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CHARACTERIZATIONS OF (SUBSTRUCTURES OF) GENERALIZED QUADRANGLES AND HEXAGONS

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Acknowledgments and Preface

While reading most prefaces or introductory notes to all kinds of papers, books, courses and manuscripts on generalized polygons and coming across the magical date 1959, it may seem that generalized polygons dropped out of the sky. But — though a gift from heaven for incidence geometry and geometers over the past five decades — they did not descend from the heavens. Nor did this work.

As geometers and geometry are closely connected, I opt to start this work with an amalgamation of acknowledgments for the former and a technical preface to the latter. Where polygons already started before Tits, my interest for mathematics started — I guess — the day that I sat staring at a blackboard on which a big circle with a tangent on the right-hand side had been drawn by my mother. In front of me on the desk was my loco-booklet of puzzles, because a 7-year old girl with a day off from school is supposed to sharpen her thoughts with such things. At the other desks in front of me were the 15-year old pupils my mother was teaching. I wonder which one of us was puzzled most by the circle.

A long time after that, having finished my masters degree at Ghent University, I began the circle of listening - reading - thinking - explaining - writing - striking - and finally proving; a circle one calls 'doctoreren' in Dutch. Let me take you through this circle, visiting on the way a range of mathematical techniques which are of use in the study of incidence geometries, and as such pop up frequently in this work.

From the very beginning, my supervisor Hendrik Van Maldeghem encouraged me to attend conferences, making the 'listening' part come through.

> At one of these early conferences, we tackled a characterization problem by translating the group theoretical concept of a Zassenhaus group (i.e. a group which is something between 2- and 3-transitive) into mere synthetic geometry. This became theorem 3.4, and the starting point of chapter 3. While the ending of this chapter would involve more joint work with Katrin Tent, we turned our attention to reading and explaining.

Though the talks I was able to give at different math meetings during the past 4 years were the most related to the work presented here, I would like to stress the additional value that was given to my work teaching the exercises accompanying the course of projective geometry to the maths-students in our department. Thanks to their many questions, (mis)understandings and need for explanations, I was forced to go through the details over and over again.

> Meanwhile, we were concentrating on the following characterization of quadrangles: *A finite generalized quadrangle of order (s,s) is isomorphic to $W(s)$ if and only if all points <u>of a geometric hyperplane</u> are regular.* As the weaker theorem (i.e. the version without the underlined words) exists also in a version for the hexagons, we handled this gap (already one third filled by previous authors) with synthetic geometry, that is basically with nothing other than pictures of points and lines, drawing conclusions from the construction of more points and lines.

As (messy) pictures often appear more promising in concept than they are in reality, I am greatly thankful to Leen Kuijken, who helped me through an up-and-down period of fuzzy symptoms. Finally, finding the right pictures enabled us to finish chapter 4.

> Substructures of geometries often give rise to new geometries or to nicely formulated characterizations, as did the geometric hyperplane mentioned above, which lumped together three related results. This is one of the reasons why new substructures keep being defined, as we did m-clouds and dense clouds of generalized hexagons respectively quadrangles in chapter 5. For these concepts, we started from the ideal planes in the classical generalized hexagon $H(q)$, and wondered what would happen if one has a point set satisfying a weaker variant of the basic property of such an ideal plane. A technique that is extensively used here, is the translation of one geometry into the terms of another one. If this second geometry is 'better known', we can use facts and theorems relating to it, and export them back to the first geometry. (As examples, results of design theory, linear spaces

> and projective planes are used.) A second way of working our way through it was the extended Higman-Sims technique. This matrix technique permits to state bounds on the size of point sets by calculating (bounds on) the eigenvalues of $(0, 1)$-matrices naturally related to an incidence geometry. The main results that are achieved show that, under certain circumstances, the extension of an affine plane to a projective plane translates into the extension of an m-cloud into an $(m + 1)$-cloud. Dense clouds are even more general, they group together ovoids, spreads and sub-n-gons in a natural way.

At this point, I would like to thank, for their company and encouragement, all fellow PhD students whom I had the pleasure of meeting during the many conferences in Ghent, Brussels and further afield. I also had great help from Ivano Pinneri, who won me over for programming. He showed me how to speed up the computer (indeed, I used Pascal, have a glimpse at appendix A), and what is back-tracking all about.

> While back-tracking was only used to achieve a side-result in chapter 5, chapter 6 is mainly computer-based. Indeed, once we decided to look for a hemisystem in the generalized hexagon $H(3)$, I spent days writing and processing dozens of subprograms, which revealed to me, one by one, parts of the structure of $H(3)$ and the underlying space. (The most difficult part was inventing appropriate names for all those subprograms.) When the puzzle was completed, I could rephrase the story as is done in chapter 6. In appendix A, one can take a look at the (cleaned) version of the Pascal program.

One thing one should know when starting programming, is that it may take the computer a while before it hands in the answer to your question, leaving you waiting, staring blankly at the flickering cursor. I imagine that I could pass for such a computer from time to time, and because of that, I would like to thank my supervisors, Prof. Dr. H. Van Maldeghem and Prof. Dr. J.A. Thas for their patience and their unremitting guidance — even when ideas did not work out the way we expected them to and had to be stopped. One of those ideas is included as a bonus in appendix B; it is a note on some characterization theorems for quadrangles.

> In fact, it was also the precursor of chapter 2. Here, we give a geometric and unifying proof of an existing characterization of quadrangles, and we replace the geometric condition with a group theoretical one. We followed different routes to reach our goal. We used counting — this is the technique I missed most in high-school geometry —, a fair amount of construction work, a nice geometrical argument, and a lot of group theoretical ones (one of the theorems was proven in

> three different ways; we included two of them). Now we are almost completely round the circle — so we return to the very first result. In the meantime, this had bloomed into characterizations of different classical quadrangles, thanks to an elaborated contribution of Katrin Tent and Hendrik Van Maldeghem. When translating group theoretical concepts into geometry, the use of coordinates turns out to be a fast and clear way of proving statements. By taking things up again, we recalculated certain of them, and altered the last part of the proof of the main theorem. This can all be found in chapter 3.

Such is the story of chapters 2 to 6, and the addenda. Chapter 1, the introductory guide, was composed during the last few weeks — for, it is written that the last shall be first.

And one may see this as a wish to the people I thank at the end of this pages. My family, for the safe haven. Triene, for keeping an eye on me for the past 8 years. And last but not least, Yves, for going on the journey with me, and Thijsje, for being my greatest fan.

Ghent, October 2000

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Chapter 1

Preliminary Results

In this chapter, we collect a number of definitions and results which are opportune for the following chapters. Once we defined generalized polygons, mentioned some frequently cited properties and described some classical examples of polygons, we turn to some characterizations of generalized quadrangles respectively hexagons, picked out for their resemblances — and which will prove useful in chapters 2 to 4. As we will give a characterization of the classical spreads of $W(q)$ in chapter 2, we list all known spreads of $W(q)$, together with all known spreads of $H(q)$, to stress again the similarities between (some) quadrangles and hexagons. Next we turn our attention to projectivities and coordinatization of generalized polygons, which appear in chapter 3. As the last two chapters are devoted to special subgeometries of generalized quadrangles and hexagons, we collect some definitions of other incidence structures than n-gons, and we recall a matrix technique to study (some of) these subgeometries.
We based this introductory guide mainly on *'Finite Generalized Quadrangles'* of Payne and Thas ([48]), *'Generalized Polygons'* of Van Maldeghem ([84]), and chapters 7 and 9 by Thas of *'The Handbook of Incidence Geometry'* edited by Buekenhout ([74], [73]).

1.1 Geometries

A **point-line incidence structure** or a **geometry of rank 2** is a triple $\Delta = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with non-empty point set $\mathcal{P}$, non-empty line set $\mathcal{L}$, and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. Points are usually denoted by small letters $(p, q, r, x, y, z, \ldots)$, lines by capital letters $(K, L, M, N, \ldots)$. Both points and lines are called **elements** of the geometry. By abuse of notation, we will sometimes write $u \in \Delta$ instead of $u \in \mathcal{P} \cup \mathcal{L}$. A **flag** is a set $\{p, L\}$ with p a point incident with the line L. Instead of the notation $p \mathrm{~I~} L$, we also use $L \mathrm{~I~} p$ (treating I as a symmetric relation), or even $p \in L$, $L \ni p$ (treating L as a subset of $\mathcal{P}$).
A **subgeometry** of a geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a geometry $(\mathcal{P}', \mathcal{L}', \mathrm{I}')$ with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathcal{P}' \times \mathcal{L}')$. The **dual** of a geometry is obtained by interchanging points and lines, i.e. the geometry $(\mathcal{L}, \mathcal{P}, \mathrm{I}^{-1})$. The **double** of the geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the geometry $(\mathcal{P} \cup \mathcal{L}, \mathcal{F}, \in)$ with $\mathcal{F}$ the set of flags, and $\in$ the set-theoretic inclusion.
The **incidence graph** of a geometry is the graph with vertex set $V = \mathcal{P} \cup \mathcal{L}$ and the flags of the geometry as edges. The adjacency relation $*$ of this graph is thus given by $x * y \Leftrightarrow x \mathrm{~I~} y$ or $y \mathrm{~I~} x$. A **(non-stammering) path** of length δ from vertex x to vertex y, is a sequence of vertices $x = x_0, x_1, \ldots, x_\delta = y$ in the graph, such that $x_{i-1} * x_i$ for $i = 1, 2, \ldots, \delta$ and such that $x_{j\pm1} \neq x_{k\pm1}$ whenever $x_j = x_k$, where indices j, k are taken modulo δ, and for all choices of signs.
Let u, v be two elements of the geometry $(\mathcal{P}, \mathcal{L}, \mathrm{I})$. The **distance** $\delta(u, v)$ between u, v is measured in the incidence graph, i.e. $\delta(u, v)$ is the length of a shortest path between u and v; if $u = v$ we put $\delta(u, v) = 0$. We always assume that every two elements can be joined by a path (this means Δ is a **connected** geometry).
If two points x, y are at distance 0 or 2, we call them **collinear** and write $x \sim y$ or $x \perp y$. The **perp** of a point x is the set of all elements collinear with x. The perp of a set X of points is the set of points collinear with every element of X. So $X^\perp = \bigcap_{x \in X} x^\perp$. If two lines L, M are at distance 0 or 2, we call them **concurrent** and use the same notations as for collinear points.
The set of all elements at distance i from an element u is denoted by $\Delta_i(u)$. The set of all elements at distance i or j from an element u is denoted by $\Delta_{i,j}(u)$. We write $\Delta(u) = \Delta_1(u)$ and call it the **point row** respectively **line pencil** for u a line respectively a point. Remark that $u^\perp = \Delta_2(u) \cup \{u\}$ for any element u of Δ. If a line L is uniquely defined by two of its points, say x and y, we write $L = xy$ and call it the line **joining** x and y. If L, M have a unique point p incident with both of them, we write $L \cap M = p$ or $L \cap M = \{p\}$, calling p the **intersection point** of L and M.

1.2 Definition of generalized polygon

An **ordinary n-gon** is a connected geometry with n points and n lines, such that any point is incident with exactly two lines, and any line is incident with exactly two points. An ordinary 2-gon has two points incident with each of the two lines. Instead of 3-gon, 4-gon, 5-gon, 6-gon, 7-gon or n-gon for some n, we also use the expressions triangle, quadrangle, pentagon, hexagon, heptagon respectively polygon.

> A **generalized n-gon Γ of order** (s,t) is an incidence structure of points and lines with $s+1$ points incident with a line and $t+1$ lines incident with a point, $s,t \geq 1$, such that Γ has no ordinary k-gons for any $2 \leq k < n$, and any two elements are contained in some ordinary $n-$gon.

The dual of a generalized n-gon is also a generalized n-gon. So all definitions and results that hold for points, can be reformulated for lines — and dually. This is called the **duality principle**.

The integers s and t are also called the **parameters** of the generalized polygon. If $s=t$, we also say that Γ has order s instead of order (s,s).
A generalized n-gon is called **thin** if $s=1$ or $t=1$, and is called **thick** if $s,t>1$. A thin n-gon can always be regarded as the (dual of) the double of a generalized $\frac{n}{2}$-gon. A **subpolygon** Δ of order (s',t') of a generalized n-gon Γ of order (s,t) is a sub-geometry of Γ which is itself a generalized n-gon. If $\Delta \neq \Gamma$, Δ is called a **proper** subpolygon. If $s'=s$, Δ is called **full**. If $t'=t$, Δ is called **ideal**. If $s'=t'=1$, Δ is an ordinary n-gon and is called an **apartment** of Γ.

Let Γ be a generalized n-gon of order (s,t). As any two elements of a generalized n-gon are inside some ordinary n-gon, n is the maximal distance between two elements. If two elements are at maximal distance n, we call them **opposite**.
If two elements u,v are opposite, the set $\Gamma_i(u)\cap\Gamma_{n-i}(v)=u^v_{[i]}$ is called the **distance-i-trace** of v with respect to u. For $i=2$, it is convenient to call the distance-2-trace $u^v_{[2]}$ simply a **trace** (with respect to u), and denote it by u^v.
If two elements u,v are at distance $k<n$, the unique element in $\Gamma_1(u)\cap\Gamma_{k-1}(v)=\mathrm{proj}_u v$ is called the **projection of v onto u**. We say that v **projects onto** u in $\mathrm{proj}_u v$. If two points are at distance 4 and $n>4$, the unique point in $\Gamma_2(x)\cap\Gamma_2(y)$ is denoted by $x \bowtie y$.

A pair of opposite elements (u,v) is **distance-i-regular** if $|u^v_{[i]} \cap u^{v'}_{[i]}| \geq 2$ implies $u^v_{[i]} = u^{v'}_{[i]}$ for every element v' opposite u, and if $|v^u_{[i]} \cap v^{u'}_{[i]}| \geq 2$ implies $v^u_{[i]} = v^{u'}_{[i]}$ for every element u' opposite v. A pair of elements (u,v) is **regular** provided (u,v) is distance-i-regular for all $2 \leq i \leq \frac{n}{2}$. An element u is **distance-i-regular** if distinct distance-i-traces $u^v_{[i]}$ have at most 1 element in common. An element u is said to be **regular** if u is regular for all $2 \leq i \leq \frac{n}{2}$. A generalized polygon Γ is said to be **point-**(distance-i-)**regular** respectively **line-**(distance-i-)**regular** if all points respectively lines of Γ are (distance-i-)regular.

A point x is said to be **span-regular**, if x is distance-2-regular and for all points p, a, b with $d(x,p)=2$, $d(p,a)=n$, $d(p,b)=n$ and $x \in p^a \cap p^b$, we have $p^a = p^b$ whenever $|p^a \cap p^b| \geq 2$. One could give the following interpretation: x is span-regular if x is distance-2-regular and every point collinear with x 'behaves as a regular point in the neighbourhood of x'.

A point x is said to be **projective** if x is distance-2-regular and $x^y \cap x^z$ is never empty, for all points y and z opposite x. The **perp-geometry** in x (of Γ) is the geometry $\Gamma^\Delta_x = (\mathcal{P}^\Delta_x, \mathcal{L}^\Delta_x, \mathrm{I}^\Delta_x)$ with point set $\mathcal{P}^\Delta_x = x^\perp$, and with lines the ordinary lines through x together with the traces x^y, y opposite x. Incidence is defined as the natural one. If x is projective, the perp-geometry Γ^Δ_x is a projective plane (see Van Maldeghem [84] page 39).

1.2.1 Generalized triangles

In the thick case, this notion coincides with the notion of projective planes. Although we mainly deal with n-gons for $n > 3$ in this thesis, generalized triangles will turn up at some places, as they appear as derived geometries of generalized n-gons for $n > 3$ (see for instance the definition of perp-geometries above, and the definition of $\Gamma(x,y)$ at page 6).

1.2.2 Generalized quadrangles

For generalized quadrangles or GQs, we have the following equivalent definition.

> A **generalized quadrangle Γ of order** (s,t), with $s,t \geq 1$, is an incidence structure of points and lines with $s+1$ points incident with a line and $t+1$ lines incident with a point, such that for every non-incident point-line pair (p,L) there is exactly one incident point-line

pair (M,q) such that $p \mathrm{I} M \mathrm{I} q \mathrm{I} L$.

If $t=1$, Γ is called a **grid**, and if $s=1$, Γ is said to be a **dual grid**. More general, a $k\times l$**-grid** is the geometry consisting of a set of mutually opposite lines $\{L_1,\ldots,L_k\}$, and a set of mutually opposite lines $\{M_1,\ldots,M_l\}$ such that every line L_i intersects every line M_j, $i=1\ldots,k$; $j=1\ldots,l$, together with their kl intersection points.
For generalized quadrangles, the set $\Gamma_2(u)\cap\Gamma_2(v)$ (u and v both points or both lines), is frequently denoted by $\{u,v\}^\perp$. If u and v are opposite, this set coincides with the **trace** $u^v=v^u$. Of course, the notion of **regular** and **distance-2-regular** also coincide. So let x be a regular point, then two traces x^y and x^z have $0,1$ or $t+1$ points in common. If L is a regular line, the intersection number of two traces L^M,L^N is $0,1$ or $s+1$.
The set of all elements at distance 2 of all elements of $\{u,v\}^\perp$ is denoted by $\{u,v\}^{\perp\perp}$. If x and y are opposite points, this set is called the **hyperbolic line** $\langle x,y\rangle$ (see also the definition in the case of the hexagons).
A **triad** is a set of three points at mutual distance 4. A **center** of a triad is an element at distance 2 of each point of the triad. A triad T is **centric**, **unicentric** or **acentric** according as T has at least one, exactly one or no centers. A triad in a generalized quadrangle of order (q,q^2), q finite and $\neq 1$, has exactly $q+1$ centers (see theorem 1.3). Such a triad $\{x,y,z\}$ is called **3-regular** if the set of points collinear with all centers of the triad (i.e. $\{x,y,z\}^{\perp\perp}$), has size $q+1$. A point x is 3-regular if each triad containing x is 3-regular. Dual notions hold for a triad of lines.
A **window** is a centric triad of lines, together with two centers and the six intersection points defined by these five lines.
A **geometric hyperplane** $\mathcal{A}$ of Γ is a set of points such that every line intersects $\mathcal{A}$ in exactly 1 or $s+1$ points. One can easily show that $\mathcal{A}$ is an ovoid (see page 14; $\forall L:|L\cap\mathcal{A}|=1$), the point set of a subquadrangle of order (s,t'), $st'=t$ (i.e. a grid if $s=t$), or the set of all points collinear with a given point.

1.2.3 Generalized hexagons

Let p,q be opposite elements of a generalized hexagon (or GH) Γ, and $x,y\in p^q$, $x\neq y$. We use the equivalent notation $p^q=(x\bowtie y)^q=\langle x,y\rangle_q$ for the trace. The **hyperbolic line** $\langle x,y\rangle$ through x and y is defined as the intersection of all traces $\langle x,y\rangle_q$, with q opposite p. The point p is called the **focus** of the hyperbolic line. For finite generalized n-gons, it is obvious that $|\langle x,y\rangle|\leq t+1$, but it also follows that $|\langle x,y\rangle|\leq s+1$ (see van Bon, Cuypers and Van Maldeghem [83]). An **ideal line** is a hyperbolic line of

length $t+1$, and it is obvious that a point p is distance-2-regular if and only if all hyperbolic lines with focus p are ideal. A generalized hexagon is said to have ideal lines if all hyperbolic lines are ideal.
Let x be a span-regular point of the generalized hexagon Γ of order (s,t). Then there is a unique thin ideal subhexagon of Γ containing x and a given point y opposite x. This subhexagon, denoted by $\Gamma(x,y)$ as in [84], is the double of a generalized triangle (being a projective plane if $t>1$). If we put $\Gamma^+(x,y)= \Gamma_0(x)\cap\Gamma_4(x)$ and $\Gamma^-(x,y)= \Gamma_2(x)\cap\Gamma_6(x)$, then the elements of $\Gamma^+(x,y)$ are identified with the points of the projective plane, and the elements of $\Gamma^-(x,y)$ can be viewed as the lines of the projective plane $(\Gamma^+(x,y),\Gamma^-(x,y),\sim)$. To make the comparison complete: each point p of $\Gamma^-(x,y)$ (or line of the projective plane) can be identified with the unique **ideal** line $\Gamma_2(p)\cap\Gamma(x,y)$. For that reason, $\Gamma(x,y)$ is also called an **ideal plane** (see Van Maldeghem and Bloemen [86]).

We say that a generalized hexagon Γ satisfies the **regulus condition** if all points (and hence all lines) are distance-3-regular. Or, equivalently: if given any pair of opposite lines L and M, and any three distinct points $x,y,z\in\Gamma_3(L)\cap\Gamma_3(M)$, then one has $\Gamma_3(x)\cap\Gamma_3(y)= \Gamma_3(y)\cap\Gamma_3(z)$. In this case we denote $\Gamma_3(x)\cap\Gamma_3(y)$ by $\mathcal{R}(L,M)$, and call it the **line regulus** defined by L and M. Dually, $\Gamma_3(L)\cap\Gamma_3(M)$ is the **point regulus** $\mathcal{R}(x,y)$ defined by x and y.
Remark: a (line) regulus $\mathcal{R}(L,M,N)$ in projective 3-space $\mathbf{PG}(3,q)$ through three mutually non-concurrent lines L,M,N is defined as the set of transversals of 3 distinct transversals of L,M and N.

1.3 Some restrictions for finite generalized n-gons

A generalized n-gon is said to be **finite** if s and t are finite. The following theorem collects a number of results found in Feit and Higman [20], Higman [35], Haemers and Roos [30] and (e.g.) Dembowski [18].

Theorem 1.1 *Let Γ be a finite generalized n-gon of order (s,t), $s>1,t>1$, with $n\geq 2$. If Γ is finite, then one of the following holds (with $|\mathcal{P}|=v$, $|\mathcal{L}|=b$):*

- $n=2$ *with* $b=t+1$, $v=s+1$;
- $n=3$ *and* $s=t$ *with* $v=b=s^2+s+1$; Γ *is a projective plane;*
- $n=4$ *and* $\frac{st(1+st)}{s+t}$ *is an integer;* $s\leq t^2$ *and* $t\leq s^2$;

- *$n = 6$ and st is a square; $s \leq t^3$ and $t \leq s^3$;*
- *$n = 8$ and $2st$ is a square, in particular $s \neq t$; $s \leq t^2$ and $t \leq s^2$.*

If n is even,

$$v = (1+s)(1+st+(st)^2+\ldots+(st)^{\frac{n}{2}-1}),$$
$$b = (1+t)(1+st+(st)^2+\ldots+(st)^{\frac{n}{2}-1}).$$

Next result (as well as many others) can be found in Thas [61], [64], [66], [68] and Van Maldeghem [84] 1.8.8.

Theorem 1.2 *Let Γ' be a proper ideal sub-n-gon of order (s', t) of a finite thick generalized n-gon Γ of order (s, t). Then one of the following cases occurs.*

- *$n = 4$ and $s \geq s't$ and $s \geq t \geq s'$;*
- *$n = 6$ and $s \geq {s'}^2 t$ and $s \geq t \geq s'$;*
- *$n = 8$ and $s \geq {s'}^2 t$.*

For generalized quadrangles, we have the following results (see respectively Bose and Shrikhande [4], Payne and Thas [48] 1.3.6(i) and Thas [63]).

Theorem 1.3 *Let Γ be a thick finite GQ of order (s, t). Then $s^2 = t \Leftrightarrow$ each triad of points has a constant number of centers, in which case this number is $s + 1$.*

Theorem 1.4 *Let Γ be a thick finite GQ of order (s, t). If Γ has a regular (pair of) point(s), then $s \geq t$.*

Theorem 1.5 *Let Γ be a thick finite GQ of order (s, t). If Γ has a regular point x and a regular pair (L_0, L_1) of non-concurrent lines for which x is incident with no line of $\{L_0, L_1\}^\perp$, then $s = t$ is even.*

For generalized hexagons, theorem 1.4 becomes (see Van Maldeghem [84] 1.9.5)

Theorem 1.6 *Let Γ be a thick finite GH of order (s, t). If Γ has a distance-2-regular point p, then $s \geq t$. Moreover, this distance-2-regular point p is projective if and only if $s = t$.*

1.4 Classical polygons

1.4.1 Classical generalized triangles

In the thick case, these are the Desarguesian planes.

1.4.2 Classical generalized quadrangles

We follow the definition of 'classical' as used in [84], and based on Bruhat-Tits [11]. For (details on) the proofs of the stated anti-isomorphisms (or dualities), we refer to Payne and Thas [48] 3.2.1 and Thas and Payne [75] for the finite case, and Van Maldeghem [84] 9.6.4 for the infinite examples.

The **classical generalized quadrangles** are the ones arising from pseudo-quadratic forms of Witt index 2 in d-dimensional projective space $\mathbf{PG}(d, \mathbb{K})$ over a (not necessarily commutative) field $\mathbb{K}$, and their duals. We will mention only those who will be of use later on.

In the <u>finite</u> thick case, i.e. if $\mathbb{K}$ is a Galois field $\mathbf{GF}(q)$ for some q; there are five classes of classical quadrangles, or three classes 'up to duality'.

- $H(4, q^2)$ is the Hermitian quadrangle arising from a non-singular Hermitian variety in $\mathbf{PG}(4, q^2)$; the order is (q^2, q^3);

- $H(3, q^2)$ is the Hermitian quadrangle arising from a non-singular Hermitian variety in $\mathbf{PG}(3, q^2)$; the order is (q^2, q);

- $Q(5, q)$ is the orthogonal[1] quadrangle arising from a non-singular elliptic quadric in $\mathbf{PG}(5, q)$; the order is (q, q^2); all lines are regular and all points are 3-regular;

- $Q(4, q)$ is the orthogonal quadrangle arising from a non-singular (parabolic) quadric in $\mathbf{PG}(4, q)$; the order is (q, q); all lines are regular;

- $W(q)$ is the symplectic quadrangle arising from a non-singular symplectic polarity in $\mathbf{PG}(3, q)$; the order is (q, q).

Dualities are as follows.

[1] The term 'orthogonal' is taken from the odd case and extended to the even case, although the polarity giving rise to the quadric $Q(5, 2^h)$ respectively $Q(4, 2^h)$ is not orthogonal.

$$\begin{aligned} H(3,q^2) &\overset{D}{\cong} Q(5,q) \\ W(q) &\overset{D}{\cong} Q(4,q) \\ W(q) &\overset{D}{\cong} W(q) \Leftrightarrow q \text{ is even} \end{aligned}$$

For <u>infinite</u> quadrangles however, polarities and quadrics of Witt index 2 are not restricted to dimension ≤ 5, so there are more families of infinite classical quadrangles than the infinite versions of the five classes mentioned above. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ for example, we have one of the following cases (if not unique, the anti-automorphism σ is specified).

	symplectic	orthogonal	Hermitian
$\mathbb{R}$	$W(\mathbb{R}), d = 3$	$Q(d, \mathbb{R}), d \geq 4$	
$\mathbb{C}$	$W(\mathbb{C}), d = 3$	$Q(4, \mathbb{C})$	$H(d, \mathbb{C}), d \geq 3$
$\mathbb{H}$			$H(d, \mathbb{H}, \mathbb{R}), d \geq 3; \sigma_1$[2]
			$H(d, \mathbb{H}, \mathbb{C}), d = 3, 4; \sigma_2$[3]

As $\mathbb{C}$ is algebraically closed, a quadric of Witt index 2 only exists in projective dimension $d = 3$ or $d = 4$. But the case $d = 3$ corresponds to a thin quadrangle, so we omitted this one.

Dualities are as follows.

$$\begin{aligned} H(3,\mathbb{H},\mathbb{C}) &\overset{D}{\cong} Q(7,\mathbb{R}) \\ H(3,\mathbb{C}) &\overset{D}{\cong} Q(5,\mathbb{R}) \\ W(\mathbb{R}) &\overset{D}{\cong} Q(4,\mathbb{R}) \\ W(\mathbb{C}) &\overset{D}{\cong} Q(4,\mathbb{C}) \end{aligned}$$

Notations for generalized quadrangles over other fields are similar; we refer to [84] and page 49.

1.4.3 Classical generalized hexagons

In analogy to the definition of some classical generalized quadrangles, i.e. consisting of all absolute points and absolute lines of a certain polarity (of appropriate witt index) in projective space, the **classical generalized hexagons** are defined as the geometries consisting of all absolute points

[2] $\sigma_1 : a + ib + jc + kd \mapsto a - ib - jc - kd$
[3] $\sigma_2 : a + ib + jc + kd \mapsto a - ib + jc + kd$

and absolute lines of a certain triality τ_σ (with σ a field automorphism) of the polar space $Q^+(7,\mathbb{K})$, and their dual geometries. For an explicit description, we refer to Van Maldeghem [84]. If $\sigma \neq \mathbf{1}$, we get the **twisted triality hexagon** $T(\mathbb{K},\mathbb{K}^{(\sigma)},\sigma)$. In the finite case (i.e. $\mathbb{K} = \mathbf{GF}(q)$), we use the notation $T(q^3,q)$, which indicates the order immediately; the notation $T(q,q^3)$ is used for the dual. If $\sigma = \mathbf{1}$, the classical hexagon is denoted by $H(\mathbb{K})$ or $H(q) = H(\mathbf{GF}(q))$, and is called the **split Cayley hexagon**. As $H(\mathbb{K})$ lies entirely in a hyperplane of $\mathbf{PG}(7,\mathbb{K})$, it can be embedded in the quadric $Q(6,\mathbb{K})$. The order of $H(q)$ is (q,q).

Explicit description of $H(q)$

Let $x(x_0,\ldots,x_6)$ and $y(y_0,\ldots,y_6)$ be two points of $\mathbf{PG}(6,\mathbb{K})$. The **Grassmann coordinates** of the line $L = xy$ of $\mathbf{PG}(6,\mathbb{K})$ are $(p_{01},p_{02},\ldots,p_{56})$ where

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$$

Let $Q(6,\mathbb{K})$ be the non-singular quadric of $\mathbf{PG}(6,\mathbb{K})$ with equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2.$$

Then it is shown in Tits [78] that $H(\mathbb{K})$ is isomorphic to the incidence structure formed by all points of $Q(6,\mathbb{K})$ and those lines on $Q(6,\mathbb{K})$ whose Grassmann coordinates satisfy the following six linear equations:

$$\begin{array}{lll} p_{12} = p_{34}, & p_{20} = p_{35}, & p_{01} = p_{36}, \\ p_{03} = p_{56}, & p_{13} = p_{64}, & p_{23} = p_{45}. \end{array}$$

The lines on the quadric $Q(6,\mathbb{K})$ which do not belong to the hexagon $H(\mathbb{K})$, are the ideal lines of $H(\mathbb{K})$. So $\delta(x,y) \leq 4 \Leftrightarrow$ the line of $\mathbf{PG}(6,\mathbb{K})$ through x and y is a line of $Q(6,\mathbb{K})$. Also, there are two kinds of planes in $Q(6,\mathbb{K})$ with respect to $H(\mathbb{K})$. The first kind is the union of $q+1$ concurrent 'ordinary' lines of the hexagon. The second kind of plane consists of q^2+q+1 ideal lines of the hexagon; it is an ideal plane of the hexagon, as defined on page 6. For each point p of the hexagon, there is exactly one plane of the first kind for which p is the intersection point of the $q+1$ lines of the hexagon.
Following duality is proved by Tits [79], by De Smet and Van Maldeghem [16] using coordinates, and by Salzberg [55]:

$$H(q) \overset{D}{\cong} H(q) \Leftrightarrow q = 3^h, h \geq 1.$$

1.5 Moufang condition

A **collineation** or **isomorphism** θ of a geometry $\Delta = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ onto a geometry $\Delta' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a bijection of $\mathcal{P}$ onto $\mathcal{P}'$, inducing a bijection of $\mathcal{L}$ onto $\mathcal{L}'$, so that incidence is preserved. A collineation of Δ is a collineation of Δ onto itself, also called an **automorphism**. Anti-isomorphisms (or correlations), anti-automorphisms, involutions and polarities of generalized polygons are also defined in the usual way.
A collineation g is said to **fix** a line L **pointwise** (or elementwise), if it fixes all points of $\Delta(L)$. If $L^g = L$ but not necessarily every point $x \in L$ is fixed under g, g is said to **stabilize** L, or to **fix** L **setwise**. In both cases however, L is said to be a **fixline** of g.

Let Γ be a generalized n-gon. A **(root)elation** or γ**-elation** of Γ is a collineation fixing setwise all elements incident with at least one element of a given path γ of length $n-2$. Let Γ be a generalized $2m$-gon. If the first and last element of the path are points, we call g a **point-elation**; in the other case a **line-elation**. A (generalized) **homology** of a generalized n-gon Γ is a collineation fixing every element incident with v or w, with v, w opposite elements of Γ.

Let $\gamma = (v_0, v_1, \ldots, v_{n-2})$ be a path of length $n-2$; so we can also use the notation $(v_0, v_1, \ldots, v_{n-2})$-elation instead of γ-elation. If the group of all γ-elations G acts transitively on the set $\Gamma(w) \setminus \{v_0\}$, with $w \mathrm{~I~} v_0$, $w \neq v_1$, the path γ is said to be a **Moufang-path** and the polygon Γ is said to be γ-transitive. A generalized n-gon satisfies the Moufang condition (or is said to be a **Moufang polygon**) if all paths of length $n-2$ are Moufang paths, or equivalently: if Γ is γ-transitive for all paths of length $n-2$.
If, for n even, all paths $(p_0, L_1, \ldots, L_{n-3}, p_{n-2})$ with $p_i \in \mathcal{P}$ and $L_j \in \mathcal{L}$ are Moufang paths, the generalized n-gon is said to be **half Moufang**.
If Γ is a Moufang polygon, then the collineation group generated by all elations is often called the **little projective group** of Γ. For any (fixed) path γ of length $n-2$ in a Moufang n-gon, the group of all γ-elations is called a **root group**.

1.6 Characterizations

In this paragraph, we assume all generalized quadrangles and hexagons to be thick.

1.6.1 Characterizations of classical quadrangles

The first part of theorem 1.8 is discovered independently by several authors (see [48] p 77), while the infinite version (i.e. theorem 1.7) is stated by Van Maldeghem in [84] 6.2.1, generalizing a result of Schroth in [58]. For more details on theorem 1.8b, we refer to Payne and Thas [48] 1.3.6(iv), 5.2.5 and 5.2.6. Theorem 1.9 and 1.10 can be found in Thas [67] (or [48]), while theorem 1.11 was proved by several authors (see [48] p 90,122–149 for more information).
The classification theorem about finite Moufang quadrangles follows from Fong and Seitz ([22],[23]) and others (see [84] p 229), while theorem 1.14 can be found in Kramer [43] and Grundh'127 ofer and Knarr [28]. The full classification of Moufang quadrangles is done in Tits and Weiss [82], and we refer to [84] p 220 for a table of the results. Part of this table is recited on page 56.

Theorem 1.7 *A GQ is isomorphic to* $W(\mathbb{K})$ *for some commutative field* $\mathbb{K}$ $\Leftrightarrow$ *all points are regular and there exists one projective point.*

Theorem 1.8 *A finite GQ of order* (s,s) *is isomorphic to* $W(s)$ $\Leftrightarrow$ *all points are regular* $\Leftrightarrow$ *all points of a geometric hyperplane are regular.*

Theorem 1.9 *A finite GQ of order* (s,s^2) *is isomorphic to* $Q(5,s)$ $\Leftrightarrow$ *each point is* 3*-regular.*

Theorem 1.10 *A finite GQ of order* (s,t) *is isomorphic to* $Q(5,s)$ $\Leftrightarrow$ *each dual window is contained in a proper subquadrangle of order* (s,t')*.*

Theorem 1.11 *Up to isomorphism, there is only one quadrangle of order* $(2,2)$*,* $(2,4)$*,* $(4,4)$*,* $(3,5)$ *respectively* $(3,9)$*. Up to duality, there is only one quadrangle of order* $(3,3)$*.*

Theorem 1.13 *A finite GQ is Moufang* $\Leftrightarrow$ *it is classical or dual classical.*

Theorem 1.14 *A compact connected GQ is Moufang* $\Leftrightarrow$ *either it is classical over* $\mathbb{R},\mathbb{C}$ *or* $\mathbb{H}$ *or it is the exceptional GQ* $Q(E_6,\mathbb{R})$*.*

Theorem 1.15 *A GQ is Moufang* $\Leftrightarrow$ *it is listed in Tits and Weiss [82].*

1.6.2 Characterizations of classical hexagons

Here we list the analogous theorems of previous paragraph for hexagons. We adjust the theorem numbering to make comparison easier.

The classification theorem (i.e. theorem 1.23) for finite Moufang hexagons follows again from Fong and Seitz ([22],[23]). Theorems 1.19and 1.22 were proved by Ronan ([54],[53]), implying theorem 1.18. A variation is given in theorem 1.17, found by Van Maldeghem in [84] 6.3.1. Remark that a generalized hexagon is point-distance-2-regular if and only if it has ideal lines (see page 5). Ronan uses the latter terminology.
Theorem 1.20 comes from De Smet and Van Maldeghem [17], and theorem 1.21 is proved by Cohen and Tits [13]. For a characterization of the dual of $H(q)$, $q \neq 3^h$, proved by Govaert in [26], we refer to page 86. For a note related to the dual theorems of numbers 1.10 and 1.20, we refer to appendix B.

Theorem 1.17 *A GH is isomorphic to $H(\mathbb{K})$ for some commutative field $\mathbb{K}$ $\Leftrightarrow$ all points are distance-2-regular and there exists one projective point.*

Theorem 1.18 *A finite GH of order (q,q) is isomorphic to $H(q)$ $\Leftrightarrow$ all points are distance-2-regular.*

Theorem 1.19 *A finite GH of order (q,q^3) respectively (q^3,q) is isomorphic to $T(q,q^3)$ respectively $T(q^3,q)$ $\Leftrightarrow$ each point (and hence each line) is distance-3-regular.*

Theorem 1.20 *A finite GH of order (s,t) is isomorphic to $T(q,q^3)$ $\Leftrightarrow$ each dual window is contained in a proper subhexagon of order (s,t').*

Theorem 1.21 *Up to isomorphism, there is only one hexagon of order $(2,2)$ respectively $(2,8)$.*

Theorem 1.22 *If Γ is a point-distance-2-regular generalized hexagon, Γ is point-distance-3-regular and Γ is Moufang.*
If Γ is Moufang, either all points or all lines of Γ are regular.

Theorem 1.23 *A finite GH is Moufang $\Leftrightarrow$ it is classical or dual classical.*

Theorem 1.25 *A GH is Moufang $\Leftrightarrow$ it is listed in Tits and Weiss [82].*

1.7 Ovoids and spreads

An **ovoid of the projective space $\mathbf{PG}(3,q)$**, $q > 2$, is a set of q^2+1 points of $\mathbf{PG}(3,q)$, no 3 of which are collinear. An **ovoid of $\mathbf{PG}(3,2)$** is a set of 5 points no four of which are coplanar.
An **ovoid of a polar space** is a set of points such that every generator (i.e. maximal totally isotropic subspace) of the polar space is incident with

exactly one point of that set.
An **ovoid of a generalized n-gon** is a set of mutually opposite points (hence $n = 2m$) such that every element of Γ is at distance at most m from at least one point of $\mathcal{O}$.

A **(line) spread of the projective space $\mathbf{PG}(3,q)$** is a set of q^2+1 mutually non-concurrent lines which necessarily partition the point set of $\mathbf{PG}(3,q)$. A spread of $\mathbf{PG}(3,q)$ is **regular** if, when L, M, N are in the spread, then the whole regulus $\mathcal{R}(L,M,N)$ is contained in the spread.
A **spread of a** line-distance-3-regular **generalized hexagon** is **Hermitian** if, when L, M are in the spread, then the whole regulus $\mathcal{R}(L,M)$ is contained in the spread. A spread of a line-distance-3-regular generalized hexagon is locally Hermitian if there exists a line L in the spread such that, when M is in the spread and $L \neq M$, then the whole regulus $\mathcal{R}(L,M)$ is in the spread.

1.7.1 Ovoids and spreads in quadrangles

Reformulating the general definition, an **ovoid of a generalized quadrangle** is a set of points such that each line of Γ is incident with a unique point of $\mathcal{O}$. A spread $\mathcal{R}$ is a set of lines such that each point of Γ is incident with a unique line of $\mathcal{R}$. It follows that $|\mathcal{O}| = |\mathcal{R}| = 1 + st$, and any set of $1 + st$ mutually opposite points is an ovoid.
For more details on the following collection of results, we refer to Payne and Thas [48] and Thas [72].

Theorem 1.26 *Let Γ be a generalized quadrangle of order (s,s). If Γ admits a polarity σ, then either $s = 1$ (and hence Γ is thin) or $2s$ is a square. Moreover, the set of all absolute points (respectively lines) of σ is an ovoid (respectively spread) of Γ.*

Theorem 1.27

- *A GQ of order (s, s^2) has no ovoids.*
- *For q even, an ovoid of $W(q)$ is an ovoid of $\mathbf{PG}(3,q)$ and vice versa.*
- *$W(q)$ has ovoids $\Leftrightarrow$ q is even.*
- *$W(q)$ always has a spread.*
- *$Q(5,q)$ has spreads but no ovoids.*
- *$H(4,q^2)$ has no ovoid. For $q = 2$ it has no spread either, but for $q > 2$ existence of a spread is still an open problem.*

1.7.2 Ovoids and spreads in hexagons

An **ovoid $\mathcal{O}$ of a generalized hexagon** Γ is a set of points of Γ at mutual distance 6, such that each point of Γ is collinear with a unique point of $\mathcal{O}$. It follows that $|\mathcal{O}| = \frac{(1+s)(1+st+s^2t^2)}{1+s+st}$, and any set of that many opposite points is an ovoid. This immediately shows the first assertion of theorem 1.29 (see Van Maldeghem [84] 7.2.4). The last assertion of theorem 1.29 is proved on page 16, where construction methods (6A) and (6A′) guarantee the existence of a spread of any $H(q)$. For more information about the various results, we refer to Thas [72] and O'Keefe and Thas [45]; theorem 1.28 can be found in Ott [46] and in Cameron, Thas and Payne [12].

Theorem 1.28 *Let Γ be a generalized hexagon of order (s, s). If Γ admits a polarity σ, then either $s = 1$ (and hence Γ is thin) or $3s$ is a square. Moreover, the set of all absolute points (respectively lines) of σ is an ovoid (respectively spread) of Γ.*

Theorem 1.29

- *A GH of order (s, s^3) or (s^3, s) has no ovoids.*
- *An ovoid of $H(q)$ is an ovoid of $Q(6, q)$ and vice versa.*
- *$H(q)$ has an ovoid for $q = 3^h$, but no ovoid for q even or $q = 5$ or $q = 7$; the other cases are not yet decided.*
- *$H(q)$ always has a spread.*

Remark the similarities between the results for $W(q)$ and $H(q)$. In the next section, we give all known examples of spreads in $W(q)$ and $H(q)$.

1.7.3 Ovoids in $Q(4, q)$ and spreads in $H(q)$

There are three general construction methods for ovoids in finite generalized quadrangles — with sometimes infinite generalizations possible. Ovoids constructed with method (4A′) can also be constructed via method (4A), but the converse is not always true.

(4A) Let Γ be a GQ of order (s, t'), embedded as a full subGQ in a GQ $\overline{\Gamma}$ of order (s, t). Let p be a point of $\overline{\Gamma} \setminus \Gamma$. Then the set of points of Γ which are collinear with p form an ovoid of Γ. (See Payne and Thas [48] 2.2.1.)

(4A′) Let Γ be an orthogonal or Hermitian Moufang quadrangle embedded in some $\mathbf{PG}(d, \mathbb{K})$. Let Π be a hyperplane of $\mathbf{PG}(d, \mathbb{K})$ not containing lines of Γ. Then the points of Γ in Π form an ovoid of the quadrangle. (See e.g. Van Maldeghem [84] 7.3.11.)

(4B) Let Γ be a GQ of order (s, s). If Γ admits a polarity, then the set of all absolute points is an ovoid of Γ. (See theorem 1.26 p 14.)

An ovoid constructed in the way of (4A) is said to be **subtended by** p. Tits [80] showed that $Q(4, q)$ admits a polarity if and only if $q = 2^{2h+1}$.

For $Q(4, q)$, the only examples of ovoids known are listed below. (The notation (4$*$) means that another construction method is used than the ones listed above.) For q even, it is conjectured that there are no other examples, and for $q = 3, 5$ and 7 it is already proved that the list is complete (see O'Keefe and Thas [45] for more references).

$q = 2^{2h}$	(4A):	classical ovoid (let $\overline{\Gamma} = Q(5, q)$)
$q = 2^{2h+1}$	(4A):	classical ovoid (let $\overline{\Gamma} = Q(5, q)$)
	(4B):	Suzuki-Tits ovoid
$q = p^h$,	(4A):	classical ovoid (let $\overline{\Gamma} = Q(5, q)$)
$p \neq 2$	(4A):	$q = 3^h, h \geq 3$; Thas-Payne ovoid [76] (let $\overline{\Gamma}$ be Roman GQ)
	(4A):	$q = p^h, h > 1$; Kantor type $\mathcal{K}_1$ [41] (let $\overline{\Gamma}$ be Kantor GQ)
	(4$*$):	$q = 3^{2h-1}, h \geq 2$; Kantor type $\mathcal{K}_2$ [41][4]
	(4A):	$q = 3^5$; Penttila-Williams ovoid [50]

An ovoid $\mathcal{O}$ of $Q(4, q)$ is **classical** (i.e. isomorphic to the elliptic quadric $Q^-(3, q)$) if and only if the corresponding spread of $W(q)$ is **regular** in $\mathbf{PG}(3, q)$ if and only if the corresponding translation plane is **Desarguesian**. An ovoid is the Suzuki-Tits ovoid if and only if the corresponding translation plane is a L‘127 uneburg plane. Kantor type $\mathcal{K}_1$ gives rise to Knuth semifield planes ([41]), while Kantor type $\mathcal{K}_2$ does not give rise to a semifield plane. The Penttila-Williams ovoid gives rise to a semifield plane. For details and proofs, see Thas [62], [76] and Tits [80].

Now we turn to the generalized hexagon $H(q)$. We discuss spreads instead of ovoids, as this will emphasize similarities between $H(q)$ and $Q(4, q)^D$. We have again some general construction methods, where (6A) and (6A$'$) give rise to isomorphic examples. Moreover, construction methods (4A) and (6A) are closely related ([85]).

(6A) Let Γ be the GH $H(q)$ of order (q, q), embedded as a subGH in $\overline{\Gamma} \cong H(q^2)$. So Γ is the substructure of fixed elements of $\overline{\Gamma}$ under the involution σ mapping every coordinate x to x^q. Let p, p' be two points of $\overline{\Gamma} \setminus \Gamma$ on lines of Γ, with $\delta(p^\sigma, p') = 4$. Then the intersection of the line set of Γ with the line set of $\overline{\Gamma}(p, p')$ (page 6) is a (Hermitian) spread of $H(q)$. (See Van Maldeghem [85] or [84] p 315.)

[4] Obtained by projecting a Ree-Tits ovoid.

(6A′) Let Γ be the GH $H(q)$ of order (q, q), embedded in $Q(6, q) \subset \mathbf{PG}(6, q)$. Let Π be a hyperplane of $\mathbf{PG}(6, q)$ meeting $Q(6, q)$ in an elliptic quadric $Q^-(5, q)$. Then the lines of Γ in Π form a spread of both the hexagon Γ and the quadrangle $Q(5, q)$. (See Thas [69].)

(6B) Let Γ be a GH of order (s, s). If Γ admits a polarity, then the set of all absolute lines is a spread of Γ. (See theorem 1.28 p 15.)

It is known that $H(q)$ admits a polarity if and only if $q = 3^{2h-1}$ (see page 10), while Thas [69] showed that a spread arising from a polarity (i.e. (6B)) is never isomorphic with a spread of type (6A′).

For $H(q)$, the known examples of spreads are listed below. Let α be an automorphism of the quadric $Q(6, q)$ but not of the hexagon $H(q)$. So α preserves the line set of $Q(6, q)$ but not necessarily the one of $H(q)$. By theorem 1.29, the image of an ovoid of the hexagon will again be an ovoid of the hexagon, but spreads of $H(q)$ need not be mapped to spreads of $H(q)$. So, for $q = 3^h$, by dualizing the image under α of the dual of the known spreads, new spreads arise. Not all of these are mentioned explicitely; only the class of the Roman spreads[5] is, as it is the unique class obtained by this construction, that is locally Hermitian but not Hermitian. The notation (6α) in following list denotes the use of this dualizing-process. Hermitian spreads are also called classical spreads, and in the following table, [H], [lH] respectively [nlH] means the spread is Hermitian, locally Hermitian, respectively not locally Hermitian.

$q = 3^{2h}; q = 3$	(6A):	classical spread	[H]
	(6α):	Roman spread	[lH]
	(6α):	others obtained from classical spread	[nlH]
$q = 3^{2h+1}; h \geq 1$	(6A):	classical spread	[H]
	(6α):	Roman spread	[lH]
	(6α):	others obtained from classical spread	[nlH]
	(6B):	dual of Ree-Tits ovoid [6]	[nlH]
	(6α):	dual of image under α of Ree-Tits ovoid	[nlH]
$q = p^h, p \neq 3$	(6A):	classical spread	[H]
	$(6*)$:	$q = 1$ mod 3; [7]([2])	[lH]

[5] These spreads ar given the name Roman as they give rise to the Thas-Payne ovoid of $Q(4, q)$, which on its turn is derived from the Roman quadrangles. See [2].

[6] For $q = 3$, these spreads coincide with the classical spreads ([69]).

[7] Obtained by modification of the coordinates of the classical spread.

1.8 Projectivities

Let Γ be a generalized n-gon and let u, w be two opposite elements of Γ. The mapping $[u; w] : \Gamma(u) \mapsto \Gamma(w)$ projecting every element of $\Gamma(u)$ onto w, is characterized by

$$[u; w](x) = y \quad \Leftrightarrow \quad \begin{cases} \delta(y, w) = 1 \\ \delta(x, y) = n - 2, \end{cases}$$

for $x \in \Gamma(u)$. We call $[u; w]$ the **perspectivity** from u to w. Let $\{w_0, w_1, \ldots, w_k\}$ be a set of elements such that $\delta(w_{i-1}, w_i) = n$ $(i = 1, 2, \ldots, k)$. The mapping

$$[w_0; w_k] := [w_0; w_1][w_1; w_2] \ldots [w_{k-1}; w_k] : \Gamma(w_0) \mapsto \Gamma(w_k)$$

is called a **projectivity** from w_0 to w_k. For $w_0 = w_k = w$, such a projectivity is a bijection of $\Gamma(w)$ onto itself, hence it is called a **self-projectivity**. The set of all self-projectivities of an element w of a generalized n-gon Γ clearly forms a group under composition, and we call it the **group of projectivities** of w. For x and y points, the groups of projectivities of x and y, viewed as permutation groups acting on $\Gamma(x)$ and $\Gamma(y)$, respectively, are isomorphic. Similarly for groups of projectivities of lines. We denote by $\Pi(\Gamma)$ the permutation group corresponding to the group of projectivities of any line of Γ, and call it **the general group of projectivities of** Γ. Dually, we denote by $\Pi^*(\Gamma)$ the permutation group corresponding to the group of projectivities of a point and call it **the general dual group of projectivities of** Γ. It turns out that for an element w of Γ, the set of self-projectivities which can be written as a composition of an even number of perspectivities forms a subgroup of index at most 2 of the full group of projectivities of w. Again, this is independent of w (but depending on the type of w, i.e., point or line) and we denote by $\Pi_+(\Gamma)$ the corresponding subgroup of $\Pi(\Gamma)$ (**the special group of projectivities of** Γ), and by $\Pi^*_+(\Gamma)$ the corresponding subgroup of $\Pi^*(\Gamma)$ (**the special dual group of projectivities of** Γ). The following theorem can be found in Knarr [42].

Theorem 1.30 *The special (dual) group of projectivities $\Pi_+^{(*)}(\Gamma)$ of a generalized polygon Γ is doubly transitive.*

1.9 Coordinatization of generalized polygons

1.9.1 Introduction of the coordinates

In this section, we recall some facts about the coordinatization of generalized quadrangles, as introduced in Hanssens and Van Maldeghem [32, 33].

Let Γ be a generalized quadrangle of order (s,t). Choose any point and label it (∞); choose any line through (∞) and label it $[\infty]$. Let R_1 and R_2 be sets of cardinalities s and t, respectively, containing the distinguished elements 0 and 1, but not containing ∞. As a general rule, we denote the coordinates of lines with square brackets; those of points by parentheses.

We complete the flag $\{(\infty),[\infty]\}$ to an apartment

$$\Sigma = (\infty)\ \mathrm{I}\ [\infty]\ \mathrm{I}\ (0)\ \mathrm{I}\ [0,0]\ \mathrm{I}\ (0,0,0)\ \mathrm{I}\ [0,0,0]\ \mathrm{I}\ (0,0)\ \mathrm{I}\ [0]\ \mathrm{I}\ (\infty).$$

We choose a set of coordinates $(a,0,0)$, $a\in R_1$, for the points x of $[0,0,0]$ distinct from $(0,0)$, with $(a,0,0)\mapsto x$ bijective (in conformity with the already defined coordinates $(0,0,0)$), and we do the same dually for the lines through $(0,0,0)$ (replacing R_1 by R_2). We label the projection of $(a,0,0)$ onto $[\infty]$ by (a) (similarly dually); we also label the projection onto $[0]$ of $\mathrm{proj}_{[1,0,0]}(a')$ by $(0,a')$ (similarly dually), and we label the projection of $(0,a')$ onto $[0,0]$ by $(0,0,a')$ (similarly dually). Furthermore, we label the projection of $(0,0,b)$ onto any line $[k]$, $k\in R_2$, by (k,b) (and dually), and we label the projection of $(0,a')$ onto the line $[a,l]$ by (a,l,a') (and dually). This way, every point and line has been given coordinates. We define the quaternary operations Φ_1 and Φ_2,

$$\begin{array}{rcrcl} \Phi_1 & : & R_1\times R_2\times R_1\times R_2 & \to & R_1, \\ \Phi_2 & : & R_1\times R_2\times R_1\times R_2 & \to & R_2, \end{array}$$

as follows:

$$\left\{\begin{array}{lcl} \Phi_1(a,k,b,k')=a' & \Leftrightarrow & \delta(\mathrm{proj}_{[k,b,k']}(a),(0,a'))=2 \\ \Phi_2(a,k,b,k')=l & \Leftrightarrow & \delta([a,l],[k,b,k'])=2. \end{array}\right.$$

Then clearly we have

$$(a,l,a')\ \mathrm{I}\ [a,l]\ \mathrm{I}\ (a)\ \mathrm{I}\ [\infty]\ \mathrm{I}\ (\infty)\ \mathrm{I}\ [k]\ \mathrm{I}\ (k,b)\ \mathrm{I}\ [k,b,k']$$

and

$$(a,l,a')\ \mathrm{I}\ [k,b,k'] \Leftrightarrow \left\{\begin{array}{rcl} \Phi_1(a,k,b,k') & = & a' \\ \Phi_2(a,k,b,k') & = & l. \end{array}\right.$$

We define the following binary operation $\oplus$ in R_1:

$$a\oplus b := \Phi_1(a,1,b,0).$$

We have the following properties (which are easy to verify):

$$\begin{array}{l} \Phi_1(a,0,b,k') = b = \Phi_1(0,k,b,0), \\ \Phi_2(a,0,0,k') = k' = \Phi_2(0,k,b,k'), \\ 0\oplus a = a = a\oplus 0. \end{array}$$

We call the quadruple $(R_1, R_2, \Phi_1, \Phi_2)$ a **coordinatizing ring for** Γ. The dual operators Ψ_1, Ψ_2 of the coordinatizing operators Φ_1, Φ_2 are defined by

$$\left.\begin{array}{l}\Psi_1(k,a,l,a') = k' \\ \Psi_2(k,a,l,a') = b\end{array}\right\} \Leftrightarrow (a,l,a')\ \mathrm{I}\ [k,b,k'].$$

We note that $[\infty]$ is a **regular line** if and only if Φ_2 is independent of its third argument. Indeed, assume $[\infty]$ is regular. The lines $[k]$ and $[0,k']$ intersect the lines $[\infty]$, $[k,b,k']$ and $[k,b^*,k']$, so any line through $(a) \in [\infty]$ intersecting $[k,b,k']$, also intersects $[k,b^*,k']$. This line has coordinates $[a, \Phi_2(a,k,b,k')] = [a, \Phi_2(a,k,b^*,k')]$, so Φ_2 is independent of its third coordinate. The proof of the converse is similar.

Also part of the **Moufang condition** can be translated into coordinates. Indeed, a quadrangle Γ is $((\infty),[\infty],(0))$-transitive if and only if

$$\left\{\begin{array}{l}\Phi_1(a,k,b\oplus B,k') = \Phi_1(a,k,b,k') \oplus B \\ \Phi_2(a,k,b\oplus B,k') = \Phi_2(a,0,B,\Phi_2(a,k,b,k')),\end{array}\right.$$

for all $a,b,B \in R_1$ and all $k,k' \in R_2$. In that case, the action of an $((\infty),[\infty],(0))$-elation on the points of the line $[0]$ is given by $(0,a) \mapsto (0, a\oplus B)$, for some (fixed) $B \in R_1$. (See [33].)

1.9.2 Mnemonic for coordinatization

Following mnemonic (including figure 1.1) of the above coordinatization method could be of use for easy labeling of points and lines at distance 3 of the flag $\{(\infty),[\infty]\}$ (which is defined as $\delta(x,\{(\infty),[\infty]\})= \min\{\delta(x,(\infty)),\delta(x,[\infty])\}$).

Suppose L is a line at distance 3 of $\{(\infty),[\infty]\}$, i.e. at distance 3 of (∞). Let L' be the projection of L onto (∞). We know that L has coordinates of the form $[Y,x,Y']$. We proceed in three steps, indicated by the three arrows in the left picture. $\boxed{1}$ First, project L' onto the point of Σ opposite (∞) (i.e. $(0,0,0)$). If the set of coordinates of this projection is $[k,0,0]$, the first coordinate of L is k. $\boxed{2}$ Secondly, project $L' \cap L$ onto the line $[0,0]$. If this point has coordinates $(0,0,b)$, the second coordinate of L is b. $\boxed{3}$ Finally, project L onto the point (0). If this projection has coordinates $[0,k']$, the third coordinate of L is k.
For points at distance 3 of the flag $\{(\infty),[\infty]\}$ (i.e. distance 3 of $[\infty]$), the same procedure holds. Let p' be the projection of p onto $[\infty]$. Then we

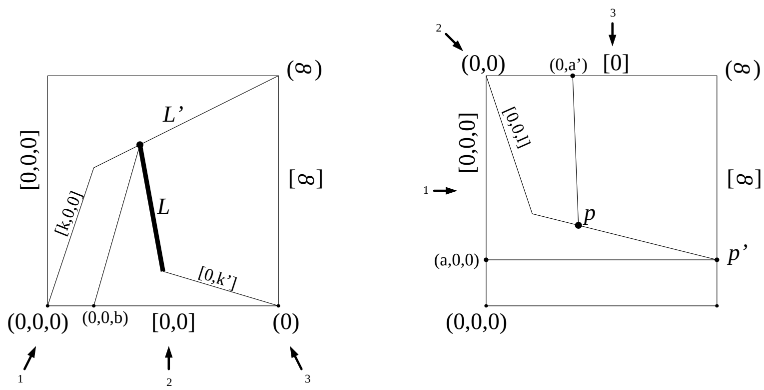


Figure 1.1: mnemonic

project the point p', the line $p'p$ and the point p respectively on the elements $[0,0,0]$, $(0,0)$ and $[0]$. The non-zero coordinates of these projections give the first, second respectively third coordinate of $p(a,l,a')$.

Coordinatization of generalized hexagons and octagons is done in exactly the same way (see Van Maldeghem [84]), ending up with 5 (respectively 7) coordinates for an element at distance 5 (respectively 7) of a fixed flag $\{(\infty),[\infty]\}$ of a generalized hexagon (respectively octagon). The same method applied to the case of generalized triangles, yields the usual coordinatization method of Hall ([31]).

1.9.3 Coordinatization of $Q(5,\mathbb{K})$

If the quadric $Q(5,\mathbb{K}) \subset \mathbf{PG}(5,\mathbb{K})$ is represented by the equation $X_0X_5 + X_1X_4 + X_2^2 - uX_3^2 = 0$, with u a non-square in $\mathbb{K}$, then the generalized quadrangle $Q(5,\mathbb{K})$ can be coordinatized as in table 1.1. The quaternary operations Φ_1, Φ_2 and Ψ_1, Ψ_2 are given by:

$$\left\{\begin{array}{lcl} \Phi_1(a,(k_1,k_2),b,(k_1',k_2')) & = & b + a(k_1^2 - uk_2^2) - 2(k_1k_1' - uk_2k_2') \\ \Phi_2(a,(k_1,k_2),b,(k_1',k_2')) & = & (k_1',k_2') - a(k_1,k_2), \end{array}\right.$$

$$\left\{\begin{array}{lcl} \Psi_1((k_1,k_2),a,(l_1,l_2),a') & = & (l_1,l_2) + a(k_1,k_2) \\ \Psi_2((k_1,k_2),a,(l_1,l_2),a') & = & a' + a(k_1^2 - uk_2^2) + 2l_1k_1 - 2ul_2k_2. \end{array}\right.$$

Coo$^{\mathrm{S}}$ in $Q(5,\mathbb{K})$	Representation in $\mathbf{PG}(5,\mathbb{K})$
(∞)	$(1,0,0,0,0,0)$
(a)	$(a,0,0,0,1,0)$
$((k_1,k_2),b)$	$(-b,1,k_1,k_2,-k_1^2+uk_2^2,0)$
$(a,(l_1,l_2),a')$	$(-l_1^2+ul_2^2+aa',-a,l_1,l_2,a',1)$
$[\infty]$	$\langle(1,0,0,0,0,0),(0,0,0,0,1,0)\rangle$
$[(k_1,k_2)]$	$\langle(1,0,0,0,0,0),(0,1,k_1,k_2,-k_1^2+uk_2^2,0)\rangle$
$[a,(l_1,l_2)]$	$\langle(a,0,0,0,1,0),(-l_1^2+ul_2^2,-a,l_1,l_2,0,1)\rangle$
$[(k_1,k_2),b,(k'_1,k'_2)]$	$\langle(-b,1,k_1,k_2,-k_1^2+uk_2^2,0),$ $(-{k'}_1^2+u{k'}_2^2,0,k'_1,k'_2,b-2(k_1k'_1-uk_2k'_2),1)\rangle$

Table 1.1: Coordinatization of $Q(5,\mathbb{K})$.

1.10 The Higman-Sims technique

Suppose A and B are square complex matrices of size n and m, respectively ($n \geq m$), having only real eigenvalues. Let $\lambda_n(A) \leq \ldots \leq \lambda_1(A)$ and $\lambda_m(B) \leq \ldots \leq \lambda_1(B)$ be the eigenvalues of A respectively B. If

$$\lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A)$$

for all $i = 1, \ldots, m$, then we say that the eigenvalues of B **interlace** the eigenvalues of A.
We recall a theorem of Haemers (see [29]), which is a generalization of a result of Sims.

Theorem 1.31 *Let A be a complete Hermitian $n \times n$-matrix partitioned into m^2 block matrices, so*

$$A = \begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix},$$

such that A_{ii} is square for $i = 1, \ldots, m$. Let b_{ij} be the average row sum of A_{ij}, for $i, j = 1, \ldots, m$. Define the $m \times m$ matrix $B = (b_{ij})$. Then the following properties hold.

1. *The eigenvalues of B interlace the eigenvalues of A.*

2. *If for some integer k, $0 \leq k \leq m$, $\lambda_i(A) = \lambda_i(B)$ for $i = 1, \ldots, k$, and $\lambda_i(B) = \lambda_{n-m+i}(A)$ for $i = k+1, \ldots, m$, then all the block-matrices of A have constant row and column sums.*

The weaker inequalities $\lambda_1(A) \geq \lambda_i(B) \geq \lambda_n(A)$ were already observed by Sims (see [34] p 144), and usually applied to the adjacency matrix of the incidence graph of certain geometries under the name Higman-Sims technique, to obtain bounds on the sizes of subgeometries. Using theorem 1.31, more results can be obtained, as well as interpretations for the case that the bounds are attained. We will aplly this theorem (under the name extended Higman-Sims technique) in chapter 5 in theorems 5.7, 5.8 and 5.10 to determine upper and lower bounds on the size of special point sets in generalized hexagons respectively quadrangles.

1.11 Some more geometries

A $t-(v,k,\lambda)$ **design** (or t-design) $\mathcal{D} = (X, \mathcal{B}, \mathrm{I})$ is an incidence structure with point set X and block set $\mathcal{B}$ such that $\mathcal{B}$ is a collection of subsets of X, with v points in total, k points in each block, and such that for each subset of t points, there are precisely λ blocks containing that subset. If there are b blocks in total, and r blocks through every point, then $vr = bk$ and $b\begin{pmatrix} k \\ t \end{pmatrix} = \lambda \begin{pmatrix} v \\ t \end{pmatrix}$. Hence, for a 2-design, $r(k-1) = \lambda(v-1)$ (see e.g. Brouwer, Cohen and Neumaier [9] p 438). A **symmetric** (or square) 2-design is a $2-(v,k,\lambda)$ design with just as many points as blocks, i.e. $b = v$ (whence $k = r$ and $k(k-1) = (v-1)\lambda$). When $\lambda = 2$, a symmetric 2-design is also called a **biplane**.

A **linear space** is a point-line geometry such that any line contains at least two points, and any two distinct points are incident with exactly one line. A (connected) linear space is non-degenerate if there exists a non-incident point-line pair.

Let π be a projective plane. An m**-secant** of a point set is a line intersecting the point set in m points. A **point set of type** (n,m) is a set of points of π such that every line is either an n-secant or an m-secant.
A $(k;n)$**-arc** $\mathcal{K}$ is a set of k points of π such that some line of the plane is an n-secant, but no line meets $\mathcal{K}$ in more than n points, where $n \geq 2$. A $(k;n)$-arc is called **maximal** if it is of type $(0,n)$. This is the case if and only if the $(k;n)$-arc has maximal size $(n-1)q+n$. (See e.g. Hirschfeld [38] p 303.)

A $(k;2)$-arc is also called a **k-arc**, so it is a set of k points of π, $k \geq 2$, such that no three points of this set are collinear. A k-arc is called **complete** if it is not contained in any $(k+1)$-arc. A $(q+1)$-arc for arbitrary q is also called an **oval**; for q even a $(q+2)$-arc is called a complete oval or a **hyperoval**. Remark that a $(q+2)$-arc for q odd does not exist. If π is Desarguesian of odd order, then, by a result of Segre, every oval of π is a **conic** (e.g. Hughes and Piper [39] p 250).
A **k-cap** in $\mathbf{PG}(n,q)$ is a set of k points, $k \geq 2$, no 3 of which are collinear (e.g. [38] p 89), so a k-cap in $\mathbf{PG}(2,q)$ is the same as a k-arc. (For more information about sizes of caps, see page 99.)

A **unital** is a $2-(q\sqrt{q}+1, \sqrt{q}+1, 1)$ design. An embedded unital or Hermitian arc is a $(q\sqrt{q}+1; \sqrt{q}+1)$-arc of type $(1, \sqrt{q}+1)$. A Hermitian curve is the canonical example of an Hermitian arc.
A **Baer subplane** is a subgeometry of a projective plane of order q which is itself a plane of order $\sqrt{q}$.

A **partial ovoid** of a generalized quadrangle is a point set which has at most one point in common with every line of the generalized quadrangle.
An **m-ovoid** of a generalized quadrangle is a set of points of the geometry such that each line of the geometry contains exactly m points of the m-ovoid. We always assume $1 \leq m \leq s$ for a generalized quadrangle of order (s,t). A 1-ovoid is an ovoid. If each line of the geometry has $s+1$ points, and $m = \frac{s+1}{2}$, then the m-ovoid is also called a **hemisystem**. In general, a hemisystem of a geometry (not necessarily of rank 2) is a point set of the geometry such that each line of the geometry has half its points inside the hemisystem, and half its points outside.
For more information on m-ovoids of partial geometries and generalized quadrangles, we refer to Thas [71]. In Thas [70], the dual notion of an m-ovoid is called a **regular system of order m**.

Chapter 2

A Characterization of $Q(5, q)$ using one Subquadrangle $Q(4, q)$

2.1 Introduction

Let Q be an elliptic quadric in the projective space $\mathbf{PG}(5, q)$. The incidence geometry consisting of the points and lines on Q is the finite orthogonal generalized quadrangle $Q(5, q)$ of order (q, q^2). If one intersects Q with a non-tangent hyperplane $\mathbf{PG}(4, q)$ of $\mathbf{PG}(5, q)$, the point-line structure on the resulting parabolic quadric is the finite orthogonal generalized quadrangle $Q(4, q)$ of order (q, q). Hence $Q(4, q)$ can be seen as a 'natural subquadrangle' of $Q(5, q)$. Now the converse question arises: if a generalized quadrangle Γ of order (q, q^2) (not known to be classical) has 'a lot of' proper classical subquadrangles Δ, is Γ on its turn classical? If one replaces the vague term 'a lot of' by asking that there is a subquadrangle through either every centric triad of lines or every dual window, the answer is 'yes' (see [48] or theorem 1.10 on page 12). But one can change the question a bit: what happens if one knows only about one classical subquadrangle Δ of Γ? What further conditions on this unique quadrangle Δ would make

the quadrangle Γ to be classical?
Let us first look at the classical $Q(5,q)$ and one of its subquadrangles isomorphic to $Q(4,q)$. The structure of $Q(5,q)$ induces some structure in $Q(4,q)$ as follows. Any point p of $Q(5,q) \setminus Q(4,q)$ subtends an ovoid $\mathcal{O}_p$ of Δ (see construction (4A) on page 15). This ovoid is classical (i.e. an elliptic quadric in three dimensions). Indeed, all points of Γ collinear with p are inside the tangent hyperplane Π_p of the quadric $Q(5,q)$ in p. The intersection of Π_p with the 4-dimensional space $\mathbf{PG}(4,q)$ that contains Δ, is a 3-dimensional space, containing the elliptic quadric mentioned.
So given a generalized quadrangle $\Gamma \cong Q(5,q)$ containing $\Delta \cong Q(4,q)$, then all ovoids of Δ subtended by points of $\Gamma \setminus \Delta$ are classical. Now we could turn the question the other way around: **if** all ovoids of $Q(4,q)$ subtended by points of a GQ of order (q,q^2) containing $Q(4,q)$ are classical, **is** Γ then automatically isomorphic to $Q(5,q)$**?** This would be a characterization of $Q(5,q)$ by only one classical subquadrangle (satisfying some extra conditions). And indeed, it *is* a characterization, as the answer on the question is positive — as stated in theorem 2.3. This theorem is nevertheless not new. For q even, this theorem was already stated in Thas and Payne [76]. For q odd, a proof using cohomology theory is given in Brown [10]. In this thesis however, we provide a purely geometrical proof, valid for any q. By doing so, we explain a step in the geometrical proof provided in [76], that was not elaborated in depth.
Theorem 2.2 gives a sufficient (and necessary) condition on the linear group acting on Δ, in order to satisfy the conditions of theorem 2.3. Here, we give two proofs. The shorter one makes use of the classification of the finite simple groups, while the longer version does not. Both theorems 2.2 and 2.3 are then combined in corollary 2.2. If we take Γ and Δ as above, and if we take conditions on triads instead of ovoids, we get (for the odd case), theorem 2.1 and corollary 2.1. They are the completion for the odd case of a theorem stated in Thas [65] (see [48] 5.3.12).
The results of this chapter are submitted for publication in *European Journal of Combinatorics*; see [7].
We will first state all theorems in the order in which the proofs must be read. As $Q(5,2)$ (respectively $Q(5,3)$) is the unique generalized quadrangle of order $(2,4)$ (respectively order $(3,9)$) (see theorem 1.11 p 12), we may assume that $q \geq 4$.

Let Δ be a subquadrangle of the generalized quadrangle Γ. A group G acting on Δ **extends to** Γ, if for all automorphisms $\alpha \in G$, there is at least one automorphism β acting on Γ such that the restriction of β to Δ is exactly α.

2.2 Collected results

Theorem 2.1 *Let Γ be a GQ of order (q, q^2) and let Δ be a subGQ of Γ of order (q, q) with the property that every triad $\{x, y, z\}$ of Δ is 3-regular in Γ and $\{x, y, z\}^{\perp\perp} \subset \Delta$. Then Δ is classical and,* **if q is odd**, *each subtended ovoid in Δ is classical.*

Theorem 2.2 *Let Γ be a GQ of order (q, q^2) and let Δ be a classical subGQ of Γ of order (q, q). Then $\Delta \cong Q(4, q)$. If the linear group G acting on Δ extends to Γ, then all subtended ovoids in Δ are classical.*

Theorem 2.3 *Let Γ be a GQ of order (q, q^2) and let Δ be a classical subGQ of Γ of order (q, q). If all subtended ovoids in Δ are classical, then Γ itself is classical (and hence isomorphic to $Q(5, q)$).*

Corollary 2.1 *Let Γ be a GQ of order (q, q^2) and let Δ be a subGQ of Γ of order (q, q) with the property that every triad $\{x, y, z\}$ of Δ is 3-regular in Γ and $\{x, y, z\}^{\perp\perp} \subset \Delta$.* **If q is odd,** *then Γ is classical.*

Corollary 2.2 *Let Γ be a GQ of order (q, q^2) and let Δ be a classical subGQ of Γ of order (q, q). If the linear group G acting on Δ extends to Γ, then Γ is classical.*

2.3 Ovoids of $Q(4, q)$, q odd, characterized by triads

We first recall some results on triads. If a generalized quadrangle has order (q, q^2), every triad has exactly $(q + 1)$ centers (see theorem 1.3). If $\Gamma \cong Q(4, q)$ with q odd, then every triad has exactly 0 or 2 centers in Γ. If $\Gamma \cong Q(4, q)$ with q even, then every triad has exactly 1 or $(q + 1)$ centers in Γ. ([48] 1.3.6.)

Proof of theorem 2.1

The theorem stated in Thas [65] reads as follows.

> *Let Γ be a GQ of order (q, q^2) and let Δ be a subGQ of Γ of order (q, q) with the property that every triad $\{x, y, z\}$ of Δ is 3-regular in Γ and $\{x, y, z\}^{\perp\perp} \subset \Delta$. Then Δ is classical and Γ has an involution θ fixing Δ pointwise.*

So the first part of theorem 2.1 was already known. To proof the assertion for q odd, we proceed as follows. Let $\mathcal{O}$ be an ovoid subtended by a point $p \in \Gamma \setminus \Delta$. We say that a **conic** of Δ is **subtended** by a point $a \in \Gamma$ if all its points are collinear with a.

- Let $x, y \in \mathcal{O}$. First we show that there are at least $\frac{q+1}{2}$ conics on $\mathcal{O}$ through x and y. The trace $\{x,y\}^{\perp}$ has $q+1$ points in common with Δ. Take a point $a \in \{x,y\}^{\perp} \cap \Delta$. As $\mathcal{O}$ is an ovoid of Δ, each line of Δ through a has a point in common with $\mathcal{O}$. Let z be such a point of $\mathcal{O} \setminus \{x,y\}$ collinear with a. As each triad has exactly 0 or 2 centers in $Q(4,q)$ (see above), the triad $\{x,y,z\}$ has a unique second center b in Δ. The trace, in $Q(4,q)$, of two non-collinear points of $Q(4,q)$ is a conic on $Q(4,q)$. We show that the conic $\{a,b\}^{\perp} \cap \Delta = C_{xyz}$ through x, y and z, is completely contained in the ovoid $\mathcal{O}$. As each point of $\{x,y,z\}^{\perp\perp}$ is — by definition — collinear with $a, b \in \{x,y,z\}^{\perp}$ and — by assumption — $\{x,y,z\}^{\perp\perp} \subset \Delta$, each point of $\{x,y,z\}^{\perp\perp}$ is in $\{a,b\}^{\perp} \cap \Delta = C_{xyz}$, with $|C_{xyz}| = |\{x,y,z\}^{\perp\perp}| = q+1$. Hence $C_{xyz} = \{x,y,z\}^{\perp\perp}$. As each point r of $\{x,y,z\}^{\perp\perp}$ is collinear with $p \in \{x,y,z\}^{\perp}$, $r (\in \Delta)$ will be a point of the ovoid $\mathcal{O}$ subtended by p. Hence the conic C_{xyz} through x, y and z is completely contained in the ovoid $\mathcal{O}$. As we can repeat the same reasoning for all points in $\{x,y\}^{\perp} \cap \Delta$, we obtain exactly $\frac{q+1}{2}$ conics on $\mathcal{O}$ through x and y which are subtended by 2 points of Δ. A conic on $\mathcal{O}$ subtended by two points of Δ will be called an **s-conic**.

- Now we show that there are $\frac{q(q+1)}{2}$ conics on $\mathcal{O}$ through a point $x \in \mathcal{O}$ which are subtended by 2 points of Δ. By the former reasoning, we constructed $\frac{q+1}{2}$ s-conics through each of the $(q^2+1)q^2$ pairs of points on $\mathcal{O}$, so there are $\frac{(\frac{q+1}{2})(q^2+1)q^2}{(q+1)q} = \frac{q(q^2+1)}{2}$ such conics on $\mathcal{O}$. Hence there will be $\frac{\frac{q(q^2+1)}{2}(q+1)}{q^2+1} = \frac{q(q+1)}{2}$ s-conics through a single point of $\mathcal{O}$.

- Thirdly, we count the number of s-conics on $\mathcal{O}$ through a point x of $\mathcal{O}$ that share exactly one point (the point x) with a given s-conic $C \subset \mathcal{O}$ through x. As there are q points on C different from x, and as there are $\frac{q-1}{2}$ s-conics different from C through x and a second point of C, there are $q(\frac{q-1}{2})$ s-conics different from C that intersect C in 2 points. Hence there are $\frac{q(q+1)}{2} - 1 - q(\frac{q-1}{2}) = q-1$ s-conics that share just the point x with C. We will denote those s-conics by C_i, $i = 1, \ldots, q-1$, and put $C = C_0$.

- Now we prove that also those $(q-1)$ conics $C_i, i > 0$, mutually share exactly one point. Suppose C is subtended by the points $a, b \in \Delta$. Take a line L of Δ through x, not through a or b. The projections y', z' on L of the points $y, z \in C \setminus \{x\}$, $y \neq z$, will never be equal, as this would imply that the triad $\{x,y,z\}$ has 3 centers (i.e. a, b and

y'). Hence there is a one-to-one correspondence between the points of C and the points on the line L through x. So every conic on $\mathcal{O}$ subtended by a point of L, will intersect C in at least 2 points (x included). So none of the points of L can subtend a conic C_i. Hence the subtending points of the $(q-1)$ conics C_i, $i = 1, 2, \ldots, q-1$, can be found on the lines xa and xb (for each conic, there is one subtending point on xa and one on xb). If two of those conics, say subtended by r respectively s, with $r, s \in xa$, would intersect each other in a point $u \neq x$, there would arise a triangle with vertices u, r and s. So we found q s-conics through x that mutually just have x in common — and hence cover all q^2+1 points of $\mathcal{O}$.

- Now the proofs of theorem 2.1 and 2.2 in Gevaert, Johnson and Thas [25] imply that all conics C_i, $i = 0, 1, \ldots, q-1$, have a common tangent line T at x. (One embeds $Q(4,q)$ in a $Q^+(5,q)$ and considers the inverse images of C_i under the Klein mapping.) As $\mathcal{O}$ contains conics different from $C_0, C_1, \ldots, C_{q-1}$, theorem 3.1 of the same paper proves that the ovoid $\mathcal{O}$ is classical, that is, belongs to a $\mathbf{PG}(3,q)$. $\square$

Remark that we only used the fact that every triad $\{x,y,z\}$ *which is centric in* Δ is 3-regular in Γ and satisfies $\{x,y,z\}^{\perp\perp} \subset \Delta$. Triads whithout center in Δ are not needed to prove the assertion for q odd.
From the previous proof, we can also deduce the following corollary.

Corollary 2.3 *Let Δ be the classical GQ $Q(4,q)$ of order (q,q), q* **odd***, and let $\mathcal{O}$ be an ovoid of Δ such that for every centric triad $\{x,y,z\}$ of $\mathcal{O}$, the set $\{x,y,z\}^{\perp\perp}$ belongs to $\mathcal{O}$. Then the ovoid $\mathcal{O}$ is classical.*

2.4 Ovoids of $Q(4,q)$ characterized by the linear group

Proof of theorem 2.2
As each point of Γ will induce an ovoid in Δ, and the classical generalized quadrangle $W(q)$ has no ovoids for q odd (see theorem 1.27 page 14), Δ is isomorphic to $Q(4,q)$. This proves the first assertion.
From now on, $\mathcal{O}$ is a subtended ovoid in Δ. The linear group G acting on $\Delta \cong Q(4,q)$ (or, equivalently, acting on the dual $W(q)$), is the group $\mathbf{PGSp}_4(q)$ of all collineations of $W(q)$ induced by $\mathbf{PGL}_4(q)$ (see e.g. [84] page 152-154), and has order $q^4(q^4-1)(q^2-1)$.
As every point in $\Gamma \setminus \Delta$ subtends exactly one ovoid, the number of points in $\Gamma \setminus \Delta$ (i.e. $q^2(q^2-1)$) is an upper bound for the size of the orbit $G(\mathcal{O})$

of a subtended ovoid $\mathcal{O}$, and hence we have a lower bound for the size of the stabilizer $G_{\mathcal{O}}$ of a subtended ovoid $\mathcal{O}$ under G.

$$\begin{array}{rcl} |G| & = & |G_{\mathcal{O}}| \cdot |G(\mathcal{O})| \\ \Rightarrow\ |G_{\mathcal{O}}| & \geq & \frac{|G|}{q^2(q^2-1)} \\ \Rightarrow\ |G_{\mathcal{O}}| & \geq & q^2(q^4-1). \end{array}$$

Now the proof is split up, according to the characteristic of $\mathbf{GF}(q)$.
For q odd, the setup is as follows. We take a triad in Δ which is centric in Δ, say $\{p_0,p_1,p_2\}$. Let p be a center of the triad in $\Gamma \setminus \Delta$, then p_0, p_1 and p_2 belong to the ovoid $\mathcal{O}_p$ subtended by p. As we know a bound for the size of the group $G_{\mathcal{O}}$ stabilizing $\mathcal{O}_p$, we can deduce that $\{p_0,p_1,p_2\}^{\perp\perp}$ is contained in $\mathcal{O}_p$, hence contained in Δ. By theorem 2.1, $\mathcal{O}_p$ is classical.
For q even, we point out that for the (self-)dual generalized quadrangle $W(q)$ in $\mathbf{PG}(3,q)$, the group stabilizing $\mathcal{O}$ is 3-transitive. This allows us to conclude that $\mathcal{O}$ is classical.

$\boxed{q \textbf{ odd}}$

The group $G_{\mathcal{O}}$ has order at least $q^2(q^4-1)$, but cannot act 3-transitive on the point set of $\mathcal{O}$. Indeed, we show that not all triads of $\mathcal{O}$ are centric, and as a centric triad will never be the image of a non-centric triad, $G_{\mathcal{O}}$ is not 3-transitive on $\mathcal{O}$.
Let X be the number of points of Δ that are centers of some triad $\{p_0,p_1,p_2\}$ of $\mathcal{O}$. As a point of $\mathcal{O}$ can never be such a center, and each point not in $\mathcal{O}$ is a center of such a triad, $X = |\Delta \setminus \mathcal{O}| = q^3+q$. So we count $X(q+1)q(q-1)/6 = q^2(q^4-1)$ pairs $(c, \{p_0,p_1,p_2\})$ with c a center of the triad $\{p_0,p_1,p_2\}$. If Y is the number of centric triads on $\mathcal{O}$, we count $2Y$ pairs $(c,\{p_0,p_1,p_2\})$ (as any triad has 0 or 2 centers, see page 27). Hence $Y = \frac{q^2(q^4-1)}{12}$, so not all triads of $\mathcal{O}$ (they are $(q^2+1)q^2(q^2-1)/6$ in total) are centric. Similarly, one shows that exactly $\frac{q^2-1}{2}$ triads $\{p_0,p_1,p_2\} \subset \mathcal{O}$, with p_0 and p_1 given, are centric.
Now we concentrate on the stabilizer $G_{\mathcal{O},x_0,x_1,x_2}$ fixing 3 points $x_0,x_1,x_2 \in \mathcal{O}$. As the orbit for $G_{\mathcal{O}}$ of x_0 has at most q^2+1 elements ($x_0^\alpha \in \mathcal{O}$), the stabilizer $G_{\mathcal{O},x_0}$ of x_0 in $G_{\mathcal{O}}$ has order at least $q^2(q^2-1)$.
As the orbit for $G_{\mathcal{O},x_0}$ of x_1 has size at most q^2 ($x_1^\alpha \in \mathcal{O} \setminus \{x_0^\alpha\}$), the group $G_{\mathcal{O},x_0,x_1}$ has order at least (q^2-1).
As $G_{\mathcal{O},x_0,x_1}$ is not transitive on the point set of $\mathcal{O} \setminus \{x_0,x_1\}$, the orbit for $G_{\mathcal{O},x_0,x_1}$ of x_2 has less than q^2-1 elements, hence the group $G_{\mathcal{O},x_0,x_1,x_2}$ has order greater than 1. Let $\{p_0,p_1,p_2\} \subset \mathcal{O}$ be a *centric* triad of Δ, with centers x and y.

- Suppose the stabilizer $G_{\mathcal{O},p_0,p_1,p_2}$ has order greater than 2. As the orbit of the center x for $G_{\mathcal{O},p_0,p_1,p_2}$ has size at most 2, the size of the stabilizer of x for $G_{\mathcal{O},p_0,p_1,p_2}$ is greater than 1. Let α be a non-identity collineation of this group $G_{\mathcal{O},p_0,p_1,p_2,x}$. As α fixes the three lines xp_0, xp_1, xp_2, this linear collineation fixes all lines through x. As also y is fixed under α, the trace x^y is pointwise fixed. Let p_3 be a point of $\mathcal{O}$ collinear with x, and suppose $p_3 \notin x^y$. As $p_3 = p_3^\alpha$, the points x, p_3 and $xp_3 \cap x^y$ would be 3 fixpoints on the line xp_3, hence all points on xp_3 are fixed and α must be the identity by Van Maldeghem [84] 4.4.2 (v). Hence $p_3 \in x^y$, and every point of $x^y = \{p_0, p_1, p_2\}^{\perp\perp}$ belongs to the ovoid. So, by corollary 2.3, $\mathcal{O}$ is classical.

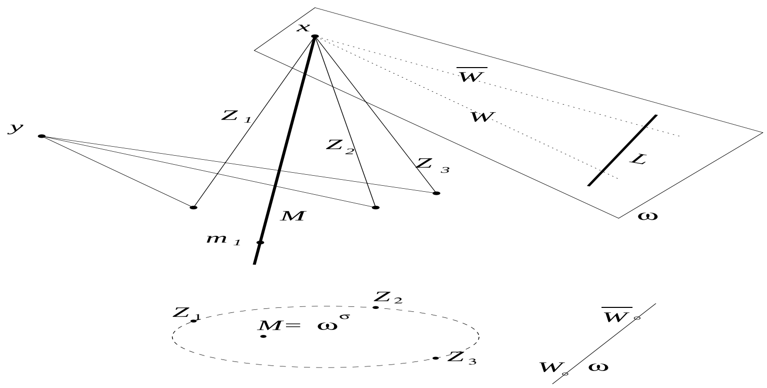
- Suppose the stabilizer $G_{\mathcal{O},p_0,p_1,p_2}$ has order exactly 2. Hence we can assume that the non-identity collineation of $G_{\mathcal{O},p_0,p_1,p_2}$ interchanges the centers x and y (otherwise, the same reasoning as above holds, to conclude that all points of $\{p_0, p_1, p_2\}^{\perp\perp}$ are inside $\mathcal{O}$).
  Also, the size of the orbit of the (ordered) triple (p_0, p_1, p_2) is at least $\frac{q^2(q^4-1)}{2}$, hence equal to $Y = \frac{q^2(q^4-1)}{2}$ since exactly Y ordered triples are centric. Hence $G_{\mathcal{O}}$ acts transitively on the set of ordered centric triads.
  At this point, we can proceed in two ways. The first completion of the proof is the fastest, but makes use of a theorem relying on the classification of finite simple groups. The second choice is much longer, but avoids the classification theorem.

Class As $G_{\mathcal{O}}$ acts transitively on the set of ordered centric triads, $G_{\mathcal{O}}$ acts 2-transitively on $\mathcal{O}$. Dually, with $\mathcal{O}$ there corresponds a spread § of $W(q)$ on which $\mathbf{PGSp}_4(q)$ acts 2-transitively. A result of Schultz [59] and Czerwinski [14] (see [40] p 181) states

> *If π is a finite translation plane with a collineation group G doubly transitive on the points of the line l_∞, then π is either Desarguesian or a L'127 uneburg plane.*

As L'127 uneburg planes only exist in the even case, the spread § is regular (see page 16), hence $\mathcal{O}$ is classical.

no Class As $G_{\mathcal{O}}$ acts transitively on the set of ordered centric triads, three points p_0, p_1, p_2 of $\mathcal{O}$ in $\Gamma_2(x)$ can be mapped to any three points q_0, q_1, q_2 of $\mathcal{O}$ in $\Gamma_2(x)$. Also, if the lines xp_0, xp_1, xp_2 are mapped to zq_0, zq_1, zq_2 respectively, then there is a collineation θ in $G_{\mathcal{O},q_0,q_1,q_2}$ mapping xp_0, xp_1, xp_2 to xq_0, xq_1, xq_2 respectively. So $G_{\mathcal{O},x}$ acts 3-transitively on $\Gamma_1(x)$; hence the action of $G_{\mathcal{O},x}$ on $\Gamma_1(x)$ (and on the $q+1$ points of $\mathcal{O}$ collinear with x) is equivalent to the action of $\mathbf{PGL}_2(q)$ on the projective line $\mathbf{PG}(1,q)$. Let x^y be the set $\{z_i\}_{i:0\to q}$, with $z_0 = p_0$, $z_1 = p_1$, $z_2 = p_2$, and put $xz_i := Z_i$. Let p_i be the unique point of $\mathcal{O}$ on Z_i. The



orbit of p_0 under the group $G_{\mathcal{O},x}$ is the set $\{p_i\}_{i:0\to q}$: as $G_{\mathcal{O},x}$ acts sharply 3-transitive on the $q+1$ lines Z_i there is at least one image of p_0 under $G_{\mathcal{O},x}$ on each Z_i, and as points of $\mathcal{O}$ are mapped onto points of $\mathcal{O}$ there is exactly one point of the orbit of p_0 on each Z_i. We will show in next paragraphs that all p_i are in the same plane $\langle p_0, p_1, p_2\rangle$, hence $\{p_i\}_{i:0\to q} = x^y$.
Let $\langle g\rangle$ be a Singer cycle of $G_{\mathcal{O},x}$, permuting all Z_i in one orbit, and fixing 2 imaginary lines W and $\overline{W}$ through x. So the plane $\omega = \langle W, \overline{W}\rangle$ is stabilized by g. In the residual geometry, i.e. the projection from x, W and $\overline{W}$ are imaginary points of the conic $\{Z_i\}_{i:0\to q}$. Hence the line ω is exterior with respect to the conic. Let M be the pole of ω with respect to the conic $\{Z_i\}_{i:0\to q}$, then M is fixed under g. Suppose there is a fixpoint N on $\omega\setminus\{W,\overline{W}\}$. Then MN is a fixline, the pole of MN is on ω, and this pole is also fixed. Hence ω is pointwise fixed, and M is linewise fixed under g. A contradiction, since the orbit of Z_0 is of order $q+1$ (look at the fixline MZ_0). Translated back to the original geometry on $Q(4,q)$, M is a line through x not in ω, stabilized under g. Now suppose x is the only fixpoint on M. Take the smallest non-trivial orbit of g on $M\setminus\{x\}$, and suppose this is of order $n\neq 1$. So g^n fixes every point of this orbit, hence g^n fixes every point on M. As n was supposed to be as small as possible, each orbit of $M\setminus\{x\}$ under g has size n. Hence $n|q$ and $n|(q+1)$, a contradiction. So either M is fixed pointwise under g, or M has exactly 2 fixpoints under g, in which case $n|(q-1)$ and $n|(q+1)$ with n the size of any orbit not containing a fixpoint; whence $n=2$ and $g^2_{|M} = \mathbf{1}$.

Now we turn back to the fixplane ω of g: as $x \in \omega$ is fixed under g, there is also a line $L \subset \omega$ which is stabilized under g, but does not contain x, as this would imply — in the residual geometry — a fixpoint L on ω.

$\boxed{\text{a}}$ Suppose $M = \{x, m_1, \ldots, m_q\}$ is fixed pointwise under g. Then all planes $\langle L, m_i \rangle$ are stabilized under g. As g permutes all Z_i in one orbit, all $q+1$ points $\{p_i\}_{i:0 \to q}$ are inside the plane $\langle L, p_0 \rangle$.

$\boxed{\text{b}}$ Suppose $M = \{x, m_1, \ldots, m_q\}$ has as only fixpoints x and m_1 under g. Then g is an involution on the set of planes $\langle L, m_i \rangle_{i:2 \to q}$. Hence the orbit of $p_0 \in Z_0$ is in the planes $\langle L, p_0 \rangle$ and $\langle L, p_0^g \rangle$. So the points $\{p_i\}_{i:0 \to q}$ are contained in either one or two planes. Suppose they are contained in 2 different planes π_1, π_2 through L. Then $\frac{q+1}{2}$ points of $\{p_i\}_{i:0 \to q}$ are in π_i, $i = 1, 2$. (So everything is in the 3-dimensional space spanned by L and M.) Let $p_0, p_1 \in \pi_1$ and $p_2, p_3 \in \pi_2$. As $G_{\mathcal{O},x}$ acts sharply 3-transitive on $\{p_i\}_{i:0 \to q}$, there is a $\theta \in G_{\mathcal{O},x}$ such that $(p_0^\theta, p_1^\theta, p_3^\theta) = (p_0, p_2, p_1)$. Then $\pi_1 \cap \pi_1^\theta$ is a line through p_0. If $\pi_1 \cap \pi_2^\theta$ would be π_1, then $\pi_1^\theta \cap \pi_2^\theta = (\pi_1 \cap \pi_2)^\theta = L^\theta$ would contain p_0, a contradiction as L^θ has no point in common with the cone consisting of the lines Z_i. Hence also $\pi_1 \cap \pi_2^\theta$ is a line. Half of the points of $\{p_i\}_{i:0 \to q}$ are in π_1^θ, the other half in π_2^θ. As all p_i are on the cone consisting of the lines Z_i, the line $\pi_1 \cap \pi_1^\theta$, respectively $\pi_1 \cap \pi_2^\theta$, contains at most 2 points of $\{p_i\}_{i:0 \to q}$. Hence there are at most 4 points of $\{p_i\}_{i:0 \to q}$ in π_1, so at most 8 points in $\pi_1 \cup \pi_2$. Hence $q \leq 7$. If $q = 3, 5$ or 7, then every ovoid of $Q(4, q)$ is an elliptic quadric (see page 16), and so all points p_i are in one plane, which means $\{p_i\}_{i:0 \to q} = x^y$ and hence $\{p_0, p_1, p_2\}^{\perp\perp}$ is part of the ovoid. So by theorem 2.1, $\mathcal{O}$ is classical.

<u>Remark</u> Without using the uniqueness result of the classical ovoid in $Q(4, 7)$, we provide the following alternative of that part of the proof. Let $q = 7$. We denote the eight lines Z_i through x by the numbers $\{\infty, 0, 1, 2, 3, 4, 5, 6\}$, and the corresponding eight points $p_i = Z_i \cap \mathcal{O}$ by $\{p_\infty, p_0, \ldots, p_6\}$. The Singer cycle $x \mapsto \frac{x+2}{-2x+1}$ fixes the imaginary lines i and $-i$ (with $(-i)^2 = -1 \in \mathbf{GF}(7)$), and corresponds to the permutation $(0, 2, 1, 4, \infty, 3, 6, 5)$. Hence the points p_0, p_1, p_∞, p_6 belong to a plane, as well as the points p_2, p_4, p_3, p_5. Now the linear collineation $x \mapsto x+1$ in $G_{\mathcal{O},x}$ fixes ∞ (and hence p_∞) and maps the four coplanar points p_2, p_3, p_4, p_5 to the points p_3, p_4, p_5, p_6

which are not coplanar, a contradiction.

Remark Another approach of the proof for q odd goes as follows: one can show that the subgroups of $\mathbf{PGL}_4(q)$ large enough to contain $G_{\mathcal{O}}$ can not contain $G_{\mathcal{O}}$ unless they are isomorphic to the stabilizer of the classical ovoid. The only cases to consider (and exclude) were the stabilizer of a point and the stabilizer of a line, using Di Martino and Wagner [19]; this was suggested to us by Tim Penttila.

$\boxed{q \textbf{ even}}$

To simplify the argumentation, we consider the symplectic quadrangle $W(q)$ in $\mathbf{PG}(3,q)$ instead of $Q(4,q)$ (which are dual, for q even). Then the ovoid $\mathcal{O}$ of $W(q)$ is also an ovoid of $\mathbf{PG}(3,q)$; see theorem 1.27 on page 14. The group $G_{\mathcal{O}}$ has order at least $q^2(q^4-1)$. Let p_0, p_1 and p_2 be three points of $\mathcal{O}$.

As the orbit for $G_{\mathcal{O}}$ of p_0 has at most q^2+1 elements ($p_0^\alpha \in \mathcal{O}$), the group $G_{\mathcal{O},p_0}$ has order at least $q^2(q^2-1)$.

As the orbit for $G_{\mathcal{O},p_0}$ of p_1 has size at most q^2 ($p_1^\alpha \in \mathcal{O} \setminus \{p_0^\alpha\}$), the group $G_{\mathcal{O},p_0,p_1}$ has order at least q^2-1.

As the orbit for $G_{\mathcal{O},p_0,p_1}$ of p_2 has at most q^2-1 elements ($p_2^\alpha \in \mathcal{O} \setminus \{p_0^\alpha, p_1^\alpha\}$), the group $G_{\mathcal{O},p_0,p_1,p_2}$ is trivial if and only if $G_{\mathcal{O}}$ acts sharply 3-transitive on $\mathcal{O}$, and $G_{\mathcal{O}}$ has order $q^2(q^4-1)$.

- Let $G_{\mathcal{O},p_0,p_1,p_2}$ be trivial. As $G_{\mathcal{O}}$ acts 3-transitively on the ovoid $\mathcal{O}$ of $\mathbf{PG}(3,q)$, the ovoid $\mathcal{O}$ is an elliptic quadric. (We refer to Dembowski [18] page 277: an ovoid of $\mathbf{PG}(3,q)$ gives rise to an egglike inversive plane. If an inversive plane I admits an automorphism group G acting 2-transitively on the points, then I is either the Miquelian plane $M(q)$ corresponding with the elliptic quadric, or it is the inversive plane $S(q)$ corresponding with the Suzuki-Tits ovoid. But by [18] page 53, $G_{\mathcal{O}}$ acts not 3-transitively on $\mathcal{O}$ if $\mathcal{O}$ is a Suzuki-Tits ovoid.)

- So we may assume that $G_{\mathcal{O},p_0,p_1,p_2}$ is not trivial. We show that in this case the order of $G_{\mathcal{O},p_0,p_1,p_2}$ is exactly 2, by pointing out that the non-identity element of $G_{\mathcal{O},p_0,p_1,p_2}$ is unique. First we remark that $p_0, p_1, p_2 \in \mathcal{O} \subset W(q)$ define a plane in $\mathbf{PG}(3,q)$. Indeed, for q even, every ovoid of the quadrangle $W(q)$ is an ovoid of the projective space $\mathbf{PG}(3,q)$ (see again theorem 1.27 p 14), so no three points of $\mathcal{O}$ are on a line of $\mathbf{PG}(3,q)$. If ζ is the symplectic polarity defining $W(q)$ and if π is the plane containing p_0, p_1, p_2, then $\pi^\zeta = x$ is the unique center of $\{p_0, p_1, p_2\}$. As $\{p_0, p_1, p_2\}$ is fixed elementwise by every $\alpha \in G_{\mathcal{O},p_0,p_1,p_2}$, also x is fixed by every such α. As p_0, p_1, p_2 and x are four linearly independent points in the plane $\pi = \langle p_0, p_1, p_2 \rangle$, α

fixes every point of this plane. Hence π is the axis of the perspectivity α. Let c be the center of α and let a be a point of $\mathcal{O}$ which is not fixed. Then $a, a^\alpha, a^{\alpha^2}$ are three points of $\mathcal{O}$ on the same line ac of $\mathbf{PG}(3,q)$, hence $a = a^{\alpha^2}$. Consequently α is an involution. As there is an odd number of points on a line, the center of the involution α should be in the axis (hence α is an elation).

Now we look for the center c of α, somewhere in the plane π. If $c \in \mathcal{O}$, there would be three points of $\mathcal{O}$ on a line of $\mathbf{PG}(3,q)$ (nl. c, a and a^α for all $a \in \mathcal{O} \setminus \pi$). If $c \neq x$, $c \in \pi \setminus \mathcal{O}$, then there are q lines of the quadrangle through c, not in π. Let L be such a line, with l the unique point of $\mathcal{O}$ on L. Then l^α also belongs to $\mathcal{O}$, lies on L, and is different from l. Hence there are 2 points of $\mathcal{O}$ on a line of the quadrangle, a contradiction. So $c = x$ is the center of the elation α. Now we show that α is unique. Suppose α' is different from α and also belongs to $G_{\mathcal{O},p_0,p_1,p_2}$. Let b be a point of $\mathcal{O}$, not in the plane π. Then $b, b^\alpha, b^{\alpha'}$ are three different points of $\mathcal{O}$ on the line xb of $\mathbf{PG}(3,q)$, a contradiction. Hence the order of $G_{\mathcal{O},p_0,p_1,p_2}$ is exactly 2. By the formula $|G_\mathcal{O}| = |G_{\mathcal{O},p_0,p_1,p_2}||G_\mathcal{O}(p_0,p_1,p_2)|$, we know that the orbit of an ordered triple (p_0,p_1,p_2) of $\mathcal{O}$ has order at least $\frac{q^2(q^4-1)}{2}$. Hence $|G_\mathcal{O}(p_0,p_1,p_2)|$ is either $\frac{q^2(q^4-1)}{2}$ or $q^2(q^4-1)$. If $|G_\mathcal{O}(p_0,p_1,p_2)| = q^2(q^4-1)$, then $G_\mathcal{O}$ acts 3-transitively on $\mathcal{O}$ and we are done by [18] page 277 and page 53. So we may assume that $|G_\mathcal{O}(p_0,p_1,p_2)| = \frac{q^2(q^4-1)}{2}$. Hence $|G_\mathcal{O}| = q^2(q^4-1)$. As $|G_\mathcal{O}| = |G_{\mathcal{O},p_0,p_1}||G_\mathcal{O}(p_0,p_1)|$ and $|G_\mathcal{O}(p_0,p_1)| \le (q^2+1)q^2$, we have $|G_{\mathcal{O},p_0,p_1}| \ge (q^2-1)$. Also, $|G_{\mathcal{O},p_0,p_1}| = |G_{\mathcal{O},p_0,p_1,p_2}||G_{\mathcal{O},p_0,p_1}(p_2)|$. We know that $|G_{\mathcal{O},p_0,p_1,p_2}| = 2$. It follows that $|G_{\mathcal{O},p_0,p_1}(p_2)| \ge \frac{q^2-1}{2}$.
Hence $|G_{\mathcal{O},p_0,p_1}(p_2)| \in \{q^2-1, \frac{q^2-1}{2}\}$. As q is even, $|G_{\mathcal{O},p_0,p_1}(p_2)| = (q^2-1)$, and so $|G_{\mathcal{O},p_0,p_1}| = 2(q^2-1)$ and $|G_\mathcal{O}(p_0,p_1)| = \frac{(q^2+1)q^2}{2}$. Now let (a,b) and (a',b') be ordered pairs, each consisting of distinct points of $\mathcal{O}$. Let $c_1, c_2 \in \mathcal{O} \setminus \{a,a'\}$, with $c_1 \neq c_2$. As $|G_{\mathcal{O},c_1,c_2}(a)| = q^2-1$, there is an element $\theta \in G_{\mathcal{O},c_1,c_2}$ for which $a^\theta = a'$; let $b^\theta = b''$. Now let $d \in \mathcal{O} \setminus \{a', b'', b'\}$. Then there is an element $\theta' \in G_{\mathcal{O},a',d}$ for which $b''^{\theta'} = b'$. Hence $a^{\theta\theta'} = a'$ and $b^{\theta\theta'} = b'$. It follows that $|G_\mathcal{O}(p_0,p_1)| = (q^2+1)q^2$, a contradiction. □

2.5 $Q(5, q)$ characterized by subtended ovoids of a subquadrangle

2.5.1 Definitions

Most of the lemma's and notions used in the following paragraphs can also be found in Thas and Payne [76] and Brown [10], but we recall them for coherency reasons.

Let Γ be a generalized quadrangle of order (q, q^2), and Δ a generalized subquadrangle of order (q, q), isomorphic to $Q(4, q)$. If L is a line of $\Gamma \setminus \Delta$, then the unique point of L in Δ will be denoted by the corresponding lowercase letter l. An ovoid $\mathcal{O}$ of Δ subtended by a point p of $\Gamma \setminus \Delta$, is denoted by $\mathcal{O}_p$. An ovoid $\mathcal{O}$ in Δ is called **doubly subtended** if there are exactly 2 points in $\Gamma \setminus \Delta$ that subtend $\mathcal{O}$.
A **rosette (of ovoids)** $\mathcal{R}$ of a $Q(4, q)$ based at a point r of $Q(4, q)$ is a set of ovoids with pairwise intersection $\{r\}$ such that $\{\mathcal{O} \setminus \{r\} | \mathcal{O} \in \mathcal{R}\}$ is a partition of the points of $Q(4, q)$ not collinear with r. The point r is called the **base point** of $\mathcal{R}$. It follows that a rosette has q ovoids.
A **rosette (of conics)** R of a $Q^-(3, q)$ based at a point r is a set of plane intersections of size $q + 1$ with pairwise intersection $\{r\}$ such that $\{C \setminus \{r\} | C \in R\}$ is a partition of the points of $Q^-(3, q)$. It follows that a rosette of conics has q elements and that the tangent at r of all conics is a fixed line.
A line L of $\Gamma \setminus \Delta$ with $L \cap \Delta = \{l\}$ will **subtend a rosette** as follows: every point of $L \setminus \{l\}$ subtends an ovoid of Δ through l. As there are no triangles in Γ, two ovoids $\mathcal{O}_x, \mathcal{O}_y$ with x, y different points of $L \setminus \{l\}$, will never share a second point. Hence $\mathcal{O}_x, \mathcal{O}_y$ have pairwise intersection l, and $\{\mathcal{O}_x\}_{x \in L \setminus \{l\}}$ is a rosette.
A **flock** $\mathcal{F}$ of an ovoid $\mathcal{O}$ of $\mathbf{PG}(3, q)$ is a partition of all but two points of $\mathcal{O}$ into $q - 1$ disjoint ovals C_i. The remaining points x, y are called the **carriers** of the flock. A flock $\mathcal{F} = \{C_1, \dots, C_{q-1}\}$ is called **linear** if all planes π_i, with $C_i \subset \pi_i$, contain a common line L. It has been proved that every flock of an ovoid is linear (see Fisher and Thas [21]).
A linear flock is uniquely defined by its two carriers, or by two of its ovals, or by an oval and a carrier. (Indeed, the line L that is common to all planes π_i of the ovals $C_i \in \mathcal{F}$, is also the intersection of the tangent planes of $\mathcal{O}$ at the carriers of $\mathcal{F}$ (equivalently, if q is odd, L is the polar line of the line xy with respect to the polarity defining $\mathcal{O}$).)

2.5.2 Lemma's

For the following lemma's, we assume Γ to be a GQ of order (q,q^2) with a classical subGQ Δ of order (q,q). We also assume that all subtended ovoids of Δ by points of $\Gamma \setminus \Delta$ are classical.

Lemma 2.1 *Each subtended ovoid in Δ is doubly subtended.*

Proof For any triad $\{x,y,z\}$ of Γ we have $|\{x,y,z\}^\perp| = q+1$, so an ovoid of Δ is subtended by at most two points of Γ. As there are $\frac{q^2(q^2-1)}{2}$ classical ovoids in $Q(4,q)$ (i.e. the number of elliptic quadrics on $Q(4,q)$, see also page 48), there are at most that much subtended classical ovoids in $Q(4,q)$. As each subtended ovoid in Δ is maximally doubly subtended, there are at most $2\frac{q^2(q^2-1)}{2}$ points in $\Gamma \setminus \Delta$ (as each point of $\Gamma \setminus \Delta$ subtends a classical ovoid). As the number of points of $\Gamma \setminus \Delta$ is equal to $q^2(q^2-1)$, each subtended ovoid is exactly doubly subtended. $\square$

If two distinct points $x, y \in \Gamma \setminus \Delta$ subtend the same ovoid, they are called **twins**, and we write $x^{\mathrm{tw}} = y$.

Lemma 2.2 *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are 2 subtended ovoids in Δ, touching at a, then there is a unique rosette of classical ovoids through $\mathcal{O}_1$ and $\mathcal{O}_2$, and moreover this rosette is subtended by a line.*

Proof Let Π_i be the 3-dimensional space containing $\mathcal{O}_i$, with $i = 1,2$. As $\mathcal{O}_1 \cap \mathcal{O}_2 = \{a\}$, the common plane π of Π_1 and Π_2 contains a. As π contains a unique point of $\mathcal{O}_i$, it is the unique tangent plane of $\mathcal{O}_i$ at a in Π_i, $i = 1,2$. Let $\mathcal{R}_* = \{\mathcal{O}_i\}_{i:1\to q}$ be the rosette we want to construct. If $\langle \mathcal{O}_3 \rangle$ would have an intersection plane with $\langle \mathcal{O}_1 \rangle$ different from π, we would have $|\mathcal{O}_1 \cap \mathcal{O}_3| = q+1$, a contradiction. So all $\langle \mathcal{O}_i \rangle$, with $\mathcal{O}_i$ in $\mathcal{R}_*$, should contain π. Hence taking the intersection of $Q(4,q)$ with the q 3-dimensional spaces through π that are not tangent to $Q(4,q)$ at a, we constructed $\mathcal{R}_*$ in a unique way.
Now we show that $\mathcal{R}_*$ is subtended. Let $\mathcal{O}_1$ be subtended by the point k_1. The rosette $\mathcal{R}_L$ subtended by $L := ak_1$ will, of course, contain $\mathcal{O}_1$. Let $\mathcal{O}'_i$ be an ovoid of $\mathcal{R}_L$ subtended by $x_i \in L \setminus \{k_1\}$, x_i collinear with a point of $\mathcal{O}_j \setminus \{a\}$. Let Π'_i be the 3-dimensional space containing $\mathcal{O}'_i$. Using the same arguments as above, we conclude that Π_1 and Π'_i intersect in the unique plane π tangent to $\mathcal{O}_1$ at a in Π_1. As this plane is the same as the one constructed above, $\mathcal{O}_j$ coincides with $\mathcal{O}'_i$. Hence $\mathcal{R}_*$ is subtended by the line L. $\square$

From this result, it follows that with each line L of $\Gamma \setminus \Delta$ subtending the rosette $\mathcal{R}_L = \{\mathcal{O}_i\}_{i:1\to q}$, one can associate the unique plane π_L being the common plane of all 3-dimensional spaces Π_i, with Π_i containing $\mathcal{O}_i$. We will refer to the plane constructed in this way as **the tangent plane π_L of Δ defined by L.**

Lemma 2.3 *If two subtended ovoids $\mathcal{O}_1$ and $\mathcal{O}_2$ of Δ are tangent, and the point k_i subtends $\mathcal{O}_i$ (i=1,2), then either k_1 and k_2 (and hence k_1^{tw} and k_2^{tw}) are collinear, or k_1^{tw} and k_2 (and hence k_1 and k_2^{tw}) are collinear.*

Proof Put $\mathcal{O}_1 \cap \mathcal{O}_2 = \{a\}$ and suppose $k_1^{\mathrm{tw}} \not\sim k_2$, $k_1 \not\sim k_2$. Then the q ovoids subtended by the q points on ak_1 form the unique rosette through $\mathcal{O}_1$ and $\mathcal{O}_2$ (lemma 2.2). But the same holds for the points on ak_1^{tw} and ak_2. Hence there are $3q$ different points defining q ovoids. This is impossible, as we know that each ovoid is doubly subtended (lemma 2.1). □

Lemma 2.4 *Let $\mathcal{R}$ be a rosette of classical ovoids with base point r, and let $\mathcal{O}$ be a classical ovoid not belonging to this rosette. If $r \notin \mathcal{O}$, then the intersection of $\mathcal{R} \cup \{\pi_r\}$, with π_r the tangent hyperplane of $Q(4,q)$ at r, and $\mathcal{O}$ consists of a flock $\mathcal{F}$ and its carriers a, b. If $r \in \mathcal{O}$, then the intersection of $\mathcal{R}$ and $\mathcal{O}$ is a rosette of q conics on $\mathcal{O}$ through r.*

Proof Obvious. □

2.5.3 Sketch of the proof of theorem 2.3

In the first part of the proof, we show that all pairs of lines of Γ are regular if they contain twins. Secondly, we show the same for lines not containing twins. These results make sure that we can use a lot of grids for constructing a lot of classical subquadrangles, as shown in the third part. In the fourth part, we show that we constructed enough classical subquadrangles (i.e. one through every dual window of Γ), so that we must conclude that Γ is classical too.

2.5.4 Part 1: regularity for line pairs containing twins

Theorem 2.4 *Let Γ and Δ be as above. Let the points l' and k' of $\Gamma \setminus \Delta$ be twins, and consider a line L through l', and a line K through k', with $L \cap K = \phi$. Then (L, K) is a regular pair of lines.*

Proof
The subtended ovoid $\mathcal{O} = \mathcal{O}_{l'} = \mathcal{O}_{k'}$ intersects L in l and K in k. The flock of $\mathcal{O}$ with carriers l and k is denoted by $\mathcal{F}$.

1. First we show that every line of $\{L,K\}^{\perp} \setminus \{l'k, lk'\}$ corresponds to the flock $\mathcal{F}$ of $\mathcal{O}$.
   Consider a line U of $\{L,K\}^{\perp}$, different from lk' and $l'k$. We put $U \cap \Delta = \{u\}$, $U \cap L = \{l''\}$, $U \cap K = \{k''\}$. Let $\mathcal{R}$ be the rosette of ovoids with base point u subtended by the line U. As $u \notin \mathcal{O}$ (avoiding triangles), $\mathcal{R}$ intersects $\mathcal{O}$ in a flock together with its two carriers (lemma 2.4). As $l'' \in U \cap L$ subtends an ovoid $\mathcal{O}_{l''}$ touching $\mathcal{O}$ in l, l'' defines the single point l on $\mathcal{O}$. Similarly for k defined by $k'' \in U \cap K$. Hence every line $U \in \{L,K\}^{\perp} \setminus \{l'k, lk'\}$ defines on $\mathcal{O}$ the flock $\mathcal{F}$ of $\mathcal{O}$ with carriers l and k.

2. Now we can show the regularity of L and K.
   Put $U_0 := lk', U_1 := l'k$ and $\{L,K\}^{\perp} := \{U_i\}_{i:0\to q}$. We claim that, if $N \notin \{L,K\}$ intersects U_2 and U_3, N will also intersect U_0 and U_1. Using this result, we show that N also intersects U_i for $i \geq 4$.
   The intersection points of N with U_2 and U_3 are respectively n_2 and n_3. As n_2 and n_3 are on lines of $\{L,K\}^{\perp}$, both conics $C_{n_2} := \mathcal{O} \cap \mathcal{O}_{n_2}$ and $C_{n_3} := \mathcal{O} \cap \mathcal{O}_{n_3}$ belong to the flock $\mathcal{F}$ of $\mathcal{O}$. Hence, by lemma 2.4, every point n_i of N will define an element $\mathcal{O}_{n_i}$ of $\mathcal{F} \cup \{l,k\}$. So one of the points of N, say n_0, will define the carrier l, or, equivalently, subtend an ovoid tangent to $\mathcal{O}$ at the point l. Hence $n_0 \sim l$. But $\mathcal{O}_{n_0}$ tangent to $\mathcal{O}$ implies $n_0 \sim l'$ or $n_0 \sim k'$ (see lemma 2.3). The first case ($n_0 \sim l'$) yields a triangle, so n_0 is collinear with k'. This implies $n_0 \in lk' = U_0$, so N and U_0 intersect.
   The same argument holds for the point $n_1 \in N$ that defines the carrier k of $\mathcal{F}$: the point n_1 belongs to $l'k = U_1$, so N and U_1 intersect. This shows our claim.
   Now we show that, if N intersects U_2 and U_3 (and hence U_0 and U_1), N also intersects U_i for $i \geq 4$. To avoid too many indices, we show this for $i = 4$. Put $\mathrm{proj}_{U_4} n_2 = p$. By our claim, the line n_2p intersects $k'l$, inducing a triangle if $n_2p \neq N$. Hence $p \mathrel{\mathrm{I}} N$. This concludes the proof. □

2.5.5 Part 2: regularity for line pairs not containing twins

Theorem 2.5 *Let Γ and Δ be as above. Let L, K be two opposite lines of $\Gamma \setminus \Delta$, such that no pair of points (l', k') can be found with $l' \in L, k' \in K$ and $l'^{\mathrm{tw}} = k'$. Then (L,K) is a regular pair of lines.*

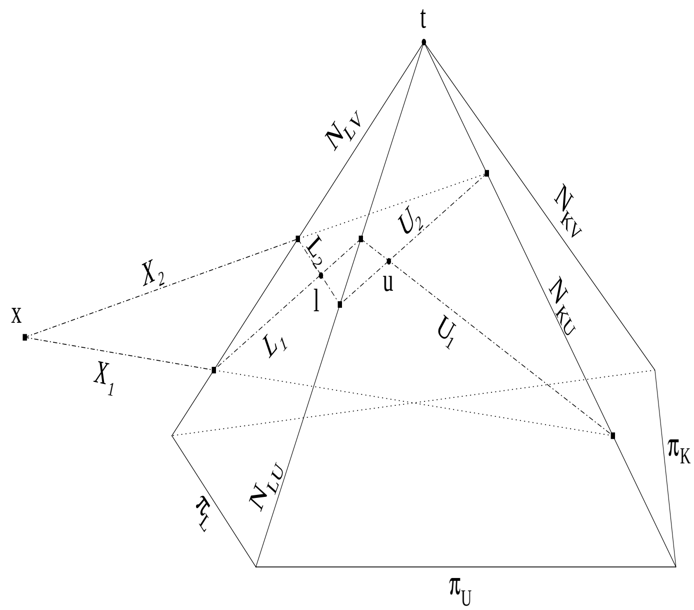
Proof

Consider 2 lines U, V of $\Gamma \setminus \Delta$ in $\{L,K\}^{\perp}$. Again, corresponding uppercase and lowercase letters are used for a line of $\Gamma \setminus \Delta$, respectively the unique point of Δ on that line. So we can consider the four points l, k, u and v in Δ, and we assume that they are all different. By theorem 2.4 we may suppose that $\{U, V\}^{\perp}$, respectively $\{L, K\}^{\perp}$, does not contain two lines A and B for which there exist points a', b' with $a' \in A$, $b' \in B$ and $a'^{\mathrm{tw}} = b'$.

1. In the first part of this proof, we show that l, k, u and v belong to a common plane.
   Consider the tangent planes π_L, π_K, π_U and π_V at Δ defined by respectively L, K, U and V (see definition following lemma 2.2).

   - Let a be the common point of U and L. As a subtends the ovoid $\mathcal{O}_a$ that belongs to the rosette $\mathcal{R}_L$ as well as to the rosette $\mathcal{R}_U$, the planes π_L and π_U both belong to the 3-dimensional space Π_a defined by $\Pi_a \cap Q(4,q) = \mathcal{O}_a$. Hence π_L and π_U share a common line (as $l \neq u$, π_L and π_U are not equal). The same result holds for each of the pairs (π_L, π_V), (π_K, π_U) and (π_K, π_V). Let $\pi_L \cap \pi_U = N_{LU}$ — with similar notations for all other pairs of planes.
   - Now we show that π_L and π_K only have a point in common. Indeed, if $\pi_L \cap \pi_K$ would be a line and $l \sim k$, then $\langle \pi_L, \pi_K \rangle$ would be a 3-dimensional space intersecting $Q(4,q)$ in the cone $Q(4,q) \cap \langle l^{\perp} \rangle$ respectively $Q(4,q) \cap \langle k^{\perp} \rangle$, yielding a contradiction. If $\pi_L \cap \pi_K$ would be a line and $l \not\sim k$, then $\langle \pi_L, \pi_K \rangle$ is a 3-dimensional space intersecting $Q(4,q)$ in an ovoid touching both Π_L and Π_K, which hence is subtended by a point of L and by a point of K. As $L \cap K = \phi$, this would imply that L and K contain a twin pair (l', k'), in contradiction with the assumptions.
     If π_U and π_V would intersect in a line, then U and V would contain a twin pair (u', v') $(u' \in U, v' \in V)$, a contradiction. So $\pi_U \cap \pi_V$ is a point. This also implies that the four lines N_{LU}, N_{LV}, N_{KU} and N_{KV} are all distinct. Since both Π_K and Π_L contain $N_{LU} \cap N_{KU}$ and $N_{LV} \cap N_{KV}$, these points coincide. Hence all lines contain a common point t.
   - Now we are ready to show that l, k, u and v belong to a common plane.[1]
     (We refer to the picture.) First we consider π_L and π_U. The 3-dimensional space $\langle \pi_L, \pi_U \rangle$ intersects $Q(4,q)$ in an ovoid tangent

[1]This is the point where the proof of theorem 7.1 of [76] is incomplete. At p 250 (a), two planes (in particular π_l and lmu, with m renamed k in our version) are supposed to intersect in a line, whereas this is not the case in general 4-dimensional setting.



to π_L at l and tangent to π_U at u. In a quadratic extension of this space, the intersection of $Q(4,q^2)$ with π_L will be a set of 2 *'imaginary'* lines through l, say L_1 and L_2. The same holds for $Q(4,q^2) \cap \pi_U$: this is the pair of *'imaginary'* lines U_1, U_2 through u. Up to choice of indices, L_1 and U_1 will intersect in an imaginary point of $N_{LU} = \pi_L \cap \pi_V$ — as will L_2 and U_2. The line through the points $L_1 \cap \pi_V$ and $U_1 \cap \pi_K$ is denoted by X_1; the line through the points $L_2 \cap \pi_V$ and $U_2 \cap \pi_K$ is denoted by X_2. Hence we obtain 2 triangles with lines respectively $\{L_1, U_1, X_1\}$ and $\{L_2, U_2, X_2\}$, that are in perspective from the point t (indeed, the vertices of both triangles are on N_{LU}, N_{KU} and N_{LV}). Hence we can apply the theorem of Desargues to conclude that l, u and x, with $\{x\} = X_1 \cap X_2$, are collinear.
Using the same arguments in the 3-dimensional space $\langle \pi_K, \pi_V \rangle$, we can conclude that k, v and x (indeed the same point x) are collinear.
Hence l, k, u and v are in the same plane $\pi_{lkuv} := \langle l, k, u, v \rangle$. This plane intersects $Q(4,q)$ in at least four points, not all on a line (avoiding triangles in Γ), hence l, k, u and v are either on an irreducible conic or on two different lines (lk and uv) of $Q(4,q)$.

2. In the second part of this proof, we show that (L, K) is a regular pair of lines.

   - Suppose the conic $\pi_{lkuv} \cap Q(4,q) = C$ defined by L, K, U, V is irreducible. Put $\{L, K\}^{\perp} = \{U, V, W_1, \ldots, W_{q-1}\}$ where $l \in W_1$, $k \in W_2$. Let w_i be the common point of W_i and Δ $(i \geq 3)$. Then L, K, U, W_i $(i \geq 3)$ also define the conic C (as a plane is defined by 3 non-collinear points), implying $w_i \in C$. Hence $C = \{l, k, u, v, w_3, \ldots, w_{q-1}\}$.
     To prove that (L, K) is regular, we have to check the following: if Y intersects $U, V \in \{L, K\}^{\perp}$, then Y will also intersect $W_i, i \in \{1, \ldots, q-1\}$. And indeed, interchanging the roles of L, K and U, V in the first part of this section, it follows that $y \in C$. Now again by this reasoning (substituting Y for K), every line containing a point of L and a point of Y, should meet $Q(4,q)$ in a point of C. Hence W_i and Y are concurrent for all i. Hence $Y \in \{L, K\}^{\perp\perp}$. It follows that the pair (L, K) is regular.

   - Secondly, consider the case that $\pi_{lkuv} \cap Q(4,q) = C$ is reducible. So lk and uv are distinct lines, and the conic $C = lk \cup uv$ is uniquely defined by any three of the points l, k, u and v. Let $\{L, K\}^{\perp} = \{U, V, W_1, \ldots, W_{q-1}\}$ with $W_1 = lk$. Let w_i be the common point of W_i and $Q(4,q)$ for $i > 1$ and let w_1 be the common point of lk and uv. Then U, W_i, L and K, $i > 1$, also define the conic C, so $w_i \in C$. Clearly $w_i \in uv$, $i > 1$. Hence $uv = \{u, v, w_1, \ldots, w_{q-1}\}$. Let $Y \in \{U, V\}^{\perp} \setminus \{L, K, uv\}$. Then, if y is the common point of Y and $Q(4,q)$, we have $y \in lk$. Now, interchanging roles of L and Y, we see that every line containing a point of uv and a point of L must contain a point of Y. Hence for $i \geq 1$, W_i and Y are concurrent. Hence $Y \in \{L, K\}^{\perp\perp}$. It follows that the pair (L, K) is regular. □

Corollary 2.4 *All lines of* Γ *are regular.*

Proof This follows from theorems 2.4 and 2.5. □

Corollary 2.5 *The intersection of* Δ *and a grid not contained in* Δ *is a conic (either irreducible or consisting of two distinct lines).*

Proof This follows from the proof of previous theorems. □

2.5.6 Part 3: construction of subGQ's

As all lines of Γ are regular, two opposite lines U, V define a $(q+1)\times(q+1)$-grid $\mathcal{G}$ in Γ. We will say $\mathcal{G}$ is the **grid based on** U, V and denote it by $\mathcal{G}(U, V)$.
In this part, we give the construction of a lot of new subGQ's of order (q, q) in Γ. Starting from an elliptic quadric (respectively a quadratic cone, a hyperbolic quadric) inside Δ, we choose an additional line of $\Gamma \setminus \Delta$ containing a point of the elliptic quadric (respectively quadratic cone, hyperbolic quadric) and construct a subGQ Δ' of order (q, q) containing this structure.

Theorem 2.6 *Let Γ and Δ be as above. Given an elliptic quadric $\mathcal{O}$ in Δ and a line L of $\Gamma \setminus \Delta$ intersecting this ovoid, with L a line not containing a point subtending $\mathcal{O}$, there exists a subGQ Δ' of order (q, q) of Γ through $\mathcal{O}$ and L.*

Proof
Construction of Δ'
Let $\mathcal{O}$ be an elliptic quadric in Δ, L a line of $\Gamma \setminus \Delta$ intersecting $\mathcal{O}$ in l, and L not through a point subtending $\mathcal{O}$. We construct Δ' as follows.

- The **basic line** of Δ' is the line L itself.

- As the ovoid $\mathcal{O}$ is not subtended by any point of L, and the base point l of the rosette $\mathcal{R}_L$ belongs to $\mathcal{O}$, the rosette $\mathcal{R}_L$ will intersect $\mathcal{O}$ in a rosette of conics (see lemma 2.4). This means that every point x of $L \setminus \{l\}$ is collinear with $q+1$ points of $\mathcal{O}$, constituting a conic C_x through l. The q lines joining this point x to the set $C_x \setminus \{l\}$, are also lines of Δ', and are said to be of the **first generation**. Hence there are q^2 lines of the first generation in Δ'. Every point of such a line will be a point of Δ', so we already defined q^3+q+1 points of Δ'. These points, including the point l, are the points of the **first generation**.

- The third set of lines belonging to Δ' is constructed as follows: take two opposite lines U, V of the first generation. As all lines of Γ are regular, we can construct the $(q+1)\times(q+1)$-grid $\mathcal{G}(U, V)$ based on these lines U, V. This grid contains L, and intersects $\mathcal{O}$ in a conic C through l, but this conic is not one of the conics in the rosette $\mathcal{R}_L \cap \mathcal{O}$. All (new) lines of the grid $\mathcal{G}(U, V)$ that are opposite L belong to the **second generation** of lines of Δ'.

- Every line that is the projection of a line of the second generation onto l, belongs to the **third generation**. In total, there will be q

such lines (this will be proved by showing that Δ' is indeed a GQ; see last part of the proof for more explanation), and the q^2 new points on these lines are the points of the **third generation**.

Note that through each conic C of $\mathcal{O}$ through l, not belonging to the rosette $\mathcal{R}_L \cap \mathcal{O}$ (i.e. not defined by one of the q points of $L \setminus \{l\}$), one can construct a unique grid $\mathcal{G}(U,V)$ based on two lines of the first generation. Indeed, choose $u, v \in C \setminus \{l\}$ and put $U :=$proj$_u L$ (so $U \cap L$ is the unique point of L collinear with u) and $V :=$proj$_v L$. Then, as C does not belong to the rosette $\mathcal{R}_l \cap \mathcal{O}$, U, V will be at distance 4 and of the first generation. By corollary 2.5, the grid $\mathcal{G}(U,V)$ intersects $\mathcal{O}$ in the conic C.

($*$) We now claim that if a line K of Γ through a point p of the first generation with $p \notin \mathcal{O}$, $p \notin L$, intersects the ovoid $\mathcal{O}$, then K is of the first or second generation.
Indeed, suppose K is not of the first generation and $K \cap \mathcal{O} = \{k\}$. If we project L onto k and put proj$_k L = V$, then V is a line of the first generation. As $p \in K$ is a point of the first generation, it belongs to a line U of the first generation. As K intersects both U and V, K belongs to the grid $\mathcal{G}(U,V)$ and hence K is of the second generation. The claim is proved.

Δ' is indeed a GQ
We show that for p a point and K a line of Δ', $p \notin K$, the line $M :=$ proj$_p K$ belongs to Δ'.

(1,1) If p and K both belong to the first generation, proj$_p K = M$ belongs — by definition of the second generation of lines — to Δ'.

(1,2) Let p be of the first, and let K be of the second generation. If $p \in L$, then clearly M belongs to Δ'. So assume $p \notin L$. Hence p belongs to a unique line S of the first generation, and K belongs to some grid $\mathcal{G}(U,V)$ with S, U, V three lines of the first generation (i.e. intersecting L and $\mathcal{O}$ in two different points). We may assume $U \neq S \neq V$. If we can show that the line $M =$proj$_p K$ intersects $\mathcal{O}$, this line M belongs to Δ'. We put $S \cap L = \{s'\}$. The line $W :=$ proj$_{s'} K$ belongs to the grid $\mathcal{G}(U,V)$, so W intersects $\mathcal{O}$ in a point w. We may assume $S \cap K = \phi$, otherwise we are done. The line W also belongs to the grid $\mathcal{G}(S,K)$, so this grid intersects $\mathcal{O}$ in the conic C_{skw} through s, k and w. As M belongs on its turn to the grid $\mathcal{G}(S,K)$, the point $\{m\} = M \cap \Delta$ belongs to the conic C_{skw} by corollary 2.5. Hence $m \in \mathcal{O}$, and this part of the proof is finished.

(3,1) Let p be of the third, and let K be of the first generation. Then p is on a line L' through l, with L' through a point u' of a line U of the second generation. So the line U intersects $\mathcal{O}$ (in the point u). The point $k'' := \mathrm{proj}_K u'$ is of the first generation (as $k'' \in K$). As $u'k''$ is a line of the second generation (taking account of case (1,2)), the line $u'k''$ meets $\mathcal{O}$ (in a point x). So the grid $\mathcal{G}(L', K)$ meets $\mathcal{O}$ in the conic C_{kxl}. As $M := \mathrm{proj}_p K$ belongs to the same grid $\mathcal{G}(L', K)$, the line M meets $\mathcal{O}$ in the same conic. Hence, by $(*)$, M is of the second generation and so it belongs to Δ'.

(1,3) Let p be of the first, and let K be of the third generation. Clearly we may assume that $p \notin L$. The line $U := \mathrm{proj}_p L$ is of the first generation and intersects $\mathcal{O}$ (in the point u). As K is of the third generation, K contains l and a point k' on a line N of the second generation. If $k' \in U$ we are done, so assume $k' \notin U$. The line $J := \mathrm{proj}_{k'} U$ is of the second generation, as it is the projection of a line of the first generation on a point of the third generation (see case (3,1)); so J intersects $\mathcal{O}$ (in the point j). Hence the grid $\mathcal{G}(K, U)$ intersects $\mathcal{O}$ in at least l, j and u, so $M = \mathrm{proj}_p K$ (belonging to $\mathcal{G}(K, U)$) will also intersect $\mathcal{O}$. By $(*)$, the line M is of the second generation, and so it belongs to Δ'.

(3,2) Let p be of the third, and let K be of the second generation. Then p is on a line L' through l, with L' through a point u' of a line U of the second generation. We may assume that $u' = p$. So U intersects $\mathcal{O}$ (in the point u). As K is of the second generation, K intersects $\mathcal{O}$ in a point k. Take a point $u'' \in U \setminus \{p\}$, which is necessarily of the first generation. We may assume that $K \cap U = \phi$, otherwise we are done. The line $V := \mathrm{proj}_{u''} K$ belongs to either the first or the second generation (by case (1,2)), so V intersects $\mathcal{O}$ (in the point v). Hence $\mathcal{G}(U, K)$ intersects $\mathcal{O}$ in a conic C_{uvk}. As $M = \mathrm{proj}_p K$ also belongs to $\mathcal{G}(U, K)$, the line M meets $\mathcal{O}$ in a point of C_{uvk}. If this point is l, M is of the third generation, so the proof is done. If this point is different from l, the point $M \cap K$ is of the first generation. (Indeed, K is of the second generation, so it has one point in $\mathcal{O}$, $q-1$ points of the first generation not in $\mathcal{O}$, and one point of the third generation; if $M \cap K$ would be of the third generation, the points $M \cap K, l$ and u' constitute a triangle.) Hence, relying on $(*)$, M is of the second generation.

(3,3) Let p as well as K be of the third generation. This case is trivial.

Hence Δ' is a generalized quadrangle. Clearly it is thick. As each line of Δ' contains $q+1$ points of Δ', and as any point of $L \setminus \{l\}$ is incident with $q+1$ lines of Δ', the quadrangle Δ' has order (q, q). □

Theorem 2.7 *Let Γ and Δ be as above. Given a quadratic cone $\mathcal{C}$ in Δ, i.e. a set of $q+1$ lines through a point p, and a line L of $\Gamma \setminus \Delta$ intersecting this cone in a point different from p, there exists a subGQ Δ' of order (q,q) of Γ through $\mathcal{C}$ and L.*

Proof
The proof is completely similar to the previous case. Let us just indicate how Δ' is defined.
Let $\mathcal{C}$ be a quadratic cone in Δ with vertex p, L a line of $\Gamma \setminus \Delta$ intersecting $\mathcal{C} \setminus \{p\}$. Put $L \cap \mathcal{C} = \{l\}$. We construct a subGQ Δ' as follows.

- The **basic lines** of Δ' are the $q+1$ lines of the cone $\mathcal{C}$ and the line L.
- The lines of the **first generation** are the q^2 lines joining a point $x \in L \setminus \{l\}$ and a point $y \in \mathcal{C} \setminus \{pl\}$. (For every point $x \in L \setminus \{l\}$, the $q+1$ points on $\mathcal{C}$ collinear with x constitute a conic C_x through l.) In this way, one obtains $q^2(q-1)$ new points of Δ'. Those points, together with the $(q+1)^2$ points on $\mathcal{C} \cup L$, constitute the **first generation of points**.
- The lines of the **second generation** are the $q^3 - q$ new lines opposite L of the q^2 grids $\mathcal{G}(U,V)$ with U, V lines of the first generation.
- The lines of the **third generation** are the lines through l intersecting a line of the second generation. The proof will imply that there are $q-1$ such lines. On these lines, we find $q(q-1)$ new points of Δ', said to be of the **third generation**. (Again, no points of the second generation are defined.) □

Theorem 2.8 *Let Γ and Δ be as above. Given a hyperbolic quadric $\mathcal{G}$ in Δ and a line L of $\Gamma \setminus \Delta$ intersecting this hyperbolic quadric, there exists a subGQ Δ' of order (q,q) of Γ through $\mathcal{G}$ and L.*

Proof
Again similar to the proof of theorem 2.6. The construction of Δ' is now as follows. Put $L \cap \mathcal{G} = \{l\}$.

- The **basic lines** of Δ' are the $2q+2$ lines of $\mathcal{G}$ and the line L.
- The lines of the **first generation** are the q^2 lines joining a point $x \in L \setminus \{l\}$ and a point $y \in \mathcal{G}$, with y not on a line of Δ containing l. (For every such point x the $q+1$ points of $\mathcal{G}$ collinear with x constitute a conic C_x through l.) Including all points of $\mathcal{G}$ we obtain in this way q^3+3q+1 points of Δ', said to be of the **first generation**.

- The lines of the **second generation** are the new lines in the grids $\mathcal{G}(U, V)$ with U, V opposite lines of the first generation. There are $q^3 - 2q$ lines of the second generation.

- The lines of the **third generation** are the lines containing l and concurrent with any line of the second generation. The points of the **third generation** are the new points incident with lines of the third generation. As the structure Δ' defined in this way turns out to be a GQ, there are $q - 2$ lines of the third generation and $q^2 - 2q$ points of the third generation. □

2.5.7 Part 4: subGQ's through every dual window

A **dual window** of a generalized quadrangle is a set of five points, two of which, say a and b, are at distance 4, while the other three are in a^b, together with the six lines through the pairs of collinear points (see page 5).

Lemma 2.5 *Let Γ be a GQ of order (q, q^2). Through every dual window of Γ, there is at most one subGQ of order (q, q).*

Proof Let Γ_1 and Γ_2 be two subquadrangles of order (q, q) of Γ. As each line of Γ_1 intersects Γ_2 ([48] 2.2.1), the intersection $\Gamma_1 \cap \Gamma_2$ of these subquadrangles is a grid of Γ_1, or an ovoid of Γ_1, or the set of all points of Γ_1 collinear with a fixed point of Γ_1. As a dual window is never contained in $\Gamma_1 \cap \Gamma_2$, we have a contradiction. □

Theorem 2.9 *Let Γ be a GQ of order (q, q^2) and let Δ be a classical subGQ of order (q, q) of Γ, such that every subtended ovoid of Δ is classical. Then one can construct a subGQ Δ' of order (q, q) through every dual window of Γ. Hence Γ is classical.*

Proof We perform a double counting on the pairs $(\mathcal{W}, \mathcal{D})$ with $\mathcal{W}$ a dual window of Γ, and $\mathcal{D}$ a subquadrangle constructed as explained in theorem 2.6, 2.7 or 2.8, such that $\mathcal{W} \subset \mathcal{D}$.
$\boxed{\text{1a}}$ Let W be the number of dual windows in Γ. We label the points and lines of a dual window $\mathcal{W}$ as follows: a, b, c_1, c_2, c_3 are the points and L, M, N, U, V, W are the lines, with $a \mathrm{~I~} L \mathrm{~I~} c_1 \mathrm{~I~} U \mathrm{~I~} b$, $a \mathrm{~I~} M \mathrm{~I~} c_2 \mathrm{~I~} V \mathrm{~I~} b$ and $a \mathrm{~I~} N \mathrm{~I~} c_3 \mathrm{~I~} W \mathrm{~I~} b$. The amount of possible choices for a,L,M,N,c_1,U respectively b in Γ is $(q^3+1)(q+1)$,(q^2+1),q^2,(q^2-1),q,q^2 respectively q. So there are $(q^3+1)(q^2+1)(q+1)^2q^6(q-1)$ ordered dual windows in Γ. The amount W of unordered dual windows is then obtained by dividing by $2 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ (these are respectively the number of possible choices for a,L,M,N,c_1,U respectively b in the fixed dual window $\mathcal{W}$). $\boxed{\text{1b}}$

By lemma 2.5, there is at most one subquadrangle of order (q,q) through every dual window. $\boxed{\text{1ab}}$ So the number of pairs $(\mathcal{W},\mathcal{D})$ is at most W.
$\boxed{\text{2a}}$ Now we count the number S of subquadrangles of order (q,q) constructed so far as follows. To that end, we need the number of classical ovoids, the number of hyperbolic quadrics and the number of cones in $Q(4,q)$. The number of hyperbolic quadrics in $Q(4,q)$ can be easily obtained as follows: there are $(q+1)(q^2+1)$ lines on $Q(4,q)$, through each of which there are q^2+q+1 threedimensional spaces. Exactly $q+1$ of them (nl. the tangent spaces at each of the $q+1$ points on that line) intersect $Q(4,q)$ in a cone, the others will intersect in a hyperbolic quadric. Hence there are $\frac{(q+1)(q^2+1)\cdot q^2}{2(q+1)} = \frac{q^2(q+1)}{2}$ hyperbolic quadrics in $Q(4,q)$. The number of elliptic quadrics is then easily seen to be $(q^4+q^3+q^2+q+1)-(q^3+q^2+q+1)-\frac{q^2(q^2+1)}{2} = \frac{q^2(q^2-1)}{2}$.
Now, through every ovoid, one constructed $q-2$ subquadrangles Δ' different from Δ (through every point p of the ovoid, there are q^2-q-2 lines to choose for starting the construction of Δ', but there are $q+1$ lines of Δ' through p). Through every cone, one constructed $q-1$ new subquadrangles Δ'. Through every grid, one constructed q new subquadrangles Δ'. This gives us a total of $S = q^5+q^2$ subquadrangles (Δ included). $\boxed{\text{2b}}$ Given a fixed subquadrangle $\mathcal{D}$ of order (q,q), one counts $x = \frac{1}{12}(q^2+1)(q+1)^2q^4(q-1)$ dual windows in $\mathcal{D}$. $\boxed{\text{2ab}}$ So there are at least xS pairs $(\mathcal{W},\mathcal{D})$.
We conclude that $W = xS$, and hence we constructed exactly one subquadrangle through every dual window. By theorem 1.10 on page 12, Γ is classical. $\square$

2.6 $Q(5,q)$, q odd, characterized by triads in a subGQ

This characterization, fully described in corollary 2.1, is an immediate corollary of theorems 2.1 and 2.3.

2.7 $Q(5,q)$ characterized by the linear group of a subGQ

Again, this characterization, fully described in corollary 2.2, is an immediate corollary of previous characterizations (theorems 2.2 and 2.3).

Chapter 3

Characterizations of $Q(d, \mathbb{K}, \kappa)$ and $H(3, \mathbb{K}, \mathbb{K}^{(\sigma)})$ by Projectivity Groups

3.1 Introduction

As noted in theorem 1.8, the finite classical orthogonal quadrangle $Q(4, q)$ is line-regular. One easily sees that this holds true for every $Q(4, \mathbb{K})$ with $\mathbb{K}$ any commutative field by dualizing: the traces in the dual quadrangle $W(\mathbb{K})$ are the lines of $\mathbf{PG}(3, \mathbb{K})$ and hence determined by two of its points. Also the classical orthogonal quadrangle $Q(5, \mathbb{K}, \kappa)$ (where κ is a suitable bilinear form, see [84]) is line-regular: as the 3-space spanned by any pair of opposite lines L, M of $Q(5, q)$ intersects its underlying quadric Q in a hyperbolic quadric, it is clear that the trace $\{L, M\}^{\perp}$ is uniquely defined by two of its elements. Conversely, the following important open question arises: is every generalized quadrangle all lines of which are regular necessarily isomorphic to $Q(4, \mathbb{K})$ or to $Q(5, \mathbb{K}, \kappa)$? A fair amount of characterizations of $Q(4, \mathbb{K})$ respectively $Q(5, q)$ exists using the assumption of regularity of all the lines, plus an extra condition. For example, theorem 1.7 asks the existence of at least one projective line (being the dual of a projective point), while theorem 1.8 puts assumptions on the order. For results on $Q(5,q)$, we refer to Payne and Thas [48] 5.3.9 (ii) and 5.3.11 (ii). Few results though characterize $Q(4,\mathbb{K})$ and $Q(5,\mathbb{K},\kappa)$ at the same time using the regularity of the lines. With a condition on the general groups of projectivities $\Pi(\Gamma)$ and $\Pi^*(\Gamma)$ (see page 18), we provide such a characterization in the finite case in theorem 3.3. The setup of theorem 3.1 respectively 3.2 is similar: assuming line-regularity and conditions on $\Pi(\Gamma)$, we characterize $Q(4,q)$ respectively $Q(d,\mathbb{K},\kappa)$ and $H^D(3,\mathbb{K},\mathbb{K}^{(\sigma)})$ (which denotes the Hermitian quadrangle in $\mathbf{PG}(3,\mathbb{K})$ over any skew field $\mathbb{K}$). Finally, the last theorem (number 3.4) makes no use of regularity assumptions, but characterizes $Q(5,q)$ by its order and a condition on the special projectivity group $\Pi^*_+(\Gamma)$.

These results, which appeared in 1998 in [5], may contribute to the connection between generalized quadrangles and projective planes. Indeed, a number of classification results exists for affine and projective planes from 'von Staudt's point of view', i.e. using properties of their groups of projectivities. For example, it was shown by Schleiermacher [56] that a projective plane is Pappian if and only if the only projectivity of any line fixing at least 5 points is the identity. Schleiermacher [57] also showed that a projective plane is Moufang if and only if the stabilizer of a point in the group of projectivities has a regular normal subgroup. A related result by Funk [24] states that an affine plane whose group of projectivities is a Zassenhaus group is a translation plane; the plane is Desarguesian unless the kernel of the plane is $\mathbf{GF}(2)$, see [51] for an overview.
The most general result classifies the finite classical projective planes by a very simple property: a finite projective plane of order $s \neq 23$, $s > 4$, is classical if and only if its group of projectivities does not contain the alternating group in its natural action, see Grundh'127 ofer [27].

So in the sections to follow, we apply von Staudt's point of view to quadrangles. While the obtained results are still far from the generality obtained for projective planes, they are the best approximation of these results for quadrangles that we are aware of.

3.2 Projectivity groups: preliminary results

Before we state and proof the theorems, we collect some facts about the projectivity groups of orthogonal quadrangles, which can be found in [84] on page 378. The representation of the general projective groups $\Pi(\Gamma)$ and $\Pi^*(\Gamma)$ of $Q(4,\mathbb{K})$ respectively $Q(5,q)$ are listed in the table below. Remark that $\Pi(\Gamma)$ respectively $\Pi^*(\Gamma)$ of both quadrangles is — in the finite case — permutation equivalent to a substructure of the representation mentioned in the last line. This remark will be made permanent (and inverted) by theorem 3.2. Concerning the special (dual) projectivity groups, we remark that these groups coincide with the general (dual) groups, except for the case $\Pi_+^*(Q(5,q))$, which is permutation equivalent to $(\mathbf{PSL}_2(q^2), \mathbf{PG}(1,q^2))$.

polygon	representation$(\Pi(\Gamma), X(\Gamma))$	representation$(\Pi^*(\Gamma), X^*(\Gamma))$
$Q(4,\mathbb{K})$	$(\mathbf{PSL}_2(\mathbb{K}), \mathbf{PG}(1,\mathbb{K}))$	$(\mathbf{PGL}_2(\mathbb{K}), \mathbf{PG}(1,\mathbb{K}))$
$Q(5,q)$	$(\mathbf{PGL}_2(q), \mathbf{PG}(1,q))$	$(\mathbf{PSL}_2^{(q)}(q^2), \mathbf{PG}(1,q^2))$
	subgroup of $(\mathbf{PGL}_2(q), \mathbf{PG}(1,q))$	subgroup of $(\mathbf{PGL}_2^{(q)}(q^2), \mathbf{PG}(1,q^2))^1$

3.3 Characterization of $Q(4,\mathbb{K})$ by regular lines and a condition on $\Pi(\mathbb{K})$

A field is **quadratically closed** if every quadratic equation over this field has at least one solution. A **separably quadratic extension** is a field extension such that a quadratic polynomial, irreducible over the original field, has two different roots in the extension. A field is **separably quadratically closed** if it has no separably quadratic extension. Every quadratically closed field is separably quadratically closed. The converse is true whenever the characteristic of the field is not equal to 2.
Indeed, let $\mathbb{K}$ be a separably quadratically closed field of odd characteristic. Let $Ax^2 + Bx + C = 0$ be an arbitrary quadratic equation over $\mathbb{K}$. If $D = B^2 - 4AC = 0$, the root is $-\frac{B}{2A}$, hence belongs to $\mathbb{K}$. If $D \neq 0$, there are two different roots, namely $\frac{-B+\sqrt{D}}{2A} \neq \frac{-B-\sqrt{D}}{2A}$. As $\mathbb{K}$ is said to be separably quadratically closed, those two different roots belong to $\mathbb{K}$. Hence every quadratic equation has one ore more solutions, and $\mathbb{K}$ is quadratically closed.
If the characteristic is equal to 2, then there are separably quadratically closed fields which are not quadratically closed (e.g. the separable quadratic closure of a non-perfect field, where every equation $Ax^2+Bx+C=0$ with $B \neq 0$ will have two roots in $\mathbb{K}$, but not every equation $Ax^2 + C = 0$ has).

Theorem 3.1 *Let Γ be a line-regular generalized quadrangle. If every element of $\Pi(\Gamma)$ has a fixed element, then $\Gamma \cong Q(4,\mathbb{K})$, for some separably quadratically closed field $\mathbb{K}$.*

Proof
We stated the theorem for $Q(4,\mathbb{K})$ to emphasize the parallel with next the-

orems, but we will prove the dual. So let Γ be a point-regular quadrangle, such that every element of $\Pi^*(\Gamma)$ has a fixed element.
First we show that Γ has at least one projective point, so that, by theorem 1.7, $\Gamma \cong W(\mathbb{K})$, for some field $\mathbb{K}$. Let p be a point of Γ, and L, M two distinct lines through p. Let p_1, p_2 be two opposite points at distance 4 of p, such that p_i projects onto L in x_i, and projects onto M in y_i, with $x_1 \neq x_2$ and $y_1 \neq y_2$. We want to show that $p^{p_1} \cap p^{p_2}$ is non-empty (i.e. p is projective). By assumption, the projectivity $[p;p_1;p_2;p]$ has at least one fixed element. Let N be a fixed element, then the projections of p_1 and p_2 onto N coincide unless Γ contains a triangle. Hence the traces p^{p_1} and p^{p_2} have at least 1 element in common. If they have 2 elements in common, they coincide by the regularity of p, leading to a contradiction. So $|p^{p_1} \cap p^{p_2}| = 1$, whence $\Gamma \cong W(\mathbb{K})$. This implies that $\Pi^*(\Gamma) \cong \mathbf{PSL}_2(\mathbb{K})$ (see table on page 51).
Secondly we show that the field $\mathbb{K}$ is separably quadratically closed.
For odd characteristic, this is equivalent to showing that $\mathbb{K}$ is quadratically closed. Let $a \in \mathbb{K}$. We show (with an argument deduced from one learned from Norbert Knarr) that a is a square in $\mathbb{K}$. Put $r = \frac{a-1}{2}$ and $s = \frac{a+1}{2}$. The element $x \mapsto \frac{2sx-r}{rx}$ of $\mathbf{PGL}_2(\mathbb{K})$ clearly belongs to $\mathbf{PSL}_2(\mathbb{K})$, hence it has a fixed point x_0 which satisfies the quadratic equation $rx_0^2 - 2sx_0 + r = 0$. Consequently the discriminant $4s^2 - 4r^2 = 4a$ is a square.
For even characteristic, we show that every quadratic equation $Ax^2 + Bx + C = 0$ over $\mathbb{K}$, with $B \neq 0$, has at least one solution in $\mathbb{K}$. We may assume that $A = 1$. As $B \neq 0$, we can put $B_1 = B^{-1}(1+C)$ and $B_2 = B + B_1$. Now the matrix defined by the element $x \mapsto \frac{B_1x+C}{x+B_2}$ of $\mathbf{PGL}_2(\mathbb{K})$ has determinant $\delta = (\frac{1+B+C}{B})^2 \neq 0$ if $B + C \neq 1$. So let $B + C \neq 1$. As δ is a square, $x \mapsto \frac{B_1x+C}{x+B_2}$ belongs to $\mathbf{PSL}_2(\mathbb{K})$, and a fixed element $x_0 \in \mathbb{K}$ satisfies $x_0^2 + Bx_0 + C = 0$, i.e. x_0 is a solution of the equation $x^2 + Bx + C = 0$. If $B + C = 1$, then $x = 1$ is a solution of the equation $x^2 + Bx + C = 0$, which finishes the proof. $\square$

3.4 Characterization of $Q(d,\mathbb{K},\kappa)$ and $H(3,\mathbb{K},\mathbb{K}^{(\sigma)})$ by regular lines and conditions on $\Pi(\Gamma)$

We introduce the following terminology. A **Zassenhaus (permutation) group** is a permutation group acting 2-transitively such that only the identity stabilizes at least 3 points.
Remark that the (special (dual)) group of projectivities acts doubly transitive (theorem 1.30 on page 18), so the first condition in following theorem (i.e. $\Pi(\Gamma)$ is a Zassenhaus group) is not too restrictive.

Theorem 3.2 *Let Γ be a generalized quadrangle all lines of which are regular. Suppose $\Pi(\Gamma)$ is a Zassenhaus group which satisfies the following additional properties:*

- (i) *the set N of all elements of $\Pi(\Gamma)$ fixing only a point p forms, together with the identity, a commutative subgroup of $\Pi(\Gamma)_p$ (the stabilizer of the point p in $\Pi(\Gamma)$);*
- (ii) *every non-identity element of $\Pi(\Gamma)$ with an involutory couple has exactly two fixed elements.*

Then Γ is either an orthogonal quadrangle $Q(d, \mathbb{K}, \kappa)$ or the dual of a quadrangle arising from a σ-hermitian form in a projective space of dimension 3 over a skew field, i.e. $H(3, \mathbb{K}, \mathbb{K}^{(\sigma)})$; in particular, Γ is a Moufang quadrangle. Moreover, the characteristic of $\mathbb{K}$ is odd in both cases. In both cases, the characteristic of $\mathbb{K}$ is odd.

If moreover

- (iii) $\Pi(\Gamma)_{p,q}$ *(the stabilizer in $\Pi(\Gamma)$ of two distinct points p, q) is abelian,*

then there is a field $\mathbb{K}$ of characteristic $\neq 2$ and with -1 a square in $\mathbb{K}$ such that Γ is isomorphic to $Q(4, \mathbb{K})$.

Proof
We put $G = \Pi(\Gamma)$ and we denote by H the (abstract) stabilizer in G of a point (so $H \cong \Pi(\Gamma)_p$). By assumption (i), the set N of elements of H which fix exactly one point, together with the identity, is an abelian (normal) subgroup of H acting on a line minus one point. The set up of the proof, which uses coordinates as introduced in paragraph 1.7.3, is as follows.
$\boxed{1}$ Using hypothesis (i), we first show that N is transitive. $\boxed{2\text{–}5}$ Then, also using hypothesis (ii), we show that Γ is $((\infty), [\infty], (0))$-transitive, so Γ is a half Moufang quadrangle. To that end, we have to show that $\Phi_1(a, k, b \oplus B, k') = \Phi_1(a, k, b, k') \oplus B$ and $\Phi_2(a, k, b \oplus B, k') = \Phi_2(a, 0, B, \Phi_2(a, k, b, k'))$ (see page 20). $\boxed{6}$ In the next part, we show that Γ is Moufang, by composing line elations with the point elations of previous part. $\boxed{7}$ We restrict further on Γ, excluding characteristic 2. This leaves us 2 possible cases for a generalized quadrangle for which (i) and (ii) hold. $\boxed{8}$ Assuming hypothesis (iii) holds, we show that there is only one choice left for Γ, i.e. Γ is an orthogonal quadrangle over a field of odd characteristic. $\boxed{9}$ At last, we restrict the possible dimension of the projective space Γ lives in, to the unique case dim$= 4$.

1. *N is transitive.*
   Let $(R_1, R_2, \Phi_1, \Phi_2)$ be an arbitrary coordinatizing ring. We show that N equals the set $\{\theta_b \,|\, b \in R_1\}$ with θ_b the projectivity

   $$\theta_b = [[\infty]; [1,b,0]; [0]; [1,0,0]; [\infty]].$$

   We may assume that H is the stabilizer of the point labeled (∞) acting on the line $[\infty]$. Let $b \in R_1$ be an arbitrary element. The projectivity $\theta_b = [[\infty]; [1,b,0]; [0]; [1,0,0]; [\infty]]$ maps (0) to (b), and, by definition of $\oplus$ on page 19, maps (a) to $(a \oplus b)$. Suppose θ_b has a fixed point (x), $x \neq \infty$. Since $[\infty]$ is a regular line, there is a line L_x through (x) meeting both $[1,b,0]$ and $[1,0,0]$. Since $(x)^{\theta_b} = (x)$, the projections onto $[0]$ of the intersections of L_x with $[1,b,0]$ and $[1,0,0]$ coincide, hence $b = 0$ or there arises a triangle. So for $b \neq 0$ there are no fixed elements besides (∞), i.e. $\theta_b \in N$. As N was supposed to be abelian and a transitive abelian subgroup is sharply transitive, $N = \{\theta_b \,|\, b \in R_1\}$.

   Considering N as an additive group with operation law $+$ and identity denoted by 0, we can now identify $\theta_b \in N$ and $b \in R_1$. Then the action of $N \leq G$ on $R_1 \cup \{\infty\}$ is given by right translation (fixing ∞) and for every projectivity ρ of the point row $[\infty]$, the mapping $\theta : R_1 \cup \{\infty\} \to R_1 \cup \{\infty\} : x \mapsto x^\theta$ defined by $(x)^\rho = (x^\theta)$ is an element of G. By projecting successively onto $[1,0,0]$ and $[0]$, it follows that, identifying (∞) with $(0,\infty)$, for every projectivity ρ of the point row $[0]$, the mapping $\theta : R_1 \cup \{\infty\} \to R_1 \cup \{\infty\} : x \mapsto x^\theta$ defined by $(0,x)^\rho = (0,x^\theta)$ is an element of G.

2. $\Phi_2(a,k,b \oplus B,k') = \Phi_2(a,0,B,\Phi_2(a,k,b,k'))$.

   As $[\infty]$ is a regular line, Φ_2 is independent of its third argument (see page 20). By the general equality $\Phi_2(a,0,0,x) = x$ (page 20), the result follows.

3. *$a \oplus b$ in R_1 is the same as $a + b$ in N.*

   Just note that $\theta_a + \theta_b = \theta_{a \oplus b}$ since they agree at 0 and N acts sharply transitively.

4. *For any $k, k' \in R_2 \setminus \{0\}$, the projectivity $[[0]; [0,0]; [k]; [0,k']; [0]]$ has no fixed points besides (∞).*

   Let x be a fixed point of the projectivity mentioned. Let x', x'' and x''' be the successive projections onto $[0,0]$, $[k]$ and $[0,k']$. Then the projectivity $[[\infty]; xx'; x''x'''; [\infty]]$ interchanges the points (∞) and (0). By assumption (ii), there are two fixed points (a_i), $i = 1, 2$.

This means that (a_i) and its projections onto xx' and $x''x'''$ are collinear (since otherwise there would be a triangle), say they are incident with M_i, $i = 1, 2$. Since $[\infty]$ is a regular line, every line meeting two elements of $\{[\infty], xx', x''x'''\}$ meets also the third, contradicting the fact that $[k]$ meets $[\infty]$ and $x''x'''$, but not xx'.

5. $\Phi_1(a, k, b \oplus B, k') = \Phi_1(a, k, b, k') \oplus B$.

   Put $a_1 = \Phi_1(a, k, b \oplus B, k')$ and $a_2 = \Phi_1(a, k, b, k')$. We have to show that $a_1 = a_2 + B$. Let L be the line that joins the point (a) to its projection x_2 onto $[k, b, k']$. Then L also meets $[k, b + B, k']$, say in the point x_1 (by regularity of the line $[\infty]$ and the fact that both $[k]$ and $[0, k']$ meet all three of $[\infty], [k, b, k'], [k, b + B, k']$). By definition, the projection of x_i onto $[0]$ is the point $(0, a_i)$, $i = 1, 2$. Now consider the projectivity $\theta = [[0]; L; [k]; [0, 0]; [0]]$. If $a = 0$, then $L = [0, k']$ and θ has no fixed points except for (∞) by the previous paragraph. Suppose now $a \neq 0$ and assume that θ has a fixed point $(0, x)$, $x \in N$. By the previous paragraph, the projectivity $\theta' = [[0]; [0, 0]; [k]; [0, k']; [0]]$ has no fixed points except (∞). Since N is a subgroup, the projectivity $\theta\theta'$ has some fixed element $(0, x')$, $x' \in N$. But $\theta\theta' = [[0]; L; [k]; [0, k']; [0]]$, and since any line joining a point of $[k]$ to its projection onto $[0, k']$ has to intersect L by the regularity of $[\infty]$, this is equal to $\theta\theta' = [[0]; L; [0, k']; [0]]$. Now if $(0, x')$ is a fixed point of $\theta\theta'$, then $[k]$ has to intersect the line defined by $(0, x')$ and its projection to L, yielding a triangle unless $k = 0$, in which case $a_1 = b + B$ and $a_2 = b$ by the identities in the introduction, so the result follows trivially. Hence we can assume that θ has no fixed points distinct from (∞) and so we can write $\theta : (0, x) \mapsto (0, x + C)$ for some $C \in K$. But $a_1^\theta = b + B$ and $a_2^\theta = b$. Hence $a_1 + C = b + B$ and $a_2 + C = b$, consequently $a_1 = a_2 + B$.

6. Γ *is a Moufang quadrangle.*

   We have already shown that Γ is a half Moufang quadrangle. In the finite case, this implies that Γ is Moufang (see Thas, Payne and Van Maldeghem [77] or [84] 5.7.4). For the infinite case though, we prove the following result (which is still more general than the result we need).

   Let Γ be a half Moufang quadrangle. If the set N of elements of $\Pi(\Gamma)$ fixing only one point p forms, together with the identity, a subgroup of $\Pi(\Gamma)$, then Γ is a Moufang quadrangle.

   Let p be a point of Γ, let L_1 and L_2 be distinct lines through p, and let x_i be incident with L_i, $i = 1, 2$, with $x_i \neq p$. Let (x_1, M_1, y, M_2, x_2)

and $(x_1, M_1', y', M_2', x_2)$ be two paths with $y \neq p \neq y'$. We prove that Γ is a Moufang quadrangle by showing that Γ is (L_1, p, L_2)-transitive for the arbitrarily chosen path (L_1, p, L_2). This means we have to find an (L_1, p, L_2)-elation mapping M_1 onto M_1'. We do this by using the composition of three point elations (see page 11 for the definition). Let z be any point on M_2, $y \neq z \neq x_2$. Let (M_1', z', P, z) and (P, z'', P', p) be two paths (which uniquely define the elements z', z'', P, P'). Let θ be the (p, L_2, x_2)-elation mapping y to z. Let θ' be the (p, P', z'')-elation mapping z to z'. Then θ' induces on the point row of L_2 an element η belonging to N (considering N with respect to L_2 and p, i.e., considering N as a subgroup of the group of self-projectivities of L_1 and each element of N fixes p). Let θ'' be the (x_1, L_1, p)-elation which maps x_2^η onto x_2 (or equivalently, z' to y'), then θ'' induces on L_2 the mapping η^{-1} because x_2 is fixed by $\theta'\theta''$. Also, if θ induces the mapping η' on the point row of L_1, then θ' similarly induces ${\eta'}^{-1}$ on L_1 since $\theta\theta'$ fixes x_1. Hence the collineation $\theta\theta'\theta''$ is an (L_1, p, L_2)-elation which maps y to $y^{\theta\theta'\theta''} = z^{\theta'\theta''} = z'^{\theta''} = y'$.

7. *Which Moufang quadrangles have regular lines?*

By the classification of Moufang quadrangles found in Tits and Weiss [82] and cited in Van Maldeghem [84] table 5.1 on page 220, the only Moufang quadrangles with regular lines are the orthogonal quadrangles, the duals of the quadrangles arising from σ-hermitian forms over a skew field $\mathbb{K}$ in projective spaces of dimension 3, the duals of some quadrangles arising from σ-hermitian forms over a skew field $\mathbb{K}$ of characteristic 2 in projective spaces of dimension > 3, and so-called mixed quadrangles (which are subquadrangles of symplectic quadrangles in characteristic 2). However, no Moufang quadrangle satisfying hypothesis (ii) of theorem 3.2 has root elations of even order (because these root elations have involutory couples without having two fixed points). This rules out all Moufang quadrangles defined over a field in characteristic 2, in particular the last two classes mentioned above. This proves the first statement of theorem 3.2.

8. *Moufang quadrangles arising from σ-hermitian forms in projective spaces of dimension* 3.

Suppose (iii) holds. We show that, if Γ is the dual of a Moufang quadrangle arising from a σ-hermitian form in a projective space of dimension 3 over the skew field $\mathbb{K}$ (i.e. the second possibility for a quadrangle satisfying (i) and (ii)), it is isomorphic with an orthogonal quadrangle. (To fix the ideas, we refer to the list of all compact connected Moufang quadrangles and their dualities on page 12 and 9.) So let $\Delta = \Gamma^D$ be a Moufang quadrangle isomorphic with $H(3,\mathbb{K},\mathbb{K}^{(\sigma)})$. It is shown in Tits [81] (10.10, page 213), that the dual group of projectivities of Δ contains all maps of the form

$$\mathbb{K}_{\sigma,-1} \rightarrow \mathbb{K}_{\sigma,-1} : x \mapsto a^{\sigma} x a,$$

where $a \in \mathbb{K} \setminus \{0\}$, and $\mathbb{K}_{\sigma,-1} = \{t + t^{\sigma} \mid t \in \mathbb{K}\}$. These projectivities all fix two elements (i.e. 0 and ∞) and hence they must commute with each other (since they are a subgroup of the stabilizer of two elements in $\mathbf{PSL}_2(\mathbb{K})$). By Tits [81] 10.5 and 10.9, Δ is the dual of an orthogonal quadrangle, and hence Γ is an orthogonal quadrangle embedded in $\mathbf{PG}(d,\mathbb{K})$ with $d = 5$ or $d = 7$. (Remember also that $\mathbb{K}$ does not have characteristic 2.)

9. *Orthogonal quadrangles.*

We show that every orthogonal quadrangle which is not isomorphic to $Q(4,\mathbb{K})$, has fixed point free involutory self-projectivities of a line, in contradiction with assumption (iii) of the theorem.

Let Γ be not isomorphic to $Q(4,\mathbb{K})$. Let $\mathbf{PSL}_2(\mathbb{K}) = \{\theta \mid \theta : x \mapsto \frac{ax+b}{cx+d}\,;\, ad - bc \text{ square in } \mathbb{K}\}$ be a (not necessarily proper) subgroup of $\Pi(\Gamma)$. As $\mathbf{PSL}_2(\mathbb{K})$ is 2-transitive, there exists at least one $\theta \in \mathbf{PSL}_2(\mathbb{K})$ mapping 0 to ∞ and ∞ to 0. Let this be $\theta : x \mapsto \frac{b}{x}$, with $-b$ a square (so we can write $\theta : x \mapsto -\frac{f^2}{x}$). Clearly, θ is an involution.

- If -1 is not a square in $\mathbb{K}$, θ has no fixed points.
- If -1 is a square in $\mathbb{K}$, we need a mapping $\theta' : x \mapsto ux$, u a non-square, such that the composition $\theta\theta' : x \mapsto -\frac{uf^2}{x}$ has no fixed points. Hence we look for an element of $\Pi(\Gamma)$ corresponding with an element of $\mathbf{PGL}_2(\mathbb{K}) \setminus \mathbf{PSL}_2(\mathbb{K})$.
  Before we do so, we should make sure $\boxed{1}$ that we may assume that there are non-squares in $\mathbb{K}$ (or, equivalently, that if $\mathbb{K}$ is quadratically closed, the result follows for some other reason) and $\boxed{2}$ that it is sufficient to find such a mapping θ' for dimension $d = 5$. We will do these things first.

$\boxed{1}$ If $\mathbb{K}$ is quadratically closed, then $d = 4$ and $\Gamma \cong Q(4,\mathbb{K})$, in contradiction with the assumptions. So the result follows.
$\boxed{2}$ The following remark holds in general: *Suppose Γ' is a subquadrangle of the generalized quadrangle Γ. Let L be a line of Γ' (note that L is also a line of Γ) and suppose that the group of projectivities*

of L in Γ is a Zassenhaus group (it suffices that the stabilizer of any $|\Gamma'(L)|$ many points is the identity). Then clearly any projectivity of L in Γ' lifts uniquely *to a projectivity of L in Γ. Hence we may consider the group of projectivities of L in Γ' as a subgroup of the group of projectivities of L in Γ.* As $Q(5,\mathbb{K}) \subset Q(n,\mathbb{K})$ for all $n > 5$, it follows that we only need a mapping θ' in $Q(5,\mathbb{K})$, and the result will follow.

Now we can go on with the search for θ'.
Using the coordinatization of $Q(5,\mathbb{K})$ (see page 21) and putting $i^2 = -1$, we show that the self-perspectivity

$$[[\infty];[(0,i),0,(0,0)];[(0,0)];[(1,0),0,(0,0)];[\infty]]$$

of the line $[\infty]$ fixes the point (∞), and maps the point (x) to (ux). Hence $\Pi(\Gamma)$ contains the element determined by $\theta' : x \mapsto ux$. The reasoning goes as follows.
The projection of a point p onto a line L, $p \notin L$, can be calculated in two ways: either by using the quaternary operations $\Phi_1, \Phi_2/\Psi_1, \Psi_2$, or by switching to projective coordinates. This last method is by far the easiest. Indeed, the projection of p onto L is the intersection of L with the tangent space of Q in p, with Q the quadric containing $Q(5,\mathbb{K})$. So we first change to projective coordinates for the lines used in the perspectivity.

L_1	$[(0,i),0,(0,0)]$	$\langle(0,1,0,i,-u,0),(0,0,0,0,0,1)\rangle$
L_2	$[(0,0)]$	$\langle(1,0,0,0,0,0),(0,1,0,0,0,0)\rangle$
L_3	$[(1,0),0,(0,0)]$	$\langle(0,1,1,0,-1,0),(0,0,0,0,0,1)\rangle$
L_4	$[\infty]$	$\langle(1,0,0,0,0,0),(0,0,0,0,1,0)\rangle$

The quadric Q is defined by $Q \leftrightarrow X_0X_5+X_1X_4+X_2^2-uX_3^2=0$ with u a non-square in $\mathbb{K}$, so the tangent space in the point $(a_0,a_1,\dots,a_5)$ of Q has equation $a_5X_0+a_4X_1+2a_2X_2-2ua_3X_3+a_1X_4+a_0X_5=0$. Let $p_0=(x)=(x,0,0,0,1,0)$. The tangent space in p_0 of Q has equation $xX_5+X_1=0$, intersecting L_1 in $p_1(0,-x,0,-ix,ux,1)$. The tangent space in p_1 of Q intersects L_2 in $p_2(-ux,1,0,0,0,0)$; the tangent space in p_2 of Q intersects L_3 in $p_3(0,-ux,-ux,0,ux,1)$. Finally, the tangent space in p_3 of Q intersects L_4 in $p_4(ux,0,0,0,1,0)$. This corresponds with the quadrangle-coordinates (ux), so θ' maps (x) to (ux), and it is easily seen that (∞) is fixed. □

Remark that the last paragraph of previous proof includes the following corollary.

Corollary 3.1 *If Γ is an orthogonal quadrangle $Q(d,\mathbb{K},\kappa)$, with $\mathbb{K}$ a field of characteristic $\neq 2$, in which -1 is a square, then $\Pi(\Gamma)$ is permutation equivalent to $\mathbf{PSL}_2(\mathbb{K})$ if and only if $d = 4$.*

More generally, we can proof next result.

Corollary 3.2 *Let Γ be a line-regular generalized quadrangle, and let $\mathbb{K}$ be some (commutative) field. If $\Pi(\Gamma)$ is permutation equivalent to $\mathbf{PSL}_2(\mathbb{K})$ acting on the projective line $\mathbf{PG}(1,\mathbb{K})$, with either $\mathbb{K}$ separably quadratically closed, or* char $\mathbb{K} \neq 2$ *and -1 is a square in $\mathbb{K}$, then $\Gamma \cong Q(4,\mathbb{K})$.*

Proof
$\boxed{1}$ First, let $\mathbb{K}$ be a field of characteristic $\neq 2$ in which -1 is a square. We show that $\mathbf{PSL}_2(\mathbb{K})$ acting on $\mathbf{PG}(1,\mathbb{K})$ satisfies the assumptions of theorem 3.2. Indeed, this action can be identified with the action of the rational transformations $x \mapsto \frac{ax+b}{cx+d}$, with $ad - bc$ a non-zero square in $\mathbb{K}$. The stabilizer of ∞ is $\mathbf{AGL}_1^+(\mathbb{K})$, its elements being the maps of the form $x \mapsto a^2x + b$. If such a function has no fixed point apart from ∞, then clearly $a^2 = 1$, and the elements with $a^2 = 1$ form a subgroup of $\mathbf{PSL}_2(\mathbb{K})$. Moreover, the stabilizer of ∞ and 0 is commutative (and consists of the elements of the form $x \mapsto a^2x$, $a \neq 0$). Also, any element with an involutory couple, say $(\infty, 0)$, has the form $x \mapsto -a^2/x$, $a \neq 0$. Since -1 is a square in $\mathbb{K}$, this has two distinct fixed points (ia) and $(-ia)$ where $i^2 = -1$.
$\boxed{2}$ On the other hand, let $\mathbb{K}$ be a separably quadratically closed field. Then obviously every rational transformation $x \mapsto \frac{ax+b}{cx+d}$ has a fixed element (as every quadratic equation has at least one solution in $\mathbb{K}$), so the conditions of theorem 3.1 are satisfied, and the result follows. $\square$
We should point out that the conditions on the permutation group $\Pi(\Gamma)$ in theorem 3.2 are a special case of a characterization of subgroups of $\mathbf{PGL}_2(\mathbb{K})$ due to Ma‘127 urer [44]. By his result, $\Pi(\Gamma)$ satisfies assumptions $(i) - (iii)$ if and only if it is permutation equivalent to $\mathbf{PSL}_2(\mathbb{K})$ for a field of characteristic $\neq 2$ with -1 a square in $\mathbb{K}$ acting on the projective line $\mathbf{PG}(1,\mathbb{K})$.

3.5 Characterization of $Q(4,q)$ and $Q(5,q)$ by regular lines and conditions on $\Pi(\Gamma)$ and $\Pi^*(\Gamma)$

We first explain some notation, already appeared in the table on page 51. Namely, for any prime power q, we denote by $\mathbf{PGL}_2^{(\sqrt{q})}(q)$ the group of all projective transformations of $\mathbf{PG}(1,q)$ generated by $\mathbf{PGL}_2(q)$ and the transformation induced by the semi-linear mapping with identity matrix and corresponding field automorphism $x \mapsto x^{\sqrt{q}}$, if q is a square. If q is not a square, then we read $\sqrt{q}$ as the identity in this definition.

Theorem 3.3 *Let Γ be a finite line-regular generalized quadrangle of order (s,t). Then Γ is isomorphic to $Q(4,s)$ or to $Q(5,s)$ if and only if $\Pi(\Gamma)$ is permutation equivalent to a subgroup of $\mathbf{PGL}_2(s)$ acting naturally on $\mathbf{PG}(1,s)$ and $\Pi^*(\Gamma)$ is permutation equivalent to a subgroup of $\mathbf{PGL}_2^{(\sqrt{t})}(t)$ acting naturally on $\mathbf{PG}(1,t)$.*

Proof

Again, we stated the theorem for orthogonal quadrangles, but we will prove the dual. So let Γ be a finite point-regular generalized quadrangle of order (s,t), such that $\Pi^*(\Gamma)$ is permutation equivalent to a subgroup of $\mathbf{PGL}_2(t)$ acting naturally on $\mathbf{PG}(1,t)$ and $\Pi(\Gamma)$ is permutation equivalent to a subgroup of $\mathbf{PGL}_2^{(\sqrt{s})}(s)$ acting naturally on $\mathbf{PG}(1,s)$.

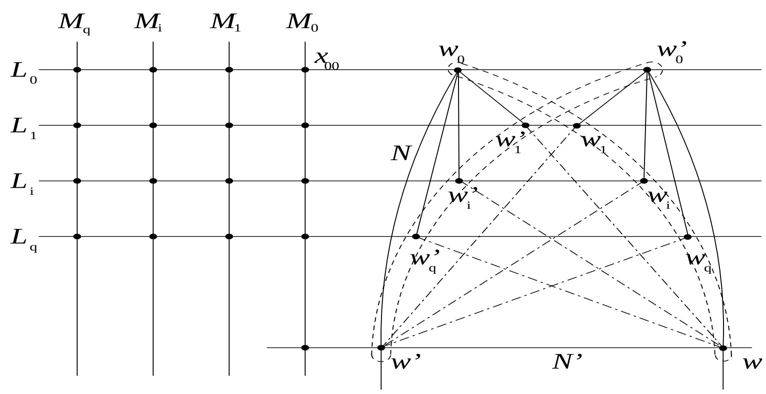
$\boxed{1}$ We first remark that $\Pi^*(\Gamma)$ satisfies condition (i) of theorem 3.2. Indeed, as N is a subset of $\Pi^*(\Gamma)$ which is a subgroup of $\mathbf{PGL}_2(q)$, every element of N corresponds with a map $x \mapsto x+b$ *for some* b. As $\Pi^*(\Gamma)$ is 2-transitive (theorem 1.30), the set N acts transitively on the point set of a line minus one point, so N is the group of projectivities corresponding with *all maps* of the form $x \mapsto x+b$.

$\boxed{2}$ Now suppose Γ does not contain any 3×3-grid. This condition may replace condition (ii) in the proof of theorem 3.2. Indeed, condition (ii) is only used in the proof of that theorem in paragraph 5. With the notation of that paragraph, the points (∞), (0), x, x', x'' and x''' form the dual of a 3×3-grid, if x is a fixed point of the projectivity under consideration.

$\boxed{3}$ Hence we assume that Γ does contain a 3×3-grid. Let $\{L_1,L_2,L_3;M_1,M_2,M_3\}$ be the six lines of this grid $\mathcal{G}$, with $L_i \perp M_j$, $i,j : 1 \to 3$. Now we turn our attention to $\Pi(\Gamma)$. $\boxed{3a}$ If s is not a square, $\Pi(\Gamma)$ is — by assumption — contained in $\mathbf{PGL}_2(s)$. But as the projectivity $[L_1;L_2;L_3;L_1]$ of $\Pi(\Gamma) \subset \mathbf{PGL}_2(s)$ has at least 3 fixpoints (i.e. $L_1 \cap M_i$, $i : 1 \to 3$), this projectivity is the identity. So $|\{L_1,L_2,L_3\}^\perp| = s+1$. We put $\{L_1,L_2,L_3\}^\perp = \{M_1,M_2,\ldots,M_{s+1}\}$. Doing the same for the projectivity $[M_1;M_2;M_i;M_1]$ with $i : 3 \to s+1$, we have a $(s+1)\times(s+1)$-grid, so Γ contains a regular pair of lines. By theorem 1.5 on page 7, $s=t$ is even. By theorem 1.8 on page 12, Γ is (dual to) $W(s)$. $\boxed{3b}$ So we may assume that s is a perfect square, say $s=q^2$, and that $\Pi(\Gamma)$ contains a non-linear semi-linear transformation. Note that, since Γ contains regular points, $s \geq t$ (see page 7).

1. *Every 3×3-grid is contained in a maximal $(q+1) \times (q+1)$-grid.* We do the analogue of the argumentation in previous paragraph. Take $\{L_1, L_2, L_3; M_1, M_2, M_3\}$ as above. The projectivity $\sigma = [L_1; L_2; L_3; L_1]$ has at least three fixed points. Identifying in $\mathbf{GF}(q^2) \cup \{\infty\}$ these points with $0, 1, \infty$, we readily see that σ is either the identity or the map $x \mapsto x^q$. If a is fixed under σ, then $\{a, \mathrm{proj}_{L_2} a, \mathrm{proj}_{L_3} a\}$ forms a triangle, hence $\mathrm{proj}_a L_2 = \mathrm{proj}_a L_3$. It follows that $|\{L_1, L_2, L_3\}^\perp| \geq q+1$. Suppose $\{M_0, M_1, \ldots, M_q\} \subseteq \{L_1, L_2, L_3\}^\perp$, with $M_i \neq M_j$ for $i \neq j$, $i, j \in \{0, 1, \ldots, q\}$. The projectivity $\sigma_i = [M_0; M_i; M_q; M_0]$, $0 < i < q$, has at least three fixed points which are independent of i. Identifying these points with $0, 1, \infty$ again, we deduce similarly as before that there are at least $q+1$ lines L_j (namely, one line through each point on M_0 corresponding to an element of $\mathbf{GF}(q) \cup \{\infty\}$) in $\{M_0, M_1, \ldots, M_q\}^\perp$. Hence we already have a $(q+1) \times (q+1)$-grid $\mathcal{G}$ containing $L_1, L_2, L_3, M_1, M_2, M_3$. We now show that this grid is maximal. In fact, we will show that whenever a line L meets three of the lines $M_0, M_1, \ldots, M_q$, then it must meet all of them and it must belong to $\mathcal{G}$. We may assume $L_1 \neq L \neq L_2$. By considering the projectivity $[L; L_1; L_2; L]$, we see as above that L must belong to $\{M_0, M_1, \ldots, M_q\}^\perp$. Now, if $L \notin \mathcal{G}$, then the projectivity $[M_0; M_i; M_q; M_0]$, $0 < i < q$, has at least $q+2$ fixed points, hence it is the identity and we easily deduce (as before) that $|\{M_0, M_1, \ldots, M_q\}^\perp| = q^2+1$. Now consider a point w on L_1 not incident with any M_i, $0 \leq i \leq q$. Let $\{M_0, M_1 \ldots, M_q\}^\perp = \{L_0, L_1, \ldots, L_{q^2}\}$. Let $N_i = \mathrm{proj}_w L_i$, $0 \leq i \leq q^2$, $i \neq 1$. $\boxed{1}$ If $N_i = N_j$ for some $i \neq j$, then N_i meets three of the lines $L_0, L_1, \ldots, L_{q^2}$ and as before, we deduce that it must meet every such line, and again as before this implies that $\mathcal{G}$ is contained in an $(s+1) \times (s+1)$-grid. Now Γ has all regular points and a regular pair of lines, so again by theorems 1.5 and 1.8, $s = t$ is even and $\Gamma \cong W(\mathbb{K})$. $\boxed{2}$ So the lines N_i, $0 \leq i \leq q^2$, $i \neq 1$, are pairwise distinct and we obtain $s \leq t$. Consequently $s = t$, a contradiction. We conclude that $\mathcal{G}$ is maximal.

2. *We have $t = q$.* Let $\mathcal{G}$ be a $(q+1) \times (q+1)$-grid with line set $\{L_0, L_1, \ldots, L_q, M_0, M_1, \ldots, M_q\}$, and with $L_i \perp M_j$, for all $i, j \in \{0, 1, \ldots, q\}$. Let w_0 be a point on L_0 not belonging to the grid. By the previous paragraph, the q lines N_i, $1 \leq i \leq q$, incident with w_0 and concurrent with L_i are mutually distinct. Now assume that $t \neq q$, i.e., $t > q$. Then there is some further line N through w_0, $N \neq N_i$, $i = 1, 2, \ldots, q$. Consider the projectivity $\sigma_i = [L_0; L_i; L_q; L_0]$. It has exactly $q+1$ fixed points, hence it is an involution σ (in fact, independent of i). Let w_0' be



the image of w_0 under σ. (We refer to the picture.) Let w_i be the projection of w_0' onto L_i, and let w_i' be the projection of w_0 onto L_i, $1 \leq i \leq q$. Using the fact that σ_i is involutory, we easily see that $w_0', w_1', \ldots, w_q' \in \{w_0, w_q\}^\perp$ and $w_0, w_1, \ldots, w_q \in \{w_0', w_q'\}^\perp$. By the regularity of points in Γ, we deduce that w_i and w_j' are collinear for all $i, j \in \{0, 1, \ldots, q\}$. Now let w' be the point of N collinear with w_q, and hence with w_i, for all $i \in \{0, 1, \ldots, q\}$. Let N' be the line through w' concurrent with M_0. Let w be the projection of w_0' onto N'. Then w is collinear with w_i', for all $i \in \{0, 1, \ldots, q\}$. Denote by x_{ij} the intersection of L_i with M_j. Then the projectivity $\theta = [L_0; L_q; N'; L_0]$ has a fixed point x_{00} and an involutory couple (w_0, w_0'). Hence it is an involution. If a semi-linear involution fixes ∞ and has an involutory couple $(0, b)$, it is of the form $x \mapsto -(b/b^q)x^q + b$. Hence if θ is semi-linear but not linear, then it coincides with the involution $\sigma = [L_0; L_1; L_q; L_0]$ since they agree on x_{00} and the involutory couple (w_0, w_0'). Hence θ has in particular the same set of fixed points as σ. As before, this implies that N' meets all M_i, $i \in \{0, 1, \ldots, q\}$, and so $\mathcal{G}$ is not maximal, a contradiction. We conclude that θ is linear. Similarly the projectivity $\theta' = [L_0; L_1; N'; L_0]$ is a linear involution. We deduce $\theta = \theta'$. Hence $\theta'\theta^{-1} = [L_0; L_1; N'; L_q; L_0]$ is the identity. It readily follows that the line M_i, $i \in \{0, 1, \ldots, q\}$, is concurrent with N'. Hence again, $\mathcal{G}$ is not maximal, a contradiction. We conclude that $t = q$.

3. Γ *is a Moufang quadrangle.*
We already have that Γ has order (q^2, q). By theorem 1.3 on page 7, every three pairwise opposite lines are contained in a $(q+1) \times 3$-grid,

in particular in a 3×3-grid. Hence, every three pairwise opposite lines are contained in a maximal $(q+1) \times (q+1)$-grid. The result now follows directly from the dual of theorem 1.9. $\square$

3.6 Characterization of $Q(5,q)$ by order and a condition on $\Pi_+^*(\Gamma)$

Theorem 3.4 *Let Γ be a finite generalized quadrangle of order (q,q^2). Then Γ is isomorphic to $Q(5,q)$ if and only if $\Pi_+^*(\Gamma)$ is a Zassenhaus group.*

Proof

Again, the proof is done for the dual $\Delta = \Gamma^\Delta$ of order (q^2,q) with $\Pi_+(\Delta)$ a Zassenhaus group. Let L_0 be any line of Δ. Let L_1 be any line opposite L_0, and let M_0, M_1, M_2 be three different lines concurrent with both L_0 and L_1. As above, we know that L_0, L_1, M_0, M_1, M_2 are contained in a $(q+1) \times 3$-grid containing $q+1$ lines $L_0, L_1, L_2, \ldots, L_q$ which are all concurrent with M_0, M_1, M_2. Similarly, there are $q+1$ lines $M_0, M_1, M_2, \ldots, M_q$ concurrent with L_0, L_1, L_2. If we show that L_j meets M_i, for $i, j \in \{3, 4, \ldots, q\}$, then as above, we are done (again using theorem 1.9). Therefore, consider the even projectivity $\theta = [L_0; L_1; L_2; L_j; L_0]$. Clearly the intersection points of L_0 with M_0, M_1, M_2, respectively, are fixed by θ. By assumption, also the intersection point x of L_0 and M_i is fixed. This yields a triangle with vertices $x, \mathrm{proj}_{L_2} x, \mathrm{proj}_{L_j} x$ if L_j does not meet M_i. $\square$

Chapter 4

A Characterization of $H(q)$ and $T(q^3, q)$ using Ovoidal Subspaces

4.1 Introduction

For finite generalized quadrangles of order (q, q), theorem 1.8 on page 12 gives a characterization of $W(q)$ by point-regularity. In fact, the second assertion tells us that we only have to be sure of the regularity of points of a geometric hyperplane. By generalizing the definition of a geometric hyperplane to that of an ovoidal subspace, we provide a similar result for hexagons. We do not require the order of the hexagon to be (q, q), but instead we assume the existence of 'a lot of' thin ideal subhexagons. (If the order is already known to be (q, q), this condition will be superfluous in 2 of the 3 cases.) So in this chapter (see also [6] in the proceedings of the Third International Conference at Deinze), we complete theorem 1.18 on page 13 in much the same way as theorem 1.8 has been completed, with that difference that we characterize both classical hexagons $H(q)$ and $T(q^3, q)$.

4.2 Definition of ovoidal subspace

An **ovoidal subspace** $\mathcal{A}$ of a generalized $2m$-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a proper non-empty set of points $\mathcal{A} \subset \mathcal{P}$, with an induced set of lines $\mathcal{A}' = \{L \in \mathcal{L} \mid \Gamma_1(L) \subset \mathcal{A}\}$, such that all elements of Γ are at distance $\leq m$ from a certain point of $\mathcal{A}$, and such that for all elements of $\Gamma \backslash (\mathcal{A} \cup \mathcal{A}')$ at distance $< m$ from a certain point p of $\mathcal{A}$, this point p is unique.
The notion 'ovoidal' is inspired by the ovoids, being special cases of ovoidal subspaces.

To show the likeness between the definition of Γ itself and the definition of an ovoidal subspace of Γ, we define the distance between a point b and a point set $\mathcal{A}$ as $\delta(b, \mathcal{A}) = \min\{\delta(b, a) | a \in \mathcal{A}\}$. Then we can formally write their respective definitions as follows (disregarding the order (s, t)):

$\boxed{\Gamma}$ (1) Given a; $\max\{\delta(a, b) | b \text{ element of } \Gamma\} = 2m$
(2) Given a; $\forall b$ element of $\Gamma : \delta(a, b) < 2m$
$\Rightarrow \exists$ unique shortest path between a, b

$\boxed{\mathcal{A}}$ (1) Given $\mathcal{A}$; $\max\{\delta(\mathcal{A}, b) | b \text{ element of } \Gamma\} = m$
(2) Given $\mathcal{A}$; $\forall b$ element of $\Gamma \backslash (\mathcal{A} \cup \mathcal{A}') : \delta(\mathcal{A}, b) < m$
$\Rightarrow \exists$ unique shortest path between $\mathcal{A}, b$

4.3 Classification of ovoidal subspaces in hexagons

For Γ a generalized quadrangle of order (s, t), an ovoidal subspace is the same as a geometric hyperplane (see page 5). We recall that this is an ovoid, the point set of a subquadrangle of order (s, t'), $st' = t$, or the set of all points collinear with a given point. For a generalized hexagon, the corresponding result is stated in theorem 4.1. First we give a lemma.

Lemma 4.1 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized hexagon of order (s, t). An ovoidal subspace $\mathcal{A}$ is a set of points such that each point of the hexagon not in $\mathcal{A}$, is collinear with a unique point of $\mathcal{A}$.*

Proof $\boxed{\Rightarrow}$ Take $x \in \Gamma \backslash \mathcal{A}$. As the distance between 2 points is even, x is at distance 2 from a certain point p of $\mathcal{A}$. By the second condition, this point p is unique. $\boxed{\Leftarrow}$ Take $x \in \Gamma$. If $x \in \mathcal{A}$, it is at distance $0 \leq 3$ from a point of $\mathcal{A}$. If $x \notin \mathcal{A}$, it is at distance $2 < 3$ from a unique point of $\mathcal{A}$. $\square$

We will use the following properties of ovoidal subspaces of generalized hexagons frequently.

- Whenever a line meets $\mathcal{A}$ in 2 points, all points of the line belong to $\mathcal{A}$ — because they are collinear with two different points of $\mathcal{A}$.
- Whenever two points x, y at distance 4 belong to $\mathcal{A}$, $x \bowtie y$ belongs also to $\mathcal{A}$ (in the other case, $x \bowtie y$ would be collinear with 2 points of $\mathcal{A}$, $x \bowtie y$ being off $\mathcal{A}$).

Theorem 4.1 *An ovoidal subspace of a generalized hexagon of order (s,t) is an ovoid, or the set of all points at distance 1 or 3 from a given line L, or the point set of a full generalized subhexagon of order $(s, \sqrt{\frac{t}{s}})$.*

Proof

1. If every point, lying inside or outside $\mathcal{A}$, is collinear with exactly one point of $\mathcal{A}$, the subspace $\mathcal{A}$ is an ovoid — by definition.
2. Suppose there is a point in $\mathcal{A}$, collinear with a second point of $\mathcal{A}$; this means, suppose $\mathcal{A}$ contains a line L.
   (a) We show that for 2 points of $\mathcal{A}$, their distance $d_{\mathcal{A}}$ measured in $\mathcal{A}$ will be the same as their distance d_Γ measured in Γ, provided we add to $\mathcal{A}$ all lines N of Γ with $\Gamma_1(N) \subseteq \mathcal{A}$. Say $x, y \in \mathcal{A}$.
   If $d_\Gamma(x,y) < 6$, the unique path of length d_Γ between x and y also belongs to $\mathcal{A}$. It follows that $d_\Gamma(x,y) = d_{\mathcal{A}}(x,y)$.
   Suppose $d_\Gamma(x,y) = 6$. $\boxed{1}$ Suppose $d_\Gamma(x,L) = 5 = d_\Gamma(y,L)$. Draw the unique path $(x, xx_2, x_2, x_2x_3, x_3, L)$. As $\delta(x, x_3) = 4$ and $x, x_3 \in \mathcal{A}$, we know that all points of this path belong to $\mathcal{A}$. As $d_\Gamma(y, xx_2) = 5$, we can project y onto xx_2, and call this projection y'. As $\delta(y,y') = 4$ and $y, y' \in \mathcal{A}$, all points of the path between y and y' belong to $\mathcal{A}$. So we constructed a path in $\mathcal{A}$ of length 6 between x and y: $d_\Gamma(x,y) = d_{\mathcal{A}}(x,y)$. $\boxed{2}$ For $d_\Gamma(x,L) \neq 5$ or $d_\Gamma(y,L) \neq 5$, the proof is completely similar.
   (b) Now we claim that there are two points of $\mathcal{A}$ at distance 6 from each other. Take a point p of Γ, at distance 5 of L and denote the joining path by $(p, pp_2, p_2, p_2p_3, p_3, L)$. $\boxed{1}$ If $p \in \mathcal{A}$, one can find s pairs (p, u), $u \in L$, with u at distance 6 from p. $\boxed{2}$ If $p \notin \mathcal{A}$, p is collinear with a unique point x of $\mathcal{A}$. $\boxed{\text{a}}$ If $x = p_2$, then take a point q collinear with p, but not on pp_2. This point q does not belong to $\mathcal{A}$ (as p is collinear with just one point of $\mathcal{A}$), so is itself collinear with a unique point $y \in \mathcal{A}$. As $\delta(x,y) = 6, x, y, \in \mathcal{A}$, the claim follows. $\boxed{\text{b}}$ If $p_2 \neq x \in pp_2$, then $(x, xp_2, p_2, p_2p_3, p_3, L)$ belongs to $\mathcal{A}$, and so does p, a contradiction. $\boxed{\text{c}}$ If $x \notin pp_2$, then $\delta(x, p_3) = 6$.

(c) At this point, we know 2 points of $\mathcal{A}$ at distance 6 (in $\mathcal{A}$), say x and y. So $\mathcal{A}$ contains at least one path $(x, xx', x', M, y', y'y, y)$ between x and y (by (a)).

If $\mathcal{A}$ contains an apartment, it is a full subhexagon of order (s, t'). By Thas [66], we know $st'^2 = t$. (See also theorem 1.2 and page 91.) If Γ has order s, $\mathcal{A}$ will be of order $(s, 1)$.

If $\mathcal{A}$ does not contain any apartment, we show that $\mathcal{A} = \Gamma_1(M) \cup \Gamma_3(M)$.

$\boxed{1}$ We show that every point of $\mathcal{A}$ is at distance ≤ 3 from M. Suppose $z \in \mathcal{A}, z \in \Gamma_5(M), \mathrm{proj}_M z = z'$. Without loss of generality, $z' \neq y'$, so $\delta(z, yy') = 5$. As $\mathrm{proj}_{yy'} z = y''$ belongs to $\mathcal{A}$, there are 2 paths of length 6 joining z and y'. This is an apartment, and hence a contradiction.

$\boxed{2}$ We show that every point of Γ at distance ≤ 3 from M belongs to $\mathcal{A}$.

Suppose $u \notin \mathcal{A}, u \in \Gamma_3(M), \mathrm{proj}_M u = u'$. Take a point z collinear with u, at distance 5 from M. As $z \notin \mathcal{A}$ (by the previous section), z is collinear with a unique point z' of $\mathcal{A}$. If $z' \in \Gamma_3(M)$, then there is a pentagon with edges $\{z', z, u, u', u' \bowtie z'\}$ (if $\delta(u', z') = 4$) or a quadrangle (if $\delta(u', z') = 2$). If $z' \in \Gamma_1(M)$, it is even worse: a quadrangle or a triangle arises. $\square$

4.4 Main result

Theorem 4.2 *Let $\mathcal{A}$ be an ovoidal subspace of the generalized hexagon Γ of order (s, t). Then $\Gamma \cong H(q)$ or $T(q^3, q)$ if and only if*
($\star$) any 2 opposite points of Γ are contained in a thin ideal subhexagon $\mathcal{D}$ and
($\star\star$) all points of $\mathcal{A}$ are span-regular.

By the previous classification, we distinguish 3 different types of ovoidal subspaces in a generalized hexagon. We will consider each of them separately, obtaining five different theorems (4.3 to 4.7), adding up to the proof of the theorem above. From these results, it will follow that condition ($\star$) becomes superfluous in certain cases. The five theorems are organized as follows:

Thm 4.3 To start with, let $\mathcal{A}$ be an ovoid. As for all known finite generalized hexagons, it are only the ones with order $s = t$ which possibly possess an ovoid, we first consider this particular case. In fact, this proof is already known. The main idea is to count the thin ideal subhexagons

$\mathcal{D}$ of the given hexagon Γ. This counting argument (1) can be written as follows:

$$X \leq \beta \leq Y$$

with

- X the number of pairs of opposite points through which there exists a $\mathcal{D}$ containing 2 points of $\mathcal{A}$;
- β the number of pairs of opposite points through which there exists a $\mathcal{D}$;
- Y the number of pairs of opposite points.

Whenever $\boxed{1}$ $X = \beta$, each $\mathcal{D}$ contains 2 points of $\mathcal{A}$. Whenever $\boxed{2}$ $\beta = Y$, we know that through each $x, y \in \mathcal{P}$, there is a $\mathcal{D}$.

For $\mathcal{A}$ being an ovoid in Γ of order s, condition $\boxed{1}$ as well as condition $\boxed{2}$ will be satisfied. Hence theorem 4.2 holds without condition $(\star)$.

<u>Thm 4.4</u> Then we consider $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, Γ of order s. In lemma 4.2 we do approximately the same counting as mentioned before, and — as $s = t$ — we conclude that $\boxed{1}$ and $\boxed{2}$ are satisfied. Hence the second part of the proof of theorem 4.2 is completely similar to the first part. Here, too, the condition $(\star)$ is redundant.

<u>Thm 4.5</u> Let $\mathcal{A}$ be the point set of a full subhexagon in the third part. Here we can prove that Γ should be of order s, while $\mathcal{A}$ has order $(s, 1)$. Indeed, if Γ of order (s, t) contains a subhexagon $\mathcal{A}$ of order (s, t'), we know $t' \leq s \leq t$ (see theorem 1.2 on page 7). As Γ has span-regular points, we know $t \leq s$ (see theorem 1.6). So $t = s$, and $t' = 1$. Unfortunately, we can not use the same counting argument (1), as X is never equal to Y if $\mathcal{A}$ is a thin full subhexagon. Nevertheless, we are able to re-arrange the proof with only half of the counting argument: we assume that $\boxed{2}$ $\beta = Y$ (this is exactly condition $(\star)$), and we do not use the (wrong) assumption $\boxed{1}$ that $X = \beta$.

<u>Thm 4.6</u> But by now, we can also re-arrange the proof in case of $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$: we do not require s to be equal to t, but we assume condition $(\star)$. So only using condition $\boxed{2}$, we are still able to complete the proof.

<u>Thm 4.7</u> At last, we can — technically — do the same for $\mathcal{A}$ being an ovoid. Suppose we do not know anything of the order (s, t) of Γ, then — assuming condition $(\star)$ — theorem 4.2 is still true. (However, it is known $T(q^3, q)$ does not have an ovoid.)

4.5 Five theorems proving the main result

4.5.1 $\mathcal{A}$ an ovoid, Γ of order s

Theorem 4.3 *Let Γ be a finite generalized hexagon of order s containing an ovoid $\mathcal{A}$. Every point of $\mathcal{A}$ is span-regular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s$.*

Proof This proof is given by De Smet and Van Maldeghem in [17]. □

4.5.2 $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, Γ of order s

For the proof of theorem 4.4, we will use a similar counting argument (lemma 4.2) as used in [17] for the proof of theorem 4.3.

Lemma 4.2 *Let Γ be a finite generalized hexagon of order (s,t), which contains a set $\mathcal{A} = \Gamma_1(M) \cup \Gamma_3(M)$ for which all points are span-regular. Then every thin ideal subhexagon of Γ contains 2 collinear points of $\mathcal{A}$* **if and only if** *$s = t$.*

Proof

1. First we count the thin ideal subhexagons containing the 'central' line M of $\mathcal{A}$. There are $\frac{(s+1)s^3t^2}{2}$ sets $\{u,v\}$ of opposite points in $\mathcal{A}$. As u is span-regular, there is a thin ideal subhexagon through u and v, named $\Gamma(u,v)$, containing M (see page 6). But in every ideal subhexagon $\Gamma(u,v)$, one can find t^2 sets $\{u',v'\}$ of opposite points in $\mathcal{A}$. So there are $\frac{s^3(s+1)}{2}$ thin ideal subhexagons containing M — and hence containing $2+2t$ points of $\mathcal{A}$.

2. Now we count the thin ideal subhexagons $\mathcal{D}$ containing two collinear points u,v of $\Gamma_3(M)$. Hence M is not a line of $\mathcal{D}$, as there are only 2 points on the line uv in $\mathcal{D}$. We count in 2 different ways the couples $(\{u,v\},\mathcal{D})$, with $\{u,v\}$ a set of collinear points in $\Gamma_3(M)$, and $\mathcal{D}$ a thin ideal subhexagon containing u and v (as u is span-regular, there will be an ideal subhexagon through u). Denoting the number of $\mathcal{D}$'s by X, it follows that

$$\frac{(s+1)s(s-1)t}{2} \cdot s^2 = 1 \cdot X$$

3. Now we compare these 2 quantities with the total number of thin ideal subhexagons in Γ. We count the pairs $(\{u,v\},\mathcal{D})$ with $\{u,v\}$ a set of opposite points in Γ, and $\mathcal{D}$ a thin ideal subhexagon containing

u and v. Denoting the total number of $\mathcal{D}$'s by α, and noting that for each set $\{u, v\}$ there is at most 1 subhexagon $\mathcal{D}$, we know

$$\frac{(1+s)(1+st+s^2t^2)s^3t^2}{2} \cdot 1 \geq \frac{2(1+t+t^2)t^2}{2} \cdot \alpha$$

The total number of thin ideal subhexagons containing 2 (collinear) points of $\mathcal{A}$ will be less than or equal to α:

$$\frac{(s+1)s^3}{2} + \frac{(s+1)s^3t(s-1)}{2} \leq \alpha \leq \frac{(1+s)(1+st+s^2t^2)s^3}{2(1+t+t^2)} \quad (4.1)$$

Equality in both cases is satisfied if and only if $t^2(t-s)(s-1) = 0$.

For $s = t$, we can conclude two things: the equality between the first and second quantity expresses that every $\mathcal{D}$ contains 2 collinear points of $\mathcal{A}$; while the second equality expresses that through every 2 points of Γ, there is a thin ideal subhexagon $\mathcal{D}$. □

Corollary

Let Γ be a finite generalized hexagon of order (s, t), which contains a set $\mathcal{A} = \Gamma_1(M) \cup \Gamma_3(M)$ for which all points are span-regular. Then, through every 2 points at distance 6, there exists 1 thin ideal subhexagon; through every 2 points at distance 4, there are s thin ideal subhexagons; through every 2 points at distance 2, there are s^2 thin ideal subhexagons; through every point, there are s^3 thin ideal subhexagons.

Theorem 4.4 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order s. Consider the set $\mathcal{A}$ consisting of all points at distance 1 or 3 of a certain line. Every point of $\mathcal{A}$ is span-regular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s$.*

Proof

$\boxed{\Leftarrow}$ This follows from Ronan [53], see theorem 1.18.

$\boxed{\Rightarrow}$ By the same theorem 1.18 and with the terminology of Ronan [53], we have to prove that all traces of Γ are ideal lines. So, for 2 points $x, y \in \mathcal{P}$ with $\delta(x, y) = 4, z = x \bowtie y$ we must prove that z^w, $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$, is independent of w.

From the corollary above, it follows that there are s thin ideal subhexagons $\mathcal{D}_i$ containing x and y. They can be obtained by choosing a point y_i on a line through y at distance 5 from x and they all contain 2 collinear points of $\mathcal{A}$. Since there is only one trace z^w in $\mathcal{D}_i$ (there are only 2 points on a line), $z^w = z^{w'}$, $\forall w, w' \in \Gamma_4(x) \cap \Gamma_4(y) \cap \mathcal{D}_i$. So we have to prove that

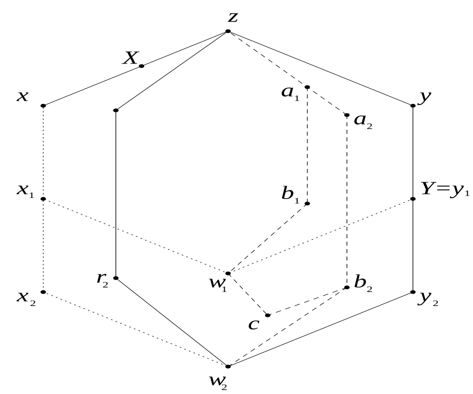


Figure 4.1: $X \neq Y, X = Z$

$z^{w_1} = \ldots = z^{w_s}$ with $w_i \in \mathcal{D}_i$.
If $x \notin \mathcal{A}$, we denote the unique point of $\mathcal{A}$ collinear with x by a capital letter X, possibly with some index i (depending on the thin ideal subhexagon $\mathcal{D}_i$ where this point belongs to). The same for $y \sim Y$ and $z \sim Z$. We denote the line yY by L.

1. If $x \in \mathcal{A}$ or $y \in \mathcal{A}$ then it is immediate that z^w is ideal.

2. If $X = Y = z$ then it is immediate that z^w is ideal.

3. Suppose $X \neq Y, X = Z$. (We refer to figure 4.1)
With every point $y_i \in L \backslash \{y\}$ (with $y_1 = Y$, without loss of generality), there corresponds a thin ideal subhexagon $\mathcal{D}_i$ through x, y and y_i. First we look at $\mathcal{D}_1$ and the hyperbolic line $\langle x, y\rangle_1$ in $\mathcal{D}_1$. We will show that the hyperbolic lines $\langle x, y\rangle_i$ in the other $\mathcal{D}_i$'s are the same.
Let y_2 be a point of $L \backslash \{y, y_1\}$ and let $\mathcal{D}_2$ be the thin ideal subhexagon through x, y and y_2. By lemma 4.2, each $\mathcal{D}_i$ contains 2 collinear points of $\mathcal{A}$, say r_i and s_i. If $\delta(r_i s_i, z) = 5$ and $\mathrm{proj}_{r_i s_i} z = r_i$, then $\delta(r_i, z) = 4$. If $\delta(r_i s_i, z) = 3$ and $\mathrm{proj}_{r_i s_i} z = r_i$, then $\delta(s_i, z) = 4$. If $\delta(r_i s_i, z) = 1$, then $r_i = s_i$ or $r_i = z$, a contradiction. So z is at distance 4 from one of these 2 points; say at distance 4 from r_i. Let $\mathcal{D}_i = \mathcal{D}_2$. Since r_2 and L are in $\mathcal{D}_2$, also the shortest path between them lies in $\mathcal{D}_2$. So the projection of r_2 onto L should be y_2 (as $\delta(r_2, y) = 6$), and we denote $r_2 \bowtie y_2$ by w_2. As $\delta(w_2, xz) = 5$, also the path between w_2 and x belongs to $\mathcal{D}_2$. Say $x_2 := w_2 \bowtie x$.
Denote $\mathrm{proj}_{xx_2} y_1$ by x_1, and $x_1 \bowtie y_1$ by w_1. Suppose that $\langle x, y\rangle_2 =$

z^{w_2} is different from $\langle x, y\rangle_1 = z^{w_1}$. So there is a line N through z on which the point a_1 at distance 4 from w_1 is different from the point a_2 at distance 4 from w_2. Denote $a_i \bowtie w_i$ by b_i. One can show (see Van Maldeghem[84] 1.9.9) that whenever a trace contains a span-regular point, this trace is an ideal line. As y_1 and r_2 are span-regular, we have ideal lines $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$. So $w_1^z = w_1^{w_2}$ and $w_2^z = w_2^{w_1}$. As $b_2 \in w_2^z = w_2^{w_1}, \delta(b_2, w_1) = 4$. Denote $b_2 \bowtie w_1$ by c. As $c \in w_1^{w_2} = w_1^z$, $\delta(c, z) = 4$. But $\delta(c, z) = 6$ as one supposed that $(z, za_2, a_2, a_2b_2, b_2, b_2c, c)$ is a path of length 6. So this is a contradiction. To solve this, a_1 should be a_2, and hence $b_1 = c$, and a_1, b_1, b_2 are collinear.

4. Suppose $X \neq Y, Y = Z$. Similar to the previous case.

5. Suppose $X \neq Y \neq Z \neq X$. If $Z \in z^w$ for some $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$ then $\langle x, y\rangle_w$ is ideal since it contains the span-regular point Z. If not, take a point $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$ and put $\text{proj}_{zZ} w = t$. By case (3.) (with x replaced by t, and with X replaced by $T = Z$), we have that $\langle t, y\rangle_w$ is ideal, so $\langle x, y\rangle_w$ is ideal. □

4.5.3 $\mathcal{A}$ a full subhexagon, and condition $(\star)$ is satisfied

Theorem 4.5 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order (s,t). Consider a proper full subhexagon $\mathcal{A}$ of Γ, and suppose there is a thin ideal subhexagon $\mathcal{D}$ through any 2 points of Γ. Then every point of $\mathcal{A}$ is span-regular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s = t$, with q a power of 3.*

Proof

$\boxed{\Leftarrow}$ This follows from Ronan [53], theorem 1.18.

$\boxed{\Rightarrow}$ By the preliminary remark on page 69, we know that Γ has order s, and $\mathcal{A}$ is thin. If $s = 2$, the result is trivially true by theorem 1.21. Hence we may assume $s > 2$. Once again, we have to prove that Γ has ideal lines. So, for 2 points $x, y \in \mathcal{P}$ with $\delta(x, y) = 4, z = x \bowtie y$ we must prove that $\langle x, y\rangle_w = z^w$, $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$, is independent of w.

As we supposed that any 2 opposite points are contained in a thin ideal subhexagon $\mathcal{D}$, there are s $\mathcal{D}_i$'s containing x and y. They can be obtained by choosing a point $y_i \neq y$ on a fixed line through y at distance 5 from x. Since there is only one trace z^w in $\mathcal{D}_i$, we know that $z^w = z^{w'}$ for all w, w' in the same $\mathcal{D}_i$ (i.e. $\forall w, w' \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z) \cap \mathcal{D}_i$). So we have to prove that $z^{w_1} = \ldots = z^{w_s}$ with all w_i in different subhexagons, say $w_i \in \mathcal{D}_i$.

If $x \notin \mathcal{A}$, we denote the unique point of $\mathcal{A}$ collinear with x by a capital letter X and some index i (depending on the thin ideal subhexagon $\mathcal{D}_i$ where this point belongs to). The same for $y \sim Y$ and $z \sim Z$.

1. If $x \in \mathcal{A}$ or $y \in \mathcal{A}$ then it is immediate that z^w is ideal.

2. If $X = Y = z$ then it is immediate that z^w is ideal.

3. Suppose $X \neq Y \neq Z \neq X$ and $\delta(X,Y) = 6$, so $\mathcal{D}_1 = \Gamma(y,X) \neq \Gamma(x,Y) = \mathcal{D}_2$.
   Attaching indices, we get $X = X_1, Y = Y_2$. We denote $\mathrm{proj}_{xX_1} Y_2$ by x_2, $\mathrm{proj}_{yY_2} X_1$ by y_1, and $w_1 := X_1 \bowtie y_1, w_2 := x_2 \bowtie Y_2$.
   Take a point $w_3 \in \Gamma_3(xX_1) \cap \Gamma_3(yY_2)$, $w_3 \neq z$, and suppose $\Gamma(z,w_3) = \mathcal{D}_3$ not equal to $\mathcal{D}_1$ or $\mathcal{D}_2$. We show that $z^{w_1} = z^{w_2} = z^{w_3}$. As $X_1 \in w_1^z$ and $Y_2 \in w_2^z$, these traces are ideal lines. So $w_1^z = w_1^{w_2} = w_1^{w_3}$ and $w_2^z = w_2^{w_1} = w_2^{w_3}$. Using the same arguments (and notations) as in the proof of theorem 4.4 case (3.), we know that $z^{w_1} = z^{w_2}$ and also $w_3^{w_1} = w_3^{w_2}$. Using this knowledge, we show that $z^{w_1} = z^{w_3}$.
   Suppose $z^{w_1} \neq z^{w_3}$; this means there is a line N through z on which the point $a_1 = a_2$ at distance 4 from w_1 (and w_2) is different from the point a_3 at distance 4 from w_3. Denote $a_i \bowtie w_i$ by b_i. In the proof of theorem 4.4, we showed already that a_1, b_1 and b_2 are collinear. As $b_1 \in w_1^z = w_1^{w_3}, \delta(b_1, w_3) = 4$. Similarly $\delta(b_2, w_3) = 4$. But then we have a pentagon, a quadrangle or a triangle, unless $w_3 \bowtie b_2 = w_3 \bowtie b_1$ and $w_3 \bowtie b_i \sim b_i$, $i = 1, 2$. Conclusion: $\delta(b_1 b_2, w_3) = 3$ and $a_1 = a_2 = a_3$.

4. Suppose $X \neq Y \neq Z \neq X$ with $\delta(X,Y) = 4$, and suppose $s \geq 4$. So the path between $X = X_1$ and $Y = Y_1$ belongs also to $\mathcal{A}$ and we can denote $X_1 \bowtie Y_1$ by the capital letter W_1. Take a point $w_3 \in (\Gamma_3(xX_1) \cap \Gamma_3(yY_1)) \backslash \mathcal{D}_1$ and say $\mathrm{proj}_{yY_1} w_3 = y_3$, $\mathrm{proj}_{xX_1} w_3 = x_3$. Take a line through z, different from zx, zy or zZ, and project w_3 onto this line. The projection is the point u. As $u \notin \mathcal{A}$ (otherwise $z \notin \mathcal{A}$ would be collinear with 2 points of the ovoidal subspace), u is collinear with a unique point U of $\mathcal{A}$. Suppose this span-regular point is also at distance 4 from X_1 and Y_1. Then we take another line through z, we project w_3 onto this line, denoting the projection and its unique collinear point of $\mathcal{A}$ by v and V, respectively. Now we show that V is at distance 6 from at least one of the three points X_1, Y_1 or U. The points X_1, Y_1, U define an ordinary hexagon in the thin full subhexagon $\mathcal{A}$. Suppose $\delta(U,V) = 4$ and $T := U \bowtie V$. As there are only 2 lines through one point in $\mathcal{A}$, T should be on the line through U and $U \bowtie Y_1$ or on the line through U and $U \bowtie X_1$. Say T is on

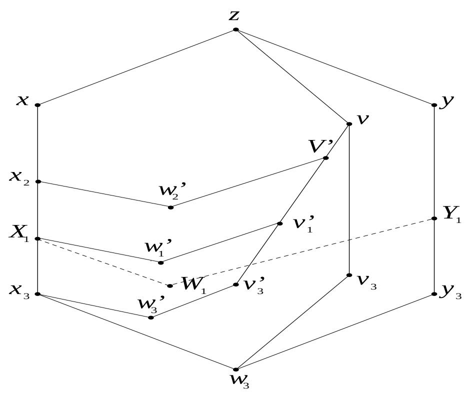


Figure 4.2: $\delta(X_1, Y_1) = 4, \delta(w_3, V') = 6$

the line through U and $U \bowtie Y_1$. If $T \neq U \bowtie Y_1$, then $\delta(V, Y_1) = 6$. If $T = U \bowtie Y_1$, then V should be on the line through $U \bowtie Y_1$ and Y_1 (as there are only 2 lines through a point in $\mathcal{A}$), hence $\delta(V, X_1) = 6$. So in this situation one can find a span-regular point $V = V'$ at distance 6 from X_1, Y_1 or U. Suppose $\delta(V', X_1) = 6$. We now use case (3.) of this proof, for $X_1 \neq V' \neq Z \neq X_1$.

$\boxed{\text{a}}$ First suppose $\delta(w_3, V') = 6$, and see figure 4.2. Put $\mathrm{proj}_{xX_1} V' = x_2$, $\mathrm{proj}_{vV'} X_1 = v_1'$, $\mathrm{proj}_{vV'} x_3 = v_3'$, $w_3 \bowtie v = v_3$, $x_3 \bowtie v_3' = w_3'$, $X_1 \bowtie v_1' = w_1'$, $x_2 \bowtie V' = w_2'$. By case (3.) of the proof, $z^{w_1'} = z^{w_2'} = z^{w_3'} = z^a$, for all $a \in \Gamma_3(xx_3) \cap \Gamma_3(vv_3) \cap \Gamma_6(z)$. As w_3 and w_3' are in the same thin ideal subhexagon $\mathcal{D}_3 = \Gamma(x_3, v)$, we know that $\langle x, v \rangle_{w_3} = \langle x, v \rangle_{w_3'}$. As $x, u, v, y \in \Gamma_2(z) \cap \mathcal{D}_3$, $\langle x, v \rangle_{w_3} = \langle x, y \rangle_{w_3}$. So $\langle x, y \rangle_{w_3} = \langle x, v \rangle_{w_3} = \langle x, v \rangle_{w_3'} = \langle x, v \rangle_a$, for all $a \in \Gamma_3(xx_3) \cap \Gamma_3(vv_3) \cap \Gamma_6(z)$. This finishes the proof if $\delta(w_3, V') = 6$.

$\boxed{\text{b}}$ Suppose $\delta(w_3, V') = 4$. Then $\langle x, v \rangle_{w_2'} = \langle x, v \rangle_{w_3}$ by case (3.) of this proof. Using $\langle x, y \rangle_{w_3} = \langle x, v \rangle_{w_3}$, we have the same result as before.

$\boxed{\text{c}}$ Suppose $\delta(w_3, V') = 2$. Then $V' = v_3$. As in case (3.) of the proof of theorem 4.4, one shows that $\langle x, v \rangle_{w_3} = \langle x, v \rangle_{w_1'}$.

4.bis Suppose $s = t = 3$.
As we assumed the existence of five lines through a point in the previous section, we now investigate the case $s = t = 3$, for $X \neq Y \neq Z \neq X$ and $\delta(X, Y) = 4$. So X_1, Y_1 are in the same $\mathcal{D}_1$, and $W_1 := X_1 \bowtie Y_1$. Take w_3 at distance 3 from xX_1 and yY_1, and define

$x_3 := \mathrm{proj}_{xX_1} w_3$ and $y_3 := \mathrm{proj}_{yY_1} w_3$. As we must prove $\langle x, y\rangle_{W_1}$ to be equal to $\langle x, y\rangle_{w_3}$, we suppose $Z \notin \langle x, y\rangle_{w_3}$ (otherwise the proof is done). As Z is span-regular, the points x and Z define an ideal line $\langle x, Z\rangle$. If y would be in $\langle x, Z\rangle$, this would imply Z to be in $\langle x, y\rangle_{w_3}$ — a contradiction. For the same reason, $x \notin \langle y, Z\rangle$.
Now we look at the fourth line through z, let's call it L. As $\langle x, Z\rangle$ and $\langle y, Z\rangle$ are different ideal lines, their intersection only contains the point Z. So their respective intersection points with L are different — and by this named t_x and t_y, respectively.
Now we consider again the traces $\langle x, y\rangle_{W_1}$ and $\langle x, y\rangle_{w_3}$. If t_x would be in $\langle x, y\rangle_{w_3}$, the trace $\langle x, y\rangle_{w_3}$ contains 2 points (x and t_x) of the ideal line $\langle x, Z\rangle$, and hence $\langle x, y\rangle_{w_3} = \langle x, Z\rangle$. This is of course a contradiction. For the same reason, $t_y \notin \langle x, y\rangle_{w_3}$. We can conclude that $|\langle x, y\rangle_{w_3} \cap L \cap \langle x, y\rangle_{W_1}| = 1$, and we call this intersection point t. We put $a_1 := t \bowtie W_1$ and $a_3 := t \bowtie w_3$.
As W_1^z contains span-regular points X_1 and Y_1, this trace is ideal . As $W_1^{w_3}$ intersects W_1^z in at least 2 points, $W_1^{w_3}$ should be equal to W_1^z. So $a_1 \in W_1^{w_3}$, which means $d(a_1, w_3) = 4$. If a_1 is not on the line ta_3, there arises an ordinary pentagon with edges $t, a_1, a_1 \bowtie w_3, w_3, a_3$. So a_1 is on ta_3.
Now we construct $s_1 := \mathrm{proj}_{zZ} W_1$; $s_3 := \mathrm{proj}_{zZ} w_3$; $b_1 := s_1 \bowtie W_1$; $b_3 := s_3 \bowtie w_3$. By a previous argument, neither s_1 nor s_3 coincide with Z (because $\langle x, Z\rangle$ is ideal and does not contain y). We know that $b_1 \in W_1^z = W_1^{w_3}$, so $d(b_1, w_3) = 4$. As there are only 4 lines through w_3, and the lines w_3x_3, w_3a_3, w_3y_3 already correspond to the respective points $X_1, a_1, Y_1 \in W_1^{w_3}$, we know that $b_1 \bowtie w_3$ is on b_3w_3. But this results in an ordinary pentagon $b_1, s_1, s_3, b_3, b_3 \bowtie b_1$ if b_1 is not on s_3b_3. Conclusion: b_1 is on s_3b_3 and $s_1 = s_3$. So $z^{W_1} = z^{w_3}$, and this part of the proof is completed.

5. Suppose $X \neq Y = Z$.
Take $w_3 \in \Gamma_3(xX) \cap \Gamma_4(y)$, and say $\mathrm{proj}_{xX} w_3 = x_3$. Take a line N through z, different from zx or zy, and say $\mathrm{proj}_N w_3 = v_3$. As $v_3 \notin \mathcal{A}, v_3$ is collinear with a unique point $V \in \mathcal{A}$, $V \notin v_3z$. At this point, we can use parts (3.) and (4.) of the proof to conclude that $\langle x, v_3\rangle_{w_3} = \langle x, v_3\rangle_{w_i}$, $i = 1, 2, 3$. As $x, y, v_3 \in \Gamma_2(z) \cap \mathcal{D}_3$, we know $\langle x, y\rangle_{w_3} = \langle x, v_3\rangle_{w_3}$, so $\langle x, y\rangle_{w_3}$ is ideal.

By now, we know $\Gamma \cong H(q)$. As Γ contains a full as well as ideal subhexagons, q must be a power of 3 by [84] 3.5.7. □

4.5.4 $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, Γ of order (s,t), and condition $(\star)$ is satisfied

Theorem 4.6 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order (s,t). Consider the set $\mathcal{A}$ consisting of all points at distance 1 or 3 from a certain line L, and suppose there is a thin ideal subhexagon $\mathcal{D}$ through any 2 points of Γ. Then every point of $\mathcal{A}$ is span-regular $\Leftrightarrow$ Γ is isomorphic to $H(s)$ or to $T(s^3, s)$.*

Proof If $s \neq t$, we can not use lemma 4.2. But by assuming $\beta = Y$ (see page 69), we can re-arrange the (provisional) proof of theorem 4.4 in the same way as in the proof of theorem 4.5: the new proof only uses the second equality in (1).

Where possible, we refer to the proof of theorem 4.5.
$\boxed{\Leftarrow}$ This follows from Ronan [53], see theorem 1.18.
$\boxed{\Rightarrow}$ By the same theorem, we have to prove that Γ has ideal lines. For $z^w = z^{w'}$, $\forall w, w' \in \Gamma_4(x) \cap \Gamma_4(y) \cap \mathcal{D}_i$ (so with w, w' in the same ideal subhexagon), we refer to theorem 4.5. For $z^{w_1} = \ldots = z^{w_s}$ with $w_i \in \mathcal{D}_i$, we refer to what follows.

1. cfr. theorem 4.5 (1.)
2. cfr. theorem 4.5 (2.)
3. cfr. theorem 4.5 (3.)
4. cfr. theorem 4.5 (4.): Suppose $X \neq Y \neq Z \neq X$ and $\delta(X,Y) = 4$. So the path between $X = X_1$ and $Y = Y_1$ belongs also to $\mathcal{A}$ and we can denote $X_1 \bowtie Y_1$ by the capital letter W_1. Take a point $w_3 \in (\Gamma_3(xX_1) \cap \Gamma_3(yY_1)) \backslash \mathcal{D}_1$ and say $\mathrm{proj}_{yY_1} w_3 = y_3$, $\mathrm{proj}_{xX_1} w_3 = x_3$. Take a line through z, different from zx, zy or zZ, and project w_3 onto this line. The projection is the point u^1. As $u^1 \notin \mathcal{A}$ (otherwise $z \notin \mathcal{A}$ would be collinear with 2 point of the ovoidal subspace), u^1 is collinear with a unique point U^1 of $\mathcal{A}$.
   New for this proof:
   We can do the same for the remaining lines through z, to obtain the points $U^1, \ldots, U^{t-2}$.
   $(\bullet)$ If we suppose that none of these points U^j is at distance 6 from X_1 or at distance 6 from Y_1, then they should all be at distance 4 from X_1 and Y_1, and hence at distance 2 from W_1 (as $\mathcal{A}$ contains no apartment). So W_1 is a point of the 'central' line L of $\mathcal{A}$. None of the t lines W_1X_1, W_1Y_1, W_1U^j is equal to L. Indeed, suppose $W_1U^1 = L$. We know $Z \in \mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, so $\delta(Z, W_1U^1) = 3$

(as Z does not belong to W_1U^1). But this results in an ordinary pentagon. Conclusion: the line L is the projection of Z onto W_1, and this completes the line pencil $\Gamma(W_1)$. So $\delta(W_1, Z) = 4$. This means: $Z \in z^{W_1} = \langle x, y\rangle_{W_1}$. By this, $\langle x, y\rangle_{W_1}$ contains a span-regular point and hence is ideal.
If on the other hand the assumption ($\bullet$) is false, i.e. if there is a point U^j at distance 6 from X_1 or Y_1, then we refer to theorem 4.5 (4.) for the remaining part of the proof.

5. cfr. theorem 4.5 (5.) □

4.5.5 $\mathcal{A}$ an ovoid, Γ of order (s, t), and condition $(\star)$ is satisfied

Theorem 4.7 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order (s, t) containing an ovoid $\mathcal{A}$. Suppose there is a thin ideal subhexagon $\mathcal{D}$ through any 2 points of Γ. Then every point of $\mathcal{A}$ is span-regular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s$.*

Proof
In a completely similar way as in the proof of theorem 4.5 — noting that all points of $\mathcal{A}$ are at distance 6 from each other (and hence case (4.) of the proof of 4.5 can not occur) —, we prove that Γ is classical. As it is known that $T(q^3, q)$ does not have an ovoid (see page 15), Γ is isomorphic to $H(q)$, $s = t = q$. □

Chapter 5

Clouds in Generalized Quadrangles and Hexagons

5.1 Introduction

In this chapter, we turn our attention towards substructures of finite generalized hexagons and generalized quadrangles. For the hexagons, we first define m-clouds, which can be used to characterize thin subhexagons of a generalized hexagon (these are important in connection with regularity conditions and for characterizations of the classical hexagons). Then a more symmetric object is derived from the definition of m-clouds, named dense clouds. We derive bounds on their size with the extended Higman-Sims technique. With a little modification for the case of the quadrangles, we define (m, f)-clouds and dense clouds. Again, bounds on the size are obtained. As in the previous chapter, where the notion of an ovoidal subspace of generalized n-gons comprises ovoids, subpolygons as well as sets of the form $\Gamma_{\frac{n}{2}-2} \cup \Gamma_{\frac{n}{2}}$, the notion of m-, (m, f)-, respectively dense clouds group together a whole set of subgeometries of hexagons and quadrangles.
The first part of this chapter will appear in [8].

CLOUDS IN HEXAGONS

5.2 m-Clouds in hexagons

Let Γ be a finite generalized hexagon of order (s,t).

> An m**-cloud** $\mathcal{C}$ of Γ, $2 \leq m \leq t$, is a subset of points of Γ at mutual distance 4, such that $\forall x, y \in \mathcal{C} : x \bowtie y$ is collinear with exactly $m+1$ points of $\mathcal{C}$.

We put $\mathcal{C}^* = \{x \bowtie y \mid x, y \in \mathcal{C}\}$, throughout.

Lemma 5.1 *Let $\mathcal{C}$ be an m-cloud of a generalized hexagon. Then the points of $\mathcal{C}$ are collinear with a constant number $f+1$ of points in $\mathcal{C}^*$.*

Proof
Take a point $x \in \mathcal{C}$, and suppose x is collinear with $f+1$ points z_i in $\mathcal{C}^*$. For each z_i there are m points y_{ij} in $\mathcal{C}$ collinear with z_i, and different from x. As $y_{ij} \neq y_{kl}$ if $i \neq k$ (otherwise there arises a quadrangle with vertex set $\{x, z_i, y_{ij} = y_{kl}, z_k\}$), $\mathcal{C}$ has at least $1+(f+1)m$ points. As all points in $\mathcal{C}$ are at mutual distance 4, we counted all points in $\mathcal{C}$, hence $|\mathcal{C}| = 1+(f+1)m$, and $f+1$ turns out to be a constant. $\square$

The parameter f is called the **index** of the m-cloud.

Corollary 5.1 *Let $\mathcal{C}$ be an m-cloud of index f of a generalized hexagon. Then the number of points in $\mathcal{C}^*$ is $\frac{(1+(f+1)m)(f+1)}{m+1}$ and $(m+1) \mid f(f+1)$.*

Proof
The geometry $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$ clearly is a $2-(1+(f+1)m, m+1, 1)$ design (see page 23), which implies the first statement in the theorem. From this fraction, the divisibility condition is derived. $\square$

Lemma 5.2 *No two distinct points of $\mathcal{C}^*$ are collinear.*

Proof
Let z, u be in $\mathcal{C}^*$ and suppose $\delta(z,u) = 2$. Take points z' and u' of $\mathcal{C}$ at distance 2 of z and u, respectively. $\boxed{1}$ If $z' = u'$ then $z' \in zu$. As $m > 1$, there are points z'' and u'' of $\mathcal{C}$ different from $z' = u'$ at distance 2 of z and u, respectively. As $\delta(z', z'') = 4$, $z'' \notin z'z$ and similarly $u'' \notin z'z$. As $z'' = u''$ is impossible, z'' and u'' are at distance 6, in contradiction with the definition of $\mathcal{C}$. $\boxed{2}$ If $z' \neq u'$, then $\delta(z', u') = 4$ by the definition of $\mathcal{C}$, hence either z' or u' is on zu. Above argument leads again to a contradiction. $\square$

Theorem 5.1 *If $\mathcal{C}$ is an m-cloud of index m, then the geometry $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$ is a projective plane of order m. The set $\mathcal{C}^*$ is also an m-cloud of index m, with $(\mathcal{C}^*)^* = \mathcal{C}$.*

Proof
As Γ' is a $2-(m^2+m+1, m+1, 1)$ design, it is a projective plane of order m. (See e.g. [9] p 439.) Hence any two distinct points of $\mathcal{C}^*$ are collinear with one common point of $\mathcal{C}$, and so these points are at distance 4. By the duality principle in projective planes, $\mathcal{C}^*$ will also be an m-cloud of index m. □

Theorem 5.2 *If $\mathcal{C}$ is an $(f-1)$-cloud of index f, then the geometry $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$ is an affine plane of order f.*

Proof
As Γ' is a $2-(f^2, f, 1)$ design, this follows again from design theory. □

For large m and f, the sizes of $\mathcal{C}$ and $\mathcal{C}^*$ are given in the following table. From this, we can deduce the nature of large clouds, as done in corollaries 5.2 and 5.3.

	$\lvert\mathcal{C}\rvert$	$\lvert\mathcal{C}^*\rvert$
$m = t$ $f = t$	t^2+t+1	t^2+t+1
$m = t-1$ $f = t$	t^2	t^2+t
$m = t-1$ $f = t-1$	t^2-t+1	t^2-t+1
$m = t-2$ $f = t-1$	t^2-2t+1	t^2-t

Corollary 5.2 *If $\mathcal{C}$ is an m-cloud with $|\mathcal{C}| \geq t^2+1$, then $\mathcal{C}$ is a t-cloud of index t, so $|\mathcal{C}| = t^2+t+1$. The geometry $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$ is a projective plane of order t. The union $\mathcal{C} \cup \mathcal{C}^*$ is the point set of a thin ideal subhexagon of Γ which is the double of the projective plane mentioned (see page 6).*

Proof This follows from the table above, and theorem 5.1. □

Corollary 5.3 *If $|\mathcal{C}| \geq t^2-t+2$, then either $|\mathcal{C}| = t^2$,$m = t-1$,$f = t$, or t^2+t+1,$m = f = t$. If $|\mathcal{C}| = t^2$, then $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$ is an affine plane of order t.*

Proof This follows from the table above, and theorems 5.1 and 5.2. □

Now theorem 5.1 and 5.2 are linked by theorem 5.3, which states that, for large m, an $(m-1)$-cloud of index m can be extended to an m-cloud of index m.

Theorem 5.3 *Let Γ be a generalized hexagon of order (s,t). For $k > t-\sqrt{t}+1$, a $(k-1)$-cloud $\mathcal{C}$ of index k is extendable to a k-cloud $\overline{\mathcal{C}}$ of index k, so that $\overline{\Gamma}' = (\overline{\mathcal{C}}, \overline{\mathcal{C}}^*, \sim)$ is a projective plane of order k.*

Proof
If $k > t-\sqrt{t}+1$, then $k > \frac{t+1}{2}$ and $k > t+1-k$. The $(k-1)$-cloud $\mathcal{C}$ defines an affine plane of order k. We introduce some notations, to make things easier to explain. A $\mathcal{C}\mathcal{C}^*$-line is a line intersecting $\mathcal{C}$ and $\mathcal{C}^*$. A $\mathcal{C}$-line only intersects $\mathcal{C}$, while a $\mathcal{C}^*$-line only intersects $\mathcal{C}^*$. We complete the geometry $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$ with some extra elements (*special* points and lines) to a projective plane.

1. First we show that 2 'parallel affine lines' in Γ' define a unique (special) point. This point is not in the affine plane, but it is in the hexagon. Take two points $u_1, u_2 \in \mathcal{C}^*$, with $\Gamma_2(u_1) \cap \mathcal{C}$ and $\Gamma_2(u_2) \cap \mathcal{C}$ disjoint. We show that $\delta(u_1, u_2) = 4$ in the hexagon. Suppose $\delta(u_1, u_2) = 6$. Hence the distance between u_2 and a line through u_1 is 5. The projection of one of the k $\mathcal{C}\mathcal{C}^*$-lines through u_1 onto u_2, should be a $\mathcal{C}^*$-line (because 2 points of $\mathcal{C}$ are at mutual distance 4 and not 6). But as the number of $\mathcal{C}\mathcal{C}^*$-lines through a point of $\mathcal{C}^*$ (that is, k) is bigger than the number of $\mathcal{C}^*$-lines through a point of $\mathcal{C}^*$ (that is, $t+1-k$), this gives a contradiction. Hence $\delta(u_1, u_2) \neq 6$. Hence $\delta(u_1, u_2) = 4$ and $u_1 \bowtie u_2 \notin \mathcal{C}$. Put $w = u_1 \bowtie u_2$ and suppose $u_1 w$ and $u_2 w$ are $\mathcal{C}\mathcal{C}^*$-lines , with $u_i w \cap \mathcal{C} = x_i$. Then $w = x_1 \bowtie x_2 \notin \mathcal{C}^*$, in contradiction with the definition of $\mathcal{C}^*$. Suppose $u_1 w$ is a $\mathcal{C}\mathcal{C}^*$-line, $u_1 w \cap \mathcal{C} = x_1$, and $u_2 w$ is a $\mathcal{C}^*$-line . Then the distance between x_1 and all points in $\Gamma_2(u_2) \cap \mathcal{C}$ is 6, again a contradiction. So w is on a $\mathcal{C}^*$-line through u_1 and on a $\mathcal{C}^*$-line through u_2.
   All points $u_i \bowtie u_j$ obtained by this construction, will be referred to as 'special points'.

2. Now we show that each parallel class defines exactly one special point. We denote this fixed parallel class by $\mathcal{C}^*_{\parallel}$, while the corresponding special points are in $(\mathcal{C}^*_{\parallel})^* = \{u_i \bowtie u_j \text{ with } u_i \neq u_j \text{ and } u_i, u_j \in \mathcal{C}^*_{\parallel}\}$. There are k elements u_i in $\mathcal{C}^*_{\parallel}$, each incident with $t+1-k$ $\mathcal{C}^*$-lines. Each $u_i \bowtie u_j$, u_i and u_j distinct points in $\mathcal{C}^*_{\parallel}$, is on a $\mathcal{C}^*$-line, and if $u_i \bowtie u_j$ and $u_i \bowtie u_l$, with $u_i, u_j, u_l \in \mathcal{C}^*_{\parallel}$ and distinct, are on the same $\mathcal{C}^*$-line, the points $u_i \bowtie u_j$ and $u_i \bowtie u_l$ must coincide (as $\delta(u_j, u_l) = 4$).

Also, if a special point belongs to a $\mathcal{C}^*$-line containing u_i, it corresponds to the parallel class of u_i. Hence $u_i \in \mathcal{C}^*_{\|}$ is collinear with at most $t+1-k$ elements of $(\mathcal{C}^*_{\|})^*$. Two points u_i, u_j of a same parallel class are collinear with a unique special point $u_i \bowtie u_j$, and two special points are collinear with at most one u_i (otherwise there arises a k-gon with $k < 6$). Hence the geometry $\Gamma_{\|} = (\mathcal{C}^*_{\|}, (\mathcal{C}^*_{\|})^*, \sim)$ is a linear space (p 23), with k points and at most $t+1-k$ lines through a point. If there exists a triangle in $\Gamma_{\|}$, there are at most $t+1-k$ points on every line.
Now we count in different ways the pairs (q, L) with q a point of $\Gamma_{\|}$, L a line of $\Gamma_{\|}$, $q \mathrm{I} L$, and $p \mathrm{I} L$, $p \neq q$ with p fixed; further we assume the existence of a triangle in $\Gamma_{\|}$. We obtain

$$\begin{aligned}(k-1) &\leq (t+1-k)(t+1-k-1) \\ 0 &\leq k^2-2k-2kt+t^2+t+1 \qquad (*)\end{aligned}$$

Solving for k, the roots of the associated equation are $k = t+1\pm\sqrt{t}$, or $t+1-k = \pm\sqrt{t}$. As we assumed $t+1-k < \sqrt{t}$ and clearly $t+1-k > -\sqrt{t}$, the right-hand side of $(*)$ is negative, hence the inequality is false, so $\Gamma_{\|}$ cannot be a non-degenerate linear space. Hence $\Gamma_{\|}$ is a unique line with k points on it. Translated to $\Gamma' = (\mathcal{C}, \mathcal{C}^*, \sim)$: each parallel class of affine lines defines a unique special point. The set of all special points constructed in this way, is denoted by W.

3. Subsequently we show that all points in $\mathcal{C} \cup W$ are at mutual distance 4 (this is a first step in proving that $\mathcal{C} \cup W$ is a cloud). $\boxed{1}$ First we look at $\delta(w, x)$, $w \in W$, $x \in \mathcal{C}$. A point $w \in W$ is at distance 2 of k points u_i of $\mathcal{C}^*$, belonging to the same parallel class of lines in Γ'. These lines u_i cover all k^2 points of Γ', hence all k^2 points of $\mathcal{C}$ are at distance 4 of w. $\boxed{2}$ Now we look at $\delta(w_1, w_2)$, $w_1, w_2 \in W$. There are k $\mathcal{C}^*$-lines through w_i, hence there are $t+1-k$ lines through w_i not intersecting $\mathcal{C}^*$. Suppose $\delta(w_1, w_2) = 6$ and suppose that the projection L_2 of a $\mathcal{C}^*$-line L_1 through w_1 onto w_2 is also a $\mathcal{C}^*$-line, with $L_2 \cap \mathcal{C}^* = y_2$ and $L_1 \cap \mathcal{C}^* = y_1$. As y_1 and y_2 are, in the terminology of the affine plane Γ', lines belonging to different parallel classes, they share a point of Γ', hence they are at distance 4 in Γ (with $y_1 \bowtie y_2 \in \mathcal{C}$). As there are no k-gons allowed for $k < 6$, either $\delta(y_1, L_2) = 2$ or $\delta(y_2, L_1) = 2$. Suppose $\delta(y_1, L_2) = 2$. As $y_1 \bowtie y_2$ belongs to $\mathcal{C}$, L_2 is a $\mathcal{C}\mathcal{C}^*$-line instead of a $\mathcal{C}^*$-line, hence we found a contradiction ($\delta(w, x)$ is already proved to be 4 for all $w \in W$, $x \in \mathcal{C}$).

So the k $\mathcal{C}^*$-lines through w_1 should all be mapped onto (different) lines through w_2 but not intersecting $\mathcal{C}^*$. As there are only $t+1-k$ of these lines, this situation is impossible, hence $\delta(w_1, w_2) \neq 6$.
Clearly, $\delta(w_1, w_2) = 2$ would imply the existence of a pair of points of $\mathcal{C}^*$ collinear to w_1, respectively w_2, that are opposite; a contradiction. Hence $\delta(w_1, w_2) = 4$, and $w_1 \bowtie w_2 \notin \mathcal{C}^*$.
Also, it is easy to show that the line N_i joining w_i and $w_1 \bowtie w_2$ is not a $\mathcal{C}^*$-line , $i = 1, 2$. So N_i is one of the $t+1-k$ lines through w_i which is not a $\mathcal{C}^*$-line, $i = 1, 2$. If we put $W^* = \{w_i \bowtie w_j | w_i, w_j \in W\}$, the geometry $\Gamma_* = (W, W^*, \sim)$ is a linear space with $k+1$ points and at most $t+1-k$ lines through a point (to verify this, one can use exactly the same arguments as used in part (2.) of this proof). By (nearly) the same counting argument, one concludes that Γ_* is degenerate, hence W^* is a singleton, containing the unique point $w_* \notin \mathcal{C}^*$.

4. At this point we can finish the proof: $\mathcal{C} \cup W$ is a k-cloud of index k, which means that all points of $\mathcal{C} \cup W$ are at mutual distance 4, and for $x, y \in \mathcal{C} \cup W, x \neq y : x \bowtie y$ is collinear with $k+1$ points of $\mathcal{C} \cup W$. Indeed, for x, y both in $\mathcal{C}$, we know that $x \bowtie y$ is collinear with k points of $\mathcal{C}$ and with 1 point of W (the unique special point on the line $x \bowtie y$ in Γ'). For x in $\mathcal{C}$ and y in W, the point $x \bowtie y$ is in Γ' the unique line through x of the parallel class corresponding with the special point y. So $x \bowtie y$ is an element of $\mathcal{C}^*$, and hence collinear with $k+1$ points of $\mathcal{C} \cup W$. For x, y both in W, we know that $x \bowtie y = w^*$, and w^* is collinear with all $k+1$ points of W; and as there should be no ordinary quadrangles, w^* cannot be collinear with any point of $\mathcal{C}$ (indeed, take $y \in \mathcal{C}$; y is collinear with some point $a \in \mathcal{C}^*$, a is collinear with a unique point $b \in W$, and b is always collinear with w^*. If $y \sim w^*$, then there arises a quadrangle).

By putting $\overline{\mathcal{C}} = \mathcal{C} \cup W$ and $\overline{\mathcal{C}}^* = \mathcal{C}^* \cup \{w^*\}$, we constructed the desired extension of Γ' to a projective plane. □

Corollary 5.4 *A $(t-1)$-cloud $\mathcal{C}$ of index t is extendable to a t-cloud $\overline{\mathcal{C}}$ of index t, so that $\overline{\Gamma}' = (\overline{\mathcal{C}}, \overline{\mathcal{C}}^*, \sim)$ is a projective plane of order t.*

5.3 m-Clouds in distance-2-regular hexagons

In the following theorem, we show that any m-cloud of a distance-2-regular hexagon of order (s, t) is contained in a t-cloud of index t. So, for any point-distance-2-regular hexagon Γ, m-clouds turn out to be well studied objects in projective planes, those planes being the geometries $(\Gamma^+(p,q), \Gamma^-(p,q), \sim)$ defined on page 6. As a finite point-distance-2-regular hexagon is classical (see theorem 1.22 p 13), every projective plane $(\Gamma^+(p,q), \Gamma^-(p,q), \sim)$ will be classical too (i.e. Desarguesian).

Theorem 5.4 *Let Γ be a generalized hexagon of order (s,t), such that all points are distance-2-regular. Let $\mathcal{C}$ be an m-cloud of Γ, with $x_1, x_2, x_3 \in \mathcal{C}$ and $x_1 \bowtie x_2 \neq x_1 \bowtie x_3$. The geometry $\Gamma_{\mathcal{C}} = (\mathcal{C}, \mathcal{C}^*, \sim)$ is a subgeometry of the projective plane $\Gamma_\pi = (\Gamma^+(x_3, x_1 \bowtie x_2), \Gamma^-(x_3, x_1 \bowtie x_2), \sim)$ of order t, such that all lines of Γ_π intersect $\Gamma_{\mathcal{C}}$ in 0, 1 or $m+1$ points. The constant $f+1$ is the number of $(m+1)$-secants of $\Gamma_{\mathcal{C}}$ through a point of $\Gamma_{\mathcal{C}}$.*

Proof
Take the unique weak ideal subhexagon $\Gamma' = \Gamma(x_3, x_1 \bowtie x_2)$. This geometry contains the ordinary hexagon with vertices $\{x_1, x_1 \bowtie x_2, x_2, x_2 \bowtie x_3, x_3, x_3 \bowtie x_1\}$. We put $y = x_1 \bowtie x_2$. Now take a point $x_4 \in \mathcal{C}$ and suppose x_4 is not contained in Γ'. $\boxed{1}$ If $x_4 \bowtie x_i$ (for $i \in \{1,2,3\}$) is different from $x_1 \bowtie x_2, x_2 \bowtie x_3, x_3 \bowtie x_1$, the unique shortest path between x_4 and x_3 is denoted by (x_4, M, z, L, x_3). As Γ' is ideal, each line of Γ through a point of Γ' is also a line of Γ'. So if z belongs to Γ', $x_4 = \mathrm{proj}_M x_1$ also belongs to Γ' — a contradiction. Hence, $u := \mathrm{proj}_L y$ is different from z. As $x_1, x_2 \in y^{x_3} \cap y^{x_4}$, $y \bowtie u \in y^{x_3}$, and y is distance-2-regular, $y \bowtie u$ should be in y^{x_4}. Hence $\delta(x_4, y \bowtie u) = 4$, and there arises a pentagon through $y \bowtie u, u, z$ and x_4. This is a contradiction. $\boxed{2}$ If on the other hand $x_4 \bowtie x_1$ is equal to $x_1 \bowtie x_2 = y$ (or some similar condition), then $x_4 \in y^{x_3}$. So x_4 belongs to Γ', again a contradiction.
Hence each point of $\mathcal{C}$ belongs to Γ'. Next, let $y_1 \in \mathcal{C}^*$, $y_1 \neq y$. Then $y_1 = x_5 \bowtie x_6$ for points $x_5, x_6 \in \mathcal{C}$. As x_5, x_6 are points of Γ', also $x_5 \bowtie x_6 = y$ belongs to Γ'. So each point of $\mathcal{C}^*$ belongs to Γ'.
This shows that all points of $\mathcal{C}$ are in $\Gamma^+(x_3, x_1 \bowtie x_2)$, and all points of $\mathcal{C}^*$ are in $\Gamma^-(x_3, x_1 \bowtie x_2)$. In particular any two distinct points of $\mathcal{C}^*$ are at mutual distance 4. If a line of Γ_π belongs to $\mathcal{C}^*$, it will be incident with $m+1$ points of $\Gamma_{\mathcal{C}}$. If a line does not belong to $\mathcal{C}^*$, it can (by definition of $\mathcal{C}^*$) only be incident with 0 or 1 point of $\Gamma_{\mathcal{C}}$. Clearly $f+1$ is the number of $(m+1)$-secants of $\Gamma_{\mathcal{C}}$ through a point of $\Gamma_{\mathcal{C}}$. □

Theorem 5.5 *Let Γ be a generalized hexagon of order (s,t) with a span-regular point p. Let q be a point opposite p and suppose $\mathcal{C}$ is a subset of the point set of the projective plane $\Gamma_\pi = (\Gamma^+(p,q), \Gamma^-(p,q), \sim)$, such that all lines of Γ_π intersect $\mathcal{C}$ in 0, 1 or $m+1$ points. Then $\mathcal{C}$ is an m-cloud of Γ.*

Proof Immediate. □

Examples

Let Γ be a generalized hexagon of order (s,t), with a span-regular point p and Γ_π as above. We refer to page 24 for definitions.
An **oval** in Γ_π corresponds with a 1-cloud of index $(t-1)$ of Γ. A **maximal arc** of type $(0,m)$ in Γ_π corresponds with an $(m-1)$-cloud of index t of Γ. **Unitals** in Γ_π correspond with $\sqrt{t}$-clouds of index $t-1$ of Γ. **Baer subplanes** in Γ_π correspond to $\sqrt{t}$-clouds of index $\sqrt{t}$ of Γ.

Baer subplanes are special subplanes of a given plane. But any subplane of Γ_π corresponds with a certain cloud, as stated in the following corollary.

Corollary 5.5 *For Γ a point-distance-2-regular hexagon of order (s,p^h), there exists a p^i-cloud of index p^i for every i dividing h, as well as a (p^i-1)-cloud of index p^i.*

If we focus on very small subplanes of a given plane, we have a result about sets of 4 points x_i at mutual distance 4, such that all $x_i \bowtie x_j$ are different. Such a set is a 1-cloud of index 2, and corresponds with the affine plane of order 2, contained in every projective plane — unlike the projective plane of order 2.

Corollary 5.6 *Let Γ be a generalized hexagon of order (s,t), such that all points are distance-2-regular, and t odd. Then a 1-cloud of index 2 in Γ is not extendable to a 2-cloud of index 2.*

Proof
If the converse were true, the Fano-plane $\mathbf{PG}(2,2)$ would be contained in a classical projective plane of odd order. $\square$

5.4 m-Clouds in anti-regular hexagons

Let Γ be a generalized hexagon with 3 distinct points p,u,v such that $\delta(p,u)=6=\delta(p,v)$. We introduce the following subset of the intersection of the traces p^u and p^v:

$$p^{\{u,v\}} = \{x \in p^u \cap p^v \mid \mathrm{proj}_x u \neq \mathrm{proj}_x v\}$$

A generalized hexagon of order q is **anti-regular** if $|p^{\{u,v\}}| \geq 2$ implies $|p^u \cap p^v| = 3$ and $|p^{\{u,v\}}| = 3$ for all traces p^u, p^v. A finite generalized hexagon Γ of order q is anti-regular if and only if Γ is isomorphic to the dual Split-Cayley hexagon $H(q)^D$ with q not divisible by 3. (This characterization can be found in Govaert and Van Maldeghem [26].)

Theorem 5.6 *Suppose Γ is a generalized hexagon of order q. If Γ is antiregular, then Γ contains no m-cloud for $m \geq 2$ with $|\mathcal{C}^*| > 1$.*

Proof
Take a point $p \in \mathcal{C}^*$ collinear with $x, y, z \in \mathcal{C}$. Let $u \in \mathcal{C}$ be at distance 6 of p. Consider $u \bowtie z \in \mathcal{C}^*$. This point is collinear with a third point of $\mathcal{C}$, say v. Put $L = \text{proj}_v x$ and $M = \text{proj}_v y$. As there are no pentagons in Γ, $\text{proj}_x v \neq \text{proj}_x u$ and $L \neq M$. But now we have $x, y, z \in p^v \cap p^u$ with $\text{proj}_x u \neq \text{proj}_x v$, $\text{proj}_y u \neq \text{proj}_y v$ and $\text{proj}_z u = \text{proj}_z v$. This is in contradiction with the antiregularity of Γ. $\square$

5.5 m-Clouds in non-classical hexagons

As the existence of $(t-1)$-clouds of index $(t-1)$ in point-distance-2-regular generalized hexagons is impossible (this would imply a subplane of order $(t-1)$ in a projective plane of order t), we could wonder whether such a cloud can exist in a non-classical generalized hexagon. As the extended Higman-Sims technique (see page 22, and page 92 for the analogous application in the case of generalized quadrangles) gives nice results for dense clouds (see following paragraph), one could hope that this technique is also applicable for proving the non-existence of $(t-1)$-clouds of index $(t-1)$ in non-classical generalized hexagons, but this does not work.

5.6 Dense clouds in hexagons

If we consider an m-cloud $\mathcal{C}$ of index m, we see that each point p of $\mathcal{C} \cup \mathcal{C}^*$ is collinear with exactly $m+1$ points of $\mathcal{C} \cup \mathcal{C}^*$, with all these points on different lines through p. By taking $\mathcal{C}$ and $\mathcal{C}^*$ together in one set $\mathcal{D}$, and generalizing the definition by allowing more than two points on a line, the notion of a dense cloud is obtained.

> A **dense cloud** $\mathcal{D}$ of index α is a set of d points such that any point p of $\mathcal{D}$ is collinear with exactly α points of $\mathcal{D} \setminus \{p\}$.

We use the extended Higman-Sims technique to obtain an upper and lower bound for dense clouds. As a dense cloud is a more symmetric object than a cloud, the technique gives better results in this case.

Lemma 5.3 *Let A be the adjacency matrix of the complement of the point graph of a generalized hexagon Γ. Then A has eigenvalues $s^2t(1+t+st)$, t, $-s+\sqrt{st}$ and $-s-\sqrt{st}$.*

Proof
Let v be the number of points of Γ, so $v = (1+s)(1+st+s^2t^2)$. The adjacency matrix $C = (c_{ij})$, $i, j = 1, \ldots, v$, of the point graph of a generalized hexagon Γ is defined by $c_{ij} = 1$ iff $w_i \sim w_j$, $i \neq j$, and $c_{ii} = 0$. The adjacency matrix $A = (a_{ij})$ is then defined by $a_{ij} = 1$ iff $w_i \not\sim w_j$, $i \neq j$, and $a_{ii} = 0$. Hence $A = J - C - I$, with J the all-one matrix and I the identity matrix. The eigenvalues of J are $j_1 = v$ (with multiplicity 1) and $j_2 = 0$. The eigenvalue of I is $i_1 = 1$ (with multiplicity v). The eigenvalues of C are $c_1 = s(t+1)$ (with multiplicity 1), $c_2 = s-1+\sqrt{st}$, $c_3 = s-1-\sqrt{st}$ and $c_4 = -t-1$ (see [9] p 203). Hence the eigenvalues λ_i, $i = 1, \ldots, 4$, of A are

$$\begin{array}{lllll}
\lambda_1 & = & j_1 - c_1 - i_1 & = & s^2t(1+t+st) \\
\lambda_2 & = & j_2 - c_2 - i_1 & = & -s-\sqrt{st} \\
\lambda_3 & = & j_2 - c_3 - i_1 & = & -s+\sqrt{st} \\
\lambda_4 & = & j_2 - c_4 - i_1 & = & t.
\end{array}$$

□

<u>**Remark**</u> The more explicit computation of the eigenvalues of A would read as follows. The eigenvalues of A satisfy the characteristic polynomial of the matrix itself. As A has in se four different classes of entries (i.e. entries a_{ij} with $\delta(x_i, x_j) = 0, 2, 4$ respectively 6), we expect the polynomial to be of degree 3. So we compute A^2 and A^3, and find

$$A = (a_{ij}) \text{ with } \begin{cases} a_{ii} = 0 & \\ a_{ij} = 0 & \text{if } \delta(x_i, x_j) = 2 \\ a_{ij} = 1 & \text{if } \delta(x_i, x_j) = 4 \\ a_{ij} = 1 & \text{if } \delta(x_i, x_j) = 6, \end{cases}$$

$$A^2 = (a'_{ij}) \text{ with } \begin{cases} a'_{ii} = s^3t^2 + st(st+s) & \\ a'_{ij} = a'_{ii} - st & \text{if } \delta(x_i, x_j) = 2 \\ a'_{ij} = a'_{ii} - st - s & \text{if } \delta(x_i, x_j) = 4 \\ a'_{ij} = a'_{ii} - st - s - 1 & \text{if } \delta(x_i, x_j) = 6, \end{cases}$$

$$A^3 = (a''_{ij}) \text{ with } \begin{cases} a''_{ii} = s^3t(st+t+1)(s^2t^2+(st-1)(t+1)) - s^3t^2 & \\ a''_{ij} = a''_{ii} + s^2t & \text{if } \delta(x_i, x_j) = 2 \\ a''_{ij} = a''_{ii} + s^2(t+1) + 2st & \text{if } \delta(x_i, x_j) = 4 \\ a''_{ij} = a''_{ii} + s^2(t+1) + 2st + 2s - t & \text{if } \delta(x_i, x_j) = 6. \end{cases}$$

The coefficients a, b, c of the characteristic polynomial $X^3 + aX^2 + bX + cI = \mu J$ of A, should satisfy the equations

$$\begin{cases} a''_{ii} + a \cdot a'_{ii} + b \cdot a_{ii} + c \cdot 1 &= \mu \cdot 1 \\ a''_{ij} + a \cdot a'_{ij} + b \cdot a_{ij} + c \cdot 0 &= \mu \cdot 1 \qquad \text{if } \delta(x_i, x_j) = 2 \\ a''_{ij} + a \cdot a'_{ij} + b \cdot a_{ij} + c \cdot 0 &= \mu \cdot 1 \qquad \text{if } \delta(x_i, x_j) = 4 \\ a''_{ij} + a \cdot a'_{ij} + b \cdot a_{ij} + c \cdot 0 &= \mu \cdot 1 \qquad \text{if } \delta(x_i, x_j) = 6. \end{cases}$$

Solving this system, A satisfies

$$A^3 + (2s - t)A^2 + s(s - 3t)A + st(t - s)I = \mu J$$

(μ being irrelevant for further calculations).
Hence the three eigenvalues λ_i of A, different from the total row sum, are given by the solutions of the equation $x^3 + (2s-t)x^2 + s(s-3t)x + st(t-s) = 0$. This gives us $t, -s - \sqrt{st}, -s + \sqrt{st}$.

Remark The row sum, λ_1, has multiplicity $m_1 = 1$. The other multiplicities m_i of λ_i are given by

$$\begin{aligned} \sum m_i &= \text{dimension of } A = |\mathcal{P}|, \\ \sum m_i \lambda_i &= \text{tr } A = \textstyle\sum_{i=1}^{|\mathcal{P}|} a_{ii} = 0, \\ \sum m_i \lambda_i^2 &= \text{tr } A^2 = \textstyle\sum_{i=1}^{|\mathcal{P}|} a'_{ii} = (s+1)(s^2t^2 + st + 1)s^2t(st + t + 1). \end{aligned}$$

Theorem 5.7 *Let Γ be a generalized hexagon of order (s,t), and let $\mathcal{D}$ be a dense cloud of index α. Then $(s+1)(\alpha + 1 - s - \sqrt{st})(st + \sqrt{st} + 1) \leq |\mathcal{D}| \leq \frac{(\alpha+t+1)(s^2t^2+st+1)}{t+1}$.*
Equality holds for the lower bound, if and only if every point outside $\mathcal{D}$ is collinear with exactly $\alpha + 1 - s - \sqrt{st}$ points of $\mathcal{D}$.
Equality holds for the upper bound, if and only if every point outside $\mathcal{D}$ is collinear with exactly $\alpha + t + 1$ points of $\mathcal{D}$.

Proof
We put $|\mathcal{D}| = d$. Let $\mathcal{P} = \{w_1, \ldots, w_v\}$ be the point set of Γ with $v = (1+s)(1+st+s^2t^2)$ and such that $w_1, \ldots, w_d \in \mathcal{D}$. Let A be the $(0,1)$-matrix (a_{ij}) over $\mathbb{R}$ defined by $a_{ij} = 1$ iff $w_i \not\sim w_j$, $i \neq j$, and $a_{ii} = 0$. So A has eigenvalues $s^2t(1+t+st)$, t, $-s+\sqrt{st}$, $-s-\sqrt{st}$ (see lemma 5.3). We write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ defined by the partition $\Delta_1 = \{1, \ldots, d\}$ and $\Delta_2 = \{d+1, \ldots, v\}$. Put $\delta_{ij} = \sum_{\substack{k \in \Delta_i \\ l \in \Delta_j}} a_{kl}$, $\delta_i = |\Delta_i|$, and define the 2×2-matrix $B = (\delta_{ij}/\delta_i)_{1 \leq i,j \leq 2}$.
As the total row sum (the number of points at distance 4 or 6 of a given point) is a constant, this number $\lambda_1 = s^2t(1+t+st)$ is eigenvalue of A and

B. Let $x \in \mathcal{D}$. Then x is at distance 0 of itself ($\in \mathcal{D}$), at distance 2 of α points of $\mathcal{D}$, and at distance 4 or 6 of $d-\alpha-1$ points of $\mathcal{D}$. Hence there are $d-\alpha-1$ non-zero entries on a row of A_{11}. As there are in total λ_1 non-zero entries on a row of A, there are $\lambda_1 - d + \alpha + 1$ non-zero entries on a row of A_{12}, and just as much on a column of A_{21}. As there are in total $v\lambda_1$ non-zero entries in A, there are $v\lambda_1 - d\lambda_1 - d(\lambda_1 - d + \alpha + 1)$ non-zero entries in A_{22}. Hence the matrix B of average row sums of A_{ij} becomes

$$B = \begin{pmatrix} d-\alpha-1 & \lambda_1 - d + \alpha + 1 \\ \frac{d(\lambda_1 - d + \alpha + 1)}{v-d} & \frac{v\lambda_1 - d(2\lambda_1 - d + \alpha + 1)}{v-d} \end{pmatrix}$$

If $\lambda_2(B)$ is the second eigenvalue of B, we know $\lambda_1(B) + \lambda_2(B) = \mathrm{tr}(B)$. With the notations of paragraph 1.9.3 on page 22 and by theorem 1.31, we have

$$\begin{cases} \lambda_{v-1}(A) & \leq & \lambda_1(B) & \leq & \lambda_1(A) \\ \lambda_v(A) & \leq & \lambda_2(B) & \leq & \lambda_2(A) \end{cases}$$
$$\Leftrightarrow \begin{cases} -s-\sqrt{st} & \leq & \lambda_1 & \leq & \lambda_1 \\ -s-\sqrt{st} & \underset{\star}{\leq} & \lambda_2(B) & \underset{\bullet}{\leq} & t \end{cases}$$

By substituting the known value of $\lambda_2(B)$, the inequality $\star$ becomes the bound $(\alpha + 1 - s - \sqrt{st})(st + \sqrt{st} + 1)(s+1) \leq d$, while the inequality $\bullet$ yields $(\alpha + t + 1)(s^2t^2 + st + 1) \geq d(t+1)$. If equality in either case is attained, by theorem 1.31, A_{ij} has constant row sum and constant column sum. The row sum of A_{11} and A_{12} was already known to be a constant. The row sum b_{21} of A_{21} is the number of points of $\mathcal{D}$ that are at distance 4 or 6 of a point $y \in \Gamma \setminus \mathcal{D}$. Hence y is collinear with $d - b_{21}$ points of $\mathcal{D}$. Substituting the lower and upper bound for d, gives the numbers stated in the theorem. □

If d attains the lower bound respectively upper bound, $\mathcal{D}$ is called a **minimal** respectively **maximal dense cloud**.

Examples

If every point outside a dense cloud is collinear with a fixed number of points of the dense cloud, we denote this number with β.
Ovoids respectively (the point set of) **spreads** of the hexagon are dense clouds of index $\alpha = 0$ respectively s, which are neither maximal nor minimal (remark that $\alpha < s + \sqrt{st}$, so the lower bound is negative). Nevertheless, every point outside the dense cloud is collinear with a constant number of points of the dense cloud; $\beta = 1$ respectively $t+1$.
A **hemisystem** of the hexagon (page 24) is a maximal dense cloud of index $\alpha = \frac{(s-1)(t+1)}{2}$, and $\beta = \frac{(s+1)(t+1)}{2}$.

The point set of a **proper full subhexagon** of order (s, t') of a generalized hexagon of order (s, t) is a dense cloud of index $s(t' + 1)$, which is never maximal. It is minimal if and only if $t' = \sqrt{t/s}$, and in that case $\beta = 1$. In Thas [66], the bound $t' \leq \sqrt{t/s}$ is derived with a variance trick, together with above interpretation (i.e. $\beta = 1$) for the equality $t' = \sqrt{t/s}$.
The point set of a **proper ideal subhexagon** of order (s', t) is a dense cloud of index $\alpha = s'(t + 1)$, which is never maximal nor minimal, and β does not exist. Indeed, let p be a point not in the subhexagon. If p is on a line of the subhexagon, it is collinear with $s' + 1$ points of the dense cloud, while if p is not on a line of the subhexagon, it is collinear with at most s' points of the dense cloud.

CLOUDS IN QUADRANGLES

5.7 (m, f)-Clouds in quadrangles

As every two points x, y at distance 4 of a generalized hexagon define a unique point $x \bowtie y$, we used these points $x \bowtie y$ to define a set $\mathcal{C}^*$ which arises naturally from (the definiton of) an m-cloud $\mathcal{C}$ in the hexagon, and we could prove that those points $x \bowtie y$ are not collinear. But in generalized quadrangles, $x \bowtie y$ is not well-defined. So we define $\mathcal{C}^*$ not by means of the elements at distance 2 of two elements of $\mathcal{C}$, but by the properties (similar to those of $\mathcal{C}$) that $\mathcal{C}^*$ turned out to have in the case of the hexagons. By doing so, it is clear that $\mathcal{C}$ and $\mathcal{C}^*$ are by definition interchangeable, which prompts us to use a more symmetric terminology than in the case of the hexagons.

> An (m, f)**-cloud**, $m, f > 1$, of a (finite) generalized quadrangle is a union of 2 non-empty disjoint sets $\mathcal{C}, \mathcal{C}^*$ such that all points in $\mathcal{C}$ respectively $\mathcal{C}^*$ are at mutual distance 4, and such that all points in $\mathcal{C}^*$ respectively $\mathcal{C}$ are collinear with $m + 1$ respectively $f + 1$ points of $\mathcal{C}$ respectively $\mathcal{C}^*$.

Still, one can not count elements of $\mathcal{C}$ and $\mathcal{C}^*$ as done in corollary 80, as the (m, f)-cloud is no 2-design. So we distinguish following two cases (among less symmetrical ones).

- If $\mathcal{C} \cup \mathcal{C}^*$ has as many quadrangles as possible, one ends up with the structure given in Payne and Thas [48] 1.4.1 of two disjoint sets of pairwise noncollinear points where each point of the first set is collinear with each point of the other set; so $|\mathcal{C}| = m+1$ and $|\mathcal{C}^*| = f+1$. Moreover, $mf \leq s^2$. This inequality is derived by using the extended Higman-Sims technique.

- On the other side of the spectrum, we define a **proper m-cloud** to be an (m,f)-cloud of minimal size such that no 4 points of $\mathcal{C} \cup \mathcal{C}^*$ form an ordinary quadrangle. So counting is possible and gives $|\mathcal{C}| = 1+(f+1)m$ and $|\mathcal{C}^*| = 1+(m+1)f$. As the geometry $(\mathcal{C}, \mathcal{C}^*, \sim)$ clearly is a 2-design, $|\mathcal{C}^*| = \frac{(1+(f+1)m)(f+1)}{m+1}$ (see corollary 5.1), hence $f = m$ and the geometry $(\mathcal{C}, \mathcal{C}^*, \sim)$ is a projective plane of order m, with $m > 1$ (cfr. theorem 5.1). The equality $f = m$ justifies the term 'proper m-cloud' instead of 'proper (m,m)-cloud'.

From now on, we assume $\mathcal{C}$ to be a proper m-cloud.

5.8 Proper m-clouds studied with the Higman-Sims technique

In the following theorem, we applied the extended Higman-Sims technique to proper m-clouds. However, the result turns out to be very weak.

Theorem 5.8 *Let Γ be a thick generalized quadrangle of order (s,t). Let $\mathcal{C}$ be a proper m-cloud.*
If $s \leq t+1$, then $m \leq \frac{st-2t-1+\sqrt{s^2t^2+2st(4t^2+10t+7)-4t^2-4t+1}}{4(t+1)}$.
If $m = t$, then $t+1 < s$ and every point w_i not in $\mathcal{C} \cup \mathcal{C}^*$ is collinear with equally many points of $\mathcal{C}$ and of $\mathcal{C}^*$; but this number is not equal for all points w_i.

Proof
Let $|\mathcal{C}| = 1+(m+1)m$ be denoted by c. Let $\mathcal{P} = \{w_1, \dots, w_v\}$ with $v = (1+s)(1+st)$ and let A be the $(0,1)$-matrix (a_{ij}) over $\mathbb{R}$ defined by $a_{ij} = 1$ iff $w_i \not\sim w_j$, $i \neq j$, and $a_{ii} = 0$. So A has eigenvalues $s^2t, t, -s$ (see Payne and Thas [48] 1.2.2 or Brouwer, Cohen and Neumaier [9] page 203). Let $\{\Delta_1, \Delta_2, \Delta_3\}$ be the partition of $\{1, \dots, v\}$ determined by the partition $\{\mathcal{C}, \mathcal{C}^*, \mathcal{P} \setminus (\mathcal{C} \cup \mathcal{C}^*)\}$ of $\mathcal{P}$. Put $\delta_{ij} = \sum_{\substack{k \in \Delta_i \\ l \in \Delta_j}} a_{kl}$, $\delta_i = |\Delta_i|$, and define the 3×3 matrix $B = (\delta_{ij}/\delta_i)_{1 \leq i,j \leq 3}$. Clearly $\delta_1 = c$, $\delta_2 = c$, $\delta_3 = v-2c$, $\delta_{11} = c(c-1)$, $\delta_{12} = c(c-m-1)$, $\delta_{13} = c(s^2t-2c+2+m)$, $\delta_{21} = c(c-m-1)$, $\delta_{22} = c(c-1)$, $\delta_{23} = c(s^2t-2c+2+m)$, $\delta_{31} = c(s^2t-2c+2+m)$, $\delta_{32} = c(s^2t-2c+2+m)$ and $\delta_{33} = (v-2c)s^2t - 2c(s^2t-2c+2+m)$.

Hence $B = \begin{pmatrix} c-1 & c-m-1 & s^2t-2c+2+m \\ c-m-1 & c-1 & s^2t-2c+2+m \\ \frac{c(s^2t-2c+2+m)}{v-2c} & \frac{c(s^2t-2c+2+m)}{v-2c} & s^2t-\frac{2c(s^2t-2c+2+m)}{v-2c} \end{pmatrix}$.

The constant row sum s^2t gives us the first eigenvalue of B. The second eigenvalue is m, belonging to the eigenvector $(1,-1,0)$. So the third eigenvalue is $\lambda_3 = \mathrm{tr}(B) - s^2t - m = \frac{v(2c-2-m)-2cs^2t}{v-2c}$, which is smaller than m. As the eigenvalues of B interlace the eigenvalues of A by theorem 1.31, we have $-s \leq \lambda_3 \leq m \leq t$.

$\boxed{-s \leq \lambda_3}$ The first inequality yields

$$\frac{2m^2+m(1-s)+s(s-1)}{v-2c} \geq 0.$$

As we supposed $\mathcal{C}$ and $\mathcal{C}^*$ to be disjoint, the denominator is negative. (If $v = 2c$, then $s = 1$ and $c = t+1 = m+1$, a contradiction.) So the nominator should be positive. Regarding this as a quadratic polynomial in m, we see that the discriminant is negative (except for $s = 1$). Hence the inequality is true for all m, but equality is never obtained, so the extended Higman-Sims technique gives no result in this case.

$\boxed{m \leq t}$ The inequality $m \leq t$ is trivial, but yet, we can obtain a smaller bound for m by the following observation. We know that $m = \lambda_2$ and λ_3 are the roots of the equation $f(x) = x^2 - (\lambda_2+\lambda_3)x + (\lambda_2\lambda_3)$, where $\lambda_2 + \lambda_3 = \mathrm{tr}(B) - s^2t$ and $\lambda_2\lambda_3 = (\det B)/s^2t$. As $-s \leq \lambda_3 \leq \lambda_2 \leq t$, we know that $f(-s) \geq 0$ (implying no new results) and $f(t) \geq 0$. This inequality yields $\frac{(t-m)}{v-2c}X \geq 0$ where

$$X = -2m^2(s+1)(t+1) + m(s^2t-st-s-2t-1) + s^2t(t+2) + st(t+1) - t.$$

$\boxed{1}$ First let $m < t$. As $(t-m)$ and $(v-2c)$ are positive, X should be nonnegative. Hence m should be between the roots m_1, m_2 of the quadratic expression X in m. We calculate those roots to be

$$m_{1,2} = \frac{st-2t-1 \pm \sqrt{s^2t^2+2st(4t^2+10t+7)-4t^2-4t+1}}{4(t+1)}$$

(with $m_1 \leq m_2$). As the nominator of the smallest root m_1 is always negative, m_1 is negative, giving no new restriction on the allowed values for m. Likewise, the biggest root m_2 will only give a restriction on m, if m_2 is smaller than t. We calculate $m_2 \leq t \Leftrightarrow s \leq t+1$. Remark that $s = t+1$ is not possible by theorem 1.1. So if $s < t+1$, then the extended Higman-Sims technique shows that m should be less than $m_2 = \frac{st-2t-1+\sqrt{s^2t^2+2st(4t^2+10t+7)-4t^2-4t+1}}{4(t+1)}$.

$\boxed{2}$ Now let $m = t$. As each line intersecting $\mathcal{C}$ should also intersect $\mathcal{C}^*$ (and vice versa), a point w_i not in $\mathcal{C} \cup \mathcal{C}^*$ is collinear with equally many points of $\mathcal{C}$ and of $\mathcal{C}^*$. Let this number be t_i. Counting the number of lines intersecting $\mathcal{C}$ (and hence $\mathcal{C}^*$), and comparing this with the total amount of lines, one sees that $t+1 \leq s$. The same result is obtained by the variance trick. (Count in two ways the couples (w_i, x) and the triples (w_i, x, y) with $w_i \notin \mathcal{C} \cup \mathcal{C}^*$, $x, y \in \mathcal{C}$ and w_i collinear with both x and y. By substituting the obtained values in the expression of the variance $\sum_i (\bar{t} - t_i)^2 \geq 0$ (with $(v - 2c)\bar{t} = \sum_i t_i$) or equivalently $(v-2c)\Sigma t_i^2 - (\Sigma t_i)^2 \geq 0$, one obtains $t+1 \leq s$.) As equality in $t+1 \leq s$ is never obtained (see theorem 1.1), we see that the number t_i is not a constant. $\square$

As mentioned, the result of previous theorem is rather weak. The next easy theorem does even better.

Theorem 5.9 *Let Γ be a generalized quadrangle of order (s,t). Let $\mathcal{C}$ be a proper m-cloud. Then $-m^3 + m^2(2t) + m(2t) + (-st^2 - st + t) \leq 0$.*

Proof

With the notations of page 82, there are $(1+m)(1+m+m^2)$ $\mathcal{C}\mathcal{C}^*$-lines, $(t-m)(1+m+m^2)$ $\mathcal{C}$-lines and just as much $\mathcal{C}^*$-lines. So in total there are $(2t-m+1)(m^2+m+1)$ lines intersecting $\mathcal{C} \cup \mathcal{C}^*$ in one or more points. As this should not exceed the total number of lines, one obtains the inequality stated in the theorem. $\square$

To illustrate the meaning of both results, we give some examples for small numbers in the following table.

s	t	max.value of m (thm 5.9)	max.value of m (thm 5.8)
2	4	1	2
3	9	3	4
4	16	5	6
5	25	8	8
6	36	10	11
7	49	13	14

5.9 Proper 2-clouds, or small hexagons in quadrangles

Instead of looking for large m, we now take a look at proper 2-clouds. They are in fact the double of a Fano-plane. By computer search, we showed that this small thin hexagon is neither contained in the classical quadrangles $Q(5,3)$ nor in $Q(4,5)$, but it is in $Q(4,7)$, $Q(4,11)$ and $Q(4,13)$. To show this, we used a back-tracking procedure in Pascal.

5.10 Dense clouds in quadrangles

As for the case of the hexagons (page 87), the extended Higman-Sims technique did not tell us a lot in the case of (proper) m-clouds. But it will do for dense clouds, as defined on page 87. Remark that the first half of next theorem was already stated in Payne [47], see Payne and Thas [48] 1.10.1.

Theorem 5.10 *Let Γ be a generalized quadrangle of order (s,t), and let $\mathcal{D}$ be a dense cloud of index α. Then $(s+1)(\alpha+1-s) \leq |\mathcal{D}| \leq \frac{(\alpha+t+1)(st+1)}{t+1}$.*
Equality holds for the lower bound, if and only if every point outside $\mathcal{D}$ is collinear with exactly $\alpha+1-s$ points of $\mathcal{D}$.
Equality holds for the upper bound, if and only if every point outside $\mathcal{D}$ is collinear with exactly $\alpha+t+1$ points of $\mathcal{D}$.

Proof
We put $|\mathcal{D}| = d$. Let $\mathcal{P} = \{w_1, \ldots, w_v\}$ be the point set of Γ with $v = (1+s)(1+st)$ and such that $w_1, \ldots, w_d \in \mathcal{D}$. Let A be the $(0,1)$-matrix (a_{ij}) over $\mathbb{R}$ defined by $a_{ij} = 1$ iff $w_i \not\sim w_j$, $i \neq j$, and $a_{ii} = 0$. So A has eigenvalues $s^2t, t, -s$ (see again [48] 1.2.2 or [9] p 203). We write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ defined by the partition $\Delta_1 = \{1, \ldots, d\}$ and $\Delta_2 = \{d+1, \ldots, v\}$. Put $\delta_{ij} = \sum_{\substack{k \in \Delta_i \\ l \in \Delta_j}} a_{kl}$, $\delta_i = |\Delta_i|$, and define the 2×2-matrix $B = (\delta_{ij}/\delta_i)_{1 \leq i,j \leq 2}$.
Let $x \in \mathcal{D}$. Then x is at distance 0 of itself ($\in \mathcal{D}$), at distance 2 of α points of $\mathcal{D}$, and at distance 4 of $(d-\alpha-1)$ points of $\mathcal{D}$. Hence there are $(d-\alpha-1)$ non-zero entries on a row of A_{11}. As there are in total s^2t points opposite to x, there are $(s^2t-d+\alpha+1)$ non-zero entries on a row of A_{12}, and just as much on a column of A_{21}. As there are in total $v(s^2t)$ non-zero entries in A, there are $vs^2t - d(2s^2t-d+\alpha+1)$ non-zero entries in A_{22}. Hence

the matrix B of average row sums of A_{ij} becomes

$$B = \begin{pmatrix} d-\alpha-1 & s^2t-d+\alpha+1 \\ \frac{d(s^2t-d+\alpha+1)}{v-d} & \frac{vs^2t-d(2s^2t-d+\alpha+1)}{v-d} \end{pmatrix}$$

If $\lambda_1(B), \lambda_2(B)$ are the eigenvalues of B, we know that $\lambda_1(B) = s^2t$ and $\lambda_1(B) + \lambda_2(B) = \mathrm{tr}(B)$. Using the notations of paragraph 1.9.3 and theorem 1.31, we have

$$\begin{cases} \lambda_{v-1}(A) & \leq & \lambda_1(B) & \leq & \lambda_1(A) \\ \lambda_v(A) & \leq & \lambda_2(B) & \leq & \lambda_2(A) \end{cases}$$

$$\Leftrightarrow \begin{cases} -s & \leq & s^2t & \leq & s^2t \\ -s & \overset{\star}{\leq} & \lambda_2(B) & \overset{\bullet}{\leq} & t \end{cases}$$

By substituting the known value of $\lambda_2(B)$, the inequality $\star$ becomes the bound $(\alpha+1-s)(s+1) \leq d$ (found in Payne [47]), while the inequality $\bullet$ yields $(\alpha+t+1)(st+1) \geq d(t+1)$. If the lower bound respectively upper bound is attained, A_{ij} has constant row sum and constant column sum by the second assertion of theorem 1.31. The row sums of A_{11} and A_{12} were already known to be a constant. The row sum b_{21} of A_{21} is the number of points of $\mathcal{D}$ that are at distance 4 of a point $y \in \Gamma \setminus \mathcal{D}$. As y is at distance 4 of b_{21} points of $\mathcal{D}$, y is collinear with $d - b_{21}$ points of $\mathcal{D}$. Substituting the lower bound gives the number $\alpha + 1 - s$, as found in [47], and substituting the upper bound, one finds the number $\alpha+t+1$. $\square$

If d attains the lower bound respectively upper bound, $\mathcal{D}$ is called a **minimal** respectively **maximal dense cloud**.

Theorem 5.11 *Let Γ be a generalized quadrangle, and let $\mathcal{D}$ be a maximal dense cloud of index α of Γ. Then every line of Γ is incident with a constant number of points of $\mathcal{D}$, this constant being equal to $\frac{\alpha}{t+1}+1$.*

Proof
Take a line L of Γ and suppose L intersects $\mathcal{D}$ in k points. Each point of $\mathcal{D}$ on L is collinear with $\alpha-k+1$ other points of $\mathcal{D}$, and as $|\mathcal{D}|$ attains the bound $\frac{(\alpha+t+1)(st+1)}{t+1}$, each point off $\mathcal{D}$ on L is collinear with $(\alpha+t+1)-k$ points of $\mathcal{D}$ not on L. As all points of Γ are at distance at most 3 of L, we counted all points of $\mathcal{D}$ in this way. Hence $k+k(\alpha-k+1)+(s+1-k)(\alpha+t+1-k) = |\mathcal{D}|$, implying that k is equal to $\frac{\alpha}{t+1}+1$. $\square$

Corollary 5.7 *The maximal dense clouds of index α of a generalized quadrangle Γ of order (s,t) are the $(\frac{\alpha}{t+1}+1)$-ovoids of Γ (see page 24).*

Examples

Let Γ be a generalized quadrangle of order (s,t).
If every point outside a dense cloud is collinear with a fixed number of points of the dense cloud, we denote this number with β.
Each m-**ovoid** of the quadrangle is a maximal dense cloud of index $\alpha = (m-1)(t+1)$, with $\beta = m(t+1)$. Each **partial ovoid** is a (non-maximal) dense cloud of index 0. Each **union of** $1+i$ **disjoint ovoids** is a maximal dense cloud of index $i(t+1)$. The point set of a **spread** is, of course, the trivial proper dense cloud of index $s(t+1)$. The point set of each **subquadrangle** Γ' of order (s',t') is a dense cloud of index $s'(t'+1)$. From the lower bound, it follows that $s = s'$ or $s \geq s't'$ for the order of a subquadrangle, and the point set of a subquadrangle is a minimal proper dense cloud if and only if $s' = s$ or $s = s't'$. In that case, $\beta = st' + 1$ respectively $s'+1$. These bounds for s' and t' and the interpretation of the equalities were already derived with the extended Higman-Sims technique by Payne, and can also be derived with the variance trick (see Payne and Thas [48] 2.2.1).
The set $\Gamma_2(x)$ of **all points at distance 2** of a given point x is a dense cloud of index $s-1$, but is never maximal.
The proper maximal dense clouds of the generalized quadrangle $Q(5,q)$ are the $\frac{q+1}{2}$-ovoids; we refer to the next chapter.

Chapter 6

Two Hill-caps but no Hemisystem

6.1 Introduction

In the previous chapter, we mentioned m-ovoids of generalized quadrangles as interesting examples of maximal dense clouds. But by a result of Segre [60], for which an easy proof can be found in Thas [70], the classical quadrangle $Q(5,q)$ has no m-ovoid except for $m = \frac{q+1}{2}$, in which case it is a hemisystem. Even worse: only for $q = 3$ such a set is really known to exist. It is the 56-cap of Hill, which is the largest possible cap in $\mathbf{PG}(5,3)$. Indeed, if we denote the largest size of any cap in $\mathbf{PG}(n,q)$ by $m_2(n,q)$, the following results are known to date:

$m_2(r,2) = 2^r$	[3]
$m_2(2,q) = q+1$, q odd	[3]
$m_2(2,q) = q+2$, q even	[3]
$m_2(3,2) = 8$	[52]
$m_2(3,q) = q^2+1$, $q > 2$	[3]
$m_2(4,3) = 20$	[49]
$m_2(5,3) = 56$	[36]

As for $q = 3$ a hemisystem and a cap are both point sets with at most 2 points on a line, we will try in this chapter to derive a hemisystem of some point-line geometry in $\mathbf{PG}(n,3)$ from large caps. Take for example the polar space $Q(6,3)$ in $\mathbf{PG}(6,3)$. This geometry has no hemisystem, as existence of such a system would imply a hemisystem of a hyperbolic quadric $Q^+(5,3)$. This set would be a 65-cap in 5 dimensions, in contradiction with $m_2(5,3) = 56$. (Or: the planes that are contained in $Q^+(5,3)$ would contain $13/2$ points, again a contradiction.) Yet, there would be another interesting hemisystem in 6 dimensions: we could look at the split Cayley hexagon $H(3)$ living on $Q(6,3)$. As the hexagon has less lines than the quadric, the restrictions on the wanted structure are less strong. So we look for a way to construct a subset $\mathcal{H}'$ of the point set of $H(3)$ (equal to the point set of $Q(6,3)$), such that each line of the hexagon contains exactly 2 points of $\mathcal{H}'$. As some nice pattern can be expected, we based our search on some symmetry-assumptions. But as a matter of fact, our construction — however promising — fails at the last (2) hurdle(s).

6.2 The Hill-cap $\mathcal{H}$

A 4-cap in $\mathbf{PG}(2,3)$, a 10-cap in $\mathbf{PG}(3,3)$ and a 56-cap in $\mathbf{PG}(5,3)$ are known to be unique (the first two being equivalent to respectively conics and elliptic quadrics, while the third, the Hill-cap, is contained in an elliptic quadric $Q^-(5,3)$.) In $\mathbf{PG}(4,3)$ there are nine inequivalent types of 20-caps. However, if a 20-cap occurs as intersection of a 56-cap in $\mathbf{PG}(5,3)$ with a hyperplane, only 2 types are possible for the 20-cap (Hill [37]).

6.2.1 Hyperplane sections of the Hill-cap $\mathcal{H}$

Let the Hill-cap $\mathcal{H}$ be contained in the elliptic quadric $Q^-(5,3)$ in $\mathbf{PG}(5,3)$. Let π be a hyperplane of $\mathbf{PG}(5,3)$.

- If π is the tangent hyperplane of $Q^-(5,3)$ at a point x of the Hill-cap $\mathcal{H}$, the intersection of π with $\mathcal{H}$ will contain 11 points of $\mathcal{H}$: the vertex x of the cone $\pi \cap Q^-(5,3)$ and one extra point on each line of the cone.

- If π is the tangent hyperplane of $Q^-(5,3)$ at a point x not on the Hill-cap $\mathcal{H}$, π will contain 20 points of $\mathcal{H}$: two on each line of the cone $\pi \cap Q^-(5,3)$, different from the vertex. This 20-cap is said to be of type Γ_1 (see [37]).

- If π is not a tangent plane of $Q^-(5,3)$, the intersection $\pi \cap Q^-(5,3)$ is a (parabolic) quadric, and as the Hill-cap is a hemisystem of $Q^-(5,3)$, it will also be a hemisystem of $\pi \cap Q^-(5,3)$. Hence π contains 20 points of the Hill-cap $\mathcal{H}$. This 20-cap is said to be of type Δ (see [37]).

Hence a hyperplane section of $\mathcal{H}$ contains either 11 or 20 points.

6.2.2 Hyperplane sections of a 20-cap of type Δ inside the Hill-cap $\mathcal{H}$

Let π_0 be a 4-dimensional projective space, intersecting $Q^-(5,3)$ in a non-degenerate quadric $Q(4,3)$. So $\mathcal{H} \cap \pi_0$ is a 20-cap C of type Δ. Let α be a hyperplane of $\mathbf{PG}(4,3)$.

- If α intersects $Q(4,3)$ in a hyperbolic quadric, α contains 8 points of C, as it is a hemisystem of the ruled quadric.
- If α is tangent to $Q(4,3)$ at a point x not on the 20-cap C, α contains 8 points of C; 2 on each line of the cone $\alpha \cap Q(4,3)$, different from the vertex x.
- If α is tangent to $Q(4,3)$ at a point x of the 20-cap C, α contains 5 points of C.
- Suppose α intersects $Q(4,3)$ in an elliptic quadric E. The elliptic quadric is subtended by 2 points of $Q^-(5,3) \setminus Q(4,3)$. If both subtending points belong to $\mathcal{H}$, α contains 2 points of C. If exactly 1 subtending point belongs to $\mathcal{H}$, α contains 5 points of C. If no subtending point belongs to $\mathcal{H}$, α contains 8 points of C. These numbers are obtained by using the result of paragraph 6.2.1, and a counting argument in the 4 hyperplanes in $\mathbf{PG}(5,3)$ through E (see Thas [70]).

Hence a hyperplane section of a 20-cap C of type Δ contains $2, 5$ or 8 points.

6.2.3 Degree with respect to a 20-cap

From now on, let C denote a 20-cap of type Δ. The **degree** of a point $x \in Q^-(5,3) \setminus Q(4,3)$ (with respect to C) is here defined as the number of points of C that are collinear on $Q^-(5,3)$ with x. In other words: it is the number of points of C in the elliptic quadric on $Q(4,3)$ subtended by x. If x is a point of $\mathcal{H}$, the degree is either 2 or 5 (see above). If x is not a point of $\mathcal{H}$, the degree is 5 or 8 (see above). (Remark that the term 'degree' is used in a different way in Hill [37].)
Now we will count the number of points of $Q^-(5,3) \setminus Q(4,3)$ of degree $2, 5$ or

8 respectively. Or, equivalently, the number of elliptic quadrics containing $2, 5$ or 8 points respectively. By [37], the coweightdistribution of a 20-cap of type Δ is $(2^{10}, 5^{36}, 8^{75})$. In other words, there are 10 hyperplanes of π_0 intersecting C in 2 points, 36 hyperplanes of π_0 intersecting C in 5 points, and 75 hyperplanes of π_0 intersecting $\mathcal{C}$ in 8 points. As there are 45 hyperbolic quadrics on $Q(4,3)$ (containing each 8 points of C) and 20 cones with vertex not in C (also containing 8 points of C), there are $75-45-20=10$ elliptic quadrics containing 8 points of C. As there are 20 cones with vertex in C (containing 5 points of C), there are $36-20=16$ elliptic quadrics containing 5 points of C. The other $36-10-16=10$ elliptic quadrics contain 2 points of C. Hence, in $Q^-(5,3) \setminus Q(4,3)$, there are 20 points of degree 2, there are 32 points of degree 5 and there are 20 points of degree 8.

6.3 Attempt to construct a hemisystem

Let π_0 be a 4-dimensional space intersecting $Q(6,3)$ in a non-degenerate quadric $Q(4,3)$. Let $\Pi_i, i=1,2$, be the two hyperplanes in $\mathbf{PG}(6,3)$ containing π_0 and intersecting $Q(6,3)$ in the elliptic quadrics $Q_i^-(5,3), i=1,2$. Let $\Pi_j, j=3,4$, be the two hyperplanes in $\mathbf{PG}(6,3)$ containing π_0 and intersecting $Q(6,3)$ in the hyperbolic quadrics $Q_j^+(5,3), j=3,4$. Let $H(3)$ be a split Cayley hexagon in $Q(6,3)$. As a hemisystem of $Q^-(5,3)$ will be a hemisystem of $Q^-(5,3) \cap H(3)$, we start our construction with a Hill-cap $\mathcal{H}_1$ in $Q_1^-(5,3)$. Then we add points of $Q_2^-(5,3)$ to $\mathcal{H}_1 \cap Q_2^-(5,3)$ such that the obtained set gives also a Hill-cap in $Q_2^-(5,3)$, say $\mathcal{H}_2$.

6.3.1 Points of $Q_2^-(5,3)$ to be added

We show that, given a Hill-cap $\mathcal{H}_1$ on $Q_1^-(5,3)$, there are only 2 possible choices for a Hill-cap $\mathcal{H}_2$ on $Q_2^-(5,3)$ such that $\mathcal{H}_2 \cap \pi_0$ equals $\mathcal{H}_1 \cap \pi_0$.
We already showed that for a point $x \in Q^-(5,3) \setminus Q(4,3)$ to belong to a Hill-cap in $Q^-(5,3)$ containing a given 20-cap C in $Q(4,3)$, x should have degree 2 or 5. So the 20 points of $Q_2^-(5,3)$ having degree 8 with respect to C should not be added to C. The 20 points of $Q_2^-(5,3)$ of degree 2 on the other hand, should all be added to C (as a point x of degree 2 with respect to C has degree 8 with respect to the complement C^c, x can not belong to the complement $\mathcal{H}_2^c$ of the Hill-cap $\mathcal{H}_2$ containing C). So we still have to choose 16 points out of 32 to obtain the Hill-cap $\mathcal{H}_2$ on $Q_2^-(5,3)$. But the set F of those 32 points of degree 5 can be divided into two subsets F_1, F_2

of the same size, as shown in next paragraph.

In order to avoid too many indices, we omit the index i of the elliptic quadrics $Q_i^-(5,3)$. The reasoning holds for any $Q^-(5,3)$ containing $Q(4,3)$. Let $L = \{l, x, y, z\}$ be a line of $Q^-(5,3)$, intersecting $Q(4,3)$ in the point l. This line subtends a rosette (see page 36) of 3 elliptic quadrics (say $\mathcal{O}_x, \mathcal{O}_y, \mathcal{O}_z$), plus a cone C_l with vertex l.

- Suppose $l \in C$. Then C_l contains 5 points of C, and as the rosette is a partition of $Q(4,3)$, the remaining 15 points of C can be found in $\mathcal{O}_x \cup \mathcal{O}_y \cup \mathcal{O}_z$. As an elliptic quadric contains 2, 5 or 8 points of the 20-cap C, we must write 15 as the sum of 3 integers out of $\{1, 4, 7\}$. The only possible cases are $15 = 7+7+1$ and $15 = 7+4+4$. Hence the line L has points of degree $8, 8, 2$ or $8, 5, 5$. In the latter case, exactly one point of those of degree 5 belongs to the hemisystem (as l belongs to C, while a point of degree 8 does not belong to the hemisystem).

- Suppose $l \notin C$. As C_l contains 8 points of C in this case, there will be 12 points of C in $\mathcal{O}_x \cup \mathcal{O}_y \cup \mathcal{O}_z$. We have $12 = 2 + 2 + 8$ and $12 = 2 + 5 + 5$, hence this line has points of degree $2, 2, 8$ or $2, 5, 5$. Again, in the latter case exactly one of the points of degree 5 belong to the hemisystem.

We see that, if a line of the quadric contains a point of degree 5, it contains a (unique) second point of degree 5. Let F_1 be a subset of F consisting of all points that are added to C in order to complete $\mathcal{H}_2$, and let $F_2 = F \setminus F_1$. Let x be a point of degree 5 that is added to C — so $x \in F_1$. Then the 10 points of $Q^-(5,3) \setminus Q(4,3)$ collinear with x and of degree 5, can not belong to $\mathcal{H}_2$, so should be in F_2. Moreover, if x subtends the elliptic quadric $\mathcal{O}_x$, the unique second point y subtending $\mathcal{O}_x$ should also belong to F_2 (as an elliptic quadric on $Q(4,3)$ containing 5 points of C is subtended by exactly one point of a Hill-cap containing C). Hence all 10 points of degree 5 collinear with y belong to F_1. So far, we found 11 points of F_1 and 11 points of F_2. Let z be a point of degree 5, different from the 22 points we already considered. Suppose none of the 10 points of degree 5 collinear with z belongs to the set of those 22 points. Then we would have $22+10+1 = 33$ points of degree 5 — a contradiction. Hence at least one point collinear with z was already considered, so we know whether z belongs to F_1 or F_2. So once we know one point of F_1, we know all points of F_1. Moreover, we know that no contradiction will arise as we know the existence of at least one Hill-cap $\mathcal{H}_2$ containing C (i.e. the image of $\mathcal{H}_1$ under a collineation of $\mathbf{PG}(6,3)$ fixing $Q(6,3)$, fixing π_0 pointwise and mapping $Q_1^-(5,3)$ to $Q_2^-(5,3)$). Hence F_1 together with the 20 points of degree 2 will extend C

to a Hill-cap. But also F_2 together with these 20 points of degree 2 will extend C to a Hill-cap. Indeed, a line of the quadric through a point of F_1 will have a unique point x of F_1 and a unique point y of F_2, while a line of the quadric not incident with a point of F_1, will neither be incident with a point of F_2. Exchanging x and y preserves the defining properties of a cap, so F_2 together with the 20 points of degree 2 extends C also to a cap of size 56.

6.3.2 Extension to $Q_3^+(5,3)$ and $Q_4^+(5,3)$

So far, we constructed $56+(56-20)=92$ points of our theoretical hemisystem $\mathcal{H}'$ of the hexagon $H(3)$, without bothering about the hexagon itself. Indeed, we constructed a hemisystem $\mathcal{H}_1 \cup \mathcal{H}_2$ for the union of the polar spaces $Q_1^-(5,3) \cup Q_2^-(5,3)$. To extend this to a hemisystem $\mathcal{H}'$ of $H(3)$, we should find some more points in the hyperbolic quadrics $Q_3^+(5,3)$ and $Q_4^+(5,3)$ ($\frac{182-92}{2}=45$ in each would be nice). We proceed as follows (more details can be found in appendix A). We exclude the point in $Q_3^+(5,3)$ and the point in $Q_4^+(5,3)$ on a line of the hexagon through a point of $\mathcal{H}_1 \subset Q_1^-(5,3)$ and a point of $\mathcal{H}_2 \subset Q_2^-(5,3)$, and then count the points that are left. As the index of the group $G_2(3)$ of $H(3)$ with respect to the group $\mathbf{PGO}_7(3)$ of $Q(6,3)$ is 2160, there are 2160 possible positions of the hexagon in $Q(6,3)$. So we have to range over all these positions. The computer showed us that, starting from $\mathcal{H}_1 \cup \mathcal{H}_2$, there were only 42 points left in $Q_i^+(5,3)$ that could belong to $\mathcal{H}'$, and only 38,41,42,43 or 46 points in $Q_j^+(5,3)$ that were still candidates for the cap $\mathcal{H}'$, with $\{i,j\}=\{3,4\}$. We choose indices in such a way that $i=3$ and $j=4$. (There are different numbers for the points left in $Q_4^+(5,3)$, depending on the position of $H(3)$ in $Q(6,3)$. The number of points in $Q_3^+(5,3)$, however, appeared to be independent of the position of $H(3)$. If we would take the set F_2 instead of F_1 in the construction of $\mathcal{H}_1 \cup \mathcal{H}_2$, the number of points that are not excluded in $Q_4^+(5,3)$ is fixed (i.e. 42), while the number of points that are left in $Q_3^+(5,3)$ varies as above.)
As $42+46<90$, the amount of candidates for points of $\mathcal{H}'$ is not large enough to reach the theoretical hemisystem $\mathcal{H}'$, having $182=92+90$ points.

This leads us to the following theorem.

6.4 Theorem

Theorem 6.1 *Suppose π_0 is a 4-dimensional subspace of $\mathbf{PG}(6,3)$ intersecting $Q(6,3)$ in a non-singular quadric $Q(4,3)$. Let Π_1 and Π_2 be the two hyperplanes through π_0 intersecting $Q(6,3)$ in two non-singular elliptic quadrics $Q_1^-(5,3)$ and $Q_2^-(5,3)$. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hill-caps on respectively $Q_1^-(5,3)$ and $Q_2^-(5,3)$, intersecting $Q(4,3)$ in the same 20-cap C. Then $\mathcal{H}_1 \cup \mathcal{H}_2$ is not extendable to a hemisystem of $H(3)$.*

6.5 $2-(16,6,2)$ Design

From the explanation in paragraph 6.3.1, one sees that a biplane (page 23) with parameters $(16,6,2)$ can be constructed starting from a Hill-cap on $Q^-(5,3)$. We first recall a result about symmetric designs, which can be found in e.g. Beth, Jungnickel and Lenz [1] II.3.14.
If a symmetric design with $v = 4n = 4(k-\lambda) \geq 2k$ exists, then it has parameters $(4u^2, 2u^2-u, u^2-u)$, and its $(1,-1)$-incidence matrix is a Hadamard matrix, being an $m \times m$-matrix H with entries in $\{1,-1\}$ such that $HH^T = kI$. If H is a Hadamard matrix, then $m = 1, 2$ or $m = 0 \bmod 4$.
To construct our example of a biplane, we take the point set X of the design equal to the set F_1 mentioned in paragraph 6.3.1 (i.e. the set of 16 points of a Hill-cap $\mathcal{H}$ in $Q^-(5,3)$ that are of degree 5 with respect to a 20-cap C of type Δ in a hypersection of $Q^-(5,3)$). The blocks of size 6 are defined as follows. If x and $y \in Q^-(5,3) \setminus Q(4,3)$ are both of degree 5 and subtend the same ovoid $\mathcal{O}_x$ in $\pi_0 \cap Q^-(5,3)$ (with $\pi_0 \cap \mathcal{H} = C$), exactly one of them belongs to F_1. Say $x \in F_1$. The ten points of degree 5 collinear with y also belong to F_1, and are by definition not in the block defined by x. The 5 remaining points of F_1 together with x itself, constitute one of the 16 blocks of the symmetric $2-(16,6,2)$ design.
The Kronecker product $H \otimes H'$ of the matrices $H = (h_{ij})$ and $H' = (h'_{kl})$ is defined by

$$H \otimes H' = \begin{pmatrix} h_{11}H' & h_{12}H' & \dots & h_{1n}H' \\ \dots & \dots & \dots & \dots \\ h_{m1}H' & h_{m2}H' & \dots & h_{mn}H' \end{pmatrix}$$

If H is the Hadamard matrix $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$ and $H' = -H$, then $H \otimes H'$ is a 16×16-Hadamard matrix, which gives (up to isomorphism) the $(1,-1)$-incidence matrix of the design considered, as we checked by computer. For the explicit enumeration of the blocks, see appendix A, page 120.

(We also refer to Beth, Jungnickel and Lenz [1] page 67.)

Appendix A

Computer Programs

The programs below are written in Pascal. There are basically three parts. As we needed a lot of sets (i.e. the point set of $Q(6,3)$, the point set of a non-singular $Q(4,3)$ in $Q(6,3)$, and the point sets of the four quadrics on $Q(6,3)$ lying in the four hyperplanes containing $Q(4,3)$), we first made these sets and stored them in different files, so that, by running the sequel, these sets could easily be read instead of computed again each time. This is done in the first program `SETS.p`, which calls for `makingofsets.p`.
In the second program, `DESIGN.p`, we only deal with incidence on the quadric and look for a Hill-cap `Ecap56` on $Q_2^-(5,3)$ intersecting $Q(4,3)$ in the same 20-cap as a given Hill-cap `Dcap56` on $Q_1^-(5,3)$. The result is written in a separate file, to be used in the third program. We also printed out the $2-(16,6,2)$ design formed by the 16 points of a Hill-cap $\mathcal{H}$ that are of degree 5 with respect to a 20-cap $\mathcal{C} \subset \mathcal{H}$ in a hypersection of $Q^-(5,3)$.
In the third program, `HEMISS.p`, we also deal with incidence on the hexagon (the hexagon in his 2160 possible positions on the quadric) and count how many points are left in $Q_3^+(5,3)$ and $Q_4^+(5,3)$ that are candidates to form a (virtual) hemisystem of the hexagon $H(3)$.
All of those programs make use of `makingofnumbers.p`, which we give first.

A.1 Subprogram `makingofnumbers.p`

To make listings easier, every point of $\mathbf{PG}(6,3)$ (denoted by its coordinates $(x_0,\ldots,x_6)$ with $x_i \in \mathbf{GF}(3)$) is labeled by a number (`function nummer_uit_coo`), while the inverse function (`function coo_uit_nummer`) is useful if one wants to compute collinearity between numbered points.

```
function nummer_uit_coo(X:cootype):integer;
var  j,k,l,qk,macht,rest:integer;
     aantal_ptn_hiervoor,eerste_niet_nul_pos:integer;
begin
 if q<>3 then
          begin
           writeln(outpt,'zie dhemi_n.p function nummer_uit_coo');
           halt;
          end;
 j:=0;
 while X[j]=0 do j:=j+1;
 eerste_niet_nul_pos:=j;
 if eerste_niet_nul_pos = 6
 then nummer_uit_coo:=1
 else
  begin
   if X[eerste_niet_nul_pos]=2
   then for l:=eerste_niet_nul_pos to 6 do
    X[l]:=(2*X[l])mod q;
   macht:=5-j;
   aantal_ptn_hiervoor:=1;
   k:=0;
   while k<>macht do
    begin
     aantal_ptn_hiervoor:=(aantal_ptn_hiervoor)*q+1;
     k:=k+1;
    end;
   rest:=0;
   qk:=1;
   for k:=6 downto eerste_niet_nul_pos+1 do
    begin
     rest:=rest+X[k]*qk;
     qk:=qk*q;
    end;
   nummer_uit_coo:=aantal_ptn_hiervoor+rest+1;
  end;
end;{of function nummer_uit_coo}

procedure coo_uit_nummer(i:integer; var X:cootype);
var  k,l,j:integer;
     aantal_ptn_hiervoor,eerste_niet_nul_pos:integer;
begin
 for j:=0 to 6 do X[j]:=0;
 if i=1
 then X[6]:=1
 else
  begin
   j:=numberofpts;
   k:=-1;
   while j>i-1 do
    begin
     j:=(j-1)div q;
     k:=k+1;
    end;
   aantal_ptn_hiervoor:=j;
   eerste_niet_nul_pos:=k;
   X[eerste_niet_nul_pos]:=1;
   l:=i-aantal_ptn_hiervoor-1;
   for k:=6 downto eerste_niet_nul_pos+1 do
    begin
     X[k]:=l mod q;
     l:=l div q;
    end;
  end;
end;{of procedure coo_uit_nummer}
```

A.2 Subprogram `makingofsets.p`

We choose the following equations:

$$\begin{array}{rcl}
Q(6,3) & \leftrightarrow & X_0X_4 + X_1X_5 + X_2X_6 = X_3^2 \\
Q_1^-(5,3) \subset \Pi_1 & \leftrightarrow & X_1 + X_2 + 2X_5 + 2X_6 = 0 \\
Q_2^-(5,3) \subset \Pi_2 & \leftrightarrow & X_0 + X_4 + X_5 = 0 \\
Q_3^+(5,3) \subset \Pi_3 & \leftrightarrow & X_0 + X_1 + X_2 + X_4 + 2X_6 = 0 \\
Q_4^+(5,3) \subset \Pi_4 & \leftrightarrow & X_0 + 2X_1 + 2X_2 + X_4 + 2X_5 + X_6 = 0
\end{array}$$

We form all different sets we will use. They are denoted as follows.

`PG`	point set of $\mathbf{PG}(6,3) \setminus Q(6,3)$	`type  0`
`Q`	point set of $Q(6,3)$ = point set of $H(3)$	
`Q5min2=D`	point set of $Q_1^-(5,3) \setminus Q(4,3)$	`type 12`
`Q5min1=E`	point set of $Q_2^-(5,3) \setminus Q(4,3)$	`type 13`
`Q5pls1=F`	point set of $Q_3^+(5,3) \setminus Q(4,3)$	`type 14`
`Q5pls2=G`	point set of $Q_4^+(5,3) \setminus Q(4,3)$	`type 15`
`Q4=C`	point set of $Q(4,3)$	`type 16`
`H`	line set of $H(3)$	

Every point (identified with its number $i \in \{1, \dots, 1093\}$) is assigned to one of those sets, by giving it the corresponding type (`procedure maak_pointtype`). The resulting array with $|\mathbf{PG}(6,3)| = 1093$ entries between 0 and 16 is named `pointtype` and stored in the file `pas_pointtype.data`. This will allow us to proceed quickly when running the main part of the program(s). Of course, storing all 1093 points of $\mathbf{PG}(6,3)$ while we only need to range over the 364 points of $Q(6,3)$, does not promote the speed of the programs. To tackle this problem, we make an array `Q` of 364 entries which stores the numbers of all points of $Q(6,3)$ (`procedure maak_Q`). The inverse procedure is `procedure maak_positiept`, which returns the position in the array `Q` if one gives the number of a point in $Q(6,3)$.

```
type  PGptarray  =array[1..numberofpts] of 0..numberofHpts;
      Qptarray   =array[1..numberofHpts] of 0..numberofpts;
      array56    =array[1..56] of 0..numberofpts;
      array112   =array[1..112] of 0..numberofpts;

var   Q:         Qptarray;
      positiept: PGptarray;
      pointtype: PGptarray;
      CcupD:     array112;
      Dcap56:    array56;

function behoort_tot_PG4(X:cootype):boolean;
begin
 if ((X[0]+X[4]+X[5]) mod q = 0 mod q)
  and ((X[1]+X[2]+2*X[5]+2*X[6]) mod q = 0 mod q)
 then behoort_tot_PG4:=true else behoort_tot_PG4:=false;
end;

function behoort_tot_PG5pls2(X:cootype):boolean;
begin
```

```
 if ((X[0]+2*X[1]+2*X[2]+X[4]+2*X[5]+X[6]) mod q =0 mod q)
 then behoort_tot_PG5pls2:=true else behoort_tot_PG5pls2:=false;
end;

function behoort_tot_PG5pls1(X:cootype):boolean;
begin
 if ((X[0]+X[1]+X[2]+X[4]+2*X[6]) mod q = 0 mod q)
 then behoort_tot_PG5pls1:=true else behoort_tot_PG5pls1:=false;
end;

function behoort_tot_PG5min1(X:cootype):boolean;
begin
 if ((X[0]+X[4]+X[5])mod q = 0 mod q)
 then behoort_tot_PG5min1:=true else behoort_tot_PG5min1:=false;
end;

function behoort_tot_PG5min2(X:cootype):boolean;
begin
 if ((X[1]+X[2]+2*X[5]+2*X[6])mod q = 0 mod q)
 then behoort_tot_PG5min2:=true else behoort_tot_PG5min2:=false;
end;

function behoort_tot_Q6(X:cootype):boolean;
begin
 if ((X[0]*X[4]+X[1]*X[5]+X[2]*X[6])mod q = (X[3]*X[3])mod q)
 then behoort_tot_Q6:=true else behoort_tot_Q6:=false;
end;

procedure maak_pointtype(var pointtype:PGptarray);
var i:integer;
begin
 for i:=1 to numberofpts do pointtype[i]:=0;
 for i:=1 to numberofpts do
 begin
  coo_uit_nummer(i,X);
  if behoort_tot_PG5min2(X) then pointtype[i]:=2
  else if behoort_tot_PG5min1(X) then pointtype[i]:=3
  else if behoort_tot_PG5pls1(X) then pointtype[i]:=4
  else if behoort_tot_PG5pls2(X) then pointtype[i]:=5;
  if behoort_tot_PG4(X)     then pointtype[i]:=6;
  if behoort_tot_Q6(X) then pointtype[i]:=pointtype[i]+10;
 end;
end;{of procedure maak_pointtype}

procedure maak_Q(var Q:Qptarray);
var i,j:integer;
begin
 j:=1;
 for i:=1 to numberofpts do
 if pointtype[i]>10
 then
  begin
   Q[j]:=i;
   j:=j+1;
  end;
end;{of procedure maak_Q}

procedure maak_positiept(var positiept:PGptarray);
var i,j:integer;
begin
 for i:=1 to numberofpts do positiept[i]:=0;
 for j:=1 to numberofHpts do
  positiept[Q[j]]:=j;
end;{of procedure maak_positiept}
```

Now we construct a 56-cap of Hill in $Q_1^-(5,3)$. We use the explicit description of the cap on $Q^-(5,3) \leftrightarrow \sum_{i=0}^{5} Y_i^2 + 2\sum_{i=0}^{4}(Y_iY_{i+1}) = 0$ that is given in Hill [36]. As the 56-cap splits into 8 orbits under the natural action of a certain orthogonal transformation t of order 7 in $PO^-(6,3)$, we only have to list 8 points (i.e. `K[1]` to `K[8]`), and calculate the other points in the 8 orbits (`procedure bereken_beeld_onder_cycl_tr`). To find the points `K[1]` to `K[8]`, we listed the 112 points of `CcupD`=`C`∪`D`= $Q_1^-(5,3)$ in Y-coordinates, compared this with the list in Hill [36], and found the 8 numbers used in `maak_D1_en_D2`. (Of course, we also could have typed in manually all 56 points from the article, but this could have lead to more typing errors.) In procedure `maak_Dcap56`, the 56 points of the Hill-cap are collected such that the first 20 points of `Dcap56` are inside $Q(4,3)$.

```
procedure maak_CcupD(var CcupD:array112);
var i,j: integer;
begin
 j:=1;
 for i:=1 to numberofpts do
  if (pointtype[i]=16) or (pointtype[i]=12)
  then begin CcupD[j]:=i;
             j:=j+1;
       end;
end;{of procedure maak_CcupD}

procedure Ycoo_uit_nummer(i:integer;var Y:cootype);
var j,k:integer;
X:cootype;
begin
 coo_uit_nummer(i,X);
 Y[0]:=(1*X[0]+1*X[1]+0*X[2]+0*X[3]+1*X[4]+0*X[5]+0*X[6])mod q;
 Y[1]:=(2*X[0]+0*X[1]+0*X[2]+0*X[3]+2*X[4]+0*X[5]+0*X[6])mod q;
 Y[2]:=(1*X[0]+1*X[1]+2*X[2]+1*X[3]+1*X[4]+0*X[5]+0*X[6])mod q;
 Y[3]:=(0*X[0]+2*X[1]+1*X[2]+0*X[3]+0*X[4]+0*X[5]+0*X[6])mod q;
 Y[4]:=(1*X[0]+1*X[1]+2*X[2]+0*X[3]+0*X[4]+2*X[5]+0*X[6])mod q;
 Y[5]:=(1*X[0]+0*X[1]+0*X[2]+0*X[3]+1*X[4]+1*X[5]+0*X[6])mod q;
 Y[6]:=(0*X[0]+1*X[1]+1*X[2]+0*X[3]+0*X[4]+2*X[5]+2*X[6])mod q;
 j:=0;
 while Y[j]=0 do j:=j+1;
 if Y[j]=2 then for k:=j to 6 do Y[k]:=(2*Y[k])mod q;
end;{of procedure Ycoo_uit_nummer}

function nummer_uit_Ycoo(Y:cootype):integer;
var X:cootype;
begin
 X[0]:=(0*Y[0]+1*Y[1]+0*Y[2]+1*Y[3]+1*Y[4]+1*Y[5]+0*Y[6])mod q;
 X[1]:=(1*Y[0]+1*Y[1]+0*Y[2]+0*Y[3]+0*Y[4]+0*Y[5]+0*Y[6])mod q;
 X[2]:=(1*Y[0]+1*Y[1]+0*Y[2]+1*Y[3]+0*Y[4]+0*Y[5]+0*Y[6])mod q;
 X[3]:=(0*Y[0]+1*Y[1]+1*Y[2]+1*Y[3]+0*Y[4]+0*Y[5]+0*Y[6])mod q;
 X[4]:=(0*Y[0]+1*Y[1]+0*Y[2]+2*Y[3]+2*Y[4]+2*Y[5]+0*Y[6])mod q;
 X[5]:=(0*Y[0]+1*Y[1]+0*Y[2]+0*Y[3]+0*Y[4]+1*Y[5]+0*Y[6])mod q;
 X[6]:=(2*Y[0]+1*Y[1]+0*Y[2]+1*Y[3]+0*Y[4]+2*Y[5]+2*Y[6])mod q;
 nummer_uit_Ycoo:=nummer_uit_coo(X);
end;{of function nummer_uit_Ycoo}

procedure bereken_beeld_onder_cycl_tr(Y:cootype;var Z:cootype);
begin
  Z[0]:=(0*Y[0]+0*Y[1]+0*Y[2]+0*Y[3]+0*Y[4]+2*Y[5]+0*Y[6])mod q;
  Z[1]:=(1*Y[0]+0*Y[1]+0*Y[2]+0*Y[3]+0*Y[4]+2*Y[5]+0*Y[6])mod q;
  Z[2]:=(0*Y[0]+1*Y[1]+0*Y[2]+0*Y[3]+0*Y[4]+2*Y[5]+0*Y[6])mod q;
  Z[3]:=(0*Y[0]+0*Y[1]+1*Y[2]+0*Y[3]+0*Y[4]+2*Y[5]+0*Y[6])mod q;
  Z[4]:=(0*Y[0]+0*Y[1]+0*Y[2]+1*Y[3]+0*Y[4]+2*Y[5]+0*Y[6])mod q;
  Z[5]:=(0*Y[0]+0*Y[1]+0*Y[2]+0*Y[3]+1*Y[4]+2*Y[5]+0*Y[6])mod q;
  Z[6]:=(0*Y[0]+0*Y[1]+0*Y[2]+0*Y[3]+0*Y[4]+0*Y[5]+1*Y[6])mod q;
end;{of procedure bereken_beeld_onder_cycl_tr}

procedure maak_D1_en_D2(var D1,D2:array56);
var i,j,k,l,m:integer;
K:array[1..16] of integer;
Y,Z:cootype;
begin
 K[ 1]:= 1; K[ 2]:= 4; K[ 3]:= 5; K[ 4]:= 7;
 K[ 5]:=10; K[ 6]:=13; K[ 7]:=17; K[ 8]:=25;
 K[ 9]:= 2; K[10]:= 3; K[11]:= 6; K[12]:= 9;
 K[13]:=18; K[14]:=22; K[15]:=34; K[16]:=44;
 m:=1;
 for k:=1 to 8 do
  begin
   i:=K[k];
   Ycoo_uit_nummer(CcupD[i],Y);
   for j:=1 to 7 do
   begin
    D1[m]:=nummer_uit_Ycoo(Y);
    m:=m+1;
    bereken_beeld_onder_cycl_tr(Y,Z);
    for l:=0 to 6 do Y[l]:=Z[l];
   end;
  end;
```

```
 m:=1;
 for k:=9 to 16 do
  begin
   i:=K[k];
   Ycoo_uit_nummer(CcupD[i],Y);
   for j:=1 to 7 do
   begin
    D2[m]:=nummer_uit_Ycoo(Y);
    m:=m+1;
    bereken_beeld_onder_cycl_tr(Y,Z);
    for l:=0 to 6 do Y[l]:=Z[l];
   end;
  end;
end;{of procedure maak_D1_en_D2}

procedure maak_Dcap56(var Dcap56:array56);
var i,j,k: integer;
    D1,D2: array56;
begin
 maak_D1_en_D2(D1,D2);
 j:=1;
 k:=21;
 for i:=1 to 56 do
  if pointtype[D1[i]]=16
  then begin Dcap56[j]:=D1[i];
             j:=j+1;
       end
  else begin Dcap56[k]:=D1[i];
             k:=k+1;
       end;
end;{of procedure maak_Dcap56}
{If one wants to run the program with the complement of this Dcap56,   }
{one has to replace D1 by D2 in the last eight lines of this procedure.}
```

In the last part (i.e. `procedure initialize_makingofsets`) of this subprogram, we call the procedures above. Once an array (e.g. `pointtype`) is explicitly calculated, it is written in a separate text-variable (e.g. `outptpointtype`). We use a variable `cursorteller` to ensure that lines are not splitted in the middle of a number, making the data unreadable.

```
procedure initialize_makingofsets;
var   i:integer;
      cursorteller:integer;
begin

 maak_pointtype(pointtype);
 cursorteller:=0;
 for i:=1 to numberofpts do
  begin
   cursorteller:=(cursorteller+1)mod 19;
   if (cursorteller=0)
   then writeln(outptpointtype,pointtype[i]:4)
   else write(outptpointtype,pointtype[i]:4);
  end;
 writeln(outptpointtype);
 flush(outptpointtype);

 maak_Q(Q);
 cursorteller:=0;
 for i:=1 to numberofQpts do
  begin
   cursorteller:=(cursorteller+1)mod 15;
   if (cursorteller=0)
   then writeln(outptQ,Q[i]:5)
   else write(outptQ,Q[i]:5);
  end;
 writeln(outptQ);
 flush(outptQ);

 maak_positiept(positiept);
 cursorteller:=0;
 for i:=1 to numberofpts do
  begin
   cursorteller:=(cursorteller+1)mod 19;
   if (cursorteller=0)
   then writeln(outptpositiept,positiept[i]:4)
   else write(outptpositiept,positiept[i]:4);
```

```
  end;
 writeln(outptpositiept);
 flush(outptpositiept);

 maak_CcupD(CcupD);

 maak_Dcap56(Dcap56);
 cursorteller:=0;
 for i:=1 to 56 do
  begin
   cursorteller:=(cursorteller+1)mod 15;
   if (cursorteller=0)
   then writeln(outptDcap56,Dcap56[i]:5)
   else write(outptDcap56,Dcap56[i]:5);
  end;
 writeln(outptDcap56);
 flush(outptDcap56);

 writeln(outpt,'Dit is type Dcap56');
 for i:=1 to 56 do write(outpt,pointtype[Dcap56[i]]:3);

end;{of procedure initialize_makingofsets}
```

A.3 Main program SETS.p

This program forms all sets needed in the programs DESIGN.p and HEMISS.p. Its subprograms makingofnumbers.p and makingofsets.p are called and all textvariables are set and written to separate data files. The same data files will be read by the main programs DESIGN.p and HEMISS.p to save computing time.

```
program SETS;

const q=3;
      qone=q-1;
      q2=q*q;
      q3=q2*q;
      q4=q3*q;
      q5=q4*q;
      q6=q5*q;
      q7=q6*q;
      numberofpts= (q7-1) div (q-1);
      numberofHpts= (q6-1) div (q-1);
      numberofQpts=numberofHpts;
      numberofQlines=(q2+q+1)*(q2+1)*(q3+1);

type  cootype=array[0..6] of 0..qone;

var   outpt:text;
      outptpointtype:text;
      outptQ:text;
      outptpositiept:text;
      outptDcap56:text;
      i,j:integer;
      X,Y,Z:cootype;

#include "makingofnumbers.p";
#include "makingofsets.p";

begin

rewrite(outpt,'SETS.data');
rewrite(outptpointtype,'pas_pointtype.data');
rewrite(outptQ,'pas_q.data');
rewrite(outptpositiept,'pas_positiept.data');
rewrite(outptDcap56,'pas_dcap56.data');

initialize_makingofsets;

end.
```

A.4 Subprogram `makingofdesign.p`

In the previous program, we constructed the point sets of the 6 quadrics ($Q(4,3)$, $Q_1^-(5,3)$, $Q_2^-(5,3)$, $Q_3^+(5,3)$, $Q_4^+(5,3)$ and $Q(6,3)$) by giving the appropriate type to each point, and a Hill-cap `Dcap56` in $Q_1^-(5,3)$. These data are read by the subprogram `makingofdesign.p`, using the four procedures `lees_*` (page 119). Now we will construct a Hill-cap `Ecap56` on $Q_2^-(5,3)$. As explained in chapter 6, we need to know the degree of a point of $Q_2^-(5,3)$ with respect to the 20-cap of $Q(4,3)$, so we need the incidence relation in $Q(6,3)$ explicitely. This is stored in a 364×364-matrix named `incidencefix` (`procedure maak_incidencefix`), with entries between 0 and 6. If 2 points `i,j` are not collinear on $Q(6,3)$, their incidence type `incidencefix[i,j]` is 0. If 2 points are collinear on $Q(6,3)$, their incidence type is positive. More specific: if they are on a line of $Q(4,3)$, their incidence type is 6 (according to the pointtype of $Q(4,3)$, see page 109). The same for lines of $Q_4^+(5,3)$, $Q_3^+(5,3)$, $Q_2^-(5,3)$ and $Q_1^-(5,3)$, which give respectively incidence type 5, 4, 3 and 2. If 2 points are on a line of $Q(6,3)$ not intersecting $Q(4,3)$, their incidence type is 1. For later convenience, we also store the point set of all lines of $Q(6,3)$ in a 3640×4-matrix called `ptson`.

Using this incidence relation (which was not calculated and written to a separate textfile by the previous program, as the computing time is not so much different from the reading time), we compute the degree of all points of `E`= $Q_2^-(5,3) \setminus Q(4,3)$ with respect to `Dcap56`$\cap Q(4,3)$ (see `procedure maak_degreeE`). In `procedure maak_helftnr`, the set of 32 points of degree 5 is divided into two equal parts. In `procedure maak_Ecap56`, the Hill-cap `Ecap56` is constructed.

The last part of the program (`maak_ovoide`, `maak_Everwant`, `maak_blok`, `schrijf_design`) computes the $2-(16,6,2)$ design mentioned on page 105 and page 107. In `maak_Everwant`, the 'twins' (see page 37) of all points of degree 5 are listed, so that we can easily deduce which set of 10 points are the complements of a block. In `maak_blok`, all blocks are listed, and by the procedures `schrijf_design`, the design is written into the outputfile `DESIGN.data`.

```
const numberoflines=3640;

type PGptarray    =array[1..numberofpts] of 0..numberofHpts;
     Qptarray     =array[1..numberofHpts] of 0..numberofpts;
     PGptmatrix   =array[1..numberofpts,1..numberofpts] of integer;
     Qptmatrix    =array[1..numberofHpts,1..numberofHpts] of -1..16;
     Qlnarray     =array[0..numberofQlines] of 0..numberofQlines;
     Qlnarray4    =array[1..numberofQlines,1..4] of 1..numberofHpts;
     array16      =array[1..16] of integer;
     array20      =array[1..20] of 0..numberofpts;
     array32      =array[1..32] of integer;
     array36      =array[1..36] of integer;
     array40      =array[1..40] of 0..numberofpts;
     array56      =array[1..56] of 0..numberofpts;
```

```
     array72      =array[1..72] of 0..numberofpts;
     array90      =array[1..90] of integer;
     array72_8    =array[1..72,1..8] of 0..numberofpts;
     array38_60   =array[38..60,38..60] of integer;
     array32_12   =array[1..32,1..12] of integer;
     matrixtype   =array[0..6,0..6] of 0..2;
     array16_6    =array[1..16,1..6] of integer;

var   Q:             Qptarray;
      C:             array40;
      D:             array72;
      E:             array72;
      degreeE:       array72;
      Esubsetdegree5:array32;
      cap20:         array20;
      incidencefix:  Qptmatrix;
      positiept:     PGptarray;
      pointtype:     PGptarray;
      linetype:      Qlnarray;
      linetypefix:   Qlnarray;
      lineon:        PGptmatrix;
      ptson:         Qlnarray4;
      cursorteller:  integer;
      i,j:           integer;
      tijd:          integer;
      Ecap56:        array56;
      Dcap56:        array56;
      A1,A,B1,B:    cootype;
      M:             matrixtype;
      Epositie:      PGptarray;
      helftnr:       array32;
      Everwant:      array32;
      blok:          array16_6;
      ovoide:        array32_12;



procedure lees_pointtype(var pointtype:PGptarray);
var A:integer;
begin
 cursorteller:=0;
 for i:=1 to numberofpts do
  begin
   cursorteller:=(cursorteller+1)mod 19;
   if (cursorteller=0)
   then readln(inptpointtype,A)
   else read(inptpointtype,A);
   pointtype[i]:=A;
  end;
end;


procedure lees_Q(var Q:Qptarray);
var A:integer;
begin
 cursorteller:=0;
 for i:=1 to numberofQpts do
  begin
   cursorteller:=(cursorteller+1)mod 15;
   if (cursorteller=0)
   then readln(inptQ,A)
   else read(inptQ,A);
   Q[i]:=A;
  end;
end;{of procedure lees_Q}


procedure lees_positiept(var positiept:PGptarray);
var  A:integer;
begin
 cursorteller:=0;
 for i:=1 to numberofpts do
  begin
   cursorteller:=(cursorteller+1)mod 19;
   if (cursorteller=0)
   then readln(inptpositiept,A)
   else read(inptpositiept,A);
   positiept[i]:=A;
  end;
end;{of procedure lees_positiept}

procedure lees_Dcap56(var Dcap56:array56);
var     A:integer;
```

```
begin
 cursorteller:=0;
 for i:=1 to 56 do
  begin
   cursorteller:=(cursorteller+1)mod 15;
   if (cursorteller=0)
   then readln(inptDcap56,A)
   else read(inptDcap56,A);
   Dcap56[i]:=A;
  end;
end;{of procedure lees_Dcap56}

procedure maak_incidencefix(var incidencefix:Qptmatrix;
                            var linetypefix:Qlnarray;
                            var lineon:PGptmatrix;
                            var ptson:Qlnarray4);
var  i,j,k,l,m:integer;
     nrz,nru,x,y,z:integer;
     incid:integer;
     lijnnr,xlijnnr:integer;
     X,Y,Z,U:cootype;
begin
 lijnnr:=0;
 xlijnnr:=3640;
 for i:=1 to numberofHpts do
 for j:=1 to numberofHpts do
  incidencefix[i,j]:=-1;
 for i:=1 to numberofHpts do
 for j:=1 to numberofHpts do
 begin{}
  if (incidencefix[i,j]=-1)
  then
   begin{+}
    if i=j
    then incidencefix[i,i]:=pointtype[Q[i]]
    else
     begin{-}
      x:=pointtype[Q[i]];
      y:=pointtype[Q[j]];
      coo_uit_nummer(Q[i],X);
      coo_uit_nummer(Q[j],Y);
      for k:=0 to 6 do Z[k]:=(X[k]+Y[k])mod q;
      nrz:=nummer_uit_coo(Z);
      z:=pointtype[nrz];
      k:=positiept[nrz];
      for m:=0 to 6 do U[m]:=(X[m]+2*Y[m])mod q;
      nru:=nummer_uit_coo(U);
      l:=positiept[nru];

      if z<10 then incid:=0;
      if z=16
      then
       if (x<>y) then writeln(outpt,'FOUT, x:',x:2,' y:',y:2,' z:',z:2)
         else incid:=x mod 10;
      if (z<16) and (z>10)
      then
       if (x<>y) and (y<>z) and (x<>z) then incid:=1
       else if (x=y) and (y=z) then incid:=x mod 10
       else if ((x=16)and(y=z)) or ((y=16)and(x=z)) then incid:=z mod 10
       else writeln(outpt,'FOUT x:',x:2,' y:',y:2,' z:',z:2);

      if (incid=0)
      then
       begin
        incidencefix[i,j]:=0;
        incidencefix[j,i]:=0;
       end
      else
       begin
        lijnnr:=lijnnr+1;

        incidencefix[i,j]:=incid;
        incidencefix[j,i]:=incid;
        incidencefix[i,k]:=incid;
        incidencefix[k,i]:=incid;
        incidencefix[j,k]:=incid;
        incidencefix[k,j]:=incid;
        incidencefix[i,l]:=incid;
        incidencefix[j,l]:=incid;
        incidencefix[k,l]:=incid;
        incidencefix[l,i]:=incid;
        incidencefix[l,j]:=incid;
        incidencefix[l,k]:=incid;
```

```
        lineon[i,j]:=lijnnr;
        lineon[i,k]:=lijnnr;
        lineon[i,l]:=lijnnr;
        lineon[j,k]:=lijnnr;
        lineon[j,l]:=lijnnr;
        lineon[k,l]:=lijnnr;
        lineon[j,i]:=lijnnr;
        lineon[k,i]:=lijnnr;
        lineon[l,i]:=lijnnr;
        lineon[k,j]:=lijnnr;
        lineon[l,j]:=lijnnr;
        lineon[l,k]:=lijnnr;

        ptson[lijnnr,1]:=i;
        ptson[lijnnr,2]:=j;
        ptson[lijnnr,3]:=k;
        ptson[lijnnr,4]:=l;

        linetypefix[lijnnr]:=incid;
       end;
     end;{-}
   end;{+}
 end;{}
if lijnnr<>numberofQlines
then
 begin
  writeln(outpt,'FOUTJE: ER ZIJN ZOVEEL LIJNEN GETELD: ',lijnnr);
  halt;
 end;
end;{of procedure maak_incidencefix}

procedure maak_cap20(var cap20:array20);
var i:integer;
begin
 for i:=1 to 20 do cap20[i]:=Dcap56[i];
end;{of procedure maak_cap20}

procedure maak_C(var C:array40);
var i,j:integer;
begin
 j:=1;
 for i:=1 to numberofpts do
  if pointtype[i]=16
  then begin C[j]:=i;
             j:=j+1;
       end;
end;{of procedure maak_C}

procedure maak_E(var E:array72);
var i,j:integer;
begin
 j:=1;
 for i:=1 to numberofpts do
  if pointtype[i]=13
  then begin E[j]:=i;
             j:=j+1;
       end;
end;{of procedure maak_E}

procedure maak_Epositie(var Epositie:PGptarray);
var i:integer;
begin
 for i:=1 to numberofpts do
 Epositie[i]:=0;
 for i:=1 to 72 do
 Epositie[E[i]]:=i;
end;{of procedure maak_Epositie}

procedure maak_degreeE(var degreeE:array72);
var i,j:integer;
begin
 for i:=1 to 72 do degreeE[i]:=0;
 for i:=1 to 72 do
 for j:=1 to 20 do
  if incidencefix[positiept[E[i]],positiept[cap20[j]]]>0
  then degreeE[i]:=degreeE[i]+1;
end;{of procedure maak_degreeE}

procedure maak_Esubsetdegree5(var Esubsetdegree5:array32);
var i,j:integer;
begin
 j:=1;
```

```
 for i:=1 to 72 do
  if degreeE[i]=5
  then begin Esubsetdegree5[j]:=E[i];
             j:=j+1;
       end;
end;{of procedure maak_Esubsetdegree5}

procedure maak_helftnr(var helftnr:array32);
var i,j,k,l,test:integer;
begin
 for i:=1 to 32 do helftnr[i]:=0;
 helftnr[1]:=1;
 i:=1;
 test:=0;
 while test=0 do
  begin{-}
   for j:=1 to 32 do
   if helftnr[j]=0
    then if ((incidencefix[positiept[Esubsetdegree5[i]],
                           positiept[Esubsetdegree5[j]]])mod 10=3)
    then begin
      helftnr[j]:=(helftnr[i]) mod 2 + 1;
          l:=j;
         end;
   test:=1;
   for k:=1 to 32 do test:=test*helftnr[k];
   i:=i+1;
   while helftnr[i]=0 do i:=i+1;
  end;{-}
end;{of procedure maak_helftnr}

procedure maak_Ecap56(var Ecap56:array56);
var i,j:integer;
begin
 for i:=1 to 20 do Ecap56[i]:=cap20[i];
 i:=21;
 for j:=1 to 72 do if degreeE[j]=2
                   then begin Ecap56[i]:=E[j];
                              i:=i+1;
                        end;
 if i<>41
 then begin write(outpt,'FFOUTT'); halt; end
 else begin for j:=1 to 32 do
             if helftnr[j]=1
             then begin Ecap56[i]:=Esubsetdegree5[j];
                        i:=i+1;
                  end;
      end;
end;{of procedure maak_Ecap56}

procedure maak_ovoide(var ovoide:array32_12);
var i,j,k:integer;
begin
 for i:=1 to 32 do
  for j:=1 to 12 do
   ovoide[i,j]:=0;
 for i:=1 to 32 do
  begin
   k:=1;
   for j:=1 to 40 do
    if incidencefix[positiept[Esubsetdegree5[i]],
                    positiept[C[j]]] > 0
    then begin ovoide[i,k]:=C[j];
              k:=k+1;
         end;
  end;
end;{of procedure_maak_ovoide}

procedure maak_Everwant(var Everwant:array32);
var     i,j,k:integer;
        same_ovoide:boolean;
begin
for i:=1 to 32 do Everwant[i]:=0;
for i:=1 to 31 do
 begin
  if Everwant[i]=0 then
  for j:=i+1 to 32 do
   begin
    same_ovoide:=true;
    for k:=1 to 10 do
      if ((ovoide[i,k])<>(ovoide[j,k]))
      then same_ovoide:=false;
    if same_ovoide=true
```

```
    then begin
          Everwant[i]:=j;
          Everwant[j]:=i;
         end;
   end;
 end;
end;{of procedure maak_Everwant}


procedure maak_blok(var blok:array16_6);
var i,j,k,l:integer;
begin
for i:=1 to 16 do for j:=1 to 7 do blok[i,j]:=0;
l:=0;
for i:=1 to 32 do if helftnr[i]=1 then
 begin
  l:=l+1;
  j:=1;
   for k:=1 to 32 do if helftnr[k]=1 then
    if ((i=k) or ((incidencefix[positiept[Esubsetdegree5[Everwant[i]]],
                                positiept[Esubsetdegree5[k]]]mod 10)<>3))
    then begin blok[l,j]:=k;
               j:=j+1;
         end;
 end;
end;{of procedure maak_blok}

procedure schrijf_design;
var i,j:integer;
    design:array16;
    designpositie:array32;
begin
 j:=1;
 for i:=1 to 32 do if helftnr[i]=1
  then begin design[j]:=i; j:=j+1; end;
 for j:=1 to 32 do designpositie[j]:=0;
 for i:=1 to 16 do designpositie[design[i]]:=i;
 for i:=1 to 16 do
  begin
   writeln(outpt);
   for j:=1 to 6 do write(outpt,designpositie[blok[i,j]]:5);
  end;
end;{of procedure schrijf_design}
```

In the last procedure of the subprogram `makingofdesign`, all procedures above are called, and the second Hill-cap `Ecap56` is written in a separate data file, which will be of use for the third program.

```
procedure initialize_makingofdesign;
var    i:integer;
begin
 lees_pointtype(pointtype);
 lees_Q(Q);
 lees_positiept(positiept);
 lees_Dcap56(Dcap56);
 maak_incidencefix(incidencefix,linetypefix,lineon,ptson);
 maak_cap20(cap20);
 maak_C(C);
 maak_E(E);
 maak_Epositie(Epositie);
 maak_degreeE(degreeE);
 maak_Esubsetdegree5(Esubsetdegree5);
 maak_helftnr(helftnr);
 maak_Ecap56(Ecap56);
 cursorteller:=0;
 for i:=1 to 56 do
  begin
   cursorteller:=(cursorteller+1)mod 15;
   if (cursorteller=0)
   then writeln(outptEcap56,Ecap56[i]:5)
   else write(outptEcap56,Ecap56[i]:5);
  end;
 writeln(outptEcap56);
 flush(outptEcap56);
 maak_ovoide(ovoide);
 maak_Everwant(Everwant);
 maak_blok(blok);
 schrijf_design;

end;{of procedure initialize_makingofdesign}
```

A.5 Main program `DESIGN.p`

This program is similar to **SETS.p**. The only difference is the replacement of rewrite-commands by reset-commands (say reading instead of writing), and of course another subprogram is called.

```
program DESIGN;

const ... [SEE SETS.p]

type  cootype=array[0..6] of 0..qone;

var   outpt:text;
      inptpointtype;text;
      inptQ:text;
      inptpositiept:text;
      inptDcap56:text;
      outptEcap56:text;
      X,Y,Z:cootype;
      teller:array[0..16]of integer;

#include "makingofnumbers.p";
#include "makingofdesign.p";

begin

rewrite(outpt,'DESIGN.data');
reset(inptpointtype,'pas_pointtype.data');
reset(inptQ,'pas_q.data');
reset(inptpositiept,'pas_positiept.data');
reset(inptDcap56,'pas_dcap56.data');
rewrite(outptEcap56,'pas_ecap56.data');

initialize_makingofdesign;

end.
```

A.6 Output of program `DESIGN.p`: `DESIGN.data`

This is the explicit enumeration of the $2-(16,6,2)$ design mentioned on pages 105 and 107.

```
 1   3   4   7  11  14
 2   3   4   5   9  16
 1   2   3   8  12  15
 1   2   4   6  10  13
 2   5   7  11  13  15
 4   6   7   9  12  15
 1   5   6   7   8  16
 3   7   8   9  10  13
 2   6   8   9  11  14
 4   8  10  11  15  16
 1   5   9  10  11  12
 3   6  11  12  13  16
 4   5   8  12  13  14
 1   9  13  14  15  16
 3   5   6  10  14  15
 2   7  10  12  14  16
```

A.7 Subprogram `lookingforhemiss.p`

In the third program, we decide whether or not the union of 2 Hill-caps can be extended to a hemisystem of the hexagon $H(3)$. This is done by eliminating points in $Q_3^+(5,3)$ and $Q_4^+(5,3)$. For the same speed-reducing reason as above (ranging over 90 elements instead of 364), we collect the

numbers of all 90 points of $Q_3^+(5,3)\setminus Q(4,3)$ respectively $Q_4^+(5,3)\setminus Q(4,3)$ in the arrays `F` respectively `G`. As we also need to range over all 2160 possible positions of $H(3)$ on $Q(6,3)$, we need 2160 transformation matrices generating all different positions. We found those matrices $\mathtt{M}_i$ with the more group-friendly package *Gap* , and made this readable for Pascal by writing the *Gap* result in a separate file `pas_versch.data` (see later on for the *Gap* -program). In `procedure maak_incidence`, we calculate the incidence-relation on (one of the 2160) hexagon(s). We proceed as follows: we calculate the Grassmann coordinates of the lines in the hexagon $H(3)_i$, with $H(3)_i$ the hexagon on $Q(6,3)$ obtained by transforming the 'original' hexagon by the coordinate transformation given in the matrix $\mathtt{M}_i$. If 2 points are on a line of the hexagon $H(3)_i$, the incidence type found in `incidencefix` is increased by 10. Remark that we also made an array `linetype` of length 364 that stores the linetype of all lines of $Q(6,3)$. (The linetype of $L = xy$ is equal to `j`, if the pair (x,y) has incidence type `j`. So every line of $H(3)_i$ will have incidence type > 10.)
Now we can start with eliminating points in $Q_3^+(5,3)$ and $Q_4^+(5,3)$. This is done in the procedure `elimineer_in_fg`, where we give all points of `F` and all points of `G` a parameter called `Fdeelname` respectively `Gdeelname`. If this parameter is 0, the point should not belong to a hemisystem of $H(3)$ since it is on a line through a point of `Ecap56` and a point of `Dcap56`. If the parameter is 1, the point is still a candidate to belong to the (virtual) hemisystem. If for one case or another, there would be at least 90 points left in `F`$\cup$`G`, the program will warn us. Once all possible positions of the hexagon are considered, the program counts how many hexagons leave i points in `F` and j points in `G`.

```
const numberoflines=3640;

type  PGptarray    =array[1..numberofpts] of 0..numberofHpts;
      PGptmatrix   =array[1..numberofpts,1..numberofpts] of integer;
      Qptarray     =array[1..numberofHpts] of 0..numberofpts;
      Qptmatrix    =array[1..numberofHpts,1..numberofHpts] of -1..16;
      Qlnarray     =array[0..numberofQlines] of 0..numberofQlines;
      Qlnarray4    =array[1..numberofQlines,1..4] of 1..numberofHpts;
      matrixtype   =array[0..6,0..6] of 0..2;
      array56      =array[1..56] of 0..numberofpts;
      array90      =array[1..90] of integer;
      array38_60   =array[38..60,38..60] of integer;


var   Q:             Qptarray;
      Ecap56:        array56;
      Dcap56:        array56;
      F,G:           array90;
      Fpositie:      PGptarray;
      Gpositie:      PGptarray;
      incidencefix:  Qptmatrix;
      incidence:     Qptmatrix;
      positiept:     PGptarray;
      pointtype:     PGptarray;
      linetypefix:   Qlnarray;
      linetype:      Qlnarray;
      lineon:        PGptmatrix;
      ptson:         Qlnarray4;
      cursorteller:  integer;
      i,j:           integer;
```

```
      tijd:          integer;
      A1,A,B1,B:     cootype;
      M:             matrixtype;
      Fdeelname,Gdeelname:array90;
      geval:         integer;
      FG:            array38_60;


procedure lees_pointtype(var pointtype:PGptarray);
... [SEE makingofdesign.p]
procedure lees_Q(var Q:Qptarray);
... [SEE makingofdesign.p]
procedure lees_positiept(var positiept:PGptarray);
... [SEE makingofdesign.p]
procedure lees_Dcap56(var Dcap56:array56);
... [SEE makingofdesign.p]
procedure lees_Ecap56(var Ecap56:array56);
var     A:integer;
begin
 cursorteller:=0;
 for i:=1 to 56 do
  begin
   cursorteller:=(cursorteller+1)mod 15;
   if (cursorteller=0)
   then readln(inptEcap56,A)
   else read(inptEcap56,A);
   Ecap56[i]:=A;
  end;
end;{of procedure lees_Ecap56}

procedure maak_F(var F:array90);
var     i,j:integer;
begin
 j:=1;
 for i:=1 to numberofpts do
 if pointtype[i]=14
 then
  begin
   F[j]:=i;
   j:=j+1;
  end;
end;{of procedure maak_F}

procedure maak_G(var G:array90);
var     i,j:integer;
begin
 j:=1;
 for i:=1 to numberofpts do
 if pointtype[i]=15
 then
  begin
   G[j]:=i;
   j:=j+1;
  end;
end;{of procedure maak_G}

procedure maak_Fpositie(var Fpositie:PGptarray);
begin
 for i:=1 to numberofpts do Fpositie[i]:=0;
 for i:=1 to 90 do Fpositie[F[i]]:=i;
end;{of procedure maak_Fpositie}

procedure maak_Gpositie(var Gpositie:PGptarray);
begin
 for i:=1 to numberofpts do Gpositie[i]:=0;
 for i:=1 to 90 do Gpositie[G[i]]:=i;
end;{of procedure maak_Gpositie}


procedure initialiseer_FG(var FG:array38_60);
var i,j:integer;
begin
 for i:=38 to 60 do
 for j:=38 to 60 do
  FG[i,j]:=0;
end;{of procedure initialiseer_FG}

procedure maak_incidencefix(var incidencefix:Qptmatrix;
                            var linetypefix:Qlnarray;
                            var lineon:PGptmatrix;
                            var ptson:Qlnarray4);
... [SEE makingofdesign.p]
```

```
procedure lees_M(var M:matrixtype);
var  i,j:integer;
     A:integer;
begin
 for i:=0 to 6 do
  begin
   readln(inptVERSCH);
   for j:=0 to 6 do
    begin
     read(inptVERSCH,A);
     M[i,j]:=A;
    end;
  end;
end;{of procedure lees_M}

procedure cootransfo(MAT:matrixtype;X:cootype;var Y:cootype);
var i:integer;
begin
 for i:=0 to 6 do
 Y[i]:=(MAT[i,0]*X[0]+MAT[i,1]*X[1]+MAT[i,2]*X[2]+MAT[i,3]*X[3]
       +MAT[i,4]*X[4]+MAT[i,5]*X[5]+MAT[i,6]*X[6])mod q;
end;{of procedure cootransfo}

function Grassmanncoo(A1,B1:cootype):boolean;
var i,j:integer;
P:array[0..5,1..6] of integer;
begin
 cootransfo(M,A1,A);
 cootransfo(M,B1,B);
 for j:=1 to 6 do
 for i:=0 to j-1 do
 P[i,j]:=(A[i]*B[j]+(q-1)*A[j]*B[i])mod q;
 if (P[1,2]=P[3,4]) and (P[0,1]=P[3,6]) and (P[0,3]=P[5,6])
    and (P[2,3]=P[4,5]) and ((P[0,2])mod q=(2*P[3,5])mod q)
                 and ((P[1,3])mod q=(2*P[4,6])mod q)
 then Grassmanncoo:=true
 else Grassmanncoo:=false;
end;{of function Grassmanncoo}

procedure maak_incidence(var incidence:Qptmatrix;
                         var linetype:Qlnarray);
var   lijn,ltype,incid,i,j:integer;
      gecontroleerd:Qptmatrix;
begin
 for i:=1 to numberofHpts do
 for j:=1 to numberofHpts do
  gecontroleerd[i,j]:=0;
 for i:=1 to numberofHpts do
 for j:=1 to numberofHpts do
 begin{}
  if (gecontroleerd[i,j]=0)
  then
   begin{+}
   if{o} i=j
   then{o} begin gecontroleerd[i,i]:=1; incidence[i,i]:=incidencefix[i,i];end
   else{o}
    if{.} incidencefix[i,j]=0
    then{.} begin incidence[i,j]:=0;
  incidence[j,i]:=0;
    end
    else{.}
     begin{-}
      coo_uit_nummer(Q[i],X);
      coo_uit_nummer(Q[j],Y);
      lijn:=lineon[i,j];
      if Grassmanncoo(X,Y)
      then begin incid:=incidencefix[i,j]+10;
                 ltype:=linetypefix[lijn]+10; end
      else begin incid:=incidencefix[i,j];
                 ltype:=linetypefix[lijn]; end;
      incidence[ptson[lijn,1],ptson[lijn,2]]:=incid;
      incidence[ptson[lijn,1],ptson[lijn,3]]:=incid;
      incidence[ptson[lijn,1],ptson[lijn,4]]:=incid;
      incidence[ptson[lijn,2],ptson[lijn,3]]:=incid;
      incidence[ptson[lijn,2],ptson[lijn,4]]:=incid;
      incidence[ptson[lijn,3],ptson[lijn,4]]:=incid;
      incidence[ptson[lijn,2],ptson[lijn,1]]:=incid;
      incidence[ptson[lijn,3],ptson[lijn,1]]:=incid;
      incidence[ptson[lijn,4],ptson[lijn,1]]:=incid;
      incidence[ptson[lijn,3],ptson[lijn,2]]:=incid;
      incidence[ptson[lijn,4],ptson[lijn,2]]:=incid;
      incidence[ptson[lijn,4],ptson[lijn,3]]:=incid;
```

```
      gecontroleerd[ptson[lijn,1],ptson[lijn,2]]:=1;
      gecontroleerd[ptson[lijn,1],ptson[lijn,3]]:=1;
      gecontroleerd[ptson[lijn,1],ptson[lijn,4]]:=1;
      gecontroleerd[ptson[lijn,2],ptson[lijn,3]]:=1;
      gecontroleerd[ptson[lijn,2],ptson[lijn,4]]:=1;
      gecontroleerd[ptson[lijn,3],ptson[lijn,4]]:=1;
      gecontroleerd[ptson[lijn,2],ptson[lijn,1]]:=1;
      gecontroleerd[ptson[lijn,3],ptson[lijn,1]]:=1;
      gecontroleerd[ptson[lijn,4],ptson[lijn,1]]:=1;
      gecontroleerd[ptson[lijn,3],ptson[lijn,2]]:=1;
      gecontroleerd[ptson[lijn,4],ptson[lijn,2]]:=1;
      gecontroleerd[ptson[lijn,4],ptson[lijn,3]]:=1;
      linetype[lijn]:=ltype;
     end;{-}
   end;{+}
 end;{}
end;{of procedure maak_incidence}

procedure test_incidence(var linetype:Qlnarray);
var   i,j:integer;
      teller:array[0..16] of integer;
begin
 for j:=0 to 16 do teller[j]:=0;
 for i:=1 to 3640 do
  teller[linetype[i]]:=teller[linetype[i]]+1;
 if teller[16]<>4 then begin writeln(outpt,'FOUT IN GEVAL ',geval:3); halt; end;
 if teller[15]<>48 then begin writeln(outpt,'FOUT IN GEVAL ',geval:3); halt; end;
 if teller[14]<>48 then begin writeln(outpt,'FOUT IN GEVAL ',geval:3); halt; end;
 if teller[13]<>24 then begin writeln(outpt,'FOUT IN GEVAL ',geval:3); halt; end;
 if teller[12]<>24 then begin writeln(outpt,'FOUT IN GEVAL ',geval:3); halt; end;
 if teller[11]<>216 then begin writeln(outpt,'FOUT IN GEVAL ',geval:3); halt; end;
 flush(outpt);
end;{of procedure test_incidence}

procedure elimineer_in_fg(var Fdeelname,Gdeelname:array90);
var i,j,k,L,priemtest:integer;
begin
 for i:=1 to 90 do Fdeelname[i]:=1;
 for i:=1 to 90 do Gdeelname[i]:=1;
 for i:=21 to 56 do
 begin{%}
  for j:=21 to 56 do
  if incidence[positiept[Dcap56[i]],positiept[Ecap56[j]]]>10
  then
   begin
    L:=lineon[positiept[Dcap56[i]],positiept[Ecap56[j]]];
    priemtest:=1;
    for k:=1 to 4 do
    if pointtype[Q[ptson[L,k]]]=14
    then begin Fdeelname[Fpositie[Q[ptson[L,k]]]]:=0;
               priemtest:=2*priemtest;
         end
    else if pointtype[Q[ptson[L,k]]]=15
         then begin Gdeelname[Gpositie[Q[ptson[L,k]]]]:=0;
                    priemtest:=3*priemtest;
              end
    else if pointtype[Q[ptson[L,k]]]=13
         then priemtest:=5*priemtest
    else if pointtype[Q[ptson[L,k]]]=12
         then priemtest:=7*priemtest;
    if priemtest <> 2*3*5*7 then begin write(outpt,'FOUT.'); halt; end;
   end;
 end;{%}
end;{of procedure elimineer_in_fg}

procedure initialize_lookingforhemiss;
var  i,j:integer;
     aantal_over_in_F,aantal_over_in_G:integer;
begin
 lees_pointtype(pointtype);
 lees_Q(Q);
 lees_positiept(positiept);
 lees_Dcap56(Dcap56);
 lees_Ecap56(Ecap56);
 maak_F(F);
 maak_G(G);
 maak_Fpositie(Fpositie);
 maak_Gpositie(Gpositie);
 maak_incidencefix(incidencefix,linetypefix,lineon,ptson);
 initialiseer_FG(FG);

 for j:=0 to 16 do teller[j]:=0;
 for i:=1 to numberofpts do
```

```
  teller[pointtype[i]]:=teller[pointtype[i]]+1;
 write(outpt,'The number of points of type i');
 writeln(outpt,' is written in the i''th place (i from 0 to 16):');
 for j:=0 to 16 do write(outpt,teller[j]:4);
 writeln(outpt);
 flush(outpt);

for geval:=1 to 2160 do
begin{**}

 lees_M(M);
 maak_incidence(incidence,linetype);
 test_incidence(linetype);
 elimineer_in_fg(Fdeelname,Gdeelname);
  aantal_over_in_F:=0;
  for i:=1 to 90 do if Fdeelname[i]=1 then aantal_over_in_F:=aantal_over_in_F+1;
  aantal_over_in_G:=0;
  for i:=1 to 90 do if Gdeelname[i]=1 then aantal_over_in_G:=aantal_over_in_G+1;
  if (aantal_over_in_F+aantal_over_in_G)>89
  then writeln(outpt,'CASE ',geval:3,' IS A VERY GOOD CANDIDATE');

  FG[aantal_over_in_F,aantal_over_in_G]:=FG[aantal_over_in_F,aantal_over_in_G]+1;
end;{**}

for i:=38 to 60 do
begin
 for j:=38 to 60 do
  begin
   if FG[i,j]>0
   then begin write(outpt,'There are ',FG[i,j]:4,' positions of the hexagon ');
              writeln(outpt,'that leave ',i:2,' candidates in F');
              write(outpt,'                                    ');
              writeln(outpt,'      and ',j:2,' candidates in G.');
        end;
  end;
end;

end;{of procedure initialize_lookingforhemiss}
```

A.8 Main program HEMISS.p

This main program calling for the subprogram **lookingforhemiss.p** is again very similar to the other main programs.

```
program HEMISS;

const ... [SEE DESIGN.p]

type  cootype=array[0..6] of 0..qone;

var   outpt:text;
      inptpointtype:text;
      inptQ:text;
      inptpositiept:text;
      inptDcap56:text;
      inptVERSCH:text;
      inptEcap56:text;    {<-> DESIGN.p}
      X,Y,Z:cootype;
      teller:array[0..16]of integer;

#include "makingofnumbers.p"; {om nummers te berekenen van punten}
#include "lookingforhemiss.p"; {om incidentie te berekenen}

begin

rewrite(outpt,'HEMISS.data');
reset(inptpointtype,'pas_pointtype.data');
reset(inptQ,'pas_q.data');
reset(inptpositiept,'pas_positiept.data');
reset(inptDcap56,'pas_dcap56.data');
reset(inptVERSCH,'pas_versch.data');
reset(inptEcap56,'pas_ecap56.data');  {<-> DESIGN.p}
```

```
initialize_lookingforhemiss; {om allerlei te berekenen}

end.
```

A.9 Output of program `HEMISS.p`: `HEMISS.data`

```
The number of points of type i is written in the i'th place (i from 0 to 16):
   0   0 171 171 153 153  81   0   0   0   0   0  72  72  90  90  40
There are   80 positions of the hexagon that leave 42 candidates in F
                                              and 38 candidates in G.
There are  640 positions of the hexagon that leave 42 candidates in F
                                              and 41 candidates in G.
There are  720 positions of the hexagon that leave 42 candidates in F
                                              and 42 candidates in G.
There are  640 positions of the hexagon that leave 42 candidates in F
                                              and 43 candidates in G.
There are   80 positions of the hexagon that leave 42 candidates in F
                                              and 46 candidates in G.
```

A.10 Side programs in *Maple* and *Gap*

If we want to range over the 2160 possible positions of the hexagon on $Q(6,3)$, we have to list a representative of each of the 2160 cosets of the full collineation group $G_2(3)$ of $H(3)$ in the automorphism group $\mathbf{PGO}_7(3)$ of $Q(6,3)$. This representative will be most handy if written as a 7×7 transformation matrix. First, we look for generators of the group $\mathbf{G}_2(3)$ respectively $\mathbf{PGO}_7(3)$. In De Smet [15] (page 31), we find an automorphism $g_{A,L,A',L',A''}$ fixing the point (∞) and acting regularly on the points of $H(3)$ which are opposite to (∞). In coordinates of the hexagon, this automorphism reads $p(a,l,a',l',a'') \mapsto p^g(a+A, l+L, a'+A'-aA'', l'+L', a''+A'')$. In projective coordinates in $\mathbf{PG}(6,3)$, this automorphism reads $x \mapsto x^g = Mx$ with $x = (-al' + a'^2 + a''l + aa'a'', -a'', -a, -a' + aa'', 1, l + 2aa' - a^2a'', -l' + a'a'')$ and $M(=$`MAT`$)$ printed below (see paragraph A.10.1). We used *Maple* to check the conversion from hexagon coordinates to projective coordinates (and we find that $x^g =$`yy:=evalm(MAT &* xx)` is indeed equal to the projective coordinate of p^g), and to determine the 5 generating matrices $\mathtt{m}_i$ of the subgroup $\{g_{A,L,A',L',A''} | A, L, A', L', A'' \in \mathbf{GF}(3)\}$ of $\mathbf{G}_2(3)$. This subgroup is called `Gm`. As (∞) has projective coordinates $(1,0,0,0,0,0,0)$, and $(0,0,0,0,0) \in \Gamma_6((\infty))$ has projective coordinates $(0,0,0,0,1,0,0)$, and the conditions on the Grassmann coordinates of the lines of the hexagon are symmetric in the triples $(0,1,2)$ and $(4,5,6)$ (see page 10), the projective transformation $x \mapsto \begin{pmatrix} . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \\ . & . & . & 1 & . & . & . \\ 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \end{pmatrix} x = Bx$ will generate an automorphism $h_{A,L,A',L',A''}$ fixing the point $(0,0,0,0,0)$ and acting regularly on the points opposite to it. The subgroup $\{h_{A,L,A',L',A''} | A, L, A', L', A'' \in$

$\mathbf{GF}(3)\}$ has generating matrices $\mathtt{n}_i = \mathtt{B}\mathtt{m}_i\mathtt{B}^{-1}$ and is called Gn. The group Gmn generated by the 10 matrices $\mathtt{m}_i$, $\mathtt{n}_i$ has order $2^6 \cdot 3^6 \cdot 7 \cdot 13$, hence it is $\mathbf{G}_2(3)$ itself. Now we still have to find an extra automorphism c of $Q(6,3)$, but not of $H(3)$, such that $\langle \mathbf{G}_2(3), \mathtt{c}\rangle$ is the group $\mathbf{PGO}_7(3)$. As the equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$ of $Q(6,3)$ is symmetric in 0 and 4 but the conditions on the Grassmann coordinates of the lines of the hexagon are not, the automorphism $x \mapsto \begin{pmatrix} . & . & . & . & 1 & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ 1 & . & . & . & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix} x$ belongs to $\mathbf{PGO}_7(3) \setminus \mathbf{G}_2(3)$. As the order of the group Gmnc$= \langle \mathtt{m}_i, \mathtt{n}_i, \mathtt{c}\rangle$ is $2160 \cdot |\mathbf{G}_2(3)|$, Gmnc equals $\mathbf{PGO}_7(3)$. In a last step, *Gap* gives us a representative of each of the 2160 cosets of $\mathbf{G}_2(3)$ in $\mathbf{PGO}_7(3)$.
We include the *Maple* - and *Gap* -program used.

A.10.1 Maple-program

Here we check the general form of the automorphism $g_{A,L,A',L',A''}$ in projective coordinates, and list the 5 generating matrices m1 to m5. We omit the output, as the 5 matrices $\mathtt{m}_i$ are the same as in the input of the *Gap* -program.

```
a:=a:
a1:=a1:
a2:=a2:
l:=l:
l1:=l1:
l2:=l2:
xx:=array([-a*l1+a1*a1+a2*l+a*a1*a2,-a2,-a,-a1+a*a2,1,l+2*a*a1-a*a*a2,-l1+a1*a2]);
A:=A:
A1:=A1:
A2:=A2:
L:=L:
L1:=L1:
MAT:=array([
[1,-L-A*A1,A1*A2+A*A2*A2+L1 ,A1-A*A2,A1*A1+A*A1*A2-A*L1+A2*L,A2,A],
[0,1      ,0                ,0      ,-A2                    ,0 ,0],
[0,0      ,1                ,0      ,-A                     ,0 ,0],
[0,-A     ,-2*A2            ,1      ,A*A2-A1                ,0 ,0],
[0,0      ,0                ,0      ,1                      ,0 ,0],
[0,A*A    ,A*A2+A1          ,A      ,L+2*A*A1-A*A*A2        ,1 ,0],
[0,-A1    ,A2*A2            ,-A2    ,A2*A1-L1               ,0 ,1]]):
yy:=evalm(MAT &* xx);

subs(A=1,L=0,A1=0,L1=0,A2=0,evalm(MAT));
subs(A=0,L=1,A1=0,L1=0,A2=0,evalm(MAT));
subs(A=0,L=0,A1=1,L1=0,A2=0,evalm(MAT));
subs(A=0,L=0,A1=0,L1=1,A2=0,evalm(MAT));
subs(A=0,L=0,A1=0,L1=0,A2=1,evalm(MAT));

quit;
```

A.10.2 Gap-program

The file `leesbaar.gap` is a small file that makes the output more readable for us (e.g. the multiplicative identity element of $\mathbf{GF}(3)$ is denoted by 1 instead of Z(3)). The matrices $\mathtt{m}_i$, $\mathtt{n}_i$ and c are defined, the order of the groups $\langle \mathtt{m}_i \rangle$, $\langle \mathtt{m}_i, \mathtt{n}_j \rangle$ and $\langle \mathtt{m}_i, \mathtt{n}_j, \mathtt{c} \rangle$ are computed, and a representative of all 2160 cosets of $\mathbf{G}_2(3)$ in $\mathbf{PGO}_7(3)$ are listed. These last 2160 matrices are then saved in `pas_versch.data`, so that they can be used by the program `HEMISS.p`.

```
Read("leesbaar.gap");

m1:=[[ 1,   0,  0,  0,   0,  0,  1 ],
     [ 0,   1,  0,  0,   0,  0,  0 ],
     [ 0,   0,  1,  0,  -1,  0,  0 ],
     [ 0,  -1,  0,  1,   0,  0,  0 ],
     [ 0,   0,  0,  0,   1,  0,  0 ],
     [ 0,   1,  0,  1,   0,  1,  0 ],
     [ 0,   0,  0,  0,   0,  0,  1 ]]*2*Z(3);;
m2:=[[ 1,  -1,  0,  0,  0,  0,  0 ],
     [ 0,   1,  0,  0,  0,  0,  0 ],
     [ 0,   0,  1,  0,  0,  0,  0 ],
     [ 0,   0,  0,  1,  0,  0,  0 ],
     [ 0,   0,  0,  0,  1,  0,  0 ],
     [ 0,   0,  0,  0,  1,  1,  0 ],
     [ 0,   0,  0,  0,  0,  0,  1 ]]*2*Z(3);;
m3:=[[ 1,  0,  0,  1,   1,  0,  0 ],
     [ 0,  1,  0,  0,   0,  0,  0 ],
     [ 0,  0,  1,  0,   0,  0,  0 ],
     [ 0,  0,  0,  1,  -1,  0,  0 ],
     [ 0,  0,  0,  0,   1,  0,  0 ],
     [ 0,  0,  1,  0,   0,  1,  0 ],
     [ 0, -1,  0,  0,   0,  0,  1 ]]*2*Z(3);;
m4:=[[ 1,  0,  1,  0,   0,  0,  0 ],
     [ 0,  1,  0,  0,   0,  0,  0 ],
     [ 0,  0,  1,  0,   0,  0,  0 ],
     [ 0,  0,  0,  1,   0,  0,  0 ],
     [ 0,  0,  0,  0,   1,  0,  0 ],
     [ 0,  0,  0,  0,   0,  1,  0 ],
     [ 0,  0,  0,  0,  -1 , 0,  1 ]]*2*Z(3);;
m5:=[[ 1,  0,   0,   0,   0,  1,  0 ],
     [ 0,  1,   0,   0,  -1,  0,  0 ],
     [ 0,  0,   1,   0,   0,  0,  0 ],
     [ 0,  0,  -2,   1,   0,  0,  0 ],
     [ 0,  0,   0,   0,   1,  0,  0 ],
     [ 0,  0,   0,   0,   0,  1,  0 ],
     [ 0,  0,   1,  -1,   0,  0,  1 ]]*2*Z(3);;

Gm:=Group(m1,m2,m3,m4,m5);;
Print("\n The group Gm has size ");
Size(Gm);

B:=[ [0,0,0, 0,1,0,0],
[0,0,0, 0,0,1,0],
[0,0,0, 0,0,0,1],
[0,0,0,-1,0,0,0],
[1,0,0, 0,0,0,0],
[0,1,0, 0,0,0,0],
[0,0,1, 0,0,0,0]]*2*Z(3);;
Binvers:=B^-1;;
n1:=B*m1*Binvers;;
n2:=B*m2*Binvers;;
n3:=B*m3*Binvers;;
n4:=B*m4*Binvers;;
n5:=B*m5*Binvers;;

Gn:=Group(n1,n2,n3,n4,n5);;
Print("\n The group Gn has size ");
Size(Gn);

Gmn:=Group(n1,n2,n3,n4,n5,m1,m2,m3,m4,m5);;
Print("\n The group Gmn has size ");
Size(Gmn);

c:=[[0,0,0,0,1,0,0],
    [0,1,0,0,0,0,0],
    [0,0,1,0,0,0,0],
    [0,0,0,1,0,0,0],
    [1,0,0,0,0,0,0],
    [0,0,0,0,0,1,0],
    [0,0,0,0,0,0,1]]*2*Z(3);;

Gmnc:=Group(n1,n2,n3,n4,n5,m1,m2,m3,m4,m5,c);;
```

```
Print("\n The group Gmnc has size");
Size(Gmnc);

Print("\n The index of Gmn in Gmnc is ");
Index(Gmnc,Gmn);

Setter(Name)(Gmnc,"Gmnc");

Setter(Name)(Gmn,"Gmn");

A:=RightCosets(Gmnc,Gmn);;
for i in [1..2160] do
 M:=Representative(A[i]);
 Print("\n");
 leesbaarmat(M);
od;

time;
quit;
```

The answer of *Gap* reads as follows.

```
 The group Gm has size  243
 The group Gn has size  243
 The group Gmn has size  4245696
 The group Gmnc has size 9170703360
 The index of Gmn in Gmnc is  2160
[ [  1,  0,  0,  0,  0,  0,  0 ],
  [  0,  1,  0,  0,  0,  0,  0 ],
  [  0,  0,  1,  0,  0,  0,  0 ],
  [  0,  0,  0,  1,  0,  0,  0 ],
  [  0,  0,  0,  0,  1,  0,  0 ],
  [  0,  0,  0,  0,  0,  1,  0 ],
  [  0,  0,  0,  0,  0,  0,  1 ] ]

[ [  1,  0,  0,  0,  0,  0,  0 ],
  [  0,  1,  0,  2,  0,  1,  0 ],
  [  0,  0,  1,  0,  0,  0,  0 ],
  [  0,  0,  0,  1,  0,  1,  0 ],
  [  0,  0,  0,  0,  1,  0,  0 ],
  [  0,  0,  0,  0,  0,  1,  0 ],
  [  0,  0,  0,  0,  0,  0,  1 ] ]

... [2157 matrices omitted]

[ [  1,  0,  0,  0,  0,  0,  0 ],
  [  0,  0,  0,  0,  0,  2,  0 ],
  [  1,  0,  2,  2,  0,  0,  2 ],
  [  0,  0,  1,  0,  0,  0,  2 ],
  [  1,  0,  0,  1,  1,  1,  0 ],
  [  1,  2,  0,  0,  0,  0,  0 ],
  [  2,  0,  2,  1,  0,  0,  2 ] ]

  7480
```

Appendix B

A Note on Characterizations by Subpolygons

In the first chapter, we gave a selection of characterizations of classical quadrangles and hexagons, which were of use in chapters 2 to 6. A variant of the dual of theorem 1.10 respectively 1.20 may be formulated as follows (we refer to the unified theorem in Van Maldeghem [84] 6.7.1):

Theorem B.1 *A finite GQ of order (s,t) is isomorphic to $H(3,q) \Leftrightarrow$ every ordinary pentagon is contained in a proper ideal subquadrangle.*

Theorem B.2 *A finite GH of order (s,t) is isomorphic to $T(q^3,q) \Leftrightarrow$ every ordinary heptagon is contained in a proper ideal subhexagon.*

In the infinite case, one can not give a classification, but point regularity is proved for the generalized hexagons (Van Maldeghem [84] 6.3.7).

Theorem B.3 *A GH is point regular $\Leftrightarrow$ every ordinary heptagon is contained in a proper subhexagon isomorphic to $H(\mathbb{K})$.*

All of these theorems give a result on polygons under assumption of the existence of 'a lot of' subpolygons of 'certain' order. We obtained a new result on quadrangles, assuming the existence of 'a lot of' subquadrangles of order 2. The counterpart for hexagons is not that straightforward, as there is no counterpart of theorem 1.3, so we ended up with a closed configuration, which leaves no way for further deductions.

Lemma B.1 *Let Γ be a generalized quadrangle of order (q^2, q). If every ordinary pentagon of Γ is contained in a unique subquadrangle $W(2)$ of Γ, then all points of Γ are regular.*

Proof
Let x, y, z be mutually opposite points. Suppose $a, b, c \in x^y$ and $a, b \in x^z$. We show that $c \in x^z$ or, equivalently, $c \sim z$.
First we show that a, b, c, x, y, z are inside a subquadrangle $W(2)$. Consider the lines xb, yc and za. Those are at mutual distance 4 and by theorem 1.3 on page 7, this triad of lines contains exactly $q+1$ centers. Take one of those centers, say L, and put $L \cap xb = p$. Now consider the pentagon with lines pb, by, ya, az, L. This pentagon is, by assumption, contained in a subquadrangle $\Gamma' \cong W(2)$. Now the following points and lines also belong to Γ': $x = \mathrm{proj}_{xb} a$, $yc = \mathrm{proj}_y L$, $c = \mathrm{proj}_{yc} x$ and $z = \mathrm{proj}_{az} b$. Hence $z \sim c$, as all points in $W(2)$ are regular. $\square$

Lemma B.2 *Let Γ be a generalized quadrangle such that every point is regular. Then a centric triad on a 3×3-grid of Γ is unicentric.*

Proof
Suppose $\{a, b, c\}$ is a triad on a 3×3-grid $\mathcal{G}$, with $x \in \mathcal{G} \cap \Gamma_2(a) \cap \Gamma_2(b)$. If y and z are different centers of $\{a, b, c\}$, then x is collinear with 2 points of the trace y^z, but not with the point c of that trace. Contradiction with regularity. $\square$

Lemma B.3 *Let Γ be a generalized quadrangle of order (q^2, q). If every ordinary pentagon of Γ is contained in a unique subquadrangle $W(2)$ of Γ, then every triad on a 3×3 grid is (uni-)centric.*

Proof
$\boxed{1}$ First we show that there are $(q-1)$ $W(2)$'s through a 2×3-grid. Let M_a, M_b be 2 opposite lines, and suppose $L_a, L_b, L_c \in \{M_a, M_b\}^\perp$. Then we show that there are $(q-1)$ subquadrangles $W(2)$ through $\mathcal{G}' = \{L_a, L_b, L_c, M_a, M_b\}$. Indeed, take a line N through $a = L_a \cap M_a$, N different from L_a, M_a. Projecting $b = L_b \cap M_b$ onto N yields a pentagon with lines $M_a, L_c, M_b, \mathrm{proj}_b N, N$. By assumption, there is a $\Gamma' \cong W(2)$ through this ordinary pentagon. As also $L_b = \mathrm{proj}_b M_a$ and $L_a = \mathrm{proj}_a M_b$ will belong to Γ', we found a subquadrangle through $\mathcal{G}'$ and the additional line N.
$\boxed{2}$ Now we show that there is a $W(2)$ through every 3×3-grid $\mathcal{G}$. Take the 2×3-grid $\mathcal{G}'$ as above. Each of the $(q-1)$ $W(2)$'s through $\mathcal{G}'$ determines a unique point x on L_a, neither on M_a nor on M_b. As all lines of $W(2)$ are regular, $M_c = \mathrm{proj}_x L_c$ will also intersect L_b. Hence $\mathcal{G}' \cup \{M_c\}$ defines

a 3×3-grid inside a $W(2)$. As $\{L_a, L_b, L_c\}$ is a triad of lines of Γ with order (q^2, q), there are $q+1$ centers of $\{L_a, L_b, L_c\}$. Hence there are $(q-1)$ 3×3-grids $\mathcal{G}$ through a 2×3-grid $\mathcal{G}'$. Let x_i,$i = 1, \ldots, q-1$, be a point on L_a and on one of the centers of $\{L_a, L_b, L_c\}$ (different from M_a, M_b). We show that all points x_i are contained in some 3×3-grid obtained by contructing a $W(2)$ through $\mathcal{G}'$ and an additional line N through $L_a \cap M_a$. Suppose this is not the case. Then there would be 2 subquadrangles Δ_1, Δ_2 isomorphic to $W(2)$ and through $\mathcal{G}'$ containing the same point x_i — and hence the same 3×3-grid. Let $c = \mathrm{proj}_{L_c} x_i$. As all triads of $W(q)$, q even, have 1 or $(t+1)$ centers (ccitePeT 1.3.6 (ii)), the triad $\{a, b, c\}$ in Δ_i, $i = 1, 2$, has a center in Δ_i — which is unique by lemma B.2. Let y_1, y_2 be the center of $\{a, b, c\}$ in Δ_1, Δ_2 respectively. As $\Delta_1 \neq \Delta_2$, also $y_1 \neq y_2$. As each centric triad is unicentric in Γ (lemma B.1 and B.2), this gives a contradiction.

$\boxed{3}$ As every 3×3-grid is contained in a $W(2)$ and every triad in $W(2)$ is centric, the proof is finished. $\square$

Appendix C

Nederlandse Samenvatting

In zijn zoektocht naar een meetkundige interpretatie van de halfenkelvoudige groepen van Lie-type, definieerde Jacques Tits in een appendix van zijn befaamde werk *'Sur la trialit'19 e et certains groupes qui s'en d'19 eduisent'* (anno 1959) de veralgemeende veelhoeken. Deze zouden niet lang verdoken blijven als 'appendix bij', maar kregen sindsdien ook een bestaansreden op zich. Inderdaad, hoewel veralgemeende drie- en vierhoeken ook v'19 o'19 or Tits' expliciete beschrijving werden bestudeerd, zijn er vooral sinds Feit en Higman (die uitplozen dat de eindige interessante n-hoeken enkel voor $n = 3, 4, 6$ en 8 bestaan) veel meetkundige, algebra'127 ısche en groepentheoretische technieken op deze structuren losgelaten. Het boek *'Finite Generalized Quadrangles'* van Payne en Thas was het eerste om een overzicht te geven van de tot dan toe (1984) gekende resultaten voor $n = 4$. Een bondige samenvatting van de resultaten voor eindige veralgemeende n-hoeken, $n = 4, 6, 8$, volgde in het *'Handbook of Incidence Geometry'*, hoofdstuk 9 van de hand van Thas. Een uitgediepte en meer lijvige versie, ook handelend over oneindige veelhoeken, vinden we dan terug bij Van Maldeghem, in het boek *'Generalized Polygons'*, anno 1998. Citering van deze werken kan echter niet zonder vermelding van andere belangrijke namen op het gebied van veralgemeende veelhoeken, waaronder Kantor, Ronan, Buekenhout, Weiss en nog vele anderen.

C.1 Inleiding

Met het inleidende hoofdstuk van deze thesis willen we de lezer een korte handleiding geven voor komende hoofdstukken. Veelgebruikte definities en gekende resultaten worden hier verzameld. De definitie van een veralgemeende veelhoek wordt gegeven als volgt.

> Een veralgemeende n-hoek Γ van de orde (s,t), $n \geq 2$, is een incidentiestructuur $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ bestaande uit de verzameling $\mathcal{P}$ van punten, de verzameling $\mathcal{L}$ van rechten en een (symmetrische) incidentierelatie I, zodanig dat elke rechte precies $s+1$ punten bevat, elk punt op precies $t+1$ rechten ligt $(s,t \geq 1)$, Γ geen deelstructuren heeft isomorf met een gewone k-hoek voor $2 \leq k \leq n$, en zodanig dat elk koppel elementen (punten en/of rechten) in ten minste '19 e'19 en gewone n-hoek bevat is.

De veralgemeende 2-hoeken zijn triviale structuren, en vallen hier buiten beschouwing. De veralgemeende driehoeken met $s \neq 1 \neq t$ zijn precies de projectieve vlakken. Voor eindige veralgemeende n-hoeken bestaan er bepaalde restricties op de parameters n, s en t. Zo zijn er voor $s \neq 1 \neq t$ enkel eindige n-hoeken voor $n = 3, 4, 6$ en 8, en gelden volgende ongelijkheden:

Stelling *Stel dat $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ een eindige veralgemeende n-hoek is van de orde (s,t), $n \geq 3$, $s,t \geq 2$. Dan geldt een van volgende uitspraken (stel $|\mathcal{P}| = v$, $|\mathcal{L}| = b$):*

- *$n = 3$ en $s = t$ met $v = b = s^2 + s + 1$; Γ is een projectief vlak;*
- *$n = 4$ en $\frac{st(1+st)}{s+t}$ is een geheel getal; $s \leq t^2$ en $t \leq s^2$;*
- *$n = 6$ en st is een kwadraat; $s \leq t^3$ en $t \leq s^3$;*
- *$n = 8$ en $2st$ is een kwadraat, waaruit volgt dat $s \neq t$; $s \leq t^2$ en $t \leq s^2$.*

Verder geldt er (voor n even):

$$v = (1+s)(1+st+(st)^2+\ldots+(st)^{\frac{n}{2}-1}),$$
$$b = (1+t)(1+st+(st)^2+\ldots+(st)^{\frac{n}{2}-1}).$$

Voor een deelstructuur Γ' van een eindige veralgemeende n-hoek Γ die op zich ook een veralgemeende n-hoek is (en dus een orde heeft, stel (s',t')), kan men eveneens restricties afleiden (zie blz 7).

Voor het vervolg spitsen we ons toe op de veralgemeende n-hoeken met $n = 4, 6$. We geven de klassieke voorbeelden van veralgemeende vierhoeken (daarmee belanden we onder andere bij de meetkunden van punten en rechten op kwadrieken of Hermitische vari‘127 eteiten van Witt-index 2 in een projectieve ruimte van aangepaste dimensie), en de veralgemeende zeshoeken (hier zijn punten en rechten absoluut t.o.v. een trialiteit van de hyperbolische kwadriek in 7 dimensies).

In paragraaf 1.4.3 gaan we over tot de definitie van de Moufang conditie. Deze voorwaarde, reeds door Moufang ingevoerd in 1933 voor de veralgemeende driehoeken, legt restricties op aan (een deelgroep van) de groep die werkt op een veralgemeende veelhoek. Met andere woorden: hier worden bepaalde symmetrie‘127 en ondersteld. De eindige zowel als oneindige veelhoeken die aan deze voorwaarde voldoen, kunnen geklasseerd worden, in tegenstelling tot de veelhoeken in hun meest algemene vorm.
Omdat we veelhoeken echter ook willen herkennen aan andere eigenschappen (om maar de puur combinatorische of meetkundige eigenschappen te vermelden), zijn er in de loop van de jaren een hele reeks karakteriseringen van veelhoeken uitgewerkt. We vermelden er in paragraaf 1.4.3 een aantal voor de vier- en zeshoeken.

Vervolgens richten we onze aandacht op ovo‘127 ıdes en spreads, en geven een selectie van de gekende voorbeelden. Een ovo‘127 ıde is als volgt gedefinieerd (waarbij links respectievelijk rechts van het ‘/’-teken gelezen moet worden):

> Een ovo‘127 ıde in een veralgemeende vier / zes-hoek is een verzameling punten die onderling op afstand 4 / 6 liggen, zodanig dat elke rechte / elk punt op afstand 1 / 2 ligt van juist ‘19 e‘19 en punt van de ovo‘127 ıde.

In paragraaf 1.7.3 defini‘127 eren we projectiviteiten van veralgemeende veelhoeken (dit is weerom een meer groepentheoretisch begrip), en om hiermee te kunnen werken in hoofdstuk 3, leggen we de algemene co‘127 ordinatisatie-methode van Hanssens en Van Maldeghem, verder uitgewerkt door De Smet en Van Maldeghem, uit aan de hand van de vierhoeken. Hier komt ook een ezelsbruggetje bij kijken (blz 20).

Vervolgens wordt de (uitgebreide) Higman-Sims techniek kort ingeleid; deze komt van pas in hoofdstuk 5.

Als afsluiter geven we nog enige definities van andere interessante meetkunden en puntenverzamelingen, die als deelstructuren of afgeleide structuren uit de veelhoeken tevoorschijn zullen komen.

C.2 Een karakterisering van $Q(5,q)$ steunend op ‘19 e‘19 en deelvierhoek $Q(4,q)$

Van de klassieke vierhoek $Q(5,q)$ is al een keur aan karakteriseringen gekend. Degene die gebruik maken van deelvierhoeken, stellen meestal een eis over het aantal deelvierhoeken dat in de grote vierhoek (lees: de vierhoek die dient geklasseerd te worden) door een bepaalde deelstructuur te vinden is (zie stellingen 1.10 and 1.20). Volgende karakterisering stelt echter maar ‘19 e‘19 en deelvierhoek voorop, en vraagt iets over de structuren die door de grote vierhoek in die klassieke deelvierhoek ge‘127 ınduceerd worden.

Stelling *Stel dat* Γ *een veralgemeende vierhoek van de orde* (q,q^2) *is, en* Δ *een klassieke deelvierhoek van* Γ *van de orde* (q,q)*. Als alle ge‘127 ınduceerde ovo‘127 ıdes in* Δ *klassiek zijn, dan is* Γ *zelf klassiek (dus* $\cong Q(5,q)$*).*

Deze stelling is reeds gekend, maar er was nog geen gemeenschappelijk bewijs dat zowel geldt voor even als oneven karakteristiek. In hoofdstuk 2 geven we zulk bewijs, dat bovendien volledig meetkundig gaat. (Het gekende bewijs voor q oneven gebruikt cohomologie-theorie, zie Brown [10]; het gekende bewijs voor q even is meetkundig maar voor ‘19 e‘19 en van de gevallen onvolledig (Thas en Payne [76]).)
We merken hierbij op dat we uiteindelijk toch toewerken naar een ge-kende karakterisering van $Q(5,q)$ aan de hand van een groot aantal deelvierhoeken, die we in het bewijs expliciet construeren, uitgaande van die ene gegeven deelvierhoek.

Een tweede stelling die $Q(5,q)$, q oneven, karakteriseert aan de hand van ‘19 e‘19 en deelvierhoek luidt als volgt:

Stelling *Stel dat* Γ *een veralgemeende vierhoek van de orde* (q,q^2) *is, en* Δ *een klassieke deelvierhoek van* Γ *van de orde* (q,q)*. Als alle triades* $\{x,y,z\}$ *van* Δ *3-regulier zijn in* Γ *en* $\{x,y,z\}^{\perp\perp} \subset \Delta$*, dan is* Δ *klassiek. Als bovendien* q *oneven is, is ook* Γ *klassiek (dus* $\cong Q(5,q)$*).*

Dit is een uitbreiding van een resultaat in Thas [65]. Tenslotte kunnen we ook met een groepentheoretische voorwaarde op $Q(4,q)$ de vierhoek $Q(5,q)$ karakteriseren.

Stelling *Stel dat* Γ *een veralgemeende vierhoek van de orde* (q,q^2) *is, en* Δ *een klassieke deelvierhoek van* Γ *van de orde* (q,q)*. Als de lineaire groep*

G die werkt op Δ uitbreidt tot Γ, dan zijn alle ge'127 ınduceerde ovo'127 ıdes klassiek, en is ook Γ klassiek.

Omdat de bewijzen van voornoemde stellingen niet volledig onafhankelijk zijn, vindt u de resultaten in het betreffende hoofdstuk gesplitst en in een ietwat gewijzigde volgorde terug. Ook een hulpresultaat dat bepaalde ovo'127 ıdes in $Q(4, q)$ klasseert, duikt op.

Opmerking In appendix B wordt een nieuw deelresultaat voor vierhoeken gegeven, dat aansluit bij de karakteriseringen van vierhoeken aan de hand van het bestaan van deelvierhoeken. We bewijzen volgende lemma's.

Lemma *Stel dat Γ een punt-reguliere veralgemeende vierhoek is. Dan heeft elke centrische triade op een 3×3-grid van Γ precies '19 e'19 en centrum.*

Lemma *Stel dat Γ een veralgemeende vierhoek is van de orde (q^2, q). Als elke gewone vijfhoek bevat is in een unieke deelvierhoek $W(2)$ van Γ, dan zijn alle punten van Γ regulier, en heeft elke triade op een 3×3-grid van Γ precies '19 e'19 en centrum.*

C.3 Karakteriseringen van $Q(d, \mathbb{K}, \kappa)$ en $H(3, \mathbb{K}, \mathbb{K}^{(\sigma)})$ a.d.h.v. projectiviteitsgroepen

Terwijl we ons in het vorige hoofdstuk vooral op het bestaan van een deelvierhoek baseerden, richten we nu onze aandacht op het klasseren van $Q(5, q)$ '19 en andere klassieke (eventueel oneindige) vierhoeken aan de hand van regulariteitseigenschappen van rechten en punten. Hier zal de groepentheoretische aanpak nog prominenter aanwezig zijn: met een voorwaarde op de projectiviteitsgroepen $\Pi(\Gamma)$ en/of $\Pi^*(\Gamma)$ verkrijgen we karakteriseringen van (verzamelingen van) klassieke vierhoeken. Een eerste resultaat klasseert enkel $Q(4, \mathbb{K})$ voor een separabel kwadratisch gesloten veld.

Stelling *Stel dat Γ een veralgemeende vierhoek is waarvan alle rechten regulier zijn. Als elk element van $\Pi(\Gamma)$ een fixelement heeft, dan is $\Gamma \cong Q(4, \mathbb{K})$, voor $\mathbb{K}$ een separabel kwadratisch gesloten veld.*

Een tweede resultaat geeft — voor oneven karakteristiek — een klassering van niet enkel $Q(4, \mathbb{K})$, maar van alle orthogonale vierhoeken $Q(d, \mathbb{K}, \kappa)$ en alle vierhoeken duaal aan de vierhoeken afkomstig van σ-hermitische vormen in $\mathbf{PG}(3, \mathbb{K})$ (dus $H(3, \mathbb{K}, \mathbb{K}^{(\sigma)})$).

Stelling *Stel dat Γ een veralgemeende vierhoek is waarvan alle rechten regulier zijn. Stel bovendien dat $\Pi(\Gamma)$ een Zassenhaus-groep is, die voldoet aan volgende bijkomende voorwaarden:*

(*i*) *de verzameling N van alle elementen van $\Pi(\Gamma)$ die enkel een punt p fixeren, vormt, tesamen met de identiteit, een commutatieve deelgroep van $\Pi(\Gamma)_p$ (de stabilisator van het punt p in $\Pi(\Gamma)$);*

(*ii*) *elk niet-triviaal element van $\Pi(\Gamma)$ met een involutorisch koppel heeft precies twee fixelementen.*

Dan is Γ ofwel een orthogonale vierhoek $Q(d, \mathbb{K}, \kappa)$ of duaal aan een vierhoek afkomstig van een σ-hermitische vorm in een projectieve ruimte $\mathbf{PG}(3, \mathbb{K})$, i.e. $H(3, \mathbb{K}, \mathbb{K}^{(\sigma)})$. Bijgevolg is Γ een Moufang vierhoek. In beide gevallen is de karakteristiek van $\mathbb{K}$ bovendien oneven.

Als bovendien

(*iii*) *$\Pi(\Gamma)_{p,q}$ (de stabilisator in $\Pi(\Gamma)$ van twee verschillende punten p, q) abels is,*

dan is er een veld $\mathbb{K}$ van karakteristiek $\neq 2$ en met -1 een kwadraat in $\mathbb{K}$ zo dat Γ isomorf is met $Q(4, \mathbb{K})$.

Vervolgens worden de eindige orthogonale vierhoeken $Q(4, q)$ en $Q(5, q)$ samen geklasseerd door de omkering van volgende vaststelling.

> *Indien $\Gamma \cong Q(4, q)$ of $\Gamma \cong Q(5, q)$ en we noteren de orde van Γ met (s, t), dan zijn alle rechten van Γ regulier, dan is $\Pi(\Gamma)$ permutatie-equivalent met een deelgroep van $\mathbf{PGL}_2(s)$ in haar natuurlijke werking op $\mathbf{PG}(1, s)$, en dan is $\Pi^*(\Gamma)$ permutatie-equivalent met een deelgroep van $\mathbf{PGL}_2^{(\sqrt{t})}(t)$ in haar natuurlijke werking op $\mathbf{PG}(1, t)$.*

Om te besluiten volgt er een korte karakterisering van $Q(5, q)$ aan de hand van zijn orde en de speciale duale projectieve groep $\Pi^*_+(\Gamma)$.

Stelling *Stel dat Γ een eindige veralgemeende vierhoek is van de orde (q, q^2). Dan is Γ isomorf met $Q(5, q)$ als en slechts als $\Pi^*_+(\Gamma)$ een Zassenhaus groep is.*

Al deze resultaten sluiten aan bij de volgende klassering van de veralgemeende driehoeken: *Een projectief vlak van de orde $s \neq 23$, $s > 4$, is klassiek $\Leftrightarrow$ zijn projectiviteitsgroep bevat de alternerende groep niet.* (Zie Grundh'127 ofer [27].) Hoewel er nog stof tot nadenken overblijft voor er een gelijkwaardig resultaat voor vierhoeken gevonden wordt, geven bovenstaande stellingen al een eerste aanzet in die richting.

C.4 Een karakterisering van $H(q)$ en $T(q^3, q)$ a.d.h.v. ovo‘127 ıdale deelruimten

In het vierde hoofdstuk komen de eindige zeshoeken aan de beurt. Voor de vierhoeken werd reeds het volgende bewezen (Payne en Thas [48] 1.3.6(iv), 5.2.5, 5.2.6):

> *Een eindige veralgemeende vierhoek van de orde (s, s) is isomorf met $W(s)$ als en slechts als alle punten van een geometrisch hypervlak regulier zijn.*

We breiden het begrip geometrisch hypervlak uit naar zeshoeken (bij deze krijgt het ook een andere naam, namelijk ovo‘127 ıdale deelruimte), en tonen de equivalente stelling voor zeshoeken aan.

Stelling *Stel dat $\mathcal{A}$ een ovo‘127 ıdale deelruimte is van een veralgemeende zeshoek Γ van de orde (s, t). Dan is $\Gamma \cong H(q)$ of $T(q^3, q)$ als en slechts als*
$(\star)$ *elke 2 punten van Γ die op afstand 6 van elkaar liggen, bevat zijn in een dunne ideale deelzeshoek $\mathcal{D}$, en*
$(\star\star)$ *alle punten van $\mathcal{A}$ spanregulier zijn.*

Uit het bewijs volgt dat de bijkomende voorwaarde $(\star)$ in sommige gevallen overbodig is.

C.5 Wolken in veralgemeende vierhoeken en zeshoeken

Voorgaande hoofdstukken handelden voornamelijk over karakteriseringen. Nu bekijken we (nieuwe) deelstructuren van veralgemeende vier- en zeshoeken — die dan misschien op hun beurt ooit opduiken in een of andere karakterisering. We defini‘127 eren een m-wolk $\mathcal{C}$ als volgt:

> *Een m-wolk $\mathcal{C}$ van een veralgemeende zeshoek van de orde (s, t), $2 \leq m \leq t$, is een verzameling van punten van Γ die onderling op afstand 4 liggen, zodanig dat elk punt dat collineair is met twee punten van $\mathcal{C}$, collineair is met precies $m + 1$ punten van $\mathcal{C}$.*

De bijhorende verzameling $\mathcal{C}^*$ defini‘127 eren we als de verzameling van alle punten die op afstand 2 liggen van $m + 1$ punten van $\mathcal{C}$. Elk punt van $\mathcal{C}$ ligt op zijn beurt ook op afstand 2 van een constant aantal punten van $\mathcal{C}^*$. Is dit aantal $k + 1$, dan zeggen we dat de m-wolk index k heeft. m-Wolken blijken veel gemeen te hebben met — in het klassieke geval — de ideale vlakken van $H(q)$; soms zijn het delen van de dubbele meetkunde van een projectief vlak. De uitbreiding van een affien vlak tot een projectief vlak vertaalt zich aldus in de theorie van de m-wolken:

Stelling *Indien $k > t - \sqrt{t} + 1$, dan is een $(k-1)$-wolk $\mathcal{C}$ van index k uitbreidbaar tot een k-wolk $\overline{\mathcal{C}}$ van index k, zodanig dat $\overline{\Gamma}' = (\overline{\mathcal{C}}, \overline{\mathcal{C}}^*, \sim)$ een projectief vlak is van de orde k.*

Vervolgens bespreken we het al dan niet bestaan van m-wolken voor kleine en grote m in verschillende soorten zeshoeken. Maken we de definitie van m-wolk algemener, i.e. spreken we enkel nog van 'elementen' in plaats van punten respectievelijk rechten in $(\mathcal{C}, \mathcal{C}^*, \sim)$, dan komen we uit bij de volgende definitie:

> *Een dichte wolk $\mathcal{D}$ van index α is een verzameling van d punten zodanig dat elk punt p van $\mathcal{D}$ collineair is met precies α punten van $\mathcal{D} \setminus \{p\}$.*

Hierop kunnen we de uitgebreide Higman-Sims techniek toepassen, met volgend resultaat.

Stelling *Stel dat Γ een veralgemeende zeshoek is van de orde (s,t), en stel dat $\mathcal{D}$ een dichte wolk is van index α. Dan geldt er*
$(s+1)(\alpha+1-s-\sqrt{st})(st+\sqrt{st}+1) \leq |\mathcal{D}| \leq \frac{(\alpha+t+1)(s^2t^2+st+1)}{t+1}$.
De ondergrens wordt bereikt als en slechts als elk punt buiten $\mathcal{D}$ collineair is met precies $\alpha+1-s-\sqrt{st}$ punten van $\mathcal{D}$.
De bovengrens wordt bereikt als en slechts als elk punt buiten $\mathcal{D}$ collineair is met precies $\alpha+t+1$ punten van $\mathcal{D}$.

Dichte wolken verenigen o.a. volgende speciale deelstructuren van veralgemeende zeshoeken: ovo‘127 ıdes, spreads, volle en ideale deelzeshoeken.

In het tweede deel van dit hoofdstuk doen we de besprekingen nog eens over voor veralgemeende vierhoeken. Daarbij moet wel enigszins gesleuteld worden aan de oorspronkelijke definitie van een m-wolk in zeshoeken. Het analogon voor de vierhoeken van de dichte wolken heeft volgende eigenschap — waarvan de helft (i.e. de ondergrens) al te lezen staat in Payne [47] (zie ook [48] 1.10.1).

Stelling *Stel dat* Γ *een veralgemeende vierhoek is van de orde* (s,t), *en stel dat* $\mathcal{D}$ *een dichte wolk is van index* α. *Dan geldt er*
$(s+1)(\alpha+1-s) \leq |\mathcal{D}| \leq \frac{(\alpha+t+1)(st+1)}{t+1}$.
De ondergrens wordt bereikt als en slechts als elk punt buiten $\mathcal{D}$ *collineair is met precies* $\alpha+1-s$ *punten van* $\mathcal{D}$.
De bovengrens wordt bereikt als en slechts als elk punt buiten $\mathcal{D}$ *collineair is met precies* $\alpha+t+1$ *punten van* $\mathcal{D}$.

Ook hier worden ovo‘127 ıdes, spreads en deelvierhoeken onder ‘19 e‘19 en en dezelfde noemer teruggevonden.

C.6 Twee Hill-kappen maar geen hemisysteem

Als afsluiter van de thesis doen we een gooi naar de constructie van een hemisysteem in de zeshoek $H(3)$. Het laatste loodje doet ons het loodje leggen, maar onderweg vinden we wel een representatie van een $2-(16,6,2)$ design op de kwadriek $Q(6,3)$.

Stelling *Stel dat* π_0 *een 4-dimensionale deelruimte is van* $\mathbf{PG}(6,3)$, *die de kwadriek* $Q(6,3)$ *in een niet-singuliere kwadriek snijdt. Stel dat* Π_1 *en* Π_2 *de twee hypervlakken door* π_0 *zijn die* $Q(6,3)$ *snijden in twee niet-singuliere elliptische kwadrieken. Stel dat op elk van deze kwadrieken een Hill-kap gedefinieerd is, zodanig dat hun gemeenschappelijke doorsnede in* π_0 *een 20-kap is. Dan is de unie van de twee Hill-kappen nooit uitbreidbaar tot een hemisysteem van een zeshoek* $H(3)$ *gedefinieerd op* $Q(6,3)$.

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Errata to 'Characterizations of...'

Comments

<u>**PAGE 97**</u> Minimal dense clouds of index α of generalized quadrangles were thoroughly studied by Payne (see [47B] and [47C]) under the name of β-tight pointsets, where $\beta = \alpha + 1 - s$ is the number of points of the minimal dense cloud $\mathcal{D}$ collinear with a point outside $\mathcal{D}$.

In the first article mentioned, the determination of all tight sets is undertaken for GQs with small parameters. For GQs with parameters $(2, 2)$ and $(2, 4)$ for example, a β-tight set is one of the following:

1. the union of β disjoint point rows,
2. the union of $\beta - 2$ disjoint point rows and the point set of a disjoint dual grid (see bottom of page 91),
3. the point set of a subGQ of order $(2, 2)$ (for GQ of order $(2, 4)$ and $\beta = 5$),
4. the complement of one of the three configurations above, where β is replaced by $st + 1 - \beta$.

For GQs with parameters $(3, 3)$, a similar result and an interesting model of the classical GQ $W(3)$ is obtained. The second paper focuses on the existence of 3-tight sets in GQs with small parameters (so far not yet handled).

Addenda to Bibliography

[**10B**] M. R. Brown. Semipartial geometries and generalized quadrangles of order (r, r^2). *Bull. Belg. Math. Soc.*, 5:187–205, 1998. *(see erratum for p 36)*

[**44B**] A. D. Offer. *Spreads and Ovoids of the Split Cayley Hexagon.* PhD thesis, University of Adelaide, 2000. *(see erratum for p 10)*

[**47B**] S. E. Payne. Tight pointsets in finite generalized quadrangles. *Cong. Num.*, 60:243–260, 1987. *(see comments in previous section; p 145)*

[**47C**] S. E. Payne. Tight pointsets in finite generalized quadrangles, II. *Cong. Num.*, 77:31–41, 1990. *(see comments in previous section; p 145)*

[7] L. Brouns, **J. A. Thas and H. Van Maldeghem**. A characterization of $Q(5,q)$ using one subquadrangle $Q(4,q)$. *Europ. J. Comb.*, submitted. 22 pp.

[8] L. Brouns, **J. A. Thas and H. Van Maldeghem**. m-Clouds in generalized hexagons. *Discrete Math.*, to appear. 10 pp.

[48] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles*, volume **110** of *Research Notes in Mathematics.* Pitman, London, 1984. 312pp.

Errata

The notation l^i points to the i^{th} line of the page respectively paragraph, the notation l_i points to the i^{th} last line of the page respectively paragraph.

p 3 'Definition of **a** generalized polygon'
p 3 par 4, l^4: 'thin **generalized** n-gon'
p 4 par 1, l^6: '...if u is **distance-i-**regular for all...'
p 5 l^3: 'More general**ly**'
p 6 l^8: '$\Gamma^+(x,y) = \Gamma_0(x)\underline{\cup}\Gamma_4(x)$' and similarly for $\Gamma^-(x,y)$
p 7 thm 1.2: '$n = 8$ *and* $s \geq t$ *and* $s' = 1$.'
p 10 l_4: For (more) geometric properties of $H(q)$, we refer to Van Maldeghem [84] 2.4,6.2 and A. Offer [44B], where a very nice overview is given in section 1.4.2.
p 11 title: '**The** Moufang condition'
p 11 The term 'fixline' should be replaced by '**fixed line**'.
p 11 l_1: 'In this **section**...'
p 12 l_2: '...**the** previous paragraph'
p 13 l^2: space after 'Theorems 1.19'
p 13 l^4: '**We note**' instead of 'Remark'
p 13 thm 1.22: '...hexagon, **then** Γ is point-distance-3-regular...'

p 14 l^4: to be added: ‘A **spread of a generalized** n**-gon** is the dual notion of an ovoid of a generalized n-gon.’

p 14 thm 1.27: ‘...is an ovoid of $\mathbf{PG}(3, q)$, **and every ovoid of** $\mathbf{PG}(3, q)$ **can be written as an ovoid of some** $W(q)$ **in** $\mathbf{PG}(3, q)$**.**’

p 16 table: e.g. ‘(**with** $\overline{\Gamma} = Q(5, q)$)’ instead of ‘(let $\overline{\Gamma} = Q(5, q)$)’

p 16 paragraph after table: For details, proofs **and references about the correspondence between ovoids and translation planes**, see Thas [62], **Thas and Payne** [76] and Tits [80].

p 17 par 4, l^7: ‘explicitly’

p 17 footnote 5: ‘are’

p 20 1.9.2 l^1: ‘**The** following’

p 23 l^9: ‘**apply**’ instead of ‘aplly’

p 25 l^1: ‘...**nonsingular** elliptic quadric’

p 25 l$_1$: Omit the word ‘unique’, as it should be clear that Δ is not really unique; it is just the only subquadrangle we know of by hypothesis.

p 26 l^9: ‘the elliptic quadric $\mathcal{O}_p$’

p 28 par 2, l^3: ‘$(q^2 + 1)q^2$ **ordered** pairs of points on $\mathcal{O}$,...’

p 29 par 3, l^2: ‘**without**’

p 30 q odd case, l^{11}: ‘(**there** are $(q^2$...’

p 34 q even case, l^2: ‘(which are **isomorphic**, for q even)’

p 36 l^1: ‘**lemmas**’ instead of ‘lemma’s’

p 36 The lemmas and notations are also in Brown [10B].

p 48 l^8: ‘**three dimensional**’

p 48 l^{11}: ‘$\frac{q^2(q^{\boxed{2}}+1)}{2}$’

p 49 l^7: ‘$Q(5, \mathbb{K}, \kappa)$’ instead of ‘$Q(5, q)$’

p 52 l^8: ‘$[p; p_1; p_2; p] := [p; p_1][p_1; p_2][p_2; p]$’

p 53 thm 3.2: last sentence is repeated

p 59 l^4: ‘**prove the** next result’

p 88 l^8: ‘with A the adjacency matrix of the complement of the point graph of Γ’

p 89 l$_1$: ‘**an** eigenvalue’

p 93 $-s \leq \lambda_3$ case: ‘denominator is **positive**’

p 93 $-s \leq \lambda_3$ case: ‘**numerator**’ instead of ‘nominator’

p 96 l^4: ‘Using the notations of **section 1.10**’

p 102 l^2: ‘**coweight distribution**’

p 104 l$_2$: ‘**putative** hemisystem’ instead of ‘theoretical hemisystem’ p 109 We choose the following equations:

$$\begin{array}{rcl} Q(6,3) & \leftrightarrow & X_0X_4 + X_1X_5 + X_2X_6 = X_3^2 \\ Q_1^-(5,3) \subset \Pi_1 & \leftrightarrow & X_1 + X_2 = X_5 + X_6 \\ Q_2^-(5,3) \subset \Pi_2 & \leftrightarrow & X_1 + X_6 = X_2 + X_5 \\ Q_3^+(5,3) \subset \Pi_3 & \leftrightarrow & X_2 = X_6 \\ Q_4^+(5,3) \subset \Pi_4 & \leftrightarrow & X_1 = X_5 \end{array}$$

p 109 l_6 `if ((X[1]) mod q = (X[5]) mod q) and ((X[2]) mod q = (X[6]) mod q)`

p 110 l^1 `if ((X[1]) mod q = (X[5]) mod q)`

p 110 l^6 `if ((X[2]) mod q = (X[6]) mod q)`

p 110 l^{11} `if ((X[1]+X[6]) mod q = (X[2]+X[5]) mod q)`

p 110 l^{16} `if ((X[1]+X[2]) mod q = (X[5]+X[6]) mod q)`

Carrying out these corrections will not change the result of the program nor the related theorem, as the errors in the thesis occured by copying the wrong subprogram into the thesis. The output on page 126 is based on data calculated with the equations given in these errata.