*Q)_\mu, (\Omega_Q)_\mu) \cong \left(T^*(Q/G) \times_{Q/G} Q/G_\mu, \pi_1^*\Omega_{Q/G} + \pi_2^*\mathcal{B}_\mu\right) ,$$

bestaat, waarbij we de gebruikelijke notaties voor de gereduceerde symplectische ruimtes hebben gebruikt. In het rechterlid staat $\mathcal{B}_\mu$ voor de zogenaamde magnetische term (geïnduceerd door de μ-component van de uitwensige differentiaal van een gekozen connectie 1-vorm) en de afbeeldingen π_1 and π_2 stellen de projecties voor in het volgende diagram:

$$\begin{tikzcd} T^*(Q/G) \times_{Q/G} Q/G_\mu \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Q/G_\mu \\ T^*(Q/G) & \end{tikzcd}$$

2) *Raakbundelreductie*: Hierbij gaan we uit van de standaard symplectische beschrijving van een regulier Lagrangiaans systeem, met Lagrangiaan L op een raabundel TQ, in termen van de symplectische variëteit (TQ, Ω_Q^L), met $\Omega_Q^L = \mathbb{F}L^*\Omega_Q$. Hierin stelt $\mathbb{F}L : TQ \to T^*Q$ de Legendretransformatie (of de vezelafgeleide) voor corresponderend met L. Als er een G-actie ϕ bestaat op Q, zo dat de gelifte actie ϕ^{TQ} naar de raakbundel de Lagrangiaan L invariant laat, dan is de gelifte actie Hamiltoniaans en bestaat er een symplectisch diffeomorfisme

$$\big((TQ)_\mu, (\Omega_Q^L)_\mu\big) \cong ((T^*Q)_\mu, (\Omega_Q)_\mu)\ .$$

Men dient echter op te merken dat over het algemeen de variëteit $(TQ)_\mu$ niet kan gerealiseerd worden als een gevezeld product, precies omdat de momentumafbeelding van deze Hamiltoniaanse actie afhangt van de vezelafgeleide van de Lagrangiaan L. Om dit probleem te omzeilen, herhalen we eerst de definitie van de notie van *G-regulariteit* van L. Ruwweg gesproken wordt een Lagrangiaan L G-regulier genoemd als L regulier is met betrekking tot de groepsvariabelen. In dat geval bekomen we met behulp van de impliciete functiestelling, de beoogde identificatie:

$$(TQ)_\mu \cong T(Q/G) \times_{Q/G} (Q/G_\mu)\,. \tag{$\star$}$$

Vervolgens blijkt er dan dat er een manier bestaat om de gereduceerde (Hamiltoniaanse) dynamica te interpreteren als zijnde Lagrangiaans. Dit vergt evenwel een veralgemening van de standaarddefinitie van een Lagrangiaans systeem (op een raakbundel) naar systemen die gedefinieerd zijn op gevezelde productruimtes, zoals de ruimte $(TQ)_\mu$ in $(\star)$, en daarbij termen kunnen bevatten van het gyroscopisch type. De rol van de gereduceerde Lagrangiaanse functie wordt gespeeld door de zogenaamde *Routhiaan*, waarvan we de definitie geven op het einde van Sectie 2.2.

De nieuwe bijdragen in deze verhandeling beginnen in Sectie 2.3, waar we de procedure van Routh reductie beschrijven in het geval dat de configuratieruimte te schrijven is als een direct product $Q = S \times G$, met S een willekeurige differentieerbare variëtiet en waarbij de Liegroep G op Q inwerkt door 'translatie' op de G-factor. We bekomen expliciete uitdrukkingen voor de gereduceerde dynamica. Vervolgens passen we deze uitdrukkingen in Sectie 2.4 toe op het geval van een star lichaam met rotors en tonen aan dat deze overeenstemmen met de gekende uitdrukkingen in de literatuur.

Het vertrekpunt van het **derde hoofdstuk** is het concept van *magnetisch Lagrangiaans systeem.* Een magnetisch Lagrangiaans systeem $(\epsilon : E \to Q, L, \mathcal{B})$ bestaat uit volgende data: (1) een vezelbundel $\epsilon : E \to Q$; (2) een differentieerbare functie L (de Lagrangiaan) op het gevezeld product $TQ \times_Q E$; (3) een gesloten 2-vorm $\mathcal{B}$ op E die we voor de eenvoud de *magnetische vorm* van het systeem zullen noemen. Binnen het kader van magnetische Lagrangiaanse systemen geven we dan een veralgemening van de gebruikelijke definities van vezelafgeleide en *energie* geassocieerd met de Lagrangiaan L.

Duiden we lokale coördinaten op Q aan met (q^i) en noteren we de corresponderende bundelcoördinaten op E als (q^i, r^a). De dynamica van een magnetisch Lagrangiaans systeem wordt bepaald door volgend stel *Euler-Lagrangevergelijkingen*:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} &= \mathcal{B}_{ij}\dot{q}^j + \mathcal{B}_{ia}\dot{r}^a\,, \\ -\frac{\partial L}{\partial r^b} &= -\mathcal{B}_{ib}\dot{q}^i + \mathcal{B}_{ab}\dot{r}^a\,, \end{aligned} \tag{$\star\star$}$$

waarbij $\mathcal{B}_{ij}$, $\mathcal{B}_{ia}$ en $\mathcal{B}_{ab}$ de componenten voorstellen van de magnetische 2-vorm $\mathcal{B}$. Onder bepaalde regulariteitsvoorwaarden op de Lagrangiaan L en de magnetische vorm $\mathcal{B}$, die besproken worden in Sectie 3.1, wordt aangetoond dat de Euler-Lagrangevergelijkingen ($\star\star$) symplectisch zijn ten opzichte van een welbepaalde symplectische vorm $\Omega^{L,\mathcal{B}}$ op $T_EQ := TQ \times_Q E$, met als Hamiltoniaan de energiefunctie E_L bepaald door de Lagrangiaan L. Een ruime klasse van dynamische systemen past in de categorie van magnetische Lagrangiaanse systemen. In het bijzonder systemen die afkomstig zijn van een Routh reductie van een Lagrangiaans systeem met symmetrie, blijken van het magnetisch Lagrangiaans type te zijn. Er bestaat ook een natuurlijk concept van *magnetisch Hamiltoniaans systeem* dat besproken wordt op het einde van Sectie 3.1.

Sectie 3.2 is gewijd aan de studie van een klasse van afbeeldingen tussen gevezelde producten die we *compatibele transformaties* zullen noemen en die gebruikt worden om bepaalde "morfismen" ψ te construeren tussen twee gegeven magnetische Lagrangiaanse systemen. Om een compatibele transformatie $\psi : T_{E_1}Q_1 \to T_{E_2}Q_2$ te construeren, vertrekken we van een koppel van submersies (F, f) die het volgende diagram commutatief maken

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\epsilon^{(1)}"'] & E_2 \arrow[d, "\epsilon^{(2)}"] \\ Q_1 & Q_2 \arrow[l, "f"] \end{tikzcd}$$

en die zodanig zijn dat al de gevezelde structuren, vezelbundels zijn. We zeggen dan dat een transformatie $\psi : T_{E_1}Q_1 \to T_{E_2}Q_2$ compatibel is (met betrekking tot de submersies F en f) als we het volgend commutatief diagram bekomen:

$$\begin{tikzcd}
T_{E_1}Q_1 \arrow[r, "F\circ\tau_2^{(1)}"] \arrow[d, "\tau_1^{(1)}"'] \arrow[dr, "\psi"] & E_2 \\
TQ_1 & T_{E_2}Q_2 \arrow[l, "Tf\circ\tau_1^{(2)}"] \arrow[u, "\tau_2^{(2)}"']
\end{tikzcd}$$

Alhoewel dit concept van compatibele transformatie vrij restricief mag lijken, zal achteraf duidelijk worden dat het toch voldoende ruim is om toepasbaar te zijn in tal van situaties die zich voordoen bij de reductie van een Lagrangiaans systeem met symmetrie.

Na de invoering van compatibele transformaties kijken we naar het geval waarbij de gevezelde producten corresponderen met de configuratieruimtes van magnetische Lagrangiaanse systemen en bestuderen we meer bepaald transformaties

$$(\epsilon^{(1)} : E_1 \to Q_1, L_1, \mathcal{B}_1) \xrightarrow{\psi} (\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$$

die de dynamica van twee dergelijke systemen met elkaar verbinden. Omwille van de toepassingen in de geometrische mechanica is het zeker nuttig om de eigenschappen van dergelijke transformaties nader te onderzoeken. Een eerste voorbeeld in de context van Routh reductie, doet zich voor bij de studie van de reductie van een Lagrangiaans systeem dat invariant is onder twee symmetriegroepen G en H. Deze situatie wordt samengevat in volgend schema

$$\begin{tikzcd}
& TQ \arrow[dl, bend right, "H-\text{red}"'] \arrow[dr, bend left, "G-\text{red}"] & \\
T(Q/H)\times_{Q/H} Q/H_\mu \arrow[rr, dashed, "\psi"'] & & T(Q/G)\times_{Q/G} Q/G_\mu
\end{tikzcd}$$

en een natuurlijke vraag is of men de dynamica van beide gereduceerde systemen kan verbinden door middel van een compatibele transformatie ψ. In Sectie 3.3 zullen we een dergelijk schema tegenkomen in verband met semi-productreductie.

We voeren dan een bepaalde klasse van compatibele transformaties $\psi_{L_2,\beta}$ in, afhankelijk van een gegeven Lagrangiaan L_2 en een gekozen afbeelding $\beta : E_1 \to V^*f$ (waarbij $Vf \subset TQ_2$ de ruimte van verticale raakvectoren voorstelt met betrekking tot de submersie f). Indien de Lagrangiaan L_1 en de magnetische vorm $\mathcal{B}_1$ voldoen aan een zekere relatie (afhankelijk van $\psi_{L_2,\beta}$, L_2 en $\mathcal{B}_2$), behoudt de geconstrueerde transformatie $\psi_{L_2,\beta}$ de symplectische structuren en de Hamiltoniaanse functies, nl.

$$(\psi_{L_2,\beta})^*\Omega^{L_2,\mathcal{B}_2} = \Omega^{L_1,\mathcal{B}_1}\,, \qquad (\psi_{L_2,\beta})^*E_{L_2} = E_{L_1}\,.$$

We combineren de voorgaande resultaten met het welbekende presymplectisch bindingsalgoritme om voorwaarden af te leiden waaronder de transformatie $\psi_{L_2,\beta}$ oplossingen van de Euler-Lagrangevergelijkingen voor $(\epsilon^{(1)} : E_1 \to Q_1, L_1, \mathcal{B}_1)$ en $(\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$ in elkaar omzet. Op het einde van Sectie 3.2 geven we een overzicht van de corresponderende transformaties in het kader van magnetische Hamiltoniaanse systemen.

Eén van de interessantse gevallen is dat waarbij $\psi_{L_2,\beta}$ een diffeomorfisme is. We kunnen dan een magnetisch Lagrangiaans systeem op $\epsilon^{(1)} : E_1 \to Q_1$ induceren dat symplectomorf is met het gegeven systeem $(\epsilon^{(2)} : E_2 \to Q_2, L_2, \mathcal{B}_2)$. Sectie 3.3 beschrijft een eerste situatie waarin dit zich voordoet, namelijk in de context van stapsgewijze semi-direct productreductie. We bestuderen het geval van een Lagrangiaans systeem dat gedefinieerd is op een product $S \times GV$, waarbij GV het semi-direct product voorstelt van de Liegroep G en een vectorruimte V.

In Sectie 3.4 gebruiken we de klasse van transformaties $\psi_{L_2,\beta}$ om Routh reductie te beschrijven van een magnetisch Lagrangiaans systeem dat gedefinieerd is op een algemeen vezelproduct. In het bijzonder wordt bewezen dat mits een geschikte keuze van de afbeelding β, de transformatie $\psi_{L_2,\beta}$ kan beschouwd worden als een restrictie tot een niveauoppervlak van de momentumafbeelding. Eens het systeem beperkt wordt tot een niveauoppervlak, verloopt de verdere reductie met betrekking tot de corresponderende isotropiedeelgroep van de symmetriegroep onmiddellijk. De resulterende *magnetishce Routh reductie* is in overeenstemming met hetgeen gevonden wordt in de literatuur in de context van stapsgewijze Routh reductie.

Het **vierde hoofdstuk** van deze scriptie houdt zich veeleer bezig met de reductietheorieën van Hamilton-Poincaré en Lagrange-Poincaré. Hierin zullen we aantonen dat de symplectische formulering van de dynamica die teruggaat op het werk van W. Tulczyjew, enerzijds, en de symplectische reductiemethodes, anderzijds, kunnen gecombineerd worden in een model voor Lagrange-Poincaré reductie en Hamilton-Poincaré reductie in het kader van de zogenaamde *gereduceerde Tulczyjew triplets*.

In Sectie 4.1 overlopen we de voornaamste elementen van de Tulczyjew triplets, waarvan we de notaties in volgend schema gebruiken:

$$\begin{tikzcd}
T^*TQ \arrow[dr, "\pi_{TQ}"] & & TT^*Q \arrow[ll, "\alpha_Q"'] \arrow[rr, "\beta_Q"] \arrow[dl, "T\pi_Q"'] \arrow[dr, "\tau_{T^*Q}"] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"'] \\
& TQ \arrow[ul, bend left, "dL"] \arrow[rr, "\mathbb{F}L"'] & & T^*Q \arrow[ur, bend right, "dH"'] &
\end{tikzcd}$$

We bespreken hoe Lagrangiaanse deelvariëteiten zich gedragen onder reductie door symmetrie. Ruwweg gesproken is het precies door de reductie van de Lagrangiaanse deelvariëteiten S_H and S_L, gegeven door

$$S_H = \beta_Q^{-1}\left(dH(T^*Q)\right), \qquad S_L = \alpha_Q^{-1}\left(dL(TQ)\right),$$

dat we, respectievelijk, de Hamilton-Poincaré en de Lagrange-Poincaré vergelijkingen zullen terugvinden. Dit wordt behandeld in Sectie 4.2, waar we eveneens de equivalentie van beide beschrijvingen zullen verklaren in het regulier geval. De algemene theorie wordt geïllustreerd aan de hand van een concreet voorbeeld: de Lie-Poisson dynamica.

Referenties. Een groot deel van het werk dat voorgesteld wordt in deze verhandeling werd ondertussen reeds gepubliceerd of is ingediend ter publicatie. Vele van de nieuwe resultaten uit hoofdstukken 2 en 3 in verband met compatibele transformaties en hun toepassingen, kunnen gevonden worden [LGTAC12] en [GTALC14]. De intrinsieke afleiding van de Hamilton-Poincaré en Lagrange-Poincaré vergelijkingen binnen de context van de Tulczyjew triplets, werd ter publicatie aanvaard in [GTAGMM14].