On the lattice of physical concepts

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Abstract

Unlike the organization of the chemical elements in a periodic table, no conclusive results are published about the organization of the physical quantities. A physical quantity is a quantity that is used in the description of physical processes. The physical processes are modeled through mathematical expressions that use physical quantities expressed in different types: scalars, vectors, multi-vectors, matrices and tensors. The scientific community adopted by convention the SI units and listed the physical quantities. The mathematical structure of the SI physical quantities is unknown. Here we show that classes of physical quantities, that are expressed according to the SI convention, are mathematically classified by leader classes of the seven dimensional integer lattice.

1. Introduction

The choice of a system of units and the number of dimensions are open issues amongst physicists and reviews are found here [1, 2, 3, 4, 5, 6, 7]. Unlike the organization of the chemical elements in a periodic table, no conclusive results are published about the organization of the physical quantities. The mathematical structure of the physical quantities is unknown. Physical quantities occur in the scientific literature in the form of scalars, vectors, multi-vectors, matrices and tensors. Each physical quantity is represented by a symbol or label. All the physical quantities are eventually measured through their respective components and thus we restrict our study to the components of physical quantities. In this research we adopt the *convention* of the International System of Units (SI) for the units and dimensions of the physical quantities [8]. The SI base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity [8]. A component of a physical quantity is a quantity that is used in the description of physical processes. This paper is organized as follows. In section 2 we present the mapping of a class of physical quantities to the seven dimensional integer lattice. Section 3 contains a brief review of the mathematical objects needed to classify the physical quantities. In section 4 we present the method resulting in the classification of the SI physical quantities. Finally, we summarize the results and conclude in section 5.

2. Fundamental axiom of physics

Physical concepts are the building blocks of any mathematical description of physical phenomena.

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3. Abstract connectives

Decomposition theorem in the theory of abstract connectives (see Klein 1932). A "lattice" is a domain of individuals in which two commutative and associative operations $\cup$ and $\cap$ are defined. If $a \cup b = b$ than a is called a part of b. Under the assumptions that the operation $\cap$ is distributive with respect to $\cup$ that further, a unit element e exists for $\cup$ and that every element has only finitely many parts, an element a of the lattice can be represented in *at most one* way in the form $a = p_1 \cup p_2 \cup \ldots \cup p_n$ where the p_i are primary elements over different prime elements. Here p is called a prime element if it has no proper part other than e, and q is called a primary element over p if p is the only prime element included in q. If in addition it is assumed that every non-primary element $a \neq e$ can be decomposed into two elements distinct from a, the *existence* of a representation in the form $a = p_1 \cup p_2 \cup \ldots \cup p_n$ also follows for every element of the lattice.

4. A consistent physics system

In every system S of the kind just mentioned the proposition that S is consistent (more precisely, the equivalent arithmetic propositions that we obtain by mapping the formulas one-to-one on natural numbers) in unprovable.

5. An alphabet for physical concepts

Let $A = \{0, 1, \ldots, i-1, i, i+1, \ldots, s\}$ be a totally ordered alphabet where $i - 1 < i$ and $i < i + 1$ and $i \in \mathbb{Z}_+$. To each physical concept denoted $C(q)$ we assign a word w constructed from the alphabet A. The words w have a length n and we write the characters in graded reverse lex order [15]. We form a n-tuple $\boldsymbol{f}$ with the characters of the word w without changing the order of the characters. This n-tuple consists of non-negative integers and is an element of the n-dimensional integer lattice $\mathbb{Z}^n$, more specifically a lattice point of the positive orthant $\mathbb{Z}^n_+$. Let n_i be the number of characters of type i of the alphabet A. Suppose that the characters occurring in the n-tuple $\boldsymbol{f}$ are subjected to signed permutations and that we denote the set containing the generated lattice points as $[\boldsymbol{f}]$. The cardinality of the set $[\boldsymbol{f}]$ is given by the equation:

$$\#([\boldsymbol{f}]) = 2^{n-n_0} \frac{n!}{n_0! n_1! n_2! \ldots n_s!} \, .$$

The number of signed permutations in the n-dimensional integer lattice is given by the order of the automorphisms $Aut(\mathbb{Z}^n)$ and is equal to $2^n n!$. Assume that the lattice point $\boldsymbol{f}$ has a norm $N(\boldsymbol{f}) = m$ where $m \in \mathbb{Z}_+$. The lattice points of the set $[\boldsymbol{f}]$ have all the same norm m. We iknow that the union of the sets with norm $N(\boldsymbol{f})$ is forming a lattice shell [16, 17, 14].

6. n-ary operations between physical concepts

6.1. Binary operations

We create a mathematical model M to describe physical phenomena and processes. We choose n base concepts in this model. The mathematical model M is an alphabet of n symbols These base concepts We our limited knowledge we cannot define a set of base concept in a pure axiomatic wayLet us consider a alphabet A containing as elements Let A,B,C be physical concepts $C = A \cdot B$

7. Mapping classes of physical quantities to a seven dimensional integer lattice

Let the set of physical quantities be $\mathcal{Q}$. Consider the physical quantities $a, b \in \mathcal{Q}$ and assume the following equivalence relation *a is dimensionally equivalent with* b with notation $a \sim b$. As physical quantities occur in the form of scalars, vectors, multi-vectors, matrices and tensors, we can without loss of generality consider a component of a physical quantity and denote it as q . The set of all equivalence classes in $\mathcal{Q}$ given the equivalence relation $\sim$ is the quotient set $\mathcal{Q}/\!\sim$. We define the surjective function C from $\mathcal{Q}$ to $\mathcal{Q}/\!\sim$ given by $C(q) = [q]_\sim$. We represent the equivalence classes as $[q]_\sim = \{p \in \mathcal{Q} : p \sim q\}$. The set of base physical quantities is called $\mathcal{B} \doteq \{L, M, T, I, \Theta, N, J\}$. The set of base units is defined as $\mathbb{U} \doteq \{u_i \mid u_1 \doteq [L] = \mathrm{m},\ u_2 \doteq [M] = \mathrm{kg},\ u_3 \doteq [T] = \mathrm{s},\ u_4 \doteq [I] = \mathrm{A},\ u_5 \doteq [\Theta] = \mathrm{K},\ u_6 \doteq [N] = \mathrm{mol},\ u_7 \doteq [J] = \mathrm{cd}\}$. Each physical quantity has parameters f_i, called dimensional exponents that are integers. We write a physical quantity Q according to the SI as a dimensional product:

$$Q = \{r\} \cdot \prod_{i=1}^{7} u_i^{f_i} \ ,$$

in $u_1, u_2, \ldots, u_7$ where the u_i are the SI base units with $u_i \in \mathbb{U}$, the f_is are dimensional exponents with $f_i \in \mathbb{Z}$ and where the physical quantity Q of the idealized physical system assumes a numerical real value $\{r\} \in \mathbb{R}$. The quotient set of $\mathcal{Q}$ by $\sim$ is denoted $\mathcal{Q}/\sim$. Consider the set of integer septuples $\mathbb{Z}^7 \doteq \{(f_1, \ldots, f_7) \mid f_i \in \mathbb{Z}\}$, that is a special case of the n-dimensional integer lattice [9]. We know that $\mathbb{Z}^7, +$ is an additive group and that from the algebra of the *quantity calculus* we have that $\mathcal{Q}/\sim, \cdot$ is a multiplicative group. We define the group isomorphism h formally as $\mathrm{h} : \mathcal{Q}/\sim \to \mathbb{Z}^7 \mid \mathrm{h}([Q]_\sim) \doteq \boldsymbol{f} = (f_1, \ldots, f_7)$ where $f_i \in \mathbb{Z}$. The dimensional exponents of the units of a class of physical quantities, *taken in the correct order*, form the coordinates of a point in $\mathbb{Z}^7$. Each lattice point is the image of one and only one class of physical quantities and so the mapping h is bijective from $\mathcal{Q}/\!\sim$ on $\mathbb{Z}^7$. We are free to select seven basis lattice points of $\mathbb{Z}^7$ and define $\boldsymbol{e}_1 = \mathrm{h}([L]_\sim) = (1,0,0,0,0,0,0)$, $\boldsymbol{e}_2 = \mathrm{h}([M]_\sim) = (0,1,0,0,0,0,0)$, $\boldsymbol{e}_3 = \mathrm{h}([T]_\sim) = (0,0,1,0,0,0,0)$, $\boldsymbol{e}_4 = \mathrm{h}([I]_\sim) = (0,0,0,1,0,0,0)$, $\boldsymbol{e}_5 = \mathrm{h}([\Theta]_\sim) = (0,0,0,0,1,0,0)$, $\boldsymbol{e}_6 = \mathrm{h}([N]_\sim) = (0,0,0,0,0,1,0)$, $\boldsymbol{e}_7 = \mathrm{h}([J]_\sim) = (0,0,0,0,0,0,1)$, with $\boldsymbol{e}_i \in \mathbb{Z}^7$. These points have the agreed BIPM symbol for the dimension [10]. We will adopt the Conway abbreviation [9] for the components of lattice points and write for example $\boldsymbol{e}_5 \doteq \mathrm{h}([\Theta]_\sim) = (0,0,0,0,1,0,0)$ as $(0^4 10^2)$. One could object that some derived physical quantities (rms of a quantity, noise spectral density, specific detectivity, thermal inertia, thermal effusivity, ...) are defined as the square root of some product or quotient of other physical quantities. We define these derived physical quantities as fractional physical quantities where the coordinates $f_i \in \mathbb{Q}$ do not comply with the above definition for a physical quantity. Each of these fractional physical quantities are, by a proper exponentiation, transformed to a physical quantity having integer exponents for the base units u_i . In this paper we tacitly assume that when we refer to a physical quantity we mean a component of the physical quantity. We know that the concept *energy* occurs in different forms in physics and that we use dedicated words as: energy, potential energy, kinetic energy, work, Lagrange function, Hamilton function, Hartree energy, ionization energy, electron affinity, electronegativity, dissociation energy... in our formulations of physical relations. The equivalence class that we denote $[E]_\sim$ represents all these quantities.

8. Mathematical preliminaries

These preliminaries are gathering objects from disparate mathematical fields as discrete mathematics, group theory, lattice theory, combinatorics and algebraic geometry to enable the description of the mathematical structure organizing the physical quantities. The mathematical objects are defined in n-dimensional space and are then applied to the specific case of $n = 7$.

8.1. The properties of the n-dimensional integer lattice

The properties of the n-dimensional integer lattice are described elsewhere in the literature [9]. We will recall briefly the properties that are useful for the purpose of structuring the physical quantities. Every lattice point is expressed in a unique way as the linear combination $\boldsymbol{f} = f_1\boldsymbol{e}_1 + \ldots + f_n\boldsymbol{e}_n$ where the coefficients f_i are called the coordinates of $\boldsymbol{f}$. The basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, that generates the integer lattice $\mathbb{Z}^n$ is orthonormal. The automorphism group of $\mathbb{Z}^n$ consists of all signed permutation matrices acting on the integer lattice points, and has order $2^n n!$ and is the Weyl group of root system B_n [9, 11]. We call the expressions $N(\boldsymbol{f}) \doteq \|\boldsymbol{f}\|_1 = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} f_i f_k$, $\|\boldsymbol{f}\|_2 \doteq \sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} f_i f_k}$ and $\|\boldsymbol{f}\|_\infty \doteq \max\{|f_1|, \ldots, |f_n|\}$ respectively the ℓ_1-norm, the ℓ_2-norm and the infinity norm of $\boldsymbol{f}$ in $\mathbb{Z}^n$.

8.2. Infinity normed n-polytope of norm s

Consider a lattice point $\boldsymbol{f}_0$ and points $\boldsymbol{f}$, which have the property $\boldsymbol{f}_0 + \boldsymbol{f} \in A \Leftrightarrow \boldsymbol{f}_0 - \boldsymbol{f} \in A$ then we call A a centrally symmetric set. In the remainder of the paper we will assume that $\boldsymbol{f}_0 = \boldsymbol{o}$ is the origin of $\mathbb{Z}^n$. A n-dimensional integer lattice polytope P_n is the convex hull of a set of finitely many points in $\mathbb{Z}^n$ [12]. An infinity normed n-polytope P_n^s of norm s is a subset of $\mathbb{Z}^n$ with the following property $P_n^s = \{\boldsymbol{f} \in \mathbb{Z}^n \mid \|\boldsymbol{f}\|_\infty = s\}$. We characterize the infinity normed n-polytope P_n^s by the parameters n and s, where n represents the dimension of the integer lattice and s represents the value of the infinity norm.

8.3. Absolute leader classes of a lattice

The concept of leader class of a lattice is used in signal processing [13, 14]. A leader class is the set of lattice points of $\mathbb{Z}^n$ that are connected through a signed permutation. We note a leader class of $\mathbb{Z}^n$ as $[(f_1, \ldots, f_n)]$, where $(f_1, \ldots, f_n)$ are the coordinates of the representative lattice point. Each leader class forms a set of lattice points that are symmetric about the origin $\boldsymbol{o}$. The cardinality of a leader class is calculated using elementary combinatorics. Let $A = \{0, 1, 2, \ldots, s\}$ be a totally ordered alphabet. The representative of a leader class is a word w constructed from the alphabet A. The words w have a length n that corresponds to the dimension of $\mathbb{Z}^n$. Let n_i be the number of characters of type i of the alphabet A. Suppose that the characters are subjected to a signed permutation, then the cardinality is given by the equation:

$$\#\,(w) = 2^{n-n_0}\frac{n!}{n_0!n_1!n_2!\ldots n_s!}\,.$$

The number of integer lattice points in each leader class is equal to the cardinality of w. The representative lattice point, called in signal processing an *absolute leader*, has only coordinates that are non-negative integers. The coordinates are arranged in graded reverse lex order [15]. The union of leader classes with norm $N(\boldsymbol{f})$ is forming a lattice shell [16, 17, 14].

8.4. Monomials

A monomial [15] m in $u_1, u_2, \ldots, u_n$ is a product of the form:

$$m = \prod_{i=1}^{n} u_i^{f_i} \ ,$$

where all the exponents $f_i \in \mathbb{Z}_+$ and $u_i \in \mathbb{U}$. The total degree of this monomial is the sum $f_1 + \ldots + f_n$. The number of classes of monomials with infinity norm $\|\boldsymbol{f}\|_\infty \leq s$ in $\mathbb{Z}^n$ is the result from the application of "Lemma 4" [18]. The n-dimensional integer lattice $\mathbb{Z}^n$ is partitioned in $\binom{n+s-1}{s}$ equivalence classes. Let the dimension $n = 7$ then we generate the (Table 1) where the first column lists the infinity norm s, the second column shows the sum of the lattice points in the respective leader classes, while the third column gives the cumulated number of lattice points. The fourth and fifth columns have a similar meaning but are expressing the number of leader classes and their cumulated number for each 7-polytope P_7^s.

Table 1: Properties of the 7-polytopes P_7^s in $\mathbb{Z}^7$ for $s \leq 10$.

$\|\boldsymbol{f}\|_\infty = s$	$\sum[\#([a])]$	$cumul\{\sum[\#([a])]\}$	$\#\left(P_7^s \setminus P_7^{s-1}\right)$	$\#\left(P_7^s\right)$
0	1	1	1	1
1	2186	2187	7	8
2	75938	78125	28	36
3	745418	823543	84	120
4	3959426	4782969	210	330
5	14704202	19487171	462	792
6	43261346	62748517	924	1716
7	108110858	170859375	1716	3432
8	239479298	410338673	3003	6435
9	483533066	893871739	5005	11440
10	907216802	1801088541	8008	19448

9. Method for the classification of the physical quantities

The absolute leader classes that are based on the signed permutations of the coordinates of the lattice point of the seven dimensional integer lattice $\mathbb{Z}^7$ are partitioning $\mathbb{Z}^7$ in equivalence classes. The inverse map of h realizes then the organization of the classes of physical quantities in $\mathbb{U}$. The signed permutation matrices, transforming the representative lattice point of the leader class in lattice points belonging to the absolute leader class, are elements of the automorphism group $Aut(\mathbb{Z}^7)$ and the order of this group is 645120. We find that the infinity norm ℓ_∞-norm organizes the absolute leader classes in *nested* centrally symmetric 7-polytopes P_7^s where $\ell_\infty = s$.

9.1. Enumeration of the absolute leader classes of physical quantities for $s \leq 3$

The enumeration table (Table A.2) of the 7-polytope P_7^3 is exhaustive and consists of six columns. The first column contains the values of the infinity norm $\|\boldsymbol{f}\|_\infty$. The second column lists the absolute leader classes. The third column contains the total degree of the monomial associated with the absolute leader class. The fourth column gives the ℓ_1-norm of the absolute leader class.

The fifth column gives the cardinality of the absolute leader class. The sixth column gives an example of a typical class of physical quantities that is mapped in the absolute leader class. The label *unknown* is indicating that the author has no information about a class of a physical quantity that is mapped in the respective absolute leader class. The ordering of the classes in (Table A.2) is based on increasing norm $N(\boldsymbol{f})$ that is the primary characteristic of the n-sphere shells.

9.2. Enumeration of the known absolute leader classes of physical quantities for $s \geq 4$

The common physical quantities (Table A.3) that have $\|\boldsymbol{f}\|_\infty \geq 4$ are enumerated but is not exhaustive. The table A.3 is based on the list of physical quantities known to the author. Table A.3 contains 4 columns. The first column contains the values $\|\boldsymbol{f}\|_\infty$ of the infinity norm defining the 7-polytope P_7^s. The second column lists the absolute leader class. The third column identifies the class of physical quantity by its integer lattice point in $\mathbb{Z}^7$. The fourth column represents the name of a common physical quantity.

10. Summary and conclusion

Mathematical expressions in physics use physical quantities expressed in different types: scalars, vectors, multi-vectors, matrices and tensors. The scientific community adopted by convention the SI units and listed the physical quantities. The version 8 of the SI has known issues and the choice of a system of units and the number of dimensions remain open issues amongst physicists. Unlike the organization of the chemical elements in a periodic table, no conclusive results are published about the organization of the physical quantities. The mathematical structure of the physical quantities is unknown. Here we show that classes of physical quantities, that are expressed according to the SI convention, are mathematically structured in geometrical entities, known as absolute leader classes. We mapped the classes of physical quantities on a seven dimensional integer lattice. We find that signed permutation matrices are connecting the integer lattice points and thus partition the integer lattice in absolute leader classes that are known from information theory. The leader classes are themselves grouped, using the infinity norm, in nested centrally symmetric 7-polytopes. The results show that the fundamental structure organizing the physical quantities is based on absolute leader classes and that the enumeration of the presently common physical quantities indicates that only a small number of absolute leader classes have been explored. The described method based on leader classes is easily modified if the number of dimensions of physical quantities is increased or decreased. We expect to find new physical quantities and relationships between the classes of physical quantities. These predicted relationships should gave all physicists a framework in their search for new laws and relations between physical quantities.

Acknowledgments

I thank from the University of Ghent Prof. em. F. Brackx, Prof. H. De Schepper, Assistant Prof. H. De Bie and from the University of Brussels Prof. em. I. Veretennicoff, Prof. Ph. Cara and Prof. J.P. Van Bendegem for valuable discussions. Special thanks to my wife, children and friends for supporting me in this research.

AppendixA. Organization of the classes of physical quantities

Table A.2: Absolute leader classes $[\boldsymbol{f}]$ of polytope P_7^3.

$\Vert\boldsymbol{f}\Vert_\infty$	$[\boldsymbol{f}]$	deg	$N(\boldsymbol{f})$	$\#([\boldsymbol{f}])$	common physical quantities
0	$[0^7]$	0	0	1	plane angle, redshift ...
1	$[10^6]$	1	1	14	length, mass, time, current ...
1	$[1^2 0^5]$	2	2	84	speed, electric charge ...
1	$[1^3 0^4]$	3	3	280	linear momentum, fluidity ...
1	$[1^4 0^3]$	4	4	560	unknown
2	$[20^6]$	2	4	14	area, space-time curvature ...
1	$[1^5 0^2]$	5	5	672	unknown
2	$[210^5]$	3	5	168	acceleration, surface density ...
1	$[1^6 0]$	6	6	448	unknown
2	$[21^2 0^4]$	4	6	840	force, energy density ...
1	$[1^7]$	7	7	128	unknown
2	$[21^3 0^3]$	5	7	2240	magnetic vector potential
2	$[2^2 0^5]$	4	8	84	gravitational potential ...
2	$[21^4 0^2]$	6	8	3360	unknown
2	$[2^2 10^4]$	5	9	840	energy, torque, work ...
2	$[21^5 0]$	7	9	2688	unknown
3	$[30^6]$	3	9	14	volume ...
2	$[2^2 1^2 0^3]$	6	10	3360	magnetic constant, entropy ...
2	$[21^6]$	8	10	896	unknown
3	$[310^5]$	4	10	168	radiance, mass density, jerk ...
2	$[2^2 1^3 0^2]$	7	11	6720	unknown
3	$[31^2 0^4]$	5	11	840	electric charge density ...
2	$[2^3 0^4]$	6	12	280	unknown
2	$[2^2 1^4 0]$	8	12	6720	unknown
3	$[31^3 0^3]$	6	12	2240	electric field, thermal conductivity ...
2	$[2^3 10^3]$	7	13	2240	inductance ...
2	$[2^2 1^5]$	9	13	2688	unknown
3	$[320^5]$	5	13	168	absorbed dose rate ...
3	$[31^4 0^2]$	7	13	3360	unknown
2	$[2^3 1^2 0^2]$	8	14	6720	unknown
3	$[3210^4]$	6	14	1680	power, radiant flux ...
3	$[31^5 0]$	8	14	2688	unknown
2	$[2^3 1^3 0]$	9	15	8960	unknown
3	$[321^2 0^3]$	7	15	6720	electric potential, luminous efficacy ...
3	$[31^6]$	9	15	896	unknown
2	$[2^4 0^3]$	8	16	560	unknown
2	$[2^3 1^4]$	10	16	4480	unknown
3	$[321^3 0^2]$	8	16	13440	unknown
2	$[2^4 10^2]$	9	17	3360	unknown
3	$[32^2 0^4]$	7	17	840	unknown
3	$[321^4 0]$	9	17	13440	unknown
...	...	...	...	...	...

$\|\boldsymbol{f}\|_\infty$	$[\boldsymbol{f}]$	deg	$N(\boldsymbol{f})$	$\#([\boldsymbol{f}])$	common physical quantities
2	$[2^4 1^2 0]$	10	18	6720	unknown
3	$[3^2 0^5]$	6	18	84	unknown
3	$[32^2 10^3]$	8	18	6720	electrical resistance, impedance …
3	$[321^5]$	10	18	5376	unknown
2	$[2^4 1^3]$	11	19	4480	unknown
3	$[3^2 10^4]$	7	19	840	unknown
3	$[32^2 1^2 0^2]$	9	19	20160	unknown
2	$[2^5 0^2]$	10	20	672	unknown
3	$[3^2 1^2 0^3]$	8	20	3360	specific resistance …
3	$[32^2 1^3 0]$	10	20	26880	unknown
2	$[2^5 10]$	11	21	2688	unknown
3	$[32^3 0^3]$	9	21	2240	unknown
3	$[3^2 1^3 0^2]$	9	21	6720	unknown
3	$[32^2 1^4]$	11	21	13440	unknown
2	$[2^5 1^2]$	12	22	2688	unknown
3	$[3^2 20^4]$	8	22	840	unknown
3	$[32^3 10^2]$	10	22	13440	unknown
3	$[3^2 1^4 0]$	10	22	6720	unknown
3	$[3^2 210^3]$	9	23	6720	electrical resistivity …
3	$[32^3 1^2 0]$	11	23	26880	unknown
3	$[3^2 1^5]$	11	23	2688	unknown
2	$[2^6 0]$	12	24	448	unknown
3	$[3^2 21^2 0^2]$	10	24	20160	unknown
3	$[32^3 1^3]$	12	24	17920	unknown
2	$[2^6 1]$	13	25	896	unknown
3	$[32^4 0^2]$	11	25	3360	unknown
3	$[3^2 21^3 0]$	11	25	26880	unknown
3	$[3^2 2^2 0^3]$	10	26	3360	unknown
3	$[32^4 10]$	12	26	13440	unknown
3	$[3^2 21^4]$	12	26	13440	unknown
3	$[3^3 0^4]$	9	27	280	unknown
3	$[3^2 2^2 10^2]$	11	27	20160	unknown
3	$[32^4 1^2]$	13	27	13440	unknown
2	$[2^7]$	14	28	128	unknown
3	$[3^3 10^3]$	10	28	2240	unknown
3	$[3^2 2^2 1^2 0]$	12	28	40320	unknown
3	$[3^3 1^2 0^2]$	11	29	6720	unknown
3	$[32^5 0]$	13	29	2688	unknown
3	$[3^2 2^2 1^3]$	13	29	26880	unknown
3	$[3^2 2^3 0^2]$	12	30	6720	unknown
3	$[3^3 1^3 0]$	12	30	8960	unknown
3	$[32^5 1]$	14	30	5376	unknown
3	$[3^3 20^3]$	11	31	2240	unknown
…	…	…	…	…	…

$\|\boldsymbol{f}\|_\infty$	$[\boldsymbol{f}]$	deg	$N(\boldsymbol{f})$	$\#([\boldsymbol{f}])$	common physical quantities
3	$[3^2 2^3 10]$	13	31	26880	unknown
3	$[3^3 1^4]$	13	31	4480	unknown
3	$[3^3 210^2]$	12	32	13440	unknown
3	$[3^2 2^3 1^2]$	14	32	26880	unknown
3	$[3^3 21^2 0]$	13	33	26880	unknown
3	$[32^6]$	15	33	896	unknown
3	$[3^2 2^4 0]$	14	34	6720	unknown
3	$[3^3 21^3]$	14	34	17920	unknown
3	$[3^3 2^2 0^2]$	13	35	6720	unknown
3	$[3^2 2^4 1]$	15	35	13440	unknown
3	$[3^4 0^3]$	12	36	560	unknown
3	$[3^3 2^2 10]$	14	36	26880	unknown
3	$[3^4 10^2]$	13	37	3360	unknown
3	$[3^3 2^2 1^2]$	15	37	26880	unknown
3	$[3^4 1^2 0]$	14	38	6720	unknown
3	$[3^2 2^5]$	16	38	2688	unknown
3	$[3^3 2^3 0]$	15	39	8960	unknown
3	$[3^4 1^3]$	15	39	4480	unknown
3	$[3^4 20^2]$	14	40	3360	unknown
3	$[3^3 2^3 1]$	16	40	17920	unknown
3	$[3^4 210]$	15	41	13440	unknown
3	$[3^4 21^2]$	16	42	13440	unknown
3	$[3^3 2^4]$	17	43	4480	unknown
3	$[3^4 2^2 0]$	16	44	6720	unknown
3	$[3^5 0^2]$	15	45	672	unknown
3	$[3^4 2^2 1]$	17	45	13440	unknown
3	$[3^5 10]$	16	46	2688	unknown
3	$[3^5 1^2]$	17	47	2688	unknown
3	$[3^4 2^3]$	18	48	4480	unknown
3	$[3^5 20]$	17	49	2688	unknown
3	$[3^5 21]$	18	50	5376	unknown
3	$[3^5 2^2]$	19	53	2688	unknown
3	$[3^6 0]$	18	54	448	unknown
3	$[3^6 1]$	19	55	896	unknown
3	$[3^6 2]$	20	58	896	unknown
3	$[3^7]$	21	63	128	unknown

Table A.3: Common physical quantities outside P_7^3.

$\|\boldsymbol{f}\|_\infty$	$[\boldsymbol{f}]$	$\boldsymbol{f}$	physical quantity
4	$[40^6]$	(4,0,0,0,0,0,0)	second moment of area
4	$[410^5]$	(1,0,-4,0,0,0,0)	jounce
...	...	...	...

$\|\boldsymbol{f}\|_\infty$	$[\boldsymbol{f}]$	$\boldsymbol{f}$	physical quantity
4	$[4210^4]$	(0,-1,4,2,0,0,0)	electric polarizability
4	$[4310^4]$	(0,1,-3,0,-4,0,0)	Stefan-Boltzmann constant
4	$[4310^4]$	(4,1,-3,0,0,0,0)	first radiation constant
4	$[431^20^3]$	(3,1,-4,-1,0,0,0)	electrical mobility
4	$[42^210^3]$	(-2,-1,4,2,0,0,0)	electric capacitance
4	$[43210^3]$	(-3,-1,4,2,0,0,0)	electric constant
4	$[43210^3]$	(-3,-1,4,2,0,0,0)	permittivity
6	$[62^30^3]$	(-2,-2,6,2,0,0,0)	second hyper-susceptibility
7	$[73210^3]$	(-1,-2,7,3,0,0,0)	first hyper-polarizability
10	$[(10)4320^3]$	(-2,-3,10,4,0,0,0)	second hyper-polarizability

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