Open Research Online

The Open University's repository of research publications
and other research outputs

Geometric configuration in robot kinematic design

Book Section

oro.open.ac.uk

Geometric Configuration in Robot Kinematic Design

Dr Joe Rooney[1]

[1]The Open University, Faculty of Technology, Milton Keynes, UK, j.rooney@open.ac.uk

Abstract. A lattice of geometries is presented and compared for representing some geometrical aspects of the kinematic design of robot systems and subsystems Three geometries (set theory, topology and projective geometry) are briefly explored in more detail in the context of three geometric configurations in robotics (robot groupings, robot connectivities and robot motion sensor patterns).

1 Introduction

Kinematics is naturally focussed on changes, with respect to time, in the relative distances and angles between mechanical systems and/or sub-systems. But distance and angle are not the only concepts involved in kinematics and a range of other entities and constructs arise in a more general treatment. Many of these are essentially geometrical, in that they can be expressed in terms of some type of geometry, although it may not be Euclidean. Characterising a particular geometry usually gives rise to one or more types of geometric configuration. In this paper the emphasis is on geometric configuration associated with robot components, spatial relationships and patterns of motion. It is assumed that a group of robot systems (or subsystems) operates, co-operates or competes kinematically within one or more geometrical regimes, and geometric configuration may be identified whenever robot systems move, change shape or evolve.

2 A Lattice of Geometries

Felix Klein, in his *Erlanger Programm*, organised the (19th century) known geometries into a very elegant hierarchical scheme (Meserve, 1955; Klein, 1939). Since then many more types of geometry have been constructed, including a major class of finite or discrete geometries. Klein's hierarchy (a tree) is now a larger and more general lattice of inter-related geometries.

Klein made sense of the diversity by treating a geometry as a set of points, a set of relations between the points, and a group of transformations on the points. For robots a 'point' might represent: a location in workspace or jointspace; a vision sensor pixel; a complete manipulator system; a robot state; a pattern of motion, etc. A relation might represent: an association of two or more pixels; distances between pairs of robots, or states; the 'connectivities between robot components or sub-systems. A transformation might represent: a rotation of a manipulator link or of a set of pixels forming a vision sensor; a permutation of robot 'names' or identities; a mapping between a robot's jointspace and workspace.

If a transformation is applied to a set of points, then certain attributes of the points and their inter-relationships do not change – they remain *invariant*. Each type of transformation and its invariants characterise the type of geometry. Typically, concepts such as magnitude, direction, dimension, orientation, connectedness etc. can be given formal geometrical meanings (as invariants under particular types of transformation).

By considering a lattice of geometries in robotics it is hoped that something akin to the Brooks hierarchical subsumption architecture (Brooks, 1999) might be achieved. The idea is that each geometry on one level of the lattice subsumes all aspects of the geometries on the adjacent lower levels. Obversely, each geometry is subsumed by the geometries on the adjacent higher levels. A corresponding situation for robot motion behaviour might be that, for example, initially the robot is being controlled to behave within the scope of a particular geometry for a period of time. Then in response to some event, fault or failure it switches to a different level in the hierarchy where it continues to function, but now by being controlled within the scope of the new geometry on the new level. The resulting new behaviour pattern replaces the original behaviour pattern in operation before the fault/failure occurred. The capabilities of the robot under the new geometrical regime might be enhanced, but probably they will be diminished. However, some aspects of the original robot motion could continue to be controlled under the new regime. Moreover, another possible scenario is that a group of robots might be controlled concurrently on a number of different (geometrical) levels in order to produce more complex behaviour and achieve multiple goals, improve efficiency or optimise performance.

3   Examples of Geometries

The methods employed by Klein to describe and analyse geometries used co-ordinates. The most general view is that a co-ordinate is merely a *label* attached to (i.e. co-ordinated with) a point. Notions or measures of distance (or of any other potential invariant) are defined in terms of co-ordinates by specifying some function of the latter. Klein introduced co-ordinate systems to label the points and then defined each geometry by a set of relationships amongst the co-ordinates that remained invariant under some group of co-ordinate transformations. The relationship $(x_2 - x_1)^2 + (y_2 - y_1)^2$ between the Cartesian co-ordinates (x_1, y_1) and (x_2, y_2) of two points in a plane, defines the square of the distance between the points, and both this Pythagorean relationship and the geometrical distance remain invariant under rotations, etc. More general types of labels (co-ordinates) than Cartesian ones, and more abstract relationships than the Pythagorean one are used in the Klein scheme (see Tables 1 and 2). However, the overall principle is the same, namely: a geometry is defined by the set of co-ordinate relations that are invariant under the action of a particular group of co-ordinate transformations, and different types of co-ordinate transformations lead to different geometries.

Klein found that the larger the group of allowed (coordinate) transformations for a geometry, the fewer the number of invariants. So, paraphrasing Klein, the most general types of transformation leave hardly anything unchanged, whereas the least general types of transformation leave many things unchanged. An example of a general transformation is a one-one mapping (essentially a permutation), and this changes everything except the *number* (cardinality) of points or objects. Conversely a Euclidean rotation (a much less general transformation) maintains *distances* between points, *angles* between lines, *areas, shapes, parallelism*, etc. Table 1 lists six different geometries and their characteristic coordinate transformations. The list includes Set Theory as a geometry as well as the more familiar types. Table 2 shows typical invariants (expressed in terms of co-ordinates) for some of these geometries and for two extra geometries - the symplectic geometry of the phase plane (Weyl, 1939), used in dynamics, and the pseudo-Euclidean geometry of relativistic kinematics (Rindler, 1966).

Table 1. Six geometries and their co-ordinate transformations.

Geometry (reference)	Characteristic Co-ordinate Transformation	Type of Transformation
		Types of Invariant
Set Theory (Birkhoff and MacLane, 1965)	$x' = f_1(x, y, z)$ $y' = f_2(x, y, z)$ $z' = f_3(x, y, z)$	One-One
		Cardinality
Topology (Porteous, 1969; Alexandroff, 1961)	$x' = f_1(x, y, z)$ $y' = f_2(x, y, z)$ $z' = f_3(x, y, z)$	One-One and Continuous
		Dimension, Connectivity, Fixed Point
Differential (Kreyszig, 1959)	$dx' = a_1(x, y, z)dx + b_1(x, y, z)dy + c_1(x, y, z)dz$ $dy' = a_2(x, y, z)dx + b_2(x, y, z)dy + c_2(x, y, z)dz$ $dz' = a_3(x, y, z)dx + b_3(x, y, z)dy + c_3(x, y, z)dz$	Continuous and Smooth
		Metric, Nearness
Projective (Klein, 1939; Modenov and Parkhomenko, 1965b)	$x' = \dfrac{a_1 x + b_1 y + c_1 z + d_1}{a_4 x + b_4 y + c_4 z + d_4}$ $y' = \dfrac{a_2 x + b_2 y + c_2 z + d_2}{a_4 x + b_4 y + c_4 z + d_4}$ $z' = \dfrac{a_3 x + b_3 y + c_3 z + d_3}{a_4 x + b_4 y + c_4 z + d_4}$	Rational Linear
		Incidence, Separation, Cross-ratio
Affine (Modenov and Parkhomenko, 1965a)	$x' = a_1 x + b_1 y + c_1 z + d_1$ $y' = a_2 x + b_2 y + c_2 z + d_2$ $z' = a_3 x + b_3 y + c_3 z + d_3$	Linear
		Parallelism, Finiteness, Between-ness
Euclidean (Yale, 1968)	$x' = (l^2(1 - \cos\theta) + \cos\theta)x + (ml(1 - \cos\theta) + n\sin\theta)y$ $\quad + (nl(1 - \cos\theta) - m\sin\theta)z + d_1$ $y' = (ml(1 - \cos\theta) - n\sin\theta)x + (m^2(1 - \cos\theta) + \cos\theta)y$ $\quad + (mn(1 - \cos\theta) + l\sin\theta)z + d_2$ $z' = (nl(1 - \cos\theta) + m\sin\theta)x + (mn(1 - \cos\theta) - l\sin\theta)y$ $\quad + (n^2(1 - \cos\theta) + \cos\theta)z + d_3$	Orthogonal
		Distance, Angle, Shape, Size

Table 2. Five geometries and some of their invariants.

Geometry (reference)	Typical Invariants and their Context
Differential (Kreyszig, 1959)	*Metric*: $$ds^2 = g_{11}dxdx + g_{12}dxdy + g_{13}dxdz$$ $$+g_{21}dydx + g_{22}dydy + g_{23}dydz$$ $$+g_{31}dzdx + g_{32}dzdy + g_{33}dzdz$$ where (x, y, z) and $(x + dx, y + dy, z + dz)$ are two neighbouring points in 3D space
Projective (Klein, 1939; Modenov and Parkhomenko, 1965b)	*Cross-ratio*: $$R_c = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}$$ where x_1, x_2, x_3 and x_4 are four points in 1D space
Symplectic (Weyl, 1939)	*Product*: $$\delta^2 = (q_2 - q_1)(p_3 - p_1) - (q_3 - q_1)(p_2 - p_1)$$ where (q_1, p_1), (q_2, p_2) and (q_3, p_3) are three points in 2D phase space
Pseudo-Euclidean (Rindler, 1966).	*Interval*: $$\tau^2 = (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$ where (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) are two points in 4D spacetime, with $c = 1$
Euclidean (Yale, 1968)	*Product*: $$\delta^2 = (x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1) + (z_2 - z_1)(z_3 - z_1)$$ *Distance*: $$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$ *Angle*: $$\cos\theta = \frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1) + (z_2 - z_1)(z_3 - z_1)}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2][(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2]}$$ where (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are three points in 3D space

4 Robot Configuration

In the following three sub-sections three different geometries from Table 1, are briefly explored in the context of geometrical configurations in robotics.

4.1 Robot Groupings (Set Theory)

Consider a set of n robots and treat it as a finite set of n points. There are a total of 2^n possible subsets into which the n robots can group themselves. Specifically there are $_nC_r$ different possible subsets containing r robots, where $_nC_r$ is the binomial coefficient Since

there are also $_nC_{n-r} = {_nC_r}$ different possible subsets containing $n-r$ robots, this enumerates the complete range of possible ways that up to n robots can partition themselves into two 'teams'.
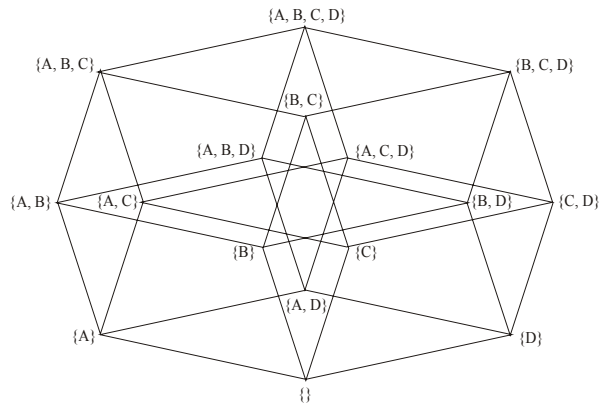

Figure 1. The lattice of subsets of a set {A, B, C, D} of four robots.

One way to represent the interrelationships amongst the subsets or 'teams' of robots is to display them in a lattice with the robots labelled, say, alphabetically. For a set {A, B, C, D} of four robots, the lattice of subsets is shown in Figure 1. The lattice may be used in various ways to characterise robot groupings and to follow changes in robot groupings. For instance this lattice can be viewed essentially as a hypercuboid with the subsets or 'teams' of robots located at its vertices. Subsets at opposite ends of a hyperspace diagonal have no robots in common and so the eight possible hyperspace diagonals represent the eight ways to partition the set of four robots into two non-overlapping 'teams'. This 'geometrical' arrangement can form the basis of a 4-bit coding scheme to label the subsets with a binary codeword (co-ordinate) rather than just using letters. The advantage gained from this code is that the codewords of adjacent subsets in the lattice differ by one bit, and those on opposite ends of a hyperspace diagonal differ by four bits, hence a simple measure is established for tracking changes in the membership of 'teams'.

The paths from one node to another in this lattice can be used as a basis for enumerating fault and failure modes. Thus suppose the top node represents the state in which all four robots are functioning, and the next level down represents states in which one robot is not functioning, and so on. Then the *number* of paths down through the hierarchy from the top to any particular node represents the number of ways (and hence modes of failure) to arrive in the state represented by that node. This leads to a crude measure of probability of ending up in that state. It may also offer modes of recovery by identifying and suggesting strategies for moving to nearby nodes.

The transformations within set theory are one-one transformations, and for a finite set of n points these transformations are permutations (a re-ordering of the points of the set). A collection of n robots might be ordered in some sequence determined by their positions, by their communication protocols, or by their 'seniority', etc. Each possible ordering may be specified by labelling the robots with a sequence of symbols. Then any re-ordering is

essentially equivalent to a permutation of the robot symbols. For a set of n robots there are a total of $n!$ different permutations of the labels. They fall into two equally sized classes, namely $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd permutations. Whatever permutation is applied to a collection of robots it does not change the number (cardinality) of robots. But the effects of even and odd permutations are different. Two types of orderings emerge. An even permutation maintains the 'orientation' of the original ordering whereas an odd permutation produces an 'opposite' orientation. The situation is akin to the difference between right-handed and left-handed co-ordinate systems. From the set theory viewpoint there is really only one invariant associated with a collection of robots and that is the number of robots in the collection. However, although the robots may be identical (i.e. indistinguishable) in construction, they might nonetheless at each instant be distinguished by labels, names, or codewords, or by an indication of their places in a chain of command. In this case the different possible labellings will fall into two equal size classes - those related by even permutations, and the remainder. Thus a second attribute of the collection of robots is the 'orientation' of the labelling scheme, but note that this is only an invariant under even permutations.

4.2 Robot Neighbours (Topology)

Given a set of n robots it is possible to introduce more structure into the set by defining a notion of 'neighbourliness' between pairs of robots. This establishes a topology for the set. In general the resulting topology can be very complicated and is often non-intuitive, but in the present context of a simple discrete set of robots the 'neighbourliness' can be specified in terms of 'connections' between pairs of robots. Each connection may be a mechanical or electrical connection, or more generally it can just be a representation of a communication channel or relation between 'neighbouring' robots. The connectedness of a discrete configuration of n robots may be described and explored in terms of graph theory (Wilson, 1996).

Within graph theory each robot is represented by a point (a vertex) and each connection or relation is represented by a line segment (an edge). In general the graph representing the connections amongst a set of n robots may be described in terms of one or more *spanning trees*. Figure 2 shows two different graphs for a set of six robots. Each graph represents one possible way for the robots to be interconnected. In both cases one possible spanning tree is shown by bold edges. A spanning tree may be used to determine two important invariants of the graph, namely its *fundamental cycles* and its *fundamental cutsets*.

In the graph on the left of Figure 2, there are three cycles: $\{e_1, e_2, e_3, e_4\}$, $\{e_3, e_7, e_6, e_5\}$ and $\{e_1, e_2, e_7, e_6, e_5, e_4\}$. But just two are *fundamental* cycles, since the third may be formed as a 'sum' of the first two (where the rule for adding two cycles is to include in the sum all edges except for those common to both of the original cycles). Given a spanning tree, the fundamental cycles are immediately identified since every cycle has an edge in common with the complement of the tree, and so the number of fundamental cycles is the same as the number of edges in the complement of the tree. If there are n vertices and m edges in the graph there are $m - n + 1$ edges in the complement and hence $m - n + 1$ fundamental cycles. The importance of fundamental cycles stems from the fact that they characterise the circuits and thus the fundamentally distinct pathways through the graph. Hence for a set of robots they therefore characterise how the robots are linked together as a group.
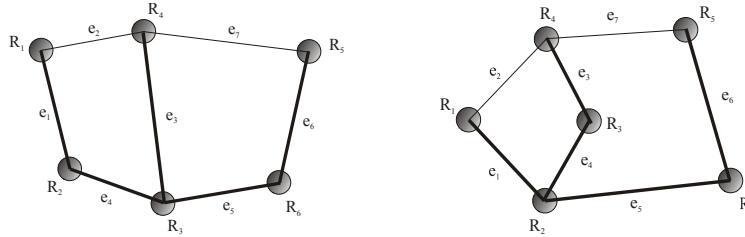
Figure 2. Two different graphs for six robots, showing spanning trees (bold edges).

In the graph on the right of Figure 2, there are sixteen cutsets: $\{e_1,e_2\}$, $\{e_3,e_2,e_7\}$, $\{e_4,e_2,e_7\}$, $\{e_5,e_7\}$, $\{e_6,e_7\}$, $\{e_6,e_5\}$, $\{e_1,e_3,e_7\}$, $\{e_2,e_4,e_5\}$, $\{e_1,e_3,e_5\}$, $\{e_2,e_3,e_6\}$, $\{e_2,e_3,e_5\}$, $\{e_1,e_4,e_6\}$, $\{e_1,e_4,e_5\}$, $\{e_1,e_4,e_7\}$, $\{e_2,e_4,e_6\}$ and $\{e_1,e_3,e_6\}$. But just five are *fundamental* cutsets, since the last eleven can be formed from sums of the first five (where the rule for adding two cutsets is to include in the sum all edges except for those common to both of the original cutsets). Given a spanning tree the fundamental cutsets are immediately identified since every cutset has one edge in common with the tree and so the number of fundamental cutsets is the same as the number of edges in the tree. If there are n vertices and m edges in the graph there are $n-1$ edges in the spanning tree and hence $n-1$ fundamental cutsets. The importance of fundamental cutsets stems from the fact that they characterise the fundamentally distinct divisions of the graph, and hence for a set of robots they therefore characterise how the robots may be divided into co-operating or competing subgroups.

4.3 Robots in Perspective (Projective Geometry)
Pixel configurations in a robot motion sensing system, such as those used in the visual morphology research at the University of Sussex (Harvey I, et al., 1996) often have their outputs grouped into special 2D patterns. The approach has proved to be successful in the interpretation of perspective views acquired by mobile robots. Their performance may be further improved by considering the projective geometry of the situation.

One particular special pattern combines the outputs from four pixels. Any non-collinear group of four pixels in the plane defines a (projective geometry) configuration of seven lines and nine points forming a *complete quadrilateral* (Coxeter, 1993). The extra five points and the seven lines on which they lie have significance in defining auxiliary pixel locations (see Figure 3(a)). An analogous dual configuration of seven points and nine lines, known as a complete quadrangle, defines another potentially useful projective construct for the vision system (see Figure 3(b)).

In Figure 3, if the four pixel positions are replaced by a group of four robots in four positions, the complete quadrilateral and complete quadrangle configurations have significance for the robot grouping. The diagonal points and the intersections of diagonal lines represent significant locations for targets, for goals or for where other associated robots may be placed.

5   Discussion and Conclusions
The notion of a hierarchy of geometries, offers a novel approach to understanding robot kinematic behaviour. A group of robots may behave in a huge range of different ways, co-operating or competing with one another, but whatever their behaviour it almost always has an external kinematic manifestation as movement or change of the spatio-temporal configuration

of the group. The geometrical approach may be used to define, characterise and classify various kinematic behaviours. Each different geometry focusses on specific (often subtle) aspects of the kinematics, and may be used either to *analyse* a possible range of kinematic behaviour or to *produce* a required kinematic behaviour. By operating solely within the scope of one level of the hierarchy, or, alternatively, by combining several levels simultaneously, various degrees of complexity of robot kinematic behaviour may be explored.
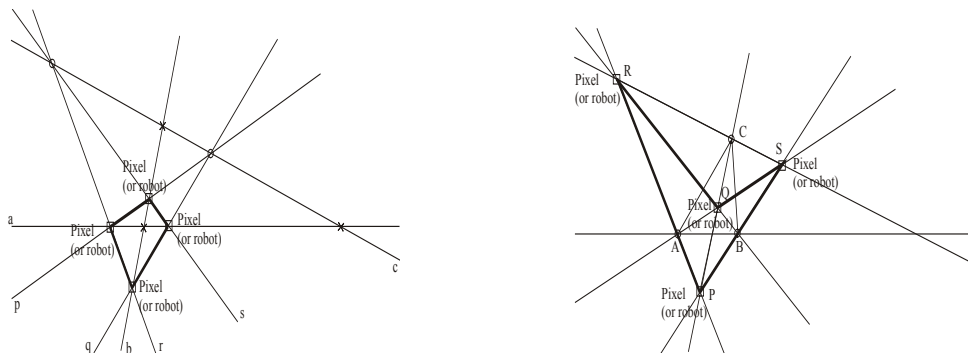

Figure 3(a) Four lines *p, q, r, s* form a complete quadrilateral, with diagonal lines *a, b, c* forming the diagonal triangle)(left); (b) Four points *P, Q, R, S* form a complete quadrangle, with diagonal points *A, B, C* forming the diagonal triangle (right).

Bibliography

Alexandroff P, (1961), *Elementary Concepts of Topology* (Dover Publications, New York)

Birkhoff G, MacLane S, (1965), *A Survey of Modern Algebra* (MacMillan, New York)

Brooks R A, (1999), *Cambrian Intelligence* (MIT Press)

Coxeter H S M, (1993), *The Real Projective Plane* (Springer-Verlag, Berlin)

Harvey I, Husbands P, Cliff D, Thomson A and Jakobi N, (1996), "Evolutionary Robotics at Sussex", in Proc. ISRAM 96: Int. Symp. on Robotics & Manufacturing, Montpelier, France

Hausner M, (1965), *A Vector Space Approach to Geometry* (Dover Publications, New York)

Klein F, (1939), *Elementary Mathematics from an Advanced Standpoint* (two volumes) (Noble) (Dover, New York edition, 1948)

Kreyszig E, (1959), *Differential Geometry*, (University of Toronto Press, Toronto) (Dover, New York edition, 1991)

Meserve B E, (1955), *Fundamental Concepts of Geometry* (Dover, New York edition, 1983)

Modenov P S and Parkhomenko A S, (1965a), *Geometric Transformations, Volume 1: Euclidean and Affine Transformations* (Academic Press, New York)

Modenov P S and Parkhomenko A S, (1965b), *Geometric Transformations, Volume 2: Projective Transformations* (Academic Press, New York)

Porteous I R, (1969), *Topological Geometry* (Van Nostrand Reinhold, London)

Rindler W, (1966), *Special Relativity* (Oliver and Boyd)

Weyl H, (1939), *The Classical Groups*, (Princeton, New Jersey)

Wilson R J, (1996), *Introduction to Graph Theory* (Longman)

Yale P B, (1968), *Geometry and Symmetry* (Holden-Day, San Francisco) (Dover, New York edition, 1988).