Asian Journal of Physics Vol. 23, No. 5 (2014) 1-10

On a Mathematical Method for Discovering Relations Between Physical Quantities: Maxwell's equations revisited

Philippe A.J.G. Chevalier
Independent researcher
De oogst 7, B-9800 Deinze, Belgium
e-mail: chevalier.philippe.ajg@gmail.com

Quantity calculus defines the rules that apply to SI physical quantities used in physics and engineering. This research aims at the development of a mathematical method for discovering the mathematical form of the relations between physical quantities. Laws of physics are *unique relations* obeying unknown mathematical selection rules. Here, we show that each SI physical quantity, that is represented by a lattice point in a seven dimensional integer lattice, has a *unique* 7D-hypersphere. The lattice points incident on the 7D-hypersphere are forming *rectangles* containing the origin $\boldsymbol{o}$, the lattice point $\boldsymbol{z}$ representing the selected physical quantity and the lattice point representations $\boldsymbol{x}, \boldsymbol{y}$ of a pair of *distinguishable* physical quantities $[x]$, $[y]$ where $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. The resulting rectangles are the geometric representations of the *realizable binary form equations* for the selected physical quantity $[z]$. The isoperimeter distribution of the resulting rectangles shows the exceptional occurrence of *unique* rectangles that can be associated with *unique relations* between physical quantities. We find *unknown* integer sequences representing the number of unique rectangles and the number of non-degenerated rectangles formed by 4 lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7$ as function of the infinity norm $\|\boldsymbol{z}\|_\infty = s$. The ratio of the number of unique rectangles to the number of rectangles is decreasing for increasing infinity norm $\|\boldsymbol{z}\|_\infty = s$. We apply the ``hypersphere method´´ on the physical quantities $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}$ to validate the mathematical method and find the integral forms of Maxwell's equations. A second method is developed for n-ary form equations based on the Gödel encoding of a leader class of a physical quantity. The canonical factorization of the Gödel number in n distinct factors generates the n-ary form equations of a physical quantity.

1. Introduction

The SI [1] is used worldwide defining the semantics and syntax in the domains of science and technology. An algebraic structure for *quantity calculus* was proposed by R. Fleischmann [2], who also introduced the concept of ``Verknüpfungsgleichung´´ that we translate as *form equation*.

This research addresses the question *What are realizable n-ary form equations?*

2. Theory

2.1 Axioms of the SI physical quantities

We posit from the 8th edition of the SI [1] a set of axioms derived from *promoting* some of the SI conventions to mathematical axioms.

Axiom 1. *The base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity.*

Axiom 2. *The base quantities are independent.*

Axiom 3. *The physical quantities are organized according to a system of dimensions.*

Axiom 4. *For each base quantity of the SI, there exists one and only one dimension.*

Axiom 5. *The product of two quantities is the product of their numerical values and units.*

Axiom 6. *The quotient of two quantities is the quotient of their numerical values and units.*

The uniqueness of the SI symbols forms an alphabet that is the base of any physical expression.

Definition 1. The dimension of a physical quantity q is expressed as a dimensional product [1] :

$$\dim q = \mathrm{L}^{\alpha}\mathrm{M}^{\beta}\mathrm{T}^{\gamma}\mathrm{I}^{\delta}\boldsymbol{\Theta}^{\epsilon}\mathrm{N}^{\zeta}\mathrm{J}^{\eta}\,;$$

where the exponents $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{Z}$ are called *dimensional exponents*.

The dimensional exponents of the *common* SI physical quantities take small *integer* values. When all the dimensional exponents are zero, we call the physical quantity *dimensionless* or a physical quantity of *dimension one*. These dimensionless quantities occur in the celebrated *Buckingham theorem* [3] also known as the Π-theorem.

2.2 Isomorphism between classes of physical quantities and the 7-dimensional integer lattice

Let the set of *all* physical quantities be denoted by Q. Physical quantities are described by tensors and we can without loss of generality consider a component of a tensor and denote it as q. We know that *concepts* in physics are labeled in many ways. The concept *energy* has the labels: potential energy, kinetic energy, work, Lagrange function, Hamilton function,... in the formulations of physics.

To cope with this multitude of *labels*, we define an equivalence relation between the physical quantities $a, b \in \mathsf{Q}$ with notation $a \sim b$ meaning ``a is *dimensionally* equivalent to b". The set of all equivalence classes in Q, given the equivalence relation $\sim$, is the quotient set $\mathsf{Q}/\sim$. The equivalence class for the *concept* energy has notation $[energy]_\sim$.

We define the surjective function $\dim(q)$ from Q to $\mathsf{Q}/\sim$ as $\dim(q) = [q]_\sim = \mathrm{L}^{\alpha}\mathrm{M}^{\beta}\mathrm{T}^{\gamma}\mathrm{I}^{\delta}\boldsymbol{\Theta}^{\epsilon}\mathrm{N}^{\zeta}\mathrm{J}^{\eta}$. In the sequel of this article we omit the symbol for the equivalence relation $\sim$ and denote the equivalence class as $[q]$. The class of dimensionless physical quantities is denoted $[I]$.

We consider a multiplicative binary operator $\{\cdot\}$ between the equivalence classes of $\mathsf{Q}/\sim$. The algebraic properties of the composition of the equivalence classes result in a *multiplicative commutative group* $\mathsf{Q}/\sim, \{\cdot\}$. We now consider the set of integer septuples $\mathbb{Z}^7 \doteq \{(f_1, \ldots, f_7) : f_i \in \mathbb{Z}\}$. We know that $\mathbb{Z}^7, \{+\}$ is an *additive commutative group*. We define a mapping dex ():

Definition 2 (Mapping dex ())**.** The mapping dex () is defined from $\mathsf{Q}/\sim$ into $\mathbb{Z}^7$ and formally as $\mathrm{dex}\,() : \mathsf{Q}/\sim \to \mathbb{Z}^7$: $\mathrm{dex}\,([q]) \doteq \boldsymbol{f} = (f_1, \ldots, f_7)$ where $f_i \in \mathbb{Z}$.

We rename f_i such that $f_1 = \alpha$, $f_2 = \beta$, $f_3 = \gamma$, $\ldots f_7 = \eta$ being the dimensional exponents *taken in the correct order* of a physical quantity q and thus associate the ordered septuple $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$ to a *lattice point* $\boldsymbol{f} = (f_1, \ldots, f_7)$. Observe that we map the unit element $[I]$ of $\mathsf{Q}/\sim, \{\cdot\}$ on the unit element $\boldsymbol{o} = (0, \ldots, 0)$ of $\mathbb{Z}^7, \{+\}$ and thus we have $\mathrm{dex}\,([I]) \doteq \boldsymbol{o} = (0, \ldots, 0)$. Each element of $\mathbb{Z}^7$ is the image of *one and only one* class $[q]$ of dimensionally equivalent physical quantities. We define the inverse mapping dex^{-1} ():

Definition 3 (Mapping dex^{-1} ())**.** The inverse of the dex () mapping is a mapping of $\mathbb{Z}^7$ into $\mathsf{Q}/\sim$, and defined as $\mathrm{dex}^{-1}\,() : \forall \boldsymbol{a} \in \mathbb{Z}^7,\ \exists [a] \in \mathsf{Q}/\sim : \mathrm{dex}^{-1}\,(\boldsymbol{a}) \doteq [a]$.

A homomorphism $\mathrm{f} : \mathsf{Q}/\sim \to \mathbb{Z}^7$ is an *isomorphism* if there exists a homomorphism $\mathrm{g} : \mathbb{Z}^7 \to \mathsf{Q}/\sim$ such that $\mathrm{f} \circ \mathrm{g}$ and $\mathrm{g} \circ \mathrm{f}$ are the identity mappings of $\mathbb{Z}^7$ and $\mathsf{Q}/\sim$ respectively [4]. We identify $\mathrm{f} = \mathrm{dex}\,()$ and $\mathrm{g} = \mathrm{dex}^{-1}\,()$ and infer that a *group isomorphism* exists between $\mathsf{Q}/\sim$ and $\mathbb{Z}^7$ that we denote $\mathbb{Z}^7 \approx \mathsf{Q}/\sim$ [4].

The set $\mathbb{Z}^d$ is known as the d-dimensional integer lattice [8] that is a discrete subgroup of the real vector space $\mathbb{R}^d$. The properties of the integer lattice $\mathbb{Z}^d$ are found in several publications [8, 9] . In the sequel of this article we choose $d = 7$. We select seven basis lattice points of $\mathbb{Z}^7$ and choose an orthonormal basis and write using the Conway notation [8]:

$$\begin{aligned}
\boldsymbol{e}_1 &\doteq \text{dex}\left([\textit{length}]\right) = (1, 0^6)\,,\\
\boldsymbol{e}_2 &\doteq \text{dex}\left([\textit{mass}]\right) = (0, 1, 0^5)\,,\\
\boldsymbol{e}_3 &\doteq \text{dex}\left([\textit{time}]\right) = (0^2, 1, 0^4)\,,\\
\boldsymbol{e}_4 &\doteq \text{dex}\left([\textit{electric current}]\right) = (0^3, 1, 0^3)\,,\\
\boldsymbol{e}_5 &\doteq \text{dex}\left([\textit{thermodynamic temperature}]\right) = (0^4, 1, 0^2)\,,\\
\boldsymbol{e}_6 &\doteq \text{dex}\left([\textit{amount of substance}]\right) = (0^5, 1, 0)\,,\\
\boldsymbol{e}_7 &\doteq \text{dex}\left([\textit{luminous intensity}]\right) = (0^6, 1)
\end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^7$.

A set of lattice points is called a *lattice constellation* [10]. An arbitrary set of physical quantities is represented by a constellation of points in $\mathbb{Z}^7$. We are interested in the properties of these constellations of points and focus on the simplest non-trivial constellation consisting of 4 integer lattice points.

Observe that the *parallelogram law* $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}$ where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{Z}^7$ is valid. We can prove [11] that binary equations $[z] = f(\Pi)[x][y]$ are geometrically represented by *parallelograms* in $\mathbb{Z}^7$. We can define [11] an inner product $\{\cdot\}$ and p-norm $\|\|_p$ in $\mathbb{Z}^7$ and write $\boldsymbol{f} = \sum_{i=1}^{7} (\boldsymbol{f} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i$.

2.3 Partitions of the d-dimensional integer lattice based on the infinity norm ℓ_∞

We calculate the *number of equivalence classes* that can be formed in a d-dimensional hypercube P_d^s [5] when the infinity norm $\ell_\infty = s$ and $s \in \mathbb{N}$. The result is known as the *multiset number* and given by:

$$\#\left(P_d^s\right) = \binom{d+s-1}{s}$$

For $d = 7$ we find the integer sequence A000579 [16]: $\#\left(P_7^s\right)$ = 1, 7 , 28, 84, 210, 462, 924, 1716, 3003, 5005, 8008 . . . where $s \in \{0, \ldots, 10\}$. The value of $s = 10$ is relevant when considering the *second hyper-polarizability* that has the largest coordinate value (-2,-3,10,4,0,0,0) of the tabulated physical quantities.

2.4 Absolute leader classes of a lattice

The representative lattice point, called in signal processing an *absolute leader*, has only coordinates that are *non-negative integers*. A *leader class* is the set of lattice points of $\mathbb{Z}^d$ that are connected through a signed permutation. Let $A = \{0, 1, 2, \ldots, s\}$ be a *totally ordered alphabet*. The representative of a leader class is a word w constructed from the alphabet A . The words w have a length d that corresponds to the dimension of $\mathbb{Z}^d$. Let d_i be the *number of characters* of type i of the alphabet A . Suppose that the characters of w are subjected to a *signed permutation*, then the cardinality of the leader class is given by the equation:

$$\#\left([w]\right) = 2^{d-d_0} \frac{d!}{d_0!d_1!d_2!\ldots d_s!}\ .$$

We note a leader class of $\mathbb{Z}^d$ as $[w] = (f_1, \ldots, f_d)$, where $(f_1, \ldots, f_d)$ are the coordinates of the representative lattice point. We write the characters in *graded reverse lex order* [6]. Each leader class forms a set of lattice points that are *centro-symmetric* about the origin $\boldsymbol{o}$ [5]. The union of all leader classes is called a *codebook* [7].

3. Methods

3.1 Method 1: Decomposition of a lattice point in pairwise orthogonal lattice points

We define *distinguishable* physical quantities as orthogonal lattice points dex $([x])$ and dex $([y])$. The decomposition of a lattice point $\boldsymbol{z}$ in two pairwise orthogonal lattice points $\boldsymbol{x}$ and $\boldsymbol{y}$ assumes the existence of a system of Diophantine equations:

$$\text{parallelogram law: } \boldsymbol{x} + \boldsymbol{y} - \boldsymbol{z} = 0 \ , \tag{1a}$$

$$\text{inner product: } \boldsymbol{x} \cdot \boldsymbol{y} = 0 \ , \tag{1b}$$

where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{Z}^7$. We eliminate $\boldsymbol{y}$ from the equation (1b) and find:

$$\boldsymbol{x} \cdot \boldsymbol{x} - \boldsymbol{x} \cdot \boldsymbol{z} = 0 \ . \tag{2}$$

We apply the method of *completing the square* and write equation (2) as:

$$(\boldsymbol{x} - \frac{\boldsymbol{z}}{2})^2 = (\frac{\boldsymbol{z}}{2})^2 \ , \tag{3}$$

that represents a 7D-hypersphere in $\mathbb{R}^7$ with center at $\frac{\boldsymbol{z}}{2}$ and radius $\|\frac{\boldsymbol{z}}{2}\|_2$.

The center of the 7D-hypersphere is only a lattice point if its coordinates are *even*. Observe that there exists a *unique* 7D-hypersphere (3) for each physical quantity $[z]$. This unique 7D-hypersphere determines the *finite* set of pairwise *distinguishable* physical quantities $[x]$ and $[y]$ that satisfy the binary realizable form equation $[z] = f(\Pi)[x][y]$. We call the above method the ``hypersphere method´´.

3.2 Method 2: Gödel encoding of physical quantities up to a signed permutation

In number theory the *atomic* parts are identified as the prime numbers [12]. The prime numbers are the *atoms* of the number system and the SI base quantities are the *atoms* of the physical quantities. We encode each integer lattice point of $\mathbb{Z}^7_+$ by using a similar scheme to the *Gödel encoding* [13].

Definition 4.

$$\phi_d(f_1, \ldots, f_d) = \prod_{i=1}^{d} p_i^{f_i} \ ,$$

where p_i is the i-th prime number, $\boldsymbol{f} = (f_1, \ldots, f_d)$ and $f_i \in \mathbb{Z}_+$.

Consider the physical quantity *energy* represented by the lattice point $(2, 1, -2, 0, 0, 0, 0)$. The corresponding leader class for the physical quantity *energy* is the lattice point with coordinates $(2, 2, 1, 0, 0, 0, 0)$ that is obtained by a signed permutation of the original coordinates. We calculate for this leader class its Gödel number.

Example 1. $\phi_7(2, 2, 1, 0, 0, 0, 0) = 2^2 \cdot 3^2 \cdot 5^1 \cdot 7^0 \cdot 11^0 \cdot 13^0 \cdot 17^0 = 180$

The encoding of the leader classes with a Gödel number allows the factorization of the Gödel number in distinct factors. Richard J. Mathar (http://home.strw.leidenuniv.nl/ mathar/) has listed in the OEIS [16] the integer series A045778 that gives the factorization of non-negative integers up to $m = 1500$. The enumeration for *common* leader classes with Gödel number ≤ 1500 of the factorization of the Gödel number in n distinct factors is given in Table 1 . The number of distinct factors is found in the respective columns Fn where $n \in [2, \ldots, 5]$. We find that there is a *finite number* of canonical form equations for each physical quantity. For the physical quantity *energy* that corresponds to the leader class $(2^2, 1, 0^4)$ we find $F2 = 8$, $F3 = 8$ and $F4 = 1$ and thus the physical quantity *energy* has $F2 + F3 + F4 = 17$ canonical form equations distributed over 8 *binary* form equations, 8 *ternary* form equations and 1 *quaternary* form equation.

Table 1: **Canonical factorization for Gödel number ≤ 1500 in n distinct factors.**

leader class	cardinality	Gödel number	$F2$	$F3$	$F4$	$F5$
(0^7)	1	1	0	0	0	0
$(1,0^6)$	14	2	0	0	0	0
$(2,0^6)$	14	4	0	0	0	0
$(1^2,0^5)$	84	6	1	0	0	0
$(3,0^6)$	14	8	1	0	0	0
$(2,1,0^5)$	168	12	2	0	0	0
$(3,1,0^5)$	168	24	3	1	0	0
$(1^3,0^4)$	280	30	3	1	0	0
$(2^2,0^5)$	84	36	3	1	0	0
$(2,1^2,0^4)$	840	60	5	3	0	0
$(3,2,0^5)$	168	72	5	3	0	0
$(3,1^2,0^4)$	840	120	7	7	1	0
$(2^2,1,0^4)$	840	180	8	8	1	0
$(1^4,0^3)$	560	210	7	6	1	0
$(3^2,0^5)$	84	216	7	8	1	0
$(3,2,1,0^4)$	1680	360	11	17	5	0
$(2,1^3,0^3)$	2240	420	11	15	4	0
$(3,1^3,0^3)$	2240	840	15	29	13	1
$(2^3,0^4)$	280	900	12	20	7	0
$(3^2,1,0^4)$	840	1080	15	33	17	1
$(2^2,1^2,0^3)$	3360	1260	17	35	16	1

4. Results

4.1 Maxwell's equations and beyond

The integral and differential representation of Maxwell's equations are:

$$\oint_{L(S)} \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t}\iint_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} \qquad \nabla\times\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$$

$$\oint_{L(S)} \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{s} = \iint_S \boldsymbol{J}\cdot \mathrm{d}\boldsymbol{S} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} \qquad \nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial}{\partial t}\boldsymbol{D}$$

$$\oint\!\!\!\oint_{S(V)} \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = \iiint_V \rho_f \,\mathrm{d}V \qquad \nabla\cdot\boldsymbol{D} = \rho_f$$

$$\oint\!\!\!\oint_{S(V)} \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = 0 \qquad \nabla\cdot\boldsymbol{B} = 0$$

The constitutive equations are:

$$\boldsymbol{D}\cdot\epsilon_0\boldsymbol{E} + \boldsymbol{P} = \epsilon\boldsymbol{E} \qquad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

We apply the ``hypersphere method´´ to the physical quantities $[H], [B], [E], [D]$ occurring in the celebrated Maxwell's equations and infer relations between the physical quantities. The SI coordinates of the physical quantities $[H], [B], [E], [D]$

are:

Magnetic field strength: dex $([H]) = (-1, 0, 0, 1, 0, 0, 0)$
Magnetic induction: dex $([B]) = (0, 1, -2, -1, 0, 0, 0)$
Electric field: dex $([E]) = (1, 1, -3, -1, 0, 0, 0)$
Electrical displacement: dex $([D]) = (-2, 0, 1, 1, 0, 0, 0)$

The results are summarized in *isoperimetric distributions* giving the frequency of occurrence of rectangles having a perimeter with value p . We denote electric current I, electric charge q, electric charge density ρ_f, volume V, area S, time t, length s, velocity v, electric current density J.

4.1.1 Magnetic field strength *[H]*

The Gödel number of the leader class is 6 with $F2 = 1$ and thus 1 *binary* canonical form equation exist for $\boldsymbol{H}$. We select the smallest non-degenerated rectangle of $[H]$ having $p = 4$. Observe that $[H]$ has a non-degenerated

Table 2: **Isoperimetric distribution for** $[H]$ **.**

Perimeter p	Frequency f
2.828	1
4	1

unique rectangle. We find the lattice points $\boldsymbol{x} = (-1, 0, 0, 0, 0, 0, 0)$ and $\boldsymbol{y} = (0, 0, 0, 1, 0, 0, 0)$. We suggest the binary realizable form equation:

$$H = f(\Pi)\left(\frac{1}{s}\right)(I)$$
$$Hs = f(\Pi)I$$
$$\oint_{L(S)} H\,\mathrm{d}s = f(\Pi)\iint_S \boldsymbol{J}_t \cdot \mathrm{d}\boldsymbol{S}$$
$$\oint_{L(S)} \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{s} = f(\Pi)\iint_S \boldsymbol{J}_t \cdot \mathrm{d}\boldsymbol{S}$$

that is one of the *integral forms of Maxwell's equations* when $\boldsymbol{J}_t = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t}$.

4.1.2 Magnetic induction *[B]*

The Gödel number of the leader class is 60 with $F2 = 5$ and $F3 = 3$ and thus 5 *binary* and 3 *ternary* canonical form equations exist for $\boldsymbol{B}$. We *restrict* the search to the binary form equations of $\boldsymbol{B}$. We select the non-degenerated

Table 3: **Isoperimetric distribution for** $[B]$ **.**

Perimeter p	Frequency f
4.489	1
6.472	2
6.828	9
6.928	8

rectangle of $[B]$ having $p = 6.828$. We find the lattice points $\boldsymbol{x} = (-1, 1, -1, -1, 0, 0, 0)$ and $\boldsymbol{y} = (1, 0, -1, 0, 0, 0, 0)$.

We suggest the binary realizable form equation:

$$B = f(\Pi) v \left(\frac{m}{sIt}\right)$$
$$\frac{Bs}{v} = f(\Pi)\frac{m}{q}$$
$$\int B dt = f(\Pi)\frac{m}{q}$$
$$\frac{\partial}{\partial x}\int B \,\mathrm{d}t = f(\Pi)\frac{\partial}{\partial x}\left(\frac{m}{q}\right)$$
$$\frac{\partial}{\partial y}\int B \,\mathrm{d}t = f(\Pi)\frac{\partial}{\partial y}\left(\frac{m}{q}\right)$$
$$\frac{\partial}{\partial z}\int B \,\mathrm{d}t = f(\Pi)\frac{\partial}{\partial z}\left(\frac{m}{q}\right)$$
$$\int (\nabla\cdot \boldsymbol{B})\,\mathrm{d}t = f(\Pi)(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z})\left(\frac{m}{q}\right) = \frac{1}{2}f(\Pi)\nabla\left(\frac{1}{\gamma}\right)$$

The right hand side of the equation is related to the *gyro-magnetic ratio* $\gamma = \frac{q}{2m}$. For an isolated electron we have $|\gamma_e| = g_e\frac{|-e|}{2m_e} = g_e\frac{\mu_\mathrm{B}}{\hbar}$ where μ_B is the Bohr magneton. The electron g-factor has been measured to twelve decimal places $g_e = 2.0023193043617(15)$. The gyro-magnetic ratio is *constant* but varies from one nucleus to another. We infer that ***if*** the gradient of the reciprocal of the gyromagnetic ratio $\nabla\left(\frac{1}{\gamma}\right) = 0$ then we find $\int (\nabla\cdot\boldsymbol{B})\,\mathrm{d}t = 0$. We find one of the *differential forms of Maxwell's equations*.

4.1.3 Electric field *[E]*

The Gödel number of the leader class is 840 with $F2 = 15$, $F3 = 29$, $F4 = 13$ and $F5 = 1$ and thus we have 15 *binary*, 29 *ternary*, 13 *quaternary* and 1 *quinternary* canonical form equations for $\boldsymbol{E}$. We *restrict* the search to the binary form equations of $\boldsymbol{E}$. We select a non-degenerated rectangle of $[E]$ having $p = 9.152$. We find the lattice points

Table 4: **Isoperimetric distribution for** $[E]$ **.**

Perimeter p	Frequency f
6.928	1
8.633	3
9.152	6
9.464	19
9.656	39
9.763	42
9.797	18

$\boldsymbol{x} = (-1,0,-1,0,0,0,0)$ and $\boldsymbol{y} = (2,1,-2,-1,0,0,0)$. We suggest the binary realizable form equation:

$$E = f(\Pi)\left(\frac{1}{st}\right)(BS)$$
$$Es = f(\Pi)\frac{1}{t}BS$$
$$\oint_{L(S)} E\,\mathrm{d}s = f(\Pi)\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$$
$$\oint_{L(S)} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{s} = f(\Pi)\frac{\partial}{\partial t}\iint_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$$

that is one of the *integral forms of Maxwell's equations* when $f(\Pi) = -1$.

4.1.4 Electrical displacement *[D]*

The Gödel number of the leader class is 60 with $F2 = 5$ and $F3 = 3$ and thus 5 *binary* and 3 *ternary* canonical form equations exist for $\boldsymbol{D}$. We *restrict* the search to the binary form equations of $\boldsymbol{D}$. We select the non-degenerated rectangle of $[D]$ having $p = 6.828$. We find the lattice points $\boldsymbol{x} = (-2, 0, 0, 0, 0, 0, 0)$ and $\boldsymbol{y} = (0, 0, 1, 1, 0, 0, 0)$. We suggest the binary realizable form equation:

Table 5: **Isoperimetric distribution for** $[D]$ **.**

Perimeter p	Frequency f
4.489	1
6.472	2
6.828	9
6.928	8

$$D = f(\Pi)\left(\frac{1}{S}\right)(q)$$

$$DS = f(\Pi)q$$

$$\underset{S(V)}{\oiint} D\,\mathrm{d}S = f(\Pi)q$$

$$\underset{S(V)}{\oiint} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = f(\Pi)\underset{V}{\iiint} \rho_f\,\mathrm{d}V$$

that is one of the *integral forms of Maxwell's equations*.

4.1.5 Discussion

We find that the physical quantities $[D]$ and $[B]$ have the same isoperimetric distributions and thus we find a matrix M such that dex $([D])^\mathsf{T} = \mathsf{M}\,$dex $([B])^\mathsf{T}$ where:

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

represents a signed permutation matrix. The automorphism group of the 7-dimensional cubic lattice Aut($\mathbb{Z}^7$) contains all permutations and sign changes of the 7 coordinates and has order $2^7\,7! = 645120$. Each signed permutation matrix is an orthogonal matrix [14]. It is known from linear vector quantization [15] that the ℓ_2-norm and the phase of a lattice point are used to partition a lattice. However, this norm and phase are not the correct classifiers for the physical quantities. If we use instead as classifier the ℓ_∞-norm we obtain equivalence classes for which the elements of the class have the *same* isoperimetric distribution [11]. In the framework of information theory we state that the lattice points dex $([D])$ and dex $([B])$ are elements of the absolute leader class $(2, 1^2, 0^4)$ that has cardinality 840.

4.2 Distribution of unique rectangles in one orthant of the 7D integer lattice

We determine the distribution of non-degenerated *unique* rectangles formed by 4 lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7$ as function of the infinity norm $\|\boldsymbol{z}\|_\infty = s$. We define a sample space Ω consisting of one orthant of the 7D-hypercube with infinity norm $\|\boldsymbol{z}\|_\infty = s$, with $s \in \mathbb{N}$ and search for the event of an unique perimeter p . Table 6 gives the result of the search for rectangles. We find in one orthant of the 7D-hypercube where $\|\boldsymbol{z}\|_\infty \leq 10$, a total of 7747 unique rectangles out of 6510466998 rectangles. The unique rectangles represent *unique realizable binary form equations* of the type $[z] = f(\Pi)[x][y]$ for the selected physical quantity $[z]$. These sequences of integers are not listed in the OEIS [16] and we suggest further research on it. We observe that the ratio of the number of unique rectangles to the number of rectangles is

Table 6: **Distribution of rectangles in $\mathbb{Z}^7$ as function of the infinity norm $\|\boldsymbol{z}\|_\infty = s$.**

Infinity norm $\|\boldsymbol{z}\|_\infty = s$	UR =# unique rectangles	R = # rectangles	$\frac{UR}{R}$
1	1	120	8.33e-03
2	7	7196	9.73e-04
3	26	162554	1.60e-04
4	79	1341957	5.89e-05
5	182	9255603	1.97e-05
6	333	40532530	8.22e-06
7	693	168302117	4.12e-06
8	1180	523421602	2.25e-06
9	1999	1637895896	1.22e-06
10	3247	4129547423	7.86e-07
Total	7747	6510466998	1.19e-06

decreasing for increasing infinity norm $\|\boldsymbol{z}\|_\infty = s$ and that for $s = 10$ the ratio is 7.86e-07 . The 7D-hypercube, where $\|\boldsymbol{z}\|_\infty \leq 10$ contains all known physical quantities and all know physical relations between these quantities.

5. Conclusion

We show that each SI physical quantity, that is represented by a lattice point in a seven dimensional integer lattice $\mathbb{Z}^7$, has a *unique* 7D-hypersphere. The lattice points incident on the 7D-hypersphere are rectangles formed by 4 lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7$ where $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. The resulting rectangles are the geometric representation of the *realizable binary form equations* of the type $[z] = f(\Pi)[x][y]$ for the selected physical quantity $[z]$. We apply the ``hypersphere method´´ on the physical quantities $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}$ and find the integral forms of Maxwell's equations. We find in one orthant of the 7D-hypercube, where $\|\boldsymbol{z}\|_\infty \leq 10$, a total of 7747 unique rectangles that represent *unique* realizable binary form equations. We observe that the ratio of the number of unique rectangles to the number of rectangles is decreasing for increasing infinity norm $\|\boldsymbol{z}\|_\infty = s$ and that for $s = 10$ the ratio is 7.86e-07 . The 7D-hypercube, where $\|\boldsymbol{z}\|_\infty \leq 10$ contains all known physical quantities and all know physical relations between these quantities. A second method is developed for n-ary form equations based on the Gödel encoding of a leader class of a physical quantity. The canonical factorization of the Gödel number in n distinct factors generates the n-ary form equations of a physical quantity.

Acknowledgment

I am grateful for the valuable discussions, the support and encouragements given by: Prof. Martin Aigner (University Berlin), Prof. Jean-Pierre Antoine (Catholic University of Louvain), Prof. P. Bienstman (Ghent University), Prof. em. F. Brackx (Ghent University), Dr. Katherine Brading (University of Notre Dame), Prof. Ph. Cara (Free University Brussels), Assistant Prof. H. De Bie (Ghent University), Prof. H. De Schepper (Ghent University), Prof. em. Michel Deza (CNRS), Prof. Martin Rees (Cambridge University), Prof. R.S. Sirohi (Tezpur University), Prof. Neil J.A. Sloane (Cornell University and AT&T), Prof. H. Steendam (Ghent University), Prof. H. Van Maldeghem (Ghent University), Prof. J.P. Van Bendegem (Free University Brussels), Prof. em. I. Veretennicoff (Free University Brussels), Prof. Doron Zeilberger (Rutger University). I thank Mr. B. Chevalier for the 7D-hypersphere software code. The computational resources (STEVIN Supercomputer Infrastructure) and services used in this work were kindly provided by Ghent University, the Flemish Supercomputer Center (VSC), the Hercules Foundation and the Flemish Government – department EWI. Special thanks to my wife, children and friends for supporting me in this research.

References

[1] Bureau International des Poids et Mesures, BIPM 2006 Le Système International d'unités The International System of Units SI, 8th edn. Paris, STEDI Media, [Internet].[cited 30 Jan 2014]. Available from http://www.bipm.org/utils/common/pdf/si_brochure_8.pdf.

[2] Fleischmann R. , Physikalisches Begriffssystem und Dimensionen. *Phys. Bl.* **9**,(1953), 301–313. (doi: 10.1002/phbl.19530090702)

[3] Buckingham E. , On physically similar systems; illustrations of the use of dimensional equations, *Phys. Rev.* **4**,(1914), 345--376.

[4] Lang S. , *Algebra, Revised Third edition.* (Springer Science+Business Media, New York, NY), 2005.

[5] Coxeter H.S.M. , Forms, Vectors, and Coordinates. In *Regular Polytopes, Third Edition*, Chapter10, 178--183. (Dover Publications, New York), 1973.

[6] Cox D., Little J., O'Shea D., Groebner Bases, In *Ideals, Varieties, and Algorithms, An Introduction to Computational Algebraic Geometry and Commutative Algebra, Third Edition*, Chapter 2, 54--61. (Springer Science+Business Media, New York), 2007.

[7] Vasilache A. , Tăbuş I., Robust indexing of lattices and permutation codes over binary symmetric channels.*Signal Processing*,**83**, (2003),1467--1486.

[8] Conway J.H., Sloane N.J.A. , Certain Important Lattices and Their Properties. In *Sphere Packings, Lattices and Groups, Third Edition*, Chapter 4, 106--108.(Springer-Verlag, Berlin Heidelberg New York), 1999.

[9] Birkhoff G. , *Lattice Theory. 3rd. edition.* (American Mathematical Society), 1979.

[10] Forney G.D., Ungerboeck G. , Modulation and Coding for Linear Gaussian Channels. *IEEE Trans. Inform. Theory* **44**(6), (1998) 2384--2415.

[11] Chevalier Ph. , On the discrete geometry of physical quantities. ResearchGate, [Internet]. [cited 30 Jan 2014].Available from http://www.researchgate.net/publication/236594822 On the discrete geometry of physical quantities, 2013.

[12] Coppel W.A. , *Number Theory, An Introduction to Mathematics, Second Edition*. (Springer Science+Business Media, New York), 2009.

[13] Feferman S. et al. , On undecidable propositions of formal mathematical systems. In *Kurt Gödel Collected Works Volume 1 Publications 1929-1936*, p355. (Oxford University Press, Clarendon Press Oxford, New York), 1986.

[14] Conway J.H., Sloane N.J.A. , Codes, Designs and Groups. In *Sphere Packings, Lattices and Groups, Third Edition*, Chapter 3, 90--93. (Springer-Verlag, Berlin Heidelberg New York), 1999.

[15] Rault P., Guillemot Ch. , Indexing Algorithms for Z_n, A_n, D_n, and D_n^{++} Lattice Vector Quantizers. *IEEE Trans. on Multimedia* **3** (4), (2001), 395--404.

[16] OEIS Foundation Inc. , The On-Line Encyclopedia of Integer Sequences, [Internet].[cited 30 Jan 2014]. Available from: http://oeis.org.