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Einstein Maxwell solutions of Newman Tamburino class and Aligned pure radiation Kundt solutions

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Wat wij weten is een druppel, wat wij niet weten een oceaan.
Isaac Newton (1642-1727)

Gent, augustus 2012

Voorwoord - Foreword

Algemene relativiteit wordt algemeen aanvaard als dé meest succesvolle theorie voor de zwaartekracht. Wiskundig wordt deze theorie beschreven door Einsteins veldvergelijking, een niet-lineaire tensoriële differentiaalvergelijking, afgeleid en op papier gezet door Albert Einstein en voor het eerst gepubliceerd in (Einstein, 1915).

Oplossingen vinden van deze vergelijkingen is geen eenvoudige taak. Een heleboel 'exacte oplossingen' zijn terug te vinden in het referentieboek Exact solutions of Einstein's field equations *(Stephani et al., 2003). Soms worden ze gezien als zuiver wiskundige objecten, zonder fysische toepassingen. Een aantal oplossingen heeft nochtans een duidelijke fysische betekenis, zoals de Schwarzschild en Kerr oplossingen voor zwarte gaten of Friedmanns oplossingen voor kosmologie.*

In deze thesis gaan we op zoek naar 'nieuwe' exacte oplossingen, die aan bepaalde fysische voorwaarden voldoen.

De integratie van de Newman Tamburino Einstein Maxwelloplossingen werd uitgevoerd in samenwerking met mijn promotor, Prof. Norbert Van den Bergh. Ook het probleem van de Petrov type D Kundtoplossingen in aanwezigheid van een zuiver stralingsveld hebben we samen aangepakt. Bij dit laatste kreeg ik ook hulp van Lode Wylleman. De onderwerpen die voorkomen in de overige hoofdstukken, namelijk het controleren en verbeteren van de Newman Tamburino vacuümoplossingen en de Petrov type D Robinson Trautmanoplossingen in een gealigneerd zuivere stalingsveld, zijn persoonlijk werk. Dit geldt ook voor de classificatie van Kundtmetrieken.

General relativity is generally accepted to be the most succesful theory of gravitation. It is mathematically described by the Einstein's field equation, a tensorial equation, which translates into a system of non-linear, coupled partial differential equations when expressed in coordinates or in a frame, which was derived by Albert Einstein and which was published for the first time in (Einstein, 1915).

Finding solutions of these equations is not an easy task. A lot of 'exact solutions' can be found in the reference book Exact solutions of Einstein's field equations *(Stephani et al., 2003). Sometimes these solutions are seen as purely mathematical objects, without physical applications. However, some of the solutions do have a clear physical meaning, such as the Schwarzschild and Kerr solutions for black holes or Friedmann's solutions for cosmology.*

In this thesis, we look for 'new' exact solutions, that satisfy certain physical conditions.

The integration of the Newman Tamburino Einstein Maxwell solutions is joint work with my supervisor, Prof. Norbert Van den Bergh. Also solving the problem of the Petroc type D Kundt solutions in the presence of an aligned pure radiation field, is something we did together. For the latter I also got some help from Lode Wylleman. The subjects that are handled in the remaining chapters, i.e. re-examining and correcting the Newman Tamburino vacuum solutions and the Petrov type D Robinson Trautman solutions in an aligned pure radiation field, are personal work of the author, as is also the case for the classification of the Kundt metrics.

Graag wil ik in dit voorwoord een aantal mensen bedanken:
In this foreword, I would also like to thank some people:
Mijn promotor en copromotor Prof. Norbert Van den Bergh en Prof. Frans Cantrijn, voor hun steun en aanmoedigingen tijdens het werken aan deze thesis. Norberts enthousiasme was steeds een bron van inspiratie. Ook wanneer het op persoonlijk vlak moeilijker ging, kon ik steeds bij hen terecht.

Het was ook steeds aangenaam om met Norbert samen te werken, samen brachten we de integratie van de Newman Tamburino Einstein Maxwell en van de Kundt oplossingen in de aanwezigheid van een zuiver stralingsveld tot een mooi resultaat.

Voor het oplossen van dit laatste probleem wil ik ook graag Lode Wylleman bedanken. Zijn bewijs, opgenomen in deze thesis (zie hoofdstuk 5), is een stuk korter dan mijn oorspronkelijke berekeningen!

I would also like to thank Jan Åman, who revealed to us the secrets of CLASSI, and who was always willing to help us with the classification of metrics.

I thank the reading committee for examining this work and for their valuable comments.

Vrienden en collega's. Om niemand te vergeten, ga ik mij niet aan een opsomming wagen, maar aan allen die ik in de voorbije jaren heb leren kennen, en die mijn leven voller en rijker hebben gemaakt: een welgemeende en dikke merci! In het bijzonder wil ik toch Brian Edgar vermelden. Hoewel we elkaar niet vaak ontmoetten, waren onze momenten samen toch tegelijk leerrijk en rustgevend. We zullen je missen, Brian!

Tenslotte wil ik ook mijn ouders en grootouders, broers, zus en vriend bedanken, omdat ze er steeds zijn als ik hen nodig heb, en ze mij opvangen wanneer het moeilijker gaat. Wij hebben samen een aantal bewogen jaren achter de rug. Moeilijke momenten, zoals het verlies van tante Trees en Femke, mijn gezondheidsproblemen en die van Henri'tje, werden even vergeten, of toch verlicht bij de feestelijkheden van het voorbije jaar: de trouw van Ans en Fre, die van Thomas en Annelies en die van Matthias en Moni en de geboorte van die twee prachtige kereltjes, Henri en Nathan. Ik weet dat ik dit niet vaak genoeg zeg of toon, maar ik zie jullie graag en kan jullie niet genoeg bedanken voor wat jullie voor mij doen.

Liselotte De Groote
Gent, 28 augustus 2012

Samenvatting

Einsteins algemene relativiteitstheorie is algemeen aanvaard als zijnde de meest succesvolle theorie voor de zwaartekracht. Wiskundig wordt deze theorie beschreven door de veldvergelijkingen

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \kappa_0 T_{ab}. \tag{1}$$

Dit is een tensoriële vergelijking die omgezet wordt in een verzameling niet-lineaire, gekoppelde partiële differentiaalvergelijkingen, wanneer ze wordt uitgedrukt in coördinaten. Het basisingrediënt hierin is de metrische tensor g_{ab}*, waaruit elk ander element kan afgeleid worden. Oplossingen vinden van deze vergelijkingen is niet eenvoudig. Eigenlijk is om het even welke metriek een oplossing van (1), zolang er geen voorwaarden opgelegd worden aan de energie-momenttensor. In dat geval is (1) immers niet meer dan een definitie voor* T_{ab}*. We zullen dus eerst veronderstellingen koppelen aan* T_{ab}*. Bovendien zullen we ook voorwaarden opleggen aan de metriek, vooraleer we de veldvergelijkingen 'oplossen'.*
Dit soort veronderstellingen en voorwaarden laat ons toe de oplossingen te verzamelen, te catalogeren in verschillende klassen. In deze thesis kijken we enerzijds naar oplossingen uit de Newman Tamburinoklasse, en anderzijds naar oplossingen die behoren tot de zogenaamde Kundtklasse.

In 1960 publiceerden Robinson en Trautman de algemene oplossing voor de familie van metrieken met hyperoppervlak orthogonale en bijgevolg niet-roterende geodetische nul-congruenties, mét expansie maar in de afwezigheid van afschuiving (shear) (Robinson and Trautman, 1960).
Newman en Tamburino (1962) poogden deze klasse van oplossingen uit te breiden door afschuiving toe te laten, in de hoop hiermee een meer algemene

oplossing te vinden. Eisen dat de afschuiving verschilt van nul, legt echter extra voorwaarden op aan het systeem, en daardoor kunnen de Robinson Trautmanoplossingen niet gevonden worden als limietgevallen van de Newman Tamburino-oplossingen.
Kundt (1961) bestudeerde oplossingen met geodetische nulcongruenties die niet expanderen, roteren noch afschuiven.

Deze thesis is opgedeeld in zes hoofdstukken: in hoofdstuk één geven we een korte inleiding met wiskundige achtergrond en woordenschat, die gebruikt zal worden in de rest van de thesis. In het bijzonder worden in dit hoofdstuk het Newman Penrose (NP) en het Geroch Held Penrose (GHP) formalisme aangebracht. Verder gaan we hier ook in op het equivalentieprobleem: gegeven twee metrieken in verschillende coördinaten, hoe kan je besluiten of ze al dan niet lokaal equivalent zijn? Een handig hulpmiddel hierbij is het classificatieprogramma CLASSI, geschreven door Jan Åman. Voorbeelden en toepassingen komen later in de tekst aan bod.

In het tweede hoofdstuk herbekijken we zuivere stralingsoplossingen van Petrov type D, die behoren tot de Robinson Trautmanfamilie. Dit probleem werd oorspronkelijk onderzocht door Frolov en Khlebnicov in 1975. Hun oplossingen zijn opgedeeld in drie verschillende klassen, A, B en C, waarbij zowel de A- als B-klasse nog verder opgedeeld zijn (in respectievelijk drie en vijf subklassen). In hoofdstuk twee van deze thesis merken we enerzijds op dat de A-klasse eigenlijk overbodig is aangezien de metrieken AI, AII en AIII gevonden kunnen worden als speciale gevallen van metrieken BI, BII en BV, respectievelijk, en anderzijds herintegreren we de C-klasse. Deze laatste is incorrect in het oorspronkelijk artikel, en de verbeterde versie vindt u terug in het tweede hoofdstuk van dit werk.

Het derde hoofdstuk van deze thesis behandelt ons onderzoek naar Newman Tamburino-oplossingen in aanwezigheid van een gealigneerd Einstein Maxwellveld. Dit probleem hebben we volledig opgelost. We tonen ook aan dat er geen zogenaamd 'cilindrische' oplossingen bestaan in de aanwezigheid van een niet-gealigneerd Einstein Maxwellveld. De uitdaging blijft bestaan om te kijken naar zogenaamde 'sferische' Newman Tamburino-oplossingen in de aanwezigheid van een niet-gealigneerd Einstein Maxwellveld.

Het vierde hoofdstuk is gewijd aan vacuümoplossingen uit de Newman Tamburinoklasse. Deze oplossingen werden oorspronkelijk gepubliceerd in (Newman and Tamburino, 1962), maar de aldaar vermelde cilindrische klasse is niet correct, en bovendien is het ook mogelijk om de (correcte) oplossing op een elegantere manier op te schrijven. Daarnaast tonen we ook aan dat de sferische oplossingen op een compactere manier kunnen voorgesteld worden. Ten slotte tonen we ook het verband aan tussen de cilindrische Einstein Maxwelloplossingen en de cilindrische vacuümoplossingen.

Vooraleer we onze conclusies geven in hoofdstuk zes, behandelen we eerst nog Petrov type D zuivere stralingsoplossingen behorend tot de Kundtfamilie (hoofdstuk vijf). Ook hier worden nieuwe metrische families voorgesteld. In dit hoofdstuk vindt u ook de volledige classificatie van deze lijnelementen; de CLASSI-input-bestanden zijn gegroepeerd in de appendix.

In hoofdstukken twee tot en met vijf introduceren we eerst het te behandelen probleem. Nadien zoeken we, m.b.v. het GHP-formalisme, alle spinrotatie- en boostinvariante eigenschappen, die we vervolgens overzetten naar het NP-formalisme. Daar herschrijven we de NP Ricci-, Bianchi- en Einstein Maxwellvergelijkingen alsook de commutatorrelaties als differentiaalvergelijkingen voor de spincoëfficiënten en tensorcomponenten. Een volgende stap is het introduceren van geschikte coördinaten en het integreren van de differentiaalvergelijkingen. Gebruik makend van de Cartanvergelijkingen vinden we zo éénvormen die de klasse van oplossingen beschrijven. Met behulp hiervan kunnen we een overeenkomstig lijnelement opstellen. Soms is het mogelijk om deze oplossingen te herschrijven door middel van coördinatentransformaties. Dit kan helpen bij het vergelijken van twee metrieken of bij het vinden van limietgevallen.

Het is ook mogelijk om bovenstaande problemen direct m.b.v. het NP-formalisme te behandelen. Het voordeel van de tweevoudige aanpak, waarbij eerst informatie wordt verzameld via GHP, vooraleer over te gaan naar NP, is onder andere dat de verzameling vergelijkingen in GHP compacter is. Daardoor is het eenvoudiger om te besluiten of er al dan niet oplossingen bestaan voor het behandelde vraagstuk. Bovendien is het in GHP niet nodig om rekening te houden met rotatie- en boostvrijheden van het *systeem. Ook kan men aan de hand van een GHP-analyse snel een besluit trekken over het aantal onafhankelijke vrije functies dat zal opteden in de uiteindelijke metriek, dit verhindert het nodeloos zoeken naar coördinaat-transformaties die* misschien *een functie zouden kunnen elimineren van het systeem. Ten slotte is het vaak eenvoudiger om 'natuurlijke' opsplitsingen binnen het probleem te vinden via GHP, in plaats van via NP (bijvoorbeeld de opsplitsing in sferische en cilindrische klasse voor Newman Tamburino metrieken).*

Alle berekeningen zijn uitgevoerd in Maple (een geregistreerd handelsmerk van Waterloo Maple Inc.), waarbij voornamelijk gebruik gemaakt werd van de NP- en GHP-pakketten, geschreven door Norbert Van den Bergh. De broncode van deze pakketten kan bekomen worden via http://users.ugent.be/~nvdbergh/rug/gr/

Summary

Einstein's theory of general relativity is generally accepted to be the most successful theory of gravitation. It is mathematically described by Einstein's field equation

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \kappa_0 T_{ab}, \tag{2}$$

a tensorial equation, which translates into a system of non-linear, coupled partial differential equations when expressed in coordinates or in a frame. The main ingredient is the metric tensor g_{ab}*, from which every other object can be constructed. Finding solutions of these equations is not an easy task. In fact any metric is a solution of (2), if no restrictions are imposed on the energy momentum tensor, since (2) then becomes just a definition of* T_{ab}*. We will thus first make some assumptions about* T_{ab}*. Furthermore, we will impose conditions on the metric, before 'solving' Einstein's field equations.*

Based on the imposed assumptions and conditions one can collect the solutions into different classes. In this thesis we examine some solutions that belong to the set of Newman Tamburino solutions, as well as some solutions that belong to Kundt's class.

In 1960 Robinson and Trautman published the general solutions for the class of metrics containing hypersurface orthogonal and thus non-rotating and geodesic null congruences with non-vanishing divergence but with vanishing shear (Robinson and Trautman, 1960).
Newman and Tamburino (1962) tried to generalise the Robinson Trautman metrics by removing the condition of vanishing shear, hoping to find a more general solution. The assumption of non-vanishing shear however, leads to additional conditions that do not appear in the non-shearing case, thus preventing the finding of the Robinson Trautman solutions as a limit case.

Kundt (1961) has considered the class of metrics containing geodesic null congruences with vanishing divergence, curl and shear.

This thesis contains six chapters: in chapter one we give a short introduction with the mathematical background and concepts that will be used in the thesis. In this chapter we introduce the Newman Penrose (NP-) and the Geroch Held Penrose (GHP-) formalisms. We also explain the equivalence problem: given two metrics in different coordinate systems, how can one decide whether or not they are (locally) equivalent? CLASSI, a classification program written by Jan Åman, is a useful tool in the answer to this kind of problems. Explicit examples will be given in later chapters.

In the second chapter we re-examine the Petrov type D pure radiation solutions of the Robinson Trautman family. These were originally examined by Frolov and Khlebnikov (1975). Their solutions are subdivided into three different classes, A, B and C, of which the A- and B-classes are divided even further into three and five subclasses, respectively. In chapter two of this thesis we remark that the A-class is in fact redundant in the sense that the A-metrics can be found as special cases of the B-metrics. Furthermore we explicitly re-integrate the C-class. The latter is incorrect in the original paper and a corrected version can be found in chapter two.

The third chapter covers our work on Newman Tamburino solutions in the presence of an aligned Einstein Maxwell field. We have completely solved this problem. We also show that there exist no so-called 'cylindrical' solutions in the presence of a non-aligned Einstein Maxwell field. Whether or not there exist 'spherical' solutions in the presence of a non-aligned Einstein Maxwell field remains an open question.

Chapter four is dedicated to vacuum solutions of the Newman Tamburino family. These solutions were published originally in (Newman and Tamburino, 1962) but the cylindrical class given there is incorrect and it is possible to write the correct version of the solution in a much more elegant way. Even the spherical solutions can be written in a more compact way, as we will show in this thesis. Finally we will also give the relation between the cylindrical Einstein Maxwell solutions and the cylindrical vacuum solutions.

Before we give our conclusions in chapter six, we first treat the Petrov type D pure radiation solutions of the Kundt family (chapter five), for which we present some new metric families. We will also give here a complete classification of the line elements for this family of solutions. The input files for CLASSI can be found in the appendix.

In chapters two to five we start with a brief introduction to the problem, after which we implement the basic assumptions in GHP. This allows us to extract all spin (or spatial) rotation and boost invariant information, which we then translate into the NP-formalism. The next step is to rewrite all NP Ricci, Bianchi and Einstein Maxwell equations and the commutator relations in a suitable form (i.e. differential equations for the spin coefficients and the tensor components). Afterwards we introduce coordinates and integrate the differential equations we previously obtained. Making use of the first Cartan structure equations, we then find the metric one forms, that correspond to the problem. These one forms lead to the metric line element in an unambiguous way. Sometimes it is possible to rewrite the solutions in a more compact way by applying a suitable coordinate transformation. This may help to find relations between metrics that have been published earlier, or to examine limit cases.

It is also possible to examine the above problems directly using the NP formalism. One of the advantages of a two-fold approach, where we first gather information through GHP, before going to NP, is that the set of equations in GHP is more compact. It is then easier to conclude whether or not there exist solutions to the given problem. Apart from that, there is no need to take into account the degrees of freedom in boost and rotation of the frame, when making use of GHP. Also, making a GHP-analysis, one can easily determine the number of distinguishing free functions that will occur in the final metric. This prevents the search for coordinate transformation which might *eliminate a function of the system. Finally GHP allows one to find the 'natural'* subclasses *of a problem more easily, compared to NP (for example the splitting in a spherical and cylindrical class of the Newman Tamburino metrics).*

All computations have been done in Maple (a registered trademark of Wa-

terloo Maple Inc.), where we extensively made use of the NP- and GHP-packages, written by Norbert Van den Bergh. The source code for those packages can be retrieved from http://users.ugent.be/˜nvdbergh/rug/gr/

Contents

Chapter 1

Introduction

The purpose of this chapter is to define the notation used in this thesis and to introduce some basic tools. For a more detailed description, we refer to Stephani et al. (2003), from which we have extracted some parts in order to make this work self contained.

1.1 Newman Penrose and Geroch Held Penrose formalisms

A useful tool in the construction of exact solutions is the null tetrad formalism due to Newman and Penrose (1962). In this formalism a set of first order differential equations has to be solved and Lorentz transformations can be used to simplify the field equations. The equations are written out explicitly without the use of the index and summation conventions, thus allowing one to concentrate on individual 'scalar' equations with — often — particular physical or geometric significance. Another interesting advantage of the formalism is that it allows one to extract invariant information about the gravitational field without using coordinates. A modified calculus, adapted to physical situations in which a pair of real null directions is naturally selected at each space-time point, was developed by Geroch et al. (1973). In this *GHP-formalism* the formulae are even simpler than in the standard Newman Penrose formalism.

1.1.1 The Newman Penrose formalism

The basic ingredient of the NP-equations is the complex null tetrad $\{\mathbf{e}_a\} = (\mathbf{m}, \overline{\mathbf{m}}, \mathbf{l}, \mathbf{k})$, with $\mathbf{k}$ and $\mathbf{l}$ real null vectors and $\mathbf{m}$ and its complex conjugate $\overline{\mathbf{m}}$ complex null vectors, which span the two-spaces orthogonal to $\mathbf{k}$ and $\mathbf{l}$. The orthogonality properties of the vectors are

$$k^a l_a = -m^a \overline{m}_a = -1,$$

$$m^a m_a = \overline{m}^a \overline{m}_a = k^a k_a = l^a l_a = 0,$$

$$k^a m_a = k^a \overline{m}_a = l^a m_a = l^a \overline{m}_a = 0.$$

In this notation the metric takes the form

$$g_{ab} = 2m_{(a}\overline{m}_{b)} - 2k_{(a}l_{b)} \qquad \text{or} \qquad g_{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

where round brackets are used to denote symmetrisation (for antisymmetrisation we will use square brackets, as in (1.14)):

$$X_{(ab)} \equiv \frac{1}{2}\left(X_{ab} + X_{ba}\right), \qquad X_{[ab]} \equiv \frac{1}{2}\left(X_{ab} - X_{ba}\right).$$

In terms of a coordinate basis, a complex null tetrad $\{\mathbf{e}_a\}$ and its dual $\{\boldsymbol{\omega}^a\}$ take the form

$$\begin{aligned} \mathbf{e_1} &= m^i \frac{\partial}{\partial x^i}, \quad \mathbf{e_2} = \overline{m}^i \frac{\partial}{\partial x^i}, \quad \mathbf{e_3} = l^i \frac{\partial}{\partial x^i}, \quad \mathbf{e_4} = k^i \frac{\partial}{\partial x^i}, \\ \boldsymbol{\omega}^\mathbf{1} &= \overline{m}_i \mathrm{d}x^i, \quad \boldsymbol{\omega}^\mathbf{2} = m_i \mathrm{d}x^i, \quad \boldsymbol{\omega}^\mathbf{3} = -k_i \mathrm{d}x^i, \quad \boldsymbol{\omega}^\mathbf{4} = -l_i \mathrm{d}x^i. \end{aligned}$$

The essential space-time structures that are used in the NP-formalism are derived, in conjunction with the null tetrad, by use of the (unique Levi Civita or Christoffel) torsion-free covariant derivative operator ∇_a, which annihilates g_{ab}. These are

- the twelve spin coefficients which are independent complex linear combinations of the connection coefficients (the tetrad components of the covariant derivatives of the tetrad vectors, see (1.1)),
- the five complex tetrad components of the Weyl tensor,

- the components of the Ricci tensor (incl. its trace).

The standard version of the Einstein equations becomes, in this formalism, a large number of complex first order partial differential equations which are grouped into three different but interacting sets: the first Cartan equations, the Newman Penrose Ricci equations and the Bianchi equations. The first Cartan equations relate the spin coefficients to the derivatives of the tetrad components, the Newman Penrose equations describe the relationship of the curvature tensor to derivatives of the connection (the spin coefficients) and the Bianchi equations relate the spin coefficients to the derivatives of the curvature tensor components. It is important to note that these equations are not integrated one set at a time, but rather together, going back and forth between the sets.
Using the connection coefficients ${\Gamma^c}_{ab}$, defined by

$$\nabla_b \mathbf{e}_a = {\Gamma^c}_{ab} \mathbf{e}_c, \tag{1.1}$$

we can now introduce the spin coefficients:

$$\begin{aligned}
-\kappa &\equiv \Gamma_{144} = k_{a;b} m^a k^b = m^a \mathrm{D} k_a, \\
-\rho &\equiv \Gamma_{142} = k_{a;b} m^a \overline{m}^b = m^a \overline{\delta} k_a, \\
-\sigma &\equiv \Gamma_{141} = k_{a;b} m^a m^b = m^a \delta k_a, \\
-\tau &\equiv \Gamma_{143} = k_{a;b} m^a l^b = m^a \Delta k_a, \\
\nu &\equiv \Gamma_{233} = l_{a;b} \overline{m}^a l^b = \overline{m}^a \Delta l_a, \\
\mu &\equiv \Gamma_{231} = l_{a;b} \overline{m}^a m^b = \overline{m}^a \delta l_a, \\
\lambda &\equiv \Gamma_{232} = l_{a;b} \overline{m}^a \overline{m}^b = \overline{m}^a \overline{\delta} l_a, \\
\pi &\equiv \Gamma_{234} = l_{a;b} \overline{m}^a k^b = \overline{m}^a \mathrm{D} l_a, \\
-\epsilon &\equiv \frac{1}{2}\left(\Gamma_{344} - \Gamma_{214}\right) = \frac{1}{2}\left(k_{a;b} l^a k^b - m_{a;b} \overline{m}^a k^b\right) = \frac{1}{2}\left(l^a \mathrm{D} k_a - \overline{m}^a \mathrm{D} m_a\right), \\
-\beta &\equiv \frac{1}{2}\left(\Gamma_{341} - \Gamma_{211}\right) = \frac{1}{2}\left(k_{a;b} l^a m^b - m_{a;b} \overline{m}^a m^b\right) = \frac{1}{2}\left(l^a \delta k_a - \overline{m}^a \delta m_a\right), \\
\gamma &\equiv \frac{1}{2}\left(\Gamma_{433} - \Gamma_{123}\right) = \frac{1}{2}\left(l_{a;b} k^a l^b - \overline{m}_{a;b} m^a l^b\right) = \frac{1}{2}\left(k^a \Delta l_a - m^a \Delta \overline{m}_a\right), \\
\alpha &\equiv \frac{1}{2}\left(\Gamma_{432} - \Gamma_{122}\right) = \frac{1}{2}\left(l_{a;b} k^a \overline{m}^b - \overline{m}_{a;b} m^a \overline{m}^b\right) = \frac{1}{2}\left(k^a \overline{\delta} l_a - m^a \overline{\delta} \overline{m}_a\right),
\end{aligned} \tag{1.2}$$

where we have used the notation

$$D \equiv k^a \nabla_a, \qquad \Delta \equiv l^a \nabla_a,$$
$$\delta \equiv m^a \nabla_a, \qquad \overline{\delta} \equiv \overline{m}^a \nabla_a,$$

for the directional derivatives D, Δ, δ, $\overline{\delta}$.

In the present notation the commutators $[\mathbf{e}_a, \mathbf{e}_b] = \mathrm{D}^c{}_{ab}\mathbf{e}_c$, with $\mathrm{D}^c{}_{ab} = -2\Gamma^c{}_{[ab]}$, are given explicitly as follows:

$$\begin{aligned}
(\Delta \mathrm{D} - \mathrm{D}\Delta) &= (\gamma + \overline{\gamma})\mathrm{D} + (\epsilon + \overline{\epsilon})\Delta - (\tau + \overline{\pi})\overline{\delta} - (\overline{\tau} + \pi)\delta, && (1.3)\\
(\delta \mathrm{D} - \mathrm{D}\delta) &= (\overline{\alpha} + \beta - \overline{\pi})\mathrm{D} + \kappa\Delta - \sigma\overline{\delta} - (\overline{\rho} + \epsilon - \overline{\epsilon})\delta, && (1.4)\\
(\delta\Delta - \Delta\delta) &= -\overline{\nu}\mathrm{D} + (\tau - \overline{\alpha} - \beta)\Delta + \overline{\lambda\delta} + (\mu - \gamma + \overline{\gamma})\delta, && (1.5)\\
\left(\overline{\delta}\delta - \delta\overline{\delta}\right) &= (\overline{\mu} - \mu)\mathrm{D} + (\overline{\rho} - \rho)\Delta - (\overline{\alpha} - \beta)\overline{\delta} - (\overline{\beta} - \alpha)\delta. && (1.6)
\end{aligned}$$

We will also introduce the connection one forms $\mathbf{\Gamma}^a{}_b \equiv \Gamma^a{}_{bc}\boldsymbol{\omega}^c$ ($\Gamma_{(ab)} = 0$), which enable us to write the exterior derivative of the basis one forms

$$\mathrm{d}\boldsymbol{\omega}^a = \boldsymbol{\omega}^a_{i,j}\mathrm{d}x^j \wedge \mathrm{d}x^i = \boldsymbol{\omega}^a_{i;j}\mathrm{d}x^j \wedge \mathrm{d}x^i = \Gamma^a{}_{bc}\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c$$

in the compact form

$$\mathrm{d}\boldsymbol{\omega}^a = -\mathbf{\Gamma}^a{}_b \wedge \boldsymbol{\omega}^b, \qquad (1.7)$$

due to Cartan (the first Cartan equations).

It is interesting to note how the spin coefficients transform under tetrad transformations. If the null direction $\mathbf{k}$ is fixed, the (special) Lorentz transformations preserving this direction

$$\mathbf{k}' = \mathbf{k}, \qquad \mathbf{m}' = \mathbf{m} + B\mathbf{k}, \qquad \mathbf{l}' = \mathbf{l} + B\overline{\mathbf{m}} + \overline{B}\mathbf{m} + B\overline{B}\mathbf{k}, \qquad (1.8)$$

and

$$\mathbf{k}' = A\mathbf{k}, \qquad \mathbf{m}' = \mathrm{e}^{i\theta}\mathbf{m}, \qquad \mathbf{l}' = A^{-1}\mathbf{l},$$

transform the spin coefficients as follows:

$$\begin{aligned}
\kappa' &= \kappa, & \tau' &= \tau + \overline{B}\sigma + B\rho + \overline{B}B\kappa,\\
\rho' &= \rho + \overline{B}\kappa, & \alpha' &= \alpha + \overline{B}(\epsilon + \rho) + \overline{B}^2\kappa,\\
\sigma' &= \sigma + B\kappa, & \beta' &= \beta + \overline{B}\sigma + B\epsilon + \overline{B}B\kappa,\\
\epsilon' &= \epsilon + \overline{B}\kappa, & \pi' &= \pi + 2\overline{B}\epsilon + \overline{B}^2\kappa + \mathrm{D}\overline{B}, \qquad (1.9)
\end{aligned}$$

$$\begin{aligned}
\gamma' &= \gamma + B\alpha + \overline{B}(\tau+\beta) + \overline{B}B(\rho+\epsilon) + \overline{B}^2\sigma + \overline{B}^2 B\kappa, \\
\lambda' &= \lambda + \overline{B}(\pi+2\alpha) + \overline{B}^2(\rho+2\epsilon) + \overline{B}^3\kappa + \overline{B}\mathrm{D}\overline{B} + \delta\overline{B}, \\
\mu' &= \mu + 2\overline{B}\beta + B\pi + \overline{B}^2\sigma + 2\overline{B}B\epsilon + \overline{B}^2 B\kappa + B\mathrm{D}\overline{B} + \delta\overline{B}, \\
\nu' &= \nu + \overline{B}(2\gamma+\mu) + B\lambda + \overline{B}^2(\tau+2\beta) + \overline{B}B(\pi+2\alpha) + \overline{B}^3\sigma \\
&+ \overline{B}^2 B(\rho+2\epsilon) + \overline{B}^3 B\kappa + \Delta\overline{B} + \overline{B}\delta\overline{B} + B\overline{\delta}\overline{B} + \overline{B}B\mathrm{D}\overline{B},
\end{aligned}$$

and

$$\begin{aligned}
&\kappa' = A^2 \mathrm{e}^{\mathrm{i}\theta}\kappa, \quad \nu' = A^{-2}\mathrm{e}^{-\mathrm{i}\theta}\nu, \quad \epsilon' = A\left[\epsilon + \frac{1}{2}\mathrm{D}\left(\ln A + \mathrm{i}\theta\right)\right], \\
&\rho' = A\rho, \quad \mu' = A^{-1}\mu, \quad \beta' = \mathrm{e}^{\mathrm{i}\theta}\left[\beta + \frac{1}{2}\delta\left(\ln A + \mathrm{i}\theta\right)\right], \\
&\sigma' = A\mathrm{e}^{2\mathrm{i}\theta}\sigma, \quad \lambda' = A^{-1}\mathrm{e}^{-2\mathrm{i}\theta}\lambda, \quad \gamma' = A^{-1}\left[\gamma + \frac{1}{2}\Delta\left(\ln A + \mathrm{i}\theta\right)\right], \\
&\tau' = \mathrm{e}^{\mathrm{i}\theta}\tau, \quad \pi' = \mathrm{e}^{-\mathrm{i}\theta}\pi, \quad \alpha' = \mathrm{e}^{-\mathrm{i}\theta}\left[\alpha + \frac{1}{2}\overline{\delta}\left(\ln A + \mathrm{i}\theta\right)\right].
\end{aligned} \tag{1.10}$$

These transformations are called *null rotations* with complex parameter B, and *boosts* and *(spin or spatial) rotations* with parameters A and θ, respectively.

To obtain the full six-parameter group of (special) Lorentz transformations, we should also mention the null rotations that leave $\mathbf{l}$ fixed. For a complex parameter E, these null rotations transform the null tetrad as follows:

$$\mathbf{l}' = \mathbf{l}, \qquad \mathbf{m}' = \mathbf{m} + E\mathbf{l}, \quad \mathbf{k}' = \mathbf{k} + E\overline{\mathbf{m}} + \overline{E}\mathbf{m} + E\overline{E}\mathbf{l}. \tag{1.11}$$

At this point, we will introduce the Riemann tensor (also 'curvature tensor'), $\mathbf{R} = R^a{}_{bcd}\mathbf{e}_a \otimes \boldsymbol{\omega}^b \otimes \boldsymbol{\omega}^c \otimes \boldsymbol{\omega}^d$. It is a tensor of type $(1,3)$, mapping an ordered set $(\boldsymbol{\sigma};\, \mathbf{u}, \mathbf{v}, \mathbf{w})$ of a one form $\boldsymbol{\sigma}$ and three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ into the real number

$$\sigma_a u^b v^c w^d R^a{}_{bcd} = \sigma_a \left(u^a{}_{;cd} - u^a{}_{;dc}\right) w^c v^d,$$

$u^a{}_{;cd}$ being short hand for $(\nabla_d \nabla_c \mathbf{u})^a$.
As the components σ_a, w^c, v^d can be chosen arbitrarily, we arrive at the Ricci identity

$$u^a{}_{;cd} - u^a{}_{;dc} = u^b R^a{}_{bcd}.$$

We will also introduce the tetrad components of the traceless Ricci tensor ($S_{ab} \equiv R_{ab} - g_{ab}R/4$) and the Weyl tensor.
The tetrad components of the Riemann tensor can be expressed in terms of the connection coefficients by

$$R^a{}_{bcd} = \Gamma^a{}_{bd|c} - \Gamma^a{}_{bc|d} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed} - \mathrm{D}^e{}_{cd}\Gamma^a{}_{be}, \tag{1.12}$$

which is equivalent to the second Cartan equation

$$\mathrm{d}\mathbf{\Gamma}^a{}_b + \mathbf{\Gamma}^a{}_c \wedge \mathbf{\Gamma}^c{}_b = \frac{1}{2} R^a{}_{bcd}\boldsymbol{\omega}^c \wedge \boldsymbol{\omega}^d. \tag{1.13}$$

The components (1.12) of the Riemann tensor satisfy the symmetry relations

$$R_{abcd} = R_{cdab}, \qquad R^a{}_{bcd} = -R^a{}_{bdc}, \qquad R^a{}_{[bcd]} = 0. \tag{1.14}$$

The covariant derivatives of the Riemann tensor obey the Bianchi identities

$$R^a{}_{b[cd;e]} = 0. \tag{1.15}$$

By contraction we obtain the identities

$$R^a{}_{bcd;a} + 2R_{b[c;d]} = 0,$$

where the components R_{bd} of the Ricci tensor are defined by $R_{bd} \equiv R^a_{\ bad}$.
According to Einstein's general theory of relativity, the curvature of spacetime is related to the distribution of matter. Specifically, components of the Ricci tensor are algebraically related to the local energy-momentum tensor T_{ab} by Einstein's field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \kappa_0 T_{ab}, \tag{1.16}$$

in which Λ is the cosmological constant. This can also be rewritten in terms of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$. The Einstein equations (1.16) together with the Bianchi identities (1.15) imply the relation

$$\kappa_0 T^{ab}_{\ ;b} = \left(R^{ab} - \frac{1}{2}Rg^{ab}\right)_{;b} = 0.$$

The relationship between the Riemann tensor, the Weyl tensor C_{abcd} and the Ricci tensor is given by

$$R_{abcd} = C_{abcd} + \frac{1}{2}\left(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}\right) + \frac{R}{12}g_{abcd}, \tag{1.17}$$

where we have used the following abbreviations:

$$S_{ab} \equiv R_{ab} - \frac{1}{4} R g_{ab}, \qquad R \equiv R^a{}_a, \qquad g_{abcd} = 2(g_{ad}g_{bc} - g_{ac}g_{bd}).$$

R and S_{ab} denote the trace and the traceless part of the Ricci tensor R_{ab}, respectively.

The decomposition (1.17) defines the completely traceless Weyl curvature (Weyl conformal tensor) C_{abcd}, which represents the part of R_{abcd} not pointwise determined by local matter; it is often referred to as gravitational radiation. C_{abcd} has the same symmetries (1.14) as the Riemann tensor, and it satisfies the trace-free property

$$C^a_{\ bad} = 0. \tag{1.18}$$

These identities reduce the number of independent components of C_{abcd} to ten. The components of the Weyl curvature are expressed by the five complex scalars Ψ_0, Ψ_1, ..., Ψ_4. The Ricci tensor components will be addressed as Φ_{ij}, $i, j = 0, 1, 2$ and R, of which Φ_{00}, Φ_{11}, Φ_{22} and R are real valued, and where (Φ_{01}, Φ_{10}), (Φ_{02}, Φ_{20}) and (Φ_{12}, Φ_{21}) are complex valued conjugate pairs:

$$\Psi_0 \equiv C_{abcd} k^a m^b k^c m^d, \qquad \Phi_{00} \equiv \frac{S_{ab}}{2} k^a k^b = \overline{\Phi}_{00} = \frac{R_{44}}{2}, \tag{1.19}$$

$$\Psi_1 \equiv C_{abcd} k^a l^b k^c m^d, \qquad \Phi_{01} \equiv \frac{S_{ab}}{2} k^a m^b = \overline{\Phi}_{10} = \frac{R_{41}}{2}, \tag{1.20}$$

$$\Psi_2 \equiv -C_{abcd} k^a m^b l^c \overline{m}^d, \qquad \Phi_{02} \equiv \frac{S_{ab}}{2} m^a m^b = \overline{\Phi}_{20} = \frac{R_{11}}{2}, \tag{1.21}$$

$$\Psi_3 \equiv C_{abcd} l^a k^b l^c \overline{m}^d, \qquad \Phi_{11} \equiv \frac{S_{ab}}{4} (k^a l^b + m^a \overline{m}^b) = \frac{R_{43} + R_{12}}{4}, \tag{1.22}$$

$$\Psi_4 \equiv C_{abcd} l^a \overline{m}^b l^c \overline{m}^d, \qquad \Phi_{12} \equiv \frac{S_{ab}}{2} l^a m^b = \overline{\Phi}_{21} = \frac{R_{31}}{2}, \tag{1.23}$$

$$\Phi_{22} \equiv \frac{S_{ab}}{2} l^a l^b = \overline{\Phi}_{22} = \frac{R_{33}}{2}. \tag{1.24}$$

The Ψ_2-component can also be written in a different (but due to (1.18) equivalent) form

$$\Psi_2 \equiv \frac{1}{2} C_{abcd} k^a l^b \left(k^c l^d - m^c \overline{m}^d \right).$$

Geometrically the Ψ_i-components can be interpreted (in vacuum space-times) as components of gravitational radiation (Pirani, 1957):

Ψ_0 : transverse component propagating in the $\mathbf{l}$−direction,

Ψ_1 : longitudinal component in the $\mathbf{l}$−direction,

Ψ_2 : so called *Coulomb* component,

Ψ_3 : longitudinal component in the $\mathbf{k}$−direction,

Ψ_4 : transverse component propagating in the $\mathbf{k}$−direction.

The Weyl and Ricci components obey the following transformation laws under null rotations about $\mathbf{k}$ and under boosts and rotations (Carmeli, 1977; Stephani et al., 2003):

$$\begin{aligned}
\Psi'_0 &= \Psi_0,\\
\Psi'_1 &= \Psi_1 + \overline{B}\Psi_0,\\
\Psi'_2 &= \Psi_2 + 2\overline{B}\Psi_1 + \overline{B}^2\Psi_0,\\
\Psi'_3 &= \Psi_3 + 3\overline{B}\Psi_2 + 3\overline{B}^2\Psi_1 + \overline{B}^3\Psi_0,\\
\Psi'_4 &= \Psi_4 + 4\overline{B}\Psi_3 + 6\overline{B}^2\Psi_2 + 4\overline{B}^3\Psi_1 + \overline{B}^4\Psi_0,\\
\Phi'_{00} &= \Phi_{00},\\
\Phi'_{01} &= \Phi_{01} + B\Phi_{00},\\
\Phi'_{02} &= \Phi_{02} + 2B\Phi_{01} + B^2\Phi_{00},\\
\Phi'_{11} &= \Phi_{11} + \overline{B}\Phi_{01} + B\Phi_{10} + B\overline{B}\Phi_{00},\\
\Phi'_{12} &= \Phi_{12} + 2B\Phi_{11} + B^2\Phi_{10} + \overline{B}\Phi_{02} + 2B\overline{B}\Phi_{01} + B^2\overline{B}\Phi_{00},\\
\Phi'_{22} &= \Phi_{22} + 2\left(\overline{B}\Phi_{12} + B\Phi_{21}\right) + \overline{B}^2\Phi_{02} + B^2\Phi_{20} + 4B\overline{B}\Phi_{11}\\
&\quad + 2B\overline{B}\left(\overline{B}\Phi_{01} + B\Phi_{10}\right) + B^2\overline{B}^2\Phi_{00},
\end{aligned}$$

and

$$\begin{aligned}
\Psi'_0 &= A^2 \mathrm{e}^{2\mathrm{i}\theta}\Psi_0, & \Phi'_{00} &= A^2\Phi_{00},\\
\Psi'_1 &= A\mathrm{e}^{\mathrm{i}\theta}\Psi_1, & \Phi'_{01} &= A\mathrm{e}^{\mathrm{i}\theta}\Phi_{01},\\
\Psi'_2 &= \Psi_2, & \Phi'_{11} &= \Phi_{11},\\
\Psi'_3 &= A^{-1}\mathrm{e}^{-\mathrm{i}\theta}\Psi_3, & \Phi'_{12} &= A^{-1}\mathrm{e}^{\mathrm{i}\theta}\Phi_{12},\\
\Psi'_4 &= A^{-2}\mathrm{e}^{-2\mathrm{i}\theta}\Psi_4, & \Phi'_{22} &= A^{-2}\Phi_{22},\\
& & \Phi'_{02} &= \mathrm{e}^{2\mathrm{i}\theta}\Phi_{02}.
\end{aligned}$$

Under null rotations about $\mathbf{l}$ (1.11) $\Psi_0,...,\Psi_4$ transform as

$$\Psi_4' = \Psi_4,$$

$$\Psi_3' = \Psi_3 + E\Psi_4,$$

$$\Psi_2' = \Psi_2 + 2E\Psi_3 + E^2\Psi_4,$$

$$\Psi_1' = \Psi_1 + 3E\Psi_2 + 3E^2\Psi_3 + E^3\Psi_4,$$

$$\Psi_0' = \Psi_0 + 4E\Psi_1 + 6E^2\Psi_2 + 4E^3\Psi_3 + E^4\Psi_4.$$

We now have all the necessary information to write down the first Cartan equations (1.7), the Ricci equations (1.13) and the Bianchi identities (1.15) in their explicit forms.

Let us start with the first Cartan equations. In terms of the basis one forms and the spin coefficients, they read

$$\begin{aligned} \mathrm{d}\boldsymbol{\omega}^1 &= (\alpha-\overline{\beta})\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2 - \lambda\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3 + \overline{\sigma}\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^4 + (\gamma-\overline{\gamma}-\mu)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3 \\ &+ (\epsilon-\overline{\epsilon}+\overline{\rho})\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^4 + (\overline{\tau}+\pi)\,\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4, \end{aligned}$$

$$\begin{aligned} \mathrm{d}\boldsymbol{\omega}^2 &= (\beta-\overline{\alpha})\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2 - \overline{\lambda}\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3 + \sigma\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^4 + (\overline{\gamma}-\gamma-\overline{\mu})\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3 \\ &+ (\overline{\epsilon}-\epsilon+\rho)\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^4 + (\overline{\pi}+\tau)\,\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4, \end{aligned}$$

$$\begin{aligned} \mathrm{d}\boldsymbol{\omega}^3 &= (\overline{\rho}-\rho)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2 + (\beta+\overline{\alpha}-\tau)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3 + (\overline{\beta}+\alpha-\overline{\tau})\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3 \\ &- \kappa\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^4 - \overline{\kappa}\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^4 - (\epsilon+\overline{\epsilon})\,\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4, \end{aligned}$$

$$\begin{aligned} \mathrm{d}\boldsymbol{\omega}^4 &= (\overline{\mu}-\mu)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2 + \overline{\nu}\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3 + \nu\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3 - (\gamma+\overline{\gamma})\,\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4 \\ &+ (\overline{\pi}-\overline{\alpha}-\beta)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^4 + (\pi-\alpha-\overline{\beta})\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^4. \end{aligned}$$

The Ricci identities, often called the Newman Penrose equations, are given by:

$$D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\,\bar{\sigma} + (\epsilon + \bar{\epsilon})\,\rho - \bar{\kappa}\tau - \left(3\,\alpha + \bar{\beta} - \pi\right)\kappa + \Phi_{00}, \tag{1.25}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho})\,\sigma + (3\,\epsilon - \bar{\epsilon})\,\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\,\beta)\,\kappa + \Psi_0, \tag{1.26}$$

$$D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{1.27}$$

$$D\alpha - \bar{\delta}\epsilon = (\rho + \bar{\epsilon} - 2\epsilon)\,\alpha + \beta\,\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + (\epsilon + \rho)\,\pi + \Phi_{10}, \tag{1.28}$$

$$D\beta - \delta\epsilon = (\alpha + \pi)\,\sigma + (\bar{\rho} - \bar{\epsilon})\,\beta - (\mu + \gamma)\,\kappa + (\bar{\pi} - \bar{\alpha})\,\epsilon + \Psi_1, \tag{1.29}$$

$$D\gamma - \Delta\epsilon = (\tau + \bar{\pi})\,\alpha + (\bar{\tau} + \pi)\,\beta - (\epsilon + \bar{\epsilon})\,\gamma - (\gamma + \bar{\gamma})\,\epsilon + \tau\,\pi - \nu\,\kappa + \Psi_2 - \frac{R}{24} + \Phi_{11}, \tag{1.30}$$

$$D\lambda - \bar{\delta}\pi = \rho\,\lambda + \bar{\sigma}\mu + \pi^2 + \left(\alpha - \bar{\beta}\right)\pi - \nu\,\bar{\kappa} + (\bar{\epsilon} - 3\,\epsilon)\,\lambda + \Phi_{20}, \tag{1.31}$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \sigma\,\lambda + \pi\,\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - (\bar{\alpha} - \beta)\pi - \nu\,\kappa + \Psi_2 + \frac{R}{12}, \tag{1.32}$$

$$D\nu - \Delta\pi = (\bar{\tau} + \pi)\mu + (\tau + \bar{\pi})\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 + \Phi_{21}, \tag{1.33}$$

$$\Delta\lambda - \bar{\delta}\nu = (\bar{\gamma} - 3\,\gamma - \mu - \bar{\mu})\,\lambda + \left(3\,\alpha + \bar{\beta} + \pi - \bar{\tau}\right)\nu - \Psi_4, \tag{1.34}$$

$$\delta\rho - \bar{\delta}\sigma = (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}, \tag{1.35}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\,\rho - \sigma\,\lambda + \alpha\,\bar{\alpha} + \beta\,\bar{\beta} - 2\,\alpha\,\beta + (\rho - \bar{\rho})\,\gamma + (\mu - \bar{\mu})\,\epsilon - \Psi_2 + \frac{R}{24} + \Phi_{11}, \tag{1.36}$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}, \tag{1.37}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\,\bar{\lambda} + (\gamma + \bar{\gamma})\,\mu - \bar{\nu}\pi + (\tau - \bar{\alpha} - 3\,\beta)\,\nu + \Phi_{22}, \tag{1.38}$$

$$\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - 2\beta)\gamma + \mu\tau - \sigma\,\nu - \epsilon\,\bar{\nu} + (\bar{\gamma} + \mu)\beta + \alpha\bar{\lambda} + \Phi_{12}, \tag{1.39}$$

$$\delta\tau - \Delta\sigma = \mu\,\sigma + \bar{\lambda}\rho + (\tau - \bar{\alpha} + \beta)\,\tau - (3\,\gamma - \bar{\gamma})\,\sigma - \kappa\,\bar{\nu} + \Phi_{02}, \tag{1.40}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\,\bar{\mu} - \sigma\,\lambda + (\gamma + \bar{\gamma})\rho - (\bar{\tau} + \alpha - \bar{\beta})\tau + \nu\,\kappa - \Psi_2 - \frac{R}{12}, \tag{1.41}$$

$$\Delta\alpha - \bar{\delta}\gamma = (\epsilon + \rho)\,\nu - (\tau + \beta)\,\lambda + (\bar{\gamma} - \bar{\mu})\,\alpha + \left(\bar{\beta} - \bar{\tau}\right)\gamma - \Psi_3. \tag{1.42}$$

The remaining set of equations to be satisfied are the Bianchi identities

$R_{ab[cd;e]} = 0$. In the Newman Penrose formalism these equations read

$$\bar{\delta}\Psi_0 - \mathrm{D}\Psi_1 + \mathrm{D}\Phi_{01} - \delta\Phi_{00} = (4\alpha - \pi)\,\Psi_0 - 2\,(2\rho + \epsilon)\,\Psi_1 + 3\kappa\Psi_2$$
$$+ (\bar{\pi} - 2\bar{\alpha} - 2\beta)\,\Phi_{00} + 2\,(\epsilon + \bar{\rho})\,\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}, \quad (1.43)$$

$$\Delta\Psi_0 - \delta\Psi_1 + \mathrm{D}\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\,\Psi_0 - 2\,(2\tau + \beta)\,\Psi_1 + 3\sigma\Psi_2$$
$$- \bar{\lambda}\Phi_{00} + 2\,(\bar{\pi} - \beta)\,\Phi_{01} + 2\sigma\,\Phi_{11} + (2\epsilon - 2\bar{\epsilon} + \bar{\rho})\,\Phi_{02} - 2\kappa\Phi_{12}, \quad (1.44)$$

$$3\,\bar{\delta}\Psi_1 - 3\,\mathrm{D}\Psi_2 + 2\,\mathrm{D}\Phi_{11} - 2\,\delta\Phi_{10} + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} = 3\,\lambda\,\Psi_0 - 9\,\rho\,\Psi_2$$
$$+ 6\,(\alpha - \pi)\,\Psi_1 + 6\,\kappa\,\Psi_3 + (\bar{\mu} - 2\,\mu - 2\,\gamma - 2\,\bar{\gamma})\,\Phi_{00} + 2\sigma\,\Phi_{20} - \bar{\sigma}\,\Phi_{02}$$
$$+ (2\alpha + 2\pi + 2\bar{\tau})\,\Phi_{01} + 2\,(\tau - 2\bar{\alpha} + \bar{\pi})\,\Phi_{10} + 2\,(2\bar{\rho} - \rho)\,\Phi_{11}$$
$$- 2\,\bar{\kappa}\,\Phi_{12} - 2\,\kappa\,\Phi_{21}, \quad (1.45)$$

$$3\,\Delta\Psi_1 - 3\,\delta\Psi_2 + 2\,\mathrm{D}\Phi_{12} - 2\,\delta\Phi_{11} + \bar{\delta}\Phi_{02} - \Delta\Phi_{01} = 6\,(\gamma - \mu)\,\Psi_1$$
$$+ 3\,\nu\,\Psi_0 - 9\,\tau\,\Psi_2 + 6\,\sigma\,\Psi_3 - \bar{\nu}\Phi_{00} + 2\,(\bar{\mu} - \mu - \gamma)\,\Phi_{01} - 2\,\bar{\lambda}\Phi_{10}$$
$$+ 2\,(\tau + 2\,\bar{\pi})\,\Phi_{11} + \left(2\,\alpha + 2\,\pi + \bar{\tau} - 2\,\bar{\beta}\right)\Phi_{02} + (2\,\bar{\rho} - 2\,\rho - 4\,\bar{\epsilon})\,\Phi_{12}$$
$$+ 2\,\sigma\,\Phi_{21} - 2\,\kappa\,\Phi_{22}, \quad (1.46)$$

$$3\,\bar{\delta}\Psi_2 - 3\,\mathrm{D}\Psi_3 + \mathrm{D}\Phi_{21} - \delta\Phi_{20} + 2\,\bar{\delta}\Phi_{11} - 2\,\Delta\Phi_{10} = 6\,\lambda\,\Psi_1 - 9\,\pi\,\Psi_2$$
$$+ 6\,(\epsilon - \rho)\,\Psi_3 + 3\,\kappa\,\Psi_4 - 2\,\nu\,\Phi_{00} + 2\,(\bar{\mu} - \mu - 2\,\bar{\gamma})\,\Phi_{10}$$
$$+ (2\pi + 4\bar{\tau})\,\Phi_{11} + (2\beta + 2\tau + \bar{\pi} - 2\bar{\alpha})\,\Phi_{20} - 2\bar{\sigma}\Phi_{12} + 2\,(\bar{\rho} - \rho - \epsilon)\,\Phi_{21}$$
$$- \bar{\kappa}\Phi_{22} + 2\,\lambda\,\Phi_{01}, \quad (1.47)$$

$$3\Delta\Psi_2 - 3\delta\Psi_3 + \mathrm{D}\Phi_{22} - \delta\Phi_{21} + 2\bar{\delta}\Phi_{12} - 2\Delta\Phi_{11} = 6\nu\,\Psi_1$$
$$- 9\mu\,\Psi_2 + 6\,(\beta - \tau)\,\Psi_3 + 3\sigma\,\Psi_4 - 2\nu\,\Phi_{01} - 2\bar{\nu}\Phi_{10} + 2\,(2\bar{\mu} - \mu)\,\Phi_{11}$$
$$+ 2\,\lambda\,\Phi_{02} - \bar{\lambda}\Phi_{20} + 2\,\left(\pi + \bar{\tau} - 2\,\bar{\beta}\right)\Phi_{12} + 2\,(\beta + \tau + \bar{\pi})\,\Phi_{21}$$
$$+ (\bar{\rho} - 2\,\epsilon - 2\,\bar{\epsilon} - 2\,\rho)\,\Phi_{22}, \quad (1.48)$$

$$\bar{\delta}\Psi_3 - \mathrm{D}\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = 3\lambda\Psi_2 - 2\,(\alpha + 2\pi)\,\Psi_3 - 2\,\nu\,\Phi_{10} - \bar{\sigma}\Phi_{22}$$
$$+ (4\epsilon - \rho)\,\Psi_4 + 2\lambda\,\Phi_{11} + (2\gamma - 2\,\bar{\gamma} + \bar{\mu})\,\Phi_{20} + 2\,(\bar{\tau} - \alpha)\,\Phi_{21}, \quad (1.49)$$

$$\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 3\nu\,\Psi_2 - 2\,(\gamma + 2\mu)\,\Psi_3 - 2\,\nu\,\Phi_{11} - \bar{\nu}\Phi_{20}$$
$$+ \,(4\beta - \tau)\,\Psi_4 + 2\lambda\,\Phi_{12} + 2\,(\gamma + \bar{\mu})\,\Phi_{21} + \left(\bar{\tau} - 2\bar{\beta} - 2\alpha\right)\Phi_{22}, \qquad (1.50)$$

$$\mathrm{D}\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + \Delta\Phi_{00} + \frac{1}{8}\,\mathrm{D}R = \bar{\sigma}\Phi_{02} + \sigma\,\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\,\Phi_{21}$$
$$(2\,\gamma - \mu + 2\,\bar{\gamma} - \bar{\mu})\,\Phi_{00} + (\pi - 2\,\alpha - 2\,\bar{\tau})\,\Phi_{01} + (\bar{\pi} - 2\,\bar{\alpha} - 2\,\tau)\,\Phi_{10}$$
$$+ \,2\,\,(\rho + \bar{\rho})\,\Phi_{11}, \qquad (1.51)$$

$$\mathrm{D}\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + \frac{1}{8}\,\delta R = (2\gamma - \mu - 2\bar{\mu})\,\Phi_{01}$$
$$+ \,\bar{\nu}\,\Phi_{00} - \bar{\lambda}\,\Phi_{10} + 2\,\,(\bar{\pi} - \tau)\,\Phi_{11} + \left(\pi + 2\,\bar{\beta} - 2\,\alpha - \bar{\tau}\right)\Phi_{02}$$
$$+ \,(2\,\rho + \bar{\rho} - 2\,\bar{\epsilon})\,\Phi_{12} + \sigma\,\Phi_{21} - \kappa\,\Phi_{22}, \qquad (1.52)$$

$$\mathrm{D}\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + \frac{1}{8}\,\Delta R = \nu\,\Phi_{01} + \bar{\nu}\,\Phi_{10}$$
$$-\,2\,(\mu + \bar{\mu})\,\Phi_{11} - \lambda\,\Phi_{02} - \bar{\lambda}\Phi_{20} + \left(2\pi - \bar{\tau} + 2\bar{\beta}\right)\Phi_{12}$$
$$+ \,(2\beta - \tau + 2\bar{\pi})\,\Phi_{21} + (\rho + \bar{\rho} - 2\,\epsilon - 2\,\bar{\epsilon})\,\Phi_{22}. \qquad (1.53)$$

If one considers the presence of a Maxwell field, also the Maxwell equations have to be taken into account. In the Newman Penrose formalism these are given by

$$\begin{aligned}
\mathrm{D}\Phi_1 - \overline{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2, &(1.54)\\
\mathrm{D}\Phi_2 - \overline{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2, &(1.55)\\
\delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2, &(1.56)\\
\delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2, &(1.57)
\end{aligned}$$

where we have used the notation

$$\begin{aligned}
\Phi_0 &\equiv F_{ab}k^a m^b,\\
\Phi_1 &\equiv \frac{1}{2}F_{ab}(k^a l^b + \overline{m}^a m^b),\\
\Phi_2 &\equiv F_{ab}\overline{m}^a l^b,
\end{aligned}$$

for the tetrad components of the electromagnetic field tensor.

Under null rotations about **k**, and under boosts and rotations these components transform as follows:

$$\Phi_0' = \Phi_0, \qquad \Phi_0' = Ae^{i\theta}\Phi_0,$$
$$\Phi_1' = \Phi_1 + \overline{B}\Phi_0, \qquad \Phi_1' = \Phi_1,$$
$$\Phi_2' = \Phi_2 + 2\overline{B}\Phi_1 + \overline{B}^2\Phi_0, \qquad \Phi_2' = A^{-1}e^{-i\theta}\Phi_2.$$

The Ricci tensor components of a Maxwell field are given by

$$\Phi_{ab} = \kappa_0\Phi_a\overline{\Phi}_b, \qquad a, b = 0, 1, 2,$$

where κ_0 is Einstein's gravitational constant. Taking into account the differential equations for Φ_0, Φ_1, Φ_2, obtained from the Maxwell equations, three of the Bianchi equations (1.51, 1.52 and 1.53) become identities.

We now give a brief discussion of the physical meaning of some of the quantities (1.2) and (1.19a − 1.24b), which will be used in the remainder of this thesis (see also Carmeli (1977); Stewart (1991); Griffiths and Podolsky (2009)). This significance becomes apparent when we consider the propagation of the basis vectors along **k** or **l**.

Before going into details, first note that the relations (1.2) are equivalent with

$$\begin{aligned}
k_{a;b} &= -(\gamma+\overline{\gamma})k_ak_b - (\epsilon+\overline{\epsilon})k_al_b + (\alpha+\overline{\beta})k_am_b + (\overline{\alpha}+\beta)k_a\overline{m}_b \\
&+ \overline{\tau}m_ak_b + \overline{\kappa}m_al_b - \overline{\sigma}m_am_b - \overline{\rho}m_a\overline{m}_b \\
&+ \tau\overline{m}_ak_b + \kappa\overline{m}_al_b - \sigma\overline{m}_a\overline{m}_b - \rho\overline{m}_am_b, \\
l_{a;b} &= (\gamma+\overline{\gamma})l_ak_b + (\epsilon+\overline{\epsilon})l_al_b - (\alpha+\overline{\beta})l_am_b - (\overline{\alpha}+\beta)l_a\overline{m}_b \\
&- \nu m_ak_b - \pi m_al_b + \lambda m_am_b + \mu m_a\overline{m}_b \\
&- \overline{\nu}\overline{m}_ak_b - \overline{\pi}\overline{m}_al_b + \overline{\lambda}\overline{m}_a\overline{m}_b + \overline{\mu}\overline{m}_am_b, \\
m_{a;b} &= (\overline{\gamma}-\gamma)m_ak_b + (\overline{\epsilon}-\epsilon)m_al_b + (\alpha-\overline{\beta})m_am_b + (\beta-\overline{\alpha})m_a\overline{m}_b \\
&- \overline{\nu}k_ak_b - \overline{\pi}k_al_b + \overline{\mu}k_am_b + \overline{\lambda}k_a\overline{m}_b \\
&+ \tau l_ak_b + \kappa l_al_b - \rho l_am_b - \sigma l_a\overline{m}_b.
\end{aligned}$$

We will use these relations to deduce some of the expressions below.

First we will introduce some important concepts. Given a vector field $\mathbf{u}$, an *integral curve* of $\mathbf{u}$ is a curve γ such that at each point p on γ the tangent vector field is $\mathbf{u}_p$. A set of integral curves is a *congruence*. Furthermore if $\mathbf{u}$ satisfies

$$\nabla_{\mathbf{u}}\mathbf{u} = f(t)\mathbf{u}, \tag{1.58}$$

then the integral curves of $\mathbf{u}$ are called *geodesics*. One can always find a parameter s such that (1.58) can be rewritten as $\nabla_{\mathbf{u}}\mathbf{u} = 0$ (affine parametrisation). In a local coordinate system the equation of the geodesic is then given by

$$\ddot{x}^i + \Gamma^i{}_{km}\dot{x}^k\dot{x}^m = 0, \qquad \text{where } \dot{x} = \mathrm{d}x/\mathrm{d}s. \tag{1.59}$$

Changing to $t = t(s)$ results in

$$\left(x''^i + \Gamma^i{}_{km}x'^k x'^m\right)\dot{t}^2 = -\ddot{t}x'^i, \qquad \text{where } x' = \mathrm{d}x/\mathrm{d}t.$$

Thus the standard form (1.59) of the equation is preserved iff $\ddot{t} = 0$, *i.e.* $t = as + b$, a, b constants. A parameter which produces the standard form for the geodesic equation is an *affine parameter*. Finally, we introduce the concept of a *connecting vector*: let $\mathbf{u}$ be the tangent vector to a congruence of curves, then any vector $\mathbf{z}$ such that $[\mathbf{u}, \mathbf{z}] = 0$ is called a connecting vector of the congruence.

It is often convenient to analyse the gravitational field in a local region in terms of the behavior of some privileged congruences of curves passing through this region; in particular, in terms of their *expansion, rotation* (or *twist*) and *shear*. Consider a beam of light rays, represented by a congruence of null geodesics γ with tangent vector $\mathbf{k}$ and connecting vector $\mathbf{z}$: $[\mathbf{k}, \mathbf{z}] = 0$. Assume $\mathrm{D}k^a = 0$, so that γ is affinely parametrised. Suppose that at a point p on γ the connecting vector is orthogonal to γ, *i.e.* $k^a z_a = 0$. Then

$$\mathrm{D}\left(k^a z_a\right) = k^a \mathrm{D} z_a = k^a \nabla_{\mathbf{k}} z_a = k^a \nabla_{\mathbf{z}} k_a = \frac{1}{2}\nabla_{\mathbf{z}}\left(k^a k_a\right) = 0.$$

Thus $\mathbf{z}$ is everywhere orthogonal to γ. Since $\mathbf{z}$ is real and orthogonal to $\mathbf{k}$, there exist a real u and complex z such that

$$z^a = uk^a + \overline{z}m^a + z\overline{m}^a.$$

Compute $\mathrm{D}z^a = \nabla_{\mathbf{k}} z^a = \nabla_{\mathbf{z}} k^a = u\mathrm{D}k^a + \overline{z}\delta k^a + z\overline{\delta} k^a$, where $\mathrm{D}k^a = 0$. Making use of the definitions of the spin coefficients $\rho = -m^a \overline{\delta} k_a$ and $\sigma = -m^a \delta k_a$ (1.2), we find that

$$\mathrm{D}z = -\rho z - \sigma \overline{z}.$$

The interpretation of $z = x + \mathrm{i}y$ is as follows: consider the projection of z^a onto the spacelike two-plane spanned by m^a, $\overline{m}^a$. Suppose that $\sigma = 0$ while $\rho = \theta$ is real:

$$\mathrm{D}x = -\theta x, \qquad \mathrm{D}y = -\theta y.$$

This is an isotropic magnification (expansion) at a rate $-\theta$. Next suppose that $\sigma = 0$ while ρ is imaginary ($\rho = -\mathrm{i}\omega$):

$$\mathrm{D}x = -\omega y, \qquad \mathrm{D}y = \omega x,$$

which corresponds to a rotation with angular velocity ω. Finally consider the case where $\rho = 0$ and σ is real, then

$$\mathrm{D}x = -\sigma x, \qquad \mathrm{D}y = \sigma y,$$

which represents a volume preserving shear at a rate σ with principal axes along the x and y axes. For a complex valued $\sigma = |\sigma| \exp \mathrm{i}\psi$ the argument $\psi/2$ denotes the orientation of the shear.

Now, since $\mathrm{D}z = -\rho z - \sigma z$ is a linear equation, the general case is a superposition of these effects. The projection of the connecting vectors of the congruence onto an orthogonal spacelike two-surface is expanded, rotated and sheared. The complex divergence $\rho = \theta + \mathrm{i}\omega$ and the shear σ are also known as the *optical parameters.*

Note that the spin coefficient κ is related to $\mathbf{k}$ by

$$k_{a;b}k^b = -\kappa \bar{m}_a - \bar{\kappa} m_a + (\epsilon + \bar{\epsilon})k_a.$$

If $\kappa = 0$, $\mathbf{k}$ is tangent to a geodesic. By a change in scale $k_a \to \phi k_a$, $\epsilon + \bar{\epsilon}$ can be made zero.

The quantity τ describes how the direction of $\mathbf{k}$ changes as we move in the direction $\mathbf{l}$, as follows from the equation

$$k_{a;b}l^b = -\tau \bar{m}_a - \bar{\tau} m_a + (\gamma + \bar{\gamma})k_a.$$

However, since τ transforms as $\tau' = \tau + \overline{B}\sigma + B\rho + B\overline{B}\kappa$ under the null rotation (1.8), which keeps $\mathbf{k}$ fixed, this interpretation is only well defined for a non-expanding, non-twisting and shear-free geodesic null congruence (*i.e.* for Kundt space-times). Again one can make $\gamma + \bar{\gamma}$ zero by the change $k_a \to \phi k_a$.

The interpretation of the spin coefficients ν, μ, λ, π is analogous, respectively, to that of κ, $-\rho$, $-\sigma$, τ, the difference being that the congruence used is given by $\mathbf{l}$ instead of $\mathbf{k}$ (*i.e.* where $\mathbf{k}$ and $\mathbf{l}$, and $\mathbf{m}$ and $\overline{\mathbf{m}}$ are both interchanged).

If $\mathbf{k}$ is taken tangent to a geodesic congruence and we propagate the tetrad parallelly along this congruence, then

$$\kappa = \pi = \epsilon = 0.$$

If in addition to being tangent to geodesics, $\mathbf{k}$ is hypersurface orthogonal (*i.e.* proportional to a gradient field), we have

$$\rho = \bar{\rho},$$

and if $\mathbf{k}$ is *equal* to a gradient field, then

$$\rho = \bar{\rho}, \qquad \tau = \bar{\alpha} + \beta.$$

We end this paragraph by explaining how the components Ψ_i, $i = 0, .., 4$, of the Weyl tensor can be used to classify gravitational fields. If the Weyl tensor vanishes, *i.e.* if $C_{abcd} = 0$, a space-time is said to be *conformally flat*. Otherwise, the gravitational fields can be classified according to the number of their distinct principal null directions. This is the so-called Petrov classification.
A null vector $\mathbf{k}$ is aligned with a *principal null direction* of the gravitational field if it satisfies

$$k_{[a}C_{f]gh[i}k_{b]}k^g k^h = 0.$$

If $\mathbf{k}$ is a member of the null tetrad defined before, then this is equivalent to the statement that $\Psi_0 = 0$. There are at most four such null vectors. To determine these, we apply the inverse of the null rotation (1.11) to an arbitrary complex null tetrad $(\mathbf{m}', \overline{\mathbf{m}}', \mathbf{l}', \mathbf{k}')$. Hereby, the null vector $\mathbf{k}'$ can be transformed into any other real null vector, except $\mathbf{l}'$. The component Ψ_0 of the Weyl tensor then transforms as

$$\Psi_0 = \Psi_0' - 4E\Psi_1' + 6E^2\Psi_2' - 4E^3\Psi_3' + E^4\Psi_4'.$$

The condition for $\mathbf{k}$ to be a principal null direction, *i.e.* $\Psi_0 = 0$, is then equivalent to the existence of a root E, such that

$$\Psi_0' - 4E\Psi_1' + 6E^2\Psi_2' - 4E^3\Psi_3' + E^4\Psi_4' = 0.$$

Since this is a quartic expression in E, there are four (complex) roots, that need not be distinct. Each root corresponds to a principal null direction and the multiplicity of each principal null direction is the same as the corresponding root. For a principal null direction $\mathbf{k}$ of multiplicity 1, 2, 3 or 4, it can be shown (Jordan et al., 1961; Petrov, 1954; Géhéniau, 1957; Penrose and Rindler, 1986) that, respectively

$$\begin{aligned} k_{[a}C_{f]gh[i}k_{b]}k^g k^h &= 0 \Leftrightarrow \Psi_0 = 0, & \Psi_1 &\neq 0,\\ C_{fgh[i}k_{b]}k^g k^h &= 0 \Leftrightarrow \Psi_0 = \Psi_1 = 0, & \Psi_2 &\neq 0,\\ C_{fgh[i}k_{b]}k^h &= 0 \Leftrightarrow \Psi_0 = \Psi_1 = \Psi_2 = 0, & \Psi_3 &\neq 0,\\ C_{fghi}k^h &= 0 \Leftrightarrow \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, & \Psi_4 &\neq 0. \end{aligned}$$

If a space-time admits four distinct principal null directions (*pnd*s), it is said to be *algebraically general*, or of type I, otherwise it is *algebraically special*. The different algebraic types can be summarised as follows:

- type I : four distinct *pnd*s,
- type II : one *pnd* of multiplicity 2, others distinct,
- type D : two distinct *pnd*s of multiplicity 2,
- type III : one *pnd* of multiplicity 3, other distinct,
- type N : one *pnd* of multiplicity 4,
- type O : conformally flat.

If the basis vector $\mathbf{k}$ is aligned with principal null directions, $\Psi_0 = 0$. If it is aligned with the repeated principal null direction of an algebraically special space-time, then $\Psi_0 = \Psi_1 = 0$. If $\mathbf{k}$ and $\mathbf{l}$ are both aligned with the two repeated principal null directions of a type D space-time, then the only non-vanishing component of the Weyl tensor is Ψ_2.

A theorem (for vacuum space-times), relating some geometrical properties of null congruences to the algebraic properties of the Weyl tensor, can be stated as follows (Goldberg and Sachs, 1962):

Theorem *A vacuum metric is algebraically special if and only if it contains a shear-free geodesic null congruence:*

$$\kappa = 0 = \sigma \qquad \Leftrightarrow \qquad \Psi_0 = 0 = \Psi_1 .$$

1.1.2 The Geroch Held Penrose formalism

A modified calculus, the GHP-formalism, was developed by Geroch et al. (1973). This formalism is especially adapted to physical situations in which a pair of real null directions is naturally picked out at each space-time point. This version of the spin coefficient method leads to even simpler formulae than the standard Newman Penrose technique.

The two-parameter subgroup of the Lorentz group preserving the two preferred null directions (boosts and spatial rotations), affects the complex null tetrad $(\mathbf{m}, \overline{\mathbf{m}}, \mathbf{l}, \mathbf{k})$ as follows:

$$\mathbf{k} \longrightarrow A\mathbf{k}, \quad \mathbf{l} \longrightarrow A^{-1}\mathbf{l}, \quad \mathbf{m} \longrightarrow \mathrm{e}^{\mathrm{i}\theta}\mathbf{m}; \quad A = C\overline{C}, \quad \mathrm{e}^{\mathrm{i}\theta} = C\overline{C}^{-1}. \tag{1.60}$$

A scalar η which undergoes the transformation

$$\eta \longrightarrow C^p \overline{C}^q \eta$$

is called a weighted scalar of *type (p,q)*. The components of the Weyl and Ricci tensors, the spin coefficients κ, λ, μ, ν, π, ρ, σ, τ, and the tetrad components of the electromagnetic field tensor are weighted scalars of types

$$\begin{array}{lllll}
\Psi_0 : (4,0), & \Psi_1 : (2,0), & \Psi_2 : (0,0), & \Psi_3 : (-2,0), & \Psi_4 : (-4,0), \\
\Phi_{00} : (2,2), & \Phi_{01} : (2,0), & \Phi_{10} : (0,2), & \Phi_{02} : (2,-2), & \Phi_{20} : (-2,2), \\
\Phi_{11} : (0,0), & \Phi_{12} : (0,-2), & \Phi_{21} : (-2,0), & \Phi_{22} : (-2,-2), & \\
\kappa : (3,1), & \lambda : (-3,1), & \mu : (-1,-1), & \nu : (-3,-1), & \\
\pi : (-1,1), & \rho : (1,1), & \sigma : (3,-1), & \tau : (1,-1), & \\
\Phi_0 : (2,0), & \Phi_1 : (0,0), & \Phi_2 : (-2,0). & &
\end{array}$$

The spin coefficients α, β, γ and ϵ transform, under the tetrad change (1.60), according to inhomogeneous laws ((1.10) containing derivatives of C). These spin coefficients do not appear directly in the modified equations, but they enter the new derivative operators acting on weighted scalars η of type (p,q):

$$\begin{array}{ll}
\text{Þ}\eta \equiv (\mathrm{D} - p\epsilon - q\overline{\epsilon})\eta, & \text{Þ}'\eta \equiv (\Delta - p\gamma - q\overline{\gamma})\eta, \\
\eth\eta \equiv (\delta - p\beta - q\overline{\alpha})\eta, & \eth'\eta \equiv (\overline{\delta} - p\alpha - q\overline{\beta})\eta.
\end{array}$$

The above operators Þ and ð ('thorn' and 'edth') respectively map a scalar of type (p,q) into scalars of types $(p+1,q+1)$ and $(p+1,q-1)$. In consequence the GHP-form of the Newman Penrose Ricci, Bianchi and Maxwell equations can be obtained by putting α, β, γ and ϵ equal to zero in the NP-equations (1.25-1.42), (1.43-1.53) and (1.54-1.57), while replacing D by Þ, Δ by Þ$'$, and δ and $\overline{\delta}$ by $\eth$ and $\eth'$, respectively. The equations involving derivatives of α, β, γ and ϵ (*i.e.* equations (1.28-1.30), (1.36), (1.39) and (1.42)) should be removed from the system. As an example, we will look at the Maxwell equation (1.54): if we switch from NP to GHP, we have to substitute D by Þ $+ p\epsilon + q\epsilon$, and $\overline{\delta}$ by $\eth' + \tilde{p}\alpha + \tilde{q}\overline{\beta}$, for D acting on a (p,q)- and $\overline{\delta}$ acting on a $(\tilde{p},\tilde{q})$-weighted quantity, respectively. As Φ_0 is of weight $(2,0)$ and Φ_1 of weight $(0,0)$, we obtain from (1.54)

$$\underbrace{\mathrm{D}\Phi_1}_{\text{Þ}\Phi_1} - \underbrace{\overline{\delta}\Phi_0}_{\eth'\Phi_0 + 2\alpha\Phi_0} = (\pi - 2\alpha)\,\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2,$$

that

$$\text{Þ}\Phi_1 - \eth'\Phi_0 = \pi\Phi_0 + \rho\Phi_1 - \kappa\Phi_2.$$

This shows one can just put α equal to zero in (1.54) and replace the NP derivation operators by their GHP-'equivalent'.

For completeness, we also give here the explicit form of the commutator relations in GHP. Acting on a (p,q)-weighted quantity η, we find

$$\begin{aligned}
\left[\eth',\eth\right]\eta &= (\bar{\mu}-\mu)\text{Þ}\eta+(\bar{\rho}-\rho)\text{Þ}'\eta+p\left(\rho\mu-\sigma\lambda-\Psi_2+\Phi_{11}+\frac{R}{24}\right)\eta \\
&- q\left(\overline{\rho\mu}-\bar{\sigma}\bar{\lambda}-\overline{\Psi_2}+\Phi_{11}+\frac{R}{24}\right)\eta, \qquad (1.61)\\
\left[\eth,\text{Þ}'\right]\eta &= \tau\text{Þ}'\eta-\bar{\nu}\,\text{Þ}\eta+\mu\,\eth\eta+\bar{\lambda}\,\eth'\eta+q\left(\overline{\rho\nu}-\bar{\tau}\bar{\lambda}-\overline{\Psi_3}\right)\eta \\
&+ p\left(\sigma\nu-\mu\tau-\Phi_{12}\right)\eta, \qquad (1.62)\\
\left[\eth,\text{Þ}\right]\eta &= \kappa\,\text{Þ}'\eta-\bar{\pi}\,\text{Þ}\eta-\bar{\rho}\,\eth\eta-\sigma\,\eth'\eta+p\left(\pi\sigma-\mu\kappa+\Psi_1\right)\eta \\
&+ q\left(\overline{\rho\pi}-\bar{\lambda}\bar{\kappa}+\Phi_{01}\right)\eta, \qquad (1.63)\\
\left[\text{Þ}',\text{Þ}\right]\eta &= -(\bar{\tau}+\pi)\eth\eta-(\tau+\bar{\pi})\eth'\eta+p\left(\tau\pi-\kappa\nu+\Psi_2+\Phi_{11}-\frac{R}{24}\right)\eta \\
&+ q\left(\overline{\tau\pi}-\overline{\nu\kappa}+\overline{\Psi_2}+\Phi_{11}-\frac{R}{24}\right)\eta. \qquad (1.64)
\end{aligned}$$

As an explicit example, we will look at (1.61). Starting from the corresponding NP-commutator $[\bar{\delta},\delta]$ acting on a (p,q)-weighted quantity η, we have that

$$\bar{\delta}\delta\eta-\delta\bar{\delta}\eta=(\bar{\mu}-\mu)\text{D}\eta+(\bar{\rho}-\rho)\Delta\eta-(\bar{\alpha}-\beta)\bar{\delta}\eta-(\bar{\beta}-\alpha)\delta\eta. \qquad (1.65)$$

The left hand side of this equation can be rewritten as follows (introducing GHP-derivative operators):

$$\bar{\delta}\left[\eth\eta+p\beta\eta+q\bar{\alpha}\eta\right]-\delta\left[\eth'\eta+p\alpha\eta+q\bar{\beta}\eta\right].$$

Notice that $\eth\eta$ is of weight $(p+1,q-1)$, whereas $\eth'\eta$ is of weight $(p-1,q+1)$ and that both α and β are *not* well-weighted quantities. We therefore obtain in the next step that the left hand side of (1.65) can also be written as

$$\eth'\eth\eta-\eth\eth'\eta+(\alpha-\bar{\beta})\eth\eta-(\bar{\alpha}-\beta)\eth'\eta-p\left(\delta\alpha-\bar{\delta}\beta\right)\eta+q\left(\bar{\delta}\bar{\alpha}-\delta\bar{\beta}\right)\eta.$$

If we replace the NP-derivative operators δ, $\bar{\delta}$, Δ and D by their GHP-equivalent in the right hand side of (1.65), we obtain

$$\begin{aligned}
&(\bar{\mu}-\mu)\left[\text{Þ}\eta+p\epsilon\eta+q\bar{\epsilon}\eta\right]+(\bar{\rho}-\rho)\left[\text{Þ}'\eta+p\gamma\eta+q\bar{\gamma}\eta\right]\\
&-(\bar{\alpha}-\beta)\left[\eth'\eta+p\alpha\eta+q\bar{\beta}\eta\right]-(\bar{\beta}-\alpha)\left[\eth\eta+p\beta\eta+q\bar{\alpha}\eta\right].
\end{aligned}$$

Equating the two previous expressions and taking into account (1.36), we see that indeed

$$\begin{aligned}\left[ð', ð\right]\eta &= (\overline{\mu}-\mu)Þ\eta + (\overline{\rho}-\rho)Þ'\eta + p\left(\rho\mu - \sigma\lambda - \Psi_2 + \Phi_{11} + \frac{R}{24}\right)\eta \\ &- q\left(\overline{\rho\mu} - \overline{\sigma}\overline{\lambda} - \overline{\Psi_2} + \Phi_{11} + \frac{R}{24}\right)\eta.\end{aligned}$$

The other commutators can be found in a similar way.

1.2 Equivalence of metrics; CLASSI

As shown in the previous sections, one can always choose a tetrad, and also coordinates, in many different ways. This freedom often results in a different form for the same metric. It is not always easy to decide whether or not two metrics are actually equivalent, meaning they describe the same gravitational field in two different coordinate systems. Therefore a coordinate invariant classification scheme is extremely important.
CLASSI, a computer program performing such a classification, following the algorithm by Karlhede (1980), has been implemented (Åman and Karlhede, 1980, 1981), using the computer algebra system SHEEP of Frick (1977). The result of this classification is a complete description of the geometry. This reduces the problem of deciding whether or not two metrics are equivalent, to the problem of deciding whether a set of algebraic equations has a solution or not.

The equivalence problem has a long history; indeed it was the motivation for Christoffel's introduction of his famous symbols. Useful reviews of the history of the problem, and its solution, are given in (Karlhede, 1980; Ehlers, 1981; MacCallum, 1983a). The problem can be stated as follows: given two line elements

$$\begin{aligned}\mathrm{d}s^2 &= g_{ab}\mathrm{d}x^a\mathrm{d}x^b, \\ \mathrm{d}s^2 &= g'_{ab}\mathrm{d}x'^a\mathrm{d}x'^b,\end{aligned}$$

in coordinates x^c and x'^c respectively, when does there exist a coordinate transformation $x'^a = x'^a(x^b)$ such that

$$g_{ab}(x^c) = \frac{\partial x'^d}{\partial x^a}\frac{\partial x'^e}{\partial x^b}g'_{de}(x'^c),$$

i.e. such that the two geometries are locally equivalent?
The answer turns out to be that there is no completely algorithmic procedure, but that it is possible to reduce the problem to that of the consistency of a set of algebraic (as opposed to differential) equations involving components of the Riemann tensor and its covariant derivatives (strictly, this applies only to metrics with appropriate differentiability conditions and only at their 'regular' points (Ehlers, 1981), but these conditions are usually satisfied in applications). The reason this is not algorithmic is that the last step, the solution of a general set of algebraic equations, is not. It should be noted that in general one needs components of tensors in a geometrically preferred frame (so called Cartan invariants), and not just scalar (polynomial) invariants.

The basic idea behind the procedure is that if a certain quantity Q is invariantly defined, and is expressed as $f(x^a)$ and $h(x'^b)$ in the two coordinate systems, then the metrics can only be the same if

$$Q = f(x^a) = h(x'^b),$$

and one has to find a set of quantities Q, for which the equality in the two metrics is sufficient to guarantee local equivalence of the metrics. The existence of such a method goes back to Christoffel. Working in four dimensions, he showed that this might involve the calculation of the twentieth covariant derivative of the Riemann tensor, which is clearly impractical. The first step towards a more practical formulation was Cartan's use of tetrad methods. He showed that the number of derivatives was reduced to ten at most. In a certain sense, Cartan's method is a special case of his technique for determining the equivalence of sets of differential forms on manifolds under appropriate transformation groups. Next Brans (1965) set out some practical ideas on how to implement Cartan's procedure. The most important idea is to separate the tetrad and coordinate variables and handle them differently. At this point the possibility of a practical computing method became apparent and Frick et al. built SHEEP. The theory of the problem was then further refined by Karlhede (1980).
His method is based on Cartan's method, but it contained some new ideas on implementation which were quite vital in bringing the calculations down

to a scale the computer could handle. Brans's idea of separate handling of the tetrad rotations and the space-time coordinates is implemented by fixing the tetrad frame at each stage of differentiation of the Riemann tensor by aligning it as far as possible with invariantly defined directions picked out by the tensors already calculated. Karlhede pointed out that the Petrov classification of the Weyl tensor implied that the frame was fixed up to at most a two-parameter family of transformations, except in the case of conformally flat metrics. Hence he was able to show that no more than the seventh derivative was required in general.

Karlhede's procedure was implemented in SHEEP by Åman, Karlhede, Joly and MacCallum (1990). Having solved the basic problem improvements were sought, mainly directed at extracting from the characterising quantities more of the invariant properties.

A particularly intriguing point is the question of the least upper bound on the number of derivatives required. Karlhede's limit of seven was low enough to encourage construction of the computer implementation. Recently it has been shown that this bound is sharp (Milson and Pelavas, 2008, 2009). Other examples can be found in (Wylleman, 2008, 2010).

The procedure for the equivalence problem can be stated as follows: put $q = 0$. Choose a constant frame metric η_{ij} and calculate the components of the Riemann tensor R_{abcd}.

- Calculate R_q, the set of components of the Riemann tensor and its derivatives up to the q^{th}.
- Find the isotropies of R_q, the Lorentz transformations which do not alter the components of R_q. These form the group H_q.
- Rotate the frame, up to H_q, so that R_q takes a canonical form.
- Determine the number t_q of functionally independent elements which are now present in R_q as functions of the space-time coordinates.
- If $t_q = t_{q-1}$ and $\dim H_q = \dim H_{q-1}$, let $N = q$ and go to the next step. If not, increase q by 1 and go to the first step.

- At this point we know all the invariants of the space and can determine the dimension and structure of the isometry and isotropy groups. For a space-time of dimension n, the isotropy group has dimension s equal to that of H_N and the isometry group has dimension $n + s - t_N$. To compare the space with another one, we first compare t_q and H_q: the two spaces can only be isomorphic if these are the same for each q. If they are the same, then one has to check the consistency of the algebraic equations derived by equating corresponding components of R_N.

Using CLASSI many of these steps are being performed automatically.

The step by step method using CLASSI works as follows: first construct a metric file. This can be done in various ways (one example is given in chapter 5). It is important here to identify real and complex coordinates. The metric can then be inserted using its tetrad vectors (one forms), or its co- or contravariant null vectors. In this file, one can also define substitutions, and it is here that one writes down the dyad transformations, necessary to obtain a standard tetrad for CLASSI. These transformations can be inserted in the following ways:

null rotation	$\begin{matrix} 1 \ \$ \ z \ \$ \\ 0 \ \$ \ 1 \ \$ \end{matrix}$	or	$\begin{matrix} 1 \ \$ \ 0 \ \$ \\ \omega \ \$ \ 1 \ \$ \end{matrix}$	$z, \omega \in \mathbb{C}$
boost	$\begin{matrix} B \ \$ \ 0 \ \$ \\ 0 \ \$ \ B^{-1}\$ \end{matrix}$			$B \in \mathbb{R}$
spin rotation	$\begin{matrix} e^{i\theta} \ \$ \ 0 \ \$ \\ 0 \ \$ \ e^{-i\theta} \ \$ \end{matrix}$			$\theta \in \mathbb{R}$

Once the metric file is ready, it can be loaded in CLASSI. A step by step procedure allows one to check if the tetrad is still in a standard form. '(CLASSIFY0)' returns the Petrov type of the metric, in addition to its Ricci spinor, curvature scalar and number of independent functions at that point. If the metric is still in its standard form (CLASSI will tell so), one can proceed to the next step '(CLASSIFY1)', which returns the first symmetrised derivatives of Ψ_i, Φ_{ij}, etc. If, for a certain step, the metric is *not* in its standard form, one should first go back to the metric file, apply the appropriate dyad transformations, reload the metric in CLASSI and start over from the first step.

During the entire procedure, CLASSI will ask whether or not certain expressions (Jacobians) can be zero. This results in subcases, that should be examined separately.

It is not necessary to follow the step by step procedure if the metric is in its standard form. It is sufficient then, to just '(CLASSIFY)' the metric, which automatically performs the step by step procedure, explained above. One can then also ask for the '(CLASSISUM)' result, which returns a very compact classification of the metric: a one line summary of the form ABC D E FGHIJKLM, where the first seven symbols have the following meaning:

A	: trace-free Ricci tensor	E	: dim. isotropy group
B	: ricciscalar Λ	F	: number of boosts still allowed
C	: Petrov type	G	: number of functionally independent elements
D	: dim. isometry group		

As an example, consider a CLASSISUM output line that reads r1D 1 0 00--12--. This corresponds to a pure radiation metric with Ricci scalar equal to one, of Petrov type D, with a one-dimensional isometry group and for which the dimension of the remaining isotropy group is zero. 00-- specifies the isotropy groups H_0 and H_1 and 12-- gives the number of functionally independent functions found up to this order of differentiation.

Chapter 2

Aligned Petrov type D pure radiation solutions of the Robinson Trautman family

2.1 Introduction

Around 1960 Robinson and Trautman investigated a large and physically important family of exact solutions, which are now known as 'Robinson Trautman solutions'. They are defined geometrically by the property that they admit a geodesic, non-twisting and shear-free but expanding null congruence. Apart from various classes of pure radiation space-times, this family also includes the Schwarzschild and Reissner Nordström black holes, the C-metric which represents accelerating black holes, the Vaidya solution with pure radiation, photon rockets and their non-rotating generalisations. As shown in the original work of Robinson and Trautman (1960), see also Stephani et al. (2003, Chapter 28), the general metric for an Einstein space with the geometric properties defined above can be written in the form

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\mathrm{d}r - 2\,H\mathrm{d}u^2 + 2\,\frac{r^2}{P^2}\,\mathrm{d}\zeta\mathrm{d}\bar{\zeta}, \tag{2.1}$$

where

$$2H \equiv 2P^2(\log P)_{,\zeta\bar{\zeta}} - 2r(\log P)_{,u} - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{2.2}$$

in which Λ is the cosmological constant. This metric contains two functions, $P = P(u,\zeta,\overline{\zeta})$ and $m = m(u)$. The coordinates employed in (2.1) are adapted to the assumed geometry which admits a geodesic, shear-free, non-twisting and expanding null congruence, generated by $\mathbf{k} = \partial_r$. Specifically, r is an affine parameter along the principal null congruence, u is a retarded time coordinate and ζ is a complex spatial stereographic type coordinate. Using the null tetrad $\mathbf{k} = \partial_r$, $\mathbf{l} = \partial_u - H\partial_r$, $\mathbf{m} = (P/r)\partial_\zeta$, the non-zero components of the Weyl tensor are

$$\begin{aligned}
\Psi_2 &= -\frac{m}{r^3}, \\
\Psi_3 &= -\frac{P}{r^2}\left(P^2(\log P)_{,\zeta\bar{\zeta}}\right)_{,\zeta}, \\
\Psi_4 &= \frac{1}{r^2}\left[P^2\left(P^2(\log P)_{,\zeta\bar{\zeta}}\right)_{,\zeta}\right]_{,\zeta} - \frac{1}{r}\left[P^2(\log P)_{,u\zeta}\right]_{,\zeta}.
\end{aligned}$$

Generally, the metric (2.1) with (2.2) admits a Ricci tensor component

$$\Phi_{22} = \frac{1}{r^2 P}\left(P^4 P_{,\zeta\zeta\overline{\zeta\zeta}} - P^3 P_{,\zeta\zeta} P_{,\overline{\zeta\zeta}} + 3mP_{,u} - Pm_{,u}\right) \tag{2.3}$$

in addition to the above mentioned Weyl tensor components and the cosmological constant. This corresponds to the presence of an aligned pure radiation field (*i.e.* a flow of matter of zero rest mass, propagating along the repeated principal null direction), with energy momentum tensor of the form $T_{ab} = \phi k_a k_b$, where $\phi \equiv \Phi_{22}/(4\pi)$ is the radiation density. For pure radiation with cosmological constant Einstein's field equation implies $4\Lambda = R$.From (2.3) it is obvious that one obtains a vacuum solution if P and m satisfy

$$P^4 P_{,\zeta\zeta\overline{\zeta\zeta}} - P^3 P_{,\zeta\zeta} P_{,\overline{\zeta\zeta}} + 3mP_{,u} - Pm_{,u} = 0.$$

Frolov and Khlebnikov (1975) investigated non-rotating aligned pure radiation metrics of Petrov type D, with cosmological constant. Using the double principal null vectors $\mathbf{k}$ and $\mathbf{l}$ to construct the null tetrad, their solutions are divided in three classes: A, B or C, according to whether $\pi = \nu = 0$, $\pi = 0 \neq \nu$ or $\pi \neq 0$, respectively. If both spin coefficients, π and ν, are

zero, the metric was written in one of three different forms:
Metric AI

$$\mathrm{d}s^2 = \left(-1 + \frac{2m}{r} + 2\Lambda r^2\right) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \left(\mathrm{d}x^2 + \sin^2 x \mathrm{d}y^2\right), \tag{2.4}$$

Metric AII

$$\mathrm{d}s^2 = \left(1 + \frac{2m}{r} + 2\Lambda r^2\right) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \left(\mathrm{d}x^2 + \sinh^2 x \mathrm{d}y^2\right), \tag{2.5}$$

Metric AIII

$$\mathrm{d}s^2 = \left(0 + \frac{2m}{r} + 2\Lambda r^2\right) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \left(\mathrm{d}x^2 + x^2 \mathrm{d}y^2\right), \tag{2.6}$$

where Λ is the cosmological constant, and where $m = m(u)$ is an arbitrary function of u.

If $\pi = 0$ but $\nu \neq 0$, the metric can take one of five different forms:
Metric BI

$$\begin{aligned} \mathrm{d}s^2 &= \left[-1 + 2ar\cos x + r^2(f^2 + g^2 \sin^2 x) + \frac{2m}{r} + 2\Lambda r^2\right] \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\ &- 2r^2 f \mathrm{d}u\mathrm{d}x - 2r^2 g \sin^2 x \mathrm{d}u\mathrm{d}y + r^2 \mathrm{d}x^2 + r^2 \sin^2 x \mathrm{d}y^2, \end{aligned} \tag{2.7}$$

$$\begin{aligned} f &= -a \sin x + b \sin y + c \cos y, \\ g &= b\,\mathrm{cotan} x \cos y - c\,\mathrm{cotan} x \sin y, \end{aligned}$$

Metric BII

$$\begin{aligned} \mathrm{d}s^2 &= \left[1 + 2ar\mathrm{cosh} x + r^2(f^2 + g^2 \mathrm{sinh}^2 x) + \frac{2m}{r} + 2\Lambda r^2\right] \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\ &- 2r^2 f \mathrm{d}u\mathrm{d}x - 2r^2 g \,\mathrm{sinh}^2 x \mathrm{d}u\mathrm{d}y + r^2 \mathrm{d}x^2 + r^2 \mathrm{sinh}^2 x \mathrm{d}y^2, \end{aligned} \tag{2.8}$$

$$\begin{aligned} f &= -a\,\mathrm{sinh} x + b \sin y + c \cos y, \\ g &= b\,\mathrm{cotanh} x \cos y - c\,\mathrm{cotanh} x \sin y, \end{aligned}$$

Metric BIII

$$\begin{aligned} \mathrm{d}s^2 &= \left[1 + 2ar\mathrm{sinh} x + r^2(f^2 + g^2 \mathrm{cosh}^2 x) + \frac{2m}{r} + 2\Lambda r^2\right] \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\ &- 2r^2 f \mathrm{d}u\mathrm{d}x - 2r^2 g \,\mathrm{cosh}^2 x \mathrm{d}u\mathrm{d}y + r^2 \mathrm{d}x^2 + r^2 \mathrm{cosh}^2 x \mathrm{d}y^2, \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
f &= -a\cosh x - b\mathrm{e}^{y} - c\mathrm{e}^{-y}, \\
g &= b\tanh x\mathrm{e}^{y} - c\tanh x\mathrm{e}^{-y},
\end{aligned}
$$

Metric BIV

$$
\begin{aligned}
\mathrm{d}s^2 &= \left[1 + 2ar\mathrm{e}^{x} + r^2(f^2 + g^2\mathrm{e}^{2x}) + \frac{2m}{r} + 2\Lambda r^2\right]\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\
&- 2r^2 f\mathrm{d}u\mathrm{d}x - 2r^2 g\mathrm{e}^{2x}\mathrm{d}u\mathrm{d}y + r^2\mathrm{d}x^2 + r^2\mathrm{e}^{2x}\mathrm{d}y^2, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
f &= a\mathrm{e}^{x} - 2by, \\
g &= b(y^2 - \mathrm{e}^{-2x}),
\end{aligned}
$$

Metric BV

$$
\begin{aligned}
\mathrm{d}s^2 &= \left[2r(\sqrt{2}x - a) + r^2(f^2 + g^2) + \frac{2m}{r} + 2\Lambda r^2\right]\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\
&- 2r^2 f\mathrm{d}u\mathrm{d}x - 2r^2 g\mathrm{d}u\mathrm{d}y + r^2\mathrm{d}x^2 + r^2\mathrm{d}y^2, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
f &= \frac{1}{\sqrt{2}}\left(y^2 - x^2\right) + ax - by, \\
g &= ay + bx - \sqrt{2}xy,
\end{aligned}
$$

where $a = a(u)$, $b = b(u)$, $c = c(u)$ and $m = m(u)$ are arbitrary functions of u. The only non-vanishing component of the Weyl tensor is $\Psi_2 = -m(u)/r^3$, while the only non-zero component of the Ricci tensor in the B-class is given by

$$
\Phi_{22} = \frac{1}{r^2}\left(3m(u)p(u,x) - \frac{\mathrm{d}m}{\mathrm{d}u}\right),
$$

with $p(u,x) = a(u)\cos x$, $a(u)\cosh x$, $a(u)\sinh x$, $a(u)\exp x$ or $a(u) - \sqrt{2}x$ for subclasses I, II, III, IV or V, respectively.

For the remaining case $\pi \neq 0$, the metric in the original paper (Frolov and Khlebnikov, 1975) reads

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left(6mx + rf'c^4 - 2r\frac{\dot{c}}{c} - \frac{2m}{r} - r^2f^2c^6 - 2\Lambda r^2\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\
&- \frac{r^2}{c}\mathrm{d}u\mathrm{d}x + \frac{r^2}{f^2c^2}\mathrm{d}x^2 + \frac{r^2f^2}{c^2}\mathrm{d}y^2, \qquad (2.12)
\end{aligned}
$$

where $c = c(u)$ is an arbitrary function of u, $'$ and $\dot{}$ denote differentiation with respect to x and u, respectively, and where

$$f = f(x) = \left(-2mx^3 + ax + b\right)^{1/2} \qquad a, b \in \mathbb{R}. \tag{2.13}$$

Note that in the above metrics we have used the opposite signature to that of Frolov and Khlebnicov.

Some remarks have to be made, when looking at the above results. First of all, it is obvious that metric AI is a special case of metric BI: by substituting $a(u) = b(u) = c(u) = 0$ in (2.7), one obtains (2.4). In the same way, one can find metric AII (2.5) as a special case of metric BII (2.8). It is even possible to find metric AIII as a special case of metric BV: in (2.11) first put the functions $a(u)$ and $b(u)$ equal to zero (note that f and g do *not* vanish in this case). Next apply the coordinate transformation $y \longrightarrow xy$. The following step is to scale the coordinates r, x and u and the function $m(u)$ by a constant factor $1/c$, c, c and $1/c^3$, respectively. Putting c equal to zero in the resulting metric, applying the coordinate transformation $x \longrightarrow x/(1+y^2)^{1/2}$ and introducing a new coordinate $\tilde{y}$ satisfying $\mathrm{d}\tilde{y} = \mathrm{d}y/(1+y^2)$, we obtain (2.6). We can thus conclude that the entire A-family is a subfamily of the B-class. In (Stephani et al., 2003) the A- and B-class are written in equations (28.71a) and (28.71b) with (28.73).

Secondly, and more importantly, note that the metric for the C-class (2.12) above is *not* a solution of the Einstein field equations. In the original paper, the directional derivatives for every metric are given by

$$\begin{aligned}
\delta &= \xi^1\partial_u + \omega\partial_r + \xi^3\partial_x + \xi^4\partial_y, \\
\Delta &= X^1\partial_u + U\partial_r + X^3\partial_x + X^4\partial_y, \\
\mathrm{D} &= \partial_r,
\end{aligned}$$

where, for the C-class, the coefficients ξ^j, X^j, ω and U are given by

$$\begin{aligned}
&\xi^1 = 0, \qquad \xi^3 = -\frac{fc}{\sqrt{2}r}, \qquad \xi^4 = -\frac{\mathrm{i}c}{\sqrt{2}rf}, \\
&X^1 = 1, \qquad X^3 = 0, \qquad X^4 = 0, \\
&\omega = \frac{rf}{\sqrt{2}}, \qquad U = -3mx - \frac{rf'c^4}{2} + \frac{r\dot{c}}{c} + \frac{m}{r} + \frac{r^2f^2c^6}{2} + \Lambda r^2,
\end{aligned}$$

with f defined in (2.13). This, however, does not correspond to a solution of the Einstein field equations. The correct coefficients should have been

$$\begin{aligned} \omega &= \frac{krf}{\sqrt{2}}, \\ U &= -3mxc^2 - \frac{r\left(f^2\right)' kc}{2} + \frac{r\dot{c}}{c} + \frac{mc^3}{rk} + \frac{r^2 f^2 k^2}{2} + \Lambda r^2, \end{aligned}$$

where $k = k(u)$ is a *new* arbitrary function of u and with ξ^j, X^j as above. In these coordinates, the non-vanishing components of the Weyl and the Ricci tensor are

$$R = 4\Lambda, \qquad \Phi_{22} = \frac{mc^3\dot{k}}{r^2k^2} \qquad \text{and} \qquad \Psi_2 = -\frac{mc^3}{r^3k}.$$

Notice that, if k is constant, Φ_{22} is zero.

It is now possible to make $c = c(u)$ equal to one, by a coordinate transformation $r \longrightarrow rc$ and by next absorbing a factor $c(u)$ in $\mathrm{d}u$, thus eliminating one of the two arbitrary functions of u (as could be expected, from the GHP-analysis, see below). It is also possible, by a scaling of x and y to make the constant m equal to one. The result (for zero cosmological constant) is then equivalent to (28.74) in (Stephani et al., 2003).

In the remainder of this chapter, we will first demonstrate how to come to the above result for the class C. The way to handle and integrate the A- and B-class of metrics is then very similar[1]. It is always possible to start directly from the general Robinson Trautman solution (2.1), but we prefer not to do so. Instead, we will first perform calculations in the GHP-formalism, in order to extract all possible rotation and boost invariant properties, after which we translate this information into the NP-formalism. This then allows us to integrate the system, and to come to a (family of) line element(s) for the given problem. Also, the calculations in GHP help us to determine the number of independent free functions to be expected in the final result.

2.2 Geroch Held Penrose analysis C-class

In this section, we will handle the metric for the C-class: a Petrov type D, non-twisting, aligned pure radiation metric for which the spin coefficient

[1] The GHP-analysis for the B-class ($\supset$ A-class) is given in section 2.4.

π is non-zero in the null tetrad constructed with the double principal null direction. The basic assumptions for aligned type D pure radiation are $\Psi_i = 0$, $i \neq 2$ and $\Phi_{ij} = 0$, $(i,j) \neq (2,2)$. So the only non-zero components of the Weyl and Ricci tensor are R, Ψ_2 and Φ_{22}. Bianchi equations (1.43) and (1.44) then show that the system is geodesic and shear-free: $\kappa = \sigma = 0$, and by (1.49) also $\lambda = 0$. For the moment, we do not assume the solutions to be non-twisting. We will prove that this is a consequence of the initial conditions. The remaining Bianchi equations can be rewritten as follows:

$$\eth\Psi_2 = 3\tau\Psi_2, \qquad \eth\Phi_{22} = \tau\Phi_{22} + 3\overline{\nu}\overline{\Psi_2},$$
$$\eth'\Psi_2 = -3\pi\Psi_2, \qquad \eth'\Phi_{22} = \overline{\tau}\Phi_{22} + 3\nu\Psi_2,$$
$$\text{Þ}'\Psi_2 = -\rho\Phi_{22} - 3\mu\Psi_2,$$
$$\text{Þ}\Psi_2 = 3\rho\Psi_2, \qquad \text{Þ}\Phi_{22} = (\rho + \overline{\rho})\Phi_{22},$$

whereas the GHP Ricci equations read

$$\eth'\pi = -\pi^2, \qquad \eth\rho = \tau(\rho - \overline{\rho}),$$
$$\eth'\mu = \pi(\overline{\mu} - \mu) + \nu(\overline{\rho} - \rho), \qquad \text{Þ}\rho = \rho^2,$$
$$\text{Þ}\mu = \Psi_2 + \eth\pi + \overline{\rho}\mu + \pi\overline{\pi} + \frac{\Lambda}{3},$$
$$\eth\nu = \text{Þ}'\mu + \mu^2 + \tau\nu + \overline{\nu}\pi + \Phi_{22}, \qquad \eth\tau = \tau^2,$$
$$\eth'\nu = \nu(\overline{\tau} - \pi), \qquad \eth'\tau = \text{Þ}'\rho + \frac{\Lambda}{3} + \rho\overline{\mu} + \tau\overline{\tau} + \Psi_2,$$
$$\text{Þ}\nu = \text{Þ}'\pi + \mu(\overline{\tau} + \pi), \qquad \text{Þ}\tau = \rho(\tau + \overline{\pi}).$$

Making use of these expressions for the directional derivatives of the spin coefficients, Weyl and Ricci tensor components, more information can be extracted from the commutator relations. From $[\eth', \text{Þ}]\Psi_2$, we obtain

$$\text{Þ}\pi = -\eth'\rho,$$

whereas $[\text{Þ}', \text{Þ}]\Psi_2$, $[\eth, \text{Þ}']\Psi_2$ and $[\eth', \text{Þ}']\Psi_2$ respectively yield

$$\begin{aligned}
\eth\pi &= -\text{Þ}'\rho - \Psi_2 - \overline{\rho}\mu - \frac{\Lambda}{3} - \tau\overline{\tau} - \frac{\rho\Phi_{22}(\overline{\rho} - \rho)}{3\Psi_2}, \\
\text{Þ}'\tau &= -\eth\mu - \frac{\overline{\nu}\rho(\overline{\Psi_2} - \Psi_2)}{\Psi_2} + \frac{\tau\Phi_{22}(2\rho + \overline{\rho})}{3\Psi_2}, \\
\text{Þ}'\pi &= \frac{\Phi_{22}\eth'\rho}{3\Psi_2} - \mu(\pi + \overline{\tau}) + \nu(\overline{\rho} - \rho) + \frac{\pi\Phi_{22}\rho}{\Psi_2}.
\end{aligned}$$

Before continuing, first note that

$$\frac{\nu\Psi_2}{\pi\Phi_{22}}$$

is $(0,0)$-weighted. Making use of the expressions for $ð'\Psi_2$, $ð'\Phi_{22}$, $ð'\pi$ and $ð'\nu$, we can prove, by taking the $ð'$-derivative of

$$\nu\Psi_2 + c\pi\Phi_{22} = 0, \tag{2.14}$$

where c is a constant, that (2.14) can only be valid for $c = 1$. To prove that this expression indeed *must* hold, we look at $[ð', Þ]\pi$ and $[ð', Þ']\pi$. From the former we find an expression for $ð'ð'\rho$

$$ð'ð'\rho = -3\pi ð'\rho, \tag{2.15}$$

which we can use to simplify the latter:

$$\frac{(5\pi\Phi_{22} - 3\nu\Psi_2)\, ð'\rho}{3\Psi_2} + \frac{4\rho\pi\,(\pi\Phi_{22} + \nu\Psi_2)}{\Psi_2} = 0. \tag{2.16}$$

It is obvious that we can rewrite (2.16) as an expression for $ð'\rho$:

$$ð'\rho = \frac{12\rho\pi\,(\pi\Phi_{22} + \nu\Psi_2)}{3\nu\Psi_2 - 5\pi\Phi_{22}}.$$

Herewith (2.15) becomes

$$\frac{24\pi^2\rho\,(\pi\Phi_{22} + \nu\Psi_2)\,(21\nu\Psi_2 + \pi\Phi_{22})}{(3\nu\Psi_2 - 5\pi\Phi_{22})^2} = 0$$

showing that (2.14), with $c = 1$, must hold (except if $\pi = 0$, which is handled in cases A and B). It follows that $\nu = -\dfrac{\pi\Phi_{22}}{\Psi_2} \neq 0$.

From the derivatives of (2.14) we find expressions for $Þ'\mu$ and $Þ'\nu$:

$$\begin{aligned} Þ'\mu &= \frac{\Phi_{22}Þ'\rho}{\Psi_2} + \frac{\rho\Phi_{22}^2(\bar{\rho} - \rho)}{3\Psi_2^2} + \left(\frac{\Lambda}{3} + \bar{\rho}\mu + \tau\bar{\tau}\right)\frac{\Phi_{22}}{\Psi_2} \\ &- \mu^2 - 3\tau\nu + \frac{\nu\bar{\nu}\left(3\overline{\Psi_2} - \Psi_2\right)}{\Phi_{22}}, \\ Þ'\nu &= \frac{\nu Þ'\Phi_{22}}{\Phi_{22}} + (\bar{\tau}\mu - \nu\,(\bar{\rho} - 3\rho))\frac{\Phi_{22}}{\Psi_2} + 2\mu\nu. \end{aligned}$$

We are now ready to prove that ρ has to be real. Assuming for a moment that ρ is not real, we can find an expression for $\text{Þ}'\rho$ from $[\eth, \text{Þ}]\pi$ and $[\text{Þ}', \text{Þ}]\rho$:

$$\text{Þ}'\rho = -\frac{\Lambda}{6} - \tau\overline{\tau} - \frac{\rho\overline{\Psi_2} - 2\rho\Psi_2 + \overline{\rho}\Psi_2}{2(\overline{\rho} - \rho)}.$$

The latter would allow us to solve $[\eth', \text{Þ}']\rho$ for τ:

$$\tau = \frac{\overline{\pi}\,(\rho - 2\overline{\rho})}{\overline{\rho}}.$$

Substituting these expressions for τ and $\text{Þ}'\rho$ in $[\text{Þ}', \text{Þ}]\pi$ leads to

$$\frac{4\pi\Phi_{22}\,(\overline{\rho} - \rho)^2}{3\Psi_2} = 0,$$

a contradiction.

Next, we calculate $[\eth', \eth]\Phi_{22}$, from which we eliminate ν by (2.14):

$$4\Phi_{22}\left(\overline{\Psi_2} - \Psi_2\right) = 0,$$

so also Ψ_2 is real. Taking the derivatives of $\overline{\Psi_2} - \Psi_2 = 0$, and again eliminating ν by (2.14), we find that μ is real and that $\tau = -\overline{\pi}$.

The only remaining information comes from $[\eth, \text{Þ}']\pi$, which yields an expression for $\text{Þ}'\text{Þ}'\rho$:

$$\text{Þ}'\text{Þ}'\rho \quad = \quad -2\mu\text{Þ}'\rho - \frac{2\rho\Phi_{22}\,(\pi\overline{\pi} - \Psi_2)}{\Psi_2} - \frac{\mu\Lambda}{3} - 2\mu\,(\pi\overline{\pi} - \Psi_2)\,.$$

Expressions for $\eth\text{Þ}'\rho$, $\eth'\text{Þ}'\rho$ and $\text{Þ}\text{Þ}'\rho$ can be obtained from applying the appropriate commutators to ρ.

As we can use $\text{Þ}'\rho$ as a new variable, of which we know all directional derivatives, we expect to see only one free function appearing in the metric, as the single unknown function in the GHP-analysis is $\text{Þ}'\Phi_{22}$.

2.3 Newman Penrose analysis C-class

In order to obtain an explicit expression for the metric line element, we first recapitulate some of the invariant information obtained in the previous section:

- basic assumptions: all Ψ_i and Φ_{ij} equal to zero, except Ψ_2 and Φ_{22},
- geodesity, shear-freeness and $\lambda = 0$,
- the fact that ρ, μ and Ψ_2 are real,
- the expressions for $\tau = -\overline{\pi}$ and $\nu = -\dfrac{\pi\Phi_{22}}{\Psi_2}$.

As we assume π to be non-zero, we can fix the rotation by making $\pi > 0$ (*i.e.* we make π real and strictly positive). From (1.27) it then follows that ϵ is real, so a boost with parameter A satisfying $2\epsilon + \mathrm{D}\ln A = 0$ exists, which makes $\epsilon = 0$. If we add the condition $\delta A = -\beta - \overline{\alpha} - \overline{\pi}$ ($\overline{\delta} A = -\overline{\beta} - \alpha - \pi$), we see, by (1.27 − 1.29), (1.31 − 1.32), (1.36) and the commutator relations (1.4) and (1.6), that this boost can also make $\beta = -\overline{\pi} - \overline{\alpha}$. From the imaginary part of (1.31 + 1.32), it then follows that also α is real, whereas the imaginary part of (1.33) implies $\overline{\gamma} = \gamma$.

The next step in our calculations is to rewrite the NP Bianchi equations:

$$\begin{aligned}
&\delta\Psi_2 = -3\pi\Psi_2, && \delta\Phi_{22} = -2\pi\Phi_{22},\\
&\overline{\delta}\Psi_2 = -3\pi\Psi_2, && \overline{\delta}\Phi_{22} = -2\pi\Phi_{22},\\
&\Delta\Psi_2 = -\rho\Phi_{22} - 3\mu\Psi_2, && \\
&\mathrm{D}\Psi_2 = 3\rho\Psi_2, && \mathrm{D}\Phi_{22} = 2\rho\Phi_{22}.
\end{aligned} \tag{2.17}$$

The NP Ricci equations can be written as follows:

$$\begin{aligned}
&\delta\mu = \pi\mu, && \Delta\mu = \frac{2\pi\Phi_{22}\,(\pi + 2\alpha)}{\Psi_2} - \mu^2 - 2\gamma\mu - \Phi_{22},\\
&\delta\pi = -2\pi(\pi + \alpha), && \Delta\pi = \frac{\rho\pi\Phi_{22}}{\Psi_2},\\
&\delta\rho = -\rho\pi, && \Delta\rho = 2\pi(\pi + 2\alpha) + \rho(2\gamma - \mu) - \Psi_2 - \frac{\Lambda}{3},\\
&\delta\gamma = -\pi\mu, && \Delta\alpha = -\frac{\rho\pi\Phi_{22}}{\Psi_2} - \mu(\alpha + \pi),\\
&\mathrm{D}\pi = 0, && \mathrm{D}\mu = \mu\rho - 2\pi(\pi + 2\alpha) + \Psi_2 + \frac{\Lambda}{3},\\
&\mathrm{D}\alpha = \rho(\alpha + \pi), && \mathrm{D}\rho = \rho^2,\\
& && \mathrm{D}\gamma = \Psi_2 - \pi^2 - \frac{\Lambda}{6}
\end{aligned} \tag{2.18}$$

and

$$\overline{\delta}\alpha = -\delta\alpha + 4\alpha^2 + 3\pi^2 + 6\pi\alpha + \mu\rho - \Psi_2 + \frac{\Lambda}{6}. \tag{2.19}$$

Applying the commutators to ρ leads to an expression for $\delta\alpha$, by which (2.19) becomes an identity:

$$\delta\alpha = \frac{3\pi^2}{2} + 3\pi\alpha + \frac{\Lambda}{12} + 2\alpha^2 + \frac{\mu\rho}{2} - \frac{\Psi_2}{2}.$$

No other information can be obtained from the set of NP-equations (Bianchi, Ricci and commutator equations).

In the next step, we look at the first Cartan equations, which give us a clue to what the one forms, corresponding to this problem, may look like. These equations read:

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^1 &= \rho\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^4 - \mu\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 + (\pi + 2\alpha)\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2, \\
\mathrm{d}\boldsymbol{\omega}^2 &= \rho\,\boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4 - \mu\,\boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 - (\pi + 2\alpha)\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2, \\
\mathrm{d}\boldsymbol{\omega}^3 &= 0, \\
\mathrm{d}\boldsymbol{\omega}^4 &= 2\pi(\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2) \wedge \boldsymbol{\omega}^4 - 2\gamma\,\boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 - \frac{\pi\Phi_{22}}{\Psi_2}(\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2) \wedge \boldsymbol{\omega}^3.
\end{aligned} \tag{2.20}$$

As $\boldsymbol{\omega}^3$ is exact, we can introduce a coordinate u such that $\boldsymbol{\omega}^3 = \mathrm{d}u$. We will not make use of the fact that $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ are hypersurface orthogonal, but prefer to write

$$\boldsymbol{\omega}^1 = H\mathrm{d}u + V\mathrm{d}r + P\mathrm{d}x + \mathrm{i}Q\mathrm{d}y,$$

where H, V, P and Q are complex valued functions of (u, r, x, y). Furthermore, we will use $-1/\rho$ as coordinate r. From the directional derivatives of ρ and

$$\mathrm{d}\rho = \delta\rho\boldsymbol{\omega}^1 + \overline{\delta}\rho\boldsymbol{\omega}^2 + \Delta\rho\boldsymbol{\omega}^3 + \mathrm{D}\rho\boldsymbol{\omega}^4,$$

we find

$$\begin{aligned}
\boldsymbol{\omega}^4 = &\left[\left(\Psi_2 - 2\pi^2 - 4\pi\alpha + \frac{\Lambda}{3}\right) r^2 + \left(2\gamma - \mu - (H + \overline{H})\pi\right) r\right] \mathrm{d}u \\
&+ \left[1 - (V + \overline{V})\pi r\right] \mathrm{d}r - \pi r \left[\left(P + \overline{P}\right) \mathrm{d}x + \mathrm{i}\left(Q - \overline{Q}\right) \mathrm{d}y\right].
\end{aligned}$$

It is obvious that we can choose the x- and y-coordinate in such a way that $V = 0$. There is still the following coordinate freedom:

$$x \longrightarrow g_1(u,x,y)\tilde{x} + g_2(u,x,y),$$
$$y \longrightarrow g_3(u,x,y)\tilde{y} + g_4(u,x,y).$$

Substituting the above expressions for the one forms in equation (2.20) we get

$$\begin{aligned}
&\left[P_y - \mathrm{i}\left(Q_x + 4(\pi+\alpha)\Re(\overline{Q}P)\right)\right]\mathrm{d}x\wedge\mathrm{d}y + H_y\mathrm{d}u\wedge\mathrm{d}y - H_x\mathrm{d}x\wedge\mathrm{d}u\\
&-\left[Q_u - Q\left(2\gamma - 2\pi(\pi+2\alpha)r + (\Psi_2+\frac{\Lambda}{3})r\right) + 4(\pi+\alpha)\Re(H\overline{Q})\right]\mathrm{i}\mathrm{d}u\wedge\mathrm{d}y\\
&+\left[P_u - P\left(2\gamma - 2\pi(\pi+2\alpha)r + (\Psi_2+\frac{\Lambda}{3})r\right) - 4(\pi+\alpha)\Re(H\overline{P})\right]\mathrm{d}x\wedge\mathrm{d}u\\
&-\frac{H-H_r r}{r}\mathrm{d}u\wedge\mathrm{d}r - \frac{P-P_r r}{r}\mathrm{d}x\wedge\mathrm{d}r - \frac{Q-Q_r r}{r}\mathrm{i}\,\mathrm{d}y\wedge\mathrm{d}r = 0, \qquad (2.21)
\end{aligned}$$

from which we see that P, Q and H are of the form $f(u,x,y)r$, so a transformation of x and y can be used to make both P and Q real. To see this, we will write

$$P = (P_0 + \mathrm{i}\tilde{P}_0)r, \qquad Q = (Q_0 + \mathrm{i}\tilde{Q}_0)r, \qquad H = (H_1 + \mathrm{i}H_2)r,$$

where P_0, $\tilde{P}_0$, Q_0, $\tilde{Q}_0$, H_1 and H_2 are real functions of (u,x,y) only. Replacing x by $x+g_2(u,x,y)$ in $\boldsymbol{\omega}^1$ (the calculations for $\boldsymbol{\omega}^2$ and $\boldsymbol{\omega}^4$ are equivalent), we find that

$$\begin{aligned}
\frac{\boldsymbol{\omega}^1}{r} &= \left(H_1 + \mathrm{i}H_2 + (P_0+\mathrm{i}\tilde{P}_0)g_{2_u}\right)\mathrm{d}u + (P_0+\mathrm{i}\tilde{P}_0)(1+g_{2_x})\mathrm{d}x\\
&+ \left(\mathrm{i}Q_0 - \tilde{Q}_0 + (P_0+\mathrm{i}\tilde{P}_0)g_{2_y}\right)\mathrm{d}y.
\end{aligned}$$

If we choose the (real) function $g_2(u,x,y)$ such that $\partial g_2/\partial y = \tilde{Q}_0/P_0$, this shows that we can make Q real (we will also replace H_1 by $H_1 - (P_0 g_{2_u})$, H_2 by $H_2 - (\tilde{P}_0 g_{2_u})$ etc.). Next we replace y by $y + g_4(u,x,y)$:

$$\frac{\boldsymbol{\omega}^1}{r} = (H_1 + \mathrm{i}(H_2 + Q_0 g_{4_u}))\,\mathrm{d}u + \left(P_0 + \mathrm{i}(\tilde{P}_0 + Q_0 g_{4_x})\right)\mathrm{d}x + \mathrm{i}Q_0(1+g_{4_y})\mathrm{d}y,$$

where we choose the real function g_4 such that $\partial g_4/\partial x = -\tilde{P}_0/Q_0$. This makes P real. We replace H_2 by $H_2 - Q_0 g_{4_u}$ and Q_0 by $Q_0/(1+g_{4_y})$ to obtain

$$\boldsymbol{\omega}^1 = (H_1 + \mathrm{i}H_2)\,r\mathrm{d}u + P_0 r\mathrm{d}x + \mathrm{i}Q_0 r\mathrm{d}y.$$

The remaining coordinate freedom is now given by

$$x \longrightarrow g_1(u,x)\tilde{x} + g_2(u,x),$$
$$y \longrightarrow g_3(u,y)\tilde{y} + g_4(u,y).$$

From (2.21) it also follows that P_0 and H_1 are independent of y. Furthermore, we see that

$$\alpha = -\pi - \frac{Q_{0x}}{4P_0Q_0r}, \tag{2.22}$$
$$\gamma = -\left(\pi^2 + \frac{\Psi_2}{2} + \frac{\Lambda}{6}\right) r - \frac{H_{2y} - Q_{0u}}{2Q_0} - \frac{Q_{0x}(\pi + H_1)}{2Q_0P_0}.$$

Substitution hereof in (2.21) leads to

$$P_{0u} = \frac{H_{1x}Q_0 - H_1Q_{0x}}{Q_0} + \frac{(Q_{0u} - H_{2y})P_0}{Q_0}, \tag{2.23}$$

and

$$H_{2x} = 0.$$

From the latter we see that H_2 can be written in the form

$$H_2 = H_2(u,y).$$

As H_1 and P_0 are independent of y and H_2 is independent of x, a transformation $x \longrightarrow xf(u,x)$ can be used to make H_1 equal to zero, and by a similar transformation $y \longrightarrow yg(u,y)$, we can make H_2 equal to zero.

No more information can be obtained from the first Cartan equations. The next step is to look at the directional derivatives of the spin coefficients and of Ψ_2 and Φ_{22}. The operators dual to the one forms we have just constructed are given by

$$\delta = \pi r\partial_r + \frac{1}{2P_0r}\partial_x - \frac{\mathrm{i}}{2Q_0r}\partial_y,$$
$$\Delta = \partial_u - r\left[2\gamma - \mu - 2\pi r(\pi + 2\alpha) + \left(\Psi_2 + \frac{\Lambda}{3}\right) r\right]\partial_r, \tag{2.24}$$
$$\mathrm{D} = \partial_r. \tag{2.25}$$

First, we will use the D-operator (2.25) and the expressions obtained from the NP-equations, to determine the r-dependence of the spin coefficients and of Ψ_2 and Φ_{22} (we will use a 0-index to denote quantities independent of r). It follows that

$$\begin{aligned}
\Psi_2 &= \frac{\psi_0}{r^3}, \\
\Phi_{22} &= \frac{\phi_0}{r^2}, \\
\pi &= \pi_0, \\
\mu &= \frac{\mu_0}{r} - \frac{\psi_0}{r^2} + \left(\pi_0^2 + \frac{\Lambda}{6}\right) r + \frac{\pi_0}{P_0 Q_0}\frac{\partial Q_0}{\partial x}.
\end{aligned}$$

Applying the δ-operator to $\psi_0(u,x,y)/r^3$ and $\phi_0(u,x,y)/r^2$, and comparing the result with the expressions for $\delta\Psi_2$ and $\delta\Phi_{22}$ (*i.e.* (2.17a and b)), we find that both ψ_0 and ϕ_0 are functions of the coordinate u only. From the directional derivatives of π, we see that $\pi_0 = \pi_1 Q_0$, where $\pi_1 = \pi_1(u,y)$ has to satisfy

$$\frac{1}{\pi_1}\frac{\partial \pi_1}{\partial y} = -\frac{1}{Q_0}\frac{\partial Q_0}{\partial y} \tag{2.26}$$

and

$$\frac{1}{\pi_1}\frac{\partial \pi_1}{\partial u} = -\frac{1}{Q_0}\frac{\partial Q_0}{\partial u} - \frac{\phi_0}{\psi_0}. \tag{2.27}$$

Next, we apply the Δ-operator (2.24) to (2.22), and compare the result with expression (2.18b). This leads to

$$\frac{\partial^2 Q_0}{\partial u \partial x} = \frac{1}{P_0}\frac{\partial Q_0}{\partial x}\frac{\partial P_0}{\partial u}.$$

We will use this expression, together with (2.23) and (2.26), to simplify the expressions for $\delta\gamma$ and $\bar{\delta}\gamma$. This leads to two more second order partial differential equations for Q_0:

$$\frac{\partial^2 Q_0}{\partial x^2} = \frac{1}{P_0}\frac{\partial Q_0}{\partial x}\frac{\partial P_0}{\partial x} + 4P_0^2 Q_0 \mu_0, \tag{2.28}$$

$$\frac{\partial^2 Q_0}{\partial x \partial y} = \frac{1}{Q_0}\frac{\partial Q_0}{\partial x}\frac{\partial Q_0}{\partial y}. \tag{2.29}$$

Making use of the above expressions for Q_{0xx} and Q_{0xy}, as well as of the expression (2.26), we can rewrite the real part of $\delta\mu$ as

$$\frac{\partial \mu_0}{\partial x} = -6\pi_1 Q_0 P_0 \psi_0, \tag{2.30}$$

whereas its imaginary part tells us that μ_0 is independent of y.
Next, we rewrite $\Delta\Psi_2$ and $\Delta\mu$:

$$\begin{aligned} \phi_0 &= \frac{\mathrm{d}\psi_0}{\mathrm{d}u} + \frac{3\psi_0}{Q_0}\frac{\partial Q_0}{\partial u}, \\ \frac{1}{\mu_0}\frac{\partial \mu_0}{\partial u} &= -\frac{2}{Q_0}\frac{\partial Q_0}{\partial u}. \end{aligned} \tag{2.31}$$

At this point, we have rewritten all the NP-equations as partial differential equations in the variables P_0, Q_0, π_1, μ_0, ψ_0 and ϕ_0. Let us now solve these equations, where, from now on, we will add the coordinate dependence of the functions.

From (2.29) we see that Q_0 is of the form $Q_0(u,x,y) = Q_1(u,x)Q_2(u,y)$. This allows us to solve (2.26): $\pi_1(u,y) = p(u)/Q_2(u,y)$. The solution for $Q_1(u,x)$ can be found from (2.30):

$$Q_1(u,x) = -\frac{1}{6p(u)\psi_0(u)P_0(u,x)}\frac{\partial \mu_0(u,x)}{\partial x}.$$

As ϕ_0 is a function of u only, it is clear from (2.31) that we can write Q_2 as $Q_2(u,y) = q_1(u)$ times a function of y, which we absorb in $\mathrm{d}y$. Then, we use (2.27) to eliminate $\mathrm{d}\psi_0(u)/\mathrm{d}u$ from (2.23):

$$\frac{\partial^2 \mu_0(u,x)}{\partial u \partial x} = -\frac{2}{P_0(u,x)}\frac{\partial P_0(u,x)}{\partial u}\frac{\partial \mu_0(u,x)}{\partial x}. \tag{2.32}$$

Substitution of this in (2.27) leads to

$$\frac{1}{\psi_0(u)}\frac{\mathrm{d}\psi_0(u)}{\mathrm{d}u} = \frac{1}{q_1(u)}\frac{\mathrm{d}q_1(u)}{\mathrm{d}u} - \frac{1}{p(u)}\frac{\mathrm{d}p(u)}{\mathrm{d}u} - \frac{4}{P_0(u,x)}\frac{\partial P_0(u,x)}{\partial u}. \tag{2.33}$$

It is then obvious that we can write P_0 as $P_0(u,x) = p_1(u)p_2(x)$. Hence we can easily solve (2.32) for $\mu_0(u,x)$:

$$\mu_0(u,x) = \frac{m_1(u) + m_2(x)}{p_1(u)^2},$$

after which (2.33) and (2.28) lead to

$$\begin{aligned} q_1(u) &= c_1\psi_0(u)p(u)p_1(u)^4, \\ m_1(u) &= c_2 \end{aligned}$$

and

$$p_2(x) = \frac{3s}{\sqrt{12m_2(x)^3 + 36c_2m_2(x)^2 - 18c_3 + 18c_4m_2(x)}} \frac{\mathrm{d}m_2(x)}{\mathrm{d}x},$$

where c_1,...,c_4 are arbitrary real constants and where $s = \pm 1$. If we now use $m_2(x) + c_2$ as coordinate x, rescale the coordinate y and the function $p_1(u)$ and introduce the constants a and b, defined by $c_3 = 2/3\left[b - c_2(c_2^2 + a)\right]$ and $c_4 = 2\left[c_2^2 + a/3\right]$, we find the following expressions for the one forms:

$$\begin{aligned}
\boldsymbol{\omega}^1 &= \frac{sp_1(u)r}{\sqrt{x^3 + ax - b}}\,\mathrm{d}x - \mathrm{i}sp_1(u)r\sqrt{x^3 + ax - b}\,\mathrm{d}y, \\
\boldsymbol{\omega}^2 &= \frac{sp_1(u)r}{\sqrt{x^3 + ax - b}}\,\mathrm{d}x + \mathrm{i}sp_1(u)r\sqrt{x^3 + ax - b}\,\mathrm{d}y, \\
\boldsymbol{\omega}^3 &= \mathrm{d}u, \\
\boldsymbol{\omega}^4 &= \left[\left(\frac{3x^2 + a}{16p_1(u)^3\psi_0(u)} + \frac{\mathrm{d}p_1(u)}{\mathrm{d}u}\right)\frac{r}{p_1(u)} - \frac{3x}{4p_1(u)^2} + \frac{\psi_0(u)}{r}\right]\mathrm{d}u \\
&\quad - \left(\frac{\Lambda}{6} + \frac{x^3 + ax - b}{64p_1(u)^6\psi_0(u)^2}\right)r^2\mathrm{d}u + \mathrm{d}r + \frac{r^2}{4p_1(u)^2\psi_0(u)}\mathrm{d}x.
\end{aligned}$$

It is obvious that the sign s doesn't play a role in the line element for this problem, so we can put s equal to 1. Furthermore a transformation

$$r \longrightarrow \tilde{r}/p_1(u),$$

followed by redefinition of u and $\psi_0(u)$, such that

$$\begin{aligned}
\mathrm{d}u &\longrightarrow p_1(u)\mathrm{d}\tilde{u}, \\
\psi_0(u) &= \frac{k(\tilde{u})}{2p_1(u)^3},
\end{aligned}$$

allows us to make $p_1(u)$ equal to one. The corresponding line element (dropping the tildes) is then given by

$$\begin{aligned}
\mathrm{d}s^2 &= \left[\left(\frac{\Lambda}{3} + \frac{x^3 + ax - b}{8k(u)^2}\right)r^2 - \frac{(3x^2 + a)r}{4k(u)} + \frac{3x}{2} - \frac{k(u)}{r}\right]\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r \\
&\quad - \frac{r^2}{k(u)}\mathrm{d}x\mathrm{d}u + \frac{2r^2}{x^3 + ax - b}\mathrm{d}x^2 + 2(x^3 + ax - b)r^2\mathrm{d}y^2, \qquad (2.34)
\end{aligned}$$

where there is indeed only one free function, $k(u)$, as we expected. In the above coordinates, the non-zero components of the Weyl and Ricci tensor (apart from the cosmological constant) are given by

$$\Psi_2 = \frac{1}{2}\frac{k(u)}{r^3}, \qquad \Phi_{22} = \frac{1}{2}\frac{\dot{k}(u)}{r^2}.$$

By rescaling of the coordinates u, r, x and y, the constants a and b and the function $k(u)$ in (2.34) and putting Λ equal to zero, we obtain the metric (28.74) from (Stephani et al., 2003).

2.4 Geroch Held Penrose analysis B-class

In this section, we will take a look at the B-class, *i.e.* the class of non-twisting, aligned pure radiation metrics of Petrov type D, for which the spin coefficient π *is* zero in the null tetrad constructed with the double principal null direction. We assume R $(= 4\Lambda)$, Ψ_2 and $\Phi_{22} \neq 0$ but all other components of the Weyl and Ricci tensor equal to zero. From (1.43), (1.44) and (1.49) we find that $\kappa = \sigma = \lambda = 0$. The remaining Bianchi and Ricci equations can be written as

$$\begin{aligned} \eth\Psi_2 &= 3\tau\Psi_2, & \eth\Phi_{22} &= \tau\Phi_{22} + 3\overline{\nu}\overline{\Psi_2}, \\ \eth'\Psi_2 &= 0, & \eth'\Phi_{22} &= \overline{\tau}\Phi_{22} + 3\nu\Psi_2, \\ Þ'\Psi_2 &= -\rho\Phi_{22} - 3\mu\Psi_2, & & \\ Þ\Psi_2 &= 3\rho\Psi_2, & Þ\Phi_{22} &= (\rho + \overline{\rho})\Phi_{22}, \end{aligned}$$

and

$$\begin{aligned} \eth'\mu &= \nu(\overline{\rho} - \rho), & \eth\rho &= \tau(\rho - \overline{\rho}), \\ Þ\mu &= \Psi_2 + \overline{\rho}\mu + \frac{\Lambda}{3}, & Þ\rho &= \rho^2, \\ \eth\nu &= Þ'\mu + \mu^2 + \tau\nu + \Phi_{22}, & \eth\tau &= \tau^2, \qquad (2.35) \\ \eth'\nu &= \nu\overline{\tau}, & \eth'\tau &= Þ'\rho + \frac{\Lambda}{3} + \rho\overline{\mu} + \tau\overline{\tau} + \Psi_2, \qquad (2.36) \\ Þ\nu &= \mu\overline{\tau}, & Þ\tau &= \rho\tau. \end{aligned}$$

From $[\eth', Þ]\Psi_2]$ we find that $\eth'\rho = 0$, after which $[\eth, Þ]\Phi_{22}$ leads to:

$$\rho\overline{\nu} - \overline{\rho\nu} - \tau\overline{\mu} = 0. \tag{2.37}$$

Taking the Þ-derivative of this expression and subtracting $(\rho + \overline{\rho}) \times (2.37)$ from the result, we find

$$\tau(3\overline{\Psi_2} + \Lambda) = 0. \tag{2.38}$$

As the $\eth'$-derivative of $3\overline{\Psi_2} + \Lambda = 0$ is equal to $36\overline{\tau}\overline{\Psi_2} = 0$, we have $\tau = 0$. This means that we can rewrite (2.36) as

$$\text{Þ}'\rho = -\left(\Psi_2 + \rho\overline{\mu} + \frac{\Lambda}{3}\right),$$

whereas (2.37) now becomes $(\rho - \overline{\rho})\overline{\nu} = 0$. We will also rewrite $[\eth', \text{Þ}']\rho$ as $\eth'\overline{\mu} = 0$.

Assuming now that ρ is not real (and thus $\nu = 0$), we have by $[\eth', \eth]\rho$ that

$$\mu = -\frac{\Lambda}{6\overline{\rho}} - \frac{\rho\overline{\Psi_2} - \overline{\rho}\Psi_2}{2\overline{\rho}(\rho - \overline{\rho})}.$$

Substituting this, together with $\nu = 0$, in (2.35) leads to $\Phi_{22} = 0$, which is impossible. We have thus proven that ρ is real. Herewith $[\eth', \eth]\rho$ shows that also Ψ_2 is real, whereas from $\text{Þ}'\rho$ we see that also μ is real.

The only unknowns in the system are (expressions for) $\text{Þ}'\nu$ (complex), $\text{Þ}'\mu$ (real) and $\text{Þ}'\Phi_{22}$ (real). We can use these quantities to construct four real indepent zero-weighted quantities of which not all directional derivatives are known (for example $\rho^3\text{Þ}'\Phi_{22}$, $\rho^2\text{Þ}'\mu$ and the real and imaginary parts of $(\rho\text{Þ}'\nu)/\nu$). We thus expect to see four arbitrary (distinguishing) real functions in the line element, and not five (as in Stephani et al. (2003, Chapter 28)). In the case where $\nu = 0$ (*i.e.* class A in the original paper) we can solve (2.35) for $\text{Þ}'\mu$ leaving only one arbitrary function (as the only unknown is then given by $\text{Þ}'\Phi_{22}$).

2.5 Conclusions

Frolov and Khlebnikov (1975) investigated non-rotating aligned pure radiation metrics of Petrov type D, with cosmological constant. Their solutions are grouped into three classes, A, B and C. In this chapter, we have shown that the A-class is in fact redundant, in the sense that it is a subfamily of the B-class. The C-class covers a family of metrics which are distinct from the metrics in the B-class. In the original paper the metric for the C-class is incorrect. In this chapter we re-integrate the latter and we give the correct line element (2.34) for this class of metrics. By rescaling the coordinates and parameters (and for zero cosmological constant[2]), this is exactly the same as metric (28.74) in (Stephani et al., 2003).

[2]The way to allow for a cosmological constant is given in paragraph 28.4 of (Stephani et al., 2003)

Chapter 3

Newman Tamburino metrics in the presence of an Einstein Maxwell field

3.1 Introduction

In 1962 Newman and Tamburino published the general empty space solutions for the class of metrics containing hypersurface orthogonal geodesic rays with non-vanishing shear and divergence (Newman and Tamburino, 1962; Carmeli, 1977). Their aim was to generalise the Robinson Trautman metrics to the shearing case.

Locally their solutions can be subdivided into two classes, called 'cylindrical' and 'spherical', according to whether respectively $\rho^2 - \sigma\overline{\sigma} = 0$ or $\rho^2 - \sigma\overline{\sigma} \neq 0$ holds in an open set of space-time. The terminology (Newman and Tamburino, 1962) refers to the geometry of the $u = constant$, $r = constant$ surfaces (using the original Newman Tamburino coordinates), which admit a single Killing vector in the 'cylindrical' class and which resemble distorted spheres in the 'spherical' class. It can be shown (Steele, 2004) that the vacuum Newman Tamburino spherical metrics always admit a Killing vector.

In the next chapter, we will show how these empty space solutions can be found without introducing coordinates from the beginning. We prefer to first rewrite the Ricci equations and Bianchi identities in the Geroch Held Penrose and Newman Penrose formalisms and only at a final stage we introduce coordinates. Also we give the empty space cylindrical metric in its correct and less complicated form.

In the present chapter, we first show that there exist no spherical solutions in the presence of an *aligned* Maxwell field, and that consistency with the field equations therefore requires the cylindrical condition $|\rho| = |\sigma|$. We will also show that the solutions are of Petrov type I (generalised Goldberg Sachs). In section 3.3, we prove that the cylindrical condition *forces* the Maxwell field to be aligned. Next we present the general solution of the Einstein Maxwell field equations: we first show that the Newman Penrose, Bianchi and Maxwell equations form an integrable system and then proceed to integrate the first Cartan structure equations. After obtaining the explicit form of the line element we will show how this solution is related to the vacuum cylindrical Newman Tamburino metric.

3.2 Geroch Held Penrose analysis

In this section, we will use the GHP-formalism to examine the Newman Tamburino metrics in the presence of an aligned Maxwell field. The Newman Tamburino metrics are characterised by the existence of a hypersurface orthogonal and geodesic principal null direction $\mathbf{k}$, for which the shear and divergence are non-vanishing. In terms of the NP-variables this translates into

$$\begin{aligned} \kappa &= 0, \\ \Psi_0 &= 0, \\ \rho - \overline{\rho} &= 0, \\ \sigma\rho &\neq 0, \end{aligned} \tag{3.1}$$

where equation (3.1) is the mathematical characterisation of the fact that $\mathbf{k}$ is a principal null direction of the Weyl tensor. We suppose that the Maxwell field is *aligned*, in the sense that $\mathbf{k}$ is not only a principal null direction of the Weyl tensor, but also that $\mathbf{k}$ is a principal null direction of the Maxwell tensor, which means that we can put Φ_0 equal to 0 too.

We will now null rotate the null tetrad $(\mathbf{m}, \overline{\mathbf{m}}, \mathbf{l}, \mathbf{k})$ about $\mathbf{k}$, such that $\pi = 0$. This fixes the null tetrad upto null rotations for which the rotation parameter B satisfies $\text{Þ}B = 0$ (*i.e.* the GHP-equivalent of (1.9b) for $\kappa = 0$). We can then rewrite the GHP Ricci equations $(1.25 - 1.42)$ as follows[1]:

$$
\begin{aligned}
\text{Þ}\mu &= \rho\mu + \sigma\lambda + \Psi_2 + \frac{R}{12}, & \eth'\mu &= \eth\lambda + \Psi_3 - \overline{\Phi_1}\Phi_2,\\
\text{Þ}\nu &= \overline{\tau}\mu + \tau\lambda + \Psi_3 + \overline{\Phi_1}\Phi_2, & \eth\nu &= \text{Þ}'\mu + \tau\nu + \mu^2 + \lambda\overline{\lambda} + \Phi_2\overline{\Phi_2},\\
\text{Þ}\rho &= \rho^2 + \sigma\overline{\sigma}, & \eth\rho &= \eth'\sigma - \Psi_1,\\
\text{Þ}\tau &= \tau\rho + \overline{\tau}\sigma + \Psi_1, & \eth\tau &= \text{Þ}'\sigma + \mu\sigma + \overline{\lambda}\rho + \tau^2,\\
\text{Þ}\lambda &= \rho\lambda + \overline{\sigma}\mu, & \eth'\nu &= \text{Þ}'\lambda + \Psi_4 + \lambda\mu + \lambda\overline{\mu} + \overline{\tau}\nu,\\
\text{Þ}\sigma &= 2\rho\sigma, & \text{Þ}'\rho &= \eth'\tau - \tau\overline{\tau} - \rho\overline{\mu} - \sigma\lambda - \Psi_2 - \frac{R}{12}.
\end{aligned}
$$

As ρ is real, it follows from the last equation that

$$
\eth'\tau - \eth\overline{\tau} - \rho\,(\overline{\mu} - \mu) - \sigma\lambda + \overline{\sigma}\overline{\lambda} - \Psi_2 + \overline{\Psi_2} = 0. \tag{3.2}
$$

We also rewrite the GHP Bianchi equations $(1.43 - 1.53)$ and the GHP Maxwell equations $(1.54 - 1.57)$. They are given by

$$
\begin{aligned}
\text{Þ}\Psi_1 &= 4\rho\Psi_1,\\
\text{Þ}\Psi_2 &= \eth'\Psi_1 + 2\rho\Phi_1\overline{\Phi_1} + 3\rho\Psi_2,\\
\text{Þ}\Psi_3 &= \eth'\Psi_2 + \rho\overline{\Phi_1}\Phi_2 + 2\rho\Psi_3 + \overline{\Phi_1}\eth'\Phi_1 - 2\lambda\Psi_1,\\
\text{Þ}\Psi_4 &= \eth'\Psi_3 - 2\lambda\overline{\Phi_1}\Phi_1 - 3\lambda\Psi_2 + \rho\Psi_4 + \overline{\Phi_1}\eth'\Phi_2,\\
\text{Þ}'\Psi_1 &= \eth\Psi_2 - 2\mu\Psi_1 - 2\rho\Phi_1\overline{\Phi_2} + 2\tau\Phi_1\overline{\Phi_1} - 3\tau\Psi_2 + 2\sigma\Psi_3,\\
\eth\Psi_1 &= 4\tau\Psi_1 - 3\sigma\Psi_2 - 2\sigma\Phi_1\overline{\Phi_1},\\
\eth\Psi_3 &= 3\mu\Psi_2 - \tau\overline{\Phi_1}\Phi_2 + \rho\Phi_2\overline{\Phi_2} - 2\nu\Psi_1 + \overline{\Phi_2}\eth'\Phi_1 + 2\tau\Psi_3 - \sigma\Psi_4\\
&\quad - \overline{\Phi_1}\text{Þ}'\Phi_1 + \text{Þ}'\Psi_2,\\
\eth\Psi_4 &= \text{Þ}'\Psi_3 + 4\mu\Psi_3 + \overline{\Phi_2}\eth'\Phi_2 - \overline{\Phi_1}\text{Þ}'\Phi_2 - 3\nu\Psi_2 + \tau\Psi_4 + 2\Phi_1(\nu\overline{\Phi_1} - \lambda\overline{\Phi_2}),
\end{aligned}
$$

and

$$
\begin{aligned}
\text{Þ}\Phi_1 &= 2\rho\Phi_1, & \eth\Phi_1 &= 2\tau\Phi_1 - \sigma\Phi_2,\\
\text{Þ}\Phi_2 &= \eth'\Phi_1 + \rho\Phi_2, & \eth\Phi_2 &= \text{Þ}'\Phi_1 + 2\mu\Phi_1 + \tau\Phi_2.
\end{aligned} \tag{3.3}
$$

[1]Note that the assumption $\rho \neq 0$ is in fact unnecessary, as follows from $\text{Þ}\rho = \rho^2 + \sigma\overline{\sigma}$ and the fact that $\sigma \neq 0$.

Note that Φ_1 cannot be zero, as then also Φ_2 would be zero, and we would have a vacuum field, the solutions of which are all known (Newman and Tamburino, 1962), see also chapter 4. Herewith we have proven that the Einstein Maxwell field is necessarily non-null, as for a null EM field the condition $16(\Phi_0\Phi_2 - \Phi_1^2) = 16\Phi_1^2 = 0$ must be satisfied.

More information can be found, when we apply the commutators (1.61-1.64) to the spin coefficients and tensor components. If we act with $[\eth, \text{Þ}]$ on Φ_1, we obtain an expression for $\eth'\Phi_1$:

$$\eth'\Phi_1 = -\frac{\Phi_1}{\sigma}\left(\eth'\sigma - 2\Psi_1 - \overline{\tau}\sigma\right).$$

We can use this expression in $[\eth', \text{Þ}]\Phi_1$, which, after eliminating $\text{Þ}\eth'\sigma$ by $[\eth', \text{Þ}]\sigma$, leads to

$$\eth\sigma = -\sigma\tau - \frac{4\sigma\eth\overline{\sigma}}{\overline{\sigma}} + \frac{\sigma^2\Phi_2}{\Phi_1} + \frac{2\left(2\sigma\overline{\Psi_1} + \rho\Psi_1\right)}{\overline{\sigma}}.$$

Expressions for $\eth'\Psi_1$ and $\eth'\sigma$ can be found from $[\eth, \text{Þ}]\Psi_1$ and $[\eth', \text{Þ}]\Psi_1$, respectively. First we find

$$\eth'\Psi_1 = \overline{\tau}\Psi_1 - \frac{\Psi_1\eth'\sigma}{\sigma} - \frac{4\sigma\rho\Phi_1\overline{\Phi_1} - 5\Psi_1^2}{2\sigma}, \tag{3.4}$$

from which it can be seen that Ψ_1 cannot be zero. This then allows us to rewrite $[\eth', \text{Þ}]\Psi_1$ as

$$\eth'\sigma = \frac{3\sigma\overline{\sigma}\overline{\Psi_2}}{2\overline{\Psi_1}} - \overline{\tau}\sigma - \frac{\sigma\overline{\sigma}\overline{\Phi_2}}{2\overline{\Phi_1}} + \frac{\rho\overline{\Psi_1} + 4\Psi_1\overline{\sigma}}{4\overline{\sigma}}.$$

Hence (3.4) becomes

$$\eth'\Psi_1 = 2\overline{\tau}\Psi_1 - 2\rho\Phi_1\overline{\Phi_1} - \frac{\overline{\sigma}\Psi_1\left(3\overline{\Psi_2\Phi_1} - \overline{\Psi_1\Phi_2}\right)}{2\overline{\Psi_1\Phi_1}} + \frac{\Psi_1\left(6\overline{\sigma}\Psi_1 - \rho\overline{\Psi_1}\right)}{4\sigma\overline{\sigma}}.$$

We then immediately find an expression for Φ_2 from $[\eth', \text{Þ}]\sigma$:

$$\Phi_2 = \frac{\Phi_1\left(2\sigma^2\overline{\sigma}\Psi_2 - \rho\Psi_1^2\right)}{\sigma^2\overline{\sigma}\Psi_1}.$$

Substituting the latter in (3.3), we find

$$\begin{aligned}\eth\Psi_2 \quad = \quad & \frac{\Psi_1\text{Þ}'\Phi_1}{\Phi_1} + 3\tau\Psi_2 + 2\mu\Psi_1 - \frac{\overline{\tau}\Psi_1^2}{2\sigma\overline{\sigma}} + \frac{\Psi_1^2\overline{\Psi_2}}{2\sigma\overline{\Psi_1}} - \frac{2\sigma\Psi_2\left(\Psi_2 + \Phi_1\overline{\Phi_1}\right)}{\Psi_1} \\ - \quad & \frac{\Psi_1\rho\left(4\Phi_1\overline{\Phi_1}\sigma^2\overline{\sigma} - 2\Psi_1^2\rho + \Psi_1\sigma\overline{\Psi_1}\right)}{4\sigma^3\overline{\sigma}^2}.\end{aligned}$$

We now calculate $[\text{Þ}', \text{Þ}]\Psi_1$, from which we eliminate $\text{ÞÞ}'\Phi_1$ by $[\text{Þ}', \text{Þ}]\Phi_1$. This leads to an expression for $\eth'\tau$:

$$\begin{aligned}
\eth'\tau &= \left(\sigma\overline{\sigma}\left(2\Psi_2 - \Phi_1\overline{\Phi_1}\right) - \frac{\Psi_1^2\rho}{\sigma}\right)\frac{\overline{\Psi_2}}{\Psi_1\overline{\Psi_1}} + \frac{\sigma\eth'\Psi_2}{\Psi_1} + \frac{2\sigma\overline{\tau}\Phi_1\overline{\Phi_1}}{\Psi_1} + \rho\overline{\mu} \\
&- \left(3\sigma\overline{\tau} - \frac{4\rho\left(\Psi_1\overline{\Psi_1} + \sigma\overline{\sigma}\Phi_1\overline{\Phi_1}\right) - 5\overline{\sigma}\Psi_1^2}{\overline{\sigma}\Psi_1}\right)\frac{\Psi_2}{\Psi_1} - \sigma\lambda + \frac{R}{24} \\
&+ \frac{4\Phi_1\overline{\Phi_1}\sigma\overline{\sigma}\left(2\Psi_1\sigma\overline{\sigma} - \rho^2\Psi_1 - \rho\sigma\overline{\Psi_1}\right) + \Psi_1^2\rho\left(3\Psi_1\overline{\sigma} - \overline{\Psi_1}\rho\right)}{8\Psi_1\sigma^2\overline{\sigma}^2}.
\end{aligned}$$

Substitution hereof in (3.2) gives

$$\begin{aligned}
&\frac{\sigma\eth'\Psi_2}{\Psi_1} - \frac{\overline{\sigma}\eth\overline{\Psi_2}}{\overline{\Psi_1}} - 2\sigma\lambda + 2\overline{\sigma}\overline{\lambda} - \frac{\overline{\sigma}\left(2\Phi_1\overline{\Phi_1} - 3\overline{\Psi_2}\right)\tau}{\overline{\Psi_1}} + \frac{\sigma\left(2\Phi_1\overline{\Phi_1} - 3\Psi_2\right)\overline{\tau}}{\Psi_1} \\
&- \frac{\left(3\Psi_1\overline{\Psi_1} + 4\Phi_1\overline{\Phi_1}\sigma\overline{\sigma}\right)\left(2\sigma\overline{\sigma}^2\Psi_1^2\overline{\Psi_2} - 2\sigma^2\overline{\sigma}\overline{\Psi_1}^2\Psi_2 + \sigma\Psi_1\overline{\Psi_1}^3 - \overline{\sigma}\Psi_1^3\overline{\Psi_1}\right)\rho}{8\sigma^2\overline{\sigma}^2\Psi_1^2\overline{\Psi_1}^2} \\
&- \frac{\left(4\Phi_1\overline{\Phi_1}\sigma\overline{\sigma} - 9\Psi_1\overline{\Psi_1}\right)\left(\overline{\Psi_2} - \Psi_2\right)}{4\Psi_1\overline{\Psi_1}} = 0. \qquad (3.5)
\end{aligned}$$

We will use this expression to eliminate $\eth'\Psi_2$ from $[\eth', \eth]\Phi_1$, which then leads to

$$\begin{aligned}
&\frac{\left(\sigma\overline{\Psi_1}^2 - 4\rho\Psi_1\overline{\Psi_1} + 3\overline{\sigma}\Psi_1^2\right)\rho}{8\sigma^2\overline{\sigma}^2} \\
&+ \left(\frac{2\sigma\Phi_1\overline{\Phi_1}}{\Psi_1} - \frac{\overline{\Psi_1}}{2\overline{\sigma}}\right)\overline{\tau} + \left(\frac{\left(\sigma\overline{\Psi_1} + 3\rho\Psi_1\right)}{4\sigma\overline{\Psi_1}} - \frac{\overline{\sigma}\overline{\Phi_1}\left(\sigma\overline{\Psi_1} - \rho\Psi_1\right)\Phi_1}{\Psi_1\overline{\Psi_1}}\right)\overline{\Psi_2} \\
&- \left(\frac{2\overline{\sigma}\overline{\Phi_1}\Phi_1}{\overline{\Psi_1}} + \frac{3\Psi_1}{2\sigma}\right)\tau + \left(\frac{\sigma\left(\overline{\sigma}\Psi_1 - \rho\overline{\Psi_1}\right)\Phi_1\overline{\Phi_1}}{\Psi_1^2\overline{\Psi_1}} + \frac{\left(3\overline{\sigma}\Psi_1 + \rho\overline{\Psi_1}\right)}{4\overline{\sigma}\Psi_1}\right)\Psi_2 \\
&+ \frac{\overline{\Phi_1}\left(\overline{\sigma}\rho\Psi_1^2 - \sigma\rho\overline{\Psi_1}^2 + 4\sigma\overline{\sigma}\Psi_1\overline{\Psi_1} - 4\rho^2\Psi_1\overline{\Psi_1}\right)\Phi_1}{2\Psi_1\overline{\Psi_1}\sigma\overline{\sigma}} = 0. \qquad (3.6)
\end{aligned}$$

Next, we calculate $[\eth', \eth]\sigma$, which we simplify by equation (3.6). This gives us an expression which we can solve for $\overline{\tau}$:

$$\begin{aligned}
&\frac{8\sigma^2\Phi_1\overline{\Phi_1}\overline{\tau}}{\Psi_1} - 2\tau\Psi_1 + \left(\frac{\Psi_1\rho}{\overline{\Psi_1}} - \frac{4\sigma^2\overline{\sigma}\Phi_1\overline{\Phi_1}}{\Psi_1\overline{\Psi_1}}\right)\overline{\Psi_2} + \left(\sigma - \frac{4\sigma^2\rho\Phi_1\overline{\Phi_1}}{\Psi_2}\right)\Psi_2 \\
&\quad + \frac{\rho\Psi_1\left(\overline{\sigma}\Psi_1 - \rho\overline{\Psi_1}\right)}{2\sigma\overline{\sigma}^2} + \frac{2\overline{\Phi_1}\left(2\sigma\overline{\sigma}\Psi_1 - \rho\sigma\overline{\Psi_1} - \rho^2\Psi_1\right)\Phi_1}{\overline{\sigma}\Psi_1} = 0. \qquad (3.7)
\end{aligned}$$

Making use of (3.5) and (3.7), we find an expression for τ from $[\eth', \eth]\Psi_1$:

$$\tau = \frac{\rho\overline{\Psi_2}}{2\overline{\Psi_1}} + \frac{\sigma\Psi_2}{2\Psi_1} + \frac{\rho\Psi_1}{4\sigma\overline{\sigma}} + \frac{2\sigma\overline{\Psi_1}\Phi_1\overline{\Phi_1}}{4\sigma\overline{\sigma}\Phi_1\overline{\Phi_1} + \Psi_1\overline{\Psi_1}} - \frac{\overline{\Psi_1}\rho^2\left(12\sigma\overline{\sigma}\Phi_1\overline{\Phi_1} + \Psi_1\overline{\Psi_1}\right)}{4\sigma\overline{\sigma}^2\left(4\sigma\overline{\sigma}\Phi_1\overline{\Phi_1} + \Psi_1\overline{\Psi_1}\right)}$$

and we find a relation between ρ^2 and $|\sigma|^2$:

$$\frac{32\sigma\Phi_1^2\overline{\Phi_1}^2\left(\sigma\overline{\sigma} - \rho^2\right)}{4\sigma\overline{\sigma}\Phi_1\overline{\Phi_1} + \Psi_1\overline{\Psi_1}} = 0.$$

The latter can only be zero for $\rho^2 - |\sigma|^2 = 0$, *i.e.* the cylindrical class.

We have thus proven that Newman Tamburino solutions in the presence of an *aligned* Einstein Maxwell field, are necessarily cylindrical.

From $\eth\tau$ and $\eth'\tau$, we then find expressions for $\eth'\Psi_2$ and $\text{Þ}'\sigma$:

$$\begin{aligned}
\eth'\Psi_2 &= \frac{\Psi_1\overline{\sigma}\text{Þ}'\overline{\Phi_1}}{\rho\overline{\Phi_1}} - \frac{\overline{\sigma}\Psi_1 R}{12\rho^2} - \left(\frac{\overline{\sigma}\Psi_2}{2\overline{\Psi_1}} - \frac{\overline{\sigma}\Psi_1\left(\overline{\sigma}\Psi_1 - 2\rho\overline{\Psi_1}\right)}{4\rho^3\overline{\Psi_1}}\right)\overline{\Psi_2} \\
&+ 2\lambda\Psi_1 + \frac{3\rho\Psi_2^2}{2\Psi_1} - \left(\frac{2\rho\Phi_1\overline{\Phi_1}}{\Psi_1} + \frac{\rho\overline{\Psi_1} - 6\overline{\sigma}\Psi_1}{4\rho^2}\right)\Psi_2 \\
&- \frac{\left(3\overline{\sigma}^2\Psi_1^2 + \overline{\sigma}\rho\Psi_1\overline{\Psi_1} - 2\rho^2\overline{\Psi_1}^2 + 8\overline{\sigma}\rho^3\Phi_1\overline{\Phi_1}\right)\Psi_1}{8\rho^5}, \qquad (3.8) \\
\text{Þ}'\sigma &= \frac{\sigma\text{Þ}'\Phi_1}{\Phi_1} - \frac{\sigma R}{24\rho} + \frac{\rho^2\overline{\Psi_2}^2}{4\overline{\Psi_1}^2} - \frac{3\sigma^2\Psi_2^2}{4\Psi_1^2} - \left(\frac{\rho\sigma\Psi_2}{2\Psi_1\overline{\Psi_1}} - \frac{\sigma\overline{\Psi_1} + \rho\Psi_1}{4\rho\overline{\Psi_1}}\right)\overline{\Psi_2} \\
&+ \frac{\sigma\left(\sigma\overline{\Psi_1} - 3\rho\Psi_1\right)\Psi_2}{4\rho^2\Psi_1} - \frac{3\sigma^2\overline{\Psi_1}^2 + 6\sigma\rho\Psi_1\overline{\Psi_1} - 9\rho^2\Psi_1^2}{16\rho^4} - \frac{\sigma\Phi_1\overline{\Phi_1}}{\rho}.
\end{aligned}$$

Until now, we have allowed the presence of a non-zero cosmological constant $\Lambda = R/4$, but we will show that $R = 0$. To obtain this result, we first look at the directional derivatives of $\rho^2 - |\sigma|^2 = 0$. The only information that can be obtained from these derivatives is

$$\begin{aligned}
&\frac{\left(\sigma\overline{\Psi_1} - \rho\Psi_1\right)\left(\sigma\Psi_1\overline{\Psi_1}^2 + \rho\Psi_1^2\overline{\Psi_1} - 2\sigma\rho^2\overline{\Psi_1}\Psi_2 - 2\rho^3\Psi_1\overline{\Psi_2}\right)}{4\sigma\rho^2\Psi_1\overline{\Psi_1}} \\
&\qquad - \frac{\rho^2\text{Þ}'\Phi_1}{\Phi_1} + \frac{\rho^2\text{Þ}'\overline{\Phi_1}}{\overline{\Phi_1}} - \frac{\rho R}{6} = 0. \qquad (3.9)
\end{aligned}$$

Taking into account this equation and $\rho^2 - |\sigma|^2 = 0$, the equations (3.5) and (3.8) allow us to conclude that $R = 0$.

Let us now look at $[\eth', \text{Þ}']\Psi_1$ and $[\eth, \text{Þ}']\Psi_1$. Before we rewrite those expressions, note that also Ψ_2 cannot be zero (by $\text{Þ}\Psi_2$). This means we can write

$$
\begin{aligned}
\eth'\Psi_3 = {} & \left(\frac{2\Psi_2}{\Phi_1} - \overline{\Phi_1}\right)\frac{\rho}{\sigma}\text{Þ}'\Phi_1 + \left(\frac{\Psi_2^2}{\Psi_1} + \frac{\Psi_1^3}{2\sigma^2\rho^2} - \frac{\Psi_1\Psi_2}{2\sigma\rho}\right)\frac{\overline{\Psi_1}}{2\rho} + \frac{2\rho\Psi_2\Psi_3}{\Psi_1} \\
& - \frac{3\Psi_1^4}{16\sigma^3\rho^2} - \frac{\Psi_1^2\Psi_2}{4\sigma^2\rho} - \frac{\Psi_1\Psi_3}{\sigma} + \frac{\Psi_2\left(12\lambda\sigma + 5\Psi_2\right)}{4\sigma} - \left(\frac{\Psi_1}{\sigma} + \frac{\Psi_2\rho}{\Psi_1}\right)\frac{\overline{\Psi_2}\Psi_2\rho}{\sigma\overline{\Psi_1}} \\
& + \left[\frac{3}{2}\left(\frac{3\Psi_1^2}{4\rho\sigma^2} - \frac{\Psi_2}{\sigma} - \frac{\rho\Psi_2^2}{\Psi_1^2}\right) - \left(\frac{\Psi_2}{\Psi_1} + \frac{\Psi_1}{2\sigma\rho}\right)\left(\frac{\overline{\Psi_1}}{4\rho} - \frac{\rho^2\overline{\Psi_2}}{2\sigma\overline{\Psi_1}}\right)\right]\Phi_1\overline{\Phi_1},
\end{aligned}
$$

where

$$
\begin{aligned}
\text{Þ}'\Phi_1 = {} & \left(3\text{Þ}'\Psi_2 - \frac{\rho\Psi_1\text{Þ}'\overline{\Psi_2}}{\sigma\overline{\Psi_1}}\right)\frac{\Phi_1}{4\Psi_2} - \frac{\sigma\Psi_4\Phi_1}{2\Psi_2} \\
& + \left[\frac{5}{8\rho} + \frac{3\sigma}{4\Psi_1}\left(\frac{\Psi_2}{\Psi_1} - \frac{\overline{\Psi_1}}{2\rho^2}\right) - \frac{\Psi_1}{2\Psi_2}\left(\frac{\Psi_1}{\rho^2\sigma} + \frac{\rho^3\overline{\Psi_2}^2}{2\sigma^2\overline{\Psi_1}^3}\right)\right]\Phi_1^2\overline{\Phi_1} \\
& + \left(\frac{\rho\left(3\sigma\overline{\Psi_1} - \rho\Psi_1\right)}{\Psi_1} + \frac{\Psi_1\left(\sigma\overline{\Psi_1} + \rho\Psi_1\right)}{\sigma\Psi_2}\right)\frac{\overline{\Psi_2}\Phi_1^2\overline{\Phi_1}}{4\sigma\overline{\Psi_1}^2} \\
& + \left[\frac{\Psi_1\sigma\overline{\Psi_1}^3}{128\rho^6\Psi_2} + \left(\frac{\Psi_1^2}{2\rho\Psi_2} - \sigma\right)\frac{3\overline{\Psi_1}^2}{64\rho^4} + \frac{17\Psi_1^4}{128\rho^3\sigma^2\Psi_2} - \frac{\left(6\overline{\Psi_2} + 43\Psi_2\right)\Psi_1^2}{64\sigma\rho^2\Psi_2}\right]\Phi_1 \\
& + \left(\frac{\sigma\Psi_2}{32\rho^2\Psi_1} - \frac{25\Psi_1^3}{128\sigma\rho^4\Psi_2} + \frac{\left(26\Psi_2 + 9\overline{\Psi_2}\right)\Psi_1}{64\rho^3\Psi_2} + \frac{13\sigma\Psi_2^2}{16\Psi_1^2\overline{\Psi_1}}\right)\overline{\Psi_1}\Phi_1 \\
& + \left(\frac{\left(2\rho^2\overline{\nu} - 6\nu\rho\sigma - \sigma\Psi_3 + \rho\overline{\Psi_3}\right)\Psi_1}{4\sigma\rho\Psi_2} + \frac{13\Psi_2 + 6\overline{\Psi_2}}{32\rho} - \frac{\sigma\Psi_3}{2\Psi_1}\right)\Phi_1 \\
& + \left(\frac{9\Psi_1^3}{4\rho\sigma^2\Psi_2} - \frac{\left(3\overline{\Psi_2} + 10\Psi_2\right)\Psi_1}{2\sigma\Psi_2} - \frac{\rho\Psi_2}{\Psi_1}\right)\frac{\overline{\Psi_2}\Phi_1}{16\overline{\Psi_1}} \\
& + \left(\frac{3\rho\overline{\Psi_2}}{8} + \frac{\rho^2\Psi_1\overline{\Psi_3}}{\sigma\Psi_2} - \frac{3\overline{\Psi_2}\Psi_1^2}{16\sigma\Psi_2} - \frac{7\rho^2\Psi_1\overline{\Psi_2}^2}{8\sigma\Psi_2\overline{\Psi_1}}\right)\frac{\rho\overline{\Psi_2}\Phi_1}{2\sigma\overline{\Psi_1}^2}
\end{aligned}
$$

It is obvious that, with the above equations, (3.9) becomes an expression containing $\text{Þ}'\Psi_2$ and $\text{Þ}'\overline{\Psi_2}$ instead of $\text{Þ}'\Phi_1$ and $\text{Þ}'\overline{\Phi_1}$. The (numerator of the) coefficient of $\text{Þ}'\overline{\Psi_2}$ is given by $\rho^2\left(\rho\Psi_1\overline{\Psi_2} + 3\sigma\overline{\Psi_1}\Psi_2\right)$, which can't be zero (as can be seen from this expression and its complex conjugate). We can then use (3.9), together with $[\text{Þ}', \text{Þ}]\Psi_2$ and $[\text{Þ}', \text{Þ}]\overline{\Psi_2}$, to simplify $[\text{Þ}', \text{Þ}]\rho$

from which we obtain an expression for $Þ'\Psi_2$:

$$
\begin{aligned}
Þ'\Psi_2 =& \left(\frac{\overline{\Psi_1}^2}{\rho^3} - \frac{2\overline{\Psi_2} - 9\Psi_2}{\sigma} - \frac{13\Psi_1^2}{4\rho\sigma^2}\right)\frac{\Psi_1^2}{16\rho^2} + \left(\frac{3\overline{\Psi_1}}{2\rho^3} - \frac{\overline{\Psi_2}}{\sigma\overline{\Psi_1}}\right)\frac{3\Psi_1^3}{16\sigma\rho} \\
&+ \left[\left(\frac{3\Psi_1}{2\rho^2\sigma} + \frac{\overline{\Psi_1}}{2\rho^3} - \frac{\overline{\Psi_2}}{\sigma\overline{\Psi_1}}\right)\frac{\Psi_1}{2} - \left(\frac{\rho\overline{\Psi_2}}{\overline{\Psi_1}} + \frac{\sigma\Psi_2}{\Psi_1}\right)\frac{\Psi_2}{\Psi_1}\right]\Phi_1\overline{\Phi_1} \\
&+ \left(\frac{\left(7\overline{\Psi_2} - 24\Psi_2\right)\overline{\Psi_1}}{32\rho^3} - \frac{5\sigma\overline{\Psi_1}^3}{64\rho^6}\right)\Psi_1 + \left(\frac{\sigma\overline{\Psi_1}}{2\rho\Psi_1} - 1\right)\frac{\Psi_2\Phi_1\overline{\Phi_1}}{\rho} \\
&+ \frac{21\Psi_2\sigma\overline{\Psi_1}^2}{32\rho^4} + \frac{3\left(4\sigma\Psi_4\rho + \Psi_2^2\right)}{16\rho} - \frac{\sigma\Psi_2^3}{\Psi_1^2} - \frac{3\rho^2\Psi_2\left(9\sigma\overline{\Psi_2}^2 - 4\rho^3\overline{\Psi_4}\right)}{8\sigma^2\overline{\Psi_1}^2} \\
&+ \frac{9\rho^2\Psi_2\overline{\Psi_3}}{2\sigma\overline{\Psi_1}} - \frac{3\rho^5\Psi_2\overline{\Psi_2}\overline{\Psi_3}}{\sigma^2\overline{\Psi_1}^3} + \frac{3\rho^5\Psi_2\overline{\Psi_2}^3}{2\sigma^2\overline{\Psi_1}^4} + \frac{\sigma\Psi_2\Psi_3}{2\Psi_1} - \frac{3\sigma\Psi_2^2\overline{\Psi_1}}{8\rho^2\Psi_1} \\
&+ \left(\frac{\sigma\Psi_3 - \rho\overline{\Psi_3} + 8\nu\rho\sigma}{4\sigma\rho} + \frac{\overline{\Psi_2}\rho^3\overline{\Psi_3}}{\sigma^2\overline{\Psi_1}^2} - \frac{\overline{\Psi_2}^2\rho^6\overline{\Psi_3}}{\sigma^3\overline{\Psi_1}^4}\right)\Psi_1 + \frac{3\rho\Psi_2^2\overline{\Psi_2}}{4\Psi_1\overline{\Psi_1}} \\
&+ \frac{4\rho^3\overline{\Psi_4} + 12\sigma\overline{\Psi_2}\Psi_2 + 3\sigma\overline{\Psi_2}^2}{16\sigma^2\overline{\Psi_1}}\Psi_1 + \frac{\rho^6\overline{\Psi_2}^4\Psi_1}{2\sigma^3\overline{\Psi_1}^5} \\
&- \frac{\rho^3\overline{\Psi_2}\Psi_1\left(7\sigma\overline{\Psi_2}^2 - 4\rho^3\overline{\Psi_4}\right)}{8\sigma^3\overline{\Psi_1}^3}.
\end{aligned}
$$

This allows us to find a solution for $\overline{\Psi_4}$ from (3.9):

$$
\begin{aligned}
\overline{\Psi_4} =& \frac{\sigma^4\overline{\Psi_1}^2\Psi_4}{\rho^4\Psi_1^2} + \left(\frac{2\overline{\Psi_2}}{\overline{\Psi_1}} - \frac{3\sigma\overline{\Psi_1}}{\rho^3}\right)\overline{\Psi_3} + \left(\frac{3\sigma^3\overline{\Psi_1}^2}{\rho^5\Psi_1} - \frac{2\overline{\Psi_1}^2\sigma^4\Psi_2}{\rho^4\Psi_1^3}\right)\Psi_3 \\
&- \frac{\overline{\Psi_2}^3}{\overline{\Psi_1}^2} + \frac{9\sigma\overline{\Psi_2}^2}{4\rho^3} + \frac{\sigma^4\Psi_2^3\overline{\Psi_1}^2}{\rho^4\Psi_1^4} - \frac{9\sigma^3\overline{\Psi_1}^2\Psi_2^2}{4\rho^5\Psi_1^2} - \frac{7\sigma\overline{\Psi_1}^2\left(\sigma^2\overline{\Psi_1}^2 - \rho^2\Psi_1^2\right)}{16\rho^9}.
\end{aligned}
$$

There are only two more commutation relations that provide us with new information. They are $[ð, Þ']\rho$ and $[ð, Þ']\Phi_1$. These expressions, together with the above equation for $\overline{\Psi_4}$ lead to

$$
\begin{aligned}
\text{Þ}'\Psi_3 = & \frac{4\sigma\Psi_3^2}{\Psi_1} + \frac{\left(6\rho\Psi_2\sigma - 2\sigma\rho\Phi_1\overline{\Phi_1} + \Psi_1^2\right)\overline{\Psi_2}\Psi_3}{2\sigma\Psi_1\overline{\Psi_1}} \\
& - \frac{\left(\Psi_1\overline{\Psi_1} + 4\rho^2\Phi_1\overline{\Phi_1}\right)\Psi_2\overline{\Psi_2}}{4\rho^2\sigma\overline{\Psi_1}} + \frac{\Psi_1^2\left(16\rho^2\sigma\Phi_1\overline{\Phi_1} + 4\sigma\Psi_1\overline{\Psi_1} - 3\rho\Psi_1^2\right)\overline{\Psi_2}}{32\sigma^3\rho^3\overline{\Psi_1}} \\
& - \frac{3\sigma\Psi_2^4}{2\Psi_1^3} + \frac{3\left(2\sigma\overline{\Psi_1} + \rho\Psi_1\right)\Psi_2^3}{8\rho^2\Psi_1^2} - \frac{3\left(8\sigma\rho^2\Phi_1\overline{\Phi_1} + 5\sigma\Psi_1\overline{\Psi_1} - 5\rho\Psi_1^2\right)\Psi_2^2}{16\sigma\rho^3\Psi_1} \\
& + \frac{\left(16\sigma\rho^2\overline{\Phi_1}\Phi_1\left(\sigma\overline{\Psi_1} + 3\rho\Psi_1\right) - 15\rho^2\Psi_1^3 + 4\sigma\Psi_1\overline{\Psi_1}\left(\sigma\overline{\Psi_1} + 3\rho\Psi_1\right)\right)\Psi_2}{32\rho^5\sigma^2} \\
& - \frac{\Psi_1^2\left(8\rho^2\sigma\Phi_1\overline{\Phi_1}\left(2\sigma\overline{\Psi_1} + 3\rho\Psi_1\right) + 3\rho\sigma\Psi_1^2\overline{\Psi_1} - 3\rho^2\Psi_1^3 + 4\sigma^2\Psi_1\overline{\Psi_1}^2\right)}{64\sigma^3\rho^6} \\
& - \frac{3\rho\Psi_2^3\overline{\Psi_2}}{2\Psi_1^2\overline{\Psi_1}} + \frac{\left(\sigma\overline{\Psi_1} + \rho\Psi_1\right)\left(2\sigma\rho\Phi_1\overline{\Phi_1} + \Psi_1^2\right)\Psi_3}{4\sigma\rho^3\Psi_1} + 3\nu\Psi_2 \\
& - \frac{\sigma\left(2\rho^2\Phi_1\overline{\Phi_1} + 3\Psi_1\overline{\Psi_1}\right)\Psi_2\Psi_3}{2\rho^2\Psi_1^2} + \frac{9\Psi_2^2\overline{\Psi_2}}{8\sigma\overline{\Psi_1}} - \frac{\sigma\Psi_2^2\Psi_3}{\Psi_1^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_4 &= \left(\frac{2\Psi_2}{\Psi_1} - \frac{\Psi_1}{\rho\sigma}\right)\Psi_3 - \frac{\Psi_2^3}{\Psi_1^2} + \frac{3\Psi_2^2}{4\rho\sigma} - \frac{\Psi_1^4}{16\rho^3\sigma^3}, \\
\overline{\Psi_3} &= \frac{\sigma^2\overline{\Psi_1}\Psi_3}{\rho^2\Psi_1} - \frac{3\left(\sigma^2\Psi_2^2\overline{\Psi_1}^2 - \rho^2\Psi_1^2\overline{\Psi_2}^2\right)}{4\Psi_1^2\overline{\Psi_1}\rho^2} - \frac{3\overline{\Psi_1}\left(\sigma^2\overline{\Psi_1}^2 - \rho^2\Psi_1\right)}{16\rho^6}.
\end{aligned}
$$

No other information can be obtained from the GHP-analysis. In summary, we have

- explicit expressions for R, τ, Ψ_0, Ψ_4 and Φ_2,
- an expression for $\overline{\Psi_3}$ as a function of Ψ_1, $\overline{\Psi_1}$, Ψ_2, $\overline{\Psi_2}$, Ψ_3, ρ and σ,

we also know the solutions belong to the cylindrical class ($\rho^2 = |\sigma|^2$), and we have expressions for the directional derivatives of zero-weighted quantities, such as Φ_1 and Ψ_2 (but also $\rho\mu$, $\lambda\sigma$, etc.). We will now use this information to continue the analysis in the Newman Penrose formalism.

3.3 Newman Penrose analysis

In this section we work in the Newman Penrose formalism. Before we copy any rotation and boost invariant information from the GHP-analysis in the previous section, we will first prove that a cylindrical Newman Tamburino metric only admits a non-null Maxwell field if the field is aligned. Note that we have shown in the previous section that an aligned Maxwell field can only occur for a cylindrical Newman Tamburino metric. Whether or not there exist spherical Newman Tamburino metrics in the presence of a non-aligned Maxwell field, remains an open problem.

In this section, we will only consider cylindrical metrics, *i.e.* those metrics for which

$$\rho^2 - \overline{\sigma}\sigma = 0. \tag{3.10}$$

The Newman Tamburino solutions are further characterised by

$$\Psi_0 = 0 = \overline{\rho} - \rho = \kappa,$$

$$\sigma\rho \neq 0.$$

At the moment, we don't assume the Maxwell field to be aligned.

We will first fix the rotational degree of freedom by making $\sigma\rho > 0$. By (3.10) we have that

$$\sigma = \rho.$$

The Newman Penrose equations (1.25) and (1.26) then show that ϵ is real and that $\Phi_0 = 0$, which means that the Maxwell field is aligned. We can then use a boost with parameter A satisfying $\epsilon' = \epsilon + 1/2\mathrm{D}\ln A = 0$ to make $\epsilon = 0$. If we add the condition for A that $\delta \ln A = \tau - \overline{\alpha} - \beta$, and take into account $(1.27 - 1.29)$, (1.36) and (1.41), we see that we can put

$$\beta = \tau - \overline{\alpha}.$$

We now null rotate such that $\pi = 0$. As the Maxwell field is aligned, we can copy the expressions for R, τ, Φ_2, $\overline{\Psi_3}$ and Ψ_4 from the previous section:

$$R = 0,$$

$$
\begin{aligned}
\tau &= \frac{\rho\Psi_2}{2\Psi_1} + \frac{\rho\overline{\Psi_2}}{2\overline{\Psi_1}} - \frac{\overline{\Psi_1} - \Psi_1}{4\rho}, \\
\Phi_2 &= \frac{\Phi_1\left(2\rho^2\Psi_2 - \Psi_1^2\right)}{2\Psi_1\rho^2}, \\
\overline{\Psi_3} &= \frac{\overline{\Psi_1}\Psi_3}{\Psi_1} - \frac{3\left(\overline{\Psi_1}^2\Psi_2^2 - \overline{\Psi_2}^2\Psi_1^2\right)}{4\overline{\Psi_1}\Psi_1^2} - \frac{3\overline{\Psi_1}\left(\overline{\Psi_1}^2 - \Psi_1^2\right)}{16\rho^4}, \qquad (3.11) \\
\Psi_4 &= \frac{\left(2\rho^2\Psi_2 - \Psi_1^2\right)\left(\Psi_1^4 + 2\Psi_1^2\Psi_2\rho^2 + 16\Psi_1\Psi_3\rho^4 - 8\Psi_2^2\rho^4\right)}{16\Psi_1^2\rho^6}.
\end{aligned}
$$

We also 'copy' the derivatives of the zero-weighted quantities Φ_1 and Ψ_2:

$$
\begin{aligned}
\delta\Phi_1 &= \left(\frac{\overline{\Psi_2}\rho}{\overline{\Psi_1}} - \frac{\overline{\Psi_1} - 2\Psi_1}{2\rho}\right)\Phi_1, \\
\bar{\delta}\Phi_1 &= \left(\frac{\Psi_1}{2\rho} + \frac{\rho\Psi_2}{\Psi_1}\right)\Phi_1, \\
\Delta\Phi_1 &= \left(\frac{2\rho\Psi_3}{\Psi_1} - \frac{3\rho\Psi_2^2}{2\Psi_1^2}\right)\Phi_1 \\
&+ \left(\left(\frac{\rho\overline{\Psi_2}}{2\Psi_1\overline{\Psi_1}} - \frac{\overline{\Psi_1} - 2\Psi_1}{4\rho\Psi_1}\right)\Psi_2 + \frac{\Psi_1\overline{\Psi_2}}{4\rho\overline{\Psi_1}} - \frac{\Psi_1\left(\overline{\Psi_1} - \Psi_1\right)}{8\rho^3}\right)\Phi_1, \\
\mathrm{D}\Phi_1 &= 2\rho\Phi_1, \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
\delta\Psi_2 &= 2\rho\Psi_3 + 2\mu\Psi_1 - \frac{\left(2\rho^2\Psi_2 + \Psi_1^2\right)\left(2\rho^2\Phi_1\overline{\Phi_1} + \Psi_1\overline{\Psi_1}\right)}{2\rho^3\Psi_1} \\
&+ \frac{\left(4\rho^2\Psi_2 + \Psi_1^2\right)\overline{\Psi_2}}{2\rho\overline{\Psi_1}} - \frac{2\rho\Psi_2^2}{\Psi_1} + \frac{3\Psi_1^3}{4\rho^3} + \frac{\Psi_1\Psi_2}{\rho}, \qquad (3.13) \\
\bar{\delta}\Psi_2 &= 2\rho\Psi_3 + 2\lambda\Psi_1 - \left(\frac{2\rho\Psi_2}{\Psi_1} + \frac{\Psi_1}{\rho}\right)\Phi_1\overline{\Phi_1} + \frac{3\Psi_1\Psi_2}{2\rho} - \frac{\Psi_1^2\overline{\Psi_1}}{4\rho^3}, \qquad (3.14) \\
\mathrm{D}\Psi_2 &= 4\rho\Psi_2 + \frac{\Psi_1^2}{\rho}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\Psi_2 &= \left(\frac{\rho\overline{\Psi_2}}{\overline{\Psi_1}} + \frac{5\rho\Psi_2}{\Psi_1} - \frac{\overline{\Psi_1} + \Psi_1}{2\rho}\right)\Psi_3 + 2\nu\Psi_1 + \frac{\Psi_1\left(\overline{\Psi_1} + 3\Psi_1\right)\Phi_1\overline{\Phi_1}}{4\rho^3} \\
&+ \left(\frac{\Psi_2\left(\overline{\Psi_1} - 2\Psi_1\right)}{2\rho\Psi_1} - \frac{\rho\Psi_2^2}{\Psi_1^2} - \left(\frac{\rho\Psi_2}{\Psi_1\overline{\Psi_1}} + \frac{\Psi_1}{2\rho\overline{\Psi_1}}\right)\overline{\Psi_2}\right)\Phi_1\overline{\Phi_1} \\
&+ \left(\frac{3\Psi_1\Psi_2}{4\rho\overline{\Psi_1}} - \frac{\Psi_1^2}{8\rho^3}\right)\overline{\Psi_2} - \frac{4\rho\Psi_2^3}{\Psi_1^2} + \frac{3\Psi_2^2}{4\rho} - \frac{3\Psi_1\left(2\overline{\Psi_1} - 3\Psi_1\right)\Psi_2}{8\rho^3} \\
&+ \frac{\Psi_1^2\left(\overline{\Psi_1} + 4\Psi_1\right)\left(\overline{\Psi_1} - \Psi_1\right)}{16\rho^5}. \qquad (3.15)
\end{aligned}
$$

More rotation and boost invariant information could be extracted from GHP, such as the derivatives of $\rho\mu$, but this does not simplify the calculations in NP.

We rewrite the NP Ricci, Bianchi and Maxwell equations: from $(1.25 - 1.33)$ and $(1.43 - 1.45)$, we get

$$
\begin{aligned}
&\mathrm{D}\mu = (\mu+\lambda)\,\rho + \Psi_2, \\
&\mathrm{D}\nu = \left(\frac{\left(\overline{\Psi_2}\Psi_1 + \Psi_2\overline{\Psi_1}\right)\rho}{2\Psi_1\overline{\Psi_1}} + \frac{\overline{\Psi_1}-\Psi_1}{4\rho}\right)(\mu+\lambda) + \Psi_3 \\
&\quad + \frac{\left(2\Psi_2\rho^2 - \Psi_1^2\right)\Phi_1\overline{\Phi_1}}{2\rho^2\Psi_1}, \\
&\mathrm{D}\rho = 2\rho^2, \\
&\mathrm{D}\alpha = \frac{\left(\overline{\Psi_2}\Psi_1 + \Psi_2\overline{\Psi_1}\right)\rho^2}{2\Psi_1\overline{\Psi_1}} + (\alpha - \overline{\alpha})\,\rho - \frac{\overline{\Psi_1}-\Psi_1}{4}, \\
&\mathrm{D}\gamma = \frac{\left(\overline{\Psi_2}\Psi_1 + \Psi_2\overline{\Psi_1}\right)^2\rho^2}{4\Psi_1^2\overline{\Psi_1}^2} - \frac{\left(\overline{\Psi_2}\Psi_1 + \Psi_2\overline{\Psi_1}\right)(\overline{\alpha}-\alpha)\,\rho}{2\Psi_1\overline{\Psi_1}} + \Psi_2 + \Phi_1\overline{\Phi_1} \\
&\quad - \frac{\left(\overline{\Psi_1}-\Psi_1\right)(\alpha+\overline{\alpha})}{4\rho} - \frac{\left(\overline{\Psi_1}-\Psi_1\right)^2}{16\rho^2}, \\
&\mathrm{D}\lambda = \rho\,(\mu+\lambda)\,, \\
&\delta\Psi_1 = \left(\frac{3\Psi_1\overline{\Psi_2}}{\overline{\Psi_1}} - 2\Phi_1\overline{\Phi_1}\right)\rho - 2\overline{\alpha}\Psi_1 - \frac{3\Psi_1\left(\overline{\Psi_1}-\Psi_1\right)}{2\rho}, \\
&\overline{\delta}\Psi_1 = \left(\Psi_2 - 2\overline{\Phi_1}\Phi_1\right)\rho + 2\alpha\Psi_1 + \frac{\Psi_1^2}{\rho}, \\
&\mathrm{D}\Psi_1 = 4\rho\Psi_1,
\end{aligned}
\tag{3.16}
$$

whereas from (1.57) we find an expression for $\delta\rho$, after which (1.35) can be solved for Ψ_2:

$$
\begin{aligned}
\delta\rho &= \frac{2\overline{\Psi_2}\rho^2}{\overline{\Psi_1}} - 2\rho\overline{\alpha} - \frac{3\left(\overline{\Psi_1}-\Psi_1\right)}{4}, \\
\Psi_2 &= \frac{2\alpha\Psi_1}{\rho} + \frac{\Psi_1\left(3\Psi_1 - \overline{\Psi_1}\right)}{4\rho^2}.
\end{aligned}
$$

If we substitute the last expression in $(3.13 - 3.15)$, we find the directional derivatives of α:

$$\delta\alpha = \frac{\Psi_3\rho^2}{\Psi_1} + \mu\rho + 2\alpha\overline{\alpha} - 4\alpha^2 + \frac{\overline{\alpha}\Psi_1 + 3\alpha\left(\overline{\Psi_1} - 2\Psi_1\right)}{2\rho} + \frac{7\Psi_1\overline{\Psi_1} - 7\Psi_1^2}{16\rho^2},$$

$$\overline{\delta}\alpha = \frac{\Psi_3\rho^2}{\Psi_1} + \lambda\rho - 2\alpha^2 + \frac{\alpha\left(\overline{\Psi_1} - \Psi_1\right)}{\rho} + \frac{3\Psi_1\left(\overline{\Psi_1} - \Psi_1\right)}{16\rho^2},$$

$$\begin{aligned}\Delta\alpha = {} & \left(\frac{\alpha}{\rho} + \frac{3\Psi_1 - \overline{\Psi_1}}{4\rho^2}\right)\Delta\rho - \left(\frac{\alpha}{\Psi_1} + \frac{6\Psi_1 - \overline{\Psi_1}}{8\Psi_1\rho}\right)\Delta\Psi_1 + \frac{\Delta\overline{\Psi_1}}{8\rho} + \nu\rho \\ & + \frac{\Psi_3\left(5\alpha + \overline{\alpha}\right)\rho}{\Psi_1} - \frac{\left(\overline{\Psi_1} - 3\Psi_1\right)\Psi_3}{2\Psi_1} - \frac{2\alpha\left(\alpha + \overline{\alpha}\right)\Phi_1\overline{\Phi_1}}{\Psi_1} - \frac{16\alpha^3}{\rho} \\ & + \frac{\left(\alpha(\overline{\Psi_1} - 9\Psi_1) + \overline{\alpha}(\overline{\Psi_1} - 5\Psi_1)\right)\Phi_1\overline{\Phi_1}}{4\Psi_1\rho} + \frac{3\alpha(4\alpha\overline{\Psi_1} - 11\alpha\Psi_1 + \overline{\alpha}\Psi_1)}{2\rho^2} \\ & + \frac{\overline{\alpha}(9\Psi_1 - 5\overline{\Psi_1})\Psi_1 + 3\alpha(21\Psi_1\overline{\Psi_1} - 4\overline{\Psi_1}^2 - 25\Psi_1^2)}{16\rho^3} \\ & + \frac{\left(\overline{\Psi_1} - \Psi_1\right)\left(4\rho^2\Phi_1\overline{\Phi_1} + 13\Psi_1^2 - 7\Psi_1\overline{\Psi_1} + \overline{\Psi_1}^2\right)}{32\rho^4}.\end{aligned}$$

The remaining NP Ricci and Bianchi equations can be written as

$$\begin{aligned}\delta\lambda &= \overline{\delta}\mu + \mu(\alpha + \overline{\alpha}) - \lambda(3\alpha - \overline{\alpha}) - \Psi_3 + \frac{\mu\overline{\Psi_1} - 3\lambda\Psi_1}{2\rho} \\ &- \frac{\Phi_1\overline{\Phi_1}\left(\overline{\Psi_1} - \Psi_1 - 8\alpha\rho\right)}{4\rho^2}, \\ \Delta\lambda &= \overline{\delta}\nu + \lambda(\overline{\gamma} - 3\gamma - \mu - \overline{\mu}) + 2\alpha\nu - \frac{4\Psi_3\alpha}{\rho} + \frac{8\Psi_1\alpha^3}{\rho^3} \\ &- \frac{3\Psi_1\alpha^2\left(\overline{\Psi_1} - 2\Psi_1\right)}{\rho^4} + \frac{3\Psi_1\alpha\left(\overline{\Psi_1} - \Psi_1\right)\left(\overline{\Psi_1} - 3\Psi_1\right)}{8\rho^5} \\ &- \frac{\Psi_1\left(\overline{\Psi_1} - 4\Psi_1\right)\left(\overline{\Psi_1} - \Psi_1\right)^2}{64\rho^6} + \frac{\left(\overline{\Psi_1} - \Psi_1\right)\Psi_3}{2\rho^2}, \\ \Delta\mu &= \delta\nu - \mu(\mu + \gamma + \overline{\gamma}) + 2\alpha\nu - \lambda\overline{\lambda} + \frac{\nu\Psi_1}{\rho} \\ &- \left(4\alpha\overline{\alpha} - \frac{\left(\overline{\Psi_1} - \Psi_1\right)\left(\overline{\alpha} - \alpha\right)}{2\rho} - \frac{\left(\overline{\Psi_1} - \Psi_1\right)^2}{16\rho^2}\right)\frac{\Phi_1\overline{\Phi_1}}{\rho^2}, \\ \Delta\rho &= \frac{2\Psi_3\rho^2}{\Psi_1} + (3\gamma - \overline{\gamma})\rho - \Phi_1\overline{\Phi_1} + \overline{\alpha}^2 - 9\alpha^2 + \frac{\Psi_1\left(5\overline{\Psi_1} - 7\Psi_1\right)}{8\rho^2} \\ &+ \frac{3\overline{\alpha}\Psi_1 + 5\overline{\alpha}\overline{\Psi_1} - 29\alpha\Psi_1 + 9\alpha\overline{\Psi_1}}{4\rho},\end{aligned}$$

and

$$\begin{aligned}
\delta\gamma &= \frac{(\Psi_1+(2\alpha+\overline{\alpha})\rho)\,\mu}{\rho}+\frac{\left(8\rho\,(\overline{\alpha}-2\alpha)+3\left(\overline{\Psi_1}-\Psi_1\right)\right)\Phi_1\overline{\Phi_1}}{8\rho^2}\\
&+ \frac{\alpha(\overline{\alpha}^2-2\alpha\overline{\alpha}-11\alpha^2)}{\rho}+\frac{\Psi_1\overline{\alpha}^2+4\overline{\Psi_1}\overline{\alpha}\alpha+6\overline{\Psi_1}\alpha^2-25\Psi_1\alpha^2}{2\rho^2}\\
&+ \frac{\Psi_1\left(29\alpha\overline{\Psi_1}+7\overline{\alpha}\overline{\Psi_1}+5\overline{\alpha}\Psi_1-61\alpha\Psi_1\right)}{16\rho^3}+\frac{11\Psi_1^2\left(\overline{\Psi_1}-\Psi_1\right)}{32\rho^4}\\
&+ \frac{\left(\Psi_1+\rho(3\alpha+\overline{\alpha})\right)\Psi_3}{\Psi_1}+\alpha\overline{\lambda}, \qquad (3.17)
\end{aligned}$$

$$\begin{aligned}
\overline{\delta}\gamma &= \Psi_3+\overline{\mu}\alpha+(\overline{\alpha}+2\alpha)\lambda+\frac{\Psi_3(3\alpha+\overline{\alpha})\rho}{\Psi_1}-\left(\frac{\alpha}{\rho}-\frac{\overline{\Psi_1}-\Psi_1}{8\rho^2}\right)\Phi_1\overline{\Phi_1}\\
&- \frac{\alpha(9\alpha^2+2\alpha\overline{\alpha}+\overline{\alpha}^2)}{\rho}-\frac{\alpha\left(7\alpha\Psi_1-3\alpha\overline{\Psi_1}+\overline{\alpha}\Psi_1\right)}{\rho^2}+\frac{\Psi_1^2(\overline{\Psi_1}-\Psi_1)}{32\rho^4}\\
&+ \frac{(17\alpha+3\overline{\alpha})(\overline{\Psi_1}-\Psi_1)\Psi_1-4\alpha\overline{\Psi_1}^2}{16\rho^3}+\frac{\lambda\Psi_1}{\rho}, \qquad (3.18)
\end{aligned}$$

$$\begin{aligned}
\Delta\Psi_1 &= 4\rho\Psi_3-2(\alpha+\overline{\alpha}+\frac{\Psi_1}{2\rho})\Phi_1\overline{\Phi_1}+2\Psi_1\gamma-\frac{2\alpha\Psi_1\,(7\alpha-\overline{\alpha})}{\rho}\\
&- \left(\frac{\overline{\Psi_1}^2-5\Psi_1\overline{\Psi_1}+5\Psi_1^2}{\rho}-41\alpha\Psi_1+15\alpha\overline{\Psi_1}+7\overline{\alpha}\Psi_1-\overline{\alpha}\overline{\Psi_1}\right)\frac{\Psi_1}{4\rho^2},
\end{aligned}$$

$$\begin{aligned}
\delta\Psi_3 &= \left(\frac{6\alpha\Psi_1}{\rho}-\frac{3\Psi_1\left(\overline{\Psi_1}-3\Psi_1\right)}{4\rho^2}\right)\mu+2\Psi_3(3\alpha+2\overline{\alpha})-\frac{24\Psi_1\alpha^3}{\rho^2}\\
&+ \left(\frac{\Psi_1\left(\overline{\Psi_1}-\Psi_1\right)}{4\rho^3}-\frac{2\rho\Psi_3}{\Psi_1}-\frac{\Psi_1\,(3\alpha+\overline{\alpha})}{\rho^2}\right)\Phi_1\overline{\Phi_1}-\frac{\Psi_3\left(\overline{\Psi_1}-5\Psi_1\right)}{2\rho}\\
&+ \frac{3\Psi_1^2\alpha\,(\overline{\alpha}-9\alpha)}{\rho^3}+\frac{\left(\overline{\Psi_1}\Psi_1-\Psi_1^2\right)\left(48\Psi_1^2-23\Psi_1\overline{\Psi_1}+3\overline{\Psi_1}^2\right)}{64\rho^5}\\
&- \frac{\Psi_1\left(9\alpha\overline{\Psi_1}^2-51\alpha\Psi_1\overline{\Psi_1}+5\overline{\alpha}\Psi_1\overline{\Psi_1}+66\alpha\Psi_1^2-9\overline{\alpha}\Psi_1^2\right)}{8\rho^4}+\frac{9\alpha^2\Psi_1\overline{\Psi_1}}{\rho^3},
\end{aligned}$$

$$
\begin{aligned}
\bar{\delta}\Psi_3 = {} & \frac{3\Psi_1\left(8\rho\alpha + 3\Psi_1 - \overline{\Psi_1}\right)\lambda}{4\rho^2} - \left(\frac{2\Psi_3\rho}{\Psi_1} - \frac{\Psi_1\left(\overline{\Psi_1} - \Psi_1 - 8\rho\alpha\right)}{2\rho^3}\right)\Phi_1\overline{\Phi_1} \\
& - \frac{(3\overline{\Psi_1} - 7\Psi_1)\Psi_3}{2\rho} + \frac{\Psi_1\left(\overline{\Psi_1} - \Psi_1 - 8\rho\alpha\right)\left(3\overline{\Psi_1}^2 - 7\Psi_1\overline{\Psi_1} + 12\Psi_1^2\right)}{64\rho^5} \\
& + \frac{\Psi_1\left(\overline{\Psi_1} - \Psi_1 - 8\rho\alpha\right)\left(24\rho^2\alpha^2 - 6\overline{\Psi_1}\rho\alpha + 15\alpha\rho\Psi_1\right)}{8\rho^5} + 10\alpha\Psi_3, \\
\Delta\Psi_3 = {} & \frac{\Psi_1\left(8\rho\alpha + \Psi_1 - \overline{\Psi_1}\right)\left(\overline{\Psi_1} - \Psi_1 - 8\rho(3\alpha + 2\overline{\alpha})\right)\Phi_1\overline{\Phi_1}}{32\rho^5} - \frac{3\Psi_1^2}{4\rho^3} \\
& - \frac{2(\alpha + \overline{\alpha})\Psi_3\Phi_1\overline{\Phi_1}}{\Psi_1} - 2\Psi_3\gamma - \frac{3\Psi_1\left(\overline{\Psi_1} - 3\Psi_1 - 8\rho\alpha\right)\nu}{4\rho^2} + \frac{4\Psi_3^2\rho}{\Psi_1} \\
& - \frac{\overline{\Psi_1}\left(3\overline{\alpha} - 5\alpha\right) + \Psi_1\left(9\alpha - 7\overline{\alpha}\right) + 8\left(\alpha - 3\overline{\alpha}\right)\alpha\rho}{2\rho^2} + \frac{\Psi_1\overline{\Psi_1}}{\rho^3} - \frac{\overline{\Psi_1}^2}{4\rho^3} \\
& + \left(\overline{\Psi_1} - \Psi_1 - 8\rho\alpha\right)\Bigg[\frac{9\Psi_1^4}{128\rho^7} + \left(\alpha^2 - \frac{\alpha\overline{\Psi_1}}{4\rho} + \frac{\overline{\Psi_1}^2}{64\rho^2}\right)\frac{3\left(\overline{\alpha} + \alpha\right)\Psi_1}{\rho^4} \\
& \quad + \left(27\alpha^2 + 15\alpha\overline{\alpha} + \frac{\overline{\Psi_1}^2}{8\rho^2} - \frac{\left(27\alpha + 7\overline{\alpha}\right)\overline{\Psi_1}}{8\rho}\right)\frac{\Psi_1^2}{8\rho^5} \\
& \quad + \left(15\alpha + 3\overline{\alpha} - \frac{11\overline{\Psi_1}}{8\rho}\right)\frac{\Psi_1^3}{16\rho^6}\Bigg], \\
\mathrm{D}\Psi_3 = {} & 4\rho\Psi_3 - \frac{\Psi_1\Phi_1\overline{\Phi_1}}{\rho} + \frac{3\alpha\Psi_1^2}{\rho^2} + \frac{\Psi_1^2\left(9\Psi_1 - 5\overline{\Psi_1}\right)}{8\rho^3}. \qquad (3.19)
\end{aligned}
$$

As ρ is real, we have from $\Delta\rho$ that

$$
\overline{\gamma} = \gamma + \frac{\overline{\alpha}^2 - \alpha^2}{\rho} + \frac{\overline{\alpha}\overline{\Psi_1} - \alpha\Psi_1}{\rho^2} + \frac{\overline{\Psi_1}^2 - \Psi_1^2}{8\rho^3}. \qquad (3.20)
$$

This is all the information that can be extracted from the Newman Penrose Bianchi and Maxwell equations. The derivatives of (3.11) and (3.20) give no new information, as is also the case for the commutators applied to the non-zero spin coefficients or Weyl and Maxwell tensor components.

We note here that Ψ_1 may be real, but not imaginary. This can be seen from the δ-derivative of $\Psi_1 + s\overline{\Psi_1} = 0$, $s = \pm 1$, from which we obtain

$$
\frac{(s+1)\left(\Psi_1^2 - 2\rho^2\Phi_1\overline{\Phi_1}s\right)}{s\rho} = 0,
$$

which excludes $s = +1$.

We proceed by isolating the ρ-dependence of all variables. Equations (3.12) and (3.16) allow us to introduce variables ϕ_1 and ψ_1, respectively defined by

$$\Phi_1 = \phi_1 \rho, \qquad \mathrm{D}\,\phi_1 = 0,$$

$$\Psi_1 = \psi_1 \rho^2, \qquad \mathrm{D}\,\psi_1 = 0.$$

We also define a variable L as

$$L = \log |\rho| \,.$$

Since

$$\mathrm{D}\,(\alpha/\rho) = \frac{1}{4}\,\psi_1\,\mathrm{D}\,L,$$

we see that

$$\alpha = \frac{1}{4}\,\psi_1\,L\,\rho + a\,\rho, \qquad \mathrm{D}\,a = 0. \tag{3.21}$$

This enables us to put a equal to zero by a null rotation with parameter $\overline{B} = \left(\frac{1}{4}\psi_1\,L\,\rho - \alpha\right) / \,\rho$ (which is compatible with the condition $\mathrm{D}\,B = 0$). Comparing the derivatives of α as given by (3.21) with those we already obtained before, we find the following expressions for λ, μ and ν:

$$\lambda = \frac{1}{4}\left(L^2 + \frac{5}{2}L + \frac{3}{2}\right)\rho\,\psi_1^2 - \frac{1}{8}\,(3\,L+1)\,\rho\,\psi_1\overline{\psi_1} - \frac{1}{2}\,\rho\,L\,\phi_1\overline{\phi_1} - \frac{\Psi_3}{\rho\,\psi_1},$$

$$\mu = \frac{1}{2}\left(\frac{1}{2}L^2 + \frac{7}{4}L + 1\right)\rho\,\psi_1^2 - \frac{1}{4}\left(\frac{3}{2}L + 1\right)\rho\psi_1\overline{\psi_1} - \frac{1}{2}\rho L\phi_1\overline{\phi_1} - \frac{\Psi_3}{\rho\psi_1},$$

$$\begin{aligned}\nu = {}& \frac{1}{8}\left(\frac{1}{2}L^3 + L^2 - \frac{7}{4}L - \frac{5}{4}\right)\rho\psi_1^3 + \frac{1}{8}\left(\frac{1}{2}L^3 + \frac{1}{4}L^2 + L + 1\right)\rho\psi_1^2\overline{\psi_1}^2 \\ & - \frac{1}{8}\left[\left(\frac{3}{4}\overline{\psi_1}^2 + \phi_1\overline{\phi_1}\right)L^2 + 2\left(\frac{1}{8}\overline{\psi_1}^2 - \phi_1\overline{\phi_1}\right)L + \left(\frac{1}{4}\overline{\psi_1}^2 + \phi_1\overline{\phi_1}\right)\right]\rho\psi_1 \\ & + \frac{1}{2}\gamma\,\psi_1 - \frac{1}{8}\left(L^2+1\right)\rho\overline{\psi_1}\phi_1\overline{\phi_1} - \frac{1}{4}\left[\left(1 + \frac{\overline{\psi_1}}{\psi_1}\right)L - 2\right]\frac{\Psi_3}{\rho}.\end{aligned}$$

Herewith we can integrate the $\mathrm{D}\,\Psi_3$-equation (3.19), which gives

$$\Psi_3 = \left(\frac{3}{16}\,(L+3)\,L\psi_1^3 - \frac{5}{16}L\psi_1^2\overline{\psi_1} - \frac{1}{2}L\psi_1\phi_1\overline{\phi_1} + \psi_3\right)\rho^2, \quad \mathrm{D}\,\psi_3 = 0.$$

We now come to a more subtle part of the integration. First we introduce a help variable $\Omega = \delta\,\rho/\mathrm{D}\,\rho = \frac{1}{8}\left([2L+3]\,\overline{\Psi_1}+\Psi_1\right)$ and define two new operators e_1 and e_2 having the property that $e_1\,\rho = e_2\,\rho = 0$ and $\mathrm{D}\,e_1\,x = \mathrm{D}\,e_2\,x = 0$ for all x obeying $\mathrm{D}\,x = 0$. One easily sees from the $[\delta, \mathrm{D}]$-commutator that this can be achieved by putting $\delta = \rho\,e_2 + e_1 + \Omega\,\mathrm{D}$ and $\overline{\delta} = \rho\,e_2 - e_1 + \overline{\Omega}\,\mathrm{D}$ and hence

$$\overline{e_1} = -e_1 \qquad \text{and} \qquad \overline{e_2} = e_2. \tag{3.22}$$

It follows that $[e_1, \mathrm{D}] = 0$ and $[e_2, \mathrm{D}] = 0$, while expressions for the commutators $[e_1, \Delta]$ and $[e_2, \Delta]$ can be derived from $[\delta, \Delta]$ and $[\overline{\delta}, \Delta]$. They read

$$[e_1, \Delta] = \frac{1}{16}\,\rho\,L\left(\overline{\psi_1}+\psi_1\right)^2\,e_1,$$

$$\begin{aligned}
[e_2, \Delta] = {} & \frac{1}{4}\left(\psi_1^2 - \overline{\psi_1}^2\right)e_1 + 2\,\gamma\,e_2 - \frac{1}{8}\Bigg[\left(L^2+\frac{7}{2}L+2\right)\psi_1^2 \\
& + \left((L+3)\,\psi_1 - \frac{1}{2}\overline{\psi_1}\right)L\overline{\psi_1} + 2(L+1)\phi_1\overline{\phi_1}\Bigg]\rho e_2 \\
& + \frac{1}{64}\Bigg[(L+7)\,\psi_1^3 - \left(2L^2+5L-3\right)\psi_1^2\overline{\psi_1} \\
& - \left(\frac{1}{4}\left(8L^2+5L+3\right)\overline{\psi_1}^2 + 4\left(2L^2+L-1\right)\phi_1\overline{\phi_1}\right)\psi_1 \\
& - 8\left(L^2+2L-2\right)\overline{\psi_1}\phi_1\overline{\phi_1} + (L+1)\,\overline{\psi_1}^3 - 32\left(\frac{\overline{\psi_1}}{\psi_1}-1\right)\psi_3\Bigg]\mathrm{D}.
\end{aligned}$$

Finally, from $[\overline{\delta}, \delta]$ we find an expression for $[e_2, e_1]$:

$$[e_2, e_1] = -\frac{1}{4}\left(\psi_1 + \overline{\psi_1}\right)e_1.$$

At this stage we have expressions for all derivatives of ρ (thus of L), ϕ_1, ψ_1 and ψ_3.

Next we integrate the $\mathrm{D}\,\gamma$-equation and obtain

$$\gamma = g_0 + \frac{1}{16}\left(\psi_1 + \overline{\psi_1}\right)\rho\,L^2\psi_1 + \frac{1}{4}\,\rho L\,{\psi_1}^2 + \left(\frac{1}{8}\,{\psi_1}^2 + \frac{1}{2}\,\phi_1\overline{\phi_1}\right)\rho,$$

with $\overline{g_0} = g_0$, $\mathrm{D}g_0 = 0$, and $\delta g_0 = \overline{\delta} g_0 = 0$ as follows from (3.17 – 3.18). Notice that we can put g_0 equal to zero by a boost: first look at the derivatives of ϕ_1, which are all zero, except for $\Delta\,\phi_1$, which equals $-4\,\phi_1\,\overline{\phi_1}\,g_0$. As a boost transforms ϕ_1 into ϕ_1/A, we can put $g_0 = 0$ by choosing $\Delta\,\ln A = -2\,g_0$, which is compatible with $\mathrm{D}A = \delta A = 0$. This also allows us to write ϕ_1 as $Q\,f$, with Q a non-zero constant, and with f on the unit circle ($\overline{f} = 1/f$).

Now we write ψ_1 as $\psi_1 = (U + V)/2$ with U real and V imaginary. Introducing the basis one forms $\mathbf{\Omega}^1$, $\mathbf{\Omega}^2$, $\mathbf{\Omega}^3$, $\mathbf{\Omega}^4$ dual to e_1, e_2, Δ, D, the derivatives of f, U and V are given by

$$\begin{aligned}
\mathrm{d}f &= e_2\,f\mathbf{\Omega}^2 + \Delta\,f\mathbf{\Omega}^3 = \frac{f\,V}{2}\left(\mathbf{\Omega}^2 + \frac{\rho L U}{4}\mathbf{\Omega}^3\right),\\
\mathrm{d}U &= e_2\,U\mathbf{\Omega}^2 + \Delta\,U\mathbf{\Omega}^3 = \frac{2V^2 - U^2 - 16Q^2}{4}\left(\mathbf{\Omega}^2 + \frac{\rho L U}{4}\mathbf{\Omega}^3\right),\\
\mathrm{d}V &= e_2\,V\mathbf{\Omega}^2 + \Delta\,V\mathbf{\Omega}^3 = \frac{UV}{4}\left(\mathbf{\Omega}^2 + \frac{\rho L U}{4}\mathbf{\Omega}^3\right),
\end{aligned} \tag{3.23}$$

showing that f, U and V are functionally dependent. Notice that U cannot be constant, as then V would be real:

$$e_2\,U = \frac{V^2}{2} - \frac{U^2}{4} - 4\,Q^2. \tag{3.24}$$

This permits us to use U as a coordinate, but we prefer to write $U = U(x)$ (hence also $V = V(x)$, $f = f(x)$ and $\phi_1 = \phi_1(x)$), with x to be specified later.

To continue with the integration, we now have to introduce appropriate coordinates.

The Cartan structure equations imply $\mathrm{d}\mathbf{\Omega}^3 = 0$ and thus $\mathbf{\Omega}^3 = \mathrm{d}u$.

Since $\mathrm{d}\rho = \Delta\,\rho\,\mathbf{\Omega}^3 + 2\,\rho^2\,\mathbf{\Omega}^4$ we have

$$\mathbf{\Omega}^4 = \frac{\mathrm{d}\rho}{2\,\rho^2} - \frac{\Delta\,\rho\,\mathrm{d}u}{2\,\rho^2}.$$

Next notice that $\mathrm{d}U = \Delta\,U\,\mathbf{\Omega}^3 + e_2\,U\,\mathbf{\Omega}^2$ and $\mathrm{d}V = \Delta\,V\,\mathbf{\Omega}^3 + e_2\,V\,\mathbf{\Omega}^2$, where, by (3.23) and (3.24), $e_2\,U$ and $e_2\,V$ cannot be 0. It follows that $\mathbf{\Omega}^2$ can be written as a linear combination of $\mathrm{d}U$ and $\mathrm{d}u$. As we have found that $\Delta\,U = L\,\rho\,U\left(2\,V^2 - U^2 - 16\,Q^2\right)/16$, we can write

$$\mathbf{\Omega}^2 = S(x)\,\mathrm{d}x - \frac{U\,L\,\rho}{4}\mathrm{d}u,$$

with $S(x)$ to be specified below. There remains $\mathbf{\Omega}^1$ which we can write as $\mathbf{\Omega}^1 = B\,\mathrm{d}u + H\,\mathrm{d}x + J\,\mathrm{d}y$, by an appropriate choice of the y-coordinate. From (3.22) we see that

$$\begin{aligned} \overline{J} &= -J, \\ \overline{S} &= S, \\ \overline{H} &= -H. \end{aligned}$$

The tetrad basis vectors are then given by

$$\begin{aligned} e_1 &= \frac{1}{J}\partial_y, \\ e_2 &= \frac{1}{S}\partial_x - \frac{H}{S\,J}\partial_y, \\ \Delta &= \partial_u + \Delta\,\rho\partial_\rho + \frac{L\,\rho\,U}{4\,S}\partial_x - \frac{4\,B\,S + H\,L\,\rho\,U}{4\,S\,J}\partial_y, \\ \mathrm{D} &= 2\,\rho^2\partial_\rho. \end{aligned}$$

Acting with the commutators on the coordinates u, ρ, x and y, we get

$$\begin{aligned} \mathrm{D}\,J &= \mathrm{D}\,S = \mathrm{D}\,H = 0 = e_1\,S, \\ \Delta\,S &= \frac{1}{4}\,L\rho U\,e_2\,S, \\ \mathrm{D}\,B &= \frac{1}{2}\,\rho\,V, \end{aligned}$$

after which it is easily seen that

$$B = \frac{1}{4}\,L\,V + B_0, \qquad \mathrm{D}\,B_0 = 0.$$

From $\mathrm{D}\,B_0 = 0 = \mathrm{D}\,H = \mathrm{D}\,J$ one deduces the existence of a new y-coordinate, as a function of u, x and y, such that H becomes 0.
From $[e_2, e_1]y$ we find the following expression for $e_2\,J$:

$$e_2\,J = \frac{JU}{4} \quad \Leftrightarrow \quad \frac{\mathrm{d}\log J}{\mathrm{d}x} = \frac{S\,U}{4},$$

from which we see that $J = J_1(x)J_2(u, y)$. Defining a new y-coordinate as a function of u and y and absorbing the $\mathrm{d}u$-part in a new B-coefficient, one can assume $J_2 = 1$. Hence $J = J(x)$ and we have that $e_1\,J = 0$ and $\Delta\,J = L\,\rho\,U^2 J/16$.
From $[e_1, \Delta]y$ we find that $e_1\,B_0 = 0$ and hence $B_0 = J\,B_1(x, u)$, after which $[e_2, \Delta]y$ results in

$$e_2\,B_1 = -UV/(4J). \tag{3.25}$$

Hence B_1 can be decomposed as $B_1 = B_2(x) + B_3(u)$. A final u-dependent y-translation allows one to transform $\mathbf{\Omega}^1$ into

$$\mathbf{\Omega}^1 = (VL/4 + JB_2(x))\,\mathrm{d}u + J\mathrm{d}y,$$

i.e. one can assume $B_1 = B_2(x)$.
We now fix $S(x)$ such that we can integrate (3.25): as $e_2\,V = UV/4$ and $e_2\,J = UJ/4$, the choice

$$S(x) = 4/(x\,U)$$

leads to

$$B_1 = -\frac{V}{J}\log x.$$

From the $e_2\,V$-equation, we see that $V = 4\,\mathrm{i}\,Q\,a\,x$, with $Q\,a$ constant (Q is assumed to be non-zero). We still need a solution for U. From the expressions for $\mathrm{D}\,U$, $\Delta\,U$, $e_1\,U$ and $e_2\,U$ it is easy to see that $U = U(x)$, with $U(x)$ determined by

$$\frac{\partial\,U}{\partial\,x} = -\frac{16\,Q^2 + 32\,Q^2a^2x^2 + U^2}{xU},$$

which integrates to

$$x^2U^2 = 16\left({c_1}^2 - a^2x^4 - x^2\right)Q^2, \tag{3.26}$$

where c_1 is a constant. From e_1, e_2, Δ and D applied to $\left(\psi_3 + \overline{\psi_3}\right)/2$ it follows that the partial derivatives of ψ (the real part of ψ_3 divided by U) are given by:

$$\frac{\partial\psi}{\partial y} = 0 = \frac{\partial\psi}{\partial u} = \frac{\partial\psi}{\partial\rho},$$

$$\frac{\partial\psi}{\partial x} = \frac{U^2 + 16\,(1 - 14\,x^2a^2)Q^2 - 64\,\psi}{32\,x}.$$

Using equation (3.26) the solution of these differential equations is given by

$$\psi = \frac{1}{8}\left(4\,c_1^2\log x + 8\,c_2 - 15\,a^2x^4\right)\frac{Q^2}{x^2}, \quad \text{with} \quad c_1, c_2 \quad \text{constant.}$$

The one forms dual to the Newman Penrose tetrad $(\mathbf{m}, \overline{\mathbf{m}}, \mathbf{l}, \mathbf{k})$ now become

$$
\begin{aligned}
\boldsymbol{\omega}^1 &= \left(\frac{1}{8}\left(4\mathrm{i}Qax - U\right)L - 2\,\mathrm{i}Qax\log x\right)\mathrm{d}u + \frac{2}{xU\rho}\mathrm{d}x + \frac{1}{2}\mathrm{i}xJ_0\mathrm{d}y,\\
\boldsymbol{\omega}^2 &= \left(-\frac{1}{8}\left(4\mathrm{i}Qax + U\right)L + 2\,\mathrm{i}Qax\log x\right)\mathrm{d}u + \frac{2}{xU\rho}\mathrm{d}x - \frac{1}{2}\mathrm{i}xJ_0\mathrm{d}y,\\
\boldsymbol{\omega}^3 &= \mathrm{d}u,\\
\boldsymbol{\omega}^4 &= \frac{1}{2\,\rho^2}\mathrm{d}\rho - \frac{1+L}{2\,x\,\rho}\mathrm{d}x - \frac{1+L}{2}J_0Qx^2a\mathrm{d}y + \left[\frac{1}{64}\left(2+L\right)LU^2 - 2\,\psi + \right.\\
&\left.\left(2\left(\log x - 2\right)x^2a^2 + \frac{1}{2}\left(1 + \left(4\log x - 1\right)x^2a^2\right)L - \frac{3}{4}x^2a^2L^2\right)Q^2\right]\mathrm{d}u,
\end{aligned}
$$

in which J_0 is a constant, that can absorbed in y.
The metric reads then

$$
\begin{aligned}
\mathrm{d}s^2 = &\left[\left(\log\rho^2 - \log x^4 + 1\right)Qx^2a\mathrm{d}y - \frac{1}{\rho^2}\mathrm{d}\rho + \frac{2}{\rho x}\mathrm{d}x\right]\mathrm{d}u + \frac{1}{2}\,x^2\mathrm{d}y^2\\
&+\left[\frac{\log^2\rho^2a^2x^2}{2} - \left(\frac{c_1^2}{2x^2} + (\log x^4 - 1)a^2x^2\right)\log\rho^2 + \left(\frac{c_1^2}{x^2} - 2a^2x^2\right)\log x^2\right.\\
&\left.+\frac{4\,c_2}{x^2} + \left(\frac{1}{2} + 2\log^2x^2\right)a^2x^2\right]Q^2\mathrm{d}u^2 + \frac{1}{2\left(c_1^2 - x^2 - a^2x^4\right)\rho^2Q^2}\mathrm{d}x^2,
\end{aligned}
\tag{3.27}
$$

with the Maxwell field given by

$$
\begin{aligned}
\Phi_0 &= 0, &(3.28)\\
\Phi_1 &= Qf\rho, &(3.29)\\
\Phi_2 &= \frac{1}{4}\left(\log|\rho|U + 4(\log|\rho| + 1)\mathrm{i}Qax\right)Qf\rho, &(3.30)
\end{aligned}
$$

where

$$
f = -\frac{2\,a\sqrt{c_1^2 - (a^2x^2+1)x^2} + (2\,a^2x^2+1)\mathrm{i}}{\sqrt{4\,a^2c_1^2+1}}
$$

is on the unit circle, as required and where a, c_1, c_2 and Q are constants. Neither u nor y appear in the components of the Riemann tensor Ψ_1, Ψ_2,

Ψ_3, Ψ_4, Φ_1, Φ_2: the latter pair is given by (3.29, 3.30) while

$$\begin{aligned}
\Psi_1 &= \frac{(U+4\,\mathrm{i}Qax)\,\rho^2}{2},\\
\Psi_2 &= \frac{(U+4\,\mathrm{i}Qax)\,\rho^2}{8}\Big[(L+1)U+4(L+2)\mathrm{i}Qax\Big],\\
\Psi_3 &= \frac{(U+4\,\mathrm{i}Qax)\,\rho^2}{128U}\Big[(3L+4)U^3L+24(L^2+3L+1)\mathrm{i}QxaU^2\\
&- 16\left(2L+(3L^2+14L+9)x^2a^2-\frac{c_1^2\mathrm{log}x^4+8c_2}{x^2}\right)Q^2U\Big],\\
\Psi_4 &= \frac{(U+4\mathrm{i}Qax)\,\rho^2}{256}\left(LU+4\,(L+1)\,\mathrm{i}Qax\right)\Bigg[(8L^2+32L+12)\mathrm{i}QxaU\\
&+ (L+1)U^2L-32\left(L+\frac{(L^2+7L+4)x^2a^2}{2}-\frac{c_1^2\mathrm{log}x^2+4c_2}{x^2}\right)Q^2\Bigg].
\end{aligned}$$

As in the canonical Petrov type I tetrad precisely two functionally independent functions remain in the curvature components, the metric will admit exactly two Killing vectors (Karlhede, 1980).

A more elegant expression for the metric (3.27), preserving its obvious symmetries, is obtained by a coordinate transformation $\rho \to r\,x^2$, which results in

$$\begin{aligned}
\mathrm{d}s^2 = &\frac{1}{2(c_1^2-x^2-a^2x^4)r^2x^4Q^2}\mathrm{d}x^2+\left(a(2\tilde{L}+1)Qx^2\mathrm{d}y-\frac{1}{r^2x^2}\mathrm{d}r\right)\mathrm{d}u\\
&+\left(\frac{a^2(2\tilde{L}+1)^2Q^2x^2}{2}-\frac{(\tilde{L}c_1^2-4c_2)Q^2}{x^2}\right)\mathrm{d}u^2+\frac{x^2}{2}\mathrm{d}y^2, \qquad (3.31)
\end{aligned}$$

in which $\tilde{L} = \mathrm{log}r = \mathrm{log}\rho - 2\,\mathrm{log}x$ and where a, Q, c_1 and c_2 are real constants.

3.3.1 Vacuum limit

After replacing c_1 by c_0/Q, the $(Q=0)$-limit of (3.27) is given by

$$\mathrm{d}s^2 = \frac{(2\mathrm{log}x-L)c_0^2}{x^2}\mathrm{d}u^2+\left(2\,\frac{\mathrm{d}x}{x\rho}-\frac{\mathrm{d}\rho}{\rho^2}\right)\mathrm{d}u+\frac{1}{2}\,\frac{\mathrm{d}x^2}{{c_0}^2\rho^2}+\frac{x^2}{2}\mathrm{d}y^2,$$

which, after the coordinate transformation $x \to c_0\,x/\sqrt{2}$, $\rho \to -1/(2\,r)$, $y \to 2\,y/c_0$, equals (26.23) of (Stephani et al., 2003). This is a special case of the cylindrical vacuum Newman Tamburino metric, the Sachs metric, which is also examined in the next chapter (4.40). It is also possible, however, to obtain the general vacuum cylindrical metric (4.39), which we present in the next chapter. After applying a coordinate transformation

$$\begin{aligned} x &\longrightarrow x^4, \\ r &\longrightarrow r^{-1} \end{aligned}$$

in (4.39) and by rescaling u and y (by a factor $-1/2$ and $1/2$, respectively), the general vacuum metric for the cylindrical class looks like this:

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{(2L-1)^2 x^2}{2} - \frac{2L-c_2}{2x^2}\right)\mathrm{d}u^2 - \frac{1}{x^2 r^2}\mathrm{d}u\mathrm{d}r + \frac{x^2}{2}\mathrm{d}y^2 \\ &+ \frac{1}{2r^2 x^4\left(1-x^4\right)}\mathrm{d}x^2 - x^2\left(2L-1\right)\mathrm{d}u\mathrm{d}y, \end{aligned}$$

where $|x| < 1$ to ensure Lorentzian signature. This metric can also be obtained from (3.31). To see this, substitute $a = 1/Q$, $c_1 = 1/Q$, replace c_2 by $c_2/(8Q^2)$ and apply the coordinate transformation $y \longrightarrow -y - 2u$. Next taking the limit $Q \to 0$ results in the line element above, which is equivalent to the general vacuum metric of the cylindrical class.

3.4 Conclusion

We have obtained the general solution of the 'aligned Newman Tamburino Maxwell' problem: if a space-time is algebraically general and possesses a hypersurface orthogonal and geodesic principal null direction, with non-vanishing shear and divergence, then a (necessarily non-null) Maxwell field will be aligned if and only if the cylindrical condition $|\rho| = |\sigma|$ is satisfied, in which case the general solution is given by (3.31). We have also shown that the cosmological constant for this class of solutions must vanish. By choosing the right values of the constants in (3.31), it is possible to find the general vacuum cylindrical Newman Tamburino metric by taking the limit for the charge $Q \to 0$.

Left as an open question is whether or not there exist spherical metrics in the presence of a *non-aligned* Einstein Maxwell field and if so, whether these solutions are a generalisation of the vacuum spherical Newman Tamburino metrics.

Chapter 4

Newman Tamburino metrics in vacuum

4.1 Introduction

In the previous chapter, we have found all Newman Tamburino solutions in the presence of an aligned Maxwell field. Also, we have shown that there exist no cylindrical solutions in the presence of a non-aligned Maxwell field. In this chapter we look at Newman Tamburino solutions in vacuum. Originally, these solutions were published by Newman and Tamburino (1962). For the calculations they refer to (Newman and Penrose, 1962) and their preceding paper (Newman and Unti, 1962). In those papers, however, many steps are not written down explicitly, making it difficult to follow how to arrive at their solutions. Also, the cylindrical metric, as published in (Newman and Tamburino, 1962), contains typing errors, and therefore is *not* a vacuum metric. We will show that it is possible, without introducing coordinates from the beginning, to find the Newman Tamburino solutions in vacuum in a very elegant form, without elliptic functions and with fewer constants compared to the original paper.

4.2 Geroch Held Penrose analysis

The Newman Tamburino metrics have non-vanishing shear and divergence, and are further characterised by the existence of a hypersurface orthogonal and geodesic principal null direction $\mathbf{k}$. These properties can be written in terms of GHP-variables as follows:

$$\sigma\rho \neq 0,$$

$$\rho - \bar{\rho} = 0 = \kappa = \Psi_0.$$

As we are now searching for *empty* space solutions, all components Φ_{ij} of the Ricci tensor are equal to zero. The Goldberg Sachs theorem then states that for an algebraically general Weyl tensor Ψ_1 cannot be zero, which can easily be checked in GHP: substituting $\Psi_1 = 0$ in (1.44) and keeping in mind $\sigma \neq 0$ leads to $\Psi_2 = 0$, after which (1.46) and (1.48) lead to $\Psi_3 = 0$ and $\Psi_4 = 0$.

Under null rotations about $\mathbf{k}$, Ψ_2 transforms as $\Psi_2' = \Psi_2 + 2\overline{B}\Psi_1 + \overline{B}^2\Psi_0$. With $\Psi_0 = 0$ and $\Psi_1 \neq 0$, we can fix the null rotation of the null tetrad $(\mathbf{m}, \bar{\mathbf{m}}, \mathbf{l}, \mathbf{k})$ by choosing any zero-weighted value for Ψ_2. In contrast with our calculations in the previous chapter, we will not use the null rotation here to put the spin coefficient π equal to zero, as this would make the calculations more difficult. We prefer to *completely fix* the null rotation by following choice:

$$\Psi_2 = -\frac{1}{4}\frac{\Psi_1\left(\rho\Psi_1 + \sigma\overline{\Psi_1}\right)}{\sigma^2\overline{\sigma}}.$$

The GHP Bianchi equations $(1.43 - 1.53)$ and Ricci equations $(1.25 - 1.42)$ can then be rewritten as follows:

$$\eth\Psi_1 = 4\tau\Psi_1 + \frac{3}{4}\frac{\Psi_1\left(\rho\Psi_1 + \sigma\overline{\Psi_1}\right)}{\sigma\overline{\sigma}}, \tag{4.1}$$

$$\eth'\Psi_1 = -2\pi\Psi_1 - \frac{1}{4}\frac{\Psi_1\left(\rho\overline{\Psi_1} + \overline{\sigma}\Psi_1\right)}{\sigma\overline{\sigma}}, \tag{4.2}$$

$$\text{Þ}\Psi_1 = 4\rho\Psi_1, \tag{4.3}$$

$$
\begin{aligned}
Þ'\Psi_1 &= \frac{\Psi_1}{4\sigma^2\overline{\sigma}}\left(\frac{\left(\rho\Psi_1+\sigma\overline{\Psi_1}\right)ð\overline{\sigma}}{\overline{\sigma}}-\Psi_1ð'\sigma+\frac{\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)ð\sigma}{\sigma}\right)\\
&- \frac{1}{4}\frac{\Psi_1\left(5\rho\Psi_1+\sigma\overline{\sigma}\right)\tau}{\sigma^2\overline{\sigma}}+\frac{1}{2}\frac{\Psi_1\overline{\Psi_1}\overline{\pi}}{\sigma\overline{\sigma}}+2\sigma\Psi_3-2\mu\Psi_1\\
&+ \frac{1}{8}\frac{\Psi_1\left(2\sigma\overline{\sigma}\Psi_1^2-4\rho\sigma\Psi_1\overline{\Psi_1}-\sigma^2\overline{\Psi_1}^2-3\rho^2\Psi_1^2\right)}{\sigma^3\overline{\sigma}^2}, \qquad (4.4)\\
ð\Psi_3 &= \frac{\Psi_1}{16\sigma^2\overline{\sigma}^2}\left(\frac{\rho\left(2\overline{\sigma}\Psi_1^2-\sigma\overline{\Psi_1}^2\right)ð'\sigma}{\sigma^2\overline{\sigma}}-\frac{\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)^2ð\sigma}{\sigma^3}\right.\\
&\quad -\frac{\rho\Psi_1\left(2\rho\Psi_1+3\sigma\overline{\Psi_1}\right)ð\overline{\sigma}}{\sigma^2\overline{\sigma}}-\frac{\overline{\Psi_1}\left(2\rho\overline{\Psi_1}+\overline{\sigma}\Psi_1\right)ð'\overline{\sigma}}{\overline{\sigma}}\\
&\quad +\frac{4\overline{\sigma}\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)Þ'\sigma}{\sigma}+4\Psi_1\left(\rho\Psi_1+\sigma\overline{\Psi_1}\right)Þ'\overline{\sigma}-4\overline{\sigma}\Psi_1Þ'\rho\\
&\quad \left.+\frac{\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)\left(5\rho\Psi_1+\sigma\overline{\Psi_1}\right)\tau}{\sigma^2}+\frac{\overline{\Psi_1}\left(5\rho\overline{\Psi_1}+\overline{\sigma}\Psi_1\right)\overline{\tau}}{\overline{\sigma}}\right)\\
&- 2\nu\Psi_1-\frac{\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)\Psi_3}{2\sigma\overline{\sigma}}-\frac{\Psi_1\overline{\Psi_3}}{2\sigma}-\frac{\Psi_1\overline{\Psi_1}\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)\overline{\pi}}{8\sigma^3\overline{\sigma}^2}\\
&- \frac{1}{4}\frac{\Psi_1\left(-\rho\Psi_1+\sigma\overline{\Psi_1}\right)\mu}{\sigma^2\overline{\sigma}}+\frac{1}{2}\frac{\Psi_1\overline{\Psi_1}\overline{\mu}}{\sigma^2\overline{\sigma}^2}-\sigma\Psi_4+2\tau\Psi_3+\frac{3\rho^2\Psi_1\overline{\Psi_1}^3}{32\sigma^3\overline{\sigma}^4}\\
&- \frac{\Psi_1\left(\Psi_1^2\sigma(4\rho\Psi_1+\overline{\Psi_1}\sigma)\overline{\sigma}+\sigma^3\overline{\Psi_1}^3-10\rho\sigma^2\Psi_1\overline{\Psi_1}^2-6\rho^3\Psi_1^3\right)\overline{\sigma}}{32\sigma^5\overline{\sigma}^4}\\
&- \frac{\Psi_1^2\overline{\Psi_1}\pi}{8\sigma^2\overline{\sigma}}+\frac{11\rho^2\Psi_1^3\overline{\Psi_1}}{32\sigma^4\overline{\sigma}3},\\
ð'\Psi_3 &= Þ\Psi_4-\frac{3}{4}\frac{\Psi_1\left(\rho\Psi_1+\sigma\overline{\Psi_1}\right)\lambda}{\sigma^2\overline{\sigma}}-4\pi\Psi_3-\rho\Psi_4,\\
Þ'\Psi_3 &= ð\Psi_4-\frac{3}{4}\frac{\Psi_1\left(\rho\Psi_1+\sigma\overline{\Psi_1}\right)\nu}{\sigma^2\overline{\sigma}}-4\mu\Psi_3-\tau\Psi_4,\\
Þ\Psi_3 &= \frac{1}{4}\frac{\Psi_1}{\sigma^2\overline{\sigma}}\left(\frac{\left(2\rho\Psi_1+\sigma\overline{\Psi_1}\right)ð'\sigma}{\sigma}-\Psi_1ð\overline{\sigma}+\frac{\left(\rho\Psi_1+\sigma\overline{\Psi_1}\right)ð'\overline{\sigma}}{\overline{\sigma}}\right)\\
&+ 2\rho\Psi_3-2\lambda\Psi_1-\frac{1}{4}\frac{\Psi_1\left(-\rho\Psi_1+\sigma\overline{\Psi_1}\right)\pi}{\sigma^2\overline{\sigma}}-\frac{\Psi_1\overline{\Psi_1}\overline{\tau}}{\sigma\overline{\sigma}}\\
&+ \frac{1}{8}\frac{\Psi_1\left(\rho^2\Psi_1\overline{\Psi_1}+\sigma\overline{\sigma}\Psi_1\overline{\Psi_1}-\sigma\rho\overline{\Psi_1}^2+\rho\overline{\sigma}\Psi_1^2\right)}{\sigma^3\overline{\sigma}^2},
\end{aligned}
$$

and

$$
\begin{aligned}
þ\lambda &= \rho\lambda + \overline{\sigma}\mu + \pi^2 + ð'\pi, \\
þ\mu &= \rho\mu + \sigma\lambda + \pi\overline{\pi} - \frac{1}{4}\frac{\Psi_1\left(\rho\Psi_1 + \sigma\overline{\Psi_1}\right)}{\sigma^2\overline{\sigma}} + \frac{R}{12} + ð\pi, \\
þ\nu &= (\tau + \pi)\,\mu + (\tau + \overline{\pi})\,\lambda + \Psi_3 + þ'\pi, \\
þ\rho &= \rho^2 + \sigma\overline{\sigma}, \\
þ\sigma &= 2\rho\sigma, \\
þ\tau &= (\tau + \overline{\pi})\,\rho + (\overline{\tau} + \pi)\,\sigma + \Psi_1, \\
þ'\lambda &= (\pi - \overline{\tau})\,\nu - (\mu + \overline{\mu})\,\lambda - \Psi_4 + ð'\nu, \\
ð\nu &= \mu^2 + \lambda\overline{\lambda} - \pi\overline{\nu} + \tau\nu + þ'\mu, \\
ð\rho &= -\Psi_1 + ð'\sigma, \\
ð\tau &= \mu\sigma + \overline{\lambda}\rho + \tau^2 + þ'\sigma, \\
ð'\tau &= \overline{\mu}\rho + \lambda\sigma + \tau\overline{\tau} - \frac{1}{4}\frac{\Psi_1\left(\rho\Psi_1 + \sigma\overline{\Psi_1}\right)}{\sigma^2\overline{\sigma}} + \frac{R}{12} + þ'\rho, \\
ð\lambda &= (\mu - \overline{\mu})\,\pi - \Psi_3 + ð'\mu.
\end{aligned}
\tag{4.5}
$$

If we apply the commutator $[ð, þ]$ to Ψ_1 we obtain

$$
ð'\sigma = (\overline{\tau} + 2\pi)\,\sigma + \frac{1}{4}\frac{\rho\overline{\Psi_1} + 11\overline{\sigma}\Psi_1}{\overline{\sigma}}.
$$

From $[ð', þ]\Psi_1$ we get

$$
þ\pi = -4\,(\tau + \overline{\pi})\,\overline{\sigma} - \frac{\rho\Psi_1 + 4\sigma\overline{\Psi_1}}{\sigma}.
$$

Next, we calculate $[ð', þ]\sigma$ which gives an expression for $ð\sigma$:

$$
ð\sigma = -3\sigma\,(3\tau + 4\overline{\pi}) + \frac{1}{4}\frac{\rho\Psi_1 - 45\sigma\overline{\Psi_1}}{\overline{\sigma}}. \tag{4.6}
$$

Making use of the previous expression (4.6) in $[ð, þ]\sigma$ we find that

$$
\Psi_1 = -\,(\overline{\tau} + \pi)\,\sigma.
$$

This shows $\tau + \overline{\pi}$ cannot be zero.

Substituting the expression for Ψ_1 into equations $(4.1 - 4.4)$ results in

$$\begin{aligned}
ð\pi &= -Þ'\rho - \frac{R}{12} - \rho\mu - \overline{\sigma}\overline{\lambda} + \frac{1}{4}(\tau+\overline{\pi})\overline{\tau} + \frac{1}{4}(5\tau+\overline{\pi})\pi \\
&+ \frac{1}{4}\frac{(\tau+\overline{\pi})^2\rho}{\sigma} - \frac{1}{2}\frac{(\overline{\tau}+\pi)^2\rho}{\overline{\sigma}}, \\
ð'\pi &= -Þ'\overline{\sigma} - \overline{\mu\sigma} - \rho\lambda + (\overline{\tau}-\pi)\pi + \overline{\tau}^2 + \frac{1}{2}\frac{(\tau+\overline{\pi})(\overline{\tau}+\pi)\rho}{\sigma}, \\
Þ'\pi &= -\frac{(\overline{\tau}+\pi)Þ'\sigma}{\sigma} - Þ'\overline{\tau} - 2(\overline{\tau}+\pi)\mu - 2\Psi_3 - \frac{1}{4}\frac{(\overline{\tau}+\pi)(\tau\overline{\pi})\overline{\pi}}{\sigma} \\
&+ \frac{1}{16}\frac{(\overline{\tau}+\pi)^2(2\rho\tau+2\rho\overline{\pi}+\sigma\pi-3\sigma\overline{\tau})}{\sigma\overline{\sigma}} - \frac{3}{16}\frac{(\overline{\tau}+\pi)^3\rho^2}{\sigma\overline{\sigma}^2}.
\end{aligned}$$

Then, we evaluate $[ð', ð]\sigma$ to obtain

$$Þ'\rho = \frac{3\overline{\tau\pi}+\tau\pi}{4} + \frac{\pi\overline{\pi}-\tau\overline{\tau}}{2} - \frac{\overline{\sigma}\overline{\lambda}+3\sigma\lambda}{2} - \frac{R}{24} + \frac{(\tau+\overline{\pi})^2\rho}{16\,\sigma} + \frac{7(\overline{\tau}+\pi)^2\rho}{16\,\overline{\sigma}}.$$

As the right hand side of this equation is real, we find an expression for $\overline{\lambda}$

$$\overline{\lambda} = \frac{\sigma\lambda}{\overline{\sigma}} + \frac{3}{8}\left(\frac{(\tau+\overline{\pi})^2}{\sigma\overline{\sigma}} - \frac{(\overline{\tau}+\pi)^2}{\overline{\sigma}^2}\right)\rho + \frac{1}{2}\frac{\tau\pi-\overline{\tau\pi}}{\overline{\sigma}}. \tag{4.7}$$

At this point it is interesting to notice that $\rho^2 = c\,\sigma\overline{\sigma}$, where c is a constant, is only possible for $c = 1$[1]. In fact

$$0 = Þ\left(\rho^2 - c\,\sigma\overline{\sigma}\right) = 2\rho\left(\rho^2 - 2\,c\,\sigma\overline{\sigma} + \sigma\overline{\sigma}\right),$$

implies $c = 1$. Also $\tau = c\,\overline{\pi}$, with c constant, is only possible for $c = 1$, as follows from

$$Þ(\tau - c\overline{\pi}) = -\rho(c-1)(\tau+\overline{\pi})$$

and the fact that $\Psi_1 = -(\tau+\overline{\pi})\sigma \neq 0$.

We now use $[Þ', Þ]\sigma$ and $[Þ', Þ]\overline{\tau}$ to eliminate $ÞÞ'\sigma$ and $ÞÞ'\overline{\tau}$ from $[Þ', Þ]\pi$. This gives us a useful algebraic expression:

$$\frac{3(\overline{\tau}+\pi)(\tau+\overline{\pi})\rho^2}{\sigma^2\overline{\sigma}} - \frac{5\overline{\tau\pi}+\tau\pi+7\overline{\tau}\tau-\overline{\pi}\pi}{\sigma} = 0. \tag{4.8}$$

[1]Notice that $\rho^2/\sigma\overline{\sigma}$ is (0,0)-weighted.

In the remainder of this section, we will often use this equation to eliminate $\overline{\sigma}$ from various expressions. From this equation, we also see that π can't be zero.

Calculating the ð- and ð′-derivatives of (4.8), and making use of (4.7) and (4.8) we find

$$\begin{aligned}
\text{Þ}'\sigma &= \left(\frac{3\sigma\,(\tau+\overline{\pi})}{\overline{\tau}+\pi} - \frac{\sigma\rho}{\overline{\sigma}}\right)\lambda - \left(\sigma + \frac{3\,(\tau+\overline{\pi})\,\rho}{\overline{\tau}+\pi}\right)\mu - \frac{1}{8}\frac{(\tau+\overline{\pi})\,R}{\overline{\tau}+\pi} \\
&+ \frac{9}{8}\frac{(\overline{\tau}+\pi)\,(\tau+\overline{\pi})\,\rho^3}{\sigma\overline{\sigma}^2} + \frac{3}{8}\frac{\left(\sigma\,(\overline{\tau}+\pi)^2 + 2\overline{\sigma}\,(\overline{\pi}^2-\tau^2)\right)\rho^2}{\sigma\overline{\sigma}^2} \\
&- \frac{1}{8}\frac{(37\,\overline{\tau}\overline{\pi} + 39\,\overline{\tau}\tau + \tau\pi - \overline{\pi}\pi)\,\rho}{\overline{\sigma}},
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= \frac{R}{24\sigma} + \frac{\rho\overline{\mu}}{\sigma} - \frac{1}{16}\frac{(\tau+\overline{\pi})\,(13\overline{\tau}\overline{\pi} + 4\pi\overline{\pi} - 8\overline{\tau}\tau - 17\tau\pi)\,\rho}{\sigma^2\,(\overline{\tau}+\pi)} \\
&+ \frac{1}{16}\frac{(7\overline{\pi} - 19\tau)\,\overline{\tau}\pi + (13\overline{\pi} + 2\tau)\,\overline{\tau}^2 + (6\overline{\pi} - 9\tau)\,\pi^2}{\sigma^2\,(\overline{\tau}+\pi)} \\
&+ \frac{1}{8}\frac{(\overline{\tau}+\pi)\,(5\overline{\tau}\overline{\pi} + \tau\pi + 7\overline{\tau}\tau - \pi\overline{\pi})}{(\tau+\overline{\pi})\,\rho}. \qquad (4.9)
\end{aligned}$$

Hence we can rewrite (4.7), to obtain an equation for $\overline{\mu}$ in terms of μ, τ, $\overline{\tau}$, π, $\overline{\pi}$, σ, $\overline{\sigma}$ and ρ:

$$\begin{aligned}
\overline{\mu} &= \mu - \frac{(\overline{\tau}\overline{\pi} - \tau\pi)\,(13\overline{\tau}\tau + 7\overline{\tau}\overline{\pi} + \pi\overline{\pi} + 7\tau\pi)}{8\,(\tau+\overline{\pi})\,(\overline{\tau}+\pi)\,\rho} \\
&+ \frac{(\overline{\tau}+\pi)(17\overline{\tau}\overline{\pi} - 13\tau\pi + 8\overline{\tau}\tau - 4\overline{\pi}\pi)}{16(\tau+\overline{\pi})\overline{\sigma}} \\
&- \frac{(\tau+\overline{\pi})(17\tau\pi - 13\overline{\tau}\overline{\pi} + 8\overline{\tau}\tau - 4\overline{\pi}\pi)}{16(\overline{\tau}+\pi)\sigma} \\
&- \frac{3\,(\tau+\overline{\pi})^2\,(\overline{\tau}\overline{\pi} - \tau\pi)}{2(5\overline{\tau}\overline{\pi} + \tau\pi + 7\overline{\tau}\tau - \overline{\pi}\pi)\sigma}, \qquad (4.10)
\end{aligned}$$

where the last term has to be purely imaginary.

In the next step, we find an interesting relation between τ and π. First we substitute (4.9) in (4.5). Eliminating from this expression λ, $\overline{\mu}$ and $\overline{\sigma}$ by

(4.9), (4.10) and (4.8) we obtain

$$(\overline{\tau}\pi - \tau\overline{\pi})\left(\frac{9\,(\tau+\overline{\pi})^3\,(\overline{\tau}+\pi)\,\rho^2}{(5\overline{\tau}\overline{\pi}+\tau\pi+7\overline{\tau}\tau-\pi\overline{\pi})^2\,\sigma^2} - \frac{5\overline{\tau}+3\pi}{4(\tau+\overline{\pi})} + \frac{\left(11\overline{\tau}^2\tau+\overline{\tau}^2\overline{\pi}-14\overline{\tau}\pi\tau-22\overline{\tau}\pi\overline{\pi}+11\pi^2\tau+13\pi^2\overline{\pi}\right)\rho}{4\,(\overline{\tau}+\pi)\,(5\overline{\tau}\overline{\pi}+\tau\pi+7\overline{\tau}\tau-\pi\overline{\pi})\,\sigma}\right) = 0. \quad (4.11)$$

Then, we evaluate $[\text{Þ}', \text{Þ}]\rho$, where we again use (4.9), (4.10) and (4.8), but also (4.11), to simplify things. This leads to

$$\begin{aligned} 0 \;&= (\overline{\tau}\pi - \tau\overline{\pi})\left[\frac{-2\pi^3\rho\left(275\tau\overline{\pi}+170\overline{\pi}^2+131\tau^2\right)\overline{\tau}}{\overline{\tau}+\pi}\right. \\ &+\frac{\rho\pi^2\left(\pi^2\left(62\tau\overline{\pi}+107\overline{\pi}^2-25\tau^2\right)+120\overline{\pi}^2\overline{\tau}^2+6\left(41\tau^2+95\tau\overline{\pi}\right)\overline{\tau}^2\right)}{\overline{\tau}+\pi} \\ &+\frac{\rho\left(\left(41\overline{\pi}^2+47\tau^2+128\tau\overline{\pi}\right)\overline{\tau}^4-2\pi\left(20\overline{\pi}^2+59\tau^2-7\tau\overline{\pi}\right)\overline{\tau}^3\right)}{\overline{\tau}+\pi} \\ &-\frac{(5\overline{\tau}+3\pi)\,(5\overline{\tau}\overline{\pi}+\tau\pi+7\overline{\tau}\tau-\overline{\pi}\pi)\left(\overline{\tau}^2-16\overline{\tau}\pi+\pi^2\right)\tau\sigma}{(\tau+\overline{\pi})} \\ &\left.-\frac{(5\overline{\tau}+3\pi)\,(5\overline{\tau}\overline{\pi}+\tau\pi+7\overline{\tau}\tau-\overline{\pi}\pi)\left(5\overline{\tau}^2-2\overline{\tau}\pi+11\pi^2\right)\overline{\pi}\sigma}{(\tau+\overline{\pi})}\right]. \quad (4.12) \end{aligned}$$

Assume for a moment that $\overline{\tau}\pi - \tau\overline{\pi} \neq 0$ (so definitely $\tau \neq \overline{\pi}$). If the coefficient of ρ in (4.12) would be zero, also the coefficient of σ in this expression would have to be zero. Eliminating $\overline{\tau}$ from those coefficients, however, we get $(\tau+\overline{\pi})\,(\tau-\overline{\pi})\,\pi^8 = 0$, which is impossible under the assumption $\overline{\tau}\pi - \tau\overline{\pi} \neq 0$. This means that, if we assume $\overline{\tau}\pi - \tau\overline{\pi} \neq 0$, we can solve (4.12) for σ. Substituting this expression for σ in (4.8) and eliminating $\overline{\tau}$ from it and its complex conjugate, we see that $\pi = 0$ or $\tau = c\,\overline{\pi}$, which is impossible. It follows that $\tau\pi$ must be real: henceforth we will often substitute

$$\overline{\tau} = \frac{\tau\pi}{\overline{\pi}}.$$

We still have to check the ð- and Þ′-derivatives of (4.9). Those equations, together with Þ′(4.8) and $[\eth, \text{Þ}']\rho$ give us expressions for $\text{Þ}'\tau$, $\eth'\mu$, $\text{Þ}'\mu$ and

$ð\mu$:

$$
\begin{aligned}
Þ'\tau &= \frac{(\bar\pi-\tau)(\bar\pi-11\tau)\mu}{3(\tau+\bar\pi)} - \frac{2\tau\overline{\Psi_3}}{\tau+\bar\pi} + \frac{(5\bar\pi-\tau)(\bar\pi-7\tau)R}{144(\tau+\bar\pi)\rho} \\
&- \frac{1}{8}\frac{\tau(\tau+\bar\pi)^2}{\sigma} + \frac{\pi(\bar\pi-7\tau)\left(9\tau^3+12\tau^2\bar\pi-19\tau\bar\pi^2+2\bar\pi^3\right)}{24\bar\pi(\tau+\bar\pi)\rho} \\
&+ \frac{\sigma\pi^2(\bar\pi-7\tau)\left(29\tau^2\bar\pi-31\tau^3-11\tau\bar\pi^2+\bar\pi^3\right)}{36\bar\pi^2(\tau+\bar\pi)\rho^2},
\end{aligned}
$$

$$
\begin{aligned}
ð'\mu &= \frac{\rho ð\mu}{\sigma} + \left(\frac{\bar\pi-3\tau}{32\sigma} - \frac{\pi(\bar\pi-7\tau)}{288\bar\pi\rho}\right)R + \left(\frac{(\bar\pi-5\tau)\rho}{2\sigma} + \frac{\pi(\bar\pi-\tau)}{6\bar\pi}\right)\mu \\
&+ \frac{\pi(\bar\pi-3\tau)\left(5\bar\pi^2+3\tau^2-4\tau\bar\pi\right)}{16\bar\pi\sigma} - \frac{\pi^2\tau(\bar\pi-7\tau)(17\bar\pi+3\tau)}{24\bar\pi^2\rho} \\
&+ \frac{\sigma\pi^3(\bar\pi-7\tau)\left(91\tau^2-42\tau\bar\pi+11\bar\pi^2\right)}{144\bar\pi^3\rho^2} + \Psi_3,
\end{aligned}
$$

$$
\begin{aligned}
Þ'\mu &= \frac{\sigma ð'\nu}{\rho} + \frac{(3\bar\pi-\tau)(\bar\pi-\tau)\pi\overline{\Psi_3}}{4\bar\pi(\tau+\bar\pi)\rho} + \frac{(\bar\pi-\tau)\pi\sigma\nu}{\bar\pi\rho} \\
&+ \left(\frac{(3\bar\pi-\tau)(\bar\pi-\tau)}{4(\tau+\bar\pi)\rho} - \frac{\pi\sigma(\bar\pi-7\tau)(\bar\pi+7\tau)}{6\bar\pi(\bar\pi+\tau)\rho^2}\right)\Psi_3 - \frac{2(\bar\pi+5\tau)\mu^2}{3(\tau+\bar\pi)} \\
&+ \frac{(\bar\pi-7\tau)\mu R}{36(\bar\pi+\tau)\rho} - \frac{(\tau+\bar\pi)^2\mu}{8\sigma} + \frac{\pi(\bar\pi-\tau)\left(17\tau^2-102\tau\bar\pi+9\bar\pi^2\right)\mu}{12\bar\pi(\tau+\bar\pi)\rho} \\
&+ \frac{\sigma\pi^2(\bar\pi-7\tau)\left(251\tau^2-78\tau\bar\pi+7\bar\pi^2\right)\mu}{72\bar\pi^2(\tau+\bar\pi)\rho^2} \\
&+ \frac{(\bar\pi-7\tau)R^2}{1728(\tau+\bar\pi)\rho^2} - \frac{(\tau+\bar\pi)^2 R}{96\sigma\rho} + \frac{\pi(\bar\pi-7\tau)\left(18\bar\pi^2-\tau^2-15\tau\bar\pi\right)R}{288\bar\pi(\tau+\bar\pi)\rho^2} \\
&- \frac{\sigma\pi^2(\bar\pi-7\tau)\left(\bar\pi^2+93\bar\pi\tau-76\tau^2\right)R}{864\bar\pi^2(\tau+\bar\pi)\rho^3} \\
&+ \frac{3\bar\pi^4\pi-22\tau\pi\bar\pi^3+4\tau^2\pi\bar\pi^2-64\bar\pi\sigma^2\Psi_4+22\tau^3\pi\bar\pi-7\tau^4\pi}{64\bar\pi\sigma\rho} \\
&+ \frac{\pi^2(\bar\pi-7\tau)\left(11\tau^4+168\tau^3\bar\pi+298\tau^2\bar\pi^2-264\bar\pi^3\tau+43\bar\pi^4\right)}{192\bar\pi^2(\tau+\bar\pi)\rho^2} \\
&+ \frac{\pi^3\sigma(\bar\pi-7\tau)\left(25\bar\pi^4-538\bar\pi^3\tau+2588\tau^2\bar\pi^2-2342\tau^3\bar\pi-501\tau^4\right)}{576\bar\pi^3(\tau+\bar\pi)\rho^3} \\
&+ \frac{\sigma^2\pi^4(\bar\pi-7\tau)^2\left(\bar\pi^3-113\tau\bar\pi^2+767\tau^2\bar\pi-847\tau^3\right)}{1728(\tau+\bar\pi)\bar\pi^4\rho^4}
\end{aligned}
$$

and

$$\begin{aligned}
\eth\mu &= \frac{\left(12\rho\tau\bar{\pi}^2 - \rho\bar{\pi}^3 - 6\bar{\pi}\pi\tau\sigma + 13\bar{\pi}\rho\tau^2 - 18\pi\tau^2\sigma\right)\mu}{6\rho\bar{\pi}\left(\tau+\bar{\pi}\right)} \\
&- \frac{\left(13\rho\bar{\pi}^3 - 9\bar{\pi}^2\pi\sigma - 30\rho\tau\bar{\pi}^2 + 66\bar{\pi}\pi\tau\sigma - 43\bar{\pi}\rho\tau^2 - 21\pi\tau^2\sigma\right)R}{576\bar{\pi}\rho^2\left(\tau+\bar{\pi}\right)} \\
&+ \frac{\overline{\Psi_3}}{4} + \frac{\left(\bar{\pi}-3\tau\right)\sigma\Psi_3}{4\rho\left(\tau+\bar{\pi}\right)} - \frac{3\left(\tau+\bar{\pi}\right)\rho\bar{\nu}}{2\left(\bar{\pi}-7\tau\right)} - \frac{\sigma\nu}{2} \\
&+ \frac{\left(\bar{\pi}+5\tau\right)\left(\tau+\bar{\pi}\right)^2}{64\sigma} - \frac{\left(19\bar{\pi}^3 - 119\bar{\pi}^2\tau + 97\bar{\pi}\tau^2 - 213\tau^3\right)\pi}{192\bar{\pi}\rho} \\
&- \frac{\pi^2\sigma\left(\bar{\pi}-7\tau\right)\left(65\bar{\pi}^3 + 71\bar{\pi}^2\tau - 413\bar{\pi}\tau^2 - 131\tau^3\right)}{576\bar{\pi}^2\left(\bar{\pi}+\tau\right)\rho^2} \\
&- \frac{\pi^3\sigma^2\left(\bar{\pi}-7\tau\right)\left(173\tau^3 - 49\tau^2\bar{\pi} - 25\tau\bar{\pi}^2 + 5\bar{\pi}^3\right)}{192\bar{\pi}^3\left(\tau+\bar{\pi}\right)\rho^3}.
\end{aligned}$$

We will now distinguish between two cases: $\tau = \bar{\pi}$ or $\tau \neq \bar{\pi}$. These respectively coincide, in the present tetrad, with the cylindrical and spherical classes of the original paper. To see this, it is sufficient to look at equation (4.8) and to take into account the fact that $\Psi_1 = -(\tau+\bar{\pi})\sigma \neq 0$.

4.2.1 The cylindrical class $\tau = \bar{\pi}$

If we substitute $\tau = \bar{\pi}$ into (4.8), we get $\rho^2 = \sigma\bar{\sigma}$, the so called cylindrical class of (Newman and Tamburino, 1962). From $\text{Þ}'(\tau - \bar{\pi}) = 0$ we readily get that $R = 0$.
From $[\eth', \text{Þ}']\rho$ we get expressions for $\bar{\nu}$ and $\overline{\Psi_3}$:

$$\bar{\nu} = \frac{\sigma\nu}{\rho} - \frac{3\left(\rho\bar{\pi} - \pi\sigma\right)\left(\sigma^2\pi^2 + \rho^2\bar{\pi}^2\right)}{4\rho^4\sigma} \tag{4.13}$$

and

$$\overline{\Psi_3} = -\frac{\sigma\Psi_3}{\rho} + \frac{2\left(\rho\bar{\pi} + \pi\sigma\right)\mu}{\rho} + \frac{\left(\rho\bar{\pi} + \pi\sigma\right)\left(5\sigma^2\pi^2 - 12\pi\sigma\bar{\pi}\rho + 3\rho^2\bar{\pi}^2\right)}{2\rho^3\sigma}.$$

The only extra information that can be obtained for this class of solutions, comes from $[\eth, \text{Þ}']\pi$ and $[\eth', \eth]\mu$. These equations together with (4.13), give explicit expressions for ν, Ψ_3 and Ψ_4:

$$\begin{aligned}
\nu &= \frac{(\rho\bar{\pi}+\pi\sigma)\,\mu}{\rho\sigma}+\frac{\pi^2\,(5\rho\bar{\pi}-\pi\sigma)}{4\rho^3},\\
\Psi_3 &= 2\pi\mu-\frac{\pi^2\,(9\rho\bar{\pi}-5\pi\sigma)}{2\rho^2},\\
\Psi_4 &= \frac{2\pi\,(3\pi\sigma+\rho\bar{\pi})\,\mu}{\rho\sigma}+\frac{\pi\,(3\pi\sigma+\rho\bar{\pi})\left(10\sigma^2\pi^2-15\pi\sigma\rho\bar{\pi}+\rho^2\bar{\pi}^2\right)}{4\rho^3\sigma^2}.
\end{aligned}$$

The subsequent integration follows in section 4.3.

4.2.2 The spherical class $\tau \neq \bar{\pi}$

If $\tau \neq \bar{\pi}$, we can find expressions for ν, Ψ_3 and Ψ_4 from $[\eth', Þ']\rho$, $[\eth, Þ']\sigma$ and $[\eth, Þ']\pi$:

$$\begin{aligned}
\nu &= \left(\frac{\tau}{\sigma}-\frac{(\bar{\pi}-7\tau)\,\tau\pi}{3\bar{\pi}\,(\tau+\bar{\pi})\,\rho}\right)\mu+\frac{(\bar{\pi}-7\tau)\left(\bar{\pi}^3-9\tau\bar{\pi}^2-3\tau^2\bar{\pi}-9\tau^3\right)\pi^2}{48\bar{\pi}^2\,(\tau+\bar{\pi})\,\rho^2}\\
&- \left(\frac{3\bar{\pi}+\tau}{192\sigma}+\frac{3\bar{\pi}^2+20\tau\bar{\pi}+\tau^2}{\bar{\pi}\,(\tau+\bar{\pi})\,\rho}\right)\frac{(\bar{\pi}-7\tau)\,R}{(\bar{\pi}-\tau)\,\rho}\\
&+ \frac{(\bar{\pi}-7\tau)^2\left(3\bar{\pi}^2-14\tau\bar{\pi}+7\tau^2\right)\sigma\pi^3}{288\bar{\pi}^3\,(\tau+\bar{\pi})\,\rho^3}-\frac{\pi\,(\bar{\pi}-\tau)^2\,(\bar{\pi}-7\tau)}{32\bar{\pi}\sigma\rho}, \qquad (4.14)\\
\Psi_3 &= \frac{\pi\,(7\tau-\bar{\pi})\,\mu}{3\bar{\pi}}-\left(\frac{(\tau+\bar{\pi})^2}{32\sigma}+\frac{\pi\,(\bar{\pi}-7\tau)\,(7\bar{\pi}-\tau)}{288\bar{\pi}\rho}\right)\frac{R}{\bar{\pi}-\tau}\\
&- \frac{\pi\,(\tau+\bar{\pi})^2\,(\bar{\pi}-\tau)}{16\bar{\pi}\sigma}-\frac{(\bar{\pi}-7\tau)^2\,\sigma\pi^3\,(\bar{\pi}-11\tau)}{144\bar{\pi}^3\rho^2}\\
&- \frac{(\bar{\pi}-7\tau)\,\pi^2\left(\bar{\pi}^2-13\tau\bar{\pi}-6\tau^2\right)}{24\bar{\pi}^2\rho},\\
\Psi_4 &= \left(\frac{\pi\,(\bar{\pi}-7\tau)}{\bar{\pi}\rho}-\frac{\tau+\bar{\pi}}{\sigma}\right)\frac{\mu\pi\,(\bar{\pi}-7\tau)}{6\bar{\pi}}-\frac{(\tau+\bar{\pi})^3\,(\bar{\pi}-3\tau)\,\pi}{64\bar{\pi}\sigma^2}\\
&+ \left(\frac{(\bar{\pi}+5\tau)\,(\tau+\bar{\pi})}{\sigma}-\frac{\pi\left(5\tau^2+74\tau\bar{\pi}+5\bar{\pi}^2\right)}{2\bar{\pi}\rho}\right)\frac{\pi\,(\bar{\pi}-7\tau)\,R}{288\,\bar{\pi}\,(\bar{\pi}-\tau)\,\rho}\\
&+ \frac{\left(19\bar{\pi}^2-118\tau\bar{\pi}-41\tau^2\right)}{576\bar{\pi}^3\rho^2}-\frac{(\bar{\pi}-7\tau)\,(\tau+\bar{\pi})\left(\bar{\pi}^2-3\tau^2-10\tau\bar{\pi}\right)}{48\sigma\bar{\pi}^2\rho}\\
&- \frac{(\bar{\pi}-3\tau)\,(\tau+\bar{\pi})^3\,\pi}{64\bar{\pi}\sigma^2}+\frac{(\bar{\pi}-11\tau)\,(\bar{\pi}-7\tau)^3\,\sigma\pi^4}{288\bar{\pi}^4\rho^3}-\frac{(\tau+\bar{\pi})^2\,R}{64\,(\bar{\pi}-\tau)\,\sigma^2}.
\end{aligned}$$

The only remaining information now comes from $\eth(4.14)$, showing that $R = 0$ also in this class.

We will now transfer all invariant information, obtained in this section, to the Newman Penrose formalism. It is interesting to note that we have completely fixed the null rotation at the beginning of the GHP-analysis and that we know all directional derivatives of the spin coefficients and tensor components. Therefore we can conclude that coordinates exist for which no arbitrary functions appear in the line elements for this problem.

4.3 Cylindrical class: Newman Penrose analysis

The cylindrical class of Newman Tamburino metrics is characterised by the existence of a principal null direction $\mathbf{k}$ of the Weyl tensor which is hypersurface orthogonal and geodesic,

$$\Psi_0 = \kappa = \rho - \overline{\rho} = 0, \tag{4.15}$$

but has non-vanishing shear and divergence, with the spin coefficients ρ and σ related by

$$\rho^2 - |\sigma|^2 = 0. \tag{4.16}$$

As in the previous section, we will fix the null rotation of the tetrad, by putting

$$\Psi_2 = -\frac{1}{4}\left(\rho\Psi_1 + \sigma\overline{\Psi_1}\right)\frac{\Psi_1}{\sigma^2\overline{\sigma}}.$$

By fixing the null rotation in the same way as in our GHP-calculations in section 4.2, we are allowed to copy the expressions for λ, τ, ν, Ψ_1, Ψ_3, Ψ_4 and R:

$$\begin{aligned}
\lambda &= \frac{\rho\mu}{\sigma} + \frac{2\pi^2}{\rho}, \\
\tau &= \overline{\pi}, \\
\nu &= \frac{(\rho\overline{\pi} + \pi\sigma)\,\mu}{\rho\sigma} + \frac{\pi^2\,(5\rho\overline{\pi} - \pi\sigma)}{4\rho^3}, \\
\Psi_1 &= -2\pi\sigma, \\
\Psi_3 &= 2\mu\pi - \frac{(9\rho\overline{\pi} - 5\pi\sigma)\,\pi^2}{2\rho^2}, \\
\Psi_4 &= \frac{2\,(\rho\overline{\pi} + 3\pi\sigma)\,\pi\mu}{\sigma\rho} + \frac{(\rho\overline{\pi} + 3\pi\sigma)\left(10\pi^2\sigma^2 - 15\sigma\pi\overline{\pi}\rho + \overline{\pi}^2\rho^2\right)\pi}{4\sigma^2\rho^3}, \\
R &= 0.
\end{aligned}$$

Also the expression for $\overline{\mu}$ (4.10) remains valid:

$$\overline{\mu} = \mu - \frac{(\rho\overline{\pi} - \pi\sigma)(\rho\overline{\pi} + \pi\sigma)}{2\sigma\rho^2}. \tag{4.17}$$

As we assume non-vanishing shear, we can fix the spatial rotation such that $\sigma\rho > 0$ and hence, by (4.16):

$$\sigma = \rho.$$

From (1.26) we see that ϵ is real, so we can boost such that $\epsilon = 0$:

$$\epsilon' = \epsilon + \frac{1}{2}\mathrm{D}\ln A = 0.$$

Taking into account (1.25), (1.27 − 1.29), (1.32) and (4.17), a further boost can be used to make $\alpha = \pi - \overline{\beta}$:

$$\pi' - \alpha' - \overline{\beta'} = \pi - \alpha - \overline{\beta} - \overline{\delta}\ln A = 0.$$

The remaining tetrad freedom now consists of boosts with (real) boost parameter A satisfying $\delta A = \overline{\delta} A = \mathrm{D}A = 0$.

As in GHP, we proceed by rewriting all NP and Bianchi equations in a nice and useful form. First, we rewrite (1.25), (1.27) and (1.29 − 1.33):

$$\begin{aligned}
\mathrm{D}\rho &= 2\rho^2, \\
\mathrm{D}\beta &= \left(\beta - \overline{\beta}\right)\rho, \\
\mathrm{D}\gamma &= 2\pi\overline{\pi} - 2\overline{\pi}\overline{\beta} + 2\pi\beta - \pi^2, \\
\mathrm{D}\pi &= 2\rho\pi, \\
\mathrm{D}\mu &= 2\mu\rho - 2\pi\overline{\beta} + \overline{\delta}\pi, \\
\delta\pi &= \pi\overline{\pi} - \pi^2 - 2\beta\pi - 2\overline{\beta}\pi + \overline{\delta}\pi, \\
\Delta\pi &= \pi\left(\overline{\gamma} - \gamma - 2\mu\right) - \frac{2\pi\left(\pi + \overline{\pi}\right)\overline{\beta}}{\rho} + \frac{3\pi^2\overline{\pi} - 3\pi^3 + \pi\overline{\delta}\pi + \overline{\pi}\overline{\delta}\pi}{\rho}.
\end{aligned} \tag{4.18}$$

Then, $(1.35 + \overline{1.35})$ shows β is imaginary.

With this information, we rewrite (1.36 − 1.40), (1.42), and (1.44):

$$\begin{aligned}
\bar{\delta}\pi &= \pi\bar{\pi} + \bar{\pi}\beta - 3\pi\beta - 4\beta^2 - \delta\beta + \bar{\delta}\beta, \\
\delta\mu &= \bar{\pi}\mu - \pi\bar{\mu} - 4\mu\beta + \bar{\delta}\mu + \frac{2\pi^2\delta\rho}{\rho^2} \\
&- \frac{\pi\left(3\pi\bar{\pi} - 3\pi^2 - 8\left(\pi - \bar{\pi}\right)\beta - 32\beta^2 + 8\delta\beta + 8\bar{\delta}\beta\right)}{2\rho}, \\
\Delta\mu &= \frac{\left(\pi + \bar{\pi}\right)\bar{\delta}\mu}{\rho} + \left(\frac{\pi\left(7\pi + 3\bar{\pi}\right)}{2\rho^2} - \frac{2\mu}{\rho}\right)\left(\delta\beta - \bar{\delta}\beta\right) + \frac{\pi^2\bar{\pi}\left(11\pi - 3\bar{\pi}\right)}{4\rho^2} \\
&- \left(\frac{\mu\left(\pi + \bar{\pi}\right)}{\rho^2} + \frac{\pi^2\left(\bar{\pi} - 5\pi\right)}{2\rho^3}\right)\delta\rho - \mu\left(\mu + \bar{\mu} + \gamma + \bar{\gamma}\right) \\
&+ \frac{2\bar{\pi}\beta\mu - \mu\pi^2 + 4\pi\bar{\pi}\mu - 8\mu\beta^2 - 6\pi\beta\mu - 2\pi^2\bar{\mu} - \mu\bar{\pi}^2}{\rho} \\
&- \frac{\pi\left(6\beta\bar{\pi}^2 - 18\beta\pi^2 + 8\pi\beta\bar{\pi} - 56\pi\beta^2 - 9\pi^3 - 24\bar{\pi}\beta^2 + \bar{\pi}^3\right)}{4\rho^2}, \qquad (4.19) \\
\delta\gamma &= \Delta\beta - \beta(\gamma - \bar{\gamma} - \mu - \bar{\mu}) - (\mu - \bar{\mu})\pi + \frac{(\pi - 5\bar{\pi})\pi^2 + 8(\beta + \pi)\bar{\pi}^2}{4\rho}, \\
\Delta\rho &= \bar{\delta}\beta - \delta\beta + \pi\bar{\pi} + \bar{\pi}\beta - 4\beta^2 - \pi\beta + (3\gamma - \bar{\gamma} - \mu - \bar{\mu})\rho - 2\bar{\pi}^2, \\
\bar{\delta}\gamma &= \beta\left(\mu + \bar{\mu} + \gamma - \bar{\gamma}\right) + \Delta\beta + \frac{\pi}{\rho}\left(\bar{\delta}\beta - \delta\beta\right) + \frac{\bar{\pi}}{\rho}\left(\bar{\delta}\beta - \delta\beta\right) \\
&+ \frac{\left(\pi + \beta\right)\bar{\pi}^2 + \pi^2\beta - 4(\pi + \bar{\pi})\beta^2}{\rho} + \frac{\left(\bar{\pi} - \pi\right)\pi^2}{4\rho} - \pi\left(\mu - \bar{\mu}\right), \\
\delta\rho &= \frac{\rho}{2\pi}\left(2\delta\beta - 2\bar{\delta}\beta - 2\bar{\pi}\beta + 8\beta^2 - \pi^2 + 10\pi\beta + \pi\bar{\pi}\right). \qquad (4.20)
\end{aligned}$$

As ρ is real, we obtain an expression for $\bar{\gamma}$ from $\Delta\rho$:

$$\bar{\gamma} = \gamma - \frac{\bar{\pi}^2 - \pi^2}{2\rho}. \qquad (4.21)$$

Next we examine (1.35) and (1.34); (1.35) can be written as follows:

$$\frac{\rho\left(\bar{\pi} - \pi\right)}{\pi\bar{\pi}}\left(\delta\beta - \bar{\delta}\beta\right) - \frac{\rho\beta}{\pi\bar{\pi}}\left(\pi^2 - 14\pi\bar{\pi} + \bar{\pi}^2 + 4\beta\left(\pi - \bar{\pi}\right)\right) = 0. \qquad (4.22)$$

Furthermore, adding (1.34) and $\left(\pi + \bar{\pi}\right)\mu/\rho^2\times$(4.22) and simplifying the resulting equation, we obtain by (4.17) and (4.21)

$$\begin{aligned}
&\left(\pi^2 - 14\pi\bar{\pi} + \bar{\pi}^2\right)\left(\bar{\delta}\beta - \delta\beta\right) \\
&\quad + \beta\left(\bar{\pi}^3 - 39\pi\bar{\pi}^2 + 39\pi^2\bar{\pi} - \pi^3 - 4\beta\left(\pi^2 - 14\pi\bar{\pi} + \bar{\pi}^2\right)\right) = 0.
\end{aligned}$$

Eliminating $\delta\beta - \overline{\delta}\beta$ from the latter equation and (4.22), we obtain

$$-\frac{12\beta\rho\left(\overline{\pi}^2 + 10\pi\overline{\pi} + \pi^2\right)}{\pi^2 - 14\pi\overline{\pi} + \overline{\pi}^2} = 0,$$

from which it is obvious that $\beta = 0$, as $\pi \neq 0$.

The only equation in the set of NP and Bianchi equations, that still contains new information, is Bianchi equation (1.48)

$$\overline{\delta}\mu = (\overline{\pi} + 3\pi)\frac{\mu}{2} + \frac{3\overline{\pi}^3 - \pi\overline{\pi}^2 - 5\pi^2\overline{\pi} + 11\pi^3}{8\rho}.$$

Herewith we have extracted all information from the NP Ricci and Bianchi equations. In summary, we have expressions for

- all directional derivatives of ρ, π and μ,

- all but one directional derivatives of γ (we do not have an expression for $\Delta\gamma$),

- the imaginary parts of μ and of γ, given by (4.17) and (4.21), respectively.

Making use of the latter, all derivatives of (4.17) and of (4.21) are identically satisfied, and the same holds for all commutators applied to ρ, π and μ. Also, acting with $[\delta, \mathrm{D}]$, $[\overline{\delta}, \mathrm{D}]$ and $[\overline{\delta}, \delta]$ on γ yields no new information. Furthermore, we note that the tetrad is fixed up to boosts with real boost parameter A satisfying $\delta A = \overline{\delta} A = \mathrm{D}A = 0$.

Remark: the spin coefficient π can be real, but not purely imaginary, as can be seen by evaluating $\delta\left(\overline{\pi} - s\pi\right)$, with $s = \pm 1$.

The Cartan equations take the following form:

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^1 &= \rho(\boldsymbol{\omega}^1+\boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^4+\pi\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2-\frac{\mu\rho+2\pi^2}{\rho}\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3\\
&+ (\gamma-\overline{\gamma}-\mu)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3+2\pi\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4,\\
\mathrm{d}\boldsymbol{\omega}^2 &= \rho(\boldsymbol{\omega}^1+\boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^4-\overline{\pi}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2-\frac{\overline{\mu}\rho+2\overline{\pi}^2}{\rho}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3\\
&+ (\overline{\gamma}-\gamma-\overline{\mu})\,\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3+2\overline{\pi}\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4,\\
\mathrm{d}\boldsymbol{\omega}^3 &= 0, \qquad (4.23)\\
\mathrm{d}\boldsymbol{\omega}^4 &= (\overline{\mu}-\mu)\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2+\frac{4\overline{\mu}\rho\,(\overline{\pi}+\pi)+\overline{\pi}^2\,(5\pi-\overline{\pi})}{4\rho^2}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3\\
&- (\gamma+\overline{\gamma})\,\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4+\frac{4\mu\rho\,(\pi+\overline{\pi})+\pi^2\,(5\overline{\pi}-\pi)}{4\rho^2}\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3.
\end{aligned}
$$

We will now introduce coordinates. From (4.23), we see that $\boldsymbol{\omega}^3$ is exact, so we will write $\boldsymbol{\omega}^3=\mathrm{d}u$. The other one forms $\boldsymbol{\omega}^1$, $\boldsymbol{\omega}^2$ and $\boldsymbol{\omega}^4$ have the most general form:

$$
\begin{aligned}
\boldsymbol{\omega}^1 &= V\mathrm{d}u+W\mathrm{d}r+P\mathrm{d}x+Q\mathrm{d}y,\\
\boldsymbol{\omega}^2 &= \overline{V}\mathrm{d}u+\overline{W}\mathrm{d}r+\overline{P}\mathrm{d}x+\overline{Q}\mathrm{d}y,\\
\boldsymbol{\omega}^4 &= H\mathrm{d}u+\tilde{R}\mathrm{d}r+X\mathrm{d}x+Y\mathrm{d}y,
\end{aligned}
$$

where V, W, Q and P are complex functions of the real coordinates (u,r,x,y) and H, $\tilde{R}$, X and Y are real functions of these coordinates. By means of a coordinate transformation of $x\longrightarrow x+f_1(u,r,x,y)$ and $y\longrightarrow y+f_2(u,r,x,y)$, we can make W, the coefficient of $\mathrm{d}r$ in $\boldsymbol{\omega}^1$, equal to zero. Then a transformation of $r\longrightarrow r*f_3(u,r,x,y)$ can be used to make $\tilde{R}$, the coefficient of $\mathrm{d}r$ in $\boldsymbol{\omega}^4$, equal to one. We can still use the following coordinate transformations:

$$
\begin{aligned}
x &\longrightarrow g_1(u,x,y)x+g_2(u,x,y),\\
y &\longrightarrow g_3(u,x,y)y+g_4(u,x,y),\\
r &\longrightarrow r+g_5(u,x,y).
\end{aligned}
$$

In the above coordinates, the directional derivatives have the following form:

$$\delta = \frac{\overline{P}Y - \overline{Q}X}{\overline{Q}P - Q\overline{P}}\partial_r + \frac{\overline{Q}}{\overline{Q}P - Q\overline{P}}\partial_x - \frac{\overline{P}}{\overline{Q}P - Q\overline{P}}\partial_y, \tag{4.24}$$

$$\overline{\delta} = -\frac{PY - QX}{\overline{Q}P - Q\overline{P}}\partial_r - \frac{Q}{\overline{Q}P - Q\overline{P}}\partial_x + \frac{P}{\overline{Q}P - Q\overline{P}}\partial_y,$$

$$\Delta = \partial_u - \left(\frac{X\left(V\overline{Q} - \overline{V}Q\right) + Y\left(\overline{V}P - V\overline{P}\right)}{\overline{Q}P - Q\overline{P}} - H\right)\partial_r$$

$$+ \frac{\overline{V}Q - V\overline{Q}}{\overline{Q}P - \overline{P}Q}\partial_x + \frac{V\overline{P} - P\overline{V}}{\overline{Q}P - Q\overline{P}}\partial_y, \tag{4.25}$$

$$\mathrm{D} = \partial_r. \tag{4.26}$$

In the next step, we will determine the r-dependence of all spin coefficients, making use of (4.26). From (4.18) we see that $\rho = (-2[r + f(u,x,y)])^{-1}$. After a transformation $r \longrightarrow r - f(u,x,y)$ we have

$$\rho = -\frac{1}{2\,r}. \tag{4.27}$$

From the D-derivatives of π, μ and γ, we get

$$\begin{aligned} \pi &= \frac{\pi_0}{r}, \\ \mu &= \frac{\mu_0}{r} + \frac{\pi_0\overline{\pi_0}\log r}{r}, \\ \gamma &= g_0 + \frac{\pi_0\left(\pi_0 - 2\overline{\pi_0}\right)}{r}, \end{aligned} \tag{4.28}$$

where π_0 and μ_0 are complex functions of (u,x,y) and g_0 is a real function of (u,x,y) (the fact that g_0 is real, comes from (4.21)).

From $[\delta,\mathrm{D}]x$, $[\overline{\delta},\mathrm{D}]x$, $[\delta,\mathrm{D}]y$ and $[\overline{\delta},\mathrm{D}]y$, we see that the real parts of P and Q are proportional to r (the proportionality factors being real functions of (u,x,y)), whereas the imaginary parts of P and Q are independent of r:

$$P = p_1 r + \mathrm{i}p_2, \quad \overline{P} = p_1 r - \mathrm{i}p_2, \qquad \mathrm{D}p_1 = \mathrm{D}p_2 = 0,$$
$$Q = q_1 r + \mathrm{i}q_2, \quad \overline{Q} = q_1 r - \mathrm{i}q_2, \qquad \mathrm{D}q_1 = \mathrm{D}q_2 = 0.$$

Combining the real and imaginary part of $[\delta,\mathrm{D}]r$ shows both X and Y are independent of r. It is obvious now, that a transformation of y exists, which makes P real, after which a transformation of x can be used to make Q purely imaginary:

$$p_2(u,x,y) = 0 = q_1(u,x,y).$$

If we compare the expression for $\delta\rho$, *i.e.* (4.20), to the one we obtain, when applying (4.24) to (4.27), we see that

$$\begin{aligned} X &= 0, \\ Y &= \mathrm{i}q_2\left(\overline{\pi_0} - \pi_0\right). \end{aligned}$$

In the same way the coefficient of r in $\delta\pi$ shows that π_0 is independent of y, whereas from the r-independent terms we deduce

$$\frac{\partial \pi_0}{\partial x} = \pi_0 p_1 \left(3\overline{\pi_0} - \pi_0\right). \tag{4.29}$$

Taking into account the latter equation and its complex conjugate, we see from $\delta\mu$ that also μ_0 is independent of y and that

$$\frac{\partial \mu_0}{\partial x} = 2\mu_0 p_1 \left(\pi_0 + \overline{\pi_0}\right) - \frac{p_1}{2}\left(3\overline{\pi_0}^3 + \overline{\pi_0}^2\pi_0 - 7\overline{\pi_0}\pi_0 + 11\pi_0^3\right). \tag{4.30}$$

It is clear from (4.29) that p_1 is independent of y too and also g_0 is independent of y (and even of x) as can be seen from $\delta\gamma$. As γ is of the form

$$\gamma = g_0 + \frac{\left(\pi_0 - 2\overline{\pi_0}\right)\pi_0}{r},$$

with g_0 now only depending on u, we can use the remaining boost freedom to make $g_0 = 0$. The only variables, of which the r-dependence is as yet unknown, are V and H. From $[\Delta, \mathrm{D}]x$, $[\Delta, \mathrm{D}]y$ and $[\Delta, \mathrm{D}]r$, we find that

$$\begin{aligned} V &= v_1 r + \mathrm{i}v_2 + \pi_0 + \overline{\pi_0} + \left(\overline{\pi_0} - \pi_0\right)\log r, \\ H &= h_0 + \left(\overline{\pi_0}^2 - 4\pi_0\overline{\pi_0} + \pi_0^2\right), \end{aligned}$$

where

- the real function $h_0(u,x,y)$ can be determined from $\Delta\rho$:

$$h_0 = \mathrm{i}\left(\overline{\pi_0} - \pi_0\right)v_2 - 2\mu_0 + 2\left(2\pi_0 - \overline{\pi_0}\right)\left(\pi_0 - \overline{\pi_0}\right),$$

- the real function $v_1(u,x)$ is independent of y, as can be seen from $\Delta\pi$,
- the real function $v_2(u,x,y)$ is for the moment completely arbitrary.

From $\Delta\pi$, we also get that

$$\frac{\partial\pi_0}{\partial u} = \pi_0\left(3\overline{\pi_0} - \pi_0\right)v_1. \tag{4.31}$$

Substituting the latter and its complex conjugate, (4.29) and its complex conjugate and (4.30) in the expressions for $\Delta\mu$ (*i.e.* equation (4.19) and the result of applying (4.25) to (4.28)), we obtain

$$\frac{\partial\mu_0}{\partial u} = 2\left(\pi_0 + \overline{\pi_0}\right)\mu_0 v_1 - \frac{1}{2}\left(3\overline{\pi_0}^3 + \overline{\pi_0}^2\pi_0 - 7\overline{\pi_0}\pi_0^2 - 11\pi_0^3\right)v_1. \tag{4.32}$$

Expressions for the partial derivatives of v_1 with respect to x and of v_2 with respect to x and to y, can be found by acting with $[\delta,\Delta]$ on x and on y:

$$\begin{aligned}
\frac{\partial v_1}{\partial x} &= \frac{\partial p_1}{\partial u}, \\
\frac{\partial v_2}{\partial x} &= \mathrm{i}\left(2\overline{\pi_0} - 2\pi_0 + \mathrm{i}v_2\right)\left(\overline{\pi_0} + \pi_0\right)p_1, && (4.33)\\
\frac{\partial v_2}{\partial y} &= \frac{\partial q_2}{\partial u} + v_1\left(\overline{\pi_0} + \pi_0\right)q_2. && (4.34)
\end{aligned}$$

In addition $[\delta,\Delta]y$ provides an expression for $\dfrac{\partial q_2}{\partial x}$:

$$\frac{\partial q_2}{\partial x} = -q_2 p_1\left(\overline{\pi_0} + \pi_0\right), \tag{4.35}$$

which shows that $q_2(u,x,y)$ can be factorised in a factor depending on u and x and one depending on u and y:

$$q_2(u,x,y) = q_3(u,x)q_4(u,y).$$

We will assume $q_3(u,x) > 0$, as one can always absorb its sign in $q_4(u,y)$. Notice also that (4.35) provides us with a solution for $p_1(u,x)$:

$$p_1 = -\frac{1}{q_3\left(\overline{\pi_0} + \pi_0\right)}\frac{\partial q_3}{\partial x}. \tag{4.36}$$

We now substitute (4.36) in (4.29), which results in a partial differential equation for π_0:

$$\frac{\pi_0\left(\pi_0 - 3\overline{\pi_0}\right)}{\pi_0 + \overline{\pi_0}}\frac{\partial q_3}{\partial x} - q_3\frac{\partial\pi_0}{\partial x} = 0,$$

the solution of which is given by

$$\pi_0(u,x) = \frac{\sqrt{f(u) - n(u)^2 q_3(u,x)^4}}{q_3(u,x)} + \mathrm{i}q_3(u,x)n(u),$$

where $f(u) > n(u)^2 q_3(u,x)^4$ is positive, and where $n(u)$ can be zero.
From (4.31) we get expressions for $v_1(u,x)$ and for $n(u)$:

$$\begin{aligned} v_1 &= -\frac{1}{4\sqrt{f - n^2 q_3^4}}\left(2\frac{\partial q_3}{\partial u} + \frac{q_3\left(2nq_3^4 \frac{\mathrm{d}n}{\mathrm{d}u} - \frac{\mathrm{d}f}{\mathrm{d}u}\right)}{f + n^2 q_3^4}\right), \\ n &= \frac{c_1}{\sqrt{f}}. \end{aligned}$$

We can also solve (4.33)

$$\left(\frac{\partial v_2}{\partial x} + 4n\frac{\partial q_3}{\partial x}\right) - \frac{v_2}{q_3}\frac{\partial q_3}{\partial x} = 0$$

from which we get following expression for v_2:

$$v_2 = q_3\left(v_3 - 4n\mathrm{log}q_3\right),$$

where v_3 is a real function of u and y.
Next, we will combine (4.17), (4.30) and (4.32) to find a solution for μ_0:

$$\mu_0 = \mathrm{i}n\sqrt{f - n^2 q_3^4} - \frac{7}{2}n^2 q_3^2 + \frac{f\left(c_2 - \mathrm{log}f + 2\mathrm{log}q_3\right)}{q_3^2},$$

with c_2 a real constant.
The only equation still remaining is (4.34):

$$\frac{1}{q_4}\left(\frac{\partial v_3}{\partial y} - \frac{\partial q_4}{\partial u}\right) - \frac{1}{2f}\frac{\mathrm{d}f}{\mathrm{d}u} = 0. \tag{4.37}$$

The metric one forms are given by

$$\begin{aligned} \boldsymbol{\omega}^1 &= \mathrm{i}q_3 q_4 \mathrm{d}y - \frac{1}{2}\frac{\frac{\partial q_3}{\partial x} r\mathrm{d}x}{\sqrt{f - n^2 q_3^4}} - \left(\frac{2\sqrt{f - n^2 q_3^4}}{q_3} + \mathrm{i}q_3\left(n\mathrm{log}(r^2 q_3^4) - v_3\right)\right)\mathrm{d}u \\ &\quad + \left(\frac{\partial q_3}{\partial u} + \frac{2nq_3^5 \frac{\mathrm{d}n}{\mathrm{d}u} - q_3 \frac{\mathrm{d}f}{\mathrm{d}u}}{2\left(f + n^2 q_3^4\right)}\right)\frac{r}{2\sqrt{f - n^2 q_3^4}}\mathrm{d}u, \\ \boldsymbol{\omega}^3 &= \mathrm{d}u, \\ \boldsymbol{\omega}^4 &= \left(\frac{f\log f^2}{q_3^2} - \frac{\left(f + 2n^2 q_3^4\right)\log\left(r^2 q_3^4\right)}{q_3^2} - \frac{2c_2 f}{q_3^2} - n(5n - 2v_3)q_3^2\right)\mathrm{d}u \\ &\quad + 2q_3^2 q_4 n\,\mathrm{d}y + \mathrm{d}r, \end{aligned}$$

which suggests the use of $q_3(u,x)$ as a new x-coordinate. Introducing $\tilde{y}$ such that $\mathrm{d}\tilde{y} = q_4 \mathrm{d}y$ and dropping the tilde, allows us to solve (4.37):

$$v_3 = c_3 + \frac{y}{2f}\frac{\mathrm{d}f}{\mathrm{d}u}.$$

Actually, we do not expect the function $f(u)$ to appear in the metric (as can be seen from our GHP-analysis, we do not expect any free function in the resulting metric). Therefore, it is necessary to look for an appropriate coordinate transformation to eliminate $f(u)$ from the system. We temporarily distinguish between the cases $c_1 = 0$ and $c_1 \neq 0$.

If $c_1 = 0 = n$, we apply the following coordinate transformation:

$$\begin{aligned} x &\longrightarrow \tilde{x}\sqrt{f}, \\ y &\longrightarrow \frac{\tilde{y}}{\sqrt{f}} - \frac{c_3}{\sqrt{f}} \int \sqrt{f}\mathrm{d}u, \\ r &\longrightarrow \frac{\tilde{r}}{\tilde{x}^2}. \end{aligned}$$

Dropping the tildes, the metric one forms simplify to

$$\begin{aligned} \boldsymbol{\omega}^1 &= \frac{2}{x}\mathrm{d}u - \frac{r}{2x^2}\mathrm{d}x + \mathrm{i}x\mathrm{d}y, \\ \boldsymbol{\omega}^2 &= \frac{2}{x}\mathrm{d}u - \frac{r}{2x^2}\mathrm{d}x - \mathrm{i}x\mathrm{d}y, \\ \boldsymbol{\omega}^3 &= \mathrm{d}u, \\ \boldsymbol{\omega}^4 &= \frac{1}{x^2}\mathrm{d}r - \frac{2r}{x^3}\mathrm{d}x - \frac{2c_2 + \log r^2}{x^2}\mathrm{d}u, \end{aligned}$$

which leads to the line element

$$\mathrm{d}s^2 = \frac{4\,(2+c_2) + \log r^4}{x^2}\mathrm{d}u^2 - \frac{2}{x^2}\mathrm{d}r\mathrm{d}u + \frac{r^2}{2x^4}\mathrm{d}x^2 + 2x^2\mathrm{d}y^2. \qquad (4.38)$$

If $c_1 \neq 0$, however, things are a bit more complicated. The required coordinate transformation is now given by

$$\begin{aligned} x &\longrightarrow \frac{\tilde{x}\sqrt{f}}{\sqrt{c_1}}, \\ y &\longrightarrow \frac{y\sqrt{c_1}}{\sqrt{f}}, \\ r &\longrightarrow \tilde{r}\sqrt{c_1}, \\ u &\longrightarrow \frac{\tilde{u}}{\sqrt{c_1}}. \end{aligned}$$

Next we replace f by $\tilde{f}\sqrt{c_1}$, c_3 by $c_3\, c_1^{3/4}$, and c_2 by $\dfrac{c_2}{2} - 2$. A second coordinate transformation

$$
\begin{aligned}
\tilde{y} &\longrightarrow y' + \int \log \tilde{f}^2 - c_3\sqrt{\tilde{f}}\mathrm{d}u,\\
\tilde{r} &\longrightarrow \frac{r'}{\tilde{x}^2},\\
\tilde{x} &\longrightarrow x'^{1/4},
\end{aligned}
$$

and dropping the dashes and tildes then leads to the one forms

$$
\begin{aligned}
\boldsymbol{\omega}^1 &= \left(\frac{2\sqrt{1-x}}{x^{1/4}} - \mathrm{i}x^{1/4}\log r^2\right)\mathrm{d}u - \frac{r}{8x^{5/4}\sqrt{1-x}}\mathrm{d}x + \mathrm{i}x^{1/4}\mathrm{d}y,\\
\boldsymbol{\omega}^3 &= \frac{1}{\sqrt{c_1}}\mathrm{d}u,\\
\boldsymbol{\omega}^4 &= \sqrt{c_1}\left[\left(\frac{4-c_2-5x}{\sqrt{x}} - \frac{(2x+1)\log r^2}{\sqrt{x}}\right)\mathrm{d}u + \frac{\mathrm{d}r}{\sqrt{x}} - \frac{r\mathrm{d}x}{2x^{3/2}} + 2\sqrt{x}\mathrm{d}y\right],
\end{aligned}
$$

which corresponds to the line element

$$
\begin{aligned}
\mathrm{d}s^2 &= \left(2\sqrt{x}\left(\log r^2\right)^2 + \frac{2\,(2x+1)\log r^2}{\sqrt{x}} + \frac{2\,(x+c_2)}{\sqrt{x}}\right)\mathrm{d}u^2 - \frac{2}{\sqrt{x}}\mathrm{d}r\mathrm{d}u\\
&- \left(4\sqrt{x}\log r^2 + 4\sqrt{x}\right)\mathrm{d}y\mathrm{d}u + \frac{r^2\mathrm{d}x^2}{32x^{5/2}\,(1-x)} + 2\sqrt{x}\mathrm{d}y^2. \qquad (4.39)
\end{aligned}
$$

Notice that the constant c_1 does not appear in the line element (indeed, as can be seen from the one forms above, we can boost such that $c_1 = 1$).

4.3.1 Comparison with the original paper

In this paragraph, we will look at the metrics (4.38) and (4.39), and we examine the relation between these and the ones in (Newman and Tamburino, 1962). First note that, due to typing mistakes, the cylindrical metrics presented in the original paper are not empty space metrics, neither the general metric, nor the limiting Sachs metric. The latter is obtained by shifting the y-origin in the general cylindrical Newman Tamburino metric and then taking the limit for $b \longrightarrow 0$. The Sachs metric in its correct form is defined by the line element

$$
\mathrm{d}s^2 = \frac{\log\left(r^2x^4\right) - g}{x^2}\mathrm{d}u^2 - \left(2\mathrm{d}r + \frac{4r}{x}\mathrm{d}x\right)\mathrm{d}u + x^2\mathrm{d}y^2 + r^2\mathrm{d}x^2, \qquad (4.40)
$$

which is equivalent to the original Sachs metric (Sachs, 1961):

$$\exp(2k - 2U)\left(\mathrm{d}r^2 + \mathrm{d}x^2\right) + r^2\exp(-2U)\mathrm{d}y^2 - \exp(2U)\mathrm{d}u^2,$$

where $U = x + \log r$ and $k = -\frac{r^2}{2} + 2x + c + \log r$, c being a constant. If we replace in the latter

$$\begin{aligned}
x &\longrightarrow \frac{g}{2} - c - \log\tilde{x},\\
r &\longrightarrow \sqrt{g - \log(\tilde{r}^2\tilde{x}^4)},\\
y &\longrightarrow \tilde{y}\tilde{x}\exp\left(\frac{g}{2} - c - \log\tilde{x}\right),\\
u &\longrightarrow \tilde{u} + h(\tilde{x},\tilde{r}),\\
c &\longrightarrow \frac{g}{2},
\end{aligned}$$

where $h(\tilde{x},\tilde{r})$ is a function satisfying

$$\begin{aligned}
\frac{\partial h}{\partial\tilde{r}} &= \frac{\tilde{x}^2}{g - \log(\tilde{r}^2\tilde{x}^4)},\\
\frac{\partial h}{\partial\tilde{x}} &= \frac{2\tilde{x}\tilde{r}}{g - \log(\tilde{r}^2\tilde{x}^4)},
\end{aligned}$$

and if we drop the tildes, this is precisely the form of the Sachs metric as in (4.40).

The empty space cylindrical metric should have had the following form:

$$\begin{aligned}
\mathrm{d}s^2 = &\left[\left(\frac{4}{cn^2} + cn^2\left(\log r^2\right)^2 + \frac{\log(r^2cn^4)}{cn^2}\right)b^2 + \frac{c}{cn^2}\right]\mathrm{d}u^2 - 2\mathrm{d}r\mathrm{d}u\\
&+ \left(\frac{r^2}{2} + 8u^2b^4\left(1 - cn^4\right)\right)\mathrm{d}x^2 + 4cn\sqrt{2}u\sqrt{1 - cn^4}b^2\mathrm{d}x\mathrm{d}y + cn^2\mathrm{d}y^2\\
&+ \left(2u\,cn^2b^2\log r^2 + r\right)\frac{2\sqrt{2}b\sqrt{1 - cn^4}}{cn}\mathrm{d}x\mathrm{d}u + 2cn^2b\log r^2\mathrm{d}y\mathrm{d}u,
\end{aligned} \tag{4.41}$$

where b and c are constants, and where $cn = cn(bx)$ is an elliptic function subject to the differential equations

$$\begin{aligned}
\frac{\mathrm{d}cn}{\mathrm{d}x} &= -b\sqrt{\frac{1 - cn^4}{2}},\\
\frac{\mathrm{d}^2cn}{\mathrm{d}x^2} &= -b^2cn^3.
\end{aligned} \tag{4.42}$$

The misprint in the original paper is situated in the coefficient of $\mathrm{d}x^2$, due to the presence of a factor $\log r/16$ in the second term. Although this mistake had been determined before, the author was unaware of that fact until after her calculations (private communication with A. Barnes and J. Åman). The calculations presented in this chapter are independent of the work of A. Barnes and J. Åman, and it is only recently that we put our results together.

It is useful to rewrite the Newman Tamburino cylindrical metric (4.41) in other coordinates: replacing y and r by

$$\begin{aligned} y &\longrightarrow \sqrt{2}y + 2bu\log cn^2, \\ r &\longrightarrow \frac{r}{cn^2}, \end{aligned}$$

and taking into account the conditions (4.42), we obtain a metric that can be compared to (4.39) more easily. In these coordinates, the line element takes the form

$$\begin{aligned} \mathrm{d}s^2 &= \left(b^2cn^2(\log r^2)^2 + \frac{b^2\log r^2}{cn^2} + \frac{4b^2+c}{cn^2}\right)\mathrm{d}u^2 - \frac{2}{cn^2}\mathrm{d}r\mathrm{d}u \\ &+ 2cn^2\sqrt{2}b\log r^2\mathrm{d}y\mathrm{d}u + \frac{r^2\mathrm{d}x^2}{2cn^4} + 2cn^2\mathrm{d}y^2. \end{aligned} \tag{4.43}$$

We will now show how one can find the metric of Newman and Tamburino (and its Sachs limit) from our metric (4.39). We will also show that metric (4.38) is a limiting metric of (4.39), which, surprisingly, coincides with the Sachs metric.

If we replace r by $r\,x^2$ and c_2 by $-\frac{g}{2} - 2 - \log 2$ in metric (4.38), and scale x and y with a factor $\sqrt{2}$ and 2^{-1} respectively, we obtain

$$\mathrm{d}s^2 = \frac{\log(r^2x^4) - g}{x^2}\mathrm{d}u^2 - \left(2\mathrm{d}r + \frac{4r}{x}\mathrm{d}x\right)\mathrm{d}u + r^2\mathrm{d}x^2 + x^2\mathrm{d}y^2,$$

which is exactly the same as the Sachs metric (Sachs, 1961).
It is also possible, by a singular limit procedure, to obtain this metric from (4.39). Therefore the latter is the most general empty space metric, satisfying the cylindrical condition (4.16), that has a hypersurface orthogonal

and geodesic principal null direction, with non-vanishing shear and divergence. The way to prove this, is as follows: in (4.39) apply the coordinate transformations

$$\begin{aligned} x &\longrightarrow \tilde{x}^4, \\ y &\longrightarrow \tilde{y} + u(1 + \log r^2), \\ r &\longrightarrow \tilde{r}\tilde{x}^2. \end{aligned}$$

Next, replace $\tilde{x}$ by $\frac{\tilde{x}}{a\sqrt{2}}$, $\tilde{r}$ by $2\,a\,\tilde{r}$, u by $\frac{u}{2a}$, $\tilde{y}$ by $a\,\tilde{y}$ and c_2 by $-g + \log a^2$. Taking the limit for a to $+\infty$ again results in the Sachs metric, in the same coordinates as before.

The more difficult task is to prove that (4.41) is a family of metrics that are covered by (4.39). To this end, we have already rewritten (4.41) in the form (4.43). If we now substitute x by cn^4 in (4.39), where cn is an elliptic function satisfying (4.42) and if we scale r and u and replace c_2 by $5 + c/b^2$, this exactly reproduces the (corrected version of the) metric of Newman and Tamburino (4.43). A CLASSI input file for both the Sachs metric (4.40) and the original (corrected version of) cylindrical Newman Tamburino metric (4.41) is given in the appendix (files by A. Barnes (modifications by J. Åman)).

4.4 Spherical class: Newman Penrose analysis

In this section we will examine the spherical empty space solutions in the same way as we did for the cylindrical solutions in section 4.3. First we substitute the basic assumptions in the set of NP Ricci and Bianchi equations:

$$\begin{aligned} \Psi_0 &= 0, \\ \overline{\rho} &= \rho, \\ \kappa &= 0, \\ \Phi_{ij} &= 0, \qquad i = 0,1,2; \quad j = 0,1,2. \end{aligned}$$

We again fix the null rotation by choosing the same value for Ψ_2 as in the GHP analysis:

$$\Psi_2 = -\frac{1}{4}\left(\rho\Psi_1 + \sigma\overline{\Psi_1}\right)\frac{\Psi_1}{\sigma^2\overline{\sigma}}.$$

This allows us to copy all the invariant information obtained in section 4.2.2 to the present situation. We thus have expressions for all the components Ψ_i of the Weyl tensor, but also the spin coefficients ν and λ are defined in an invariant manner.

$$\begin{aligned}
\Psi_1 &= -\left(\overline{\tau}+\pi\right)\sigma,\\
\Psi_2 &= -\frac{1}{4}\frac{\left(\overline{\tau}+\pi\right)^2\rho}{\overline{\sigma}} - \frac{1}{4}\left(\overline{\tau}+\pi\right)\left(\tau+\overline{\pi}\right),\\
\Psi_3 &= \frac{1}{3}\frac{\left(7\tau-\overline{\pi}\right)\pi\mu}{\overline{\pi}} + \frac{1}{16}\frac{\left(\tau-\overline{\pi}\right)\left(\tau+\overline{\pi}\right)^2\pi}{\overline{\pi}\sigma}\\
&\quad - \frac{1}{144}\frac{\pi^2\left(\overline{\pi}-7\tau\right)^2}{\overline{\pi}^2\rho}\left(\frac{6\left(\overline{\pi}^2-13\overline{\pi}\tau-6\tau^2\right)}{\left(\overline{\pi}-7\tau\right)} + \frac{\sigma\pi\left(\overline{\pi}-11\tau\right)}{\overline{\pi}\rho}\right),\\
\Psi_4 &= \frac{1}{6}\frac{\pi\left(\overline{\pi}-7\tau\right)^2}{\overline{\pi}}\left(\frac{\pi}{\overline{\pi}\rho} - \frac{\tau+\overline{\pi}}{\sigma\left(\overline{\pi}-7\tau\right)}\right)\mu\\
&\quad - \frac{1}{192}\frac{\pi\left(\tau+\overline{\pi}\right)^3}{\overline{\pi}\sigma}\left(\frac{3\left(\overline{\pi}-3\tau\right)}{\sigma} + \frac{4\pi\left(\overline{\pi}-7\tau\right)\left(\overline{\pi}^2-10\overline{\pi}\tau-3\tau^2\right)}{\overline{\pi}\rho\left(\overline{\pi}+\tau\right)^2}\right)\\
&\quad + \frac{1}{576}\frac{\pi^3\left(\overline{\pi}-7\tau\right)^3}{\overline{\pi}^3\rho^2}\left(\frac{19\overline{\pi}^2-118\overline{\pi}\tau-41\tau^2}{\left(\overline{\pi}-7\tau\right)} + \frac{2\pi\sigma\left(\overline{\pi}-11\tau\right)}{\overline{\pi}\rho}\right),
\end{aligned}$$

$$\begin{aligned}
\nu &= \left(\frac{1}{\sigma} - \frac{1}{3}\frac{\pi\left(\overline{\pi}-7\tau\right)}{\overline{\pi}\left(\tau+\overline{\pi}\right)\rho}\right)\tau\mu - \frac{1}{32}\frac{\pi\left(\overline{\pi}-\tau\right)^2\left(\overline{\pi}-7\tau\right)}{\overline{\pi}\sigma\rho}\\
&\quad + \frac{\pi^3\left(\overline{\pi}-7\tau\right)^2}{288(\tau+\overline{\pi})}\left(\frac{6\left(\overline{\pi}^3-3\overline{\pi}\tau^2-9\tau(\tau^2+\overline{\pi}^2)\right)}{\pi\overline{\pi}^2\left(\overline{\pi}-7\tau\right)\rho^2} + \frac{\sigma\left(3\overline{\pi}^2-14\overline{\pi}\tau+7\tau^2\right)}{\overline{\pi}^3\rho^3}\right),\\
\lambda &= \frac{\rho\mu}{\sigma} + \frac{\pi\left(3\overline{\pi}-\tau\right)\left(\overline{\pi}-\tau\right)}{8\overline{\pi}\sigma} - \frac{\pi^2\left(\overline{\pi}-7\tau\right)\left(\overline{\pi}+7\tau\right)}{24\overline{\pi}^2\rho}.
\end{aligned}$$

In addition, we have two algebraic equations

$$\overline{\tau\pi} = \tau\pi, \tag{4.44}$$

$$\rho^2 + \frac{1}{3}\frac{\sigma\overline{\sigma}\left(\overline{\pi}-7\tau\right)}{\tau+\overline{\pi}} = 0 \tag{4.45}$$

and an expression for the imaginary part of μ, given by

$$\overline{\mu} = \mu + \frac{1}{4}\frac{\left(\tau+\overline{\pi}\right)\left(\overline{\pi}-2\tau\right)\left(\overline{\sigma\pi^2}-\pi^2\sigma\right)}{\sigma\overline{\sigma\pi}^2}. \tag{4.46}$$

As we assume non-vanishing shear, we can again fix the rotational freedom of the null tetrad $(\mathbf{m},\bar{\mathbf{m}},\mathbf{l},\mathbf{k})$ by making σ real. This in turn allows us to boost such that $\epsilon = 0$ (by (1.26)) and such that

$$\alpha = \overline{\tau} - \overline{\beta}.$$

The required boost parameter A has to satisfy

$$\begin{aligned} \epsilon' &= 0 = \epsilon + \frac{1}{2}\mathrm{D}\ln A, \\ \overline{\alpha}' + \beta' - \tau' &= 0 = \overline{\alpha} + \beta - \tau + \delta \ln A, \end{aligned}$$

the integrability conditions of which are identically satisfied under the Newman Penrose equations (1.27 − 1.29), (1.36) and (1.41).

In the spherical case, it is not so easy to first 'solve' the NP-equations for directional derivatives of the spin coefficients, subsequently do the same for the Bianchi equations and the directional derivatives of the Weyl tensor components and next apply the commutators to the spin coefficients and Weyl tensor components. Doing so, the equations become very lengthy and hard to solve. We will therefore, in this section, switch between the sets, in order to extract all information in a faster, more elegant way.

First we write (1.25 − 1.30) as

$$\begin{aligned} \mathrm{D}\rho &= \rho^2 + \sigma^2, \\ \mathrm{D}\sigma &= 2\rho\sigma, \\ \mathrm{D}\tau &= (\tau + \overline{\pi})\,\rho, \\ \mathrm{D}\beta &= \rho\beta - \overline{\beta}\sigma, \\ \mathrm{D}\gamma &= \frac{1}{4}\left(3\overline{\tau}\,(\tau + \overline{\pi}) + \pi\,(3\tau - \overline{\pi})\right) + \beta\,(\overline{\tau} + \pi) - \overline{\beta}\,(\tau + \overline{\pi}) - \frac{(\overline{\tau} + \pi)^2\,\rho}{4\sigma}. \end{aligned} \tag{4.47}$$

Then, we solve (1.43) for $\mathrm{D}\pi$:

$$\mathrm{D}\pi = (\overline{\tau} + \pi)\,\rho.$$

At this point, we can find a solution for τ: notice first that $\rho^2 - c\,\sigma^2 = 0$, where c is a constant, is only possible for $c = 1$, the cylindrical case, which we have already studied. This follows directly by evaluating $\mathrm{D}\left(\rho^2 - c\,\sigma^2\right) = 0$. It is thus allowed to solve equation (4.45) for τ:

$$\tau = -\frac{\overline{\pi}\left(3\rho^2 + \sigma^2\right)}{3\rho^2 - 7\sigma^2}.$$

The equations (4.44) and (4.47) are then identically satisfied.

Next we proceed by rewriting (1.31 − 1.32), (1.35 − 1.36) and (1.40):

$$
\begin{aligned}
\mathrm{D}\mu &= \frac{\sigma\overline{\delta}\pi}{\rho} + 2\mu\rho - \frac{2\sigma\pi\overline{\beta}}{\rho} \\
&+ \frac{3\left(\rho^2-\sigma^2\right)\left(9\overline{\pi}\rho^3 + 12\pi\sigma\rho^2 - 11\overline{\pi}\sigma^2\rho - 14\pi\sigma^3\right)\pi}{\rho\left(3\rho^2-7\sigma^2\right)^2}, \\
\delta\pi &= \frac{\sigma\overline{\delta}\pi}{\rho} - 2\beta\pi - \frac{2\sigma\pi\overline{\beta}}{\rho} \\
&+ \frac{2\sigma\pi\left(21\sigma^4\pi + 12\sigma\overline{\pi}\rho^3 - 38\sigma^2\pi\rho^2 - 4\overline{\pi}\sigma^3\rho + 9\pi\rho^4\right)}{\rho\left(3\rho^2-7\sigma^2\right)^2}, \\
\delta\rho &= \overline{\delta}\sigma + 4\overline{\beta}\sigma - \frac{3\overline{\pi}\rho^3 + \sigma^2\overline{\pi}\rho - 9\rho^2\pi\sigma + 5\sigma^3\pi}{3\rho^2-7\sigma^2},
\end{aligned}
$$

$$
\begin{aligned}
\overline{\delta}\pi = & \frac{48\rho^2\pi}{\left(3\rho^2+\sigma^2\right)\left(3\rho^2-7\sigma^2\right)}\left(\sigma\overline{\delta}\sigma - \rho\delta\sigma\right) - \frac{\rho\left(3\rho^2-7\sigma^2\right)}{\left(3\rho^2+\sigma^2\right)\sigma}\left(\overline{\delta}\beta + \delta\overline{\beta}\right) \\
& - \frac{\left(9\overline{\pi}\rho^5 - 18\sigma\pi\rho^4 - 18\sigma^2\overline{\pi}\rho^3 - 156\sigma^3\pi\rho^2 - 7\overline{\pi}\sigma^4\rho + 14\sigma^5\pi\right)\overline{\beta}}{\sigma\left(3\rho^2+\sigma^2\right)\left(3\rho^2-7\sigma^2\right)} \\
& - \frac{2\pi\sigma\left(153\overline{\pi}\rho^5 - 255\sigma\pi\rho^4 + 21\sigma^5\pi + 138\sigma^3\pi\rho^2 - 11\overline{\pi}\sigma^4\rho - 78\sigma^2\overline{\pi}\rho^3\right)}{\left(3\rho^2-7\sigma^2\right)^2\left(3\rho^2+\sigma^2\right)} \\
& - \frac{4\rho\left(3\rho^2-7\sigma^2\right)\beta\overline{\beta}}{\left(3\rho^2+\sigma^2\right)\sigma} - \frac{\rho\pi\beta}{\sigma},
\end{aligned}
$$

$$
\begin{aligned}
\Delta\sigma = & \frac{\rho}{\sigma}\left(\delta\overline{\beta} + \overline{\delta}\beta\right) + \frac{48\overline{\pi}\rho\left(\rho^2+\sigma^2\right)\overline{\delta}\sigma}{\left(3\rho^2-7\sigma^2\right)^2} - \frac{96\overline{\pi}\rho^2\sigma\delta\sigma}{\left(3\rho^2-7\sigma^2\right)^2} + \frac{4\rho\overline{\beta}\beta}{\sigma} \\
& - \frac{\left(7\sigma^4\pi + 192\sigma^3\overline{\pi}\rho + 18\sigma^2\pi\rho^2 - 9\rho^4\pi\right)\rho\beta}{\left(3\rho^2-7\sigma^2\right)^2\sigma} - \frac{\left(18\rho^2-185\sigma^2\right)\rho\sigma\overline{\pi}\overline{\beta}}{\left(3\rho^2-7\sigma^2\right)^2} \\
& - \frac{\rho\pi\overline{\pi}\left(27\rho^6 - 441\sigma^2\rho^4 - 63\sigma^4\rho^2 + 157\sigma^6\right)}{\left(3\rho^2-7\sigma^2\right)^3\sigma} + 3\sigma\gamma - \sigma\overline{\gamma} - \mu\sigma - \frac{\rho^2\overline{\mu}}{\sigma} \\
& + \frac{2\overline{\pi}^2\left(27\rho^6 + 285\sigma^2\rho^4 - 163\sigma^4\rho^2 - 21\sigma^6\right)}{\left(3\rho^2-7\sigma^2\right)^3} + \frac{9\rho^5\overline{\pi}\overline{\beta}}{\left(3\rho^2-7\sigma^2\right)^2\sigma}. \qquad (4.48)
\end{aligned}
$$

As we have rotated the tetrad so that σ is real, also $\Delta\sigma$ is real. Taking into account (4.46), we get from the imaginary part of (4.48) that:

$$\frac{\left(\rho^2\overline{\pi}+2\sigma\pi\rho+\sigma^2\overline{\pi}\right)\overline{\delta}\sigma}{\left(3\rho^2-7\sigma^2\right)^2\sigma}-\frac{\left(\rho^2\pi+2\sigma\overline{\pi}\rho+\pi\sigma^2\right)\delta\sigma}{\left(3\rho^2-7\sigma^2\right)^2\sigma}-\frac{4\sigma\left(\rho\overline{\pi}+\sigma\pi\right)\beta}{\left(3\rho^2-7\sigma^2\right)^2}$$
$$+\frac{4\sigma\left(\rho\pi+\sigma\overline{\pi}\right)\overline{\beta}}{\left(3\rho^2-7\sigma^2\right)^2}+\frac{\gamma-\overline{\gamma}}{12\rho}-\frac{\sigma\left(7\sigma^4-138\sigma^2\rho^2+195\rho^4\right)\left(\overline{\pi}^2-\pi^2\right)}{12\rho\left(3\rho^2-7\sigma^2\right)^3}=0. \tag{4.49}$$

Notice that we can solve this equation for $\delta\sigma$ (or $\overline{\delta}\sigma$), as $\rho^2\pi+2\sigma\overline{\pi}\rho+\pi\sigma^2$ cannot be zero. Proof: substitute

$$\overline{\pi}=-\frac{\pi\left(\rho^2+\sigma^2\right)}{2\sigma\rho}$$

in $\mathrm{D}(\rho^2\pi+2\sigma\overline{\pi}\rho+\pi\sigma^2)=0$. This leads to

$$\frac{\left(\rho^2-\sigma^2\right)^2\pi}{\rho}=0,$$

which is impossible in the spherical class.

Another relation between $\delta\sigma$ and $\overline{\delta}\sigma$ can be found by calculating D(4.46). The equation so obtained is given by

$$\frac{\rho\pi+\sigma\overline{\pi}}{\sigma^2\left(2\rho^2-\sigma^2\right)}\delta\sigma-\frac{\rho\overline{\pi}+\sigma\pi}{\sigma^2\left(2\rho^2-\sigma^2\right)}\overline{\delta}\sigma+\frac{4\left(\overline{\pi}\beta-\pi\overline{\beta}\right)}{2\rho^2-\sigma^2}+\frac{4\left(\overline{\pi}^2-\pi^2\right)}{3\rho^2-7\sigma^2}=0. \tag{4.50}$$

Again one can solve this equation for $\delta\sigma$ (or $\overline{\delta}\sigma$). To show that $(\rho\pi+\sigma\overline{\pi})\neq 0$, it suffices to take the D-derivative of $\rho\pi+\sigma\overline{\pi}$.

Next, we will rewrite the couple of Bianchi equations (1.44) and (1.45). The first gives us an expression for $\delta\overline{\beta}$, which we will use, together with (4.50) to simplify the second:

$$\begin{aligned}\delta\overline{\beta} &= -\overline{\delta}\beta+\frac{\left(9\rho^2+\sigma^2\right)\pi\delta\sigma}{\sigma\left(3\rho^2-7\sigma^2\right)}-\frac{6\pi\rho\overline{\delta}\sigma}{3\rho^2-7\sigma^2}\\ &- 4\overline{\beta}\beta-\frac{5\pi\left(3\rho^2+\sigma^2\right)\beta}{3\rho^2-7\sigma^2}-\frac{\left(3\rho^2\overline{\pi}+24\sigma\rho\pi+\sigma^2\overline{\pi}\right)\overline{\beta}}{3\rho^2-7\sigma^2}\\ &+ \frac{2\left(27\overline{\pi}\rho^4-27\sigma\pi\rho^3-9\overline{\pi}\rho^2\sigma^2+11\pi\sigma^3\rho-2\overline{\pi}\sigma^4\right)\pi}{\left(3\rho^2-7\sigma^2\right)^2},\end{aligned} \tag{4.51}$$

$$\left(9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}\right)\overline{\delta}\sigma = \frac{4\left(3\pi\rho^3 + 12\overline{\pi}\sigma\rho^2 - 13\pi\sigma^2\rho - 14\sigma^3\overline{\pi}\right)\rho\sigma\beta}{\rho^2 - \sigma^2}$$
$$+ \frac{4\left(9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}\right)\sigma^3\overline{\beta}}{\rho^2 - \sigma^2}$$
$$+ \frac{2\sigma\left(2\pi\sigma^2 + \sigma\overline{\pi}\rho - 3\pi\rho^2\right)\left(9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}\right)}{3\rho^2 - 7\sigma^2}.$$

We can solve this last equation for $\overline{\delta}\sigma$, as $\left(9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}\right)$ cannot be zero. It is again sufficient to look at $\mathrm{D}\left(9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}\right)$, which after substituting $\overline{\pi} = \dfrac{\pi\left(9\rho^2 - 7\sigma^2\right)}{6\sigma\rho}$ equals

$$-\frac{\left(\rho^2 - \sigma^2\right)\left(9\rho^2 + 7\sigma^2\right)\pi}{\rho}.$$

It follows that

$$\begin{aligned}\overline{\delta}\sigma &= \frac{4\left(3\pi\rho^3 + 12\rho^2\overline{\pi}\sigma - 13\sigma^2\pi\rho - 14\sigma^3\overline{\pi}\right)\rho\sigma\beta}{(\rho^2 - \sigma^2)\left(9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}\right)} + \frac{4\sigma^3\overline{\beta}}{\rho^2 - \sigma^2} \\ &+ \frac{2\left(2\pi\sigma^2 + \sigma\overline{\pi}\rho - 3\pi\rho^2\right)\sigma}{3\rho^2 - 7\sigma^2},\end{aligned} \tag{4.52}$$

with a similar expression for $\delta\sigma$, as σ is real.

We now return to equation (4.49), from which we eliminate $\delta\sigma$ by (4.50) and $\overline{\delta}\sigma$ by (4.52). This gives us the imaginary part of γ:

$$\overline{\gamma} = \gamma - \frac{\left(\overline{\pi}^2 - \pi^2\right)\sigma}{3\rho^2 - 7\sigma^2}\left(\frac{48\rho^2\beta}{9\rho^2\overline{\pi} - 6\sigma\pi\rho - 7\sigma^2\overline{\pi}} + \frac{9\rho^2 - \sigma^2}{3\rho^2 - 7\sigma^2}\right). \tag{4.53}$$

From (4.50), we get an expression for $\overline{\beta}$:

$$\overline{\beta} = \frac{9\pi\rho^2 - 6\sigma\overline{\pi}\rho - 7\pi\sigma^2}{9\overline{\pi}\rho^2 - 6\sigma\pi\rho - 7\overline{\pi}\sigma^2}\beta, \tag{4.54}$$

the D-derivative of which tells us that β has to be zero, unless π is real or purely imaginary. Substituting (4.54) in D(4.54) leads to

$$\beta\sigma\left(27\rho^4 - 150\rho^2\sigma^2 + 91\sigma^4\right)(\pi - \overline{\pi})(\pi + \overline{\pi}) = 0. \tag{4.55}$$

We can now prove that $\beta = 0$. To this end, we first look at $[\delta, \mathrm{D}]\pi$, from which we eliminate $\mathrm{D}\overline{\delta}\beta$ and $\mathrm{D}\delta\overline{\beta}$ by $[\overline{\delta}, \mathrm{D}]\beta$ and $[\delta, \mathrm{D}]\overline{\beta}$. From the result we then eliminate $\delta\overline{\beta}$ by (4.51), and $\overline{\beta}$ by (4.54), yielding

$$\pi\rho^2\sigma^3\beta\Big[\rho^2\left(81\rho^6-1035\rho^4\sigma^2+1971\rho^2\sigma^4-889\sigma^6\right)\overline{\pi}^3$$
$$-\sigma\pi\rho(189\rho^6-1527\rho^4\sigma^2+1959\rho^2\sigma^4-749\sigma^6)\overline{\pi}^2$$
$$+\left(1215\rho^8-5229\rho^6\sigma^2+7989\rho^4\sigma^4-6063\sigma^6\rho^2+1960\sigma^8\right)\pi^2\overline{\pi}$$
$$-\pi^3\rho\sigma\left(891\rho^6-3537\rho^4\sigma^2+4545\rho^2\sigma^4-1771\sigma^6\right)\Big]=0.$$

If β were *not* equal to zero, then, by (4.55) and by the fact that ρ^2/σ^2 cannot be constant, π is real or purely imaginary. Substituting $\overline{\pi}=s\pi$, $s=\pm1$, in the previous equation, however, we get

$$\beta\sigma\rho\pi\left(\rho^2-\sigma^2\right)\left(162\rho^6-621\sigma^2\rho^4+624\sigma^4\rho^2-245\sigma^6\right)s$$
$$-\beta\sigma\rho\pi\left(\rho^2-\sigma^2\right)\left(135\rho^4-498\rho^2\sigma^2+315\sigma^4\right)\rho\sigma=0,$$

so that $\beta=0$ anyway.

We now proceed by solving the remaining NP Ricci and Bianchi equations for the directional derivatives of the spin coefficients ρ, π, μ and γ:

$$\Delta\rho=2\rho\left(\gamma-\mu\right)-\frac{\pi\overline{\pi}\left(27\rho^4-42\rho^2\sigma^2-\sigma^4\right)}{\left(3\rho^2-7\pi^2\right)^2}$$
$$-\frac{27\sigma\pi^2\rho^3+9\sigma\overline{\pi}^2\rho^3+7\sigma^3\overline{\pi}^2\rho-11\sigma^3\pi^2\rho}{\left(3\rho^2-7\pi^2\right)^2},$$

$$\Delta\pi=\frac{\left(3\rho^4-2\rho^2\sigma^2+7\sigma^4\right)\pi\mu}{\sigma^2\left(3\rho^2-7\sigma^2\right)}+\frac{\pi^3\left(54\rho^6-111\rho^4\sigma^2+224\rho^2\sigma^4-7\sigma^6\right)}{\sigma\left(3\rho^2-7\sigma^2\right)}$$
$$+\frac{\left(9\rho^2-\sigma^2\right)\pi\sigma\overline{\pi}^2}{\left(3\rho^2-7\sigma^2\right)^2}+\frac{\left(27\rho^6-135\rho^4\sigma^2+21\rho^2\sigma^4-169\sigma^6\right)\rho\pi^2\overline{\pi}}{\sigma^2\left(3\rho^2-7\sigma^2\right)^3},$$

$$\delta\mu=\frac{2\left(3\rho\overline{\pi}+\sigma\pi\right)\rho\mu}{7\sigma^2-3\rho^2}-\frac{24\sigma^5\overline{\pi}^3+3\rho\pi\left(27\rho^4-66\rho^2\sigma^2+47\sigma^4\right)\overline{\pi}^2}{\left(3\rho^2-7\sigma^2\right)^3}$$
$$+\frac{3\sigma\pi^2\left(45\rho^4-106\rho^2\sigma^2+37\sigma^4\right)\overline{\pi}+4\pi^3\sigma^2\rho\left(45\rho^2-23\sigma^2\right)}{\left(3\rho^2-7\sigma^2\right)^3},$$

$$
\begin{aligned}
\bar{\delta}\mu =& -\frac{2\rho\left(\sigma\bar{\pi}+3\pi\rho\right)\mu}{3\rho^2-7\sigma^2}-\frac{\pi\sigma\left(31\sigma^4-66\rho^2\sigma^2+27\rho^4\right)\bar{\pi}^2}{\left(3\rho^2-7\sigma^2\right)^3}\\
&-\frac{24\sigma^4\rho\bar{\pi}^3}{\left(3\rho^2-7\sigma^2\right)^3}-\frac{\pi^2\rho\left(81\rho^4-306\rho^2\sigma^2+185\sigma^4\right)\bar{\pi}}{\left(3\rho^2-7\sigma^2\right)^3}\\
&-\frac{4\pi^3\sigma\left(27\rho^4+9\pi^2\sigma^2-14\sigma^4\right)}{\left(3\rho^2-7\sigma^2\right)^3},
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mu =& -2\mu\gamma-\frac{3\pi^2\left(81\rho^8-630\rho^6\sigma^2+1356\rho^4\sigma^4-842\rho^2\sigma^6-29\sigma^8\right)\bar{\pi}^2}{\left(3\rho^2-7\sigma^2\right)\sigma^2}\\
&-\frac{\left(\rho^2+\sigma^2\right)\mu^2}{\sigma^2}-\frac{2\rho\pi^3\left(405\rho^6-1755\rho^4\sigma^2+1671\rho^2\sigma^4-257\sigma^6\right)\bar{\pi}}{\sigma\left(3\rho^2-7\sigma^2\right)^4}\\
&+\frac{24\left(3\rho^2+\sigma^2\right)\sigma^4\bar{\pi}^4}{\left(3\rho^2-7\sigma^2\right)^4}-\frac{8\pi^4\left(81\rho^6-180\rho^4\sigma^2+48\rho^2\sigma^4+7\sigma^6\right)}{\left(3\rho^2-7\sigma^2\right)}\\
&+\frac{3\sigma^3\left(\rho^2-\sigma^2\right)\bar{\pi}^2-12\rho^3\pi\left(3\rho^2-11\sigma^2\right)\bar{\pi}-\pi^2\left(54\rho^2-13\rho^2\sigma^2-9\sigma^4\right)}{\sigma^2\left(3\rho^2-7\sigma^2\right)^2}\mu,
\end{aligned}
$$

$$
\begin{aligned}
\delta\gamma =& \frac{3\pi\left(\left(\bar{\pi}^2+8\pi^2\right)\sigma^2-6\left(4\pi^2+7\bar{\pi}^2\right)\rho^2\right)\rho\sigma^2}{\left(3\rho^2-7\sigma^2\right)^3}\\
&+\frac{\pi\bar{\pi}\left(27\bar{\pi}\rho^5+9\pi\sigma\rho^4+78\pi\rho^2\sigma^3-7\pi\sigma^5\right)}{\left(3\rho^2-7\sigma^2\right)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\delta}\gamma =& \frac{\pi\left(\left(8\pi^2+\bar{\pi}^2\right)\sigma^2-6\left(4\pi^2+7\bar{\pi}^2\right)\rho^2\right)\sigma^3}{\left(3\rho^2-7\sigma^2\right)^3}\\
&+\frac{\pi\bar{\pi}\left(27\pi\rho^5+9\bar{\pi}\rho^4\sigma-54\pi\sigma^2\rho^3+11\pi\sigma^4\rho\right)}{\left(3\rho^2-7\sigma^2\right)^3}.
\end{aligned}
$$

No further information can be obtained from the NP Ricci equations, the Bianchi equations, the commutator relations applied to any of the spin coefficients or from the directional derivatives of (4.53) or (4.46). The next step in the integration is to introduce coordinates.

We first look at the first Cartan equations:

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^1 = &-\frac{\pi\left(3\rho^2+\sigma^2\right)}{3\rho^2-7\sigma^2}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2+\left(\gamma-\overline{\gamma}-\mu\right)\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3+\rho\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^4\\
&-\frac{\mu\left(9\rho^5-42\rho^3\sigma^2+49\rho\sigma^4\right)+3\overline{\pi}\pi\left(3\rho^4-8\rho^2\sigma^2+5\sigma^4\right)}{\sigma\left(3\rho^2-7\sigma^2\right)^2}\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3\\
&-\frac{2\pi^2\rho\left(9\rho^2+7\sigma^2\right)}{\left(3\rho^2-7\sigma^2\right)^2}\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3+\sigma\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^4-\frac{8\pi\sigma^2}{3\rho^2-7\sigma^2}\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^2 = &\frac{\overline{\pi}\left(3\rho^2+\sigma^2\right)}{3\rho^2-7\sigma^2}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2+\left(\overline{\gamma}-\gamma-\overline{\mu}\right)\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3+\rho\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^4\\
&-\frac{\overline{\mu}\left(9\rho^5-42\rho^3\sigma^2+49\sigma^4\rho\right)+3\overline{\pi}\pi\left(3\rho^4-8\rho^2\sigma^2+5\sigma^4\right)}{\sigma\left(3\rho^2-7\sigma^2\right)^2}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3\\
&-\frac{2\overline{\pi}^2\rho\left(9\rho^2+7\sigma^2\right)}{\left(3\rho^2-7\sigma^2\right)^2}\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3+\sigma\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^4+\frac{8\pi\sigma^2}{3\rho^2-7\sigma^2}\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4,
\end{aligned}$$

$$\mathrm{d}\boldsymbol{\omega}^3 = 0, \tag{4.56}$$

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^4 = &\left(\overline{\mu}-\mu\right)\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2-\left(\gamma+\overline{\gamma}\right)\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4+\frac{6\left(\rho^2-\sigma^2\right)}{3\rho^2-7\sigma^2}\left(\overline{\pi}\boldsymbol{\omega}^1+\pi\boldsymbol{\omega}^2\right)\wedge\boldsymbol{\omega}^4\\
&-\left(\frac{\left(3\rho^2+\sigma^2\right)\left(\overline{\pi}\rho+\sigma\pi\right)\overline{\mu}}{\left(3\rho^2-7\sigma^2\right)\sigma^2}+\frac{2\rho\left(27\rho^4-42\rho^2\sigma^2+7\sigma^4\right)\overline{\pi}^3}{\left(3\rho^2-7\sigma^2\right)^3\sigma}\right)\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3\\
&-\left(\frac{\pi\left(27\rho^6-54\rho^4\sigma^2+99\rho^2\sigma^4+8\sigma^6\right)\overline{\pi}^2+27\pi^2\sigma\rho\left(\rho^2-\sigma^2\right)^2\overline{\pi}}{\left(3\rho^2-7\sigma^2\right)^3\sigma^2}\right)\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^3\\
&-\left(\frac{\left(3\rho^2+\sigma^2\right)\left(\pi\rho+\overline{\pi}\sigma\right)\mu}{\left(3\rho^2-7\sigma^2\right)\sigma^2}+\frac{2\rho\left(27\rho^4-42\rho^2\sigma^2+7\sigma^4\right)\pi^3}{\left(3\rho^2-7\sigma^2\right)^3\sigma}\right)\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3\\
&-\left(\frac{\overline{\pi}\left(99\rho^2\sigma^4+8\sigma^6+27\rho^6-54\rho^4\sigma^2\right)\pi^2+27\overline{\pi}^2\rho\sigma\left(\rho^2-\sigma^2\right)^2\pi}{\left(3\rho^2-7\sigma^2\right)^3\sigma^2}\right)\boldsymbol{\omega}^2\wedge\boldsymbol{\omega}^3.
\end{aligned}$$

By (4.56), we have $\boldsymbol{\omega}^3 = \mathrm{d}u$. Furthermore, we have proven that $-\rho/\sigma$ cannot be constant, and we will use this expression as coordinate r. From

$$\mathrm{d}r = -\mathrm{d}(\rho/\sigma) = -\delta(\rho/\sigma)\boldsymbol{\omega}^1-\overline{\delta}(\rho/\sigma)\boldsymbol{\omega}^2-\Delta(\rho/\sigma)\boldsymbol{\omega}^3-\mathrm{D}(\rho/\sigma)\boldsymbol{\omega}^4,$$

we have that

$$\begin{aligned}
\boldsymbol{\omega}^4 &= \left(\frac{\left(9\overline{\pi}r^4 - 18\pi r^3 - 48\overline{\pi}r^2 + 10\pi r + \overline{\pi}\right)\pi}{\left(3r^2-7\right)^2\sigma^2} - \frac{r\mu}{\sigma}\right)\mathrm{d}u \\
&+ \frac{1}{\sigma\left(r^2-1\right)}\mathrm{d}r - \frac{\left(3\overline{\pi}r - \pi\right)}{\sigma\left(3r^2-7\right)}\boldsymbol{\omega}^1 - \frac{\left(3\pi r - \overline{\pi}\right)}{\sigma\left(3r^2-7\right)}\boldsymbol{\omega}^2,
\end{aligned}$$

where $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ have the general form

$$\begin{aligned}
\boldsymbol{\omega}^1 &= P\mathrm{d}x + Q\mathrm{d}y + V\mathrm{d}u + W\mathrm{d}r, \\
\boldsymbol{\omega}^2 &= \overline{P}\mathrm{d}x + \overline{Q}\mathrm{d}y + \overline{V}\mathrm{d}u + \overline{W}\mathrm{d}r.
\end{aligned}$$

In the above expressions P, Q, V and W are complex functions of (u, r, x, y). It is obvious now that we can choose the coordinates x and y in such a way that W is equal to zero. The derivative operators then become

$$\begin{aligned}
\delta &= \frac{\left(3\overline{\pi}r - \pi\right)\left(r^2-1\right)}{3r^2-7}\partial_r + \frac{\overline{Q}}{\overline{Q}P - \overline{P}Q}\partial_x - \frac{\overline{P}}{\overline{Q}P - \overline{P}Q}\partial_y, \\
\overline{\delta} &= \frac{\left(3\pi r - \overline{\pi}\right)\left(r^2-1\right)}{3r^2-7}\partial_r - \frac{Q}{\overline{Q}P - \overline{P}Q}\partial_x + \frac{P}{\overline{Q}P - \overline{P}Q}\partial_y, \\
\Delta &= \partial_u + \left(r\mu - \frac{\left(9\overline{\pi}r^4 - 18\pi r^3 - 48\overline{\pi}r^2 + 10\pi r - \overline{\pi}\right)\pi}{\sigma\left(3r^2-7\right)^2}\right)\left(r^2-1\right)\partial_r \\
&+ \frac{\overline{V}Q - \overline{Q}V}{\overline{Q}P - Q\overline{P}}\partial_x + \frac{V\overline{P} - \overline{V}P}{\overline{Q}P - \overline{P}Q}\partial_y, \\
\mathrm{D} &= \sigma\left(r^2-1\right)\partial_r.
\end{aligned}$$

We again proceed by examining the r-dependence of the spin coefficients, making use of their D-derivatives. From $\mathrm{D}\sigma$, for example, we find that

$$\sigma = \frac{\sigma_0}{r^2-1},$$

where $\sigma_0 = \sigma_0(u, x, y)$ is a real r-independent function. We then examine the r-dependence of π, γ and μ, making use of the expressions for $\mathrm{D}\pi$, $\mathrm{D}\gamma$ and $\mathrm{D}\mu$. The linear differential equations so obtained lead to

$$\begin{aligned}
\pi &= \frac{\pi_0\left(3r^2-7\right)}{r^2-1}, \\
\gamma &= \gamma_0 + \frac{2\pi_0\overline{\pi_0}}{\sigma_0}\log\left(\frac{r-1}{r+1}\right)^2 - \frac{\pi_0\left(9\overline{\pi_0}r^3 + 7\overline{\pi_0}r + 8\pi_0\right)}{\sigma_0\left(r^2-1\right)}, \\
\mu &= \frac{\mu_0}{r^2-1} + \frac{4\pi_0\overline{\pi_0}}{\sigma_0\left(r^2-1\right)}\log\left(\frac{r-1}{r+1}\right)^2 + \frac{3\pi_0 r\left(3\overline{\pi_0}r^2 - 6\pi_0 r - 13\overline{\pi_0}\right)}{\sigma_0\left(r^2-1\right)},
\end{aligned}$$

where π_0 is a complex function of (u, x, y), which we will write as $\pi_1 + \mathrm{i}\pi_2$ (π_1 and π_2 are real functions), and where γ_0 and μ_0 can be determined from (4.53) and (4.46), respectively. This leads to

$$\begin{aligned}\gamma_0 &= \gamma_1 - \frac{18\mathrm{i}\pi_1\pi_2}{\sigma_0}, \\ \mu_0 &= \mu_1 + \frac{20\mathrm{i}\pi_1\pi_2}{\sigma_0}.\end{aligned}$$

The functions π_1, π_2, γ_1 and μ_1 are all real and r-independent. We now know the r-dependence of all spin coefficients. We will also determine the r-dependence of the functions P, Q and V. First note that we can assume Q to be *not* real as if it were real, then P must have a non-zero imaginary part (else $\boldsymbol{\omega}^i$, $i = 1..4$, is not a basis) and by a transformation $x \longrightarrow x + f(y)$ we can make the imaginary part of Q non-zero.

Applying the commutators $[\delta, \mathrm{D}]$ and $[\overline{\delta}, \mathrm{D}]$ to x and to y, we find that P and Q are of the form

$$\begin{aligned}P &= p_1\,(r+1) + \mathrm{i}p_2\,(r-1)\,, \\ Q &= q_1\,(r+1) + \mathrm{i}q_2\,(r-1)\,,\end{aligned}$$

where p_1, p_2, q_1 and q_2 are real functions of (u, x, y), and where $q_2(u, x, y)$ is definitely non-zero. Next, we apply $[\Delta, \mathrm{D}]$ to the coordinates x and y to find an expression for V:

$$\begin{aligned}V &= (r+1)\left[v_1 + \frac{\pi_1}{\sigma_0}\log\left(\frac{r-1}{r+1}\right)^2\right] + \mathrm{i}\,(r-1)\left[v_2 - \frac{\pi_2}{\sigma_0}\log\left(\frac{r-1}{r+1}\right)^2\right] \\ &+ \frac{4}{\sigma_0}\left(\pi_1 - \mathrm{i}\pi_2\right),\end{aligned}$$

where we have again introduced two real r-independent functions v_1 and v_2. It is obvious now, that a transformation of x can be used to make $q_1(u, x, y)$ equal to zero while a transformation of y can be used to make $p_2(u, x, y)$ equal to zero.

We proceed by looking at the remaining directional derivatives of the spin coefficients. From $\delta\sigma$ we find the following expressions:

$$\frac{\partial\sigma_0}{\partial x} = -8\pi_1\sigma_0 p_1, \qquad \frac{\partial\sigma_0}{\partial y} = 8\pi_2\sigma_0 q_2.$$

These expressions, in combination with the information that can be extracted from $\delta\pi$:

$$\frac{\partial\pi_1}{\partial x} = 4p_1\left(2\pi_2^2 - \pi_1^2\right), \qquad \frac{\partial\pi_2}{\partial x} = -12p_1\pi_1\pi_2, \tag{4.57}$$

$$\frac{\partial\pi_2}{\partial y} = 4q_2\left(\pi_2^2 - 2\pi_1^2\right), \qquad \frac{\partial\pi_1}{\partial y} = 12q_2\pi_1\pi_2, \tag{4.58}$$

lead to

$$\begin{aligned} \frac{\partial\pi_1}{\partial y} &= \frac{3}{2}\frac{\pi_1}{\sigma_0}\frac{\partial\sigma_0}{\partial y}, \\ \frac{\partial\pi_2}{\partial x} &= \frac{3}{2}\frac{\pi_2}{\sigma_0}\frac{\partial\sigma_0}{\partial x}. \end{aligned}$$

We can solve the above equations to obtain solutions for π_1 and π_2:

$$\begin{aligned} \pi_1 &= \pi_4(u,x)\sigma_0(u,x,y)^{\frac{3}{2}}, \\ \pi_2 &= \pi_3(u,y)\sigma_0(u,x,y)^{\frac{3}{2}}. \end{aligned}$$

Note that π_1 and π_2 (and therefore π_4 and π_3) cannot be zero, as follows from the set $(4.57 - 4.58)$ and the fact that π is non-zero. We also find expressions for $p_1(u,x,y)$ and $q_2(u,x,y)$:

$$\begin{aligned} p_1(u,x,y) &= -\frac{1}{8}\frac{1}{\pi_4(u,x)\sigma_0(u,x,y)^{5/2}}\frac{\partial\sigma_0(u,x,y)}{\partial x}, \\ q_2(u,x,y) &= \frac{1}{8}\frac{1}{\pi_3(u,y)\sigma_0(u,x,y)^{5/2}}\frac{\partial\sigma_0(u,x,y)}{\partial y}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial\sigma_0(u,x,y)}{\partial x} &= -\frac{\pi_4(u,x)\sigma_0(u,x,y)}{\pi_4(u,x)^2 + \pi_3(u,y)^2}\frac{\partial\pi_4(u,x)}{\partial x}, \\ \frac{\partial\sigma_0(u,x,y)}{\partial y} &= -\frac{\pi_3(u,y)\sigma_0(u,x,y)}{\pi_4(u,x)^2 + \pi_3(u,y)^2}\frac{\partial\pi_3(u,y)}{\partial y}. \end{aligned}$$

Substitution of the above expressions for $p_1(u,x,y)$ and $q_2(u,x,y)$ in the basis one forms shows that we can use $-\pi_4(u,x)/4$ as coordinate x and $\pi_3(u,y)/4$ as coordinate y. From the partial differential equations for σ_0 we then see that

$$\sigma_0(u,x,y) = \frac{\sigma_1(u)}{\sqrt{x^2+y^2}}.$$

We can use the remaining boost freedom to put $\sigma_1(u)$ equal to one.

We then proceed by looking at $\delta\mu$ and $\delta\gamma$, which give us expressions for μ_1 and γ_1:

$$\begin{aligned}\mu_1(u,x,y) &= m(u)+\frac{x^2}{2\left(x^2+y^2\right)},\\ \gamma_1(u,x,y) &= g(u)-\frac{9}{8}\frac{x^2}{x^2+y^2}.\end{aligned}$$

The only information still remaining, comes from the expressions for the Δ-derivatives of the spin coefficients. From $\Delta\sigma$ and $\Delta\pi$ we find that

$$\begin{aligned}v_1(u,x,y) &= \frac{x\left(1-2m(u)\right)}{2\left(x^2+y^2\right)^{1/4}},\\ v_2(u,x,y) &= -\frac{y\left(1+m(u)\right)}{\left(x^2+y^2\right)^{1/4}},\\ g(u) &= \frac{11}{16}+\frac{1}{2}m(u),\end{aligned}$$

whereas $\Delta\mu$ tells us that $m(u)$ is a constant. We will write

$$m(u)=\frac{1}{4}\left(2c-1\right), \qquad c=\text{constant}.$$

The basis one forms now look like this:

$$\begin{aligned}\boldsymbol{\omega}^1 &= -\frac{1}{2\left(x^2+y^2\right)^{1/4}}\left[(r+1)\,\mathrm{d}x+\mathrm{i}\,(r-1)\,\mathrm{d}y\right]\\ &+ \frac{\left[2\left(\mathrm{i}y(1-r)-x(1+r)\right)(c-2L)+x(3r-1)-\mathrm{i}y(3r+1)\right]}{4\left(x^2+y^2\right)^{1/4}}\mathrm{d}u,\\ \boldsymbol{\omega}^3 &= \mathrm{d}u,\\ \boldsymbol{\omega}^4 &= \frac{\left[x(r+1)(1-3r)\mathrm{d}x+y(r-1)(1+3r)\mathrm{d}y\right]}{4\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}\mathrm{d}r\\ &+ \frac{1}{4\sqrt{x^2+y^2}}\Bigg[\left((1-4r-3r^2)x^2-(1+4r-3r^2)y^2\right)(c-2L)\\ &\qquad +\frac{(3r-1)^2x^2+(3r+1)^2y^2}{4}\Bigg]\,\mathrm{d}u,\end{aligned}$$

where we have introduced $2L=\log\left(\frac{r+1}{r-1}\right)$. The corresponding line element is then given by

$$\begin{aligned}
\mathrm{d}s^2 = & \frac{c-2L}{2\sqrt{x^2+y^2}}\Big[2r(x^2+y^2)+(c-2L)\Big((r+1)^2x^2+(r-1)^2y^2\Big)\Big]\mathrm{d}u^2 \\
& - \left[2\sqrt{x^2+y^2}\mathrm{d}r - \frac{(r+1)^2 x}{\sqrt{x^2+y^2}}\,(c-2L)\,\mathrm{d}x - \frac{(r-1)^2\,y}{\sqrt{x^2+y^2}}(c-2L)\mathrm{d}y\right]\mathrm{d}u \\
& + \frac{1}{2}\frac{(r+1)^2}{\sqrt{x^2+y^2}}\mathrm{d}x^2 + \frac{1}{2}\frac{(r-1)^2}{\sqrt{x^2+y^2}}\mathrm{d}y^2. \qquad (4.59)
\end{aligned}$$

4.4.1 Comparison with the original paper

In this section, we will compare the spherical metric (4.59) with the spherical metrics in the original paper (Newman and Tamburino, 1962). First note that no free functions appear in the solution (4.59), as was expected from the GHP-analysis in section 4.2.2. Furthermore we see the presence of one constant c, as is also the case in the original paper. More interesting is to note that we have only one line element, and not two. This suggests that, using the notation of the paper (Newman and Tamburino, 1962), the metric for which $A = B$, where B is a constant can be found as a special case of the metric for which $A = B\,u$.

Before presenting the relation between (4.59) and the metrics in the original paper, we will first rewrite the latter. In the paper, the metric is presented in the following way:

$$\begin{aligned}
g^{12} &= 1, \\
g^{22} &= \frac{2rL}{A} - \frac{2r^2\sqrt{\zeta\bar{\zeta}}}{r^2-a^2} + \frac{2r^2A\left(r\left(\zeta^2+\bar{\zeta}^2\right)-2A\left(\zeta\bar{\zeta}\right)^{3/2}\right)}{\left(r^2-a^2\right)^2}, \\
g^{33} &= \frac{-2\left(\zeta\bar{\zeta}\right)^{3/2}}{(r+a)^2}, \qquad g^{44} = \frac{-2\left(\zeta\bar{\zeta}\right)^{3/2}}{(r-a)^2}, \\
g^{23} &= 4A^2x\left(\zeta\bar{\zeta}\right)^{3/2}\left(\frac{L}{2a^3} - \frac{r-2a}{2a^2\left(r^2-a^2\right)} - \frac{r-a}{\left(r^2-a^2\right)^2}\right), \\
g^{24} &= 4A^2y\left(\zeta\bar{\zeta}\right)^{3/2}\left(\frac{L}{2a^3} - \frac{r+2a}{2a^2\left(r^2-a^2\right)} - \frac{r+a}{\left(r^2-a^2\right)^2}\right),
\end{aligned}$$

where A is either constant or proportional to u ($A \in \{B, Bu\}$, where B is a real constant) and where

$$\begin{aligned} a &= A\sqrt{\zeta\bar{\zeta}}, \qquad \zeta = x + \mathrm{i}y, \\ L &= \frac{1}{2}\log\left(\frac{r+a}{r-a}\right). \end{aligned}$$

First note that it is interesting to apply the coordinate transformation $r \to ar$. A set of one forms corresponding to this metric is then given by

$$\begin{aligned} \boldsymbol{\omega}^1 &= \frac{1}{2}\frac{A\,(r+1)}{(x^2+y^2)^{1/4}}\mathrm{d}x + \frac{\mathrm{i}}{2}\frac{A\,(r-1)}{(x^2+y^2)^{1/4}}\mathrm{d}y, \\ \boldsymbol{\omega}^2 &= \frac{1}{2}\frac{A\,(r+1)}{(x^2+y^2)^{1/4}}\mathrm{d}x - \frac{\mathrm{i}}{2}\frac{A\,(r-1)}{(x^2+y^2)^{1/4}}\mathrm{d}y, \\ \boldsymbol{\omega}^3 &= \mathrm{d}u, \\ \boldsymbol{\omega}^4 &= \left(\left(\dot{A}+L\right) r\sqrt{x^2+y^2} - \frac{\left((r+1)^2x^2+(r-1)^2y^2\right)L^2}{\sqrt{x^2+y^2}}\right)\mathrm{d}u \\ &+ \frac{AxL\,(t+1)^2}{\sqrt{x^2+y^2}}\mathrm{d}x + \frac{AyL\,(t-1)^2}{\sqrt{x^2+y^2}}\mathrm{d}y + A\sqrt{x^2+y^2}\mathrm{d}r, \end{aligned}$$

where $\dot{}$ denotes differentiation to u and where L is now given by

$$L = \frac{1}{2}\log\left(\frac{r+1}{r-1}\right).$$

If A is a constant we can apply the following coordinate transformation in order to put $A = B$ equal to one:

$$\begin{aligned} x &\longrightarrow x/A^2, \\ y &\longrightarrow y/A^2, \\ u &\longrightarrow uA. \end{aligned}$$

The line element for this family is then given by

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{2\left((r+1)^2x^2+(r-1)^2y^2\right)L^2}{\sqrt{x^2+y^2}} - 2rL\sqrt{x^2+y^2}\right)\mathrm{d}u^2 \\ &- 2\sqrt{x^2+y^2}\mathrm{d}r\mathrm{d}u - \frac{2xL(r+1)^2}{\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}u - \frac{2yL(r-1)^2}{\sqrt{x^2+y^2}}\mathrm{d}y\mathrm{d}u \\ &+ \frac{1}{2}\frac{(r+1)^2}{\sqrt{x^2+y^2}}\mathrm{d}x^2 + \frac{1}{2}\frac{(r-1)^2}{\sqrt{x^2+y^2}}\mathrm{d}y^2. \end{aligned} \tag{4.60}$$

If A is proportional to u we will apply a coordinate transformation

$$\begin{aligned} x &\longrightarrow \frac{64}{xu^2c^2}, \\ y &\longrightarrow \frac{64}{yu^2c^2}. \end{aligned}$$

Next, we replace u by $\exp(-3u/64)$ and we substitute $c = 3c/2$. The corresponding line element is then identical to (4.59).

The only question that remains is to find a relation between (4.60) and (4.59). As the latter is the most general vacuum Newman Tamburino metric belonging to the spherical class, the former should be a special case of (4.59). Indeed, by putting c equal to zero in (4.59), we obtain (4.60).

4.5 Conclusion

In this chapter we have re-examined the Newman Tamburino empty space metrics. It is shown here that the cylindrical metrics published by Newman and Tamburino (1962) are incorrect, and that the correct version can be simplified considerably. By first making some calculations in the Geroch Held Penrose formalism, we can find rotation and boost invariant information concerning the Weyl tensor components. If we use the same fixation for the null rotation in the Newman Penrose formalism, we are allowed to copy this invariant information, helping us to find an integrable set of differential equations rather quickly. Next, we introduce coordinates, which allow us to integrate the system, resulting in an elegant form for the metric one forms.

We also compared our metrics and those of Newman and Tamburino. In the cylindrical case, we were able to correct the mistakes which occur in the original paper. Also we found a line element (4.39) which looks much nicer than the (corrected version of) the original one. In the spherical case, we were able to prove that the two distinct metrics which occur in (Newman and Tamburino, 1962) are actually related to one another, in the sense that the metric with $A = B$, $B \in \mathbb{R}$, is a special case of the metric with $A = B\,u$. The latter, which is equivalent to our solution (4.59) is the most general vacuum Newman Tamburino metric for the spherical class.

Chapter 5

Aligned Petrov type D pure radiation solutions of Kundt's class

5.1 Introduction

As mentioned in the introduction, a space-time is said to belong to Kundt's class if it admits a null congruence, generated by a vector field $\tilde{\mathbf{k}}$, which is non-diverging (*i.e.* which is both non-expanding and non-rotating and therefore also geodesic). In the case of a pure radiation field, which, by definition, has an energy momentum tensor of the form $T_{ab} = \phi k_a k_b$, with $k_a k^a = 0$, $\phi \neq 0$, one can furthermore show (Stephani et al., 2003), making use of the null energy condition $R_{ab}k^a k^b \geq 0$ and the Raychaudhuri equation

$$\Theta_{,a}\tilde{k}^a - \omega^2 + \Theta^2 + \sigma\overline{\sigma} = -\frac{1}{2}R_{ab}\tilde{k}^a\tilde{k}^b,$$

that $\mathbf{k}$ and $\tilde{\mathbf{k}}$ are aligned and that the associated null congruence is shear-free:

$$\begin{aligned} &R_{ab}\tilde{k}^a\tilde{k}^b = -2\sigma\overline{\sigma} && \Rightarrow \sigma = 0 \quad \text{and} \quad R_{ab}\tilde{k}^a\tilde{k}^b = 0, \\ &R_{ab} = \phi k_a k_b + \Lambda g_{ab} && \Rightarrow \mathbf{k} \propto \tilde{\mathbf{k}}, \end{aligned}$$

where Λ is the cosmological constant. The Goldberg Sachs theorem implies then that the Kundt space-times must be algebraically special (type II, D,

III or *N* or conformally flat) and that $\tilde{\mathbf{k}}$ is aligned with a repeated principal null direction of the Weyl tensor.
Conversely, for aligned *Petrov type D* pure radiation fields, it follows immediately from the Bianchi equations (without invoking the energy conditions) and from

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \qquad \Psi_2 \neq 0, \tag{5.1}$$

$$\Phi_{00} = \Phi_{11} = \Phi_{01} = \Phi_{12} = \Phi_{02} = 0, \qquad \Phi_{22} \neq 0, \tag{5.2}$$

that

$$\kappa = \sigma = \lambda = 0, \tag{5.3}$$

such that the principal null direction $\mathbf{k}$ aligned with the radiative vector is geodesic and shear-free, *i.e.* the converse part of the Goldberg Sachs theorem is generalised to this situation. Moreover, it is known that aligned Petrov type D pure radiation fields cannot be of Maxwell type (Debever et al., 1989) nor of neutrino or scalar field type (Wils and Van den Bergh, 1990), while Wils (1990) showed that $\mathbf{k}$ is necessarily non-twisting ($\overline{\rho} = \rho$). The diverging solutions ($\rho \neq 0$) then belong to the Robinson Trautman class and are all explicitly known (see chapter 2 and Stephani et al., 2003). In this chapter we present the non-diverging case $\rho = 0$, for which only a few type D examples have appeared in the literature (Wils and Van den Bergh, 1990). In De Groote et al. (2010) we presented an exhaustive list of the solutions belonging to the latter class, thus solving completely the problem of aligned Petrov type D pure radiation fields. This work extends in a natural way the complete classifications of pure radiation fields of Petrov type 0 and N (Edgar and Machado Ramos, 2005, 2007; Edgar and Ramos, 2007; Podolsky and Prikryl, 2009).

As shown by Kundt (1961) the line elements admitting a geodesic, shear-free and non-diverging null congruence can all be expressed in the form:

$$\mathrm{d}s^2 = 2P^{-2}\mathrm{d}\zeta\mathrm{d}\overline{\zeta} - 2\mathrm{d}u(\mathrm{d}r + H\mathrm{d}u + W\mathrm{d}\zeta + \overline{W}\mathrm{d}\overline{\zeta}), \tag{5.4}$$

in which P is a real function of $(\zeta, \overline{\zeta}, u)$ and H, W are respectively real and complex functions of $(\zeta, \overline{\zeta}, u, r)$, to be determined by appropriate field equations. The vacuum solutions of this type have been known for a very long time (Kinnersley, 1969). A procedure (based on Theorem 31.1 in Stephani et al. (2003)) is also available allowing one to generate non-vacuum solutions of Kundt's class, from vacuum solutions. However, in this way the Petrov type of the metric generally will be changed from D to II. Insisting that the Petrov type remains the same, constrains the function H_0 of the 'background metric' and it was not certain whether *all* pure radiation solutions could be generated in this way. This technique was used in Wils and Van den Bergh (1990), where the authors managed to construct a family of type D pure radiation fields and where they conjectured that these solutions were the only aligned type D pure radiation fields of Kundt's class. Below we show that the solutions obtained in Wils and Van den Bergh (1990) only cover a small part of the entire family, which is presented in this chapter.

The most obvious way to construct type D pure radiation fields of Kundt's class, is to start from the general Kundt metric (5.4), express the condition that the solutions one is looking for are of Petrov type D, and to make sure that the field equations for pure radiation are satisfied. This however introduces a hard to solve system of non-linear conditions. In section 5.2 we derive a classification theorem within the GHP-formalism, the use of which makes the proof geometrically more clear and much more compact. For the integration in section 5.3 we switch to the NP-formalism.

5.2 Geroch Held Penrose analysis

Introducing the set of conditions $(5.1 - 5.3)$ and the non-diverging condition $\rho = 0$ within the GHP Bianchi equations, one arrives at

$$\begin{aligned} Þ\Psi_2 &= 0, & Þ\Phi_{22} &= 0, \\ ð\Psi_2 &= 3\tau\Psi_2, & ð\Phi_{22} &= 3\overline{\nu}\overline{\Psi}_2 + \tau\Phi_{22}, \\ ð'\Psi_2 &= -3\pi\Psi_2, & ð'\Phi_{22} &= 3\nu\Psi_2 + \overline{\tau}\Phi_{22}. \\ Þ'\Psi_2 &= -3\mu\Psi_2, \end{aligned} \tag{5.5}$$

The GHP Ricci equations then reduce to

$$
\begin{aligned}
\text{Þ}\tau &= 0, & \eth'\mu &= (\overline{\mu}-\mu)\pi,\\
\eth\tau &= \tau^2, & \eth'\nu &= (\overline{\tau}-\pi)\nu,\\
\eth'\tau &= \tau\overline{\tau}+\Psi_2+\frac{\Lambda}{3}, & \eth'\pi &= -\pi^2
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
\text{Þ}\nu &= \text{Þ}'\pi+(\overline{\tau}+\pi)\mu,\\
\text{Þ}'\mu &= \eth\nu-\mu^2+\overline{\nu}\pi-\nu\tau-\Phi_{22},\\
\text{Þ}\mu &= \eth\pi+\pi\overline{\pi}+\Psi_2+\frac{\Lambda}{3}.
\end{aligned}
\tag{5.7}
$$

The remaining basic variables at this stage are $\mu, \nu, \pi, \tau, \Psi_2$ and their complex conjugates, together with Φ_{22}. From the integrability conditions for Ψ_2 one obtains:

$$
\begin{aligned}
\text{Þ}\nu &= 0, & \text{Þ}\pi &= 0,\\
\text{Þ}\mu &= \pi\overline{\pi}-\tau\overline{\tau}, & \eth\pi &= -(\tau\overline{\tau}+\Psi_2+\frac{\Lambda}{3}),\\
\eth\mu &= -\text{Þ}'\tau, & \text{Þ}'\pi &= -(\overline{\tau}+\pi)\mu.
\end{aligned}
\tag{5.8}
$$

$[\eth', \text{Þ}]\,\mu$ now yields the algebraic equation

$$
\left(2\tau\overline{\tau}+\overline{\Psi}_2+\frac{\Lambda}{3}\right)\pi+\left(2\tau\overline{\tau}+\Psi_2+\frac{\Lambda}{3}\right)\overline{\tau}=0. \tag{5.9}
$$

From $[\eth, \eth']\mu$ we can extract an expression for $\eth'\text{Þ}'\tau$:

$$
\eth'\text{Þ}'\tau=\left(\overline{\Psi}_2-2\Psi_2-2\overline{\tau}\tau-\frac{\Lambda}{3}\right)\mu+\left(\Psi_2+2\overline{\tau}\tau+\frac{\Lambda}{3}\right)\overline{\mu}-\pi\text{Þ}'\tau,
$$

after which evaluation of $[\eth, \eth']\,\text{Þ}'\tau$, using $[\text{Þ}', \text{Þ}]\tau$, $[\eth, \text{Þ}']\tau$ and (5.9), leads to

$$
\text{Þ}'\tau=\overline{\mu}\tau-\mu\overline{\pi}-2\mu\tau.
$$

Herewith $[\text{Þ}', \eth]\,\pi$ yields a second algebraic equation

$$
\mu Q=0, \qquad Q=6(\overline{\tau\pi}+\tau\pi)+3(\tau\overline{\tau}+\pi\overline{\pi})-\Lambda+3(\Psi_2+\overline{\Psi}_2). \tag{5.10}
$$

From $\eth(5.9)-2\tau(5.9)$ one readily infers

$$
\pi\overline{\pi}=\tau\overline{\tau}.
$$

In the case where $\pi = \tau = 0$, $\eth\pi = 0$ and $\text{Þ}'\eth\pi = 0$ yield

$$\Psi_2 = -\frac{\Lambda}{3}, \quad \mu = 0 \tag{5.11}$$

and all the above equations are identically satisfied.

When $\pi\tau \neq 0$ we introduce the zero-weighted quantities

$$A \equiv |\pi\overline{\pi}|^{1/2} = |\tau\overline{\tau}|^{1/2} \qquad (\overline{A} = A > 0),$$
$$T \equiv \frac{\pi}{\overline{\tau}} = \frac{\tau}{\overline{\pi}} \qquad \left(\overline{T} = T^{-1}\right).$$

In this way π and τ are conveniently parametrised by

$$\pi = \frac{A}{B}, \quad \tau = ATB,$$

the weight of both being absorbed in the (1,-1)-weighted quantity

$$B \equiv \left(\frac{\overline{\pi}}{\pi}\right)^{1/2} = \left(\frac{\tau}{\overline{\tau}}\right)^{1/2} \qquad (\overline{B} = B^{-1}).$$

Herewith equations (5.9) and (5.10) translate into

$$(T+1)(2A^2 + \Lambda/3) + \Psi_2 + T\overline{\Psi}_2 = 0, \tag{5.12}$$

$$\mu Q = 0, \qquad Q \equiv 2(T^2+T+1)A^2 + T(\Psi_2 + \overline{\Psi}_2 - \Lambda/3), \tag{5.13}$$

respectively. Defining the (-2,-2)-weighted quantities $X \equiv \eth\nu$, $N \equiv B\nu$ and the zero-weighted derivative operators $\mathcal{D} \equiv B^{-1}\eth$, $\mathcal{D}' \equiv B\,\eth'$, one deduces from $(5.5-5.7)$, $[\text{Þ}', \eth]\mu$ and $[\eth, \eth']\nu$ the following autonomous first-order system for the zero-weighted fields Ψ_2, $\overline{\Psi}_2$, T, A and the (-2,-2)-weighted fields X, $\overline{X}$, N, $\overline{N}$, Φ_{22}:

$$\mathcal{D}\Psi_2 = 3AT\Psi_2, \qquad \mathcal{D}\overline{\Psi}_2 = -3A\overline{\Psi}_2, \tag{5.14}$$

$$\mathcal{D}T = AT(T+1), \qquad \mathcal{D}A = -A^2 - \frac{3\Psi_2 + \Lambda}{6}, \tag{5.15}$$

$$\begin{aligned}\mathcal{D}X = {} & A(4T+1)X - AT\overline{X} - 2A^2T(T+1)N + 2A^2(T+2)\overline{N}A^2 \\ & + (3\overline{\Psi}_2 + \Psi_2 + \Lambda/3)\overline{N} - A(T+1)\Phi_{22},\end{aligned} \tag{5.16}$$

$$\mathcal{D}\overline{X} = A(T-1)\overline{X} + (4\overline{\Psi}_2 + \Lambda/3 + 2A^2)\overline{N}, \tag{5.17}$$

$$\mathcal{D}N = X + \frac{(3\Psi_2 + \Lambda)}{6A}N, \qquad \mathcal{D}\overline{N} = \left(\frac{A(T-1)}{T} - \frac{3\overline{\Psi}_2 + \Lambda}{6A}\right)N, \tag{5.18}$$

$$\mathcal{D}\Phi_{22} = AT\Phi_{22} + 3\overline{\Psi}_2\overline{N}. \tag{5.19}$$

For later use, one immediately observes from (5.15) that T *constant implies* $T = -1$. Also notice that (5.14 − 5.15) forms a zero-weighted subsystem, whereas the equations (5.16 − 5.19) are (-2,-2)-weighted, such that their right hand sides are necessarily linear and homogeneous in $X, \overline{X}, N, \overline{N}, \Phi_{22}$. The complex conjugates of (5.14 − 5.19) yield an analogous $\mathcal{D}'$-system. However, as deduced from $[\eth, \eth']\Phi_{22}$, both systems are constrained by

$$3T(\Psi_2 X - \overline{\Psi}_2\overline{X}) + 6A(T^2\Psi_2 N - \overline{\Psi}_2\overline{N}) + T(\Psi_2 - \overline{\Psi}_2)\Phi_{22} = 0, \quad (5.20)$$

which is again linear and homogeneous in $X, \overline{X}, N, \overline{N}, \Phi_{22}$.
We are now ready to prove:

Theorem *With respect to a Weyl canonical tetrad (5.1 − 5.2) any aligned Petrov type D pure radiation field, for which the principal null direction, aligned with the radiative direction is non-diverging, satisfies the following boost and rotation invariant properties:*

$$\kappa = \sigma = \rho = 0, \quad \lambda = \mu = 0, \quad \nu \neq 0, \quad (5.21)$$

$$\pi + \overline{\tau} = 0, \quad \overline{\Psi}_2 = \Psi_2, \quad \pi\,\Phi_{22} + \nu\left(\Psi_2 + \frac{\Lambda}{3}\right) = 0. \quad (5.22)$$

Proof It immediately follows from (5.7) that $\mu = \nu = 0$ is inconsistent with $\Phi_{22} \neq 0$, and from (5.9) that $\pi + \overline{\tau} = 0$ implies Ψ_2 to be real. By (5.11) the theorem is true in the case where $\pi = \tau = 0$, and it remains to establish $\mu = 0$ and $\pi = -\overline{\tau}$ ($T = -1$ in the above notation) when $\pi\tau \neq 0$. Referring to (5.13), we first treat the case $Q = 0$. Two linear and homogeneous systems in $X, \overline{X}, N, \overline{N}$ and Φ_{22} are obtained from [(5.20), $\mathcal{D}$(5.20), $\mathcal{D}'$(5.20), $\mathcal{D}\mathcal{D}'$(5.20)] and $\mathcal{D}\mathcal{D}$(5.20), respectively $\mathcal{D}'\mathcal{D}'$(5.20). As $\Phi_{22} \neq 0$, the determinants of these systems must vanish. This yields two polynomial relations in A^2, Λ, Ψ_2, $\overline{\Psi}_2$ and T. Eliminating A^2 and Λ by means of (5.12) and $Q = 0$, and appropriately scaling the results with Ψ_2, yields two polynomial relations in the zero-order variables T and $k \equiv \overline{\Psi}_2/\Psi_2$ of the form $(T + 1)^3 T^7 P_i(T, k) = 0$, $i = 1, 2$, where P_1 and P_2 are irreducible and non-proportional. Calculating the resultant of P_1 and P_2 with respect to k, necessarily yields a non-trivial polynomial relation $P(T) = 0$. Thus T is constant, whence $T = -1$ and $\overline{\Psi}_2 = \Psi_2$. Substituting this into $Q = 0$ yields $\pi\overline{\pi} - \Psi_2 + \Lambda/6 = 0$, and calculating the Þ′-derivative hereof gives $\mu = 0$.

Thus $\mu = 0$ in any case, and (5.7) implies

$$\eth'\overline{\nu} = \eth\nu = -\pi\overline{\nu} - \overline{\pi}\nu + \Phi_{22}, \quad i.e., \quad \overline{X} = X = -A(\overline{N} - TN) + \Phi_{22}. \quad (5.23)$$

Three linear and homogeneous systems in N, $\overline{N}$ and Φ_{22} can now be obtained, by substituting the above expressions in [(5.20),$\mathcal{D}$(5.20),$\mathcal{D}'$(5.20)], [(5.20),$\mathcal{D}$(5.20),$\mathcal{D}\mathcal{D}$(5.20)] and [(5.20),$\mathcal{D}'$(5.20),$\mathcal{D}'\mathcal{D}'$(5.20)]. Eliminating A^2 and Λ from the three determinants and (5.12) leads to two polynomial relations in T and k with $(T+1)^3$ as the only common factor, such that again $T = -1$ and $\overline{\Psi}_2 = \Psi_2$. Inserting this and (5.23) into (5.20) implies $\overline{N} = N$ (*i.e.*, ν/π is real), a final $\mathcal{D}$-derivative of which yields

$$A\Phi_{22} = N(\Psi_2 + \Lambda/3),$$

the translation of the last equation of (5.22). □

One checks that, given the specifications (5.1 − 5.2) and (5.21 − 5.22), the remaining GHP-equations are consistent, so that corresponding solutions exist. Referring to (5.13) and the above proof, we finally remark that all derivatives of $Q\,\Psi_2^{-2/3}$ are zero, so that one has the following important relation:

$$\pi\overline{\pi} = \Psi_2 - \frac{\Lambda}{6} + c_1\Psi_2^{2/3}, \qquad c_1 \quad \text{constant}. \quad (5.24)$$

Because of (5.11), this only yields new information when $\pi \neq 0$; the case $c_1 = 0 \Leftrightarrow Q = 0$ will be distinguished from $c_1 \neq 0$ for the corresponding integration in § 5.3.2.

5.3 Newman Penrose analysis

In order to construct the metrics, we now switch to the NP-formalism, where we first introduce the invariant information, as obtained above:

$$
\begin{aligned}
\Psi_2 &\neq 0, & \Psi_i &= 0, \quad i \neq 2,\\
\Phi_{2,2} &\neq 0, & \Phi_{i,j} &= 0, \quad [i,j] \neq [2,2],\\
\overline{\Psi_2} &= \Psi_2, & \Phi_{2,2}\ \pi &+ \nu\left(\Psi_2 + \frac{\Lambda}{3}\right) = 0,\\
\kappa &= \sigma = \lambda = 0, & \rho &= \mu = 0,\\
\nu &\neq 0, & \pi &= -\overline{\tau}.
\end{aligned}
$$

As ν is non-zero, we can eliminate the rotational degree of freedom of the null tetrad, by putting $\overline{\nu} = \nu$. From (5.22) it follows that also π (and thus τ) is real. A boost allows one to put $\beta = -(\overline{\alpha} - \tau)$, fixing the tetrad up to boosts satisfying $\delta A = \overline{\delta} A = 0$.

From Newman-Penrose equations (1.27) and (1.33), we get

$$
\begin{aligned}
\mathrm{D}\tau &= -\tau(\epsilon - \overline{\epsilon}),\\
\mathrm{D}\nu &= -\Delta\tau - \tau(\gamma - \overline{\gamma}) - \nu(3\epsilon + \overline{\epsilon}),
\end{aligned}
$$

showing ϵ is real and $\tau(\gamma - \overline{\gamma}) = 0$. (1.28) and (1.29) then yield expressions for $\mathrm{D}\alpha$ and $\delta\epsilon$:

$$
\begin{aligned}
\mathrm{D}\alpha &= 0,\\
\delta\epsilon &= 2\epsilon\tau,
\end{aligned}
$$

from which we see one can boost such that $\epsilon = 0$. Notice that boosts satisfying $\mathrm{D}A = \delta A = \overline{\delta} A = 0$ are still allowed.

The solutions can now be subdivided into two families: one in which $\tau = 0$ and one in which $\tau \neq 0$. We also introduce γ_0 and γ_1/ν, the real and imaginary part of γ respectively.

5.3.1 The family $\tau = 0$

In the case $\tau = -\pi = 0$ equation (5.8) shows $\Psi_2 = -\frac{\Lambda}{3} \neq 0$. The NP Ricci equations can then be rewritten in the form:

$$
\Phi_{22} = -4\alpha\nu, \tag{5.25}
$$

$$\mathrm{D}\nu = 0, \qquad \delta\alpha + \overline{\delta}\alpha = 4\alpha^2 + \frac{\Lambda}{2}, \tag{5.26}$$

$$\mathrm{D}\alpha = 0, \qquad \nu\Delta\alpha - \mathrm{i}\overline{\delta}\gamma_1 = 0, \tag{5.27}$$

$$\mathrm{D}\gamma_1 = 0, \qquad \mathrm{D}\gamma_0 = -\frac{\Lambda}{2}, \tag{5.28}$$

$$\delta\nu = -2\alpha\nu, \qquad \overline{\delta}\nu = -2\alpha\nu, \tag{5.29}$$

$$\delta\gamma_0 = 0.$$

The Bianchi equations now yield an expression for $\delta\alpha$ (under which (5.26) becomes an identity):

$$\delta\alpha = 2\alpha^2 + \frac{\Lambda}{4}.$$

From the last equation and $\Lambda \neq 0$ it follows that $\alpha \neq 0$. Hence, by (5.25) and (5.29) ν cannot be constant and this allows one to use ν as a coordinate. As is clear from (5.28), γ_0 provides a second independent function and hence can be used to define a coordinate $r = -2\gamma_0/\Lambda$. As $\delta\gamma_0 = 0$ it follows that $\boldsymbol{\omega}^4 = \mathrm{d}r + H\boldsymbol{\omega}^3$ ($H = -\Delta r$, $\boldsymbol{\omega}^i$ representing the basis one forms). From (5.29) we see that we can write $\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 = (\nu Y\boldsymbol{\omega}^3 - \mathrm{d}\nu)/(2\alpha\nu)$, where $Y = \Delta\nu/\nu$. The Cartan equations read

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^1 &= 2\alpha\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 + 2\mathrm{i}\gamma_1/\nu\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3, \\
\mathrm{d}\boldsymbol{\omega}^3 &= 0, \\
\mathrm{d}\boldsymbol{\omega}^4 &= \nu(\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2) \wedge \boldsymbol{\omega}^3 + r\Lambda\,\boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4,
\end{aligned} \tag{5.30}$$

showing that a function u exists, such that $\boldsymbol{\omega}^3 = \mathrm{d}u$. A fourth coordinate x is defined by writing $\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2 = \mathrm{i}N\mathrm{d}\nu + 4\mathrm{i}\nu(U\mathrm{d}u + X\mathrm{d}x)$, with, because of (5.30), N, U, X real functions depending on (u, x, ν), only. This allows one to put $N = 0$ and we obtain the following structure for the basis one forms:

$$\begin{aligned}
\boldsymbol{\omega}^1 &= \frac{1}{4\alpha}\left(Y\mathrm{d}u - \frac{\mathrm{d}\nu}{\nu}\right) + 2\mathrm{i}\nu(U\mathrm{d}u + X\mathrm{d}x), \\
\boldsymbol{\omega}^3 &= \mathrm{d}u, \\
\boldsymbol{\omega}^4 &= \mathrm{d}r + H\mathrm{d}u,
\end{aligned} \tag{5.31}$$

where H is a real function of (u, r, ν, x). The corresponding directional derivatives are then given by:

$$\begin{aligned} \delta &= -2\alpha\nu\partial_\nu - \frac{\mathrm{i}}{4\nu X}\partial_x, \\ \Delta &= \partial_u - H\partial_r + \nu Y\partial_\nu - \frac{U}{X}\partial_x, \\ \mathrm{D} &= \partial_r. \end{aligned}$$

Notice that, because of the remaining boost freedom $A = A(u)$, r is fixed only up to transformations of the form $r \to A^{-1}(u)r + B(u)$.
From the directional derivatives of α it follows that α can be written in the form

$$\alpha = \frac{\pm\sqrt{-2\Lambda\nu^2 + sm(u)^2}}{4\nu}, \qquad s = \pm 1. \tag{5.32}$$

Applying $[\Delta, \mathrm{D}]$ and $[\delta, \Delta]$ to r, we obtain $H = -\Lambda r^2/2 + 4\alpha\nu/\Lambda + n(u)$, while $[\overline{\delta}, \delta]x$ and $[\delta, \Delta]x$ show that

$$X = X(u, x),$$

$$\frac{\partial U}{\partial x} = \frac{\partial X}{\partial u} + YX \tag{5.33}$$

and

$$\frac{\partial U}{\partial \nu} = -\frac{\gamma_1}{4\alpha\nu^3}. \tag{5.34}$$

From (5.27) and $\mathrm{D}\gamma_1 = 0$ we see that $\gamma_1 = \gamma_1(u, x)$ and

$$\frac{\mathrm{d}m^2}{\mathrm{d}u} = 2Ym^2 - \frac{8\alpha}{sX}\frac{\partial\gamma_1}{\partial x}. \tag{5.35}$$

Hence evaluation of $[\delta, \Delta]\nu$ implies

$$\frac{\partial Y}{\partial x} = -16\gamma_1\alpha X \tag{5.36}$$

and

$$\frac{\partial Y}{\partial \nu} = -\frac{\partial\gamma_1/\partial x}{4\alpha\nu^3 X}. \tag{5.37}$$

We now distinguish two subclasses, corresponding to $m = 0$, respectively $m \neq 0$.

The class $m(u) = 0$

Notice from (5.32) that this case is only consistent for $\Lambda < 0$, where we put $\alpha_0 = \alpha = \pm\sqrt{-2\Lambda}/4$ for convenience. Equations (5.35) and (5.37) are equivalent to $\gamma_1 = \gamma_1(u)$ and $Y = Y(u, x)$. At this stage we make a distinction between the subcases $\gamma_1 = 0$ and $\gamma_1 \neq 0$.

If $\gamma_1 = 0$, then (5.34) and (5.31) imply that a transformation of x exists which makes $U = 0$. The remaining equations (5.33) and (5.36) combine to $\frac{1}{X}\frac{\partial X}{\partial u} = -Y(u)$, such that $X(u, x) = f(x)/b(u)$, where we can put $f(x)$ equal to one by a further transformation of x. Using $\nu = y^2 b(u)$ as a coordinate, together with the coordinate transformation

$$\begin{aligned} r &\longrightarrow \tilde{r}\, p(\tilde{u}) + \frac{1}{\Lambda p(\tilde{u})}\frac{\mathrm{d}p(\tilde{u})}{\mathrm{d}\tilde{u}}, \\ x &\longrightarrow \frac{\tilde{x}}{2}, \end{aligned}$$

where $p(\tilde{u})$ is a function of $\tilde{u}$ defined by

$$2\,p\,\frac{\mathrm{d}^2 p}{\mathrm{d}\tilde{u}^2} - 3\left(\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}\right)^2 + 2\,n\,\Lambda\,p^2 = 0,$$

and replacing

$$\mathrm{d}\tilde{u} \longrightarrow \frac{1}{p(\grave{u})}\mathrm{d}\grave{u},$$

we are able to eliminate the function $n(u)$. Absorbing a factor $\pm p(\grave{u})^{-2}$ in $b(\grave{u})$ and dropping the tildes and dashes, we obtain the following line element for this class[1]:

$$\mathrm{d}s^2 = \left(-\Lambda r^2 + \frac{2\sqrt{-2\Lambda}\,b\,y^2}{\Lambda}\right)\mathrm{d}u^2 + 2\mathrm{d}r\mathrm{d}u - 2y^4\mathrm{d}x^2 + \frac{4}{\Lambda y^2}\mathrm{d}y^2. \quad (5.38)$$

The only free function in this metric is $b(u)$.

If $\gamma_1 \neq 0$, we can use Y as coordinate x (as $[\delta, \Delta]\nu$ and $[\overline{\delta}, \Delta]\nu$ show that $(\delta - \overline{\delta})\Delta\nu \neq 0$, it follows that ν, u and Y are functionally independent). From (5.36) we get $X(u, x) = -1/(16\gamma_1\alpha_0)$, after which (5.33) and (5.34) yield

$$U(u, \nu, x) = \pm\frac{\gamma_1}{2\sqrt{-2\Lambda}\nu^2} - \frac{x^2}{8\gamma_1\sqrt{-2\Lambda}} + \frac{x}{4\gamma_1^2\sqrt{-2\Lambda}}\frac{\mathrm{d}\gamma_1}{\mathrm{d}u} + q,$$

[1] There was an error in this line element in (De Groote et al., 2010)

with $q(u)$ an arbitrary function of u. Applying the following coordinate transformations

$$\begin{aligned} \nu &\longrightarrow \frac{y^2}{p^2}, \\ x &\longrightarrow 2\sqrt{-2\Lambda}\tilde{x}\gamma_1 p^2 - \frac{2}{p^2}\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}, \\ r &\longrightarrow \frac{1}{p}\tilde{r} - \frac{1}{\Lambda p^2}\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}, \\ \mathrm{d}u &\longrightarrow p\,\mathrm{d}\tilde{u}, \end{aligned}$$

with p a function of $\tilde{u}$ defined by

$$n = 4\frac{qg}{p^3\sqrt{-2\Lambda}} - \frac{3}{2p^4\Lambda}\left(\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}\right)^2 + \frac{1}{p^3\Lambda g}\frac{\mathrm{d}g}{\mathrm{d}\tilde{u}}\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}},$$

and where we have put $\gamma_1(u) = g(\tilde{u})/p(\tilde{u})^3$, we are able to eliminate the function $n(u)$. Introducing $b(\tilde{u})$, defined by

$$q = \frac{\sqrt{-2\Lambda}}{4}bpg + \frac{1}{2\sqrt{-2\Lambda}g^2}\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}\frac{\mathrm{d}g}{\mathrm{d}\tilde{u}} - \frac{1}{2\sqrt{-2\Lambda}g}\frac{\mathrm{d}^2p}{\mathrm{d}\tilde{u}^2},$$

and dropping the tildes then leads to the line element:

$$\begin{aligned} \mathrm{d}s^2 = &\left[-\Lambda r^2 + \left(\frac{1}{\Lambda y^4} - 4x^2 + \Lambda y^4(b(u) - 2x^2)^2\right)g(u)^2 - \frac{4y^2}{\sqrt{-2\Lambda}}\right]\mathrm{d}u^2 \\ &+ 2\left[\mathrm{d}r + \frac{8g(u)x\mathrm{d}y}{y\sqrt{-2\Lambda}} + \left(\sqrt{-2\Lambda}(b(u) - 2x^2)y^4 + \frac{2}{\sqrt{-2\Lambda}}\right)g(u)\mathrm{d}x\right]\mathrm{d}u \\ &+ \frac{4\mathrm{d}y^2}{y^2\Lambda} - 2y^4\mathrm{d}x^2, \end{aligned} \tag{5.39}$$

where $[b(u), g(u)]$ is a pair of distinguishing free functions (meaning that two metrics of this class, described by pairs $[b_1(u), g_1(u)] \neq [b_2(u), g_2(u)]$, are necessarily different). The fact that we find two distinguishing free functions could be predicted from the GHP-analysis: in the case where $m(u) = 0$, we find from (5.25) and $\alpha = \pm\sqrt{-2\Lambda}/4$, that $\Phi_{22} = \mp\sqrt{-2\Lambda}\nu$. To obtain a well-weighted expression in GHP, we should write $\Phi_{22}^2 = -2\Lambda\nu\overline{\nu}$. The only unknown function in GHP is then $\text{Þ}'\nu$. As the latter is complex, we can construct from it two zero-weighted quantities of which not all directional derivatives are known.

The class $m(u) \neq 0$

The general solution of (5.34) is now

$$U = \frac{4\alpha\gamma_1}{sm^2} + B(u,x).$$

As in the case $m = \gamma_1 = 0$ one can make $B(u,x) = 0$ by redefinition of x. Hence equations (5.33) and (5.35) can be solved to give

$$X = \frac{X_1(x)}{m(u)}, \qquad Y = \frac{1}{m}\left(\frac{\mathrm{d}m}{\mathrm{d}u} + \frac{4\alpha}{sX_1}\frac{\partial\gamma_1}{\partial x}\right).$$

Equation (5.37) is now identically satisfied. Applying the coordinate transformation $x \to \tilde{x}$, with $\frac{\mathrm{d}\tilde{x}}{\mathrm{d}x} = 2X_1$, the only remaining equation (5.36) becomes

$$\frac{\partial^2\gamma_1}{\partial\tilde{x}^2} + s\gamma_1 = 0.$$

The general solution of this equation is $\gamma_1 = \frac{f(u)}{2\sqrt{2}}\cos[(\tilde{x} + q(u))]$ for $s = 1$ and $\gamma_1 = \frac{f(u)}{2\sqrt{2}}\cosh[(\tilde{x} + q(u))]$ for $s = -1$. One can now show that a coordinate transformation of the form

$$\begin{aligned} \nu &\longrightarrow \pm\frac{y\,m}{\sqrt{2}}, \\ r &\longrightarrow \tilde{r}\,p + \frac{1}{\Lambda}\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}, \\ \mathrm{d}u &\longrightarrow \frac{\mathrm{d}\tilde{u}}{p}, \end{aligned}$$

where p is a function of $\tilde{u}$ defined by

$$n = \frac{1}{2\Lambda}\left(\frac{\mathrm{d}p}{\mathrm{d}\tilde{u}}\right)^2 - \frac{p}{\Lambda}\frac{\mathrm{d}^2p}{\mathrm{d}\tilde{u}^2},$$

in combination with an appropriate boost to make $f = m\,p$ and redefinition of $m = \pm\tilde{m}\,p^2$, allows one to write the line element in the form

$$\begin{aligned} \mathrm{d}s^2 &= \left[-\Lambda r^2 - (s - y^2\Lambda)g^2 + \frac{2m}{\Lambda}\sqrt{s - y^2\Lambda} - \left(\frac{\partial g}{\partial x}\right)^2\right]\mathrm{d}u^2 \\ &+ 2\left[\mathrm{d}r - s\,y\sqrt{s - y^2\Lambda}\,g\mathrm{d}x + \frac{s}{\sqrt{s - y^2\Lambda}}\frac{\partial g}{\partial x}\mathrm{d}y\right]\mathrm{d}u \\ &- y^2\mathrm{d}x^2 - \frac{1}{s - y^2\Lambda}\mathrm{d}y^2, \end{aligned} \tag{5.40}$$

where $g(u,x) = \cos(x + q(u))$ for $s = 1$ and $g(u,x) = \cosh(x + q(u))$ for $s = -1$ and where $[m(u), q(u)]$ is a couple of distinguishing free functions.

5.3.2 The family $\tau \neq 0$

In this case (5.22) yields

$$\Phi_{22} = \frac{\nu(\Psi_2 + \Lambda/3)}{\tau}.$$

As we have shown above $\tau(\gamma - \overline{\gamma}) = 0$, so γ is now real. The Bianchi equations can be written as follows:

$$\begin{aligned} \delta\Psi_2 &= 3\tau\Psi_2, \\ \overline{\delta}\Psi_2 &= 3\tau\Psi_2, \\ \Delta\Psi_2 &= 0, \\ \mathrm{D}\Psi_2 &= 0, \end{aligned} \qquad \begin{aligned} \delta\tau &= \frac{\tau(\delta\nu + \nu\tau)}{\nu}, \\ \overline{\delta}\tau &= \frac{\tau(\overline{\delta}\nu + \nu\tau)}{\nu}, \\ \Delta\tau &= 0, \end{aligned}$$

while the NP Ricci equations yield an expression for α:

$$\alpha = -\frac{3\Psi_2 + \Lambda - 6\tau^2}{12\tau},$$

and expressions for some of the directional derivatives of ν and γ:

$$\begin{aligned} \delta\nu &= \frac{\nu(3\Psi_2 + \Lambda)}{6\tau}, \\ \overline{\delta}\nu &= \frac{\nu(3\Psi_2 + \Lambda)}{6\tau}, \end{aligned} \qquad \begin{aligned} \delta\gamma &= 0, \\ \overline{\delta}\gamma &= 0, \\ \mathrm{D}\gamma &= -c_1\Psi_2^{2/3}. \end{aligned}$$

We thus have the total derivatives:

$$\begin{aligned} \mathrm{d}\gamma &= \Delta\gamma\,\boldsymbol{\omega}^3 - c_1\Psi_2^{2/3}\boldsymbol{\omega}^4, \\ \mathrm{d}\nu &= \frac{\nu(3\Psi_2 + \Lambda)}{6\tau}\,(\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2) + \Delta\nu\boldsymbol{\omega}^3, \\ \mathrm{d}\tau &= (\tau^2 + \frac{\Psi_2}{2} + \frac{\Lambda}{6})\,(\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2), \\ \mathrm{d}\Psi_2 &= 3\,\tau\,\Psi_2\,(\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2). \end{aligned}$$

We can use τ (or Ψ_2) as a coordinate, but for the moment we write $\tau = \tau(x)$, $\Psi_2 = \Psi_2(x)$ and define a function $S(x)$ such that

$$\frac{\mathrm{d}\Psi_2}{\mathrm{d}x} = 6S\tau\Psi_2, \tag{5.41}$$

or, equivalently,

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = \frac{1}{3}S(6\tau^2 + 3\Psi_2 + \Lambda).$$

The first Cartan equations read

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^1 &= -\frac{3\Psi_2+\Lambda}{6\tau}\,\boldsymbol{\omega}^1\wedge\boldsymbol{\omega}^2,\\
\mathrm{d}\boldsymbol{\omega}^3 &= 0,\\
\mathrm{d}\boldsymbol{\omega}^4 &= \nu(\boldsymbol{\omega}^1+\boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^3-2\gamma\,\boldsymbol{\omega}^3\wedge\boldsymbol{\omega}^4-2\tau\,(\boldsymbol{\omega}^1+\boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^4.
\end{aligned}$$

These equations, together with the directional derivatives of τ, show that we can write the basis one forms as:

$$\begin{aligned}
\boldsymbol{\omega}^1 &= S\mathrm{d}x+\mathrm{i}(U\mathrm{d}u+X\mathrm{d}x+f\mathrm{d}y),\\
\boldsymbol{\omega}^3 &= \mathrm{d}u,\\
\boldsymbol{\omega}^4 &= \mathrm{d}r+H\mathrm{d}u+F\mathrm{d}x+G\mathrm{d}y,
\end{aligned}$$

where f, F, G, H, U and X are real functions of (u, r, x, y). The corresponding directional derivatives are given by

$$\begin{aligned}
\delta &= \frac{1}{2Sf}\left((XG-fF+\mathrm{i}SG)\partial_r+f\partial_x-(X+\mathrm{i}S)\partial_y\right),\\
\Delta &= \frac{1}{f}\left(f\partial_u+(UG-fH)\partial_r-U\partial_y\right),\\
\mathrm{D} &= \partial_r.
\end{aligned}$$

Applying $[\Delta, \mathrm{D}]$ and $[\delta, \mathrm{D}]$ to y shows that f, U and X are independent of r, so a transformation of y exists making $U = 0$. From $[\delta, \Delta]y$ it then follows that f and X are independent of u so a coordinate transformation exists, allowing one to put $X = 0$. Next $[\delta, \overline{\delta}]y$, (5.24) and (5.41) imply that $f = -\tau\Psi_2^{-1/3}$. From the directional derivatives of γ, we get $\gamma = -c_1\Psi_2^{2/3}r + \gamma'(u, x, y)$, $F(u, r, x, y) = 4\tau(x)rS(x) + F_1(u, x, y)$ and $G(u, r, x, y) = G_1(u, x, y)$, with F_1 and G_1 defined by

$$F_1 = -\frac{1}{c_1\Psi_2^{2/3}}\frac{\partial\gamma'}{\partial x},$$

$$G_1 = -\frac{1}{c_1\Psi_2^{2/3}}\frac{\partial\gamma'}{\partial y}.$$

Notice also that, in case $c_1 = 0$, γ' is a function of u only, such that we can use the remaining boost freedom to put γ' equal to zero. This allows us to write $\gamma' = c_1^2 b$.

The directional derivatives of ν, together with (5.41), lead to an expression for ν:

$$\nu = -\frac{n(u)}{\tau(x)\Psi_2^{1/3}(x)},$$

and from $[\Delta, \mathrm{D}]r$, $[\delta, \Delta]r$ and $[\delta, \overline{\delta}]r$ we find an expression for H:

$$H = \frac{1}{\Psi_2^{2/3}}\left[c_1\left(-r^2\Psi_2^{4/3} + 2rc_1 b\Psi_2^{2/3} - c_1^2 b^2 - \frac{\partial b}{\partial u}\right) + \left(m - n\Psi_2^{1/3}\right)\right],$$

where $m(u)$, $n(u)$ and $b(u, x, y)$ are real functions.
If we use Ψ_2 as coordinate x, we find expressions for S and τ:

$$S = \frac{1}{6\tau x},$$
$$\tau = \pm\sqrt{x - \frac{\Lambda}{6} + c_1 x^{2/3}}.$$

Performing a translation $r \to (r + q(u) + c_1 b(u, x, y))/x^{2/3}$, where $\mathrm{d}q/\mathrm{d}u = -m$, the line element for this family is given by

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{2}{x^{2/3}}\left(c_1(r + q(u))^2 + n(u)x^{1/3}\right)\mathrm{d}u^2 + \frac{2}{x^{2/3}}\mathrm{d}r\mathrm{d}u \\ &- \frac{1}{3x^2\left(6x - \Lambda + 6c_1 x^{2/3}\right)}\mathrm{d}x^2 - \frac{6x - \Lambda + 6c_1 x^{2/3}}{3x^{2/3}}\mathrm{d}y^2. \end{aligned} \quad (5.42)$$

(Notice that the sign of τ has no influence on the metric).

This metric contains only one arbitrary function of u, namely $n(u)$ if $c_1 = 0$ (as q does not appear in the line element then) and $q(u)$ if $c_1 \neq 0$ (as one can then fix the boost such that $n = 1$). Alternatively, for $c_1 \neq 0$, one could use a transformation $r \to \tilde{r}g(u) + f(u)$ followed by $\mathrm{d}u = g(u)^{-1}\mathrm{d}\tilde{u}$, where we choose the functions f and g such that $f(\tilde{u}) = -q(\tilde{u}) + \frac{\dot{g}(\tilde{u})}{2c_1}$ and $\ddot{g}(\tilde{u}) = \frac{\dot{g}(\tilde{u})^2}{2g(\tilde{u})} + 2c_1\dot{q}(\tilde{u})$. This also eliminates $q(u)$ from the system and as we then obtain exactly the same line element as before, but with $q(u) = 0$, this shows we can always put $q(u) = 0$, leaving $n(u)$ as the only free function.

The fact that we find only one free function in this case could again be easily predicted from the GHP-analysis: the only unknown is now $\text{Þ}'\nu$, which is real, as can be deduced from $\tau\nu - \overline{\tau\nu} = 0$ and $\text{Þ}'\tau = 0$ $(\tau \neq 0)$.

5.4 Classification of the Kundt metrics

In this section we present the classification of the Kundt metrics. For each metric, and for some special cases where specific choices for the free functions are made, we give the number of isometries and the components of the Weyl and Ricci tensor. To obtain a valid classification of each of the metrics, it is necessary to (null-) rotate and boost the tetrad, such that it is in a 'standard tetrad form' for CLASSI. We will give an explicit example for one of the Kundt metrics (5.38), without its special cases (see below). The other input files can be found in the appendix.

5.4.1 Classification of metric 5.38

In order to obtain a standard CLASSI tetrad for metric 5.38, it is necessary to apply following dyad transformations:

- boost with parameter $A = (-bLy^2)^{1/4}$,
- rotation with parameter $\theta = \pi/4$.

Also we have replaced the function $b(u)$ by $-b(u)$ to make the value of Φ_{22}, returned by CLASSI, positive, and we use $L = \sqrt{-2\Lambda}$ to simplify the calculations. The non-zero components of the Weyl and Ricci tensor are then given by $\Psi_2 = L^2/6 = -\Lambda/3$ and $\Phi_{22} = 1$. This metric generally has three functionally independent elements:

$$
\begin{aligned}
f_1 &= \Re \mathrm{D}\Phi_{33} = -\frac{r\,L\sqrt{L}}{y\sqrt{b}} - \frac{\dot{b}}{y\sqrt{b}\,L\,b}, \\
f_2 &= \Re \mathrm{D}^2\Phi_{44} = 1 - \frac{5}{12}\frac{L^3 r^2}{b y^2} - \frac{5}{6}\frac{L r \dot{b}}{b^2 y^2} - \frac{1}{3}\frac{\ddot{b}}{b^2 L y^2}, \\
f_3 &= \Re \mathrm{D}^3\Phi_{55} = \frac{L\sqrt{L}r}{y\sqrt{b}} - \frac{15}{104}\frac{L^4\sqrt{L}r^3}{y^3 b\sqrt{b}} + \frac{\dot{b}}{y b\sqrt{bL}} - \frac{45}{104}\frac{L^2\sqrt{L}r^2\dot{b}}{y^3 b^2\sqrt{b}} \\
&\quad - \frac{9}{26}\frac{\sqrt{L}r\ddot{b}}{y^3 b^2\sqrt{b}} - \frac{1}{13}\frac{\dddot{b}}{y^3 b^2 L\sqrt{bL}}.
\end{aligned}
$$

As the function $b(u)$ appears in those elements explicitly, no coordinate or tetrad transformation exists eliminating this function from the metric. The

metric is of Petrov type D and of Segre type A_3 $[(11,2)]$, pure radiation (including electromagnetic null). The remaining isotropy group is zero-dimensional. The *classisum* result reads r1D 1 0 s0000123.
The general input in CLASSI looks like this:

Input	Comment
(PRELOAD DYTRSP)	
(TITLE "KUNDT5.38.SPI")	Name of the metric
(OFF ALL) (ON NOZERO PSUBS)	
(NAMLC U R X Y) (VARS U R X Y)	 The variables (u, r, x, y)
(FUNS (B U) (L))	b is in general a function of u, L is a constant
(RPL IZUD) 1 \$ 0 \$ 0 \$ 0 \$ L^2/4*R^2+2*B*Y^2/L \$ 1 \$ 0 \$ 0 \$ 0 \$ 0 \$ Y^2*I \$ -2/Y/L \$ 0 \$ 0 \$ -Y^2*I \$ -2/Y/L \$	Input of the metric one forms
(NULLT IFRAME)	Null tetrad
(RPL DYTR1) 0 \$ I \$ I \$ 0 \$	A first dyad transformation
(RPL DYTR2) (-B*L*Y^2)^(1/4) \$ 0 \$ 0 \$ (-B*L*Y^2)^(-1/4) \$	A second one (boost)
(RPL DYTR3) sqrt(2)/2*(1+I) \$ 0 \$ 0 \$ sqrt(2)/2*(1-I) \$	And a third (spatial rotation)

During the classification of the metric, CLASSI asks to verify whether or not some expressions can be equal to zero. If so, we obtain special cases.

If $b(u)$ is a non-zero *constant*, only one functionally independent quantity survives:

$$f_1 = \Re \mathrm{D}\Phi_{33} = \frac{rL\sqrt{L}}{y\sqrt{b}},$$

and the corresponding *classisum* result reads r1D 3 0 s0000111.

Putting the function $b(u)$ equal to *zero*, all functionally independent quantities disappear, as this corresponds to a (Bertotti) vacuum field (all Φ_{ij}-components are equal to zero). The fact that the metric no longer presents a pure radiation field can be easily predicted: the function $b(u)$ is proportional to the (non-zero) spin coefficient ν. The Segre type is now A_1 $[(111,1)]$, with Λ-term and the remaining isotropy group consists of boosts and rotations (two-dimensional). This leads to the *classisum* result 01D 6 2 eeee0000.

5.4.2 Classification of metric 5.39

To obtain a standard tetrad, one now has to

- boost with parameter $A = (-Ly^2)^{1/4}$,
- rotate with parameter $\theta = \pi/4$,

where we have again replaced $\sqrt{-2\Lambda}$ by L in order to simplify the calculations. We find the same expressions for the components of the Weyl and Ricci tensor, as in the previous case:

$$\begin{aligned} \Psi_2 &= \frac{L^2}{6}, \\ \Phi_{22} &= 1. \end{aligned}$$

The metric is of Petrov type D, Segre type A_3 $[(11,2)]$, pure radiation (including electromagnetic null). The independent quantities are now given by

$$\begin{aligned} f_1 &= \Re \mathrm{D}\Phi_{33} = \frac{g\,x\sqrt{L}}{y} - \frac{1}{2}\frac{r\,L\sqrt{L}}{y}, \\ f_2 &= \Im \mathrm{D}^2\Psi_{52} = \frac{g\sqrt{L}}{y^3}, \\ f_3 &= \Re \mathrm{D}^2\Phi_{44} = \frac{g^2Lx^2}{y^2} + \frac{g^2}{Ly^6} - \frac{5}{2}\frac{gL^2rx}{y^2} + \frac{5}{8}\frac{L^3r^2}{y^2} - \frac{x\dot{g}}{y^2} - \frac{3}{2}, \\ f_4 &= \Im \mathrm{D}^3\Psi_{63} = \frac{g^2Lx}{y^4} - \frac{5}{4}\frac{gL^2r}{y^4} - \frac{1}{2}\frac{\dot{g}}{y^4}. \end{aligned}$$

The *classisum* result for this metric is r1D 0 0 s0000134. The isotropy group is zero-dimensional.

It is a bit surprising that $b(u)$ does not appear in the free functions, giving the impression that a transformation should exist, which eliminates this function from the line element. However, as we have explained in section 5.3.1, we do expect *two* distinguishing free functions in the metric. Indeed, as will be shown in the appendix, $b(u)$ will appear in the free functions for some special cases. The fact that it is not present in the free elements for the general case is probably due to the fact that CLASSI also counts (relations between) coordinates as independent quantities.

5.4.3 Classification of metric 5.40

The necessary transformation to obtain a CLASSI standard tetrad is a boost with parameter

$$A = \left(m\sqrt{\frac{L^2y^2}{2} + s} \right)^{1/4}.$$

In this choice of tetrad the components of the Weyl and Ricci tensor are given by

$$\begin{aligned} \Psi_2 &= \frac{L^2}{6}, \\ \Phi_{22} &= 1, \end{aligned}$$

where again, we have replaced Λ by $-L^2/2$.

The metric is of Petrov type D and of Segre type A_3 $[(11,2)]$, pure radiation (including electromagnetic null). For both cases, $s = +1$ and $s = -1$, the functionally independent elements are given by $\Re \mathrm{D}\Psi_{41}$, $\Re \mathrm{D}\Phi_{33}$, $\Re \mathrm{D}^2\Psi_{52}$

and $\Im \mathrm{D}^2\Psi_{52}$. The explicit forms of these elements are given below:

$$f_1 = \Re \mathrm{D}\Psi_{41} = \frac{L^2 y}{\sqrt{\frac{L^2y^2}{2}+s}},$$

$$f_2 = \Re \mathrm{D}\Phi_{33} = \frac{\sqrt{m}}{m\left(\frac{L^2y^2}{2}+s\right)^{1/4}}\left(L^2 r + \frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}u} - \frac{L^2 y\sqrt{s(1-H^2)}}{2\left(\frac{L^2y^2}{2}+s\right)}\right),$$

$$f_3 = \Re \mathrm{D}^2\Psi_{52} = \frac{L^2 y\sqrt{m}}{\left(\frac{L^2y^2}{2}+s\right)^{3/4}}\left(\frac{L^2 r}{m} + \frac{1}{m^2}\frac{\mathrm{d}m}{\mathrm{d}u} - \frac{\sqrt{(1-H^2)\left(\frac{sL^2y^2}{2}+1\right)}}{ym}\right),$$

$$f_4 = \Im \mathrm{D}^2\Psi_{52} = \frac{HL^2\sqrt{m}}{m\left(\frac{L^2y^2}{2}+s\right)^{3/4}},$$

where $H = \cos(x+q(u))$ or $\cosh(x+q(u))$ and $L = \sqrt{-2\Lambda}$. The corresponding *classisum* result for these metrics is r1D 0 0 s0000244. The remaining isotropy group is zero-dimensional.

5.4.4 Classification of metric 5.42

We will now classify metric 5.42, where $\pi \neq 0$. To obtain a standard CLASSI tetrad we use a boost with parameter

$$A^{1/4} = \left(-n(u)x^{2/3}\left(\frac{\Lambda}{3x}+1\right)\right)^{1/4}.$$

In general we can find the following functionally independent elements:

$$\begin{aligned}
f_1 &= \Re\Psi_2 = x,\\
f_2 &= \Re \mathrm{D}\Phi_{33} = \frac{c_1 r}{\sqrt{A}} - \frac{A_u}{4A^{3/2}},\\
f_3 &= \Re \mathrm{D}^2\Phi_{44} = \frac{3(L^3x-(5-8x^2)L^2-12x(2+2x^2)L-108x^2)}{2(L+3x)^2(Lx+3)}\\
&+ \ddot{A}2A^2 - \frac{5c_1\dot{A}r}{A^2} + \frac{10c_1^2r^2}{A}.
\end{aligned}$$

The non-zero components of Weyl and Ricci tensor for this choice of tetrad are $\Psi_2 = x$ and $\Phi_{22} = 1$. The metric is of Petrov type D and of Segre type A_3 $[(11,2)]$, pure radiation (including electromagnetic null). The dimension of the remaining isotropy group is zero. The *classisum* result is r1D 1 0 s0001233.

A first special case arises when simultaneously $c_1 = 0$ and $n = (c_2 u + c_3)^{-2}$, where c_2 and c_3 are constants (not both equal to zero). In this case there is only one functionally independent element, $f_1 = \Psi_2 = x$. The metric remains of Petrov type D and Segre type A_3 $[(11,2)]$, pure radiation (including electromagnetic null). The *classisum* result however is given by r1D 3 0 s0001111.

A second case that has to be examined separately, arises in two situations:

- when $c_1 = 0$, but $n(u)$ is not of the form $(c_2 u + c_3)^{-2}$
- when $n(u)$ is subject to the differential equation

$$-4n^2 \left(\frac{\mathrm{d}^3 n}{\mathrm{d}u^3}\right) - 15 \left(\frac{\mathrm{d}n}{\mathrm{d}u}\right)^3 + 18n \left(\frac{\mathrm{d}n}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}^2 n}{\mathrm{d}u^2}\right) = 0.$$

In the first situation the functionally independent elements are given by

$$\begin{aligned} f_1 &= \Re\Psi_2 = x, \\ f_2 &= \Re\mathrm{D}\Phi_{33} = -\frac{n_u}{n\sqrt{A}}. \end{aligned}$$

Also in the second situation only two functionally independent elements survive:

$$\begin{aligned} f_1 &= \Re\Psi_2 = x, \\ f_2 &= \Re\mathrm{D}\Phi_{33} = \frac{1}{\sqrt{A}}\left[c_1 r - \frac{n_u}{4n}\right]. \end{aligned}$$

For both subcases the components of Weyl and Ricci tensor are $\Psi_2 = x$, $\Phi_{22} = 1$, all others being zero. The corresponding metrics are of Petrov type D and of Segre type A_3 $[(11,2)]$, pure radiation (including electromagnetic null). The dimension of the remaining isotropy group is zero. This leads to the *classisum* result r1D 2 0 s0001222, valid for both subcases.

Notice also that it is again not necessary to examine the case $n(u) = 0$, as $n(u)$ is introduced as a function proportional to the non-zero spin coefficient ν. Putting $n(u)$ equal to zero in metric 5.42 makes all Φ-components equal to zero (so we no longer have a non-zero pure radiation field). The metric remains of Petrov type D, but the Segre type changes to A_1 $[(111,1)]$, with Λ-term. The only functionally independent element is $f_1 = \Re\Psi_2 = x$ and the *classisum* result is now 01D 4 1 ebbb111, from which we see that the remaining isotropy group exists of boosts (one-dimensional) and an interchange of null directions.

5.5 Discussion and conclusions

We have shown that all Petrov type D pure radiation metrics, with or without cosmological constant, which admit a non-diverging null congruence, can be written in one of the four forms (5.38), (5.39), (5.40) or (5.42). Metric 5.42 reduces to the metric of Wils and Van den Bergh (1990) for $n(u) = 0$.

We have also classified the metrics, using the program CLASSI (Åman, 2002). The dimension of the isotropy group is zero for all cases. The dimension of the isometry group depends on particular choices of the parameters and the free functions. The Kundt solutions presented here, together with the already known Robinson Trautman solutions, constitute the complete class of aligned pure radiation fields of Petrov type D.

Chapter 6

Conclusion and perspectives

In this thesis we have examined some previously known metrics and some new metric families. After introducing the terminology and notation used in this thesis in chapter one, we first re-examined the Petrov type D pure radiation metrics belonging to the Robinson Trautman family in chapter two. These pure radiation metrics, which are defined geometrically by the property that they admit a geodesic, non-twisting and shear-free but expanding null congruence had first been investigated by Frolov and Khlebnicov. They organised their solutions into three classes, A, B and C, from which the A- and B-classes were further subdivided into three and five subclasses, respectively. In this thesis we showed that their A-class is a subfamily of the B-class, thus removing some redundancy in the published solutions. Furthermore we noted that their C-class is incorrect and we presented the full calculations and integration of this class explicitly. Both results can also be found in Stephani et al. (2003, Chapter 28). In section 2.4 we gave the GHP-analysis of the B-class. It is shown there that we only expect *four* distinguishing functions in the line element for this class, and not five, as in the exact solutions book. Therefore a coordinate transformation should exist, which eliminates one function from expression (28.73) in Stephani et al. (2003).

In chapters three and four we examined solutions belonging to the Newman Tamburino family. These are defined geometrically by the presence of a hypersurface orthogonal, geodesic null congruence with non-vanishing shear and divergence. We were able to show that, in vacuum, the cylindrical metric in the original paper is incorrect (a fact that had been known before, although the author was not aware of that) and we wrote the metric line element for this class in a much more elegant way, without elliptic functions. This results in a metric which can be used more easily (for example for classification, limit cases, equivalence of metrics,...). The corrected version of the original metric for this class was classified by Alan Barnes and Jan Åman and the input file is added in the appendix. We also re-integrated the spherical class and proved that also here there was some redundancy in the original article: both spherical vacuum metrics from the article can be written in one single form. As far as the author knows, this fact was not determined before.

In chapter three, we examined Newman Tamburino solutions in the presence of a Maxwell field. We were able to prove that the Maxwell field is aligned if and only if the metric belongs to the cylindrical class. This Einstein Maxwell solution can be shown to be a generalisation of the vacuum cylindrical metric. Whether or not spherical solutions in the presence of a non-aligned Maxwell field exist remains an open problem.

Chapter five was dedicated to Kundt solutions in the presence of a pure radiation field. These solutions are defined by having a non-diverging (*i.e.* non-rotating and non-expanding and therefore geodesic) null congruence. In the case of a pure radiation field, the solutions are shear-free. We integrated this problem and found four different line elements. It is possible that some of these are related to one another, but this remains to be examined. We also classified the different line elements and their special cases, and the input files can be found in the appendix. A discussion on how to find the number of distinguishing free functions in the line element, based on the GHP-analysis, is also demonstrated in this chapter.

Appendix A

Input files for CLASSI

In this appendix, we present the input files for CLASSI.

A.1 Input file for the Sachs metric

In this section we give the input file for the Sachs metric, which was presented to us by Alan Barnes.

(PRELOAD IZUD)

(TITLE "NTS.NUL Special limit of cylindrical Newman-Tamburino metric in null tetrad. Collinson & French JMP 8, 701 (1967). KSMH p.244: Two errors corrected (by Koutras). This version by A Barnes. empty space")

(OFF ALL) (ON NOZERO)

(NAM X Y R U C)
x $ y $ r $ u $ c $

(VARS X Y R U)

(RPL A)
(C/2 + log(R*X^2))/X^2 $

(FUNS (C) A)

(NEWSUL RIESUL)
A $:A $
(USESUL RIESUL RIEF SPCURV RICC)

```
(RPL IZUD)
0 $ 0 $ 0 $ 1 $
2*R/X $ 0 $ 1 $ -A $
R*2^(-1/2) $ I*X*2^(-1/2) $ 0 $ 0 $
R*2^(-1/2) $ -I*X*2^(-1/2) $ 0 $ 0 $

(NULLT IFRAME)
```

A.2 Input file for the cylindrical Newman Tamburino empty space metric

In this section we give the input file, as presented to us by Alan Barnes, for the general vacuum cylindrical Newman Tamburino metric. In the present file a form of the metric containing elliptic functions is being classified. As we have shown that our solution (4.39) is equivalent to the one below, it was not necessary to write a new input file. Also the file below can serve as a nice example on how to insert more difficult functions.

```
(PRELOAD IZUD)

(TITLE "NTC.NUL Cylindrical Newman-Tamburino metric in null tetrad. Newman & Tamburino JMP 3, 902 (1962). This version by A Barnes, modified by Jan E. Aman. empty space")

(OFF ALL) (ON NOZERO)
% Jacobi's elliptic functions
% sn(x,k), cn(x,k), dn(x,k).
% The modulus and complementary modulus are denoted by K and K!'
% usually written mathematically as k and k' respectively


% Special case of modulus k=2^(-1/2) appearing in Newman Tamburino solutions

(NAMLC SN CN DN)

(DS DFEFUN (FN VRL !& REST BDY) (DC FN EFUN VRL . BDY))

% Define derivatived of sn, cn and dn.
(DFEFUN SN (X) (SMTIMES2 (LIST 'CN (CADR X)) (LIST 'DN (CADR X))))
```

```
(DFEFUN CN (X) (SMMINUS (SMTIMES2 (LIST 'SN (CADR X)) (LIST 'DN
(CADR X)))) )

(DFEFUN DN (X) (SMTIMES2 MIHALF!* (SMTIMES2 (LIST 'SN (CADR X))
(LIST 'CN (CADR X)))) )

% Symmetry simplification for Sheep simplifier.
(DEFLIST '((SN ODDXX) (CN EVENXX) (DN EVENXX)) 'SIMPXFN)

% To make Reduce accept them, no particular simplification.
(DEFLIST '((SN SIMPIDEN) (CN SIMPIDEN) (DN SIMPIDEN)) 'SIMPFN)

(NAMLC U R X Y B C)

(VARS U R X Y)

(RPL H W)
2*B^2*CN(B*X)^2 +(C/2 +B^2*LOG(R*CN(B*X)^2))/CN(B*X)^2 $
B*(1 - CN(B*X)^2)^(1/2)*(1+CN(B*X)^2)^(1/2)/(2*2^(1/2)*CN(B*X)) $

(FUNS (B) (C))

(NEWSUL 2 RIESUL)
SN(B*X) $ (1 -CN(B*X)^2)^(1/2) $
DN(B*X) $ 2^(-1/2)*(1 +CN(B*X)^2)^(1/2) $
(USESUL RIESUL RIEF)

(RPL IZUD)
1 $ 0 $ 0 $ 0 $
-:H $ 1 $ 0 $ 0 $
4*:W+I*2^(1/2)*B*CN(B*X)*LOG(R) $ 0 $
R/2+I*4*2^(1/2)*B*U*:W*CN(B*X) $ I*2^(-1/2)*CN(B*X) $
4*:W-I*2^(1/2)*B*CN(B*X)*LOG(R) $ 0 $
R/2-I*4*2^(1/2)*B*U*:W*CN(B*X) $ -I*2^(-1/2)*CN(B*X) $

(NULLT IFRAME)
```

A.3 Input file for metric 5.38

The input file for the classification of metric 5.38 (not for its special cases) has been given in section 5.4.1. As mentioned there, a first special case arises, when the function $b(u)$ is a non-zero constant. For the classification of this subclass, it is sufficient to replace the input line (FUNS (B U) (L)) by (FUNS (B) (L)).

A second special case occurs when $b(u) = 0$. In fact we knew in advance that this would no longer represent a pure radiation metric, as the function $b(u)$ is proportional to the non-zero spin coefficient ν. To obtain the input file for this metric, replace b by 0 in the one forms and remove dyad transformations two and three.

A.4 Input file for metric 5.39

```
(PRELOAD DYTRSP)
(TITLE "KUNDT5.39.SPI")

(OFF ALL) (ON NOZERO PSUBS)
(NAMLC U R X Y)
(VARS U R X Y)

(FUNS (B U) (L) (G U))

(RPL IZUD)
1 $ 0 $ 0 $ 0 $
L^2/4*R^2 + B*G^2 + 2*Y^2/L $ 1 $ 0 $ 0 $
(-I*L/2*(B-2*X^2)*Y^2+2*X-I/L/Y^2)*G $ 0 $ -I*Y^2 $ 2/L/Y $
(I*L/2*(B-2*X^2)*Y^2+2*X+I/L/Y^2)*G $ 0 $ I*Y^2 $ 2/Y/L $

(NULLT IFRAME)
(RPL DYTR1)
0 $ I $
I $ 0 $

(RPL DYTR2)
(-L*Y^2)^(1/4) $ 0 $
0 $ (-L*Y^2)^(-1/4) $

(RPL DYTR3)
SQRT(2)/2*(1+I) $ 0 $
0 $ SQRT(2)/2*(1-I) $
```

A special case arises if the function $g(u)$ satisfies

$$\frac{3}{4}g\frac{\mathrm{d}g}{\mathrm{d}u}\frac{\mathrm{d}^2g}{\mathrm{d}u^2}-\left(\frac{\mathrm{d}g}{\mathrm{d}u}\right)^3=0. \tag{A.1}$$

Let us first examine the case $g(u) = constant$. It is sufficient to remove the u-dependence of g in the input lines above. We again find four functionally independent functions of which f_1, f_2 and f_3 are the same as in the general case, but where f_4 is now given by

$$\begin{aligned} f_4 = \Im \mathrm{D}^3\Phi_{55} &= \frac{bg^3xL\sqrt{L}}{y^3}+\frac{3g^2L^2rx^2\sqrt{L}}{y^3}+\frac{3g^2r\sqrt{L}}{y^7}-\frac{15}{4}\frac{gr^2xL^3\sqrt{L}}{y^3} \\ &+ \frac{26}{3}\frac{gx\sqrt{L}}{y}+\frac{5}{8}\frac{r^3L^4\sqrt{L}}{y^3}-\frac{13}{3}\frac{rL\sqrt{L}}{y}. \end{aligned}$$

If *both* $g(u)$ and $b(u)$ are constants, only the first three functionally independent elements survive, and the *classisum* result reads r1D 1 0 s0000133.

Now assume that $g(u)$ is not a constant, but still satisfies (A.1). It is not necessary to solve this differential equation. Instead we will give the following input in CLASSI:

```
(FUNS (B U) (L) (G U) (G SPEC U) (GU U) (GU SPEC U))
GU $
4*GU^2/3/G $
```

Classifying the metric then leads to four functionally independent functions, of which f_1, f_2 and f_3 are again the same as in the general metric 5.39, but where f_4 is now given by

$$\begin{aligned} f_4 = \Re \mathrm{D}^3\Phi_{55} &= \frac{bg^3xL\sqrt{L}}{y^3}+\frac{1}{3}\frac{bg\dot{g}\sqrt{L}}{y^3}+\frac{3g^2rx^2L^2\sqrt{L}}{y^3}+\frac{3g^2r\sqrt{L}}{y^7} \\ &+ \frac{2g\dot{g}x^2\sqrt{L}}{y^3}+\frac{2g\dot{g}}{L\sqrt{L}y^7}-\frac{15}{4}\frac{gr^2xL^3\sqrt{L}}{y^3}+\frac{26}{3}\frac{gx\sqrt{L}}{y} \\ &- \frac{3\dot{g}rxL\sqrt{L}}{y^3}+\frac{5}{8}\frac{r^3L^4\sqrt{L}}{y^3}-\frac{13}{3}\frac{rL\sqrt{L}}{y}-\frac{8}{9}\frac{\dot{g}^2x}{g\sqrt{L}y^3}. \end{aligned}$$

The *classisum* result is again given by r1D 0 0 s0000134.

A.5 Input file for metric 5.40

In the input file below, replace s by ± 1.

```
(PRELOAD DYTRSP)
(TITLE "KUNDT5.40.SPI")

(OFF ALL) (ON NOZERO PSUBS)
(NAMLC U R X Y)
(VARS U R X Y)

(FUNS (M U) (M SPEC U) (MU U) (A U Y) (A SPEC Y) (A SPEC U) (H U X)
(H SPEC X) (L))
MU $
A*L^2*Y/(s*2+L^2*Y^2) $
A/M*MU $
-s*SQRT (s*(1-H^2)) $

(RPL IZUD)
1 $ 0 $ 0 $ 0 $
L^2/4*R^2 + M*SQRT(s+Y^2*L^2/2)/L^2*2 $ 1 $ 0 $ 0 $
s*I/SQRT(2)*SQRT(s+L^2*Y^2/2)*H-SQRT (s*(1-H^2) )/SQRT(2) $ 0 $
I*Y/SQRT(2) $ -1/SQRT(2)/SQRT(s+L^2*Y^2/2) $
-s*I/SQRT(2)*SQRT(s+L^2*Y^2/2)*H-SQRT (s*(1-H^2) )/SQRT(2) $ 0 $
-I*Y/SQRT(2) $ -1/SQRT(2)/SQRT(s+L^2*Y^2/2) $

(NULLT IFRAME)
(RPL DYTR1)
0 $ I $
I $ 0 $

(NEWSUL EEN)
M $ A/SQRT(L^2*Y^2/2+s) $
(USESUL EEN PHI DPSI XI)

(NEWSUL TWEE)
A $ M*SQRT(L^2*Y^2/2+s) $
(USESUL TWEE DPHI D2PSI D2PHI)

(RPL DYTR2)
A^(1/4) $ 0 $
0 $ 1/A^(1/4) $
```

A.6 Input file for metric 5.42

```
(PRELOAD DYTRSP)
(TITLE "KUNDT5.42.SPI")

(OFF ALL) (ON NOZERO PSUBS)
(NAMLC U R X Y)
(VARS U R X Y)

(RPL A)
-L*N/X^(1/3)/3-N*X^(2/3) $

(FUNS (L) (N U) (C) (g) (h) (A U X) (A SPEC X))
-A/3/X*(L-6*X)/(L+3*X) $

(RPL IZUD)
-1 $ 0 $ 0 $ 0 $
(R)^2*C/X^(2/3)+N/X^(1/3) $ -1/X^(2/3) $ 0 $ 0 $
0 $ 0 $ -1/6/X/(X+X^(2/3)*C-L/6)^(1/2) $
I*(X+X^(2/3)*C-L/6)^(1/2)/X^(1/3) $
0 $ 0 $ -1/6/X/(X+X^(2/3)*C-L/6)^(1/2) $
-I*(X+X^(2/3)*C-L/6)^(1/2)/X^(1/3) $

(NEWSUL EEN)
L $ -X^(1/3)/N*(A+N*X^(2/3))*3 $
(USESUL EEN PHI)

(NEWSUL TWEE)
N $ -A/X^(2/3)/(L/3/X+1) $
(USESUL TWEE DPSI XI D2PSI APSI D2PHI)

(NULLT IFRAME)
(RPL DYTR1)
0 $ I $
I $ 0 $

(RPL DYTR2)
A^(1/4) $ 0 $
0 $ 1/A^(1/4) $
```

For the first special case, where $c_1 = 0$ and $n(u)$ is of the form $(c_2 u + c_3)^{-2}$, the input is given by

```
(PRELOAD DYTRSP)
(TITLE "KUNDT5.42.SPI")

(OFF ALL) (ON NOZERO PSUBS)
(NAMLC U R X Y)
(VARS U R X Y)

(funs (L) (G) (H) (A U X) (A SPEC X) (A SPEC U) (N U) (N SPEC U))
-A/3/X*(L-6*X)/(L+3*X) $
-2*G*A*N^(1/2) $
-2*G*N^(3/2) $

(RPL IZUD)
-1 $ 0 $ 0 $ 0 $
1/X^(1/3)*N $ -1/X^(2/3) $ 0 $ 0 $
0 $ 0 $ -1/6/X/(X-L/6)^(1/2) $ I*(X-L/6)^(1/2)/X^(1/3) $
0 $ 0 $ -1/6/X/(X-L/6)^(1/2) $ -I*(X-L/6)^(1/2)/X^(1/3) $

(NEWSUL EEN)
L $ -X^(1/3)/N*(A+N*X^(2/3))*3 $
(USESUL EEN PHI)

(NEWSUL TWEE)
N $ -A/X^(2/3)/(L/3/X+1) $
(USESUL TWEE DPSI XI D2PSI APSI D2PHI)

(NULLT IFRAME)
(RPL DYTR1)
0 $ I $
I $ 0 $

(RPL DYTR2)
A^(1/4) $ 0 $
0 $ 1/A^(1/4) $
```

The input file for the second special case, $c_1 = 0$ but $n(u)$ not of the form $(c_2 u + c_3)^{-2}$ looks like this:

```
(PRELOAD DYTRSP)
(TITLE "KUNDT5.42.SPI")

(OFF ALL) (ON NOZERO PSUBS)
```

```
(NAMLC U R X Y)
(VARS U R X Y)

(RPL A)
-L*N/X^(1/3)/3-N*X^(2/3) $

(FUNS (Q U) (L) (N U) (N SPEC U) (NU SPEC U) (C) (A U X) (A SPEC X)
(A SPEC U))
NU $
-A/3/X*(L-6*X)/(L+3*X) $
A/N*NU $

(RPL IZUD)
-1 $ 0 $ 0 $ 0 $
N/X^(1/3) $ -1/X^(2/3) $ 0 $ 0 $
0 $ 0 $ -1/6/X/(X-L/6)^(1/2) $ I*(X-L/6)^(1/2)/X^(1/3) $
0 $ 0 $ -1/6/X/(X-L/6)^(1/2) $ -I*(X-L/6)^(1/2)/X^(1/3) $

(NEWSUL EEN)
L $ -(A+N*X^(2/3))/N*3*X^(1/3) $
(USESUL EEN PHI)

(NEWSUL TWEE)
A $ :A $
(USESUL TWEE DPHI)

(NULLT IFRAME)
(RPL DYTR1)
0 $ I $
I $ 0 $

(RPL DYTR2)
A^(1/4) $ 0 $
0 $ 1/A^(1/4) $
```

Finally, if $n(u)$ is subject to the differential equation

$$4n^2\left(\frac{\mathrm{d}^3n}{\mathrm{d}u^3}\right)-15\left(\frac{\mathrm{d}n}{\mathrm{d}u}\right)^3+18n\left(\frac{\mathrm{d}n}{\mathrm{d}u}\right)\left(\frac{\mathrm{d}^2n}{\mathrm{d}u^2}\right)=0$$

the input looks like this:

```
(PRELOAD DYTRSP)
(TITLE "KUNDT5.42.SPI")

(OFF ALL) (ON NOZERO psubs)
(NAMLC U R X Y)
(VARS U R X Y)

(FUNS (L) (N U) (N SPEC U) (NU U) (NU SPEC U) (NUU U) (NUU SPEC U)
(C) (A U X) (A SPEC X) (A SPEC U))
NU $
NUU $
(-15*NU^3+18*N*NU*NUU)/4/N^2 $
-A/3/X*(L-6*X)/(L+3*X) $
A/N*NU $

(RPL IZUD)
-1 $ 0 $ 0 $ 0 $
R^2*C/X^(2/3)+N/X^(1/3) $ -1/X^(2/3) $ 0 $ 0 $
0 $ 0 $ -1/6/X/(X+X^(2/3)*C-L/6)^(1/2) $
I*(X+X^(2/3)*C-L/6)^(1/2)/X^(1/3) $
0 $ 0 $ -1/6/X/(X+X^(2/3)*C-L/6)^(1/2) $
-I*(X+X^(2/3)*C-L/6)^(1/2)/X^(1/3) $

(NEWSUL EEN)
L $ -X^(1/3)/N*(A+N*X^(2/3))*3 $
(USESUL EEN PHI)


(NULLT IFRAME)
(RPL DYTR1)
0 $ I $
I $ 0 $

(RPL DYTR2)
A^(1/4) $ 0 $
0 $ 1/A^(1/4) $
```

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