

Clustergedrag in systemen van gekoppelde oscillatoren
en netwerken van elkaar aantrekkende agenten

Clustering Behavior in Systems of Coupled Oscillators
and Networks of Mutually Attracting Agents

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Preface

The research I performed in the previous years can be categorized in three main subjects: area contraction of multi-dimensional surfaces, the Kuramoto-Sakaguchi-model of coupled oscillators, and a model for cluster formation. The latter two are closely related, while the first one is rather loosely connected to the other two, and has therefore been left out of this dissertation. The results on this subject can be found in [8, 10].

The results on the Kuramoto-Sakaguchi model and the clustering model, which are presented in this thesis (this includes all of the lemmas, propositions and theorems), are original (except of course for the introductory material, for which the appropriate references have been included) unless specified otherwise in the text. They are (to be) published in several conference papers [9, 4, 7] and journal articles [20, 21, 6, 18, 19, 22, 5].

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Nederlandse samenvatting

Het Kuramoto-Sakaguchi-model

Het Kuramoto-Sakaguchi-model is een uitbreiding van het Kuramoto-model, dat geïntroduceerd werd om synchronisatie in systemen van gekoppelde oscillatoren te beschrijven. Voorbeelden van zulke systemen zijn zwermen flikkerende vuurvliegjes, groepen pacemakercellen in het hart, en serieschakelingen van Josephson-juncties.

Het Kuramoto-Sakaguchi-model wordt beschreven door de volgende differentiaalvergelijkingen:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j(t) - \theta_i(t) - \alpha), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N \triangleq \{1, \dots, N\}, \quad (1)$$

met $N > 1$ het aantal oscillatoren, K de koppelingssterkte, $|\alpha| < \frac{\pi}{2}$, en θ_i en ω_i de fase en de natuurlijke (hoek)frequentie van oscillator i . De natuurlijke frequenties ω_i worden willekeurig gekozen uit een verdeling g .

Kuramoto en Sakaguchi beschouwden de limiet $N \rightarrow \infty$ en onderzochten een oplossing waarvoor een groep oscillatoren aan één enkele constante frequentie Ω_0 beweegt, terwijl de andere oscillatoren ongeordend bewegen, met een (tijds)gemiddelde frequentie die ligt tussen hun natuurlijke frequentie en Ω_0 . Ze toonden aan dat dit gedrag (*partiële synchronisatie* genaamd) enkel kan optreden boven een minimale waarde K_k van de koppelingssterkte. Voor K -waarden kleiner dan K_k is er geen partiële synchronisatie en oscillatoren met verschillende natuurlijke frequenties bewegen aan verschillende gemiddelde frequenties.

Eindige N

Het systeem (1) met N eindig vertoont een gedrag dat lijkt op partiële synchronisatie. Alhoewel oscillatoren met verschillende natuurlijke frequenties niet kunnen bewegen aan exact dezelfde frequentie, kunnen er deelverzamelingen

zijn van oscillatoren met begrensde faseverschillen. Dit verschijnsel wordt *partiële meevoering* genoemd en de betreffende deelverzamelingen worden meegevoerde deelverzamelingen genoemd.

We kunnen een ongelijkheid opstellen in de parameters van het model die een voldoende voorwaarde vormt voor partiële meevoering t.o.v. een gegeven deelverzameling S_m . Het bewijs hiervan impliceert dat partiële meevoering blijft bestaan onder kleine storingen op de beginvoorwaarde. De resultaten blijven ook niet-triviaal in de limiet $N \rightarrow \infty$.

De interactie tussen oscillatoren met sterk verschillende natuurlijke frequenties is, uitgemiddeld over de tijd, zeer klein. Dit laat ons toe om de kritische K -waarde af te schatten die het begin van partiële meevoering bepaalt voor een gegeven deelverzameling S_m , door interacties te verwaarlozen tussen oscillatoren uit S_m met oscillatoren die niet behoren tot S_m .

Voor bepaalde configuraties is het mogelijk dat partiële meevoering van een deelverzameling verdwijnt met toenemende koppelingssterkte. Als de koppelingssterkte verder toeneemt zal de partiële meevoering terugkeren. Een gelijkaardig verschijnsel kan waargenomen worden in serieschakelingen van Josephson-juncties.

Oneindige N

Voor het Kuramoto-Sakaguchi-model met een oneindig aantal oscillatoren beschouwen we een storing van de oplossing onderzocht door Kuramoto en Sakaguchi. De deelverzameling die overeenkomt met de partieel gesynchroniseerde deelverzameling van de ongestoorde oplossing is niet meer partieel gesynchroniseerd, maar wel nog partieel meegevoerd, en gaat gepaard met een aantal kleinere partieel meegevoerde deelverzamelingen (met groottes van de orde van de grootte van de storing). (We herdefiniëren het begrip ‘partiële meevoering’ voor een oneindig aantal oscillatoren zodanig dat partieel meegevoerde deelverzamelingen steeds minstens twee oscillatoren moeten bevatten met verschillende natuurlijke frequenties.)

We kunnen vergelijkingen opstellen die de natuurlijke frequenties van de oscillatoren in de meegevoerde deelverzamelingen bepalen in eerste orde in de grootte van de storing.

Zoals bij het model met eindige N , is het mogelijk een verdeling van de natuurlijke frequenties te vinden waarvoor partiële meevoering van de kleinere meegevoerde deelverzamelingen verdwijnt als de koppelingssterkte toeneemt boven een bepaalde kritische waarde.

Een ander verschijnsel dat kan afgeleid worden uit de analytische resultaten bestaat uit het ontstaan van partiële meevoering in intervallen waar de frequentiedichtheid te laag is om deze te kunnen verklaren. De partiële meevoering is een gevolg van resonanties met andere partieel meegevoerde deelverzamelingen, die wel veroorzaakt zijn door hoge frequentiedichtheden.

Het clustermodel

Het basismodel

De analytische resultaten voor het Kuramoto-Sakaguchi-model uit het eerste deel van deze thesis kennen een aantal beperkingen: de resultaten voor het model met een eindig aantal oscillatoren behelzen enkel voldoende voorwaarden; voor het model met een oneindig aantal oscillatoren kunnen enkel kleine storingen t.o.v. een oplossing met één meegevoerde deelverzameling onderzocht worden. Ook kan de omslachtige formulering van sommige van de resultaten een intuïtieve interpretatie in de weg staan en hun toepassing bemoeilijken.

We stellen een systeem voor met een gedrag dat kan vergeleken worden met het partiële-meevoeringsgedrag van het Kuramoto-Sakaguchi-model, dat echter ook toepassingen heeft op systemen die niets met gekoppelde oscillatoren te maken hebben. We vervangen daarom de term ‘oscillatoren’ door ‘agenten’, ‘partiële-meevoeringsgedrag’ door ‘clustergedrag’, en in de wiskundige beschrijving vervangen we θ_i door x_i , ω_i door b_i en de sinusoidale interactie door f :

$$\dot{x}_i(t) = b_i + \frac{K}{N} \sum_{j=1}^N f(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (2)$$

met $K \geq 0$, f oneven, niet-dalend en Lipschitz-continu, en f bereikt een saturatiewaarde: $f(\pm x) = \pm F$, $\forall x \geq d$, voor een bepaalde $d > 0$.

Veronderstel dat, voor een specifieke oplossing x van (2), het gedrag van de agenten gekenmerkt wordt als volgt via een *geordende* verzameling *clusters* $G \triangleq (G_1, \dots, G_M)$ die overeenkomt met een partitie van \mathcal{I}_N :

- De afstanden tussen agenten van dezelfde cluster blijven begrensd (m.a.w. $|x_i(t) - x_j(t)|$ is begrensd voor alle $i, j \in G_k$, voor elke $k \in \mathcal{I}_M$, voor $t \geq 0$).
- Voor elke $D > 0$ bestaat er een tijdstip waarna de afstanden tussen agenten van verschillende clusters tenminste D bedragen en tenminste D blijven.
- Na voldoende tijd zijn de agenten geordend volgens de volgorde van de clusters: $\exists T > 0 : \forall t \geq T, \forall k, l \in \mathcal{I}_M, \forall i \in G_k, \forall j \in G_l, k < l \Rightarrow x_i(t) < x_j(t)$.

Als de oplossing x voldoet aan deze voorwaarden, dan zeggen we dat x *clustergedrag* vertoont ten opzichte van G .

We kunnen ongelijkheden formuleren in de parameters van het model die een nodig en voldoende stel voorwaarden vormen voor clustergedrag t.o.v. een gegeven geordende partitie G .

Ook kunnen we aantonen dat er voor elke keuze van de parameters een clusterstructuur G bestaat die voldoet aan deze voorwaarden, zodat voor elke keuze van de parameters de oplossingen van het model steeds clustergedrag zullen vertonen t.o.v. een bepaalde clusterstructuur.

Onder milde extra voorwaarden kan aangetoond worden dat de snelheden van de agenten en de verschillen in $x_i(t)$ -waarden van agenten van dezelfde cluster naderen naar constante waarden, die onafhankelijk zijn van de beginvoorwaarden. De asymptotische snelheden zijn vanzelfsprekend dezelfde voor agenten van dezelfde cluster.

Uitbreidingen van het model

Het basismodel kan uitgebreid worden door een algemene interactiestructuur te beschouwen met verschillende interactiefuncties voor elk paar agenten, en door strikt positieve gewichtsfactoren γ_i en strikt positieve gevoeligheidsfactoren A_i te associëren met de agenten:

$$\dot{x}_i(t) = b_i + K A_i \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (3)$$

waarbij de functies f_{ij} Lipschitz-continu en niet-stijgend zijn, en voldoen aan $f_{ji}(x) = -f_{ij}(-x)$, $\forall x \in \mathbb{R}$, en $f_{ij}(x) = F_{ij}$, $\forall x \geq d_{ij}$, voor bepaalde $F_{ij}, d_{ij} \in \mathbb{R}$, $\forall i, j \in \mathcal{I}_N$. De matrix F wordt verondersteld symmetrisch en niet-reduceerbaar te zijn.

De resultaten voor het basismodel kunnen veralgemeend worden tot het model (3). De algemene netwerkstructuur heeft een belangrijk verschil met het basismodel als gevolg: clusters kunnen opsplitsen met stijgende koppelingssterkte, waardoor meer dan N verschillende clusterstructuren kunnen optreden met veranderende koppelingssterkte.

Gelijkaardige resultaten kunnen verkregen worden voor het systeem (3) waarbij de interactiefuncties niet noodzakelijk hun saturatiewaarden bereiken, maar wel voldoen aan $\lim_{x \rightarrow +\infty} f_{ij}(x) = F_{ij}$, $\forall i, j \in \mathcal{I}_N$.

Als de parameters b_i en de functies f_{ij} in (3) tijdsafhankelijk zijn kan er nog steeds een nodig en voldoende stel voorwaarden geformuleerd worden voor clustergedrag van alle oplossingen van het systeem.

We kunnen ook de limiet $N \rightarrow \infty$ beschouwen in (2), wat aanleiding geeft tot een partiële differentiaalvergelijking. De clusters worden gedefinieerd door intervallen I_k van b -waarden. Vanuit het nodige en voldoende stel voorwaarden voor eindige N kunnen we voorwaarden opstellen voor de intervallen I_k waarvan (onder extra voorwaarden) kan aangetoond worden dat ze nodig zijn voor het optreden van het overeenkomstige clustergedrag.

Toepassingen

Het clustergedrag van de verschillende versies van het clustermodel is gelijkaardig aan het meevoeringsgedrag van het Kuramoto-Sakaguchi-model. Er bestaan echter ook verschillende toepassingen van het model die niets te maken hebben met systemen van oscillatoren.

Geïnterconnecteerde waterbekkens. Beschouw N verschillende waterbekkens, onderling verbonden door horizontale pijpleidingen, waarbij elk bekken onderhevig is aan een constante externe toevoer of afvoer van water. Veronderstellen we dat de leidingen slechts een bepaald debiet aankunnen, dan kunnen we het systeem beschrijven via (3), waarbij $x_i(t)$ de hoogte van het waterniveau voorstelt in waterbekken i .

Opinievorming. Stellen we de mening van agent i voor door de afgeleide $\dot{x}_i(t)$ in (3), dan verkrijgen we een model voor opinievorming waar de finale meningen overeenkomen met de asymptotische snelheden van de agenten. Afhankelijk van de modelparameters zijn er verschillende eindsituaties mogelijk: verschillende groepen met elk hun eigen mening kunnen naast elkaar bestaan, er kan een polarisatie ontstaan tussen twee tegengestelde meningen, of er kan volledige overeenstemming bereikt worden.

Wielervedstrijden. Interacties tussen wielrenners leidt tot de vorming van (één of meerdere) groep(en) of cluster(s). De clusterstructuur die wordt waargenomen aan de aankomstlijn kan gemodelleerd worden via een aangepaste versie van (3). Door een gepaste methode aan te wenden om de modelparameters vast te leggen, kunnen resultaten verkregen worden die zowel kwalitatief als kwantitatief overeenstemmen met resultaten verkregen via databases.

Het minimale-kost-stroomprobleem (minimum cost flow problem)

Beschouw een volledig verbonden netwerk, met een externe toevoer b_i in knooppunt i en een stroom \mathbf{f}_{ij} door de tak die de knooppunten i en j verbindt ($\mathbf{f}_{ji} = -\mathbf{f}_{ij}$). De stroom \mathbf{f}_{ij} brengt een kost $U_{ij}(\mathbf{f}_{ij}) = U_{ji}(\mathbf{f}_{ji})$ met zich mee. Het minimale-kost-stroomprobleem komt overeen met de minimalisatie van de totale kost (de som van de kosten over alle takken), waarbij de stromen \mathbf{f}_{ij} voldoen aan continuïteitsvoorwaarden in de knooppunten en $\sum_{i=1}^N b_i = 0$. Als de kostfuncties U_{ij} strikt convex zijn en de afgeleiden U'_{ij} bijectief, dan kan de oplossing voor dit probleem asymptotisch geïmplementeerd worden door het

dynamische systeem

$$\dot{x}_i(t) = b_i + \sum_{\substack{j=1 \\ j \neq i}}^N f_{ij}(x(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (4)$$

waarbij de functies f_{ij} — overeenkomend met de stromen \mathbf{f}_{ij} — voldoen aan

$$U'_{ij}(f_{ij}(x)) = \tilde{\lambda}_j(x) - \tilde{\lambda}_i(x), \quad \forall x \in \mathbb{R}^N, \quad \forall i, j \in \mathcal{I}_N, \text{ met } i \neq j, \quad (5)$$

voor functies $\tilde{\lambda}_i : \mathbb{R}^N \rightarrow \mathbb{R}$ ($i \in \mathcal{I}_N$). Elk evenwichtspunt van (4) implementeert de oplossing van het minimale-kost-stroomprobleem. Onder voorwaarden op de vectorfunctie $\tilde{\lambda}$ kunnen we aantonen dat elke oplossing van (4) convergeert naar een evenwichtoplossing waarbij de overeenkomstige stromen de oplossing vormen van het minimale-kost-stroomprobleem.

Een gedecentraliseerde implementatie van de optimale stromen is mogelijk, waarbij elke stromingsvariabele enkel afhangt van lokale informatie. De keuze $\tilde{\lambda}_i(x) \triangleq x_i$, $\forall x \in \mathbb{R}^N$, leidt tot het systeem (3) met $K = 1$, $A_i = \gamma_i = 1$, $\forall i \in \mathcal{I}_N$, maar met interactiefuncties f_{ij} die niet satureren.

De aanpak blijft bruikbaar als de kost lineair is en er harde begrenzingen geïntroduceerd worden op de stromen \mathbf{f}_{ij} , en kan uitgebreid worden naar het probleem waarbij elke tak meerdere soorten stromen draagt, met continuïteitsvoorwaarden en externe toevoer in de knooppunten voor elke soort.

English summary

The Kuramoto-Sakaguchi model

The Kuramoto-Sakaguchi model is an extension of the Kuramoto model, which was introduced to describe synchronization in systems of coupled oscillators. Examples include swarms of flashing fireflies, groups of pacemaker cells in the heart, and arrays of Josephson-junctions.

The Kuramoto-Sakaguchi model consists of the following differential equations:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j(t) - \theta_i(t) - \alpha), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N \triangleq \{1, \dots, N\}, \quad (1)$$

with $N > 1$ the number of oscillators, K the coupling strength, $|\alpha| < \frac{\pi}{2}$, and θ_i and ω_i the phase and natural (angular) frequency of oscillator i . The natural frequencies ω_i are chosen randomly from a distribution g .

Kuramoto and Sakaguchi considered the limit $N \rightarrow \infty$ and investigated a solution for which a group of oscillators is moving at a single constant frequency Ω_0 and the other oscillators are moving incoherently, with long term average frequencies between their natural frequencies and Ω_0 . They showed that this behavior (called *partial synchronization*) can only occur above a minimal value K_c of the coupling strength. For values of K smaller than K_c there is no partial synchronization and oscillators with different natural frequencies have different long term average frequencies.

Finite N

The model (1) with finite N exhibits a phenomenon similar to partial synchronization. Although oscillators with different natural frequencies cannot move at exactly the same frequency, there may be subsets of oscillators having mutually bounded phase differences. This phenomenon is called *partial entrainment* and the subsets are called entrained subsets.

We can formulate an inequality in terms of the parameters of the model, constituting a sufficient condition for partial entrainment w.r.t. a given subset. The proof of this result implies persistence of partial entrainment under perturbations: for a sufficiently small perturbation of the initial condition, it follows that the entrainment of the corresponding solution will be maintained. Furthermore the result also remains non-trivial in the limit $N \rightarrow \infty$.

Oscillators which differ largely in natural frequency will have small (long term average) interactions with each other. This allows us to estimate the critical values for the coupling strength, defining the onset of entrainment of a given subset S_e , by neglecting the interactions between oscillators from S_e with oscillators not belonging to S_e .

For some configurations it is possible that partial entrainment disappears with increasing coupling strength. When the coupling strength is increased further the partial entrainment will reappear. A similar phenomenon can be observed in arrays of Josephson junctions.

Infinite N

For the Kuramoto-Sakaguchi model with an infinite number of oscillators we consider a perturbation of the solution Kuramoto and Sakaguchi investigated. The subset corresponding to the partially synchronized subset in the unperturbed solution is no longer partially synchronized, but still partially entrained, and is accompanied by smaller entrained subsets (with sizes of the order of the perturbation size). (For convenience we redefine the concept ‘partial entrainment’ for an infinite number of oscillators such that a partially entrained subset always includes at least two oscillators with different natural frequencies.)

We can formulate equations which characterize the natural frequencies of the oscillators in the entrained subsets in first order of the perturbation size.

As in the model with finite N , there exists a distribution of the natural frequencies, for which partial entrainment of the smaller entrained subsets disappears when the coupling strength is increased in a particular interval.

Another phenomenon following from the analytical results consists of the occurrence of partial entrainment in intervals where the frequency density is too low to account for it. The partial entrainment is a consequence of resonances with other partially entrained subsets, originating as a result of high frequency densities.

The clustering model

The basic model

The analytical results for the Kuramoto-Sakaguchi model from the first part of this thesis are restricted: the results for the model with a finite number

of oscillators only concern sufficient conditions; for the model with an infinite number of oscillators only small perturbations from a solution with one entrained subset can be investigated. Furthermore the formulation of the results can be complicated, which may obstruct an intuitive interpretation and makes it difficult to apply them.

We introduce a system which captures the partial entrainment behavior of the Kuramoto-Sakaguchi model, but also has applications to systems not related to coupled oscillators. We therefore replace the term ‘oscillators’ by ‘agents’, ‘partial entrainment behavior’ by ‘clustering behavior’, and in the mathematical notation θ_i is replaced by x_i , ω_i by b_i and the sinusoidal interaction function by f :

$$\dot{x}_i(t) = b_i + \frac{K}{N} \sum_{j=1}^N f(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (2)$$

with $K \geq 0$, and f odd, non-decreasing, Lipschitz continuous, and reaching a saturation value: $f(\pm x) = \pm F$, $\forall x \geq d$, for some $d > 0$.

Assume that, for a particular solution x of (2), the behavior of the agents can be characterized as follows by an *ordered* set of *clusters* $G \triangleq (G_1, \dots, G_M)$ representing a partition of \mathcal{I}_N :

- The distances between agents in the same cluster remain bounded (i.e. $|x_i(t) - x_j(t)|$ is bounded for all $i, j \in G_k$, for any $k \in \mathcal{I}_M$, for $t \geq 0$).
- For any $D > 0$ there exists a time after which the distances between agents in different clusters are and remain at least D .
- After some time, the agents are ordered by their membership to the clusters: $\exists T > 0 : \forall t \geq T, \forall k, l \in \mathcal{I}_M, \forall i \in G_k, \forall j \in G_l, k < l \Rightarrow x_i(t) < x_j(t)$.

If the solution x satisfies these conditions, then x is said to exhibit *clustering behavior* with respect to G .

We can formulate inequalities in the parameters of the model which constitute a necessary and sufficient set of conditions for clustering behavior of all solutions of the model w.r.t. a given ordered partition G .

Furthermore we can show that for any choice of the parameters, a cluster structure G exists, satisfying these conditions, implying that for any choice of the parameters all solutions of (2) will exhibit clustering behavior w.r.t. some cluster structure.

Under some mild extra conditions the velocities of the agents and the differences between $x_i(t)$ -values of agents from the same cluster can be shown to approach constant values which are independent of the initial condition. The asymptotic velocity is of course the same for agents from the same cluster.

Extensions of the model

The basic model can be extended to include a general interaction structure with different interaction functions for each pair of agents, while associating positive weighting factors γ_i and positive sensitivity factors A_i to the agents:

$$\dot{x}_i(t) = b_i + KA_i \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (3)$$

where the functions f_{ij} are Lipschitz continuous, non-decreasing, and satisfy $f_{ji}(x) = -f_{ij}(-x)$, $\forall x \in \mathbb{R}$, and $f_{ij}(x) = F_{ij}$, $\forall x \geq d_{ij}$, for some $F_{ij}, d_{ij} \in \mathbb{R}$, $\forall i, j \in \mathcal{I}_N$. The matrix F is assumed to be symmetric and irreducible.

The results for the basic model can be generalized to the model (3). As a result of the general network structure, there is an important difference with the basic model: clusters may split up with increasing coupling strength, allowing more than N different cluster structures to appear when the coupling strength is varied.

Similar results can be obtained when the interaction functions do not necessarily reach their saturation values, but still satisfy $\lim_{x \rightarrow +\infty} f_{ij}(x) = F_{ij}$, $\forall i, j \in \mathcal{I}_N$.

For (3) with time-dependent parameters b_i and functions f_{ij} , it is still possible to formulate a necessary and sufficient set of conditions for clustering behavior of all solutions of the system.

We can also consider the limit $N \rightarrow \infty$ in the model (2), leading to a formulation by a partial differential equation. The clusters are defined by intervals I_k of b -values. Based on the necessary and sufficient set of conditions for the model with finite N we can formulate conditions on the intervals I_k which (under some extra conditions) can be shown to be necessary for the corresponding clustering behavior.

Applications

The clustering behavior of the different versions of the clustering model is similar to the entrainment behavior of the Kuramoto-Sakaguchi model. However, there are also several applications of the model which are not related to oscillator systems.

Interconnected water basins. Consider N separate basins connected by horizontal pipes, with each basin furthermore subject to a constant external inflow or outflow of water. Assuming that the pipes have a maximal throughput the system equations can be cast into the model (3), where $x_i(t)$ represents the water height of basin i .

Opinion formation. Representing the opinion of agent i by the time-derivative $\dot{x}_i(t)$ in (3), we obtain a model for opinion formation where the final opinions correspond to the asymptotic velocities of the agents. Dependent on the parameters of the model, different outcomes are possible: a coexistence of several groups, each characterized by its opinion, a polarization of two opposite opinions, or total consensus.

Cycling races. Interactions between riders, on both physical and psychological grounds, leads to the formation of (one or more) platoon(s) or cluster(s). The platoon structure observed at the finish line can be modeled by a modification of (3). By considering an appropriate procedure for determining the model parameters, results can be obtained which agree both qualitatively and quantitatively with results obtained from databases.

The minimum cost flow problem

Consider an all-to-all connected network, with an external inflow b_i in node i and a flow \mathbf{f}_{ij} through the arc with end nodes i and j ($\mathbf{f}_{ji} = -\mathbf{f}_{ij}$). The flow \mathbf{f}_{ij} entails a cost $U_{ij}(\mathbf{f}_{ij}) = U_{ji}(\mathbf{f}_{ji})$. In the minimum cost flow problem the objective is to minimize the total cost (summing the costs over all arcs), while satisfying flow conservation constraints in the nodes. There is no net external inflow: $\sum_{i=1}^N b_i = 0$. If the cost functions U_{ij} are strictly convex with the derivatives U'_{ij} bijective, then the solution to this problem can be asymptotically implemented by the system

$$\dot{x}_i(t) = b_i + \sum_{\substack{j=1 \\ j \neq i}}^N f_{ij}(x(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (4)$$

where the functions f_{ij} — corresponding to the flows \mathbf{f}_{ij} — satisfy

$$U'_{ij}(f_{ij}(x)) = \tilde{\lambda}_j(x) - \tilde{\lambda}_i(x), \quad \forall x \in \mathbb{R}^N, \quad \forall i, j \in \mathcal{I}_N, \quad \text{with } i \neq j, \quad (5)$$

for some functions $\tilde{\lambda}_i : \mathbb{R}^N \rightarrow \mathbb{R}$ ($i \in \mathcal{I}_N$). Any equilibrium point of the system (4) implements the solution of the minimum cost flow problem. Under some conditions on the vector function $\tilde{\lambda}$ it can be shown that any solution of (4) converges to an equilibrium solution, and therefore asymptotically implements the solution of the minimum cost flow problem.

A decentralized implementation of the optimal flows is possible, with each flow variable dependent on local information only. Choosing $\tilde{\lambda}_i(x) \triangleq x_i$, $\forall x \in \mathbb{R}^N$, results in the system (3) with $K = 1$, $A_i = \gamma_i = 1$, $\forall i \in \mathcal{I}_N$, but with non-saturating interaction functions f_{ij} .

The approach remains useful when the cost is linear and hard bounds are introduced on the flows f_{ij} , and can be extended to the multi-commodity minimum cost flow problem, where an arc carries different types of flows, with continuity equations and external inflow in the nodes for each commodity type.

Chapter 1

Introduction

1.1 Coupled oscillators

The concept ‘oscillator’ refers to any system exhibiting periodic behavior. Examples of periodic systems can be found in several research areas, such as astrophysics (e.g. the planets orbiting the sun, or revolving about their axis), mechanics (vibrations, pendulums), electronics (inductor-capacitor-networks), chemistry (the Belousov-Zhabotinsky reaction), physics (Josephson-junctions), biology (flashing fireflies, the menstruation cycle, pacemaker cells in the heart).

In general the frequencies of similar isolated systems (i.e. their natural frequencies) will be different. Interactions will cause the oscillators to deviate from their natural frequencies. If the interaction is attractive and large enough compared to the differences in natural frequencies it may constrain the long term average frequencies to coincide. For instance, interaction between pacemaker cells may lead to simultaneous firing, resulting in a strong and unambiguous signal and a regular heart beat. Various other examples of this phenomenon can be found in [48].

When the difference in natural frequencies is too large with respect to the coupling strength, other relations between the long term average frequencies of the oscillators may arise. Regarding the orbits of the planets and their moons, numerous examples are known for which the ratio of the orbital periods can be written as the ratio of (fairly small) natural numbers. The orbital period of Neptune e.g. equals $2/3$ of the orbital period of Pluto. This phenomenon is called orbital resonance. Some ratios are stabilizing, others destabilizing; this provides an explanation for the density variation of the asteroid belts according to their distance to the sun (which unambiguously correlates with their orbital period).

The Kuramoto-Sakaguchi model is an extension of the Kuramoto model, which was introduced to allow for a mathematical description of the former

phenomenon, where the interaction between oscillators with different natural frequencies leads to the same long term average frequency for a subset of the oscillator population. As will be shown in this dissertation, the Kuramoto(-Sakaguchi) model also exhibits the latter phenomenon, for which the oscillators are stimulated to follow long term average frequencies with rational ratios.

1.2 The Kuramoto-Sakaguchi model

For a class of oscillators, for which the associated periodic solution is asymptotically stable, the behavior of weakly coupled oscillators can be modeled by

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \Lambda(\theta_j(t) - \theta_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N \triangleq \{1, \dots, N\},$$

with $N > 1$ the number of oscillators, K the coupling strength (with $|K|$ assumed to be small), Λ the coupling function, and θ_i and ω_i the phase and angular frequency (i.e. 2π divided by the period) of oscillator i . (In the remainder of this dissertation the word ‘angular’ will be omitted and ‘frequency’ will always denote angular frequency.) The phase θ_i can be seen as a parameter indicating the position on the trajectory of the periodic solution corresponding to the (isolated) system associated with oscillator i . An increase of θ_i with 2π corresponds to going through one cycle of this periodic solution. For $K = 0$ the associated differential equation reduces to $\dot{\theta}_i(t) = \omega_i$, leading to $\theta_i(t) = \theta_i(0) + \omega_i t$, $\forall t \in \mathbb{R}$, which exactly models the isolated system i . If a sufficiently weak interaction with other systems is introduced, the resulting trajectory will remain in a neighborhood of the original periodic trajectory, allowing us to parametrize the state of the system by the same phase variable θ_i through a mapping of this neighborhood onto the original trajectory. Details can be found in [44].

Notice that some of the examples from section 1.1 do not have asymptotically stable solutions and they will therefore not satisfy the previous explanation. It is for instance possible for asteroids, due to the fact that their orbits are not asymptotically stable, to be thrown out of their orbit by a (relatively) weak interaction with the planet Jupiter.

For the Kuramoto model [29] Λ is set equal to the sine function, resulting in the following differential equations:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j(t) - \theta_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N. \quad (1.1)$$

The ω_i are chosen randomly from a distribution g .

Kuramoto considered the limit $N \rightarrow \infty$ and assumed g to be unimodal (i.e. g attains exactly one maximum value) and even about the mode Ω_0 , i.e. $g(\Omega_0 - \omega) = g(\Omega_0 + \omega)$, $\forall \omega \in \mathbb{R}$. He discovered that there is a critical value K_c for the coupling strength K , above which there is a solution with a group of oscillators moving at the same frequency Ω_0 . The remaining oscillators are moving incoherently, with long term average frequencies between their natural frequencies and Ω_0 . This behavior is also called *partial synchronization*. For values of K smaller than K_c there is no partial synchronization and each oscillator follows its natural frequency.

The model (1.1) was extended by Kuramoto and Sakaguchi [45] as follows:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j(t) - \theta_i(t) - \alpha), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (1.2)$$

with $|\alpha| < \frac{\pi}{2}$. The distribution g of the natural frequencies is no longer assumed to be even. The behavior is similar to the behavior of the Kuramoto model, except for the fact that the frequency of the synchronized oscillators is harder to calculate and depends on the shape of g and the value of K .

Most research on the Kuramoto model with a finite number of oscillators focuses on simulations [35, 42] and/or is concerned with modifications such as an alternative interaction structure [25, 39, 16, 43, 38, 37], although some analytical results have been derived for special cases such as identical natural frequencies [50] or the case of complete phase locking behavior [11, 26]. For an infinite number of oscillators analytical results have been obtained for e.g. the noisy model [15], the periodically driven model [14], and the model with general periodic interaction functions [17]. For an overview see e.g. [47, 3].

In the first part of this dissertation we will show that the behavior of the system (1.2) with a finite number of oscillators resembles the behavior of the system with an infinite number of oscillators. For sufficiently large values of K one or more groups arise consisting of oscillators with *bounded* mutual phase differences. There is no partial synchronization, since in general the phase differences are time-varying. The system is said to exhibit *partial entrainment*. A sufficient condition can be formulated for the occurrence of partial entrainment with respect to a given set of oscillators. In general, increasing the coupling strength results in stronger entrainment: the different groups will grow and merge at distinct transition values for K , until there is entrainment with respect to the entire population (i.e. *full entrainment*). The transition values for the coupling strength in the Kuramoto model can be estimated using analytical results from [11]. However, we will also show that partial entrainment may disappear with increasing coupling strength, and that a similar phenomenon may be observed in arrays of Josephson junctions.

The analysis of the Kuramoto-Sakaguchi model with an infinite number of oscillators in chapter 3 starts from the solution Kuramoto and Sakaguchi investigated, corresponding to a unimodal distribution of the natural frequencies.

This allows us to study the influence of small modifications of the unimodal distribution on the solution of (1.2), leading to solutions exhibiting partial entrainment w.r.t. different groups of oscillators. (Partial synchronization cannot occur in more than one group of oscillators since phase differences between oscillators cannot be constant if there is more than one entrained group.) We will show that again partial entrainment can disappear with increasing coupling strength. Another phenomenon following from the analytical results is similar to orbital resonance of planets and asteroids. If there is partial entrainment w.r.t. several groups of oscillators, this will induce partial entrainment in other groups of oscillators for which the average frequencies are related to the average frequencies of the former groups. The phenomenon is related to results from [14], where the periodically driven Kuramoto model is considered.

1.3 Approximating the Kuramoto-Sakaguchi model

Although we have been able to derive analytical results for the behavior of the system (1.2), the difficulty level of a complete analytical treatment puts restrictions on these results. For the model with a finite number of oscillators we have only been able to derive sufficient conditions; since they are not necessary, this implies that there are sets of parameters for which our results cannot be applied. For the model with an infinite number of oscillators the distribution of the natural frequencies is restricted to perturbations of a unimodal distribution, and the results are only correct up to first order in the size of the perturbation. Furthermore the formulation of the results can be complicated, which may hinder an intuitive interpretation and makes it difficult to apply them.

We will introduce a system with a behavior similar to the partial entrainment behavior of the oscillators in the Kuramoto-Sakaguchi model. The sinusoidal interaction between different oscillators is approximated by a constant attractive interaction, which is no longer periodic; it is positive for lagging oscillators, and negative for leading oscillators. As a result, the model cannot exhibit phenomena typical for oscillators, such as the resonances described in the previous section. Furthermore we will consider different applications for this model, which are not related to oscillator systems. We will therefore refer to the elementary subsystems as *agents* instead of oscillators, and the partial entrainment behavior will be referred to as the *clustering* behavior. The mathematical notation is also altered; θ_i and ω_i are replaced by x_i and b_i , the interaction function is denoted by f .

1.4 The clustering model

The basic version of the clustering model with N interacting agents is described by

$$\dot{x}_i(t) = b_i + \frac{K}{N} \sum_{j=1}^N f(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (1.3)$$

where the interaction function f is Lipschitz continuous, odd, non-decreasing, and reaches a saturation value:

$$f(\pm x) = \pm F, \quad \forall x \geq d,$$

for some $d > 0$. In chapter 4 we will provide necessary and sufficient conditions characterizing the clustering behavior of the model, which is independent of the initial condition. When the coupling strength is increased from 0 on, N different cluster structures arise, starting with each agent in a separate cluster, and ending in one cluster containing all agents. In the generic case, each transition of the coupling strength between different cluster structures corresponds to the joining of two clusters. We offer an intuitive explanation, and postpone the mathematical details until the next chapter, where a more general version of the model is considered.

The model allows several extensions, while retaining analytical tractability. In chapter 5 we consider a general network structure with different interaction functions for each pair of agents, and we associate weights γ_i and sensitivity factors A_i with the agents:

$$\dot{x}_i(t) = b_i + K A_i \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (1.4)$$

where the functions f_{ij} satisfy $f_{ji}(x) = -f_{ij}(-x)$, $\forall x \in \mathbb{R}$, $\forall i, j \in \mathcal{I}_N$ with $i \neq j$. As a result of the general network structure, there is an important difference with the basic model: clusters may split up with increasing coupling strength, allowing more than N different cluster structures to appear when the coupling strength is varied.

In chapter 6 we consider (1.4), but the requirement for the interaction functions to reach their saturation values is relaxed to

$$\lim_{x \rightarrow \infty} f_{ij}(x) = F_{ij}.$$

Another extension consists in introducing time-dependence in the parameters and the interaction functions. Results similar to those for the time-invariant model can be derived. When there is periodicity, these results can be

obtained by replacing the time-variant parameters with their long term averages.

In chapter 8 we consider the limit $N \rightarrow \infty$ in the model (1.3), resulting in a formulation in terms of a partial differential equation. We derive necessary conditions for clustering behavior, and we describe how the cluster structure varies with varying coupling strength.

In the first section of chapter 9 the clustering model is compared to the Kuramoto-Sakaguchi model. To illustrate the broader potential of the clustering model, we also discuss applications that are not related to oscillator systems.

- We show that a system of water tanks interconnected by pipes with a maximal throughput can be described by (1.4). The results obtained for this model are useful for investigating flooding in this system.
- We indicate how the dynamics of opinion formation can be described by an adaptation of (1.4).
- We investigate the relation between platoon forming in cycling races and clustering behavior.

Compartmental systems, such as the aforementioned system of water tanks, are related to the minimum cost flow problem [12]. For the minimum cost flow problem the objective is to minimize a cost function associated with the flows through the arcs of a network, while satisfying continuity equations in the nodes. In chapter 10, we will discuss an approach to solving the minimum cost flow problem, based on the relation with compartmental systems; for well-chosen interaction functions the corresponding system (1.4) (with $K = A_i = \gamma_i = 1$ and f_{ij} not necessarily saturating) will asymptotically implement the solution to the minimum cost flow problem. This technique can be extended to more general optimization problems [5].

Part I

**The Kuramoto-Sakaguchi
model**

Chapter 2

Partial entrainment for a finite number of oscillators

In this chapter we investigate partial entrainment in the finite Kuramoto-Sakaguchi model. In the next section we illustrate and define the concept of partial entrainment. In section 2.2 we describe the general behavior of the Kuramoto-Sakaguchi model as it is observed in simulations, and in support of these observations we formulate a sufficient condition for the entrainment of a given subset of the population of oscillators in section 2.3. The proof implies persistence of the entrainment behavior under perturbations in the initial condition and the result remains non-trivial in the limit $N \rightarrow \infty$. Section 2.4 deals with an estimation of the critical values of the coupling strength defining the transitions between different forms of partial entrainment for the case $\alpha = 0$. In section 2.5 we illustrate the phenomenon for which, for both the Kuramoto-Sakaguchi model and a system of Josephson junction arrays, entrainment may disappear with increasing coupling strength.

2.1 Partial entrainment

For $N = 2$, (1.2) can be solved exactly. From

$$\begin{aligned}\dot{\theta}_1(t) - \dot{\theta}_2(t) &= \omega_1 - \omega_2 + \frac{K}{2} (\sin(\theta_2(t) - \theta_1(t) - \alpha) - \sin(\theta_1(t) - \theta_2(t) - \alpha)) \\ &= \omega_1 - \omega_2 + K \cos \alpha \sin(\theta_2(t) - \theta_1(t)),\end{aligned}$$

$\forall t \in \mathbb{R}$, and solving for $\theta_1(t) - \theta_2(t)$ one distinguishes three cases:

1. $K \cos \alpha > |\omega_1 - \omega_2|$: Restricting $\theta_1 - \theta_2$ to $[-\pi, \pi)$, there are two equilibrium values for $\theta_1 - \theta_2$, satisfying $K \cos \alpha \sin(\theta_1 - \theta_2) = \omega_1 - \omega_2$. The

one with $|\theta_1 - \theta_2| < \frac{\pi}{2}$ is asymptotically stable, the other one is unstable. It follows that the phase difference $\theta_1 - \theta_2$ will approach a constant value. This phenomenon is called *phase-locking*.

2. $K \cos \alpha = |\omega_1 - \omega_2|$: There is one attracting equilibrium point and the system will again approach a phase-locking state.
3. $K \cos \alpha < |\omega_1 - \omega_2|$: There are no equilibrium points and $\dot{\theta}_1 - \dot{\theta}_2$ has a fixed sign. The oscillators have a long term average frequency:

$$\lim_{t \rightarrow \infty} \frac{\theta_i(t)}{t} = \omega_i^{\text{lim}}, \quad \text{for some } \omega_i^{\text{lim}} \in \mathbb{R}, \quad \forall i \in \{1, 2\},$$

with

$$\omega_1^{\text{lim}} - \omega_2^{\text{lim}} = (\omega_1 - \omega_2) \sqrt{1 - \left(\frac{K \cos \alpha}{\omega_1 - \omega_2} \right)^2}.$$

For general N , phase-locking between oscillators will not occur unless there is phase-locking for *all* oscillators. However it is possible for oscillators to have bounded mutual phase differences, without these phase differences being or approaching a constant value. This is called *partial entrainment*.

Definition 2.1. Let θ be a solution of (1.2) (with N finite) and let $S_e \subset \mathcal{I}_N$ be non-empty. If

$$\exists C > 0 : \forall i, j \in S_e, \forall t \geq 0, |\theta_i(t) - \theta_j(t)| < C,$$

then the solution θ exhibits partial entrainment with respect to S_e , and S_e is called an entrained subset.

Notice that according to this definition there is always a trivial form of entrainment with respect to the singletons $\{i\} \subset \mathcal{I}_N$. Partial entrainment with respect to the entire population is called *full entrainment*. We will consider a different definition of partial entrainment for an infinite number of oscillators; this will be motivated in the next chapter.

Simulations of (1.2) reveal that each oscillator has a long term average frequency:

$$\forall i \in \mathcal{I}_N, \exists \omega_i^{\text{lim}} \in \mathbb{R} : \lim_{t \rightarrow \infty} \frac{\theta_i(t)}{t} = \omega_i^{\text{lim}},$$

and that for most choices of the parameters both the ω_i^{lim} -values and the entrainment behavior are independent of the initial condition. From the definition of partial entrainment it follows that all oscillators in the same entrained subset have equal ω_i^{lim} .

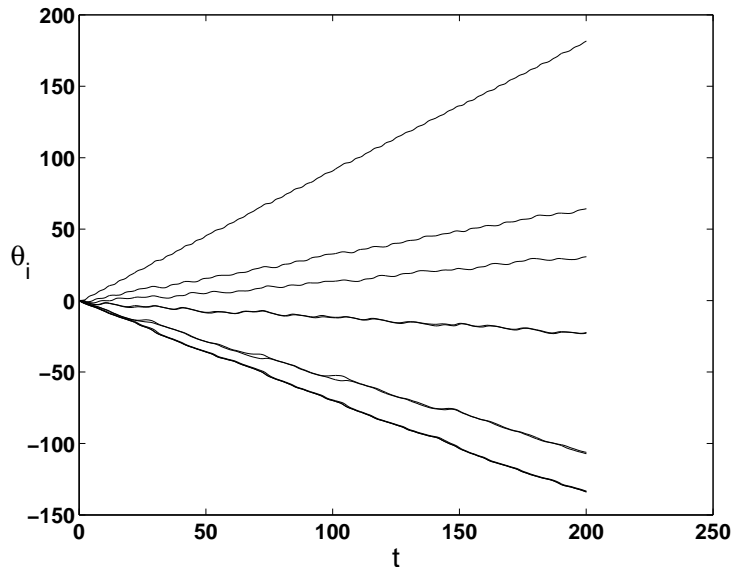


Figure 2.1: A simulation of (1.2) with $N = 10$ and $\alpha = \frac{\pi}{4}$.

Remark 2.1. If $\omega_i \leq \omega_j$ and $\theta_i(0) \leq \theta_j(0) + k2\pi$, for some $k \in \mathbb{Z}$, then $\theta_i(t) \leq \theta_j(t) + k2\pi$, $\forall t \in \mathbb{R}^+$, since if for some $t^* \geq 0$, $\theta_i(t^*) = \theta_j(t^*) + k2\pi$, then $\dot{\theta}_i(t^*) - \dot{\theta}_j(t^*) = \omega_i - \omega_j \leq 0$. It follows that $\omega_i \leq \omega_j$ implies that $\omega_i^{\text{lim}} \leq \omega_j^{\text{lim}}$.

In figure 2.1 an example is given of a simulation of a system with 10 oscillators and $\alpha = \frac{\pi}{4}$. If we order the oscillators such that $\omega_1 \leq \dots \leq \omega_{10}$, then there is partial entrainment with respect to the subsets $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7\}$, $\{8\}$, $\{9\}$ and $\{10\}$.

2.2 The general scenario

For most simulations of (1.2) with N small and $|\alpha|$ small ($\lesssim 0.5$) the entrainment behavior in terms of the coupling strength can be described as follows. If all ω_i are different then for $K = 0$ the only entrained subsets are trivially the singletons $\{i\}$, $1 \leq i \leq N$.

With increasing K , oscillators start to become entrained, enlarging sets already entrained. In general there are $N - 1$ bifurcation values $K_{c,k}$ ($k \in \{1, \dots, N-1\}$), each value representing a coupling strength where two entrained subsets (which subsets depends on the actual values of the ω_i) merge. After $N - 1$ transitions full entrainment occurs, which is investigated for $\alpha = 0$ in [11].

This scenario is clearly illustrated in figure 2.2, where the different long term average frequencies of the oscillators are plotted (horizontal axis) for varying coupling strength (vertical axis), for a system with $\alpha = 0$, consisting of four oscillators with natural frequencies given by $\omega_1 = -2.32$, $\omega_2 = -0.89$, $\omega_3 = 0.68$ and $\omega_4 = 1.23$.

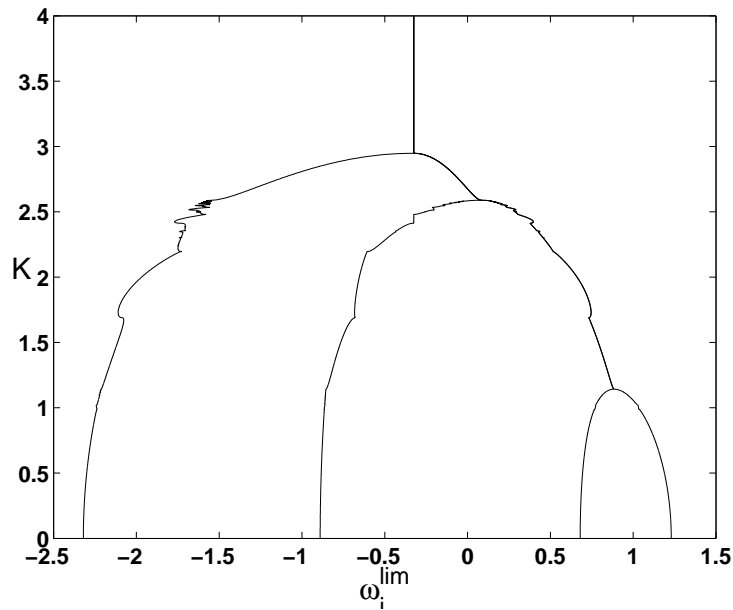


Figure 2.2: Long term average frequencies for varying coupling strength for a system of four oscillators. Different forms of partial entrainment can be distinguished: from no entrainment for small K , to full entrainment for large K .

2.3 Analytical results

2.3.1 A sufficient condition for partial entrainment

In support of the scenario described in the previous section, which is based on simulations, we provide an analytical result, proving that the Kuramoto-Sakaguchi model is able to exhibit partial entrainment: we derive a sufficient condition for the existence of a solution exhibiting partial entrainment with respect to a given set of oscillators. This result is not to be considered as an attempt to estimate critical values for the coupling strength — this will be dealt with in section 2.4 — but as an analytical proof of the existence of partial entrainment.

First we recall some results from [9], which concern the system (1.2) with $\alpha = 0$. Stronger but analytically more technical results follow.

Proposition 2.1. *Let S_e be a subset of \mathcal{I}_N with M elements and such that $M > \frac{N}{2}$. Assume that*

$$|\omega_i - \omega_j| < K \sqrt{\frac{N}{M}} \left(\frac{4M - 2N}{3N} \right)^{\frac{3}{2}}, \quad \forall i, j \in S_e.$$

Then there exists a solution of (1.1) that exhibits partial entrainment with respect to S_e .

Proposition 2.2. *Assume that $\omega_i = \omega_j = \tilde{\omega}$ for some $i \neq j$, both in \mathcal{I}_N . Set $S \triangleq \mathcal{I}_N \setminus \{i, j\}$. If $\omega_k \neq \tilde{\omega}$, $\forall k \in S$, then there exists an $\epsilon > 0$, such that $\forall K \in (0, \epsilon)$ the submanifolds defined by $\theta_i = \theta_j + 2\pi m$, $m \in \mathbb{Z}$, are locally asymptotically stable under the flow of (1.1).*

In this section we provide an extension to these propositions, based on the following proof of proposition 2.1.

Proof of proposition 2.1. For any $a \in (0, \frac{\pi}{2})$ let R_a denote the region

$$R_a = \{\theta \in \mathbb{R}^N : |\theta_i - \theta_j| \leq a, \forall i, j \in S_e\}.$$

We will determine a value for a for which R_a is a trapping region for (1.1). Assume that for some $t \in \mathbb{R}$ the solution of (1.1) at time t is located at the boundary of R_a : $\theta(t) \in R_a$ and $\theta_i(t) - \theta_j(t) = a$ for some $i, j \in S_e$. From (1.1) it follows that

$$\begin{aligned} \dot{\theta}_i(t) - \dot{\theta}_j(t) &= \omega_i - \omega_j - 2\frac{K}{N} \sin\left(\frac{\theta_i(t) - \theta_j(t)}{2}\right) \\ &\quad \times \sum_{k=1}^N \cos\left(\theta_k(t) - \frac{\theta_i(t) + \theta_j(t)}{2}\right). \end{aligned} \quad (2.1)$$

In the summation we can bound the terms for which $k \in S_e$ by

$$\cos\left(\theta_k(t) - \frac{\theta_i(t) + \theta_j(t)}{2}\right) \geq \cos a$$

since $\left|\theta_k(t) - \frac{\theta_i(t) + \theta_j(t)}{2}\right| \leq a$. (In fact $\left|\theta_k(t) - \frac{\theta_i(t) + \theta_j(t)}{2}\right| \leq \frac{a}{2}$, but this bound would result in more complicated calculations.)

If $k \notin S_e$ then $\cos\left(\theta_k(t) - \frac{\theta_i(t) + \theta_j(t)}{2}\right) \geq -1$, and thus

$$\dot{\theta}_i(t) - \dot{\theta}_j(t) \leq \omega_i - \omega_j - 2\frac{K}{N} \sin \frac{a}{2} (M \cos a - (N - M)).$$

For R_a to be a trapping region we need the right hand side to be negative. Minimizing this expression by choosing a appropriately leads to $\sin \frac{a}{2} = \sqrt{\frac{2M-N}{6M}}$, resulting in

$$\dot{\theta}_i(t) - \dot{\theta}_j(t) \leq \omega_i - \omega_j - K \sqrt{\frac{N}{M}} \left(\frac{4M-2N}{3N} \right)^{\frac{3}{2}} < 0,$$

and thus for this value of a R_a is a trapping region. Since R_a is non-empty we can choose an initial condition in R_a and the resulting solution of (1.1) will exhibit partial entrainment with respect to S_e . \square

To extend proposition 2.1 we will invoke extra knowledge about oscillators not in S_e to provide a better bound for the term $\cos \left(\theta_k(t) - \frac{\theta_i(t) + \theta_j(t)}{2} \right)$ in (2.1) with $k \notin S_e$. Although this term can attain its minimal value of -1 , if ω_k differs at least $2K$ from $\frac{\omega_i + \omega_j}{2}$, then it cannot remain -1 and it will also attain positive values. Using this property, we will provide a condition for which a solution of (1.2), starting within a region $R_{a'}$, with $a' \in (0, a)$, cannot leave R_a . For the proof we refer to section A.1.1 of the appendix.

Proposition 2.3. *Let $\{S_1, S_2, S_3\}$ be a partition of \mathcal{I}_N , with S_2 and S_3 possibly empty. Pick $m, M \in S_1$ such that $\omega_m = \min_{i \in S_1} \omega_i$ and $\omega_M = \max_{i \in S_1} \omega_i$ and assume that $|\omega_i - \frac{\omega_m + \omega_M}{2}| > 2K > 0, \forall i \in S_2$. Define $\tilde{\Theta}_j, \gamma_1, \tilde{\gamma}_2, \gamma_3$ and T by*

$$\begin{aligned} \tilde{\Theta}_j &\triangleq \arccos \left(\frac{4K}{4K + \pi \left(|\omega_j - \frac{\omega_m + \omega_M}{2}| - 2K \right)} \right), \quad \forall j \in S_2, \\ \gamma_i &\triangleq \frac{|S_i|}{N}, \quad \forall i \in \{1, 3\}, \\ \tilde{\gamma}_2 &\triangleq \frac{1}{N} \sum_{j \in S_2} \cos \tilde{\Theta}_j, \\ T &\triangleq \exp \left(\frac{2K}{N} \sum_{j \in S_2} \frac{\sin \tilde{\Theta}_j - \tilde{\Theta}_j \cos \tilde{\Theta}_j}{|\omega_j - \frac{\omega_m + \omega_M}{2}| - 2K} \right), \end{aligned}$$

assume that $|\alpha| < \frac{\pi}{6}$, $\delta \triangleq \gamma_1^2 \cos^2 \alpha - (\tilde{\gamma}_2 + \gamma_3)^2 > 0$ and that the inequality

$$\frac{\omega_M - \omega_m}{K} \leq \frac{4\sqrt{3}T \left(\delta^{\frac{3}{2}} - \sqrt{3} \gamma_1 |\sin \alpha| \delta \right)}{(3T^2(\gamma_1 \cos \alpha + \tilde{\gamma}_2 + \gamma_3) + \gamma_1 \cos \alpha - \tilde{\gamma}_2 - \gamma_3)(2\gamma_1 \cos \alpha + \tilde{\gamma}_2 + \gamma_3)}$$

holds. Then there exists a solution of (1.2) exhibiting partial entrainment with respect to S_1 .

2.3.2 Discussion

- If the set S_1 contains a considerable fraction of the oscillators, α is sufficiently close to zero, and the other oscillators have natural frequencies that differ largely from those in S_1 , such that δ is positive, then for sufficiently small frequency differences of the oscillators in S_1 a solution exhibiting partial entrainment with respect to S_1 is guaranteed to exist. If $|\alpha| \geq \frac{\pi}{6}$ a similar sufficient condition can still be derived by choosing \tilde{s} — defined in the proof — in a different way.
- To verify that proposition 2.3 is an extension of proposition 2.1, set $\alpha = 0$, $|S_1| = M$, $S_2 = \emptyset$ and thus $\tilde{\gamma}_2 = 0$ and $T = 1$ and assume that $M > \frac{N}{2}$ to obtain the condition (from proposition 2.3)

$$\frac{\omega_M - \omega_m}{K} \leq \frac{4\sqrt{3}\sqrt{N}(2M - N)^{\frac{3}{2}}}{2(M + N)^2}.$$

Since

$$\frac{4\sqrt{3}\sqrt{N}(2M - N)^{\frac{3}{2}}}{2(M + N)^2} = \frac{2\sqrt{3}\sqrt{M}(3N)^{\frac{3}{2}}}{2^{\frac{3}{2}}(M + N)^2} \sqrt{\frac{N}{M}} \left(\frac{4M - 2N}{3N} \right)^{\frac{3}{2}},$$

with

$$\frac{2\sqrt{3}\sqrt{M}(3N)^{\frac{3}{2}}}{2^{\frac{3}{2}}(M + N)^2} \geq \frac{2\sqrt{3}\sqrt{\frac{N}{2}}(3N)^{\frac{3}{2}}}{2^{\frac{3}{2}}(2N)^2} = \frac{9}{8},$$

it follows that proposition 2.3 entails proposition 2.1.

- This result also implies that entrainment will persist under perturbations: for a perturbed initial condition $\tilde{\theta}(0)$, with $\tilde{\theta}(0) \in R_{a'}$ — defined in the proof — and $\tilde{\theta}_m(0) \leq \tilde{\theta}_i(0) \leq \tilde{\theta}_M(0)$, $\forall i \in S_1$, it follows from the proof that the entrainment with respect to S_1 will be maintained.
- Notice also that the proposition remains non-trivial for $N \rightarrow \infty$: if S_1 , S_2 and S_3 contain oscillators with natural frequencies in prescribed intervals, then in general $\omega_M - \omega_m$, γ_1 , $\tilde{\gamma}_2$, γ_3 , δ and T will approach non-zero constant values for $N \rightarrow \infty$.
- The condition $|\omega_i - \omega_j| > 2K$ for some $i, j \in S_1$ is obviously sufficient to exclude partial entrainment with respect to S_1 since this implies that $|\theta_i(t) - \theta_j(t)|$ will grow unbounded. For some configurations this condition together with proposition 2.3 allows us to determine a maximal entrained subset, i.e. a subset of \mathcal{I}_N for which partial entrainment can occur and which is not included in another entrained subset. If, based on

the analytical results, the population can be partitioned in maximal entrained subsets, then the entire entrainment behavior can be determined on analytical grounds and all entrained subsets can be identified.

2.3.3 An asymptotic stability result

We will show that, in case the entrained subset S_1 contains oscillators with equal natural frequencies, the submanifolds where the oscillators in S_1 have equal phases are asymptotically stable for small values of the coupling strength, under a mild extra condition.

Assume all oscillators in S_1 have the same natural frequency: $\omega_i = \tilde{\omega}$, $\forall i \in S_1$, for some $\tilde{\omega} \in \mathbb{R}$. Let S_3 contain the oscillators with natural frequency equal to $\tilde{\omega}$ that are not included in S_1 ; then S_2 contains all oscillators with natural frequency different from $\tilde{\omega}$. (Notice that, by remark 2.1, every solution of (1.2) will exhibit partial entrainment with respect to S_1 .) Assume that K is sufficiently small for the condition of proposition 2.3 on the oscillators in S_2 to be valid. As is shown in section A.1.2 of the appendix we can adapt the proof of proposition 2.3 to obtain the following extension of proposition 2.2.

Proposition 2.4. *Assume that $\omega_{i_1} = \dots = \omega_{i_P} = \tilde{\omega}$ for some $i_1, \dots, i_P \in \mathcal{I}_N$, with $2 \leq P \leq N$, and that no other oscillators have an ω_i -value equal to $\tilde{\omega}$. Then for any $M \in \mathbb{Z}$, with $M > \frac{P}{1 + \cos \alpha}$, there exists an $\epsilon > 0$, such that $\forall K \in (0, \epsilon)$ the submanifolds defined by $\theta_{i_1} + 2\pi m_{i_1} = \dots = \theta_{i_M} + 2\pi m_{i_M}$, $(m_{i_1}, \dots, m_{i_M}) \in \mathbb{Z}^M$, are locally asymptotically stable under the flow of (1.2).*

2.4 Estimation of the transition values for $\alpha = 0$

The value of the coupling strength calculated from proposition 2.3, guaranteeing partial entrainment of a given subset, may be quite conservative. In this section a better estimation for the actual transition value is given for the case $\alpha = 0$.

Proposition 2.3 (and its proof) suggest(s) that oscillators which differ largely in natural frequency will have small mutual influence as to the entrainment behavior. Simulations confirm this and indicate that this is already true for much smaller frequency differences than suggested by the analytical results. We estimate the critical values for the coupling strength, defining the transitions between different forms of partial entrainment behavior, by neglecting the interactions between oscillators from different entrained subsets and using analytical results from [11]. Since the latter results are concerned with the case $\alpha = 0$, we will restrict α to be zero throughout this section.

We estimate the entrainment behavior with respect to a subset S_e by disregarding all oscillators not belonging to S_e , and we determine the coupling strength K_c for which full entrainment of S_e would occur. This is done by

numerically solving the following equations for K and r , obtained from [11]. (See also (2.3) and (2.4) in section 2.5.1.)

$$r = \frac{1}{N} \sum_{j \in S_e} \sqrt{1 - \left(\frac{\omega'_j}{Kr}\right)^2}, \quad \sum_{j \in S_e} \frac{1 - 2\left(\frac{\omega'_j}{Kr}\right)^2}{\sqrt{1 - \left(\frac{\omega'_j}{Kr}\right)^2}} = 0. \quad (2.2)$$

In these equations $\omega'_j \triangleq \omega_j - \frac{1}{|S_e|} \sum_{i \in S_e} \omega_i$.

This procedure is also supported by observations from the model with an infinite number of oscillators. For $\alpha = 0$ and with the distribution g of the natural frequencies even, the partially synchronized subset in the solution mentioned in the introduction is independent of the shape of the distribution g outside the region corresponding to the frequencies of the oscillators in the partially synchronized subset. (See e.g. [47] for mathematical details.)

2.4.1 Comparison and discussion

We compare the critical values for K resulting from the above analytical estimation based on equalities (2.2) with the values for K derived from propositions 2.1 and 2.3, and from simulations. We considered (1.2) for $\alpha = 0$ and with $N = 100$, and we randomly picked 100 natural frequencies from the distribution g defined by

$$g(\omega) = \begin{cases} C \frac{1-0.0001|\omega|}{0.0001+|\omega|}, & |\omega| \leq 10000, \\ 0, & |\omega| > 10000, \end{cases}$$

where $C > 0$ is such that $\int_{-\infty}^{+\infty} g(\omega) d\omega = 1$. (The expression for $g(\omega)$ is a slight modification of $1/\omega$, to guarantee that g can be normalized to 1.) The frequencies were ordered by their absolute values.

For the calculation of the K -values according to propositions 2.1 and 2.3 and equation (2.2), we then considered the entrainment of the subsets $S_e = \{1, \dots, M\}$, with $M > 1$. For each value of M , propositions 2.1 and 2.3 provide a minimal value of K for which entrainment of S_e is guaranteed. Together with the estimation resulting from the equations (2.2) and the value given by the simulations, these are shown in figure 2.3 for varying M . In the simulations, the entrained subsets may differ from the sets $\{1, \dots, M\}$, but in most cases they are equal. (In the case represented by figure 2.3 this holds for 95% of the M -values.)

Since proposition 2.1 imposes that $M > N/2$, the corresponding curve only starts at $M = 51$. Also proposition 2.3 does not generate results for any value of M , but the condition is less stringent and the corresponding curve starts for a value of $M < 51$. (Although for several other distributions of the natural

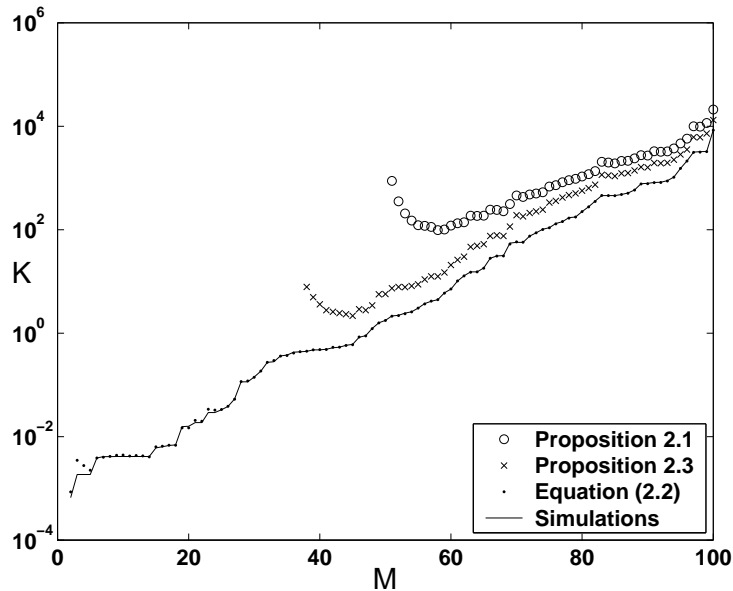


Figure 2.3: Comparison of values for the coupling strength related to entrainment of the subset $\{1, \dots, M\}$ for different procedures.

frequencies this curve also starts at $M = 51$, the present form of g shows that this does not always hold.) For the curve associated with proposition 2.3 the K -values are closer to those obtained from the simulations than for the curve associated with proposition 2.1, but, considering the logarithmic scale, both are unsuitable as estimations for the transition values between different forms of partial entrainment.

For this choice of the natural frequencies, the estimation based on equations (2.2) corresponds quite well with the simulation results, justifying the assumption that oscillators outside an entrained subset have little influence on the entrainment of this subset.

2.5 Entrainment break up with increasing K

The general scenario does not always hold; one of the points we want to draw attention to is that entrained oscillators may break up with increasing K , a phenomenon reported before in [36, p. 46] for the case $\alpha = 0$. We consider a particular system with four oscillators, and also $\alpha = 0$. Observe in figure 2.4 that there is a critical value for the coupling strength (≈ 0.313) above which the entrainment of a subset breaks up. Further increase of K reestablishes the entrainment.

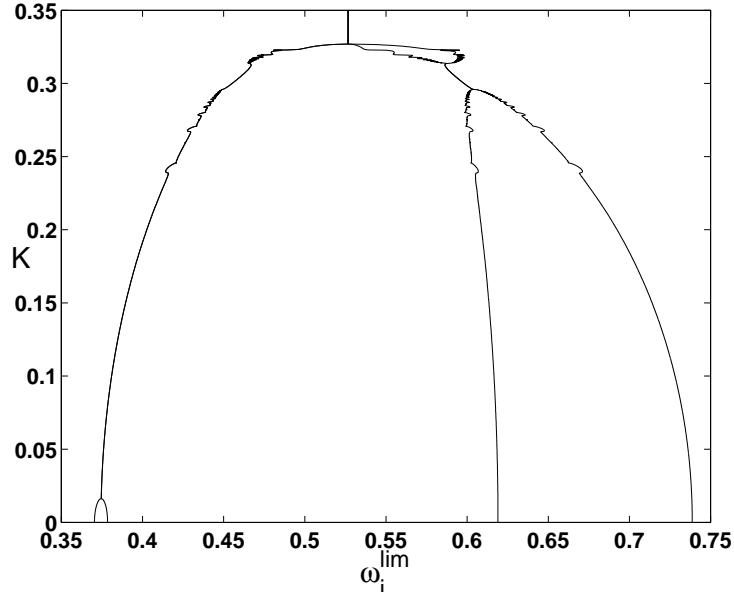


Figure 2.4: Long term average frequencies for varying coupling strength for a system of four oscillators. When K is increased from 0.3 to 0.35 there is a transition in which entrainment of two oscillators disappears. It reappears at the transition to full entrainment.

We offer an intuitive explanation. Denote the oscillators by 1, 2, 3 and 4, according to the order of their natural frequencies ω_i (i.e. $\omega_1 < \omega_2 < \omega_3 < \omega_4$).

As we have already mentioned in previous sections, the interaction between two oscillators appears to decrease with increasing difference in their natural frequencies. This implies that oscillator 3 will be subject to a stronger attraction towards oscillators 1 and 2 than oscillator 4. With increasing coupling strength K this attraction, and also the difference in attraction from oscillators 1 and 2 on oscillator 3 and on oscillator 4, will increase. For some value of K oscillators 3 and 4 become entrained, but when K is increased further, the increase in attraction difference becomes more important than the increased attraction between oscillators 3 and 4, making it possible for partial entrainment to disappear.

For $\alpha = 0$ simulation results indicate that entrainment of two oscillators in a system of only three oscillators cannot disappear with increasing coupling strength. The probability of entrainment disappearing with increasing coupling strength seems to increase with $|\alpha|$, and for $|\alpha|$ sufficiently large it can also be observed in a system of three oscillators.

2.5.1 Persistence of full entrainment with increasing K

The explanation offered in the previous paragraph implies that a given entrained subset can break up with increasing coupling strength only if other oscillators are present, and suggests that entrainment of the entire population cannot disappear with increasing coupling strength. For $\alpha = 0$ this is confirmed as follows by analytical results that can be derived from [11]. In that paper, it was proven that the existence of a locally stable phase-locked solution (which corresponds to full entrainment) is equivalent to the existence of a solution $r \in (0, 1]$ of (see also (2.2) in section 2.4)

$$\varphi_1(r, K) \triangleq r - \frac{1}{N} \sum_{i=1}^N \sqrt{1 - \left(\frac{\omega'_i}{Kr}\right)^2} = 0 \quad (2.3)$$

(with $\omega'_i \triangleq \omega_i - \frac{1}{N} \sum_{j=1}^N \omega_j$; $Kr \geq \max_i \omega'_i$), satisfying

$$\varphi_2(r, K) \triangleq r - \frac{1}{N} \sum_{i=1}^N \frac{\left(\frac{\omega'_i}{Kr}\right)^2}{\sqrt{1 - \left(\frac{\omega'_i}{Kr}\right)^2}} > 0. \quad (2.4)$$

Assume that r and K satisfy both (2.3) and (2.4) and take $K' > K$. It follows that $\varphi_1(r, K') \leq 0$, while clearly $\varphi_1(1, K') \geq 0$, implying the existence of an $r' \in [r, 1]$ with $\varphi_1(r', K') = 0$. From $r' \geq r$, $K'r' > Kr$, and $\varphi_1(r', K') = 0$ it then follows that

$$\varphi_2(r', K') \geq \varphi_2(r, K) > 0,$$

implying the existence of a solution of (1.2) for a coupling strength K' which exhibits entrainment of the entire population. This result implies that the solution of (1.2) corresponding to entrainment of the entire population will persist with increasing coupling strength (for the case $\alpha = 0$).

2.5.2 Josephson junctions

The relation between the Kuramoto-Sakaguchi model and arrays of Josephson junctions [51, 52] suggests that the phenomenon of destruction of entrainment with increasing K may also be observed in Josephson junction arrays. Simulations confirm this. For a Josephson junction characterized by a phase difference ϕ the voltage and current across the junction are given by $\frac{\hbar}{2e} \dot{\phi}$ and $I_C \sin \phi$ respectively, where \hbar is Planck's constant divided by 2π , e denotes the elementary electrical charge and the constant I_C is the critical current of the junction. We consider the same circuit as in [52] (see figure 2.5): a (parallel) connection of a bias current I_B , N different junctions in series, and a load with inductance L ,

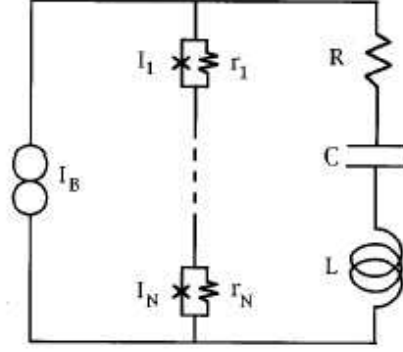


Figure 2.5: Parallel connection of a bias current, a series of Josephson junctions, and an inductive, resistive and capacitive load. The figure was taken from [52].

resistance R and capacitance C , with the charge on the capacitor denoted by Q . For junction i the phase difference, resistance and critical current are denoted by ϕ_i , r_i and I_i respectively; we assume its capacitance can be neglected. The system equations can then be written as

$$\frac{\hbar}{2er_i} \dot{\phi}_i + I_i \sin \phi_i + \dot{Q} = I_B, \quad \forall i \in \mathcal{I}_N,$$

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \frac{\hbar}{2e} \sum_{i=1}^N \dot{\phi}_i.$$

In [52] it was shown that in the limit of weak coupling and weak disorder this system can be cast into the model (1.2). Setting $N = 4$, $I_B = 1.5$ mA, $R = 50 \Omega$, $L = 25$ pH, $C = 0.04$ pF, $r_i = 0.5 \Omega$, $\forall i \in \mathcal{I}_N$ and $I_i = \left(0.5 + \frac{I'_i}{\beta_I}\right)$ mA, with $I' = (-0.057, -0.021, 0.0075, 0.0165, 0.021)$, (most values are taken from [52]), we calculate the long term average frequencies (i.e. the values $\omega_i^{\text{lim}} \triangleq \lim_{t \rightarrow \infty} \frac{\phi_i(t) - \phi_i(0)}{t}$) for varying β_I , representing different levels of disorder. The result is shown in figure 2.6. In spite of the irregular shape of the graph (which persists when simulating with different time steps or over different time intervals, excluding simulation errors as the cause of the irregularity), different phenomena can be observed. Enumerating the junctions from 1 to 5, such that $\omega_1^{\text{lim}} \leq \omega_2^{\text{lim}} \leq \omega_3^{\text{lim}} \leq \omega_4^{\text{lim}} \leq \omega_5^{\text{lim}}$ (with strict inequalities e.g. for $\beta_I = 9.6$, notice that for $\beta_I = 9.2$ junctions 2 and 3 have already become entrained), the following transitions (with increasing β_I) are clearly visible:

- $\beta_I \approx 9.36$ junction 2 leaves junction 3, and joins junction 1 at $\beta_I \approx 9.71$;
- $\beta_I \in (10.03, 10.16)$: entrainment of junctions 1 and 2 temporarily disap-

pears;

- $\beta_I \in (9.31, 10)$: junctions 4 and 5 are entrained but for several subintervals the entrainment disappears;
- $\beta_I \approx 10.17$: junction 4 (having left junction 5 at $\beta_I \approx 10$) becomes entrained with junction 3.

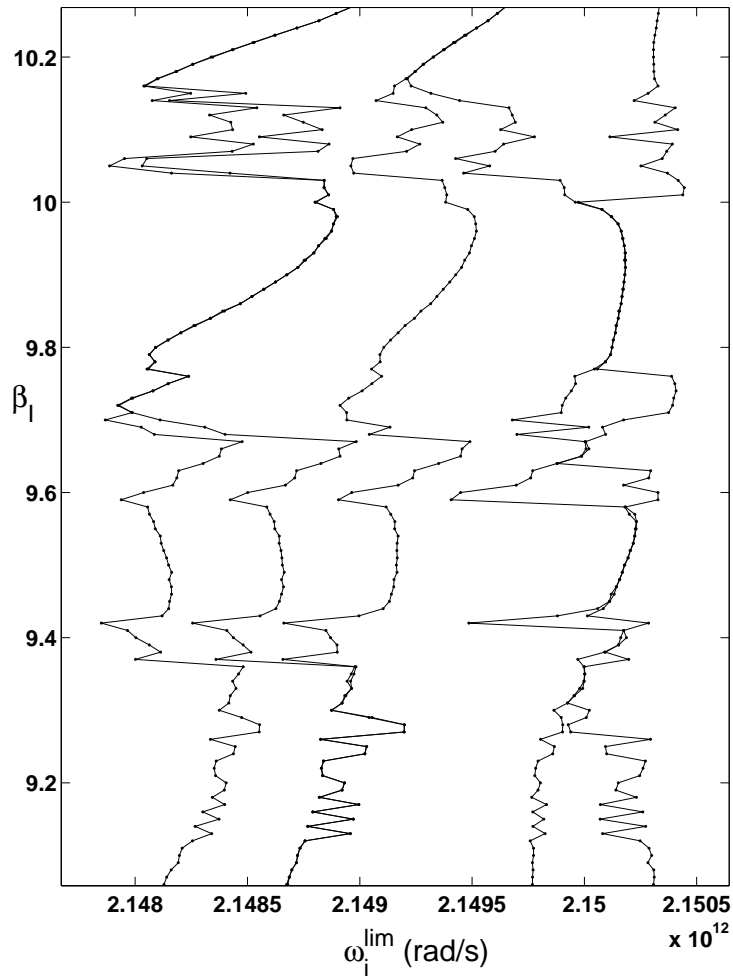


Figure 2.6: Long term average frequencies for varying levels of disorder for an array of 5 Josephson junctions. When β_I is increased there are different transitions in which entrainment of two junctions disappears.

2.6 Conclusion

For the Kuramoto-Sakaguchi model we can formulate a sufficient condition for partial entrainment of a given subset of oscillators; the result implies persistence of the entrainment behavior under perturbations and remains non-trivial in the limit $N \rightarrow \infty$.

Considering that the long term average interaction between oscillators decreases with increasing difference in their natural frequencies, we can estimate the critical value of the coupling strength defining the onset of partial entrainment of a given subset by neglecting oscillators which do not belong to this subset.

Simulations indicate that entrainment can disappear with increasing coupling strength in the Kuramoto-Sakaguchi model, and that a similar phenomenon can be observed in arrays of Josephson junctions, where it is also possible that a junction leaving one entrained subset joins another entrained subset.

Chapter 3

Partial entrainment for an infinite number of oscillators

In this chapter we investigate a perturbation of the partially synchronized solution of the Kuramoto-Sakaguchi model with an infinite number of oscillators. The perturbation consists of adding a small fraction of oscillators with natural frequencies in small intervals, corresponding to some narrow extra peaks in the distribution, and leading to extra entrained subsets (for appropriate values of the coupling strength K). In a first order approximation with respect to their size, the extra entrained subsets can be characterized analytically by a set of equalities.

For a particular choice of the distribution we show that these extra entrained subsets may disappear when the coupling strength is increased, a phenomenon which is described in the previous chapter for a finite number of oscillators. The analysis also predicts that the extra entrained subsets may induce entrainment in other subsets of oscillators, which are not necessarily associated with high frequency densities. Both phenomena are also mentioned in [36, p. 46-47], but without further elaboration. The second phenomenon has similarities with the occurrence of Shapiro steps in the periodically driven Kuramoto model [14].

We will express the order parameter r in terms of the oscillator density ρ . In sections 3.3 and 3.4 a particular solution for the density ρ is expressed in terms of r , allowing us to write down a self-consistency equation for r , which is investigated in section 3.5. Since the mathematical analysis is somewhat tedious, some of the calculations have been relegated to the appendix. We conclude the chapter with two examples.

3.1 The system equations for $N = \infty$

In the previous chapter a solution was said to exhibit partial entrainment with respect to a set S_e if the mutual phase differences of all oscillators in S_e remain bounded. For an infinite number of oscillators, two possibilities can be distinguished. Either the set S_e can be expanded to an entrained set containing all oscillators with natural frequencies in an interval with non-zero length (see also remark 2.1), or such an expansion is not possible and then the set S_e only contains oscillators with the same natural frequency. In this chapter, the latter case will not be considered as partial entrainment (see also definition 3.1 below); the corresponding oscillators will be called *drifting* oscillators. The set containing the frequencies of all entrained (i.e. non-drifting) oscillators will be denoted by \mathcal{E} .

We will describe the model (1.2) with infinite N in terms of a population density $\rho(\theta, \omega, t)$, which we consider to be periodic in θ with period 2π : the fraction of the population of oscillators with a natural frequency equal to ω and a phase value in $\cup_{m \in \mathbb{Z}} [\theta + 2\pi m, \theta + d\theta + 2\pi m]$ at time t equals $\rho(\theta, \omega, t)d\theta$ for infinitesimally small values of $d\theta$. This implies that $\int_{-\pi}^{\pi} \rho(\theta, \omega, t)d\theta = 1$, $\forall \omega, t \in \mathbb{R}$.

Let $\tilde{\theta}(\theta_i, \omega, t)$ denote the phase of an oscillator with initial condition θ_i (i.e. $\tilde{\theta}(\theta_i, \omega, 0) = \theta_i$, $\forall \theta_i, \omega \in \mathbb{R}$) and natural frequency ω ; then

$$\frac{\partial \tilde{\theta}}{\partial t}(\theta_i, \omega, t) = \omega + \int_{-\infty}^{+\infty} d\omega' g(\omega') \int_{-\pi}^{\pi} d\theta' \rho(\theta', \omega', t) \sin(\theta' - \tilde{\theta}(\theta_i, \omega, t) - \alpha),$$

$$\forall \theta_i, \omega, t \in \mathbb{R}. \quad (3.1)$$

If we set

$$v(\tilde{\theta}(\theta_i, \omega, t), \omega, t) \triangleq \frac{\partial \tilde{\theta}}{\partial t}(\theta_i, \omega, t), \quad \forall \theta_i, \omega, t \in \mathbb{R},$$

such that $v(\theta, \omega, t)$ represents the velocity of an oscillator with natural frequency ω and phase θ at time t , then the evolution of ρ is determined by the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial \theta}, \quad (3.2)$$

with

$$v(\theta, \omega, t) = \omega + \int_{-\infty}^{+\infty} d\omega' g(\omega') \int_{-\pi}^{\pi} d\theta' \rho(\theta', \omega', t) \sin(\theta' - \theta - \alpha), \quad \forall \theta, \omega, t \in \mathbb{R}. \quad (3.3)$$

Similarly as in remark 2.1, it follows that, if $\theta_1 \leq \theta_2 + k2\pi$, for some $k \in \mathbb{Z}$, and $\omega_1 \leq \omega_2$, then $\tilde{\theta}(\theta_1, \omega_1, t) \leq \tilde{\theta}(\theta_2, \omega_2, t) + k2\pi$, $\forall t \in \mathbb{R}^+$.

Definition 3.1. Let $\tilde{\theta}$ be a solution of (3.1), and let $I \subset \mathbb{R}$ be a non-empty interval. If, $\forall \theta_1, \theta_2 \in \mathbb{R}$, and $\forall \omega_1, \omega_2 \in I$, $\exists C > 0 : \forall t \geq 0$,

$$\left| \tilde{\theta}(\theta_1, \omega_1, t) - \tilde{\theta}(\theta_2, \omega_2, t) \right| < C,$$

then the solution $\tilde{\theta}$ exhibits partial entrainment with respect to I , and I is called an entrained subset.

3.2 The order parameter

We will consider a perturbation of the solution Kuramoto and Sakaguchi investigated. The unperturbed solution involves one partially synchronized group of oscillators (i.e. partially entrained but with constant mutual phase differences; notice that other definitions of synchronization do exist which demand the phase differences to be zero) moving at a constant velocity Ω_0 , while the other (drifting) oscillators are moving with average velocities different from Ω_0 . The perturbation consists in a (small) fraction of the drifting oscillators becoming entrained.

For the perturbed solution, define Ω_0 as the long term average frequency of the largest entrained subset (the one that corresponds to the synchronized subset for the unperturbed solution). Switching to a rotating frame with constant frequency Ω_0 results in similar system equations but with ω replaced by $\omega - \Omega_0$. In the rotating frame, the largest entrained subset is moving on average at zero velocity, which allows us to look for an almost stationary solution. Repeating the reasoning from the previous section in this rotating frame then results in a density function ρ_R , a velocity function v_R , and a distribution of the natural frequencies g_R given by

$$\begin{aligned} \rho_R(\theta, \omega, t) &= \rho(\theta - \Omega_0 t, \omega, t), \\ v_R(\theta, \omega, t) &= v(\theta - \Omega_0 t, \omega, t) - \Omega_0, \\ g_R(\omega) &= g(\Omega_0 + \omega), \end{aligned}$$

$\forall \theta, \omega, t \in \mathbb{R}$. For notational convenience we will drop the index R of ρ_R and v_R (not for g_R , since we will need both the functions g and g_R later on; all further occurrences of ρ and v are thus w.r.t. the rotating frame).

Define the real-valued functions r and ψ by

$$r(t)e^{i\psi(t)} = \int_{-\infty}^{+\infty} d\omega' g_R(\omega') \int_{-\pi}^{\pi} d\theta' \rho(\theta', \omega', t) e^{i(\theta' - C_0)}, \quad \forall t \in \mathbb{R}, \quad (3.4)$$

where $C_0 \in \mathbb{R}$ will be determined later. By multiplying (3.4) with $e^{i(C_0 - \theta - \alpha)}$ and considering the imaginary part, (3.3) can be rewritten as

$$v(\theta, \omega, t) = \omega + Kr(t) \sin(\psi(t) + C_0 - \theta - \alpha), \quad \forall \theta, \omega, t \in \mathbb{R}. \quad (3.5)$$

The parameter $r(t)$ can be seen as an order parameter, since when all oscillators are close together, $r(t)$ will be close to one, and when they are spread uniformly over the interval $[-\pi, \pi]$ $r(t)$ will be zero. Since we consider a perturbation of a stationary solution we set $r(t)e^{i\psi(t)} = (r_0 + r'(t))e^{i\psi_0}$, where $r_0 e^{i\psi_0}$ (with $r_0 \in \mathbb{R}_0^+$) equals the long term average value of $r(t)e^{i\psi(t)}$ (which we assume to exist and to be different from zero), and the complex-valued perturbation r' has real part r'_R and imaginary part r'_I . We set $C_0 \triangleq \alpha - \psi_0$, allowing us to rewrite (3.5) as

$$v(\theta, \omega, t) = \omega - K(r_0 + r'_R(t)) \sin \theta + Kr'_I(t) \cos \theta, \quad \forall \theta, \omega, t \in \mathbb{R}, \quad (3.6)$$

and (3.4) as

$$r_0 + r'_R(t) + ir'_I(t) = e^{-i\alpha} \int_{-\infty}^{\infty} d\omega' g_R(\omega') \int_{-\pi}^{\pi} d\theta' \rho(\theta', \omega', t) e^{i\theta'}, \quad \forall t \in \mathbb{R}. \quad (3.7)$$

We assume that r'_R , r'_I and r' can be represented by Fourier-sums as follows:

$$\begin{aligned} r'_R(t) &= \sum_{\gamma \in \Gamma} R'_R(\gamma) e^{i\gamma t}, \\ r'_I(t) &= \sum_{\gamma \in \Gamma} R'_I(\gamma) e^{i\gamma t}, \\ r'(t) &= \sum_{\gamma \in \Gamma} R'(\gamma) e^{i\gamma t}, \\ &= \sum_{\gamma \in \Gamma} (R'_R(\gamma) + iR'_I(\gamma)) e^{i\gamma t}, \end{aligned}$$

$\forall t \in \mathbb{R}$, where $|R'_R|^2 + |R'_I|^2 : \mathbb{R} \rightarrow \mathbb{C}$ is zero everywhere, except on the set $\Gamma \in \mathbb{R}$, for which we assume that $\Gamma \cap [a, b]$ is a finite set, for any $a, b \in \mathbb{R}$ with $a < b$. Furthermore we can assume that $0 \notin \Gamma$ because of the definition of r' , and since r'_R and r'_I are real-valued functions, $\gamma \in \Gamma$ implies $-\gamma \in \Gamma$. We will find a self-consistency equation for $r'(t)$ by expressing the right hand side of (3.7) up to first order in $r'_{\text{RMS}} \in \mathbb{R}^+$, with

$$r'_{\text{RMS}}{}^2 \triangleq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |r'(t')|^2 dt' = \sum_{\gamma \in \Gamma} |R'(\gamma)|^2;$$

r'_{RMS} is assumed to be bounded and small with respect to r_0 and $\min_{\gamma \in \Gamma} \frac{|\gamma|}{K}$.

We separately consider two contributions to the density ρ :

$$\rho(\theta, \omega, t) = \rho_e(\theta, \omega, t) + \rho_d(\theta, \omega, t), \quad \forall \theta, \omega, t \in \mathbb{R}.$$

The first contribution ρ_e corresponds to oscillators belonging to an entrained subset, and is restricted to satisfy

$$\rho_e(\theta, \omega, t) = \delta\left(\theta - \tilde{\theta}(\theta_i(\omega), \omega, t)\right), \quad \forall \theta, t \in \mathbb{R}, \forall \omega \in \mathcal{E},$$

where the initial phase $\theta_i(\omega)$ will be determined later. The second contribution ρ_d corresponds to drifting oscillators. It follows that $\rho_e(\theta, \omega, t) = 0, \forall \omega \in \mathbb{R} \setminus \mathcal{E}, \forall \theta, t \in \mathbb{R}$, and $\rho_d(\theta, \omega, t) = 0, \forall \omega \in \mathcal{E}, \forall \theta, t \in \mathbb{R}$.

We split ρ_d in a stationary part $\rho_{d,0}$ and a perturbation ρ'_d :

$$\rho_d(\theta, \omega, t) = \rho_{d,0}(\theta, \omega) + \rho'_d(\theta, \omega, t), \quad \forall \theta, t \in \mathbb{R}, \quad \forall \omega \in \mathbb{R} \setminus \mathcal{E}.$$

The density $\rho_{d,0}$ corresponds to a stationary solution of the continuity equation (3.2), with v replaced with v_0 , where $v_0(\theta, \omega) \triangleq \omega - Kr_0 \sin \theta, \forall \theta, \omega \in \mathbb{R}$:

$$\rho_{d,0}(\theta, \omega) \triangleq \frac{\sqrt{1 - \left(\frac{Kr_0}{\omega}\right)^2}}{2\pi \left(1 - \frac{Kr_0}{\omega} \sin \theta\right)}, \quad \forall \theta \in \mathbb{R}, \forall \omega \in \mathbb{R} \setminus \mathcal{E}.$$

(It will be shown later on that oscillators with $|\omega| < Kr_0$ belong to the entrained subset moving at zero average velocity, implying that $Kr_0 \leq |\omega|, \forall \omega \in \mathbb{R} \setminus \mathcal{E}$.)

In the next sections we will calculate the first order approximations of the contributions of ρ_d and ρ_e to the right hand side of (3.7). The solution $\tilde{\theta}$ corresponding to the entrained oscillators will be derived from (3.6). The expression for $\tilde{\theta}$ must be correct up to first order for oscillators in the entrained subset at zero average velocity. The other entrained subsets only contain a fraction of the oscillators of the order of r'_{RMS} and therefore first order terms in $\tilde{\theta}$ will appear as second order terms when substituted in (3.7), and can be neglected. The density ρ'_d will be derived from the first order approximation of (3.2). The resulting function ρ'_d itself will not necessarily be correct up to first order for all possible values of its arguments, but we assume that this will still hold for its contribution to (3.7).

3.3 The drifting oscillators

Regarding ρ'_d , we write down the continuity equation (3.2) for $\omega \in \mathbb{R} \setminus \mathcal{E}$, with v given by (3.6):

$$\begin{aligned} \frac{\partial \rho'_d}{\partial t}(\theta, \omega, t) &= -\frac{\partial}{\partial \theta} ((\rho_{d,0}(\theta, \omega) + \rho'_d(\theta, \omega, t)) \\ &\quad \times (\omega - K(r_0 + r'_R(t)) \sin \theta + Kr'_I(t) \cos \theta)) \\ &= -\frac{\partial}{\partial \theta} (\rho'_d(\theta, \omega, t)(\omega - Kr_0 \sin \theta)) \\ &\quad - \frac{\partial}{\partial \theta} ((\rho_{d,0}(\theta, \omega) + \rho'_d(\theta, \omega, t))(-Kr'_R(t) \sin \theta + Kr'_I(t) \cos \theta)), \end{aligned}$$

and we consider the first order approximation

$$\begin{aligned} \frac{\partial \rho'_d}{\partial t}(\theta, \omega, t) &= -\frac{\partial}{\partial \theta} (\rho'_d(\theta, \omega, t)(\omega - Kr_0 \sin \theta)) \\ &\quad - \frac{\partial}{\partial \theta} (\rho_{d,0}(\theta, \omega)(-Kr'_R(t) \sin \theta + Kr'_I(t) \cos \theta)). \end{aligned} \quad (3.8)$$

For $\omega \in \mathbb{R} \setminus [-Kr_0, Kr_0]$, set $\varphi_\omega \triangleq \arcsin\left(\frac{Kr_0}{\omega}\right)$, $\gamma_\omega \triangleq \omega \cos \varphi_\omega$ and let $\tilde{\theta}_\omega$ be a continuous function satisfying

$$\tan\left(\frac{\tilde{\theta}_\omega(t)}{2}\right) = \sin \varphi_\omega + \cos \varphi_\omega \tan\left(\frac{\gamma_\omega t}{2}\right), \quad \forall t \in \mathbb{R}.$$

One can verify that $\tilde{\theta}_\omega$ satisfies $\dot{\tilde{\theta}}_\omega(t) = \omega - Kr_0 \sin \tilde{\theta}_\omega(t)$, $\forall t \in \mathbb{R}$. Define $\tilde{\rho}'_d$ as

$$\tilde{\rho}'_d(\omega, t_0, t) \triangleq \rho'_d(\tilde{\theta}_\omega(t - t_0), \omega, t), \quad \forall \omega \in \mathbb{R} \setminus \mathcal{E}, \forall t_0, t \in \mathbb{R}.$$

Then, using (3.8), it follows that

$$\begin{aligned} \frac{\partial \tilde{\rho}'_d}{\partial t}(\omega, t_0, t) &= \frac{\partial \rho'_d}{\partial \theta}(\tilde{\theta}_\omega(t - t_0), \omega, t) \frac{d\tilde{\theta}_\omega}{dt}(t - t_0) + \frac{\partial \rho'_d}{\partial t}(\tilde{\theta}_\omega(t - t_0), \omega, t) \\ &= \tilde{\rho}'_d(\omega, t_0, t) Kr_0 \cos \tilde{\theta}_\omega(t - t_0) \\ &\quad + \frac{\partial}{\partial \theta} (\rho_{d,0}(\theta, \omega)(Kr'_R(t) \sin \theta - Kr'_I(t) \cos \theta)) \Big|_{\theta=\tilde{\theta}_\omega(t-t_0)}. \end{aligned}$$

For a well-chosen initial condition, this differential equation for $\tilde{\rho}'_d$ is solved in section A.2.1 of the appendix, resulting in

$$\begin{aligned} \tilde{\rho}'_d(\omega, t_0, t) &= \frac{K\rho_{d,0}(\theta, \omega)}{1 - \sin\theta \sin\varphi_\omega} \sum_{\gamma \in \Gamma} \frac{e^{i\gamma t}}{\gamma_\omega^2 - \gamma^2} \\ &\quad \times \left(\frac{R'_R(\gamma)}{\cos\varphi_\omega} (i\gamma \cos\theta \cos\varphi_\omega + \gamma_\omega(\sin\theta - \sin\varphi_\omega)) \right. \\ &\quad \left. + \frac{R'_I(\gamma)}{i} (-\gamma(\sin\theta - \sin\varphi_\omega) - i\gamma_\omega \cos\theta \cos\varphi_\omega) \right) \Bigg|_{\theta=\tilde{\theta}_\omega(t-t_0)}. \end{aligned}$$

This is also equal to $\rho'_d(\tilde{\theta}_\omega(t-t_0), \omega, t)$ for all t_0 and t in \mathbb{R} , implying that

$$\begin{aligned} \rho'_d(\theta, \omega, t) &= \frac{K\rho_{d,0}(\theta, \omega)}{1 - \sin\theta \sin\varphi_\omega} \sum_{\gamma \in \Gamma} \frac{e^{i\gamma t}}{\gamma_\omega^2 - \gamma^2} \left(\left(\frac{\gamma R'_R(\gamma)}{\cos\varphi_\omega} + i\gamma_\omega R'_I(\gamma) \right) i \cos\theta \cos\varphi_\omega \right. \\ &\quad \left. + \left(\frac{\gamma_\omega R'_R(\gamma)}{\cos\varphi_\omega} + i\gamma R'_I(\gamma) \right) (\sin\theta - \sin\varphi_\omega) \right). \end{aligned}$$

3.4 The entrained oscillators

Regarding the entrained oscillators, we first consider the system equations for an oscillator belonging to the largest entrained subset, moving (on average) at zero velocity:

$$\frac{\partial \tilde{\theta}}{\partial t}(\theta_i, \omega, t) = \omega - K(r_0 + r'_R(t)) \sin(\tilde{\theta}(\theta_i, \omega, t)) + Kr'_I(t) \cos(\tilde{\theta}(\theta_i, \omega, t)),$$

which, for $\omega \in (-Kr_0, Kr_0)$, has the following first order approximation (setting $\tilde{\theta}(\theta_i, \omega, t) \triangleq \theta_0 + \theta'(t)$, with $\theta_0 = \arcsin(\frac{\omega}{Kr_0})$, and assuming that $\theta'(t) = \mathcal{O}(r'_{\text{RMS}})$):

$$\dot{\theta}'(t) = -Kr_0\theta'(t) \cos\theta_0 - Kr'_R(t) \sin\theta_0 + Kr'_I(t) \cos\theta_0,$$

from which $\theta'(t)$ can be calculated as

$$\begin{aligned} \theta'(t) &= e^{-Kr_0 t \cos\theta_0} \left(\theta'(0) + \int_0^t e^{Kr_0 t' \cos\theta_0} \text{Im}(Kr'(t') e^{-i\theta_0}) dt' \right) \\ &= e^{-Kr_0 t \cos\theta_0} \left(\theta'(0) + \text{Im} \sum_{\gamma \in \Gamma} KR'(\gamma) \frac{e^{(Kr_0 \cos\theta_0 + i\gamma)t - i\theta_0} - e^{-i\theta_0}}{Kr_0 \cos\theta_0 + i\gamma} \right) \end{aligned}$$

$$= e^{-Kr_0 t \cos \theta_0} \theta'(0) + \text{Im} \sum_{\gamma \in \Gamma} KR'(\gamma) \frac{e^{i\gamma t - i\theta_0} - e^{-Kr_0 t \cos \theta_0 - i\theta_0}}{Kr_0 \cos \theta_0 + i\gamma}.$$

Choosing

$$\theta'(0) = \theta_i - \theta_0 \triangleq \text{Im} \sum_{\gamma \in \Gamma} KR'(\gamma) \frac{e^{-i\theta_0}}{Kr_0 \cos \theta_0 + i\gamma},$$

we obtain

$$\theta'(t) = \text{Im} \sum_{\gamma \in \Gamma} KR'(\gamma) \frac{e^{i(\gamma t - \theta_0)}}{Kr_0 \cos \theta_0 + i\gamma}$$

(which is indeed of first order in r'_{RMS}), implying that oscillators with $\omega \in (-Kr_0, Kr_0)$ will have a long term average velocity equal to zero and will belong to the corresponding entrained subset.

For an oscillator with $\omega \notin [-Kr_0, Kr_0]$, moving on average at velocity $\gamma_{\omega'} \gg Kr'(t)$, with $\omega' \notin [-Kr_0, Kr_0]$, we set $\theta(\theta_i, \omega, t) \triangleq \tilde{\theta}_{\omega'}(t + s(t))$, resulting in the following differential equation for $s(t) = \tilde{\theta}_{\omega'}^{-1}(\tilde{\theta}(\theta_i, \omega, t)) - t$:

$$\dot{s}(t) = \frac{v(\tilde{\theta}_{\omega'}(t + s(t)), \omega, t) - \dot{\tilde{\theta}}_{\omega'}(t + s(t))}{\dot{\tilde{\theta}}_{\omega'}(t + s(t))}, \quad \forall t \in \mathbb{R}.$$

In section A.2.2 of the appendix, we rewrite this differential equation and discuss its solution. Three cases are distinguished:

- If $\gamma_{\omega'} \notin \Gamma$, then $\omega \notin \mathcal{E}$; if also $r'_{\text{RMS}} \ll |\gamma - \gamma_{\omega'}|, \forall \gamma \in \Gamma$, then $\omega \approx \omega'$.
- If $\gamma_{\omega'} \in \Gamma$ and $r'_{\text{RMS}} \ll |\gamma - \gamma_{\omega'}|, \forall \gamma \in \Gamma$ with $\gamma \neq \gamma_{\omega'}$, then $\omega \in \mathcal{E}$, and the corresponding entrained subset will consist of all oscillators for which

$$|\omega - \omega'| \leq K |R'_C(\gamma_{\omega'})|,$$

with

$$R'_C(\gamma_{\omega'}) \triangleq R'_R(\gamma_{\omega'}) + i \cos \varphi_{\omega'} R'_1(\gamma_{\omega'}), \quad \forall \omega' \in \mathbb{R} \setminus [-Kr_0, Kr_0],$$

and with the zeroth order approximation of $s(t)$ settling into a constant value ($\sim \mathcal{O}(1)$), which (for a generic initial condition) can be calculated as

$$\lim_{t \rightarrow +\infty} s(t) \approx \frac{1}{\gamma_{\omega'}} \left(\arcsin \left(\frac{\omega - \omega'}{K |R'_C(\gamma_{\omega'})|} \right) + \arg(R'_C(\gamma_{\omega'})) - \varphi_{\omega'} \right).$$

- If there is more than one $\gamma \in \Gamma$ with $|\gamma - \gamma_{\omega'}| = \mathcal{O}(r'_{\text{RMS}})$, then, for simplicity, we assume that the fraction of oscillators involved in the entrainment is small enough to neglect its contribution to the equation (3.7). From the results in the next section it follows that if $\gamma \in \Gamma$, then $k\gamma \in \Gamma, \forall k \in \mathbb{Z}_0$. If there exist $\gamma_1, \gamma_2 \in \Gamma$, with $\frac{\gamma_1}{\gamma_2}$ irrational, then the existence of $\omega' \in \mathbb{R}$ with two (or more) $\gamma \in \Gamma$ with $|\gamma - \gamma_{\omega'}| = \mathcal{O}(r'_{\text{RMS}})$ is inevitable. However, if $\frac{\gamma_1}{\gamma_2}$ is ‘sufficiently irrational’ and g is decaying fast enough, then $g(\omega')$ will be small and the above condition will be satisfied.

3.5 The self-consistency equation

In section A.2.3 of the appendix the results of sections 3.3 and 3.4 are inserted in (3.7), and the coefficients of the complex exponentials are identified, resulting in

$$\begin{aligned}
r_0 e^{i\alpha} &= K r_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta_0 d\theta_0 g_R(K r_0 \sin \theta_0) e^{i\theta_0} \\
&\quad + \sum_{\gamma_{\omega'} \in \Gamma} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K |R'_C(\gamma_{\omega'})| \cos \theta_0 d\theta_0 g_R(\omega' + K |R'_C(\gamma_{\omega'})| \sin \theta_0) \\
&\quad \times i \tan\left(\frac{\varphi_{\omega'}}{2}\right) + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{i(1 - \cos \varphi_{\omega})}{\sin \varphi_{\omega}}, \\
R'(\gamma_{\omega'}) e^{i\alpha} &= K r_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta_0 d\theta_0 g_R(K r_0 \sin \theta_0) e^{i\theta_0} \frac{K}{2(K^2 r_0^2 \cos^2 \theta_0 + \gamma_{\omega'}^2)} \\
&\quad \times \left(e^{-i\theta_0} (R'_R(\gamma_{\omega'}) + i R'_I(\gamma_{\omega'})) (K r_0 \cos \theta_0 - i \gamma_{\omega'}) \right. \\
&\quad \left. - e^{i\theta_0} (R'_R(\gamma_{\omega'}) - i R'_I(\gamma_{\omega'})) (K r_0 \cos \theta_0 - i \gamma_{\omega'}) \right) \\
&\quad + \sum_{\gamma_{\omega''} \in \Gamma \cap \{\frac{\gamma_{\omega'}}{k} : k \in \mathbb{N}_0\}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K |R'_C(\gamma_{\omega''})| \cos \theta_0 d\theta_0 \\
&\quad \times g_R(\omega'' + K |R'_C(\gamma_{\omega''})| \sin \theta_0) (-2i) \cot \varphi_{\omega''} \\
&\quad \times \left(i e^{i(\theta_0 + \arg(R'_C(\gamma_{\omega''})))} \tan\left(\frac{\varphi_{\omega''}}{2}\right) \right)^{\frac{\gamma_{\omega'}}{\gamma_{\omega''}}} \\
&\quad + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{iK(1 - \cos \varphi_{\omega})}{\sin^2 \varphi_{\omega} (\gamma_{\omega} - \gamma_{\omega'})} (R'_R(\gamma_{\omega'}) + i \cos \varphi_{\omega} R'_I(\gamma_{\omega'})),
\end{aligned}$$

for all $\omega' \in \mathbb{R}$ with $\gamma_{\omega'} \in \Gamma$. There is always the solution $r_0 = 0$ and $R'(\gamma) = 0, \forall \gamma \in \mathbb{R}$, which corresponds to total incoherence with every oscillator moving at its natural frequency ω . If K is larger than some critical value K_c there is also a non-zero solution for r_0 , corresponding to an entrained subset of oscillators with ω -values in $[-K r_0, K r_0]$. The zeroth order values for r_0 and Ω_0 can be

deduced from the equations

$$K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta_0 d\theta_0 g_R(Kr_0 \sin \theta_0) = \cos \alpha, \quad (3.9a)$$

$$Kr_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \theta_0 \cos \theta_0 d\theta_0 g_R(Kr_0 \sin \theta_0) + \int_{\mathbb{R} \setminus [-Kr_0, Kr_0]} d\omega g_R(\omega) \frac{(1 - \cos \varphi_\omega)}{\sin \varphi_\omega} = r_0 \sin \alpha, \quad (3.9b)$$

in agreement with the results from [45]. For K equal to K_c , at the onset of the entrainment, r_0 equals 0 and it can be derived that

$$\frac{\pi}{2} K_c g(\Omega_0) = \cos \alpha$$

$$K_c \int_0^\infty \frac{g(\Omega_0 + \omega) - g(\Omega_0 - \omega)}{2\omega} d\omega = \sin \alpha.$$

For $\alpha = 0$ and g even and unimodal it immediately follows that $\Omega_0 = 0$. For general α and g (but g having a finite number of maximums) it can be easily seen that for small Ω_0 the left hand side of the second equation will be positive, while for large Ω_0 it will be negative. It follows that for $|\alpha|$ small enough there always exists a solution Ω_0 for this equation. The coupling strength K needs to be at least $\frac{2 \cos \alpha}{\pi \max(g)}$ for the existence of an entrained subset.

A non-zero solution for $R'(\gamma_{\omega'})$, for some $\gamma_{\omega'} \in \mathbb{R}$, corresponds to another entrained subset of oscillators with ω -values in a neighborhood of ω' and moving at an average velocity $\gamma_{\omega'}$. In general, this entrained subset will induce other entrained subsets at frequencies $-\gamma_{\omega'}$ and $k\gamma_{\omega'}$, with $k \in \mathbb{N}_0$, as follows from the corresponding equations for R' , in a similar way as for the periodically driven Kuramoto model [14]. This will lead to entrainment at all frequencies $k\gamma_{\omega'}$, with $k \in \mathbb{Z}_0$.

From now on assume that g is the sum of a smooth distribution, which does not change abruptly in intervals with a size of the order r'_{RMS} , and a distribution with one (or two if g is even) narrow peak(s) (with width of the order of r'_{RMS}), with (a) maximum value(s) of the same order of magnitude as for the first (smooth) contribution. (An example is presented in figure 3.1 on page 37.) Then we can set $\Gamma = \{k\gamma_{\tilde{\omega}} : k \in \mathbb{Z}_0\}$, for some $\gamma_{\tilde{\omega}} \in \mathbb{R}_0^+$, where $\tilde{\omega}$ and/or $-\tilde{\omega}$ will correspond to the narrow peak(s) in g . In this case we only need to calculate r_0 and Ω_0 in zeroth order to obtain an equation in first order for $R'(k\gamma_{\tilde{\omega}})$. (The average velocities in the original system (i.e. with respect to the initial non-rotating frame) $\Omega_0 \pm \gamma_{\tilde{\omega}}$ can be calculated up to first order; consequently the value of $\gamma_{\tilde{\omega}}$ will also be correct only in zeroth order. Because of the assumption on g , the value of $g_R(k\gamma_{\tilde{\omega}})$, with $k \notin \{-1, 0, 1\}$, will be correct in zeroth order, guaranteeing the correctness of the equations up to first order for this value of $\gamma_{\tilde{\omega}}$.) We can thus first calculate r_0 and Ω_0 from (3.9), and

then use these values to determine the values of $R'(k\gamma_{\tilde{\omega}})$, with $k \in \mathbb{Z}_0$. Notice that the values $R'(k'\gamma_{\tilde{\omega}})$, with $k' \in \mathbb{Z}_0$, are needed to know the set \mathcal{E} , which appears in one of the integrals, but because of the assumptions on g only the values for $k' = \pm 1$ will be important, and for the first order approximation of this integral \mathcal{E} can be put equal to \mathcal{E}_1 , with

$$\begin{aligned} \mathcal{E}_1 \triangleq & [-\tilde{\omega} - K |R'_C(-\gamma_{\tilde{\omega}})|, -\tilde{\omega} + K |R'_C(-\gamma_{\tilde{\omega}})|] \\ & \cup [-Kr_0, Kr_0] \cup [\tilde{\omega} - K |R'_C(\gamma_{\tilde{\omega}})|, \tilde{\omega} + K |R'_C(\gamma_{\tilde{\omega}})|]. \end{aligned}$$

Then $R'(k'\gamma_{\tilde{\omega}})$ does not appear in the equation for $R'(k\gamma_{\tilde{\omega}})$ if $|k'| > |k|$, and the values $R'(k\gamma_{\tilde{\omega}})$ can be calculated recursively.

For $\alpha = 0$ and g even, performing the substitutions $\omega \leftrightarrow -\omega$, $\theta \leftrightarrow -\theta, \dots$ and $R'_1 \leftrightarrow -R'_1$ in the summations and integrals, followed by complex conjugation, corresponds to interchanging the equations for $R'(\gamma_{\omega'})$ and $R'(-\gamma_{\omega'})$. It then follows that R'_1 can be put equal to zero everywhere, while for each positive $\gamma_{\omega'}$ only one (complex) equation remains.

For $\alpha = 0$ and g even we can therefore consider the equations

$$\begin{aligned} 1 &= K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(Kr_0 \sin \theta_0) \cos^2 \theta_0 d\theta_0, \\ \Omega_0 &= 0, \\ 1 &= Kr_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta_0 d\theta_0 g(Kr_0 \sin \theta_0) \frac{K \sin^2 \theta_0}{K^2 r_0^2 \cos^2 \theta_0 + k^2 \gamma_{\tilde{\omega}}^2} (Kr_0 \cos \theta_0 - ik\gamma_{\tilde{\omega}}) \\ &+ \sum_{\substack{k' \in \mathbb{Z}_0: \frac{k'}{k} \in \mathbb{N}_0 \\ \gamma_{\omega'} \triangleq k' \gamma_{\tilde{\omega}}} } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K \frac{|R'_R(k'\gamma_{\tilde{\omega}})|}{R'_R(k\gamma_{\tilde{\omega}})} \cos \theta_0 d\theta_0 g(\omega' + K |R'_R(k'\gamma_{\tilde{\omega}})| \sin \theta_0) \\ &\times (-2i) \cot \varphi_{\omega'} \left(i e^{i(\theta_0 + \arg(R'_R(k'\gamma_{\tilde{\omega}}))} \right) \tan \left(\frac{\varphi_{\omega'}}{2} \right) \Big)^{\frac{k}{k'}} \\ &+ \int_{\mathbb{R} \setminus \mathcal{E}_1} d\omega g(\omega) \frac{iK(1 - \cos \varphi_{\omega})}{\sin^2 \varphi_{\omega}(\gamma_{\omega} - k\gamma_{\tilde{\omega}})}, \\ R'_1(k\gamma_{\tilde{\omega}}) &= 0, \end{aligned}$$

$\forall k \in \mathbb{Z}_0$. For $k = 1$, the summation reduces to the term corresponding to $k' = 1$ (and thus $\omega' = \tilde{\omega}$), and real and imaginary part of the corresponding equality can then be rewritten as follows:

$$\begin{aligned} 1 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(Kr_0 \sin \theta_0) \frac{K^3 r_0^2 \sin^2 \theta_0 \cos^2 \theta_0}{K^2 r_0^2 \cos^2 \theta_0 + \gamma_{\tilde{\omega}}^2} d\theta_0 \\ &+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(\tilde{\omega} + K |R'_R(\gamma_{\tilde{\omega}})| \sin \theta_0) \frac{2K \cos \varphi_{\tilde{\omega}} \cos^2 \theta_0}{1 + \cos \varphi_{\tilde{\omega}}} d\theta_0 \end{aligned}$$

$$\begin{aligned}
0 = & - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(Kr_0 \sin \theta_0) \frac{K^2 r_0 \gamma_{\tilde{\omega}} \cos \theta_0 \sin^2 \theta_0}{K^2 r_0^2 \cos^2 \theta_0 + \gamma_{\tilde{\omega}}^2} d\theta_0 \\
& + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(\tilde{\omega} + K |R'_R(\gamma_{\tilde{\omega}})| \sin \theta_0) \frac{2K \cos \varphi_{\tilde{\omega}} \sin \theta_0 \cos \theta_0}{1 + \cos \varphi_{\tilde{\omega}}} d\theta_0 \\
& + \int_{\mathbb{R} \setminus \mathcal{E}_1} \frac{K g(\omega) d\omega}{(1 + \cos \varphi_{\omega})(\gamma_{\omega} - \gamma_{\tilde{\omega}})}.
\end{aligned}$$

These equations can be numerically solved to find the value of $R'_R(\gamma_{\tilde{\omega}}) = \overline{R'_R(-\gamma_{\tilde{\omega}})}$.

3.6 Examples

For a first example we consider the distribution g_1 defined by

$$g_1(\omega) = \frac{1}{1.04\sqrt{\pi}} \left(e^{-\omega^2} + e^{-50^2(\omega-1.2)^2} + e^{-50^2(\omega+1.2)^2} \right), \quad \forall \omega \in \mathbb{R},$$

which is shown in figure 3.1. Figure 3.2(a) shows simulation results for a system of 1000 oscillators. To achieve a better approximation of the continuous distribution g_1 , the ω -values are not chosen randomly, but in such a way that, when applying the cumulative density distribution associated with g_1 , the values $\frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N}$ are obtained. In figure 3.2(b) the intervals for ω corresponding to the entrained subsets are compared to analytical predictions. As is clearly visible for both simulations and analysis, the length of the interval, corresponding to the entrained subset, first increases but then decreases again with increasing K , until the entrained subset has disappeared. (Notice that the long term average frequencies decrease (in absolute value) with increasing K .) This phenomenon, for which entrainment disappears with increasing coupling strength, has also been discussed in the previous chapter for a finite number of oscillators. The explanation from the previous chapter can be interpreted as follows for infinite N .

As can be seen in figure 3.2(a), the graph depicting the long term average frequencies as a function of ω has a large slope for drifting oscillators with ω -values just outside the interval $[-Kr_0, Kr_0]$, which obstructs entrainment of a smaller subset in this region since entrainment corresponds to this slope being zero. As K increases, the boundary of this interval approaches the intervals associated with the smaller entrained subsets, which tends to reduce entrainment in these intervals.

In a second example we illustrate the emergence of entrained subsets which cannot be accounted for by the distribution of the natural frequencies. (In the previous example this phenomenon can only be observed after a substantial increase in the number of oscillators, since the associated intervals for ω are

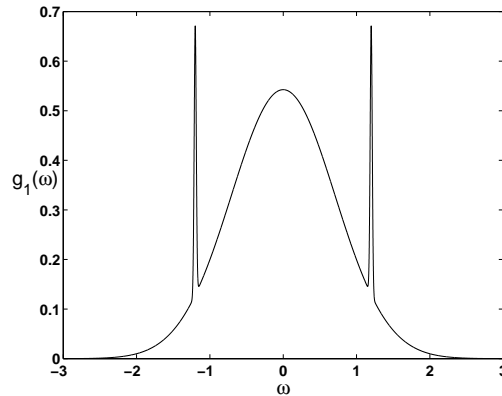


Figure 3.1: The distribution g_1 .

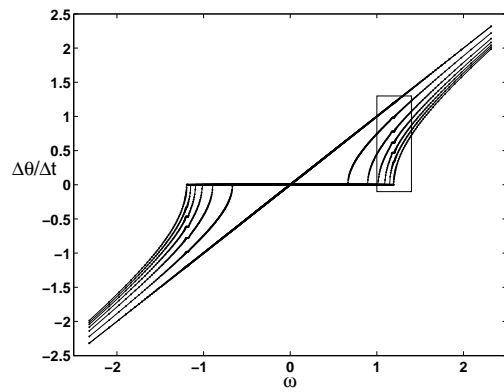
quite small due to choosing the extra peaks in g_1 narrow enough for a good agreement with the analysis.) The distribution g_2 is defined by

$$g_2(\omega) = \frac{1}{0.6\pi} \left(\frac{1}{1 + \left(\frac{\omega-1}{0.3}\right)^2} + \frac{1}{1 + \left(\frac{\omega+1}{0.3}\right)^2} \right), \quad \forall \omega \in \mathbb{R},$$

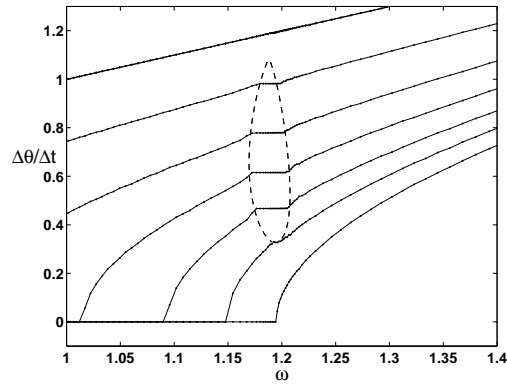
and depicted in figure 3.3. Instead of considering g_2 as an even distribution with no middle peak one has to treat one of the two peaks as the unperturbed distribution and the other as the perturbation. Of course the mathematical analysis cannot provide quantitative results in this case, but the prediction regarding the existence of induction of entrained subsets is still valid, as can be observed from figure 3.4. In this figure, which is obtained in a similar way as figure 3.2(a), only the largest two entrained subsets arise from the shape of the distribution g_2 . For different values of the coupling strength other entrained subsets can be distinguished, unaccounted for by the shape of g_2 . For a fixed K , the differences in long term average frequencies of the various entrained subsets are multiples of the frequency difference of the largest two entrained subsets. The corresponding steps in figure 3.4 are also referred to as Shapiro steps [46, 14].

3.7 Conclusion

When the distribution of the natural frequencies in the Kuramoto-Sakaguchi model with an infinite number of oscillators can be considered as a perturbation of a unimodal distribution, it is possible to derive analytical results. The natural frequency intervals of the entrained oscillators can be characterized by a set of equations, which is exact up to first order in the perturbation size.



(a)



(b)

Figure 3.2: The long term average frequencies (vertical axis) are shown as a function of ω (horizontal axis) for different values of the coupling strength, resulting from a simulation of a system of 1000 oscillators with frequencies determined by g_1 . Figure (b) is obtained by zooming in on the rectangular region indicated in figure (a). The dashed line shows an analytical prediction of the boundaries of the entrained subsets.

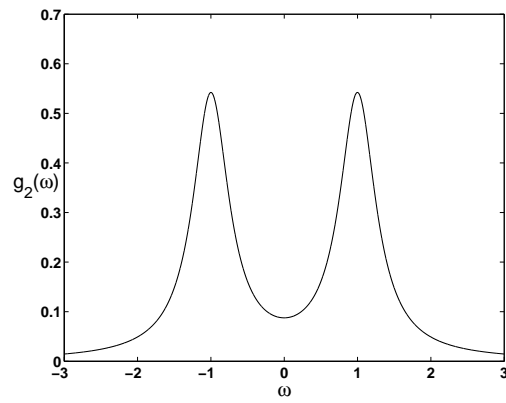


Figure 3.3: The distribution g_2 .

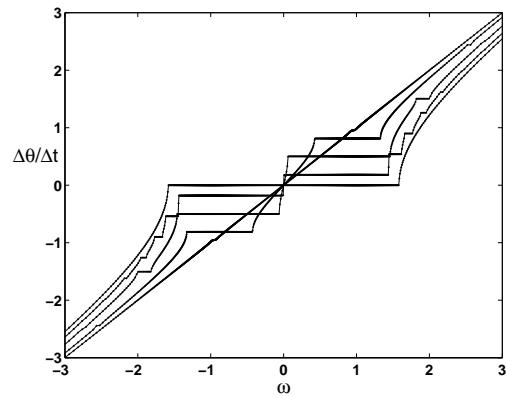


Figure 3.4: The long term average frequencies (vertical axis) are shown as a function of ω (horizontal axis) for different values of the coupling strength, resulting from a simulation of a system of 1000 oscillators with frequencies determined by g_2 .

Similarly as in the model with a finite number of oscillators, entrained subsets may break up with increasing coupling strength.

Furthermore, entrainment does not always result from high frequency densities for the corresponding oscillators, but resonances with other entrained subsets may be responsible for the entrainment of oscillators in intervals where the frequency density is too low to account for the entrainment.

Part II

The clustering model

Chapter 4

A basic model for clustering

This chapter treats the basic version of the clustering model; the analytical results will be discussed without going into the mathematical details. For a finite number of agents the mathematical analysis is given in chapter 5, where a generalization of the model will be considered. In this chapter we formulate necessary and sufficient conditions for clustering behavior with respect to a given cluster structure. We also indicate how the cluster structure varies with varying coupling strength, and how the agents behave within a cluster.

4.1 The model

The differential equations for the model consisting of N agents ($N > 1$) are

$$\dot{x}_i(t) = b_i + \frac{K}{N} \sum_{j=1}^N f(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (4.1)$$

with $K \geq 0$. The attraction exerted by the other agents on each agent depends on their mutual distances. The interaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be odd and non-decreasing. This implies a *symmetric attraction* between any two agents. We assume that the interaction intensifies with separation up to a certain saturation level:

$$\exists d > 0 : \forall x \geq d, \quad f(\pm x) = \pm F.$$

A Lipschitz condition on f is introduced for technical reasons; it guarantees a unique solution to the differential equation with respect to a set of initial conditions.

Due to the similarity with the Kuramoto-Sakaguchi model, a property similar to the one in remark 2.1 can be stated concerning the order of the agents:

Remark 4.1. If $b_i \leq b_j$ and $x_i(0) \leq x_j(0)$, then $x_i(t) \leq x_j(t)$, $\forall t \in \mathbb{R}^+$, since if for some $t^* \geq 0$, $x_i(t^*) = x_j(t^*)$, then $\dot{x}_i(t^*) - \dot{x}_j(t^*) = b_i - b_j \leq 0$.

Definition 4.1. Assume that, for a particular solution x of (4.1), the behavior of the agents can be characterized as follows by an *ordered* set of *clusters* $G \triangleq (G_1, \dots, G_M)$ representing a partition of \mathcal{I}_N :

- The distances between agents in the same cluster remain bounded (i.e. $|x_i(t) - x_j(t)|$ is bounded for all $i, j \in G_k$, for any $k \in \mathcal{I}_M$, for $t \geq 0$).
- For any $D > 0$ there exists a time after which the distances between agents in different clusters are and remain at least D .
- After some time, the agents are ordered by their membership to the clusters: $\exists T > 0 : \forall t \geq T, \forall k, l \in \mathcal{I}_M, \forall i \in G_k, \forall j \in G_l, k < l \Rightarrow x_i(t) < x_j(t)$.

Then x exhibits *clustering behavior* with respect to G .

Remark 4.2. In its formulation as a linear model (but with an additional bias term) where a static nonlinearity f is introduced, the model (4.1) may resemble some well-known systems such as Lur'e systems and artificial neural networks [49]. However, research on these latter systems is mostly concentrated on bounded solutions and/or convergence to equilibrium points (see e.g. [30] and references therein), while we are interested in the clustering behavior of (4.1), which (in the generic case) corresponds to *unbounded* solutions.

4.2 Results and discussion

For any non-empty set $G_0 \subset \mathcal{I}_N$, with the number of elements denoted by $|G_0|$, and an arbitrary vector $w \in \mathbb{R}^N$, we introduce the notation $\langle w \rangle_{G_0}$ for the average value of w_i over G_0 :

$$\langle w \rangle_{G_0} \triangleq \frac{1}{|G_0|} \sum_{i \in G_0} w_i.$$

Theorem 4.1. *The following set of conditions is necessary and sufficient for clustering behavior of all solutions of the system (4.1), with the cluster structure (G_1, \dots, G_M) independent of the initial condition:*

$$\langle b \rangle_{G_{k+1}} - \langle b \rangle_{G_k} > \frac{KF}{N} (|G_{k+1}| + |G_k|), \quad \forall k \in \mathcal{I}_{M-1}, \quad (4.2a)$$

$$\langle b \rangle_{G_{k,2}} - \langle b \rangle_{G_{k,1}} \leq \frac{KF}{N} |G_k|, \quad (4.2b)$$

$$\forall G_{k,1}, G_{k,2} \subsetneq G_k, \text{ with } G_{k,2} = G_k \setminus G_{k,1}, \quad \forall k \in \mathcal{I}_M.$$

The characteristics of the interaction play a key role in the proof. Since f is odd all internal interactions (i.e. interactions between agents in the same cluster) cancel when calculating the velocity of the ‘center of mass’ of a cluster, similar to the cancellation of internal interactions in mechanics. The saturation of f implies that the interactions between agents from different clusters reduce to KF/N or $-KF/N$ whenever agents from different clusters are separated over at least a distance d . The monotonicity of f guarantees that the resulting clustering behavior is independent of the initial condition.

Under the assumption of clustering behavior and taking into account the previous considerations (see also formula (4.3) on page 46), the ordering of the agents and distances growing unbounded with time for agents in different clusters will lead to the condition (4.2a). Similarly, since distances between agents from the same cluster remain bounded the condition (4.2b) can be derived. This implies the necessity of the inequalities (4.2a) and (4.2b) for the existence of a solution of (4.1) satisfying clustering behavior. Next we give an outline of the proof of sufficiency.

The main idea is to pick an initial condition for which agents from different G_k will always (i.e. for all $t \geq 0$) be separated over at least a distance d , with their interaction saturated as a consequence. Invoking the condition (4.2b) it can then be shown that the differences in $x_i(t)$ -values will be bounded for agents in the same G_k . From this boundedness together with the condition (4.2a), it will follow that the differences in $x_i(t)$ -values for agents from different G_k will grow unbounded. The solution of (4.1) corresponding to this particular initial condition will exhibit clustering behavior (with $T = 0$, and the clusters equal to the G_k). Any other solution \hat{x} of (4.1) will exhibit the same clustering behavior (i.e. identical clusters, possibly a different value for T). This follows by observing that the distance in the state space \mathbb{R}^N between x and \hat{x} is a non-increasing function of time:

$$\frac{d}{dt} \left(\sum_{i=1}^N (x_i(t) - \hat{x}_i(t))^2 \right) \leq 0.$$

The above reasoning is the starting point of the mathematical treatment of the extended model in chapter 5.

The next result implies that for any choice of the parameters, the solution x of (4.1) will always satisfy clustering behavior w.r.t. some cluster structure.

Theorem 4.2. *For each $b \in \mathbb{R}^N$ and each $K \geq 0$, there exists a unique ordered set partition G of \mathcal{I}_N , such that (4.2) holds.*

The proof of theorem 4.2 is based on the following reasoning. For large values of K , $G = (\mathcal{I}_N)$ (corresponding to only one cluster) satisfies (4.2). Whenever for a given cluster structure — satisfying (4.2) — one of the conditions (4.2b) ceases to hold with decreasing K (obviously (4.2a) will remain

satisfied), one can show that there is always another unique ordered set partition for which (4.2) will hold again. In the generic case, where only one of the conditions (4.2b) ceases to hold for a transition value of K , the new set partition will correspond to the old set partition in which one of the clusters has been split in two. This results in $N - 1$ bifurcation values for the intensity of attraction K , defining N intervals for K ; each interval corresponds to a particular cluster configuration, and transitions to new cluster configurations take place at these bifurcation points.

For a solution satisfying clustering behavior with clusters G_1, \dots, G_M , it is easy to verify that the average velocity $\langle \dot{x}(t) \rangle_{G_k}$ over cluster G_k will be constant after some time T :

$$\langle \dot{x}(t) \rangle_{G_k} = \langle b \rangle_{G_k} + \frac{KF}{N} \left(\sum_{k' > k} |G_{k'}| - \sum_{k' < k} |G_{k'}| \right), \quad \forall t \geq T. \quad (4.3)$$

For any $i \in G_k$, set v_i equal to the right hand side of the above equation. From the boundedness of the distances between agents of cluster G_k we can derive that (4.2a) and (4.2b) are necessary and sufficient for every solution x of (4.1) to satisfy:

$$\exists l > 0 : |x_i(t) - v_i t| \leq l, \quad \forall t \geq 0, \quad \forall i \in \mathcal{I}_N. \quad (4.4)$$

Furthermore, from (4.3) together with the following theorem, which concerns the internal behavior of the clusters, it follows that for any $i \in G_k$, with $k \in \mathcal{I}_M$, $x_i(t)$ tends asymptotically to $v_i t$ plus a constant, or equivalently: $\lim_{t \rightarrow +\infty} (x_i(t) - v_i t)$ exists.

Theorem 4.3. *Let x be a solution of (4.1) with cluster structure $G = (G_1, \dots, G_M)$, and let v_i ($i \in G_k$) be equal to the right hand side of (4.3). For each $k \in \mathcal{I}_M$, if $i, j \in G_k$, then*

$$\begin{aligned} \lim_{t \rightarrow +\infty} (x_i(t) - x_j(t)) \text{ exists,} \\ \lim_{t \rightarrow +\infty} \dot{x}_i(t) \text{ exists and equals } v_i. \end{aligned}$$

Furthermore, if the inequalities (4.2b) hold with strict inequality and f is increasing in the interval $(-d, d)$, then $\lim_{t \rightarrow +\infty} (x_i(t) - x_j(t))$ is independent of $x(0)$, $\forall i, j \in G_k$, $\forall k \in \mathcal{I}_M$.

In other words, within a cluster all differences between $x_i(t)$ -values will approach constants which are, in the generic case, independent of the initial condition, and the velocities of the agents will approach the asymptotic average cluster velocity. The second part of theorem 4.3 does not follow from the corresponding extension in chapter 5 (i.e. theorem 5.3); for the proof we refer to [6].

Given the model (4.1) and faced with the question of describing the eventually emerging cluster structure, our analysis offers two options: one can check the inequalities (4.2) or one can simply run a simulation of the model: the mathematical analysis guarantees convergence to a cluster structure, irrespective of the initial condition.

4.2.1 Cluster structure for varying coupling strength

When decreasing K from a sufficiently large value, a tree structure relating K to asymptotic cluster velocities can be identified; see figure 4.1 for a system of 6 agents, with

$$b = [-2.4 \quad -2 \quad -1 \quad 1 \quad 1.3 \quad 4]^T, \quad \text{and} \quad F = 1.$$

In the generic case there will be $N - 1$ bifurcation values for K , each corresponding to a separation of a cluster into two new clusters. The tree structure

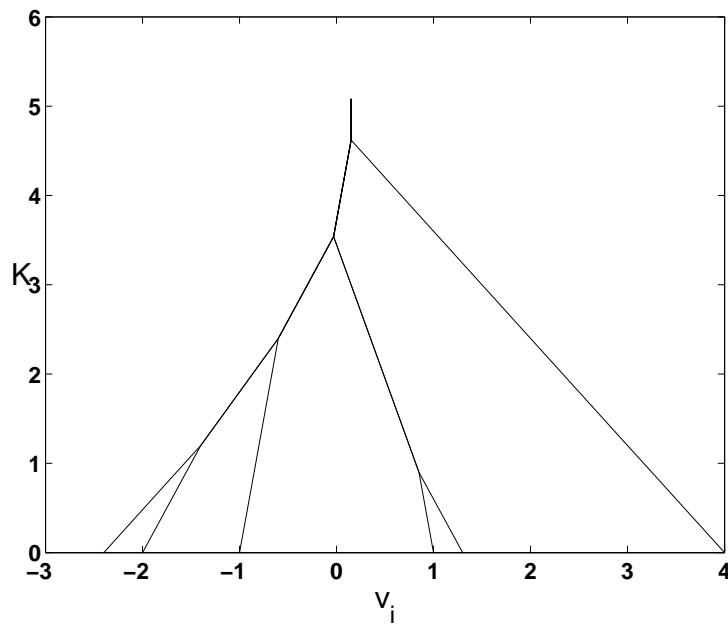


Figure 4.1: Cluster structures represented by the long term average velocities v_i for varying values of the coupling strength K , for a system of 6 agents.

can also be interpreted as a dendrogram, indicating that this model may lead to a hierarchical procedure for data clustering (see e.g. [27] for more information on this subject).

4.3 Conclusion

Replacing the coupling of the Kuramoto-Sakaguchi model by a saturating interaction function leads to a system which allows for a thorough mathematical analysis. Its (clustering) behavior is similar to the partial entrainment behavior of the Kuramoto-Sakaguchi model and can be characterized by a set of necessary and sufficient conditions. It can be shown that the model exhibits clustering behavior for any value of the coupling strength, and that each agent asymptotically moves at a constant velocity, with distances between agents of the same cluster approaching constant values.

Chapter 5

General network structures

This chapter contains an extension of the basic model to include a general network structure and weighting factors. The results of the basic model can also be generalized for this extension.

5.1 The model

The model is described by the following differential equations:

$$\dot{x}_i(t) = b_i + KA_i \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (5.1)$$

with $A_i, \gamma_j > 0, K \geq 0, N > 1$. The functions f_{ij} are non-decreasing, Lipschitz continuous and satisfy

$$\begin{aligned} f_{ji}(x) &= -f_{ij}(-x), & \forall x \in \mathbb{R}, \\ f_{ij}(x) &= F_{ij}, & \forall x \geq d_{ij}, \quad \text{and thus} \quad f_{ij}(x) = -F_{ji}, & \forall x \leq -d_{ji}, \end{aligned}$$

for some F_{ij}, d_{ij} in \mathbb{R} , for all i, j in \mathcal{I}_N . (It follows that the functions f_{ii} ($i \in \mathcal{I}_N$) are odd, and since they are only evaluated in zero they have no contribution to the system. They only exist for notational convenience.) The interval $[-F_{ji}, F_{ij}]$ covers the range of the attraction of agent j on agent i . Notice that d_{ij} is not necessarily positive (but of course $d_{ij} > -d_{ji}$ if $F_{ij} > -F_{ji}$). The extent to which each individual agent j tends to attract other agents is denoted by γ_j . The parameter A_i reflects the sensitivity of agent i to interactions with other agents; K is the global coupling strength.

Since we can rewrite the system (5.1) in the form

$$\dot{x}_i(t) = b_i + K \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (5.2)$$

by setting $f'_{ij} \triangleq A_i f_{ij} A_j$, $\gamma'_j \triangleq \frac{\gamma_j}{A_j}$, and then omitting the accents, we will only consider the system (5.2) for the mathematical analysis, corresponding to putting $A_i = 1$, $\forall i \in \mathcal{I}_N$. Furthermore, redefining the parameters b_i and the functions f_{ij} appropriately (by adding/subtracting appropriate constants) allows us to assume that $F_{ij} = F_{ji} \geq 0$, i.e. the matrix F is symmetric. We also assume that F is irreducible.

The interaction structure may be represented in graph theoretical terms: F_{ij} corresponds to the weight of the edge of the associated undirected graph with vertices i and j representing agents i and j . If edges corresponding to $F_{ij} = 0$ are omitted, then the irreducibility of the matrix F is equivalent to the connectedness of this graph. The parameters γ_j correspond to weights attached to the vertices.

5.2 Some notation

For an ordered set partition $G = (G_1, \dots, G_M)$ of \mathcal{I}_N , let $G_k^<$ be a shorthand notation for $\bigcup_{k' < k} G_{k'}$, and analogously set $G_k^> \triangleq \bigcup_{k' > k} G_{k'}$.

For any set S , let $\mathcal{P}_2(S)$ denote the set of ordered set partitions of S in two subsets:

$$\mathcal{P}_2(S) \triangleq \{(S_1, S_2) : S_1, S_2 \subsetneq S \text{ with } S_2 = S \setminus S_1\}.$$

Consider a non-empty set $G_0 \subset \mathcal{I}_N$. For a vector $w \in \mathbb{R}^N$ we redefine $\langle w \rangle_{G_0}$ as the weighted average of w_i over G_0 , with weighting factors γ_i :

$$\langle w \rangle_{G_0} \triangleq \frac{\sum_{i \in G_0} \gamma_i w_i}{\sum_{i \in G_0} \gamma_i}.$$

Because of the properties of the interaction functions f_{ij} , all internal interactions (i.e. interactions between agents in G_0) will cancel in the expression for $\langle \dot{x}(t) \rangle_{G_0}$. Assume that, at some time instance t_0 , the agents in G_0 are separated by at least a distance $d \triangleq \max_{i \neq j} d_{ij}$ from all other agents (i.e. $|x_i(t_0) - x_j(t_0)| \geq d$ whenever $i \in G_0$ and $j \notin G_0$). Then the interactions of agents in G_0 with agents not in G_0 are all saturated. Assume furthermore that at t_0 each agent not belonging to G_0 has an $x(t_0)$ -value either smaller or larger than *all* $x(t_0)$ -values of agents in G_0 . Denote by G_- , resp. G_+ , the set

of agents with $x(t_0)$ -values smaller, resp. larger, than the $x(t_0)$ -values of the agents in G_0 . Then for any $i \in G_0$ it follows that

$$\begin{aligned}\dot{x}_i(t_0) &= b_i + K \sum_{j \in \mathcal{I}_N} \gamma_j f_{ij}(x_j(t_0) - x_i(t_0)) \\ &= b_i - K \sum_{j \in G_-} \gamma_j F_{ij} + K \sum_{j \in G_0} \gamma_j f_{ij}(x_j(t_0) - x_i(t_0)) + K \sum_{j \in G_+} \gamma_j F_{ij},\end{aligned}$$

and thus

$$\langle \dot{x}(t_0) \rangle_{G_0} = \tilde{v}(G_-, G_0, G_+), \quad (5.3)$$

where we define the function \tilde{v} as

$$\tilde{v}(G_-, G_0, G_+) \triangleq \langle b \rangle_{G_0} + \frac{K}{\sum_{i \in G_0} \gamma_i} \sum_{i \in G_0} \gamma_i \left(\sum_{j \in G_+} \gamma_j F_{ij} - \sum_{j \in G_-} \gamma_j F_{ij} \right),$$

for all $G_-, G_0, G_+ \subset \mathcal{I}_N$ with G_0 non-empty.

For any $i, j \in \mathcal{I}_N$, let $\tilde{F}_{ijm_{ij}} \in [-F_{ji}, F_{ij}]$ ($m_{ij} \in \mathcal{I}_{n_{ij}}$ for some $n_{ij} \in \mathbb{N} \cup \{+\infty\}$) denote the function values of f_{ij} which constitute images of intervals of non-zero length. (Notice that the values $\tilde{F}_{ijm_{ij}}$ are indeed countable, since \mathbb{R} is a countable union of bounded intervals, and a partition of a bounded interval in intervals of non-zero length is countable by ordering the intervals (in reverse order) by their length.) Enumerate the values $\tilde{F}_{ijm_{ij}}$ such that $\tilde{F}_{ijm_{ij}} = -\tilde{F}_{jim_{ij}}$, $\forall m_{ij} \in \mathcal{I}_{n_{ij}} = \mathcal{I}_{n_{ji}}$. Let $G = (G_1, \dots, G_M)$ be an ordered set partition of \mathcal{I}_N . By (\mathcal{A}) , we will refer to the following assumption on the parameters of the model (5.2):

$$\begin{aligned}\tilde{v}(G_k^<, G_{k,1}, G_k^>) + \frac{\sum_{i \in G_{k,1}} \sum_{j \in G_{k,2}} \gamma_i \gamma_j \tilde{F}_{ijm_{ij}}}{\sum_{i \in G_{k,1}} \gamma_i} &\neq \\ \tilde{v}(G_k^<, G_{k,2}, G_k^>) + \frac{\sum_{i \in G_{k,2}} \sum_{j \in G_{k,1}} \gamma_i \gamma_j \tilde{F}_{ijm_{ij}}}{\sum_{i \in G_{k,2}} \gamma_i}, & \\ \forall m \in \mathbb{N}^{N \times N}, \text{ with } m_{ij} = m_{ji} \in \mathcal{I}_{n_{ij}}, \forall (i, j) \in G_{k,1} \times G_{k,2}, & \\ \forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \forall k \in I_M. &\end{aligned}$$

(This assumption is considered in theorem 5.3 below, where it is further explained.) Notice that for generic choices of the parameters, this condition is satisfied. E.g. if b is chosen randomly in \mathbb{R}^N from some continuous probability distribution, while all other parameters are fixed, then (\mathcal{A}) is satisfied with probability one, since (\mathcal{A}) not being satisfied would imply that b is lying on one of a countable number of hyperplanes in \mathbb{R}^N .

5.3 Results

The proofs are given later on.

Theorem 5.1. *The following set of conditions is necessary and sufficient for clustering behavior of all solutions of the system (5.2), with the cluster structure (G_1, \dots, G_M) independent of the initial condition:*

$$\tilde{v}(G_k^<, G_k, G_k^>) < \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>), \quad \forall k \in \mathcal{I}_{M-1}, \quad (5.4a)$$

$$\begin{aligned} \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) &\leq \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}), \\ \forall (G_{k,1}, G_{k,2}) &\in \mathcal{P}_2(G_k), \quad \forall k \in \mathcal{I}_M. \end{aligned} \quad (5.4b)$$

Theorem 5.2. *For each $b \in \mathbb{R}^N$, each $K \geq 0$, and each symmetric and irreducible matrix $F \in (\mathbb{R}^+)^{N \times N}$, there exists a unique ordered set partition G of \mathcal{I}_N , satisfying (5.4).*

As opposed to the model with all-to-all coupling, it is possible for clusters to split up when the coupling strength is increased, since for some cluster structure (5.4a) may be violated for large K (while (4.2a) will always hold for K sufficiently large). This is also related to the fact that there is no analog to remark 4.1 for (5.2), since the cluster structure is no longer restricted to follow the order of the b_i -values.

Theorem 5.3. *Let x be a solution of (5.2) with cluster structure $G = (G_1, \dots, G_M)$. For each $k \in \mathcal{I}_M$, if $i, j \in G_k$, then*

$$\begin{aligned} \lim_{t \rightarrow +\infty} (x_i(t) - x_j(t)) &\text{ exists and is independent of } x(0), \\ \lim_{t \rightarrow +\infty} \dot{x}_i(t) &\text{ exists and equals } \tilde{v}(G_k^<, G_k, G_k^>). \end{aligned}$$

Furthermore, if (A) holds, then $\lim_{t \rightarrow +\infty} (x_i(t) - x_j(t))$ is independent of $x(0)$, $\forall i, j \in G_k, \forall k \in \mathcal{I}_M$.

As for the basic model, within a cluster all differences between $x_i(t)$ -values will approach constants which are, in the generic case, independent of the initial condition, and the velocities of the agents will approach the asymptotic average cluster velocity.

For $b \in \mathbb{R}^N$ for which (A) is not satisfied, there exists a cluster G_k which can be partitioned in two subsets $G_{k,1}$ and $G_{k,2}$ with all interactions between agents from $G_{k,1}$ and agents from $G_{k,2}$ involving function values for which the corresponding functions f_{ij} are constant in some interval. As a consequence the interactions between agents from $G_{k,1}$ and agents from $G_{k,2}$ are insufficient to determine the relative position of the agents in G_k (e.g. adding some sufficiently small value $\epsilon > 0$ to the x_i -values of all agents in $G_{k,1}$ will result in the same values for the interactions). This may lead to dependence of the asymptotic values of the differences between agents in G_k on the initial condition.

Proposition 5.1. *The condition (A) is always satisfied if all functions f_{ij} for which $F_{ij} > 0$ are increasing in \mathbb{R} .*

Proof. Because of the irreducibility of the matrix F , for any $k \in G_k$, and for any $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, there exist $i \in G_{k,1}$ and $j \in G_{k,2}$ for which $F_{ij} > 0$, implying that $n_{ij} = 0$, such that (A) is satisfied. \square

Notice that theorems 5.1 and 5.2 reduce to 4.1 and 4.2 respectively if one sets $\gamma_i \triangleq \frac{1}{N}$, $\forall i \in \mathcal{I}_N$, and $f_{ij} \triangleq f$, $\forall i, j \in \mathcal{I}_N$. (Theorem 5.3 does not include some non-generic cases of theorem 4.3, since (A) is not necessarily satisfied under the conditions of theorem 4.3.)

5.3.1 Cluster structure for varying coupling strength

As an example, figure 5.1 shows the asymptotic velocities $v_i \triangleq \lim_{t \rightarrow +\infty} \dot{x}_i(t)$ of the agents for varying coupling strength K , with $\gamma_i = \frac{1}{6}$, $\forall i \in \mathcal{I}_6$, and

$$b = \begin{bmatrix} -2.4 \\ -2 \\ -1 \\ 1 \\ 1.3 \\ 4 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

5.3.2 Proof of theorem 5.1

Necessity of the conditions (5.4)

Assume there is a solution x of (5.2), exhibiting clustering behavior w.r.t. G . Choose $T > 0$ such that the distances between agents in different clusters are and remain at least $d = \max_{i \neq j} d_{ij}$. It follows that for any $k \in \mathcal{I}_{M-1}$ and $t > T$ we can apply (5.3) with (G_-, G_0, G_+) equal to resp. $(G_{k+1}^<, G_{k+1}, G_{k+1}^>)$ and $(G_k^<, G_k, G_k^>)$:

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} = \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>) - \tilde{v}(G_k^<, G_k, G_k^>).$$

Considering that $\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k}$ cannot be non-positive for all $t > T$ because of the clustering behavior of x , the condition (5.4a) follows.

Similarly one derives, for any $k \in \mathcal{I}_M$ and $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, and taking into account that the functions f_{ij} are non-decreasing, that

$$\langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}} \geq \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) - \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}),$$

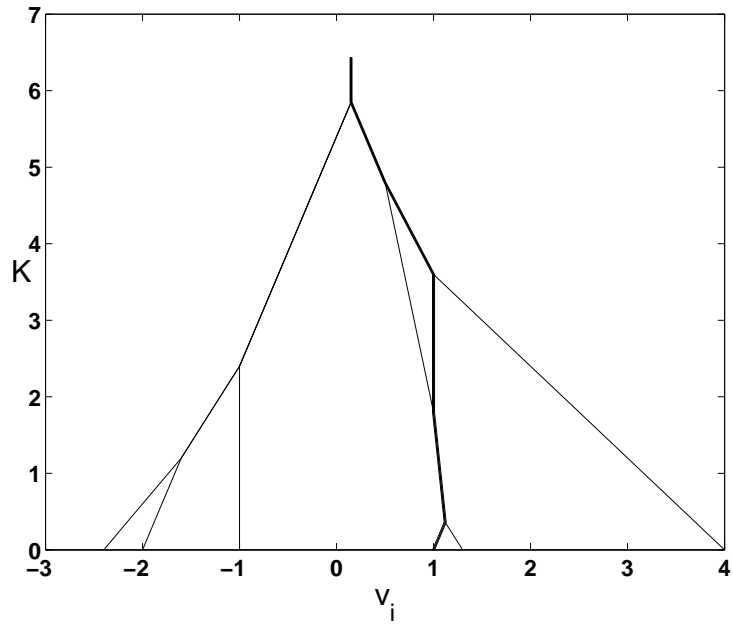


Figure 5.1: Asymptotic velocities v_i for varying coupling strength K . For clarity the curve corresponding to agent 4 is indicated with a thicker line.

and since $\langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}}$ cannot be positive for all $t > T$ because of the clustering behavior of x , (5.4b) follows.

Sufficiency of the conditions (5.4) for the existence of a solution exhibiting clustering behavior w.r.t. G

Assume (5.4) holds. Set $\gamma_{\min} \triangleq \min_{i \in \mathcal{I}_N} \gamma_i$. Consider the following region $R_d \subset \mathbb{R}^N$:

$$x \in R_d \Leftrightarrow \begin{cases} \langle x \rangle_{G_{k+1}} - \langle x \rangle_{G_k} \geq \frac{d \sum_{i \in G_k \cup G_{k+1}} \gamma_i}{2\gamma_{\min}}, & \forall k \in \mathcal{I}_{M-1}, \\ \langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}} \leq \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \\ \quad \forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), & \forall k \in \mathcal{I}_M. \end{cases}$$

We will show that R_d is a trapping region, i.e. if $x(t_0) \in R_d$, for some $t_0 \in \mathbb{R}$, then $x(t) \in R_d, \forall t \geq t_0$.

The second set of inequalities characterizing R_d can be rewritten as

$$\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_k} \leq \frac{d \sum_{i \in G_{k,1}} \gamma_i}{2\gamma_{\min}} \quad \text{or} \quad \langle x \rangle_{G_k} - \langle x \rangle_{G_{k,1}} \leq \frac{d \sum_{i \in G_{k,2}} \gamma_i}{2\gamma_{\min}},$$

$\forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \forall k \in \mathcal{I}_M$. Considering the inequalities for which $|G_{k,1}| = 1$ or $|G_{k,2}| = 1$ together with the first set of inequalities characterizing R_d , it follows that if $x \in R_d$ then for any $k \in \mathcal{I}_{M-1}$, $i \in G_k$, and $j \in G_{k+1}$, $x_j - x_i \geq \frac{d(\gamma_i + \gamma_j)}{2\gamma_{\min}} \geq d$. As a consequence we can derive that, if $x(t) \in R_d$, for some $t \in \mathbb{R}$, then (again applying (5.3))

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} = \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>) - \tilde{v}(G_k^<, G_k, G_k^>) > 0. \quad (5.5)$$

It follows that $x(t)$ cannot leave R_d by changing the sign of

$$\langle x(t) \rangle_{G_{k+1}} - \langle x(t) \rangle_{G_k} - \frac{d \sum_{i \in G_k \cup G_{k+1}} \gamma_i}{2\gamma_{\min}}$$

for some $k \in \mathcal{I}_{M-1}$.

We will now show that for d large enough, $x(t)$ cannot leave R_d by changing the sign of

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}},$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$.

Assume that $x(t) \in R_d$, for some $t \in \mathbb{R}$, with $x(t)$ at the boundary of R_d where

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} = \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \quad (5.6)$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$. Assume that $|G_{k,1}| \neq 1 \neq |G_{k,2}|$ and pick $i_1 \in G_{k,1}$ and $i_2 \in G_{k,2}$. Since $x(t) \in R_d$,

$$\langle x(t) \rangle_{G_{k,2} \cup \{i_1\}} - \langle x(t) \rangle_{G_{k,1} \setminus \{i_1\}} \leq \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \quad (5.7a)$$

$$\langle x(t) \rangle_{G_{k,2} \setminus \{i_2\}} - \langle x(t) \rangle_{G_{k,1} \cup \{i_2\}} \leq \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}. \quad (5.7b)$$

Multiplying (5.7a) with $\left(\sum_{i \in G_{k,1} \setminus \{i_1\}} \gamma_i\right) \left(\sum_{i \in G_{k,2} \cup \{i_1\}} \gamma_i\right)$, (5.7b) with $\left(\sum_{i \in G_{k,1} \cup \{i_2\}} \gamma_i\right) \left(\sum_{i \in G_{k,2} \setminus \{i_2\}} \gamma_i\right)$, and for each product subtracting (5.6) multiplied with $\left(\sum_{i \in G_{k,1}} \gamma_i\right) \left(\sum_{i \in G_{k,2}} \gamma_i\right)$, results in

$$\begin{aligned} x_{i_1} \gamma_{i_1} \sum_{i \in G_k} \gamma_i - \gamma_{i_1} \sum_{i \in G_k} x_i \gamma_i &\leq \gamma_{i_1} \left(\sum_{i \in G_{k,1}} \gamma_i - \sum_{i \in G_{k,2}} \gamma_i - \gamma_{i_1} \right) \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \\ -x_{i_2} \gamma_{i_2} \sum_{i \in G_k} \gamma_i + \gamma_{i_2} \sum_{i \in G_k} x_i \gamma_i &\leq \gamma_{i_2} \left(\sum_{i \in G_{k,2}} \gamma_i - \sum_{i \in G_{k,1}} \gamma_i - \gamma_{i_2} \right) \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}. \end{aligned}$$

After dividing by $\gamma_{i_1} \sum_{i \in G_k} \gamma_i$, resp. $\gamma_{i_2} \sum_{i \in G_k} \gamma_i$, we add both expressions, and obtain

$$x_{i_1} - x_{i_2} \leq -\frac{d(\gamma_{i_1} + \gamma_{i_2})}{2\gamma_{\min}} \leq -d. \quad (5.8)$$

If $|G_{k,1}| = 1$ or $|G_{k,2}| = 1$, similar results can be derived. It follows that if $x(t) \in R_d$ satisfies (5.6), then

$$\begin{aligned} \langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}} &= \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) - \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}) \\ &\leq 0, \end{aligned}$$

because of (5.4b). It follows that $x(t)$ cannot leave R_d by changing the sign of

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - \frac{d \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}},$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$.

Consequently, R_d is a trapping region for (5.2). For any solution x with $x(t_0) \in R_d$, for some $t_0 \in \mathbb{R}$, it follows that distances between agents of the same set G_k are bounded, agents are ordered by their membership to the different sets G_k , and taking into account (5.5), distances between agents in different sets G_k will grow unbounded in such a way that, for any $D > 0$, there exists a time after which these distances are and remain at least D . Since R_d is non-empty we conclude that there exist solutions of (5.2) exhibiting clustering behavior with respect to $G = (G_1, \dots, G_M)$.

Sufficiency of (5.4) for clustering behavior (w.r.t. G) of *all* solutions of (5.2)

Let x^* be a solution exhibiting clustering behavior w.r.t. G and let x be any other solution of (5.2). Consider the function

$$V : \mathbb{R} \rightarrow \mathbb{R} : t \mapsto V(t) = \sum_{i \in \mathcal{I}_N} \gamma_i (x_i^*(t) - x_i(t))^2.$$

Then, $\forall t \in \mathbb{R}$,

$$\begin{aligned} \dot{V}(t) &= 2 \sum_{i \in \mathcal{I}_N} \gamma_i (x_i^*(t) - x_i(t)) K \sum_{j \in \mathcal{I}_N} \gamma_j (f_{ij}(x_j^*(t) - x_i^*(t)) - f_{ij}(x_j(t) - x_i(t))) \\ &= K \sum_{i \in \mathcal{I}_N} \sum_{j \in \mathcal{I}_N} \gamma_i \gamma_j (x_i^*(t) - x_j^*(t) + x_j(t) - x_i(t)) \\ &\quad \times (f_{ij}(x_j^*(t) - x_i^*(t)) - f_{ij}(x_j(t) - x_i(t))) \\ &\leq 0, \end{aligned}$$

since the functions f_{ij} are non-decreasing. It follows that V is non-increasing, and therefore $|x_i^*(t) - x_i(t)|$ is bounded, for any $i \in \mathcal{I}_N$. This implies that x exhibits the same clustering behavior as x^* .

Remark 5.1. If we introduce a metric in \mathbb{R}^N such that the expression for $V(t)$ equals the square of the distance between $x^*(t)$ and $x(t)$, then the implication that $\dot{V} \leq 0$ leads to a unique clustering behavior for all initial conditions is strongly related to the notions of ‘convergent systems’ [40] and ‘contracting systems’ [31]. However, the system (5.2) exhibits a weaker property than *uniform* contraction (which is imposed in [40, 31] and related literature), and consequently the function V does not necessarily approach zero.

5.3.3 Proof of theorem 5.2

From the irreducibility of F it follows that for K sufficiently large, (5.4) is satisfied for $G = (\mathcal{I}_N)$. When decreasing K , there will be a lower bound K' below which (5.4) will not be satisfied anymore. In the following lemma we

will show that we can find another partition G' for which (5.4) holds for all $K \in (K'', K')$, for some $K'' < K'$. An important difference with the all-to-all coupled case is that, when lowering K , not only (5.4b) can cease to hold, but also (5.4a). This means that clusters may also grow with decreasing coupling strength. In the all-to-all coupled case, if (4.2a) holds for some $K^* > 0$, then it also holds for all $K \in [0, K^*]$, and agents in separate clusters for $K = K^*$ will remain separate for lower values of K .

Lemma 5.1. *Assume G^1 and G^2 are ordered set partitions of \mathcal{I}_N , with sizes M and $M + 1$ respectively ($M \in \mathbb{N}_0$), and that, for some $k' \in \mathcal{I}_M$,*

$$\begin{aligned} G_k^1 &= G_k^2, & \forall k < k', \\ G_{k'}^1 &= G_{k'}^2 \cup G_{k'+1}^2, \\ G_k^1 &= G_{k+1}^2, & \forall k > k', \end{aligned}$$

and therefore $(G_{k',1}^1, G_{k',2}^1) \triangleq (G_{k'}^2, G_{k'+1}^2) \in \mathcal{P}_2(G_{k'}^1)$.

Then

$$\tilde{v}(G_k^{1<}, G_k^1, G_k^{1>}) \leq \tilde{v}(G_{k+1}^{1<}, G_{k+1}^1, G_{k+1}^{1>}), \quad \forall k \in \mathcal{I}_{M-1}, \quad (5.9a)$$

$$\begin{aligned} \tilde{v}(G_k^{1<} \cup G_{k,1}^1, G_{k,2}^1, G_k^{1>}) &\leq \tilde{v}(G_k^{1<}, G_{k,1}^1, G_k^{1>} \cup G_{k,2}^1), \\ \forall (G_{k,1}^1, G_{k,2}^1) &\in \mathcal{P}_2(G_k^1), \quad \forall k \in \mathcal{I}_M, \end{aligned} \quad (5.9b)$$

with

$$\tilde{v}(G_{k'}^{1<} \cup G_{k',1}^1, G_{k',2}^1, G_{k'}^{1>}) = \tilde{v}(G_{k'}^{1<}, G_{k',1}^1, G_{k'}^{1>} \cup G_{k',2}^1), \quad (5.9c)$$

if and only if

$$\tilde{v}(G_k^{2<}, G_k^2, G_k^{2>}) \leq \tilde{v}(G_{k+1}^{2<}, G_{k+1}^2, G_{k+1}^{2>}), \quad \forall k \in \mathcal{I}_M, \quad (5.10a)$$

$$\begin{aligned} \tilde{v}(G_k^{2<} \cup G_{k,1}^2, G_{k,2}^2, G_k^{2>}) &\leq \tilde{v}(G_k^{2<}, G_{k,1}^2, G_k^{2>} \cup G_{k,2}^2), \\ \forall (G_{k,1}^2, G_{k,2}^2) &\in \mathcal{P}_2(G_k^2), \quad \forall k \in \mathcal{I}_{M+1}, \end{aligned} \quad (5.10b)$$

with

$$\tilde{v}(G_{k'}^{2<}, G_{k'}^2, G_{k'}^{2>}) = \tilde{v}(G_{k'+1}^{2<}, G_{k'+1}^2, G_{k'+1}^{2>}). \quad (5.10c)$$

The strict inequalities from (5.4a) have been replaced by non-strict inequalities to include non-generic cases where more than one inequality becomes an equality in (5.4) for a particular value of K . The lemma then has to be applied several times. The lemma is proven in section A.3.1 of the appendix. (Figure

A.1 on page 145 may be helpful to grasp the relation between cluster structures G^1 and G^2 from the lemma.)

We now give the proof of theorem 5.2. Let $[K', +\infty)$ denote the interval containing the K -values for which (5.4) is satisfied with $G = (\mathcal{I}_N)$. Then for $K = K'$, (at least) one of the inequalities in (5.4) will have become an equality, and we can apply the previous lemma for each equality, eventually resulting in a new ordered set partition G' . For this G' the inequalities from (5.4) corresponding to equalities at $K = K'$ will have the opposite sign of the corresponding inequalities from (5.4) with respect to G . Furthermore, if these inequalities are strict for G' , then they are non-strict for G and vice versa. This implies that for $K = K'$ either G or G' satisfies (5.4), and when decreasing K below K' , (5.4) will be satisfied for G' , and there exists a $K'' < K'$ such that (5.4) is satisfied for all $K \in (K'', K')$ with respect to the set partition G' .

We can repeat this argument until K reaches zero, proving that there exists an ordered set partition for which (5.4) is satisfied for all K in \mathbb{R}^+ . The ordered set partition is unique since a solution x of (5.2) can only exhibit clustering behavior w.r.t. at most one cluster structure and theorem 5.1 then implies that (5.4) can only hold for at most one cluster structure.

5.3.4 Proof of theorem 5.3

We consider a solution of the system (5.2) exhibiting clustering behavior w.r.t. the cluster structure (G_1, \dots, G_M) . We assume that $K > 0$. (The case $K = 0$ is trivial.) Choose $T > 0$ such that, for $t > T$, the distances between agents in different clusters are and remain at least d , with all agents ordered by their membership to a cluster. Fix k ; for $t > T$, the interaction $f_{ij}(x_j(t) - x_i(t))$ with an agent i in G_k from an agent j in another cluster equals its saturation value $\pm F_{ij}$. We will show that the distances between x -values in the cluster G_k will approach constant values, which are independent of the initial condition under the assumption (A). Let σ be a bijection from $\mathcal{I}_{|G_k|}$ to G_k and set

$$\begin{aligned} x'_i &\triangleq x_{\sigma(i)} - \langle x \rangle_{G_k}, \\ b'_i &\triangleq \tilde{v}(G_k^<, \{\sigma(i)\}, G_k^>) - \tilde{v}(G_k^<, G_k, G_k^>), \\ \gamma'_i &\triangleq \gamma_{\sigma(i)}, \\ f'_{ij} &\triangleq f_{\sigma(i)\sigma(j)}, \\ F'_{ij} &\triangleq F_{\sigma(i)\sigma(j)}, \\ n'_{ij} &\triangleq n_{\sigma(i)\sigma(j)}, \\ \tilde{F}'_{ijm_{ij}} &\triangleq \tilde{F}_{\sigma(i)\sigma(j)m_{ij}}, \quad \forall m_{ij} \in \mathcal{I}_{n'_{ij}}, \end{aligned}$$

$\forall t > T, \forall i, j \in \mathcal{I}_{|G_k|}$. Then we obtain

$$\begin{aligned} \dot{x}'_i(t) &= b_{\sigma(i)} + K \sum_{j \in \mathcal{I}_N} \gamma_j f_{\sigma(i)j}(x_j(t) - x_{\sigma(i)}(t)) \\ &\quad - \langle b \rangle_{G_k} - \frac{K}{\sum_{m \in G_k} \gamma_m} \sum_{m \in G_k} \gamma_m \sum_{j \in \mathcal{I}_N} \gamma_j f_{mj}(x_j(t) - x_m(t)) \\ &= \tilde{v}(G_k^<, \{\sigma(i)\}, G_k^>) + K \sum_{j \in \mathcal{I}_{|G_k|}} \gamma_{\sigma(j)} f_{\sigma(i)\sigma(j)}(x_{\sigma(j)}(t) - x_{\sigma(i)}(t)) \\ &\quad - \tilde{v}(G_k^<, G_k, G_k^>), \end{aligned}$$

$\forall t > T, \forall i \in \mathcal{I}_{|G_k|}$, or

$$\dot{x}'_i(t) = b'_i + K \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j f'_{ij}(x'_j(t) - x'_i(t)), \quad \forall t > T, \quad \forall i \in \mathcal{I}_{|G_k|}, \quad (5.11)$$

The system (5.11) describes the dynamics of the agents belonging to G_k . The definition of x' implies that (5.11) is to be considered on the state-space $L \triangleq \{x' \in \mathbb{R}^{|G_k|} : \langle x' \rangle'_{\mathcal{I}_{|G_k|}} = 0\}$, where $\langle \cdot \rangle'$ denotes the average with respect to the weighting factors γ'_i . Consider the function

$$W_k : L \rightarrow \mathbb{R} : x' \mapsto W_k(x') = - \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i b'_i x'_i + \frac{K}{2} \sum_{i, j \in \mathcal{I}_{|G_k|}} \int_0^{x'_j - x'_i} \gamma'_i \gamma'_j f'_{ij}(\tau) d\tau.$$

Then

$$\frac{\partial W_k}{\partial x'_i}(x'(t)) = -\gamma'_i \dot{x}'_i(t), \quad \forall t > T, \quad \forall i \in \mathcal{I}_{|G_k|},$$

and (5.11) is the gradient system corresponding to the function W_k w.r.t. the metric represented by the diagonal matrix G , with $G_{ii} = \frac{1}{\gamma'_i}$, $\forall i \in \mathcal{I}_{|G_k|}$ (see section A.3.2 of the appendix). Taking into account that a solution x' of (5.11) is bounded because of the clustering behavior and that the function W_k is convex (as it is a sum of convex functions) we can apply theorem A.1 (see again section A.3.2 of the appendix) and conclude that

$$\lim_{t \rightarrow +\infty} x'(t) = x'^e$$

for some equilibrium point x'^e of (5.11). It follows that, $\forall i, j \in \mathcal{I}_{|G_k|}$,

$$\lim_{t \rightarrow +\infty} (x_{\sigma(i)}(t) - x_{\sigma(j)}(t)) = \lim_{t \rightarrow +\infty} (x'_i(t) - x'_j(t)) = x'^e_i - x'^e_j,$$

and that, $\forall i \in \mathcal{I}_{|G_k|}$,

$$\lim_{t \rightarrow +\infty} \dot{x}_{\sigma(i)}(t) = \lim_{t \rightarrow +\infty} (\dot{x}'_i(t) + \langle \dot{x}(t) \rangle_{G_k}) = \tilde{v}(G_k^<, G_k, G_k^>).$$

For the remainder of this section we will show that, under the assumption (\mathcal{A}) , the equilibrium point x^{e} of (5.11) is *unique*, implying that $\lim_{t \rightarrow +\infty} (x_i(t) - x_j(t))$ (with $i, j \in G_k$ for some $k \in \mathcal{I}_M$) is independent of the initial condition.

An equilibrium point of (5.11) solves the equations

$$b'_i + K \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j f'_{ij}(x'_j - x'_i) = 0, \quad \forall i \in \mathcal{I}_{|G_k|}. \quad (5.12)$$

Assume that $x^{e,1}, x^{e,2} \in L$ are two solutions of (5.12). Then it follows that

$$\sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j \left(f'_{ij}(x_j^{e,1} - x_i^{e,1}) - f'_{ij}(x_j^{e,2} - x_i^{e,2}) \right) = 0, \quad \forall i \in \mathcal{I}_{|G_k|}.$$

Multiplying with $\gamma'_i(x_i^{e,1} - x_i^{e,2})$, summing over i and then switching the summation indices i and j in $\frac{1}{2}$ of the result (inspired by the expression for \dot{V} in section 5.3.2), we obtain

$$\begin{aligned} \frac{1}{2} \sum_{i \in \mathcal{I}_{|G_k|}} \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_i \gamma'_j \left((x_j^{e,2} - x_i^{e,2}) - (x_j^{e,1} - x_i^{e,1}) \right) \\ \times \left(f'_{ij}(x_j^{e,1} - x_i^{e,1}) - f'_{ij}(x_j^{e,2} - x_i^{e,2}) \right) = 0, \quad \forall i \in \mathcal{I}_{|G_k|}. \end{aligned}$$

Since the functions f'_{ij} are non-decreasing it follows that each separate term in the left hand side is non-positive, and therefore they all equal zero. It follows that

$$f'_{ij}(x_j^{e,1} - x_i^{e,1}) = f'_{ij}(x_j^{e,2} - x_i^{e,2}), \quad \forall i, j \in \mathcal{I}_{|G_k|}. \quad (5.13)$$

Consider the undirected graph $\mathcal{G}(x^{e,1})$, defined by connecting i and j ($i \neq j$) if and only if $f'_{ij}(x_j^{e,1} - x_i^{e,1}) \neq \tilde{F}'_{ijm_{ij}}$, $\forall m_{ij} \in \mathcal{I}_{m'_{ij}}$. If $\mathcal{G}(x^{e,1})$ would not be connected, then denoting by $G'_{k,1} \subsetneq \mathcal{I}_{|G_k|}$ one of the connected components of $\mathcal{G}(x^{e,1})$, and setting $G'_{k,2} \triangleq \mathcal{I}_{|G_k|} \setminus G'_{k,1}$, we would obtain a contradiction with

the assumption (A) by expressing that

$$\begin{aligned} & \frac{1}{\sum_{i \in G'_{k,1}} \gamma'_i} \sum_{i \in G'_{k,1}} \gamma'_i \left(b'_i + K \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j f'_{ij}(x'_j - x'_i) \right) \\ &= \frac{1}{\sum_{i \in G'_{k,2}} \gamma'_i} \sum_{i \in G'_{k,2}} \gamma'_i \left(b'_i + K \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j f'_{ij}(x'_j - x'_i) \right). \end{aligned}$$

The equation (5.13) implies that $x'_j{}^{e,2} - x'_i{}^{e,2} = x'_j{}^{e,1} - x'_i{}^{e,1}$ if $\{i, j\} \in \mathcal{G}(x'^{e,1})$. Since $\mathcal{G}(x'^{e,1})$ is connected, for any i and j in $\mathcal{I}_{|G_k|}$ ($i \neq j$) there exist agents $l_0 = i, l_1, \dots, l_{P-1}, l_P = j$ (where P denotes the corresponding path length) such that subsequent agents l_{m-1} and l_m have an edge of $\mathcal{G}(x'^{e,1})$ in common and thus $x'_{l_m}{}^{e,2} - x'_{l_{m-1}}{}^{e,2} = x'_{l_m}{}^{e,1} - x'_{l_{m-1}}{}^{e,1}$. Consequently

$$x'_j{}^{e,2} - x'_i{}^{e,2} = \sum_{m=1}^P x'_{l_m}{}^{e,2} - x'_{l_{m-1}}{}^{e,2} = \sum_{m=1}^P x'_{l_m}{}^{e,1} - x'_{l_{m-1}}{}^{e,1} = x'_j{}^{e,1} - x'_i{}^{e,1}.$$

This implies $x'_i{}^{e,2} - \langle x'^{e,2} \rangle'_{\mathcal{I}_{|G_k|}} = x'_i{}^{e,1} - \langle x'^{e,1} \rangle'_{\mathcal{I}_{|G_k|}}$, $\forall i \in \mathcal{I}_{|G_k|}$, and since $\langle x'^{e,2} \rangle'_{\mathcal{I}_{|G_k|}} = \langle x'^{e,1} \rangle'_{\mathcal{I}_{|G_k|}} = 0$, it follows that $x'^{e,2} = x'^{e,1}$.

5.4 Conclusion

The results pertaining to the basic model can be generalized for the extension where the interaction is determined by a general network structure and weighting factors. The main difference — due to the general network structure — is the possibility for clusters to split up when the coupling strength is increased.

Chapter 6

General saturating and non-decreasing interaction functions

In this chapter we show that the results from the previous chapter regarding the clustering model with a finite number of agents can be extended to more general interaction functions. The interaction functions are still non-decreasing and saturating, but do not necessarily reach their saturation values.

6.1 The model

We consider the system equations (5.2):

$$\dot{x}_i(t) = b_i + K \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (6.1)$$

where the functions f_{ij} are non-decreasing, Lipschitz continuous and satisfy

$$f_{ji}(x) = -f_{ij}(-x), \quad \forall x \in \mathbb{R},$$
$$\lim_{x \rightarrow +\infty} f_{ij}(x) = F_{ij}, \quad \text{and thus} \quad \lim_{x \rightarrow -\infty} f_{ij}(x) = -F_{ji}.$$

for some F_{ij} in \mathbb{R} , for all i, j in \mathcal{I}_N .

6.2 Results

The proofs are given later on.

Theorem 6.1. *Consider the following inequalities*

$$\tilde{v}(G_k^<, G_k, G_k^>) < \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>), \quad \forall k \in \mathcal{I}_{M-1}, \quad (6.2a)$$

$$\begin{aligned} \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) < \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}), \\ \forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \quad \forall k \in \mathcal{I}_M. \end{aligned} \quad (6.2b)$$

Denote by (6.2a'), resp. (6.2b'), the inequalities (6.2a), resp. (6.2b), in which the strict inequalities are replaced by non-strict inequalities. Denote by (6.2') the inequalities (6.2a') together with (6.2b'). Then the conditions (6.2), resp. (6.2'), are sufficient, resp. necessary, for clustering behavior w.r.t. G of all solutions of the system (6.1).

Since the saturation values of the functions f_{ij} are not necessarily reached, (5.4b) (i.e. (6.2b) with non-strict inequalities) is insufficient to guarantee boundedness of the clusters. For the same reason (6.2a) is not necessary for the distances between agents from different clusters to tend to infinity.

Set $d_{ij} \triangleq \inf f_{ij}^{-1}(F_{ij})$ (where $\inf \emptyset = +\infty$).

Theorem 6.2. *Let x be a solution of (6.1) with cluster structure $G = (G_1, \dots, G_M)$ for some $b \in \mathbb{R}^N$, and assume that (\mathcal{A}) holds in some neighborhood (in \mathbb{R}^N) of b (i.e. (\mathcal{A}) remains satisfied for small perturbations in b). For each $k \in \mathcal{I}_M$, if $i, j \in G_k$, then*

$$\begin{aligned} \lim_{t \rightarrow +\infty} (x_i(t) - x_j(t)) \text{ exists and is independent of } x(0), \\ \lim_{t \rightarrow +\infty} \dot{x}_i(t) \text{ exists and equals } \tilde{v}(G_k^<, G_k, G_k^>). \end{aligned}$$

In other words, if (\mathcal{A}) holds in some neighborhood of b , then within a cluster all differences between x_i -values will approach constants, which are independent of the initial condition, and the velocities of the agents will approach the asymptotic average cluster velocity.

Although the formulation of this theorem is similar to the formulation of theorem (5.3), its proof will be more involved, since the interaction between different clusters does not necessarily become constant, and therefore the analog of the subsystem (5.11) will not necessarily be time-invariant.

6.2.1 Proof of theorem 6.1

Set

$$\varphi(D) \triangleq \max_{i \in \mathcal{I}_N} \sum_{j \in \mathcal{I}_N} \gamma_j \max(F_{ij} - f_{ij}(D), F_{ij} + f_{ij}(-D)).$$

(Notice that $\lim_{D \rightarrow +\infty} \varphi(D) = 0$.)

For a partition $\{G_-, G_0, G_+\}$ of \mathcal{I}_N , the function value $\tilde{v}(G_-, G_0, G_+)$ now represents the limit value of the average velocity $\langle \dot{x}(t) \rangle_{G_0}$ of the agents in G_0 when the agents in G_- , resp. G_+ , have x -values smaller than, resp. larger than, the x -values of the agents in G_0 , with the differences tending to infinity.

Assume that, at some time instance t_0 , the agents in G_0 are separated by at least a distance D from all other agents, and denote by G_- , resp. G_+ , the set of agents with $x(t_0)$ -values smaller, resp. larger, than the $x(t_0)$ -values of the agents in G_0 . Then, for all $i \in G_0$, it follows that

$$\left| \dot{x}_i(t_0) - b_i + \sum_{j \in G_-} \gamma_j F_{ij} - \sum_{j \in G_0} \gamma_j f_{ij}(x_j(t_0) - x_i(t_0)) - \sum_{j \in G_+} \gamma_j F_{ij} \right| \leq \varphi(D),$$

and thus

$$|\langle \dot{x}(t_0) \rangle_{G_0} - \tilde{v}(G_-, G_0, G_+)| \leq \varphi(D). \quad (6.3)$$

Necessity of the conditions (6.2')

Let x be a solution of (6.1) exhibiting clustering behavior w.r.t. G . Pick a $D > 0$ and then choose $T > 0$ such that the distances between agents in different clusters are and remain at least D .

For any $k \in \mathcal{I}_{M-1}$ and $t > T$

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} \leq \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>) - \tilde{v}(G_k^<, G_k, G_k^>) + 2\varphi(D).$$

Considering that $\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k}$ cannot be negative for all $t > T$ because of the clustering behavior of x , and then letting $D \rightarrow \infty$ and therefore $\varphi(D) \rightarrow 0$, the condition (6.2a') follows.

For any $k \in \mathcal{I}_M$ and $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, and taking into account that the functions f_{ij} are non-decreasing,

$$\begin{aligned} \langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}} \\ \geq \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) - \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}) - 2\varphi(D), \end{aligned}$$

and since $\langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}}$ cannot be positive for all $t > T$ because of the clustering behavior of x , (6.2b') follows if we again consider the limit $D \rightarrow \infty$.

Sufficiency of the conditions (6.2) for the existence of a solution exhibiting clustering behavior w.r.t. G

Assume (6.2) holds. For arbitrary $D > 0$, consider the region $R_D \subset \mathbb{R}^N$:

$$x \in R_D \Leftrightarrow \begin{cases} \langle x \rangle_{G_{k+1}} - \langle x \rangle_{G_k} \geq \frac{D \sum_{i \in G_k \cup G_{k+1}} \gamma_i}{2\gamma_{\min}}, & \forall k \in \mathcal{I}_{M-1}, \\ \langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}} \leq \frac{D \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \\ \quad \forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), & \forall k \in \mathcal{I}_M, \end{cases}$$

where $D > 0$ has to be determined yet. We will show that for D sufficiently large R_D is a trapping region.

The second set of inequalities characterizing R_D can be rewritten as

$$\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_k} \leq \frac{D \sum_{i \in G_{k,1}} \gamma_i}{2\gamma_{\min}} \quad \text{or} \quad \langle x \rangle_{G_k} - \langle x \rangle_{G_{k,1}} \leq \frac{D \sum_{i \in G_{k,2}} \gamma_i}{2\gamma_{\min}},$$

$\forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \forall k \in \mathcal{I}_M$. Considering the inequalities for which $|G_{k,1}| = 1$ or $|G_{k,2}| = 1$ together with the first set of inequalities characterizing R_D , it follows that if $x \in R_D$ then for any $k \in \mathcal{I}_{M-1}$, $i \in G_k$, and $j \in G_{k+1}$, $x_j - x_i \geq \frac{D(\gamma_i + \gamma_j)}{2\gamma_{\min}} \geq D$. As a consequence we can derive that, if $x(t) \in R_D$, for some $t \in \mathbb{R}$, then

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} \geq \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>) - \tilde{v}(G_k^<, G_k, G_k^>) - 2\varphi(D),$$

and because of (6.2a) there exist $D_1 > 0$ and $\epsilon > 0$, such that

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} > \epsilon, \quad \forall t \geq t_0, \quad \forall k \in \mathcal{I}_M, \quad (6.4)$$

whenever $x(t) \in R_D$, with $D \geq D_1$. It follows that $x(t)$ cannot leave R_D ($D \geq D_1$) by changing the sign of

$$\langle x(t) \rangle_{G_{k+1}} - \langle x(t) \rangle_{G_k} - \frac{D \sum_{i \in G_k \cup G_{k+1}} \gamma_i}{2\gamma_{\min}}$$

for some $k \in \mathcal{I}_{M-1}$.

We will now show that for D large enough, $x(t)$ cannot leave R_D by changing the sign of

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - \frac{D \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}},$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$.

Assume that $x(t) \in R_D$, for some $t \in \mathbb{R}$, with $x(t)$ at the boundary of R_D where

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} = \frac{D \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \quad (6.5)$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$.

Similarly as in the proof of theorem 5.1 we can derive that for $i_1 \in G_{k,1}$ and $i_2 \in G_{k,2}$

$$x_{i_1}(t) - x_{i_2}(t) \leq -D,$$

and therefore

$$\langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}} \leq \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>) - \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}) + 2\varphi(D).$$

Because of (6.2b) this is negative for $D \geq D_2$, for some sufficiently large D_2 . It follows that $x(t)$ cannot leave R_D (with $D \geq D_2$) by changing the sign of

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - \frac{D \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}},$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$.

Consequently, if $D \geq \max(D_1, D_2)$ then R_D is a trapping region for (6.1). For any solution x with $x(t_0) \in R_D$, for some $t_0 \in \mathbb{R}$, it follows that distances between agents of the same set G_k are bounded, agents are ordered by their membership to the different sets G_k , and taking into account (6.4), distances between agents in different sets G_k will grow unbounded in such a way that, for any $D > 0$, there exists a time after which these distances are and remain at least D . Since R_D is non-empty we conclude that there exist solutions of (6.1) exhibiting clustering behavior with respect to $G = (G_1, \dots, G_M)$.

Sufficiency of (6.2) for clustering behavior (w.r.t. G) of *all* solutions of (6.1)

For a solution x^* exhibiting clustering behavior w.r.t. G and any other solution x of (6.1), we can again (as in section 5.3.2) derive that the function

$$V : \mathbb{R} \rightarrow \mathbb{R} : t \mapsto V(t) = \sum_{i \in \mathcal{I}_N} \gamma_i (x_i^*(t) - x_i(t))^2$$

satisfies

$$\dot{V}(t) \leq 0,$$

$\forall t \in \mathbb{R}$, and therefore x exhibits the same clustering behavior as x^* .

6.2.2 Proof of theorem 6.2

We consider a solution of the system (6.1) exhibiting clustering behavior w.r.t. the cluster structure (G_1, \dots, G_M) , and we assume that (\mathcal{A}) is satisfied in a neighborhood of b . We again restrict to the case $K > 0$. Pick $D > 0$ and choose $T > 0$ such that, for $t > T$, the distances between agents in different clusters are and remain at least D , with all agents ordered by their membership to a cluster. Fix k ; for $t > T$, the interaction with an agent i in G_k from an agent j in another cluster will deviate at most $\max(F_{ij} - f_{ij}(D), F_{ij} + f_{ij}(-D))$ from the corresponding saturation value $\pm F_{ij}$. By increasing D , which is possible because of the clustering behavior, we will show that the distances between x -values in the cluster G_k will approach constant values, which are independent of the initial condition. Let σ be a bijection from $\mathcal{I}_{|G_k|}$ to G_k and set

$$\begin{aligned} x'_i &\triangleq x_{\sigma(i)} - \langle x \rangle_{G_k}, \\ b'_i &\triangleq \tilde{v}(G_k^<, \{\sigma(i)\}, G_k^>) - \tilde{v}(G_k^<, G_k, G_k^>), \\ \gamma'_i &\triangleq \gamma_{\sigma(i)}, \\ f'_{ij} &\triangleq f_{\sigma(i)\sigma(j)}, \\ F'_{ij} &\triangleq F_{\sigma(i)\sigma(j)}, \end{aligned}$$

$\forall i, j \in \mathcal{I}_{|G_k|}$. Then, repeating the calculations from section 5.3.4, we obtain

$$\dot{x}'_i(t) = b'_i + \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j f'_{ij}(x'_j(t) - x'_i(t)) + \mathcal{R}_i(t), \quad \forall i \in \mathcal{I}_{|G_k|}, \quad (6.6)$$

for some functions \mathcal{R}_i which satisfy

$$|\mathcal{R}_i(t)| \leq 2\varphi(D), \quad \forall t \geq T, \quad \forall i \in \mathcal{I}_{|G_k|},$$

and

$$\langle \mathcal{R}(t) \rangle'_{\mathcal{I}_{|G_k|}} = 0, \quad \forall t \in \mathbb{R}.$$

The time-varying system (6.6), with state space $L = \{x' \in \mathbb{R}^{|G_k|} : \langle x' \rangle'_{\mathcal{I}_{|G_k|}} = 0\}$, describes the dynamics of the agents belonging to G_k . The function

$$W_k : L \rightarrow \mathbb{R} : x' \mapsto W_k(x') = - \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i b'_i x'_i + \frac{1}{2} \sum_{i, j \in \mathcal{I}_{|G_k|}} \int_0^{x'_j - x'_i} \gamma'_i \gamma'_j f'_{ij}(\tau) d\tau,$$

satisfies

$$\frac{\partial W_k}{\partial x'_i}(x'(t)) = -\gamma'_i(\dot{x}'_i(t) - \mathcal{R}_i(t)) = -\gamma'_i \mathcal{F}_i(x'(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_{|G_k|}$$

with

$$\mathcal{F}_i(x') \triangleq b'_i + \sum_{j \in \mathcal{I}_{|G_k|}} \gamma'_j f'_{ij}(x'_j - x'_i), \quad \forall x' \in L, \quad \forall i \in \mathcal{I}_{|G_k|},$$

and along a solution x' of (6.6) it follows that

$$\begin{aligned} \frac{d(W_k \circ x')}{dt}(t) &= - \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i (\mathcal{F}_i(x'(t)))^2 - \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i \mathcal{F}_i(x'(t)) \mathcal{R}_i(t) \\ &\leq - \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i (\mathcal{F}_i(x'(t)))^2 + 2\mathcal{M}\varphi(D), \end{aligned}$$

with

$$\mathcal{M} \triangleq \sup_{x' \in L} \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i |\mathcal{F}_i(x')|.$$

Set

$$\Omega_D \triangleq \{x' \in L : \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i (\mathcal{F}_i(x'))^2 \leq 2\mathcal{M}\varphi(D) + \frac{1}{1+D^2}\}.$$

Then, for all $t \in \mathbb{R}$ for which $x'(t) \in L \setminus \Omega_D$,

$$\frac{d(W_k \circ x')}{dt} \leq -\frac{1}{1+D^2} < 0, \quad (6.7)$$

and since x' is bounded because of the clustering behavior, $W_k \circ x'$ is also bounded, and therefore, for any $t_0 \in \mathbb{R}$, $x'(t)$ cannot remain in $L \setminus \Omega_D$ for all $t > t_0$. For any $t_0 \in \mathbb{R}$, it follows that there exists a $t > t_0$ for which $x'(t) \in \Omega_D$, and thus, setting $D \triangleq n \in \mathbb{N}_0$, there exist $t_1 < t_2 < \dots$ with $x'(t_n) \in \Omega_n$, $\forall n \in \mathbb{N}_0$, and thus

$$\lim_{n \rightarrow \infty} x'(t_n) = x'^e,$$

for some $x'^e \in L$, satisfying (because of the continuity of the functions \mathcal{F}_i)

$$\mathcal{F}_i(x'^e) = 0, \quad \forall i \in \mathcal{I}_{|G_k|}.$$

As in the proof of theorem 5.3, we can again use (\mathcal{A}) to derive that x'^e is the unique solution of the equations $\mathcal{F}_i(x') = 0, \forall i \in \mathcal{I}_{|G_k|}$, with $x' \in L$. From the expression for $\frac{\partial W_k}{\partial x'_i}$, it follows that x'^e also constitutes a unique extremum point for the function W_k , and since the function W_k is convex (as it is a sum of convex functions), it attains its absolute minimum in x'^e . Therefore, given an $\epsilon_0 > 0$ there exists an $\epsilon_1 > 0$ such that

$$\|x' - x'^e\| < \epsilon_0, \quad \forall x' \in L \text{ with } W_k(x') \leq W_k(x'^e) + \epsilon_1,$$

and because of the continuity of W_k there exists an $\epsilon_2 > 0$ such that

$$W_k(x') \leq W_k(x'^e) + \epsilon_1, \quad \forall x' \in L \text{ with } \|x' - x'^e\| < \epsilon_2.$$

The proof of the uniqueness of x'^e as a solution of $\mathcal{F}(x') = 0$ and the assumption that (\mathcal{A}) holds in a neighborhood of b together with the definition of \mathcal{F} imply that $\mathcal{F}^{-1} : L \rightarrow L$ is well-defined in a neighborhood of 0 (with $\mathcal{F}^{-1}(0) = x'^e$), and therefore there exists an $\epsilon_3 > 0$ such that

$$\|x' - x'^e\| < \epsilon_2, \quad \forall x' \in L \text{ with } \sum_{i \in \mathcal{I}_{|G_k|}} \gamma'_i(\mathcal{F}_i(x'))^2 < \epsilon_3.$$

Choose $D > 0$ sufficiently large such that $2\mathcal{M}\varphi(D) + \frac{1}{1+D^2} < \epsilon_3$. It then follows that $W_k(x') \leq W_k(x'^e) + \epsilon_1, \forall x' \in \Omega_D$, and because of (6.7), the set $\{x' \in L : W_k(x') \leq W_k(x'^e) + \epsilon_1\}$ is a trapping region for the system (6.6). Pick $n > D$ ($n \in \mathbb{N}$) such that $x'(t_n) \in \Omega_D$ and therefore

$$W_k(x'(t)) \leq W_k(x'^e) + \epsilon_1, \quad \forall t \geq t_n,$$

and thus

$$\|x'(t) - x'^e\| < \epsilon_0.$$

Since for any $\epsilon_0 > 0$ there exists a $t_n > 0$ with the above property, we can conclude that

$$\lim_{t \rightarrow +\infty} x'(t) = x'^e.$$

It follows that, $\forall i, j \in \mathcal{I}_{|G_k|}$,

$$\lim_{t \rightarrow +\infty} (x_{\sigma(i)}(t) - x_{\sigma(j)}(t)) = \lim_{t \rightarrow +\infty} (x'_i(t) - x'_j(t)) = x'^e_i - x'^e_j,$$

and that, $\forall i \in \mathcal{I}_{|G_k|}$,

$$\lim_{t \rightarrow +\infty} \dot{x}_{\sigma(i)}(t) = \lim_{t \rightarrow +\infty} (\dot{x}'_i(t) + \langle \dot{x}(t) \rangle_{G_k}),$$

with $\lim_{t \rightarrow +\infty} \dot{x}'_i(t) = 0$ and

$$|\langle \dot{x}(t) \rangle_{G_k} - \tilde{v}(G_k^<, G_k, G_k^>)| \leq \varphi(D),$$

or, considering the limits $D \rightarrow +\infty$ and $t \rightarrow +\infty$:

$$\lim_{t \rightarrow +\infty} \dot{x}_{\sigma(i)}(t) = \tilde{v}(G_k^<, G_k, G_k^>).$$

6.3 Conclusion

When the conditions on the interaction functions are relaxed, such that they are still saturating but do not necessarily reach their saturation values, one can obtain similar results as for the model where the saturation values are reached. Since it is not known whether or not the interaction functions reach their saturation values, the necessary conditions for clustering behavior are different from the sufficient conditions.

Chapter 7

The time-dependent model

We reconsider the model (5.2) but we assume that the b_i -values and the interaction functions are time-dependent. We derive necessary and sufficient conditions for clustering behavior.

7.1 The model

The system equations are:

$$\dot{x}_i(t) = b_i(t) + K \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t), t), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad (7.1)$$

with $\gamma_j > 0$, $K \geq 0$, b_i continuous, and $f_{ij}(x, t)$ is continuous in t and Lipschitz continuous, non-decreasing and saturating in x . We assume that the functions f_{ij} attain their saturation values, i.e.

$$f_{ij}(x, t) = F_{ij}(t), \quad \forall x \geq d, \quad \forall t \in \mathbb{R}, \quad \forall i, j \in \mathcal{I}_N,$$

for some functions F_{ij} and some $d \in \mathbb{R}^+$, and satisfy

$$f_{ji}(x, t) = -f_{ij}(-x, t), \quad \forall x, t \in \mathbb{R}, \quad \forall i, j \in \mathcal{I}_N.$$

From their definition it follows that the functions F_{ij} are continuous.

7.2 Results

Redefine the function \tilde{v} to include the time-dependence:

$$\begin{aligned} \tilde{v}(G_-, G_0, G_+, t) &\triangleq \langle b(t) \rangle_{G_0} \\ &+ \frac{K}{\sum_{i \in G_0} \gamma_i} \sum_{i \in G_0} \gamma_i \left(\sum_{j \in G_+} \gamma_j F_{ij}(t) - \sum_{j \in G_-} \gamma_j F_{ij}(t) \right), \end{aligned}$$

$\forall t \in \mathbb{R}, \forall G_-, G_0, G_+ \subset \mathcal{I}_N$ with G_0 non-empty. When $\{G_-, G_0, G_+\}$ partitions \mathcal{I}_N , $\tilde{v}(G_-, G_0, G_+, t)$ again equals the average velocity $\langle \dot{x}(t) \rangle_{G_0}$ of the agents in G_0 when the agents in G_- , resp. G_+ , have x -values which are at least d smaller than, resp. d larger than, the x -values of the agents in G_0 .

Theorem 7.1. *The conditions*

$$\lim_{t \rightarrow +\infty} \int_0^t (\tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>, t') - \tilde{v}(G_k^<, G_k, G_k^>, t')) dt' = +\infty, \quad (7.2a)$$

$$\forall k \in \mathcal{I}_{M-1},$$

$$\begin{aligned} \exists c \in \mathbb{R} : \int_{t_0}^t (\tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>, t') - \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}, t')) dt' < c, \\ \forall t, t_0 \in \mathbb{R}^+ \text{ with } t \geq t_0, \quad \forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \quad \forall k \in \mathcal{I}_M. \end{aligned} \quad (7.2b)$$

are necessary and sufficient for clustering behavior w.r.t. G of all solutions of the system (7.1).

There is no counterpart to theorem 5.2, since a solution of (7.1) does not necessarily exhibit clustering behavior. Consider for instance the case $N = 2$, $K = 0$, and $b_1(t) = 0$, $b_2(t) = t \cos(t)$, $\forall t \in \mathbb{R}$. Then

$$x_2(t) - x_1(t) = t \sin(t) + \cos(t) - 1 + x_2(0) - x_1(0), \quad \forall t \in \mathbb{R},$$

which is not bounded for $t \geq 0$, while $x_2(t) - x_1(t)$ will switch sign infinitely often.

If all functions b_i and f_{ij} are periodic in time with period T_p , then, setting

$$\tilde{v}(G_-, G_0, G_+) \triangleq \frac{1}{T_p} \int_0^{T_p} \tilde{v}(G_-, G_0, G_+, t) dt,$$

$\forall G_-, G_0, G_+ \subset \mathcal{I}_N$ with G_0 non-empty, the conditions (7.2) are equivalent with (5.4), and consequently, theorem 5.2 can be applied.

7.2.1 Proof of theorem 7.1

Necessity of the conditions (7.2)

Assume there is a solution x of (7.1), exhibiting clustering behavior w.r.t. G . Choose $T > 0$ such that the distances between agents in different clusters are

and remain at least d . For any $k \in \mathcal{I}_{M-1}$ and $t > T$

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} = \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>, t) - \tilde{v}(G_k^<, G_k, G_k^>, t).$$

From the clustering behavior of x immediately follows that

$$\lim_{t \rightarrow +\infty} \langle x(t) \rangle_{G_{k+1}} - \langle x(t) \rangle_{G_k} = +\infty,$$

implying (7.2a).

For any $k \in \mathcal{I}_M$, any $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, and $t > T$,

$$\langle \dot{x}(t) \rangle_{G_{k,2}} - \langle \dot{x}(t) \rangle_{G_{k,1}} \geq \tilde{v}(G_k^< \cup G_{k,1}, G_{k,2}, G_k^>, t) - \tilde{v}(G_k^<, G_{k,1}, G_k^> \cup G_{k,2}, t),$$

and since $\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - (\langle x(t_0) \rangle_{G_{k,2}} - \langle x(t_0) \rangle_{G_{k,1}})$ is bounded for all $t, t_0 \in \mathbb{R}$ with $t \geq t_0 \geq 0$, (7.2b) follows.

Sufficiency of the conditions (7.2) for the existence of a solution exhibiting clustering behavior w.r.t. G

Assume (7.2) holds. Set $\gamma_{\min} \triangleq \min_{i \in \mathcal{I}_N} \gamma_i$. For any $D_1 \geq D_2 > 0$, consider the following region $R_{D_1, D_2} \subset \mathbb{R}^N$:

$$x \in R_{D_1, D_2} \Leftrightarrow \begin{cases} \langle x \rangle_{G_{k+1}} - \langle x \rangle_{G_k} \geq \frac{D_1 \sum_{i \in G_k \cup G_{k+1}} \gamma_i}{2\gamma_{\min}}, & \forall k \in \mathcal{I}_{M-1}, \\ \langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}} \leq \frac{D_2 \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}}, \\ \quad \forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \quad \forall k \in \mathcal{I}_M. \end{cases}$$

We will show that a solution x with $x(t_0) \in R_{D_1, d}$, satisfies $x(t) \in R_{D_1', D_2'}$, $\forall t \geq t_0$, for some $D_1, D_1', D_2' \in \mathbb{R}$, with $D_1 \geq D_1' \geq D_2' \geq d$.

The second set of inequalities characterizing R_{D_1, D_2} can be rewritten as

$$\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_k} \leq \frac{D_2 \sum_{i \in G_{k,1}} \gamma_i}{2\gamma_{\min}} \quad \text{or} \quad \langle x \rangle_{G_k} - \langle x \rangle_{G_{k,1}} \leq \frac{D_2 \sum_{i \in G_{k,2}} \gamma_i}{2\gamma_{\min}},$$

$\forall (G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k), \forall k \in \mathcal{I}_M$. Considering the inequalities for which $|G_{k,1}| = 1$ or $|G_{k,2}| = 1$ together with the first set of inequalities characterizing R_{D_1, D_2} , it follows that if $x \in R_{D_1, D_2}$ then for any $k \in \mathcal{I}_{M-1}$, $i \in G_k$, and $j \in G_{k+1}$, $x_j - x_i \geq \frac{D_1(\gamma_i + \gamma_j)}{2\gamma_{\min}} \geq D_1$. As a consequence we can derive that, if $x(t) \in R_{D_1, D_2}$, with $D_1 \geq D_2 \geq d$, for some $t \in \mathbb{R}$, then

$$\langle \dot{x}(t) \rangle_{G_{k+1}} - \langle \dot{x}(t) \rangle_{G_k} = \tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>, t) - \tilde{v}(G_k^<, G_k, G_k^>, t). \quad (7.3)$$

Fixing $t_0 \in \mathbb{R}$, it follows that for the choice

$$D_1 \geq D'_1 - \min_{k \in \mathcal{I}_{M-1}} \frac{2\gamma_{\min}}{\sum_{i \in G_k \cup G_{k+1}} \gamma_i} \\ \times \inf_{t \geq t_0} \int_{t_0}^t (\tilde{v}(G_{k+1}^<, G_{k+1}, G_{k+1}^>, t') - \tilde{v}(G_k^<, G_k, G_k^>, t')) dt', \quad (7.4)$$

(the right hand side exists because of (7.2a)) and $D'_1 = D'_2 \geq d$, a solution with $x(t_0) \in R_{D_1, d}$ cannot leave $R_{D'_1, D'_2} \supset R_{D_1, d}$ by changing the sign of

$$\langle x(t) \rangle_{G_{k+1}} - \langle x(t) \rangle_{G_k} - \frac{D'_1 \sum_{i \in G_k \cup G_{k+1}} \gamma_i}{2\gamma_{\min}},$$

for some $k \in \mathcal{I}_{M-1}$.

In what follows we will show that, for $D'_1 = D'_2$ sufficiently large, it is also impossible for this solution to leave $R_{D'_1, D'_2}$ by changing the sign of

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - \frac{D'_2 \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}},$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$. This implies that $x(t) \in R_{D'_1, D'_2}$, $\forall t \geq t_0$, and because of (7.3) and (7.2a) we can conclude that x exhibits clustering behavior with respect to G .

Fix a $k \in \mathcal{I}_M$ and define $\tilde{D}_{2,k} : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\tilde{D}_{2,k}(x) \triangleq \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} \max_{(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)} (\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}}), \quad \forall x \in \mathbb{R}^N.$$

For any $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, set

$$W_{G_{k,1}, G_{k,2}} \triangleq \left\{ x \in \mathbb{R}^N : \tilde{D}_{2,k}(x) \geq 2d, \text{ and } \frac{2\gamma_{\min}(\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}})}{\sum_{i \in G_k} \gamma_i} = \tilde{D}_{2,k}(x) \right\}, \\ B_{G_{k,1}, G_{k,2}}(\delta) \triangleq \left\{ x \in \mathbb{R}^N : \tilde{D}_{2,k}(x) = 2d \text{ and } \exists x' \in W_{G_{k,1}, G_{k,2}} : \|x - x'\| \leq \delta \right\}, \\ \tilde{W}_{G_{k,1}, G_{k,2}} \triangleq \left\{ \lambda x \in \mathbb{R}^N : x \in B_{G_{k,1}, G_{k,2}}\left(\frac{d}{2}\right), \lambda \geq 1 \right\}.$$

Lemma 7.1. *For any $k \in \mathcal{I}_M$, and any $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$,*

$$x_i - x_j \leq -\tilde{D}_{2,k}(x), \quad \forall i \in G_{k,1}, \forall j \in G_{k,2}, \forall x \in W_{G_{k,1}, G_{k,2}}.$$

The proof follows from the reasoning in the proof of theorem 5.1 leading to (5.8).

It follows that for any $x \in W_{G_{k,1}, G_{k,2}}$,

$$x_i - x_j \leq -2d, \quad \forall i \in G_{k,1}, \forall j \in G_{k,2},$$

and thus for any $x \in B_{G_{k,1}, G_{k,2}}(\frac{d}{2})$,

$$x_i - x_j \leq -d = -\frac{\tilde{D}_{2,k}(x)}{2}, \quad \forall i \in G_{k,1}, \forall j \in G_{k,2},$$

and since $\tilde{D}_{2,k}(\lambda x) = |\lambda| \tilde{D}_{2,k}(x)$, $\forall \lambda \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$, any $x \in \tilde{W}_{G_{k,1}, G_{k,2}}$ satisfies

$$x_i - x_j \leq -\frac{\tilde{D}_{2,k}(x)}{2} \leq -d, \quad \forall i \in G_{k,1}, \forall j \in G_{k,2}. \quad (7.5)$$

Furthermore

$$\Delta(G_{k,1}, G_{k,2}) \triangleq \sup_{x \in \tilde{D}_{2,k}^{-1}(2d) \setminus B_{G_{k,1}, G_{k,2}}(\frac{d}{2})} \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} (\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}}) < 2d.$$

If this did not hold, then $\Delta(G_{k,1}, G_{k,2}) = 2d$, and there would exist a limit point $x \in \mathbb{R}^N$ with $\tilde{D}_{2,k}(x) = 2d$, $x \notin B_{G_{k,1}, G_{k,2}}(\delta)$ with $0 < \delta < \frac{d}{2}$, and

$$\frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} (\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}}) = 2d.$$

It follows that $x \in W_{G_{k,1}, G_{k,2}}$, and therefore $x \in B_{G_{k,1}, G_{k,2}}(\delta)$, for any $\delta > 0$, leading to a contradiction.

As a consequence, for any $(G'_{k,1}, G'_{k,2}) \in \mathcal{P}_2(G_k)$,

$$\sup_{x \in W_{G'_{k,1}, G'_{k,2}} \setminus \tilde{W}_{G_{k,1}, G_{k,2}}} \frac{\langle x \rangle_{G_{k,2}} - \langle x \rangle_{G_{k,1}}}{\langle x \rangle_{G'_{k,2}} - \langle x \rangle_{G'_{k,1}}} \leq \frac{\Delta(G_{k,1}, G_{k,2})}{2d} < 1. \quad (7.6)$$

Consider a solution x of (7.1), with $x(t_0) \in R_{D_1, d}$, where D_1 satisfies (7.4), with $D'_1 = D'_2 > 2d$ to be determined later. Assume that x leaves the region $R_{D'_1, D'_2}$ by changing the sign of

$$\langle x(t) \rangle_{G_{k,2}} - \langle x(t) \rangle_{G_{k,1}} - \frac{D'_2 \sum_{i \in G_k} \gamma_i}{2\gamma_{\min}},$$

for some $(G_{k,1}, G_{k,2}) \in \mathcal{P}_2(G_k)$, for some $k \in \mathcal{I}_M$.

Then there exist $t_1, t_2 \in \mathbb{R}$ with $t_0 < t_1 < t_2$, with $\tilde{D}_{2,k}(x(t_1)) = 2d$, $\tilde{D}_{2,k}(x(t_2)) = D'_2$, and $x(t) \in R_{D'_1, D'_2}$ and $2d \leq \tilde{D}_{2,k}(x(t)) \leq D'_2$, $\forall t \in [t_1, t_2]$.

Since $\tilde{D}_{2,k}(x(t)) \geq 2d$, $\forall t \in [t_1, t_2]$, there exists a sequence $(G_{k,1}^1, G_{k,2}^1), \dots, (G_{k,1}^n, G_{k,2}^n)$ and times $\tau_0 = t_1, \tau_1, \dots, \tau_n = t_2$, with the property that

$$\begin{aligned} x(t) &\in \tilde{W}_{G_{k,1}^l, G_{k,2}^l}, \quad \forall t \in (\tau_{l-1}, \tau_l), \quad \forall l \in \mathcal{I}_n, \\ x(\tau_l) &\in W_{G_{k,1}^l, G_{k,2}^l} \cap \partial \tilde{W}_{G_{k,1}^{l+1}, G_{k,2}^{l+1}}, \quad \forall l \in \mathcal{I}_{n-1}, \\ x(\tau_n) &\in W_{G_{k,1}^n, G_{k,2}^n}. \end{aligned}$$

This sequence can be constructed easily by starting at time t_2 and reversing to t_1 . First $(G_{k,1}^n, G_{k,2}^n)$ is chosen such that $x(t_2) \in W_{G_{k,1}^n, G_{k,2}^n}$. Then τ_{n-1} follows as the value for t for which $x(t)$ leaves $\tilde{W}_{G_{k,1}^n, G_{k,2}^n}$ (when reversing in time). The pair $(G_{k,1}^{n-1}, G_{k,2}^{n-1})$ is chosen such that $x(\tau_{n-1}) \in W_{G_{k,1}^{n-1}, G_{k,2}^{n-1}}$, and the procedure can be repeated until t_1 is reached.

For each $l \in \mathcal{I}_n$, it follows from (7.5) that $x_i(t) \leq x_j(t) - d$, $\forall (i, j) \in G_{k,1}^l \times G_{k,2}^l$, $\forall t \in (\tau_{l-1}, \tau_l)$, and since also $x(t) \in R_{D'_1, D'_2}$ with $D'_2 \geq 2d$, we can derive that

$$\langle \dot{x}(t) \rangle_{G_{k,2}^l} - \langle \dot{x}(t) \rangle_{G_{k,1}^l} = \tilde{v}(G_k^< \cup G_{k,1}^l, G_{k,2}^l, G_k^>, t) - \tilde{v}(G_k^<, G_{k,1}^l, G_k^> \cup G_{k,2}^l, t),$$

$\forall t \in (\tau_{l-1}, \tau_l)$, and therefore, from (7.2b), that

$$\langle x(\tau_l) \rangle_{G_{k,2}^l} - \langle x(\tau_l) \rangle_{G_{k,1}^l} \leq \langle x(\tau_{l-1}) \rangle_{G_{k,2}^l} - \langle x(\tau_{l-1}) \rangle_{G_{k,1}^l} + c.$$

From (7.6) (and the continuity of x) it follows that for $l \in \mathcal{I}_{n-1}$

$$\frac{\langle x(\tau_l) \rangle_{G_{k,2}^{l+1}} - \langle x(\tau_l) \rangle_{G_{k,1}^{l+1}}}{\langle x(\tau_l) \rangle_{G_{k,2}^l} - \langle x(\tau_l) \rangle_{G_{k,1}^l}} \leq \Delta_M,$$

with

$$\Delta_M \triangleq \max_{(G_{k,2}, G_{k,1}) \in \mathcal{P}_2(G_k)} \frac{\Delta(G_{k,1}, G_{k,2})}{2d} < 1.$$

Consequently, $\forall l \in \mathcal{I}_n$,

$$\begin{aligned} \tilde{D}_{2,k}(x(\tau_l)) &= \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} \left(\langle x(\tau_l) \rangle_{G_{k,2}^l} - \langle x(\tau_l) \rangle_{G_{k,1}^l} \right) \\ &\leq \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} \left(\langle x(\tau_{l-1}) \rangle_{G_{k,2}^l} - \langle x(\tau_{l-1}) \rangle_{G_{k,1}^l} + c \right), \end{aligned}$$

and if $l > 1$

$$\begin{aligned}\tilde{D}_{2,k}(x(\tau_l)) &\leq \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} \left(\Delta_M \left(\langle x(\tau_{l-1}) \rangle_{G_{k,2}^{l-1}} - \langle x(\tau_{l-1}) \rangle_{G_{k,1}^{l-1}} \right) + c \right) \\ &= \Delta_M \tilde{D}_{2,k}(x(\tau_{l-1})) + \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} c,\end{aligned}$$

while, for $l = 1$,

$$\tilde{D}_{2,k}(x(\tau_1)) \leq \tilde{D}_{2,k}(x(\tau_0)) + \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} c,$$

and thus

$$\begin{aligned}\tilde{D}_{2,k}(x(t_2)) &\leq (\Delta_M)^{n-1} \tilde{D}_{2,k}(x(\tau_1)) + \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} c (1 + \Delta_M + \cdots + \Delta_M^{n-2}) \\ &\leq (\Delta_M)^{n-1} \tilde{D}_{2,k}(x(t_1)) + \frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} c (1 + \Delta_M + \cdots + \Delta_M^{n-1}) \\ &\leq 2d + \frac{\frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} c}{1 - \Delta_M}.\end{aligned}$$

Choosing D'_2 larger than this latter value will result in a contradiction, and therefore we have shown that a solution x of (7.1), with $x(t_0) \in R_{D_1,d}$, for some $t_0 \in \mathbb{R}$, D_1 satisfying (7.4), and

$$D'_1 = D'_2 > 2d + \frac{\frac{2\gamma_{\min}}{\sum_{i \in G_k} \gamma_i} c}{1 - \Delta_M},$$

will satisfy $x(t) \in R_{D'_1,D'_2}$, $\forall t \geq t_0$.

We conclude that x exhibits clustering behavior with respect to G .

Sufficiency of (7.2) for clustering behavior (w.r.t. G) of *all* solutions of (7.1)

Again the function

$$V : \mathbb{R} \rightarrow \mathbb{R} : t \mapsto V(t) = \sum_{i \in \mathcal{I}_N} \gamma_i (x_i^*(t) - x_i(t))^2,$$

with x^* a solution exhibiting clustering behavior w.r.t. G and x any other solution of (7.1), satisfies

$$\dot{V}(t) \leq 0, \quad \forall t \in \mathbb{R},$$

implying that x exhibits the same clustering behavior as x^* .

7.3 Conclusion

When the parameters b_i and the interaction functions in the clustering model are time-dependent, one can still formulate necessary and sufficient conditions for clustering behavior of the solutions of the model. When the model is periodic in time, the conditions can be expressed in terms of the time averages of the parameters and interaction functions, leading to the same conditions as in chapter 5.

Chapter 8

An infinite number of agents

We will investigate how the formulation of the conditions for clustering behavior can be adapted when considering an infinite number of agents. Under some extra assumptions on the interaction function, we show that these conditions are necessary for the clustering behavior of the model. We also give an expression for the asymptotic velocities of the agents, and we present an example corresponding to a unimodal distribution of the b -values for which the clustering behavior is quantitatively described in terms of the coupling strength.

8.1 The limit $N \rightarrow \infty$

We assume that the b -values are chosen randomly from a continuous distribution g with

$$\int_{-\infty}^{+\infty} g(b)db = 1,$$

and that the system (4.1) can be transformed into the following partial differential equation for an infinite number of agents:

$$\frac{\partial x}{\partial t}(b, t) = b + K \int_{-\infty}^{\infty} g(b')f(x(b', t) - x(b, t)) db', \quad \forall b, t \in \mathbb{R}, \quad (8.1)$$

where we let x be a continuous function of b to guarantee that the integral is well-defined. The assumptions on the interaction function are relaxed: f is odd, non-decreasing, Lipschitz continuous and satisfies

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm F.$$

The approach for finite N cannot be applied to prove similar results for (8.1), since in general there may be an (uncountably) infinite number of clusters, excluding the existence of an initial condition for which all clusters are separated over a distance d for all $t \geq 0$. Furthermore clusters may no longer be bounded, since they may contain an infinite number of agents. Another complication is the possibility that $x(b, t)$, for a fixed t , is unbounded in b . This can cause the transient time (for the ‘clustering regime’) to become infinitely large. However, if we impose some restrictions on the initial condition and the interaction function f , we are still able to obtain analytical results. We first reformulate the conditions (4.2) for $N \rightarrow \infty$.

Since agents with equal b -values belong to the same cluster, we can characterize the clusters by sets of b -values. From remark 4.1 it follows that these sets will be either singletons or intervals. We will denote the intervals (with non-zero length) by I_k , ($k \in N_I \subset \mathbb{Z}$), while the set containing the singletons will be denoted by J . From now on, the term ‘clusters’ in the context of $N = \infty$ will only refer to the intervals I_k , for a better correspondence with intuition (and in analogy with entrained subsets in the infinite Kuramoto-Sakaguchi model). Define c_k and d_k ($c_k < d_k$) as the endpoints of interval I_k .

For an arbitrary interval I with endpoints $b_1 < b_2$ (possibly infinite) let α_I denote the fraction of agents lying in the interval I :

$$\alpha_I \triangleq \int_{b_1}^{b_2} g(b') db',$$

and let $\langle b \rangle_I$ denote the average value of b in the interval I :

$$\langle b \rangle_I \triangleq \frac{\int_{b_1}^{b_2} b' g(b') db'}{\alpha_I}.$$

The asymptotic velocity $v(b)$ is characterized by the following equations (derived from the right hand side of (4.3)).

$$v(b) = \begin{cases} b - KF\alpha_{(-\infty, b)} + KF\alpha_{(b, +\infty)}, & b \in J; \\ \langle b \rangle_{I_k} - KF\alpha_{(-\infty, c_k)} + KF\alpha_{(d_k, +\infty)}, & b \in I_k \text{ for some } k \in N_I. \end{cases} \quad (8.2)$$

We will adapt the conditions (4.2) for $N = \infty$ under the assumption that all clusters I_k are separated by intervals of non-zero length, allowing us to apply the following reasoning.

For b in the interior of J it follows from (4.2a) that

$$\langle b \rangle_{(b, b+\epsilon)} - \langle b \rangle_{(b-\epsilon, b)} > (\alpha_{(b, b+\epsilon)} + \alpha_{(b-\epsilon, b)}) KF,$$

for some sufficiently small $\epsilon > 0$. Dividing by ϵ and taking the limit $\epsilon \rightarrow 0$ then results in

$$1 \geq 2g(b)KF, \quad \forall b \in J,$$

by the continuity of g . For $k \in N_I$ we obtain

$$\langle b \rangle_{I_k} - \langle b \rangle_{(c_k - \epsilon, c_k)} > (\alpha_{I_k} + \alpha_{(c_k - \epsilon, c_k)}) KF,$$

leading to

$$\langle b \rangle_{I_k} - c_k \geq \alpha_{I_k} KF.$$

Similarly

$$d_k - \langle b \rangle_{I_k} \geq \alpha_{I_k} KF.$$

The equations (4.2b) can be reformulated as

$$\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} \leq \alpha_{I_k} KF, \quad \forall p \in (c_k, d_k), \quad \forall k \in N_I.$$

The conditions on the sets I_k and J , defining the clusters, can then be written as

$$1 \geq 2g(b)KF, \quad \forall b \in J, \quad (8.3a)$$

$$\langle b \rangle_{I_k} - c_k = \alpha_{I_k} KF, \quad \forall k \in N_I, \quad (8.3b)$$

$$d_k - \langle b \rangle_{I_k} = \alpha_{I_k} KF, \quad \forall k \in N_I, \quad (8.3c)$$

$$\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} \leq \alpha_{I_k} KF, \quad \forall p \in (c_k, d_k), \forall k \in N_I. \quad (8.3d)$$

The equations (8.3b) and (8.3c) result from the corresponding inequalities together with the limits $p \rightarrow c_k$ and $p \rightarrow d_k$ in (8.3d).

Theorem 8.1. *Assume that $K > 0$ and that f is odd, non-decreasing and satisfies $\lim_{x \rightarrow \pm\infty} f(x) = \pm F$, with $F > 0$. Furthermore assume that f is twice differentiable with $\frac{df}{dx}$ unimodal and $\frac{d^2f}{dx^2}$ bounded. Consider a solution x of (8.1) with $0 \leq \frac{\partial x}{\partial b}(b, 0)$, $\forall b \in \mathbb{R}$ and $\sup_{b \in \mathbb{R}} \frac{\partial x}{\partial b}(b, 0) < \frac{1}{K \frac{df}{dx}(0)}$, with x at least twice differentiable.*

Then there exist intervals $I_k \subset \mathbb{R}$, $k \in N_I \subset \mathbb{Z}$, such that $x(b_1, t) - x(b_2, t)$, with $b_1 \neq b_2$, is bounded in t for $t \geq 0$ if and only if $b_1, b_2 \in I_k$, for some $k \in N_I$, and the sets I_k and $J \triangleq \mathbb{R} \setminus \bigcup_{k \in N_I} I_k$ satisfy (8.3).

Furthermore, defining v by (8.2),

$$\lim_{t \rightarrow +\infty} \frac{\partial x}{\partial t}(b, t) = v(b), \quad \forall b \in \mathbb{R}.$$

(The extra assumptions on f may not be necessary, but they simplify the mathematical analysis.) The proof is given in section A.4.1 of the appendix. Theorem 8.1 implies that the conditions (8.3) are *necessary* for the associated clustering behavior.

The equations (8.3b) and (8.3c) are equivalent with

$$\begin{cases} d_k - c_k = 2\alpha_{I_k}KF, \\ \frac{c_k + d_k}{2} = \langle b \rangle_{I_k}, \end{cases} \quad \forall k \in N_I,$$

or

$$\begin{cases} \frac{\int_{I_k} g(b')db'}{d_k - c_k} = \frac{1}{2KF}, \\ \frac{c_k + d_k}{2} = \langle b \rangle_{I_k}, \end{cases} \quad \forall k \in N_I. \quad (8.4)$$

In other words, the average b -value in a cluster defined by I_k equals the center of I_k , while the average value of the density g over the interval I_k is equal to $1/(2KF)$.

If g has a finite number of local maxima, the development of the clusters can be described as follows. For $K < 1/(2 \max(g)F)$ there are no clusters, and one can take $J = \mathbb{R}$. For $K \gtrsim 1/(2 \max(g)F)$ a cluster arises in an interval I_0 containing the value where the maximum of g is attained. When increasing K , this cluster increases in such a way that (8.4) remains satisfied; the average value of g over the interval I_0 equals $1/(2KF)$ and the average value of b over the cluster is the center of I_0 . If the value of $1/(2KF)$ reaches another local maximum of g then a new cluster arises there. This guarantees that outside the intervals I_k the equations (8.3a) will hold. In section A.4.2 of the appendix we prove that for a fixed p , (8.3d) will continue to hold when increasing K . Since for a given p , the first occurrence of (8.3d) with increasing K appears for $p = c_k$ in (8.3b) or $p = d_k$ in (8.3c), (8.3d) will always hold when applying this procedure. This means that — while increasing K — we only need to consider the equations (8.4). Of course, if two clusters collide (i.e. $c_k - d_{k-1} \rightarrow 0$ for some k) with increasing K , they become one cluster. In section A.4.2 of the appendix we verify that the new cluster still satisfies (8.4).

Remark 8.1. Consider the solution x^* corresponding to the initial condition $x^*(b, 0) = 0$, $\forall b \in \mathbb{R}$, and let x be an arbitrary solution for which $\int_{-\infty}^{+\infty} (x(b, 0))^2 g(b) db$ is finite. Then the function V , defined by

$$V(t) \triangleq \int_{-\infty}^{+\infty} (x(b, t) - x^*(b, t))^2 g(b) db, \quad \forall t \in \mathbb{R},$$

exists for $t = 0$ and satisfies

$$\begin{aligned} \frac{dV}{dt}(t) &= 2K \int_{-\infty}^{+\infty} (x(b, t) - x^*(b, t))g(b)db \\ &\quad \times \int_{-\infty}^{+\infty} (f(x(b', t) - x(b, t)) - f(x^*(b', t) - x^*(b, t)))g(b')db' \\ &= -K \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ((x(b', t) - x(b, t)) - (x^*(b', t) - x^*(b, t))) \\ &\quad \times (f(x(b', t) - x(b, t)) - f(x^*(b', t) - x^*(b, t)))g(b)g(b')dbdb' \\ &\leq 0. \end{aligned}$$

Intuitively one might be tempted to conclude that $x(b, t) - x^*(b, t)$ is bounded in t for all b , but (if this is true) this cannot be inferred from the fact that $\frac{dV}{dt} \leq 0$.

8.2 A unimodal and symmetric distribution

If g is symmetric and has a unique local maximum in zero then there will be at most one cluster I_k ($|N_I| \leq 1$), since g cannot be strictly monotone in I_k as can be derived from (8.4). If the corresponding interval is denoted by I_0 , the second equation of (8.4) is automatically satisfied by setting $c_0 = -d_0$, and then the first equation can be written as

$$d_0 = 2KF \int_0^{d_0} g(b)db. \quad (8.5)$$

For a distribution g which is three times differentiable in a neighborhood of zero, we can investigate the onset of clustering for small values of d_0 by the equation (notice that $g'(0) = 0$ and $g''(0) \leq 0$)

$$d_0 \approx 2KF \left(g(0)d_0 + \frac{g''(0)}{6}d_0^3 \right),$$

resulting in either $d_0 = 0$ (no clustering) or

$$d_0 \approx \sqrt{\frac{6}{-g''(0)} \left(g(0) - \frac{1}{2KF} \right)}.$$

Example 8.1. Consider the distribution g , defined by

$$g(b) = \frac{1}{2\sqrt{1+b^2}(\sqrt{1+b^2}+1)}, \quad \forall b \in \mathbb{R},$$

for which

$$\int_0^{d_0} g(b)db = \frac{1}{2d_0} \left(\sqrt{1 + d_0^2} - 1 \right) = \frac{d_0}{2(\sqrt{1 + d_0^2} + 1)},$$

and thus the relation between d_0 and K for $d_0 \neq 0$ is given by

$$d_0 = KF \sqrt{1 - \frac{2}{KF}},$$

for $K > \frac{1}{2g(0)F} = \frac{2}{F}$. It also follows that

$$\alpha_{I_0} = \sqrt{1 - \frac{2}{KF}}.$$

8.3 Generalizations

The model (8.1) can also be extended to include weighting factors and a general interaction structure, but, besides the observations below, we will not elaborate on this.

8.3.1 Weighting factors

Introducing a weighting function $\gamma : \mathbb{R} \rightarrow \mathbb{R}_0^+$ (where $\int_{-\infty}^{+\infty} g(b')\gamma(b')db'$ is assumed to exist), results in the following system:

$$\frac{\partial x}{\partial t}(b, t) = b + \frac{K}{\int_{-\infty}^{\infty} g(b')\gamma(b')db'} \int_{-\infty}^{\infty} g(b')\gamma(b')f(x(b', t) - x(b, t)) db'. \quad (8.6)$$

Obviously, an appropriate redefinition of g leads again to (8.1), and the mathematical treatment is not affected by the introduction of weighting factors.

8.3.2 General coupling structure

Introducing a dependency of the interaction on the b -values of the interacting agents results in:

$$\frac{\partial x}{\partial t}(b, t) = b + K \int_{-\infty}^{\infty} g(b')f(b, b', x(b', t) - x(b, t)) db'. \quad (8.7)$$

If $f(b, b', x)$ is symmetric in b and b' , i.e. $f(b_1, b_2, x) = f(b_2, b_1, x)$, $\forall b_1, b_2, x \in \mathbb{R}$, then results from chapter 6 may be used to obtain conditions analogous to

(8.3) for the different clusters that will arise. However, a different approach will be needed to support these conditions by mathematical analysis. An important difference consists of the fact that v will not necessarily be non-decreasing, and agents with smaller b -values may have larger velocities than agents with larger b -values.

8.4 Conclusion

Considering the limit for an infinite number of agents in the necessary and sufficient conditions pertaining to the basic clustering model (with a finite number of agents) results in conditions which, under some extra conditions on the interaction function, can be shown to be necessary for clustering behavior of solutions (with some restrictions on the initial conditions) of the model. When the coupling strength is increased, clusters arise at local maxima of the density function. The growth of the clusters with increasing coupling strength can be derived from the necessary conditions.

Chapter 9

Applications

Although the equations of the clustering model are inspired by the Kuramoto model, we also want to relate the model to clustering phenomena which are not related to systems of coupled oscillators.

In section 9.1 we compare the clustering model with the Kuramoto model (i.e. (1.1)) for a finite as well as for an infinite number of agents/oscillators. In section 9.2 we investigate compartmental systems, and for a configuration of connected water basins where the connections have a maximal throughput, we show that the system is naturally modeled by (5.1). Section 9.3 deals with an application to opinion formation. In section 9.4 we discuss platoon formation in cycling races.

9.1 The Kuramoto model

9.1.1 Finite N

The partial entrainment behavior of the Kuramoto model is very similar to the behavior of the clustering model, where the clusters are independent of the initial condition and the transitions between the different clusters for varying K are similar. For comparison figure 9.1 shows the time evolution for a particular configuration for the Kuramoto model (figure 9.1(a)) and the analog for the model (4.1) (figure 9.1(b)), as well as the evolution of the characteristic frequencies/velocities for both models in terms of the coupling strength (figures 9.1(c) and 9.1(d)).

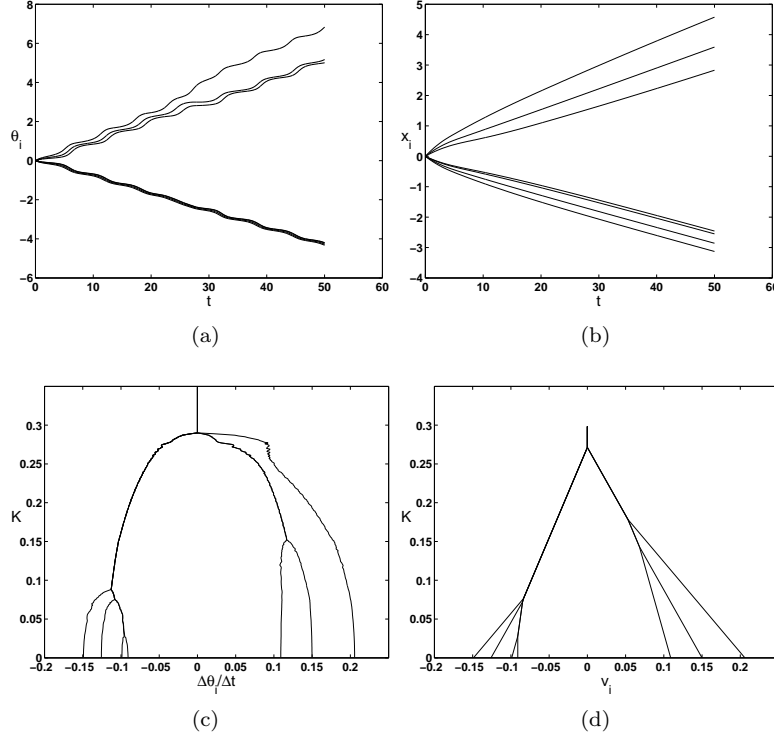


Figure 9.1: Comparison of the Kuramoto model (left column) with the model (4.1) (right column): The first row shows the time evolution of the oscillators/agents. The second row shows the evolution of the long term average frequencies/velocities for varying coupling strength. The parameters are: $F = 1$, $b = \omega = (-0.149, -0.126, -0.099, -0.091, 0.109, 0.150, 0.206)$, $K = 0.2$ (figure (a)) and $K = 0.15$ (figure (b)).

9.1.2 Infinite N

The order parameter r_0 in the Kuramoto model satisfies (3.9a), and with $\alpha = 0$ and g even and unimodal this can be written as (see also [47])

$$Kr_0 = 2K \int_0^{Kr_0} g(\omega) \sqrt{1 - \frac{\omega^2}{K^2 r_0^2}} d\omega,$$

the form of which has been chosen to emphasize the similarity with equation (8.5). There is always the solution $r_0 = 0$, corresponding to total incoherence, while for K above the critical value of $K_c = \frac{2}{\pi g(0)}$ a non-zero solution exists, with the corresponding partially synchronized cluster consisting of all oscillators

with $\omega \in [-Kr_0, Kr_0]$. For $K \gtrsim K_c$ the product Kr_0 can be approximated by

$$Kr_0 \approx \sqrt{\frac{8}{-g''(0)} \left(g(0) - \frac{2}{K\pi} \right)}.$$

Example 9.1. For the distribution g , defined by

$$g(\omega) = \frac{C}{\pi(C^2 + \omega^2)}, \quad \forall \omega \in \mathbb{R},$$

with $C > 0$, it follows that $K_c = 2C$ and one can derive that

$$r_0 = \sqrt{1 - \frac{2C}{K}}.$$

This has to be compared to the expression for α_{I_0} in example (8.1) (although the distribution g is different in both cases).

For non-unimodal or asymmetric distributions of the parameters b and ω the qualitative similarities of the development of the clusters remain, although there are of course differences:

- Clusters of the model (8.1) always start in a local maximum, while for the Kuramoto model this is no longer true for a general distribution of the natural frequencies, and an analytical treatment of the general Kuramoto model is much harder.
- Although (8.1) can (qualitatively) generate the clustering process of the Kuramoto model, other phenomena characteristic of the latter model such as frequency locking [35] or induction of clusters by resonances (see chapter 3) are not present in the former model, which might explain why the Kuramoto model is harder to investigate analytically.

9.2 Compartmental systems

Compartmental systems consist of different compartments, subject to an external inflow or outflow of a commodity, and with mutual connections which allow transportation of the commodity between the compartments. To illustrate the relation of the clustering model with compartmental systems we will focus on a system of interconnected water basins.

9.2.1 Interconnected water basins

We consider N separate basins connected by horizontal pipes, each basin furthermore subject to either a constant external inflow or outflow of water. For laminar flow through a pipe connecting two basins, the fluid velocity is proportional to the pressure difference (Hagen-Poiseuille law). For larger velocities the flow becomes turbulent and the relation between velocity and pressure difference is described by the Darcy-Weisbach equation [41]:

$$\Delta p = \lambda \frac{L}{D} \frac{\rho v^2}{2},$$

with Δp the pressure difference, L and D length and diameter of the pipe, ρ the fluid density, v the mean fluid velocity (i.e. the ratio of the volume flow rate and the cross-section area), and λ the friction factor. Although the friction factor depends on the Reynolds number (which is proportional to the fluid velocity) its variation is small for large values of the Reynolds number. We will *approximate* the resulting relation ($v \sim \sqrt{\Delta p}$) by a saturating function, keeping in mind that the flow rate cannot grow unbounded (because of the finite basin heights and finite resulting pressures and because of possible safety precautions taken to avoid damage to the pipes). In other words, we assume that the pipes have a maximal throughput, which is denoted by F_{ij} for the pipe connecting basin i with basin j . (The maximal throughput may depend on the direction, i.e. F_{ij} may be different from F_{ji} , although it is physically acceptable in most situations to assume that $F_{ij} = F_{ji}$.) Representing the water height of basin i by x_i , the pressure difference between basins i and j will be proportional to $x_j - x_i$, and thus the volume flow rate through the connecting pipe can be represented by $f_{ij}(x_j - x_i)$. The absence of a pipe between basins i and j corresponds to $F_{ij} = F_{ji} = 0$.

Denoting the inflow for basin i by Q_i and its surface area — which we assume to be water level independent — by S_i , one derives

$$\dot{x}_i(t) = \frac{Q_i}{S_i} + \frac{1}{S_i} \sum_{j=1}^N f_{ij}(x_j(t) - x_i(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N,$$

which is the model (5.1) with $K = 1$, $b_i = Q_i/S_i$, $\gamma_i = 1$, and $A_i = 1/S_i$ for all i . The asymptotic velocity v_i associated with basin $i \in G_k$ (where G denotes the corresponding cluster structure) can be calculated as

$$v_i = \frac{1}{\sum_{m \in G_k} S_m} \sum_{m \in G_k} \left(Q_m + \sum_{j \in G_k^>} F_{mj} - \sum_{j \in G_k^<} F_{mj} \right).$$

We are interested in checking whether a network of connected basins is prone to flooding. For simplicity we assume that the total external inflow equals the

total external outflow, implying that the desired behavior corresponds to one cluster $G_1 = \mathcal{I}_N$ with velocity $v_1 = \sum_{i \in G_1} Q_i / \sum_{i \in G_1} S_i = 0$. The model will hold as long as no basins overflow and all pipes remain below the water level of the basins they connect. We assume that the basins and initial water level heights are such that these conditions are satisfied during the transient behavior. This implies that they will remain satisfied for a solution consisting of one cluster at zero velocity, since then each x_i will approach a constant value. For any solution with more than one cluster, one of the inequalities (5.4b) will be violated, expressing that there exists a partition of the set of basins into two non-empty subsets, for which the production in one subset cannot be transported to the other subset by the interconnections between them. Basins will overflow and the model will no longer hold in this situation. In this case a simulation of the model will only be valid in a short time interval, but from the long term behavior the existence of multiple clusters can be inferred and the overflowing basins can be identified as the basins with a positive v_i -value.

As an illustration, consider a configuration of $N = 10$ basins all having the same surface area $S_i = 1$, implying that b_i equals the external inflow rate of basin i . (For simplicity we will omit units.) The vector b containing the b_i -values is given by

$$b = [-5 \quad 4 \quad 1 \quad -2 \quad -3 \quad 0 \quad 6 \quad -3 \quad 0 \quad 2]^T.$$

(Notice that the net inflow in the configuration is zero.) The pipes all have a maximal throughput equal to 2 and are connected in a ring structure, as shown in figure 9.2(a). A simulation (see figure 9.3(a)) reveals that this interconnection structure is not able to prevent basins from overflowing (i.e. there are different clusters and at least one of them has a positive velocity). Notice that any simulation with arbitrary initial condition will settle into the same cluster structure. The objective is to alter the connection structure by adding a minimal number of pipes (of maximal throughput 2) in order to avoid flooding, i.e. in order to obtain a single cluster at zero velocity.

Figure 9.3(a) shows that there are 6 different clusters: $G_1 = \{1\}$, $G_2 = \{4, 5\}$, $G_3 = \{8, 9, 10\}$, $G_4 = \{6\}$, $G_5 = \{2, 3\}$, $G_6 = \{7\}$. Adding an extra pipe will not affect the v_i -values of the basins in the cluster with largest (resp. smallest) velocity unless the pipe is connected to one of the basins in this cluster. Therefore if one extra connection would be sufficient it would have to connect basins 1 and 7. Simulation with this extra connection results in 3 clusters: $G'_1 = \{4, 5\}$, $G'_2 = \{1, 6, 7, 8, 9, 10\}$, $G'_3 = \{2, 3\}$ (see figure 9.3(b)), implying that we still need (at least) one extra connection between a basin from G'_1 and a basin from G'_3 . Simulations show that any extra connection between either 4 or 5 and 2 or 3 lead to one cluster at zero velocity, solving the problem. A possible solution is shown in figure 9.2(b).

This example shows that the model (5.1) can be helpful for analyzing compartmental systems and systematizing the search for solutions to associated

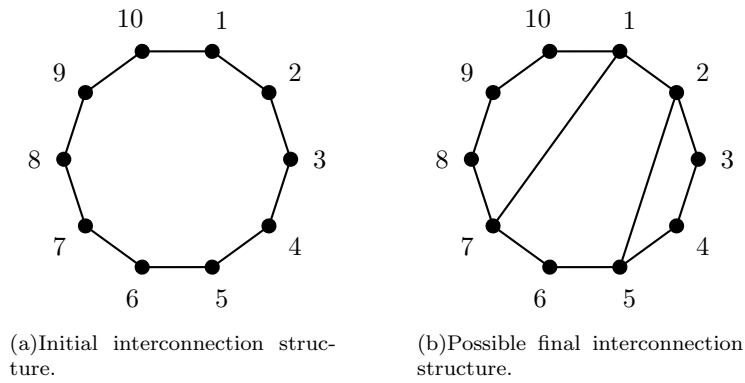


Figure 9.2: Interconnection structures.

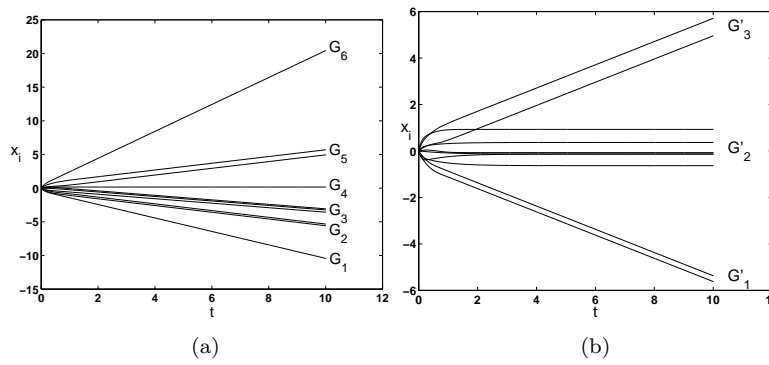


Figure 9.3: Compartmental systems: Figure (a) shows the time evolution of the x_i -values resulting from the initial configuration. Figure (b) shows the time evolution of the x_i -values when an extra pipe is added between basins 1 and 7.

problems.

9.2.2 Other examples

Other applications of compartmental systems may require specific adaptations of the model and may suggest further extensions.

- Consider a system of lakes interconnected by rivers or channels, for which the lakes, rivers and channels are not allowed to overflow. In the case of *artificial* lakes there may be control actions, regulating the volume flow rate through the rivers to prevent them from flooding. The main objective is then to avoid the reservoirs from overflowing. Rivers emanating from *natural* lakes have no control mechanism to restrict the water supply to their maximal throughput, imposing a second objective of preventing the *rivers* from overflowing. In both cases the volume flow rate of a river flowing from one lake to another will not be a mere function of the water level difference between the lakes, and the model (5.1) will not be able to describe the behavior of the water level heights. However, the inequalities (5.4b) still express that the capacity of the channels and rivers has to be sufficient for transporting the water excess out of (the lakes in) regions with a positive net supply rate. Therefore these inequalities are still *necessary* to prevent flooding. They can be checked directly or by simulation of the differential equations (5.1). To obtain a set of *sufficient* conditions for flood prevention, additional conditions (e.g. to express that rivers should not overflow) may need to be formulated.
- A major issue regarding road traffic is to avoid congestions. Different regions can be considered as compartments interconnected by highways which have limited transporting capacities. Again the expression for the flow rate needs to be modified, but the main property — the saturation — still holds. Regarding regular traffic the model will need to incorporate the fact that cars are not interchangeable since each driver has its own destination. This aspect can be neglected in special cases (e.g. during holidays) where there is a common destination for the majority of the drivers, and then the remark on necessity of the inequalities in the previous item still applies.
- Data traffic over the internet has many similarities to road traffic and can be treated analogously.

9.3 Opinion formation

We represent opinions on a particular matter by real numbers, with zero corresponding to a neutral position. Since opinions cannot grow unbounded, x_i in (5.1) is not an appropriate quantity to represent an opinion. Instead we will

take the derivatives $y_i = \dot{x}_i$ as a measure of someone's opinion. The equations for y_i can be written as (assuming $x_i(0) = 0$, $\forall i \in \{1, \dots, N\}$, without loss of generality regarding the long term behavior)

$$y_i(t) = b_i + \frac{KA_i}{\sum_{j=1}^N \gamma_j F_{ij}} \sum_{j=1}^N \gamma_j f_{ij} \left(\int_0^t (y_j(t') - y_i(t')) dt' \right), \quad (9.1)$$

$\forall i \in \mathcal{I}_N$, where we have redefined the sensitivity factors A_i to explicitly include a normalization of the interaction, such that each agent deviates at most KA_i from its inherent opinion b_i . With $y_i(t)$ representing the opinion of agent i at time t , the absolute value of the integral $\int_0^t (y_j(t') - y_i(t')) dt'$ may reflect the level of disagreement accumulated over time, or the amount of discussions taking place between agents i and j , proportional with time and with difference in opinion. When there is no interaction the opinion of agent i is represented by b_i .

In general, everyone starts with his own opinion (b_i) while with time and through interaction, different groups are formed, each group characterized by a final opinion v_i obtained through discussion. The pressure to reach a decision, or the tendency to adapt one's opinion by paying attention to each other's arguments is reflected by the coupling strength K .

In figure 9.4(a) we show the evolution of the opinions v_i eventually reached as a function of K . (The other parameters are left unchanged.) The v_i were calculated by means of an algorithm based on the inequalities (5.4), not by a simulation of the integral equation (9.1). We considered 100 agents with b_i chosen from a Gaussian distribution with zero mean and standard deviation one. The parameters A_i , γ_i , and F_{ij} ($i \neq j$) were all taken equal to one. Notice a steady convergence to complete agreement as a function of K . In figure 9.4(b) the time evolution of the opinions y_i for $K = 1.5$, as obtained by numerical integration of the mathematical model, is shown.

In a second simulation (figure 9.4(c)) we kept the same parameters b_i , but the values for A_i and γ_i were changed to account for the fact that people with extreme opinions are reluctant to change their point of view (smaller A_i) while making more efforts to persuade other people (larger γ_i). Also F_{ij} decreases with increasing values of $|b_j - b_i|$, reflecting the idea that people tend to pay more attention to people with similar opinions:

$$\gamma_i = 1 + 2b_i^2, \quad (9.2)$$

$$A_i = \frac{1}{1 + b_i^2}, \quad (9.3)$$

$$F_{ij} = \exp(-2|b_j - b_i|). \quad (9.4)$$

In figure 9.4(d) we show the time evolution of the y_i for $K = 15$, again obtained by numerical integration. (For the numerical integration in figures 9.4(b) and

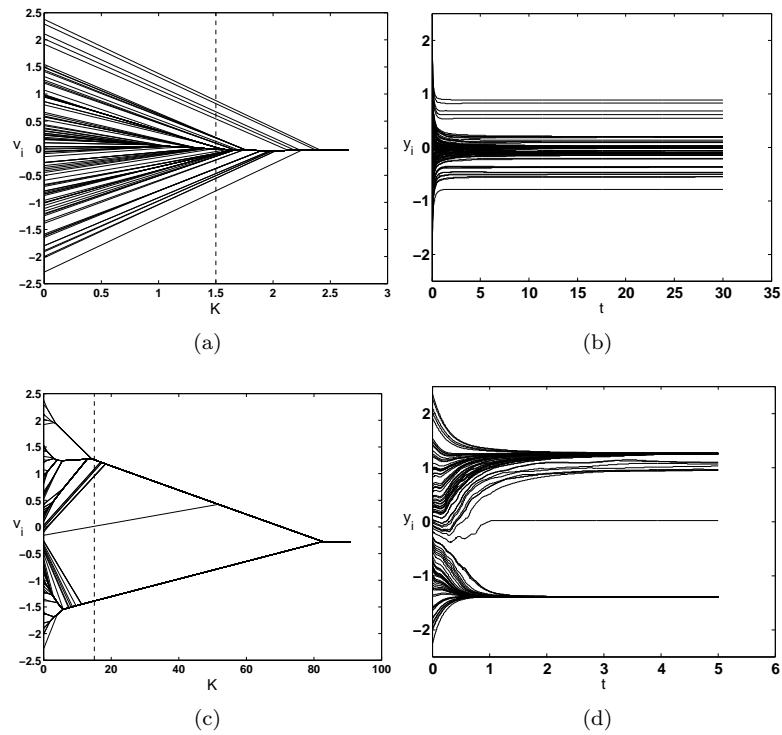


Figure 9.4: Opinion formation: Figures (a) and (c) show the opinions v_i as a function of the coupling strength K . Figures (b) and (d) show the time evolution for a constant K (1.5 and 15 respectively). For figures (a) and (b) the parameters A_i , γ_i and F_{ij} are all equal to one (except $F_{ii} = 0, \forall i \in \mathcal{I}_N$), for figures (c) and (d) the parameter values are given by equations (9.2) to (9.4).

9.4(d) the Euler method was used with a time step of $0.03/K$.)

While in the first case it seems possible to take a decision by a unanimous consent, in the second case — which is more realistic — it is far more favorable to let a majority vote decide. One notices a deadlock of extreme opinions for K around 15. Total consensus can only be reached under much higher pressure compared to the pressure needed for reaching a decision by a majority vote and might require unreasonable concessions from all parties involved.

As an important distinction with other existing models [24] we want to emphasize that the model (9.1) allows the coexistence of several groups, each characterized by its opinion (as opposed to models focusing on total consensus or the coexistence of only two opinions), while still allowing analytical exploration.

9.4 Cycling races

9.4.1 Introduction

In cycling races, the contenders cover a fixed distance; the cyclist crossing the finish line first is declared the winner. There are basically two types of races: tours or multi-stage races are organized as a sequence of races or stages (e.g. the Giro d'Italia, the Tour de France); another type are one-day races, with top status for the so-called classics, e.g. the Ronde van Vlaanderen (Tour of Flanders) or Milano-San Remo. Disregarding time trials, cyclists arrive at the finish line in groups (or platoons), the outcome of course highly dependent on the individual talent, strength and stamina of each contender, on the profile of the race (e.g. mountainous or flat race) but also on the interaction between participants during the contest. We intend to show that the distribution of arrival times of a race and in particular the platoon (cluster) structure observed at the finish line can be modeled by the following modification of (5.1), where we have again (as in (9.1)) included an explicit normalization factor,

$$\dot{x}_i(t) = b_i + \frac{KA_i}{\sum_{j=1}^N \gamma_j F_{ij}} \sum_{j=1}^N \gamma_j f_{ij}(x_j(t) - x_i(t)), \quad \forall i \in \mathcal{I}_N. \quad (9.5)$$

We will compare data retrieved from databases [1] with information provided by time solutions generated by (9.5). The data concern races within the period July 25, 2004 - June 19, 2005. Time trials data were omitted for obvious reasons.

The parameters b_i in (9.5) represent the intrinsic strengths of the riders; in the model b_i stands for the (average) speed rider i is capable of when there is no interaction. We will also add noise to b_i , reflecting incomplete information regarding external factors or related to the rider's shape of the day.

We now discuss the interaction between the riders. Interaction due to tactical maneuvering at the team level will be ignored; neither will the influence of the riders' rankings in multi-stage races be incorporated. Riders are clearly attracted to those ahead. Front riders are also attracted to the ones lagging, mainly for tactical or psychological reasons: e.g. an attempt to escape depends on the reaction of the other riders, or an escape may eventually be deemed ill-fated by its members. By choosing the weights γ_j in (9.5) appropriately, we can incorporate that attraction towards the front riders may be stronger than attraction to the ones in the back. The fact that riders are not attracted towards others who are too far behind or ahead can be dealt with by the choice of the F_{ij} . For further details we refer to section 9.4.3.

9.4.2 Results comparing data and model simulations

As indicated before, we are interested in comparing *arrival times* and *cluster structure* at the finish line as given by databases and as predicted by the model. Neglecting transient behavior, the model predicts that the long term average value of \hat{x}_i is given by v_i . These numerical values can be derived from the inequalities (5.4a) and (5.4b) but may also be obtained by simulation of (9.5) as will be done for the present example. The arrival times τ_i follow then by dividing the distance (which we assume to be 200 km) by the corresponding v_i .

The parameters of the model define how the riders interact and therefore depend on the development of the race. We do not consider a continuous change of the parameters in order to keep the model simple. The dependence of the parameters on the race development is dealt with in two steps, i.e. with two sets of parameters: a first simulation of (9.5) will provide a particular outcome which we regard as a race situation from which a new parameter set of (9.5) is calculated, to be used in the second simulation. (Details can be found in section 9.4.3.)

Races characterized by arrival times

All simulations were performed with 150 riders; the length of the races was kept constant. For each race (be it real data or simulation results) we calculated two variables. In figure 9.5 the two variables are plotted, each dot representing a race. The variable on the horizontal axis is the standard deviation (in seconds) of the arrival times ($\sigma = \sqrt{\langle(\tau - \langle\tau\rangle)^2\rangle}$), the variable on the vertical axis is the skewness ($\eta_1 = \langle(\tau - \langle\tau\rangle)^3\rangle/\sigma^3$). (For the calculation of the average $\langle\tau\rangle$, the riders are assumed to have equal weights, i.e. $\langle\tau\rangle = \frac{1}{N} \sum_{i \in \mathcal{I}_N} \tau_i$.) The vector τ lists the arrival times of the riders.

For flat races the standard deviation can be expected to be small since often the majority of the riders arrive in one group, while the skewness is usually positive since there is a long tail to the right in the distribution of

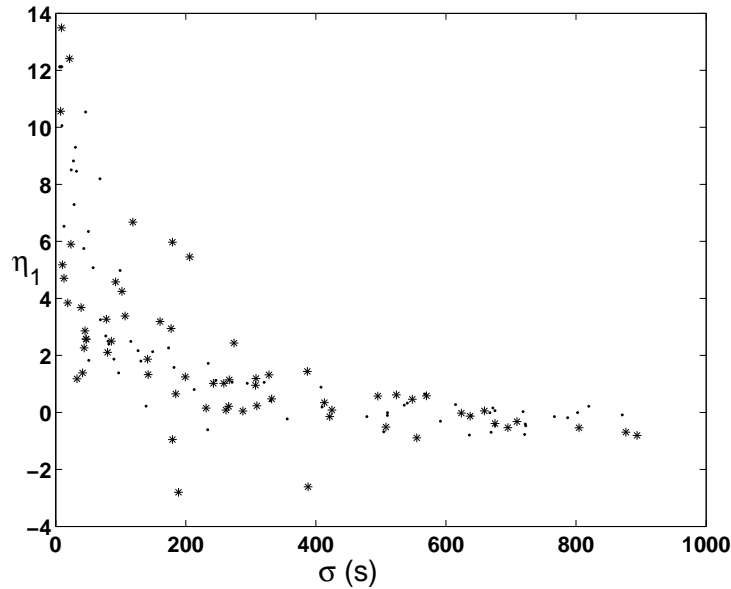


Figure 9.5: Cycling: Races represented by their skewness and standard deviation for both simulation results (dots) and data (stars).

the arrival times. Often the race ends in a mass finish, or the few riders that have managed to escape have a smaller lead with respect to the average arrival time than the time delays of those coming behind. For mountain stages (in multi-stage races) the arrival times are more dispersed which results in a larger standard deviation. The majority are still arriving in one group (the “bus”), only interested in making it to the finish in time (and as such avoiding of being excluded from further participation), while the leaders may lead by several minutes; the tail to the left is longer, leading to a smaller (possibly negative) skewness. All this is clearly reflected in figure 9.5 by the data and the simulation results; notice the correspondence between both.

Platoon structure of race results

Regarding cluster formation we have considered some region in figure 9.5 and for all simulation and data points in this region we have considered plots of the corresponding arrival times (after subtraction of the winner’s arrival time) against the positions of the riders. We selected two qualitatively similar figures (one corresponding to a simulation run and one corresponding to data), and repeated this procedure for several other regions. Three of those resulting pairs of plots are presented in figure 9.6 and show a remarkable qualitative correspondence between data and simulations of race results, and in particular

platoon (cluster) structure at the finish line. (Notice that a data point from figure 9.5 is not just compared against its nearest simulation point because the two parameters σ and η_1 do not completely determine the distribution of the arrival times.)

Classification of races according to difficulty level

To allow for a classification of actual races, we will associate to a given set of arrival times a value for the parameter K , which measures the difficulty level of the race (smaller K -values means tougher races since there is less interaction and one cannot just ride along with the flow). After calculating the (σ, η_1) -value of the arrival times, the K -values corresponding to neighboring simulation points in figure 9.5 are averaged and the result is considered to be a K -value of the race corresponding to the given set of arrival times. In table 9.1 the K -values for all stages (except the time trial stages 1, 4 and 20, and stage 21 of which the data were inconsistent: the arrival times were not ordered correctly, suggesting the occurrence of typing errors) of the 2005 Tour de France are presented [1]. While the first stages are rather flat, corresponding to a high K , the passage through the Vosges (stages 8 and 9), the Alps (stages 10 and 11) and the Pyrenees (stages 14, 15 and 16) can be clearly distinguished by a decrease in the value of K . The small K -value of stage 17 may be explained by the combination of fatigue setting in and the length of the stage (239.5 km). (Details about the procedure for calculating K and the data on which table 9.1 is based can be found in section 9.4.3.)

Table 9.1: K -values of the stages of the 2005 Tour de France.

Stage	Departure - Arrival	K (km/h)
2	Challans - Les Essarts	6.3
3	La Châtaigneraie - Tours	7.2
5	Chambord - Montargis	6.6
6	Troyes - Nancy	5.6
7	Lunéville - Karlsruhe	5.9
8	Pforzheim - Gérardmer	3.5
9	Gérardmer - Mulhouse	3.8
10	Grenoble - Courchevel	2.4
11	Courchevel - Briançon	2.1
12	Briançon - Digne-les-Bains	4.8
13	Miramas - Montpellier	6.0
14	Agde - Ax-3 Domaines	1.3
15	Lézat-sur-Lèze - Saint-Lary Soulan	2.2
16	Mourenx - Pau	3.0
17	Pau - Revel	2.2
18	Albi - Mende	3.8
19	Issoire - Le Puy-en-Velay	5.5

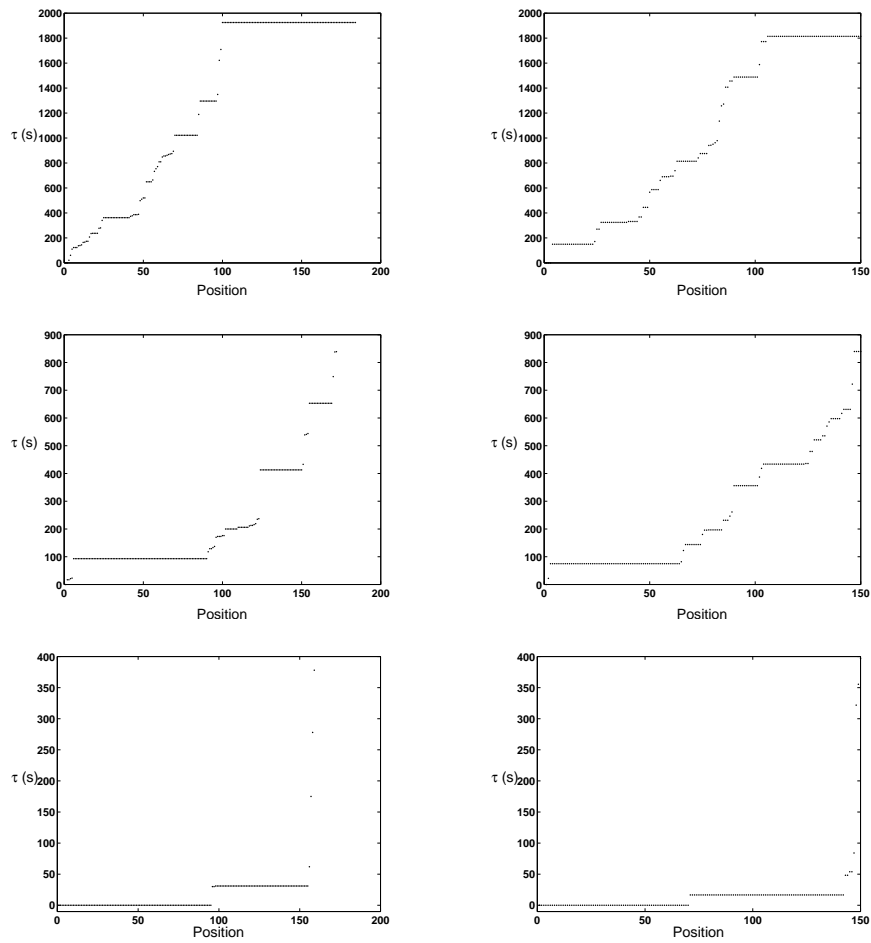


Figure 9.6: Cycling: Comparison of some data and simulation results, corresponding to sets of two neighboring points in figure 9.5. In the left column the data arrival times are shown for three races, in the right column the associated simulation result is shown. The (σ, η_1) -values are resp. $(709, -0.32)$ and $(668, -0.01)$, $(199, 1.25)$ and $(213, 0.81)$, and $(41, 6.33)$ and $(50, 6.35)$.

Quadratic mean cluster size versus difficulty level

As a final validation of our cycling race model we consider figure 9.7, with on the horizontal axis the K -value (calculated as described in the previous section) and on the vertical axis the quadratic mean ζ of the cluster sizes; for a particular race outcome resulting in z_i clusters of size i and a total of $Z = \sum_i z_i$ clusters, ζ is defined as

$$\zeta \triangleq \sqrt{\frac{1}{Z} \sum_i z_i i^2}.$$

The quadratic mean, compared to e.g. the arithmetic mean, emphasizes the contribution of larger clusters, which we consider to be more important for the platoon structure. (For instance, random changes in the b_i -values may easily cause a cluster of two riders to split in two separate singletons, while splitting up a large platoon is more difficult. However, both result in the same change of the arithmetic mean, while the quadratic mean will be less affected by the first modification.) In figure 9.7 we compare ζ -values for data (stars) and for simulations (dots), plotted against K . Notice again the correspondence between both.

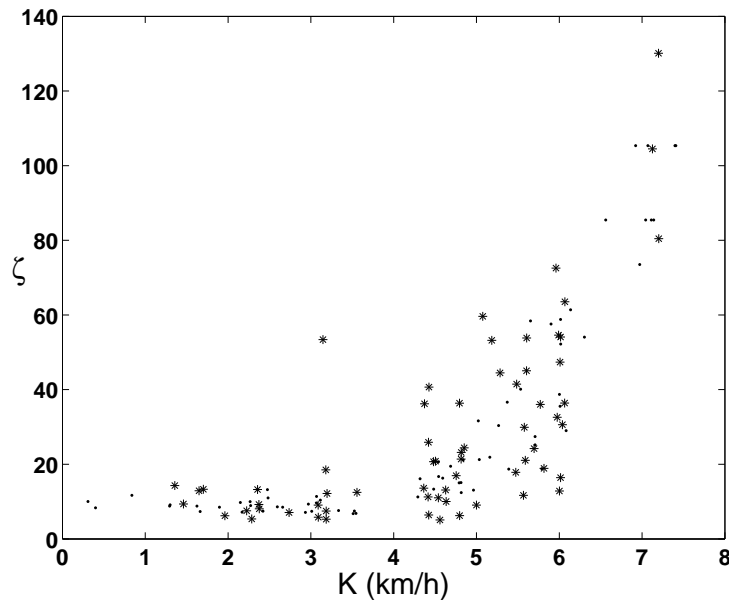


Figure 9.7: Cycling: Races represented by their K -value and quadratic mean cluster size for both simulation results (dots) and data (stars).

9.4.3 Simulation details

In this paragraph we explain how the parameters in the model are chosen and how the simulations are performed. The number of agents N was set equal to 150. The b_i -values were obtained by adding Gaussian noise to some base value $b_{i,0}$. Although there are several options to pick values $b_{i,0}$ from data (they could e.g. be drawn from time trial results, or derived from the UCI-ranking; for multi-stage races they could be related to the final standings obtained in the same tour during the previous year), for the present case we decided to draw the $b_{i,0}$ from a uniform distribution on the interval $[40,50]$ (in km/h). While $b_{i,0}$ corresponds to the long-term average capabilities of rider i , the added noise reflects daily fluctuations. The noise term has zero mean and standard deviation equal to 1 km/h.

For each simulation of a race, we have picked K from a distribution. The support of this distribution determines the different race types (mountainous, flat, . . .) and defines the extent of the region in figure 9.5 in which the simulation points will be located. The actual form of the distribution of K is less important since it only determines the relative occurrence of the different race types and therefore only affects the spatial density of the simulation points in figure 9.5 within the aforementioned region. This leads us to choose the distribution of K in the interest of the presentation of the results: the distribution of K is such that simulation points cover the region determined by the data points and such that data points and simulation points have similar spatial density in figure 9.5. We eventually picked a distribution on the interval $[0,7.5]$ (in km/h) in such a way that K^2 is uniformly distributed, thus considering more races with large values of K . (The corresponding probability density for K is $2K/(7.5)^2$ for $K \in [0, 7.5]$, and zero outside $[0, 7.5]$.) The particular form of f_{ij} in the interval $(-d_{ji}, d_{ij})$ is not important; this will be discussed later on.

The simulation is carried out in two steps, i.e. with two sets of parameters. The choice of the second set of parameters is based on the results of the first simulation, and reflects the modified interconnection structure between the riders, resulting from the course of the race. In a first simulation we assume all-to-all coupling ($F_{ij} = 1$, $i \neq j$) and all A_i equal to 1. In general it is hard for anyone to take a lead; in order to have this reflected in the choice of the parameters, we first note that for all-to-all coupling the order of b_i can be shown to determine the order of v_i (see also remark 4.1, which can be extended when the agents are assigned different weighting factors), and thus we only need to take account of the rider i_l with the largest b_i -value. He is assigned a large weighting factor $\gamma_{i_l} = 100$, while all other γ_i are equal to 1. If rider i_l would take a lead, then the others' interaction will be mainly directed at catching up with him. The values v_i are obtained by numerical integration of the differential equations and then the arrival times τ_i (in seconds) are calculated as $\tau_i = 200\text{km}/v_i$. These arrival times offer a first prediction of the arrival times at the end of the race. They are used to redefine the parameters of the model in order to run a second simulation. The new parameters defining the interaction

between riders will depend on the time differences between riders. Since the parameters are assumed constant, we consider the average value of the time differences during a race; with the assumption of linear divergence of the riders this amounts to considering $\tau_i/2 - \tau_j/2$ for riders i and j .

In the second simulation the parameter K is left unchanged. Furthermore riders are only attracted to riders not too far ahead or behind:

$$F_{ij} \triangleq \exp\left(-\left(\frac{\tau_j/2 - \tau_i/2}{500}\right)^4\right).$$

The formula for F_{ij} specifies that riders separated in time by more than 10 minutes have almost no interaction, and that there exists almost full interaction for riders within 5 minutes' time separation. Denote by $p_i \in \mathcal{I}_N$ the position of rider i (determined by the order of the τ_i , which is the reverse order of the b_i). The weighting factors γ_i consist of two parts:

$$\gamma_i \triangleq \frac{\gamma_{i,1}}{\sum_{j=1}^N \gamma_{j,1}} + \frac{\gamma_{i,2}}{\sum_{j=1}^N \gamma_{j,2}},$$

$\gamma_{i,1}$ corresponding again to increased attraction to the leaders:

$$\gamma_{i,1} = \frac{1}{\sqrt{p_i}};$$

this time the focus is not only on the first rider, but also on the following riders, according to a decaying function of the position. Both $\gamma_{i,2}$ and the new expression for A_i relate to the fact that riders who are too far behind and are no longer concerned about their arrival times, tend to form larger groups in which the velocity is determined by the slower riders. To be able to model this adaptation we introduce the factor

$$\phi_i \triangleq 1 + \tanh\left(\frac{\tau_i/2 - \min_j \tau_j/2 - 1000}{250}\right), \quad \forall i \in \mathcal{I}_N,$$

which will be close to zero and have no importance if rider i is less than 10 minutes behind the leading group, while if he is more than 20 minutes behind, it will be close to 2. The factor ϕ_i is then introduced to affect the parameters of the riders far behind. Adapting A_i changes the coupling of rider i with the other riders. We define A_i as

$$A_i \triangleq 1 + \frac{5}{K} \phi_i;$$

it follows that the coupling strength between riders who are more than 20 minutes behind is increased from K to $K + 10$ (km/h). This allows them

to form larger groups. However, it also introduces a larger deviation of $\dot{x}_i(t)$ from b_i in (9.5) for these riders, while a large deviation might seem physically unrealistic if it were positive. By defining $\gamma_{i,2}$ as

$$\gamma_{i,2} \triangleq 1 + \frac{p_i}{N} \phi_i,$$

such physical inconsistencies are avoided, as will now be explained. Because of the formula for F_{ij} , riders for whom the factor ϕ_i is important will mostly interact with riders for whom this factor is also important. For those riders $\gamma_{i,2}$ increases linearly with position; as a consequence interactions with riders behind are stronger than interactions with riders ahead, resulting in a decrease of the deviation $\dot{x}_i - b_i$. Notice that, by the choice of F_{ij} it is guaranteed that almost no riders have high interaction terms with both the leaders and the ones far behind, which makes sure that for rider i the relevant terms in the expression of $\dot{x}_i(t)$ correspond either to $\gamma_{i,1}$ or to $\gamma_{i,2}$. With the parameters as defined above the model was simulated a second time.

The simulations only serve to calculate the v_i -values and the arrival times, implying that there is no need to be specific about some of the simulation parameters (the behavior of f_{ij} in the interval $(-d_{ji}, d_{ij})$, the value of d_{ij} and the duration of the race) and their relation to the characteristics of real cycling races. For both simulations the (continuous) functions f_{ij} were taken linear between $-d_{ji}$ and d_{ij} with $d_{ij} = 0.3$, $\forall i, j \in \mathcal{I}_N$ with $i \neq j$. We have applied the Euler method with a time step of $0.3/K$ and a total time of $T_{\text{tot}} = 1000/K$; the initial conditions were $x_i(0) = 0$, $\forall i \in \mathcal{I}_N$. Assuming that the transient time is less than $T_{\text{tot}}/3$, v_i was calculated as $(x_i(T_{\text{tot}}) - x_i(T_{\text{tot}}/3))/(2/3T_{\text{tot}})$. From the values τ_i obtained from the second simulation we computed two variables: the standard deviation

$$\sigma \triangleq \sqrt{\frac{1}{N} \sum_{i=1}^N \left(\tau_i - \frac{1}{N} \sum_{j=1}^N \tau_j \right)^2},$$

and the skewness

$$\eta_1 \triangleq \frac{1}{\sigma^3} \frac{1}{N} \sum_{i=1}^N \left(\tau_i - \frac{1}{N} \sum_{j=1}^N \tau_j \right)^3.$$

Races for which $\sigma \approx 0$ were disregarded to avoid problems in the calculation of η_1 ; since those concern races with all riders having the same arrival time, they are not of interest to us. Also data concerning time trials, inconsistent data (i.e. with the arrival times not ordered correctly), or incomplete data (e.g. containing only the first 10 riders) were disregarded. Eventually 64 from 83 data results and 67 from 77 simulation results were withheld. For every run a

point (σ, η_1) was plotted in a 2D-graph; the data were represented by stars in the same graph, yielding figure 9.5.

In order to construct table 9.1 we assign a parameter K to a given outcome (be it a simulation result or an actual race result), characterized by a standard deviation σ_0 and a skewness $\eta_{1,0}$, by the following procedure. Denote by K_i , σ_i and $\eta_{1,i}$ ($i > 0$) the K -value, standard deviation and skewness of (the second part of) the simulation run i , and denote by ξ_σ , resp. ξ_{η_1} , the standard deviation of all σ_i , resp. $\eta_{1,i}$, with $i > 0$. For $\alpha \in \mathbb{R}_0^+$ define $K_\alpha(\sigma_0, \eta_{1,0})$ as the following weighted average of the K_i :

$$K_\alpha(\sigma_0, \eta_{1,0}) \triangleq \frac{\sum_i K_i \exp\left(-\alpha^2 \left(\frac{(\sigma_i - \sigma_0)^2}{\xi_\sigma^2} + \frac{(\eta_{1,i} - \eta_{1,0})^2}{\xi_{\eta_1}^2}\right)\right)}{\sum_i \exp\left(-\alpha^2 \left(\frac{(\sigma_i - \sigma_0)^2}{\xi_\sigma^2} + \frac{(\eta_{1,i} - \eta_{1,0})^2}{\xi_{\eta_1}^2}\right)\right)},$$

where the summations are over all simulation runs, except if $(\sigma_i, \eta_{1,i}) = (\sigma_0, \eta_{1,0})$ for some i ; then this simulation run is skipped in the summation. For smaller values of α more data results are taken into account in the estimate for K , smoothing out local random fluctuations in the data; for α too small all data results are taken into account and the dependence of K on σ_0 and $\eta_{1,0}$ is lost. An appropriate value for α is found by minimizing

$$\sum_i (K_\alpha(\sigma_i, \eta_{1,i}) - K_i)^2;$$

the summation is over all simulation runs. (If the summations in the definition of K_α would also be taken over *all* simulation runs, then the minimization procedure would result in $\alpha \rightarrow \infty$, or picking the K -value of the simulation run corresponding to the nearest dot in figure 9.5. This is less preferable since the corresponding function K_∞ would be highly discontinuous in σ_0 and $\eta_{1,0}$.) From our simulation results we obtained an optimal value for α of 6.1.

9.5 Conclusion

The clustering behavior of the basic clustering model is qualitatively comparable to the partial entrainment behavior of the Kuramoto model, as well for a finite as an infinite number of agents/oscillators.

A system of interconnected water tanks, with the interconnections having a maximal throughput, leads to the same system equations as the clustering model with a general network structure and weighting factors. The necessary and sufficient conditions for clustering are useful for investigating whether the system of water tanks is prone to flooding.

The clustering model is also applicable to opinion formation by an appropriate representation of the opinions. The cluster structure corresponds to the

fragmentation of the opinions.

Concerning platoon formation in cycling races, the simulation results of the clustering model agree both qualitatively and quantitatively with results obtained from databases. The model allows us to associate a coupling strength with each outcome of a cycling race, thereby characterizing cycling races by their difficulty level.

Chapter 10

Relation to the minimum cost flow problem

We consider the minimum cost flow problem, which consists of finding optimal flows (w.r.t. a cost function on the flows) in a network, where the flows satisfy continuity equations in the nodes. The optimal flow distribution can be seen as the asymptotic behavior of a compartmental system (see section 9.2 in chapter 9), where the pipe connections do not necessarily have a maximal throughput. Inspired by this relation, we propose a dynamical system, with system equations similar to those of the clustering model, and we formulate conditions which guarantee that the asymptotic behavior of the system implements the solution to the minimum cost flow problem. The stability properties follow from a Lyapunov approach. We will show that the Lyapunov function is related to the cost function of the Lagrange dual problem to the minimum cost flow problem.

An important observation is the possibility of a decentralized implementation of the optimal flows based on local information only. The system equations can be chosen equal to (5.1) with $\gamma_i = 1, \forall i \in \mathcal{I}_N$ and $K = 1$, but with different interaction functions f_{ij} . In practical situations (e.g. compartmental systems) the state variables may correspond to physical quantities (e.g. water level in a water distribution system or the queue length of the buffer of packages waiting to be processed by a router in a computer network). The decentralized nature of the optimizing strategy makes for a convenient practical implementation.

To illustrate the potential of the method the cases of linear or non-convex cost functions are also discussed, as well as the extension to the multi-commodity minimum cost flow problem.

The results from this chapter can be extended to more general optimization problems [5], but we will restrict our attention to the minimum cost flow problem. Some aspects of our approach may be compared to [33, 34].

10.1 Introduction and problem formulation

Research on the minimum cost flow problem [12, p. 9] — also referred to as the optimal distribution problem — and its relatives frequently focuses on the search for efficient algorithms. In general the cost function is assumed to be linear, although more general cases such as convex cost functions have been considered [12, p. 16].

Consider an (undirected) all-to-all connected network, with $N > 2$ nodes. Node i is assigned a value b_i , corresponding to inflow ($b_i \geq 0$) or outflow ($b_i \leq 0$) of the commodity under consideration. The arc connecting nodes i and j ($(i, j) \in \mathcal{I}_N \triangleq \{(i, j) \in \mathcal{I}_N \times \mathcal{I}_N : i \neq j\}$) carries a flow \mathbf{f}_{ij} from j to i (and thus $\mathbf{f}_{ji} = -\mathbf{f}_{ij}$), such that no accumulation takes place in the nodes:

$$\sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ij} + b_i = 0, \quad \forall i \in \mathcal{I}_N. \quad (10.1)$$

To guarantee the existence of flows \mathbf{f}_{ij} satisfying (10.1) we impose that $\sum_{i=1}^N b_i = 0$.

The arc flow \mathbf{f}_{ij} is assigned a cost $U_{ij}(\mathbf{f}_{ij}) = U_{ji}(\mathbf{f}_{ji})$. The function U_{ij} is assumed to be well-defined on \mathbb{R} , continuously differentiable, strictly convex (i.e. U'_{ij} is increasing), and has a minimal value. The problem is to minimize

$$\frac{1}{2} \sum_{(i,j) \in \mathcal{I}_N} U_{ij}(\mathbf{f}_{ij}), \quad (10.2)$$

subject to the constraints (10.1). Since $\sum_{i=1}^N b_i = 0$ there exists a feasible flow distribution, and since the restriction of the cost function to the set of feasible flow distributions is strictly convex and has a minimum, there will be a *unique* set of flows \mathbf{f}_{ij} solving the minimum cost flow problem.

Remark 10.1. The formulation admits more general problems; deleting the arc between i and j is obtained e.g. by setting $U_{ij}(\mathbf{f}_{ij}) \triangleq a\mathbf{f}_{ij}^2$ and letting $a \rightarrow \infty$. Hard bounds on the flow rates \mathbf{f}_{ij} can be obtained in a similar manner. For instance, setting $U_{ij}(\mathbf{f}_{ij}) = \mathbf{f}_{ij}^2$, $\forall \mathbf{f}_{ij} > 0$, and $U_{ij}(\mathbf{f}_{ij}) = a\mathbf{f}_{ij}^2$, $\forall \mathbf{f}_{ij} \leq 0$, and then again considering the limit $a \rightarrow \infty$, will result in the bound $\mathbf{f}_{ij} \geq 0$. In both cases, however, there may be no feasible flow distribution w.r.t. (10.1), and for the analysis we will maintain the restriction on U_{ij} to be well-defined on \mathbb{R} .

Determining necessary conditions for the solution to this minimization problem is straightforward. Introducing Lagrange multipliers λ_i and μ_{ij} ,

$j \in \mathcal{I}_N \setminus \{i\}$, $i \in \mathcal{I}_N$, and taking the partial derivative of

$$\mathcal{U}(\mathbf{f}, \lambda, \mu) = \frac{1}{2} \sum_{(i,j) \in \mathcal{J}_N} U_{ij}(\mathbf{f}_{ij}) + \sum_{i=1}^N \lambda_i \left(\sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ij} + b_i \right) + \sum_{(i,j) \in \mathcal{J}_N} \mu_{ij} (\mathbf{f}_{ij} + \mathbf{f}_{ji})$$

with respect to \mathbf{f}_{ij} ($i \neq j$) leads to

$$\frac{1}{2} U'_{ij}(\mathbf{f}_{ij}) + \lambda_i + \mu_{ij} + \mu_{ji} = 0, \quad (10.3)$$

which implies that

$$U'_{ij}(\mathbf{f}_{ij}) + \lambda_i - \lambda_j = 0, \quad \forall (i, j) \in \mathcal{J}_N, \quad (10.4)$$

since $U'_{ij}(\mathbf{f}_{ij}) = -U'_{ji}(\mathbf{f}_{ji})$.

Since the minimum cost flow problem has a unique solution, this solution will correspond to a unique set of flows \mathbf{f}_{ij} , satisfying (10.1), (10.4) for some $\lambda \in \mathbb{R}^N$, and $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$, $\forall (i, j) \in \mathcal{J}_N$.

10.2 Dynamical system description

We introduce auxiliary variables x_i , $i \in \mathcal{I}_N$, each x_i representing the level of the commodity at node i . With $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ we associate a distribution of the flows \mathbf{f}_{ij} over the network, through functions $f_{ij} : \mathbb{R}^N \rightarrow \mathbb{R}$, such that

$$\mathbf{f}_{ij} = f_{ij}(x), \quad \forall (i, j) \in \mathcal{J}_N,$$

with f_{ij} Lipschitz continuous, and $f_{ji}(x) = -f_{ij}(x)$, $\forall x \in \mathbb{R}^N$. The functions f_{ij} will be determined shortly. If the constraints (10.1) do not hold, then x is time-varying, with its dynamics described by

$$\dot{x}_i(t) = b_i + \sum_{\substack{j=1 \\ j \neq i}}^N f_{ij}(x(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N. \quad (10.5)$$

Notice that the constraints (10.1) are satisfied in equilibrium points of (10.5). If for some equilibrium point x^e , there should exist $\lambda^{x^e} \in \mathbb{R}^N$ such that equations (10.4) hold with $\mathbf{f}_{ij} = f_{ij}(x^e)$ and $\lambda = \lambda^{x^e}$, then the optimal distribution of the minimum cost flow problem is attained in x^e . To make sure that (10.4) is satisfied, pick Lipschitz continuous functions $\tilde{\lambda}_i : \mathbb{R}^N \rightarrow \mathbb{R}$, $i \in \mathcal{I}_N$, such

that the image of $\tilde{\lambda}_j - \tilde{\lambda}_i$ is contained in the image of U_{ij} , $\forall (i, j) \in \mathfrak{J}_N$, and determine f_{ij} ($i \neq j$) by

$$U'_{ij}(f_{ij}(x)) = \tilde{\lambda}_j(x) - \tilde{\lambda}_i(x), \quad \forall x \in \mathbb{R}^N, \quad \forall (i, j) \in \mathfrak{J}_N, \quad (10.6)$$

or

$$f_{ij} = U'^{-1}_{ij} \circ (\tilde{\lambda}_j - \tilde{\lambda}_i), \quad \forall (i, j) \in \mathfrak{J}_N. \quad (10.7)$$

For any choice of $b \in L_0$, with $L_C \triangleq \{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = C\}$, $\forall C \in \mathbb{R}$, the solution of the minimum cost flow problem satisfies (10.4) for some $\lambda^* \in \mathbb{R}^N$ and unique optimal flows \mathbf{f}_{ij}^* . It follows that any $x^e \in \mathbb{R}^N$ satisfying $\tilde{\lambda}_j(x^e) - \tilde{\lambda}_i(x^e) = \lambda_j^* - \lambda_i^*$, $\forall (i, j) \in \mathfrak{J}_N$, is an equilibrium point of (10.5), which implements the solution of the minimum cost flow problem.

10.3 Stability

In this section we introduce additional conditions to guarantee that any solution of the dynamical system (10.5) converges to an equilibrium point, and therefore asymptotically implements the solution of the minimum cost flow problem.

For the remainder of this chapter we will assume that the functions $\tilde{\lambda}_i$ are continuously differentiable and that the functions $U'_{ij} : \mathbb{R} \rightarrow \mathbb{R}$ are surjective, removing the restriction on the image of $\tilde{\lambda}_j - \tilde{\lambda}_i$ ($(i, j) \in \mathfrak{J}_N$). Denote by $\mathcal{S}_{\tilde{\lambda}}$ the symmetric part of the Jacobian matrix of the vector function $\tilde{\lambda}$, i.e.

$$(\mathcal{S}_{\tilde{\lambda}}(x))_{ij} \triangleq \frac{1}{2} \left(\frac{\partial \tilde{\lambda}_i}{\partial x_j}(x) + \frac{\partial \tilde{\lambda}_j}{\partial x_i}(x) \right), \quad \forall x \in \mathbb{R}^N,$$

and set

$$\Delta \tilde{\lambda}(x) \triangleq \max_{(i,j) \in \mathfrak{J}_N} \left| \tilde{\lambda}_j(x) - \tilde{\lambda}_i(x) \right|, \quad \forall x \in \mathbb{R}^N.$$

We demand that

- (C1) $\mathcal{S}_{\tilde{\lambda}}(x) \geq 0$ (i.e. $\mathcal{S}_{\tilde{\lambda}}(x)$ is positive semi-definite), $\forall x \in \mathbb{R}^N$,
- (C2) the set $\{x \in \mathbb{R}^N : \det(\mathcal{S}_{\tilde{\lambda}}(x)) = 0\}$ consists of isolated points,
- (C3) the function $\Delta \tilde{\lambda}$ is radially unbounded in L_C for each $C \in \mathbb{R}$.

Consider the function V defined by

$$V(\lambda) \triangleq -\sum_{i=1}^N b_i \lambda_i + \frac{1}{2} \sum_{(i,j) \in \mathcal{I}_N} \int_0^{\lambda_j - \lambda_i} U'_{ij}(\sigma) d\sigma, \quad \forall \lambda \in \mathbb{R}^N. \quad (10.8)$$

Since

$$\frac{\partial V}{\partial \lambda_i}(\lambda) = -b_i - \sum_{\substack{j=1 \\ j \neq i}}^N U'_{ij}(\lambda_j - \lambda_i), \quad \forall \lambda \in \mathbb{R}^N, \quad \forall i \in \mathcal{I}_N,$$

and thus, with x a solution of (10.5),

$$\frac{\partial V}{\partial \lambda_i}(\tilde{\lambda}(x(t))) = -\dot{x}_i(t), \quad \forall t \in \mathbb{R},$$

it follows that the derivative of $V \circ \tilde{\lambda} \circ x$ satisfies

$$\begin{aligned} \frac{d(V \circ \tilde{\lambda} \circ x)}{dt}(t) &= -\sum_{i=1}^N \sum_{j=1}^N \dot{x}_i(t) \frac{\partial \tilde{\lambda}_i}{\partial x_j}(x(t)) \dot{x}_j(t) \\ &= -\sum_{i=1}^N \sum_{j=1}^N \dot{x}_i(t) (\mathcal{S}_{\tilde{\lambda}}(x(t)))_{ij} \dot{x}_j(t) \leq 0, \quad \forall t \in \mathbb{R}, \end{aligned}$$

and $V \circ \tilde{\lambda} : \mathbb{R}^N \rightarrow \mathbb{R}$ is a Lyapunov function for the system (10.5).

Remark 10.2. If $\mathcal{S}_{\tilde{\lambda}}(x)$ is positive definite everywhere in \mathbb{R}^N , one can introduce a new metric, represented by the matrix function $\mathcal{S}_{\tilde{\lambda}}$ (i.e. $\langle v(x), w(x) \rangle = \sum_{i=1}^N \sum_{j=1}^N v_i(x) (\mathcal{S}_{\tilde{\lambda}}(x))_{ij} w_j(x)$, for any two vector functions v and w in \mathbb{R}^N), and then $V \circ \tilde{\lambda}$ is a gradient function for the system (10.5) with respect to this metric.

Notice that, since $\sum_{i=1}^N \dot{x}_i(t) = \sum_{i=1}^N b_i = 0$, the set L_C (with $C \in \mathbb{R}$ an arbitrary constant) is invariant under the flow of (10.5).

Proposition 10.1. *For each $C \in \mathbb{R}$, the restriction of the function $V \circ \tilde{\lambda}$ to the set L_C is radially unbounded.*

The proof is given in section A.5.1 of the appendix.

Since the Lyapunov function $V \circ \tilde{\lambda}$ is radially unbounded, every solution x of (10.5), with $x(0) \in L_C$, has a non-empty, bounded positive limit set, which is contained in the largest invariant subset S of

$$\left\{ x \in L_C : -\sum_{i=1}^N \sum_{j=1}^N \frac{\partial V}{\partial \lambda_i}(\tilde{\lambda}(x)) \frac{\partial \tilde{\lambda}_i}{\partial x_j}(x) \frac{\partial V}{\partial \lambda_j}(\tilde{\lambda}(x)) = 0 \right\}.$$

If there are $x \in S$ which are not equilibrium points of (10.5), then they satisfy $\det(\mathcal{S}_{\tilde{\lambda}}(x)) = 0$, and thus by (C2) these points are isolated from each other. They are also isolated from other points in S . If they would not be isolated in S , then there would exist equilibrium points $x^i \in S$, $i \in \mathbb{N}$, converging to x . By continuity of the interaction functions f_{ij} , x would also be an equilibrium point of (10.5), leading to a contradiction. Therefore x is also isolated in S , and by the invariance of S it follows again that x is an equilibrium point of (10.5), again leading to a contradiction. Consequently S consists of the equilibrium points of (10.5). We now show that (10.5) cannot have more than one equilibrium point.

Assume there are two different equilibrium points $x^{e,1}$ and $x^{e,2}$ of (10.5), both belonging to L_C . Since both $x^{e,1}$ and $x^{e,2}$ implement the *unique* solution of the minimum cost flow problem, it follows that $\tilde{\lambda}_i(x^{e,1}) - \tilde{\lambda}_i(x^{e,2}) = \gamma$, $\forall i \in \mathcal{I}_N$, for some $\gamma \in \mathbb{R}$, and thus $\sum_{i=1}^N (x_i^{e,1} - x_i^{e,2})(\tilde{\lambda}_i(x^{e,1}) - \tilde{\lambda}_i(x^{e,2})) = \gamma \sum_{i=1}^N (x_i^{e,1} - x_i^{e,2}) = \gamma C - \gamma C = 0$. On the other hand

$$\begin{aligned} & \sum_{i=1}^N (x_i^{e,1} - x_i^{e,2}) (\tilde{\lambda}_i(x^{e,1}) - \tilde{\lambda}_i(x^{e,2})) \\ &= \sum_{i=1}^N \sum_{j=1}^N \int_0^1 (x_i^{e,1} - x_i^{e,2}) \frac{\partial \tilde{\lambda}_i}{\partial x_j} ((1-t)x^{e,2} + tx^{e,1}) (x_j^{e,1} - x_j^{e,2}) dt \\ &> 0, \end{aligned}$$

because of the properties of $\mathcal{S}_{\tilde{\lambda}}(x)$, leading to a contradiction.

Consequently, the restriction of the system (10.5) to the state space L_C has a *unique* equilibrium point, which is globally asymptotically stable. Summing up, we obtain the following theorem:

Theorem 10.1. *Under the assumptions (C), every solution of the system (10.5), with f_{ij} defined by (10.7), will converge to an equilibrium solution, for which the corresponding set of flows solves the minimum cost flow problem.*

An important observation is that, in practical problems where the x_i correspond to physical quantities which can be measured, and the flows can be controlled, one can implement (10.7) as a control law based on observation of x . By choosing $\tilde{\lambda}$ independent of the vector b , the functions f_{ij} are also independent of b , and in order to attain the minimizing flow distribution, the in- and outflow in the different nodes do not have to be known, and may also slowly (i.e. w.r.t. the transient time of the system (10.5)) vary in time. In other words, the corresponding practical implementation of the solution of the minimum cost flow problem is robust w.r.t. the inflow vector b .

10.4 Relation with the Lagrange dual problem

Notice that our approach also implies that the solution of the minimum cost flow problem corresponds to the minimum of the restriction $V|_{L_C}$, with $C \in \mathbb{R}$ arbitrary. We will show that this result is related to the Lagrange dual problem to the minimum cost flow problem [13].

The dual problem can be formulated as follows: maximize

$$\mathcal{V}(\lambda, \mu) \triangleq \min_{\mathbf{f}} \mathcal{U}(\mathbf{f}, \lambda, \mu),$$

where \mathcal{U} is minimized over all \mathbf{f}_{ij} ($(i, j) \in \mathfrak{I}_N$) independently. (One could also leave out the term in the expression for \mathcal{U} related to the asymmetry of \mathbf{f} and assume that $\mathbf{f}_{ij} + \mathbf{f}_{ji} = 0$, $\forall (i, j) \in \mathfrak{I}_N$. Then $\mathcal{V}(\lambda, \mu)$ does not depend on μ . As will be clear shortly, this leads to the same result.) Minimizing \mathcal{U} over \mathbf{f}_{ij} ($(i, j) \in \mathfrak{I}_N$) leads to (10.3) and thus to

$$\mathbf{f}_{ij} = U'_{ij}{}^{-1}(-2\lambda_i - 2\mu_{ij} - 2\mu_{ji}). \quad (10.9)$$

The maximization over μ can also be accomplished. After substituting the previous expression in \mathcal{U} and taking the partial derivative to μ_{ij} ($(i, j) \in \mathfrak{I}_N$), we derive that

$$\mathbf{f}_{ij} + \mathbf{f}_{ji} = 0,$$

retrieving the asymmetry of \mathbf{f} . It follows that $U'_{ij}(\mathbf{f}_{ij}) = -U'_{ji}(\mathbf{f}_{ji})$, and using (10.9)

$$-2\lambda_i - 2\mu_{ij} - 2\mu_{ji} = 2\lambda_j + 2\mu_{ji} + 2\mu_{ij},$$

and thus,

$$\mathbf{f}_{ij} = U'_{ij}{}^{-1}(\lambda_j - \lambda_i),$$

and we retrieve (10.4). Setting

$$V_{\text{dual}}(\lambda) \triangleq -\mathcal{U}\left(\left(U'_{ij}{}^{-1}(\lambda_j - \lambda_i)\right)_{(i,j) \in \mathfrak{I}_N}, \lambda, \mu\right), \quad \forall \lambda \in \mathbb{R}^N,$$

we have reduced the dual problem to the minimization of V_{dual} , with

$$\begin{aligned} V_{\text{dual}}(\lambda) = & -\sum_{i=1}^N \lambda_i b_i - \frac{1}{2} \sum_{(i,j) \in \mathfrak{I}_N} U_{ij}\left(U'_{ij}{}^{-1}(\lambda_j - \lambda_i)\right) \\ & - \frac{1}{2} \sum_{(i,j) \in \mathfrak{I}_N} (\lambda_i - \lambda_j) U'_{ij}{}^{-1}(\lambda_j - \lambda_i), \quad \forall \lambda \in \mathbb{R}^N. \end{aligned}$$

Since the derivative of the function mapping $\lambda_j - \lambda_i \in \mathbb{R}$ to

$$U_{ij} (U'_{ij}{}^{-1} (\lambda_j - \lambda_i)) + (\lambda_i - \lambda_j) U'_{ij}{}^{-1} (\lambda_j - \lambda_i)$$

equals

$$-U'_{ij}{}^{-1} (\lambda_j - \lambda_i),$$

it follows that V_{dual} is, up to a constant term, equal to V , and the minimization of the Lyapunov function V is equivalent to the minimization of V_{dual} .

10.5 Decentralized implementation

We restrict the functions f_{ij} to depend only on x_i and x_j :

$$f_{ij}(x) = \bar{f}_{ij}(x_i, x_j), \quad \forall x \in \mathbb{R}^N, \quad \forall (i, j) \in \mathfrak{I}_N,$$

with $\bar{f}_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}$ continuous, and satisfying $\bar{f}_{ji}(x_j, x_i) = -\bar{f}_{ij}(x_i, x_j)$, $\forall x_i, x_j \in \mathbb{R}$, $\forall (i, j) \in \mathfrak{I}_N$. In (10.6), the left hand side now only depends on x_i and x_j , and together with lemma A.2, which is formulated and proven in the appendix, this implies that there exist functions $\bar{\lambda}_i : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\tilde{\lambda}_j(x) - \tilde{\lambda}_i(x) = \bar{\lambda}_j(x_j) - \bar{\lambda}_i(x_i), \quad \forall x \in \mathbb{R}^N, \quad \forall (i, j) \in \mathfrak{I}_N,$$

resulting in the condition

$$U'_{ij} (\bar{f}_{ij}(x_i, x_j)) = \bar{\lambda}_j(x_j) - \bar{\lambda}_i(x_i), \quad \forall x_i, x_j \in \mathbb{R}, \quad \forall (i, j) \in \mathfrak{I}_N. \quad (10.10)$$

The conditions (\mathcal{C}) can be reformulated as follows. Conditions $(\mathcal{C}1)$ and $(\mathcal{C}2)$ are equivalent to the derivatives of the functions $\bar{\lambda}_i$ satisfying $\frac{d\bar{\lambda}_i}{dx_i}(x_i) > 0$, $\forall x_i \in \mathbb{R}$, $\forall i \in \mathfrak{I}_N$. Under this condition, $(\mathcal{C}3)$ is equivalent to the realization of (at least) one of the following conditions. (The proof is given in the appendix, section A.5.2.)

(D1) The functions $\bar{\lambda}_i$ all satisfy $\lim_{x_i \rightarrow +\infty} \bar{\lambda}_i(x_i) = +\infty$.

(D2) The functions $\bar{\lambda}_i$ all satisfy $\lim_{x_i \rightarrow -\infty} \bar{\lambda}_i(x_i) = -\infty$.

(D3) All but one of the functions $\bar{\lambda}_i$ satisfy $\lim_{x_i \rightarrow \pm\infty} \bar{\lambda}_i(x_i) = \pm\infty$.

An important observation is that for a decentralized implementation the functions \bar{f}_{ij} , which are determined by the ‘local costs’ U_{ij} , determine the flow \mathbf{f}_{ij} in terms of the ‘local variables’ x_i and x_j .

Notice that for the choice $\bar{\lambda}_i(x_i) = x_i, \forall x_i \in \mathbb{R}, \forall i \in \mathcal{I}_N$, we can set $f_{ij}(x) = \bar{f}_{ij}(x_i, x_j) = f_{ij}^*(x_j - x_i)$, with $f_{ij}^* \triangleq U'_{ij}{}^{-1}, \forall (i, j) \in \mathcal{J}_N$. The formulation of the system (10.5) is then equal to (5.1), but with different interaction functions. In the two following sections, we will assume this formulation for (10.5).

10.5.1 Linear cost functions

In the classical formulation of the minimum cost flow problem the cost is linear and the flows are subject to hard bounds. Because of the hard bounds a solution does not necessarily exist, and since the cost is not strictly convex, a solution may not be unique. Throughout this section we will only consider the case for which a solution exists, and we assume that it is unique (which corresponds to the generic case). In this case the previous results are still useful. The corresponding cost functions, implementing the hard bounds, take the form

$$U_{ij}(\mathbf{f}_{ij}) = \begin{cases} a_{ij}\mathbf{f}_{ij}, & \mathbf{f}_{ij} \in [c_{ij}, d_{ij}]; \\ \infty, & \mathbf{f}_{ij} \notin [c_{ij}, d_{ij}], \end{cases}$$

for some $a_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$, with $a_{ji} = -a_{ij}$ and $c_{ji} = -d_{ij}$. The functions U'_{ij} are not invertible, but by approximating U_{ij} by strictly convex functions, we are led to the following definition of f_{ij}^* :

$$f_{ij}^*(x) = \begin{cases} c_{ij}, & \text{if } x < a_{ij}; \\ d_{ij}, & \text{if } x > a_{ij}. \end{cases}$$

The corresponding differential equations (10.5) have discontinuous right hand sides, which raises issues with regard to the existence of solutions of the resulting system. Although approaches exist to extend the solution concept to discontinuous differential equations (see [23]), we favor a heuristic approach. Notice that there is a gradient function V_{lin} (assuming that $\tilde{\lambda}(x) = x, \forall x \in \mathbb{R}^N$) with

$$V_{\text{lin}}(x) \triangleq - \sum_{i=1}^N b_i x_i + \frac{1}{2} \sum_{(i,j) \in \mathcal{J}_N} \left(\frac{c_{ij} + d_{ij}}{2} (x_j - x_i - a_{ij}) + \frac{d_{ij} - c_{ij}}{2} |x_j - x_i - a_{ij}| \right), \quad \forall x \in \mathbb{R}^N,$$

which corresponds to the definition of V in (10.8), (after replacing $U'_{ij}{}^{-1}$ by f_{ij}^*), except for a constant term (the lower bounds of the integrals in (10.8) were replaced by a_{ij}). The function V_{lin} is convex and piecewise linear and by the assumptions on the existence of a unique solution, V_{lin} has an absolute

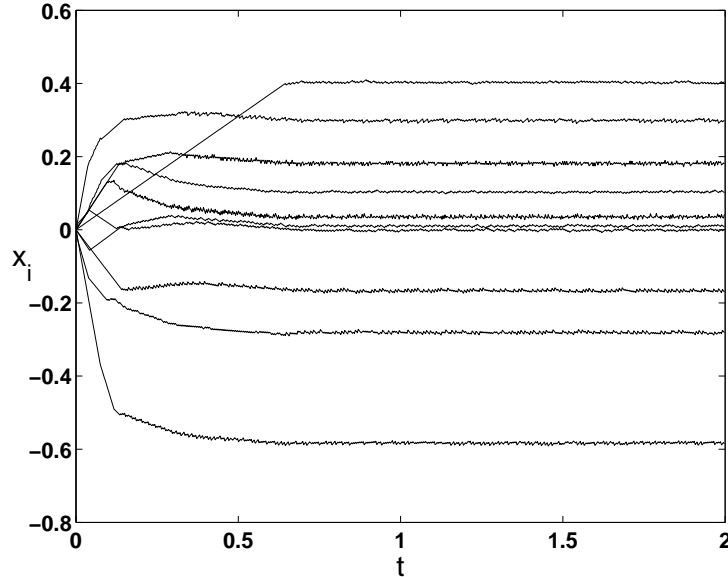


Figure 10.1: Simulation of the time evolution of the x_i for a linear cost function with time step 0.003, $N = 10$, a_{ij} and b_i chosen randomly from a uniform distribution on $[-1, 1]$ and $[-\frac{1}{2}, \frac{1}{2}]$ respectively (the values b_i were shifted to guarantee that they sum up to zero), $c_{ij} = -1$ and $d_{ij} = 1$.

minimum in some $x^e \in \mathbb{R}^N$. The point x^e is no longer an equilibrium point in the classical sense, since the right hand sides in (10.5), which in the present case can only take a finite number of values, will not be equal to zero in x^e (in the generic case we are considering). In a simulation with a constant time step the trajectory will approach x^e at first, but since the right hand sides in (10.5) will not approach zero, chattering occurs around x^e and as a consequence there is an error proportional to the size of the time step (see figure 10.1 for an illustration). If the time step is sufficiently small, this results in a good approximation of x^e . The error can be reduced by using a variable time step. Since the right hand sides of (10.5) are piecewise constant along a solution x , the time instances at which $\dot{x}(t)$ changes can be exactly calculated, and by adjusting the time step appropriately the exact trajectory of x can be obtained, as well as the exact value of x^e , resulting in an algorithmic approach.

Notice that x^e does not completely determine all \mathbf{f}_{ij} , since $f_{ij}^*(x)$ is not properly defined if $x = a_{ij}$. In the generic case at most $N-1$ equations $x_j^e - x_i^e = a_{ij}$ ($i < j$) can be satisfied, since for each $C > 0$, $x^e \in L_C$ has only $N-1$ degrees of freedom. If less than $N-1$ components were satisfied, there would be infinitely many solutions for $x^e \in L_C$ satisfying the corresponding equalities, contradicting the fact that (in the generic case) the minimum of V_{lin} is unique. It follows

that there are exactly $N - 1$ unknown values of $\mathbf{f}_{ij} = f^*(x_j - x_i)$ ($i < j$), which can be derived from the constraints (10.1), or they can be calculated as long term averages of $f^*(x_j(t) - x_i(t))$ resulting from simulations. Considering that a simulation trajectory x will remain bounded (even though there is no convergence to x^e), it follows that $\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \dot{x}(t') dt' = 0$, implying that the flows $\mathbf{f}_{ij} \triangleq \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t f^*(x_j(t') - x_i(t')) dt'$ satisfy (10.1).

10.5.2 Non-convex cost functions

If U_{ij} is not convex, then U'_{ij} is not monotone and therefore possesses no continuous inverse. Each monotone branch can be associated to an inverse f^*_{ij} , only defined in some part of \mathbb{R} . Each combination of convex branches from different U_{ij} may lead to (at most) one local minimum of (10.2) under (10.1), which can be implemented by the system (10.5) (but is not guaranteed to be an absolute minimum). However, the realization of the minimum cost may also involve a strictly concave branch, and then the corresponding equilibrium solution of (10.5) may be unstable. In this case a stabilizing decentralized implementation is still possible but will depend on the values of the b_i , and will therefore not be robust w.r.t. changes of the b_i .

Example 10.1. Consider a network of three nodes with associated cost functions

$$\begin{aligned} U_{12}(\mathbf{f}_{12}) &= \frac{1}{2} \mathbf{f}_{12}^2, & \forall \mathbf{f}_{12} \in \mathbb{R}, \\ U_{23}(\mathbf{f}_{23}) &= \frac{1}{2} \mathbf{f}_{23}^2, & \forall \mathbf{f}_{23} \in \mathbb{R}, \\ U_{31}(\mathbf{f}_{31}) &= \frac{1}{2} \mathbf{f}_{31}(-2 + |\mathbf{f}_{31}|), & \forall \mathbf{f}_{31} \in \mathbb{R}. \end{aligned}$$

For $\mathbf{f}_{31} < 0$ the cost is not convex, but for all choices of $b \in \mathbb{R}^3$, with $b_1 + b_2 + b_3 = 0$, the cost (10.2) still has a minimal value under the constraints (10.1). The minimizing flows can be written as

$$\begin{aligned} \mathbf{f}_{12} &= \frac{1}{3} (b_2 - b_3 + 2 - |b_1 - b_3 + 1|) \\ \mathbf{f}_{23} &= \frac{1}{3} (b_1 - b_2 + 2 - |b_1 - b_3 + 1|) \\ \mathbf{f}_{31} &= \frac{1}{3} (2(b_1 - b_3 + 1) - |b_1 - b_3 + 1|). \end{aligned}$$

For $b_1 - b_3 + 1 \geq 0$ the minimum only involves convex branches and can be implemented by the system (10.5) with $f^*_{12}(x) = x$, $f^*_{23}(x) = x$, and $f^*_{31}(x) = 1 + x$. The eigenvalues associated with its Jacobian matrix are 0, -3 and -3 .

For $b_1 - b_3 + 1 < 0$ the minimum involves the concave branch of U_{31} and the corresponding system (10.5) with $f_{12}^*(x) = x$, $f_{23}^*(x) = x$, and $f_{31}^*(x) = -1 - x$, has eigenvalues 0, -3 and 1, and is unstable.

An implementation by a stable linear system, irrespective of the sign of $b_1 - b_3 + 1$, can be achieved e.g. by the choice $f_{12}^*(x) = x$, $f_{23}^*(x) = x$, $f_{31}^*(x) = 1 + x + 2 \min(0, b_1 - b_3 + 1)$, but due to the dependence on the value of $b_1 - b_3$, this control law is not robust.

10.6 The multi-commodity minimum cost flow problem

Without going into mathematical details, we will indicate how the approach introduced in this chapter can be applied to the multi-commodity minimum cost flow problem.

Let K denote the number of commodity types, and \mathbf{f}_{ijk} ($(i, j) \in \mathcal{J}_N$, $k \in \mathcal{I}_K$) the flow of commodity k from node j to node i ($\mathbf{f}_{ijk} = -\mathbf{f}_{jik}$, $\forall (i, j) \in \mathcal{J}_N$, $\forall k \in \mathcal{I}_K$). For each commodity the corresponding flows satisfy conservation constraints:

$$\sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{f}_{ijk} + b_{ik} = 0, \quad \forall i \in \mathcal{I}_N, \forall k \in \mathcal{I}_K.$$

The cost is no longer separable but can be written as

$$\frac{1}{2} \sum_{(i,j) \in \mathcal{J}_N} U_{ij}(\mathbf{f}_{ij1}, \dots, \mathbf{f}_{ijK}),$$

with

$$U_{ij}(\mathbf{f}_{ij1}, \dots, \mathbf{f}_{ijK}) = U_{ji}(\mathbf{f}_{ji1}, \dots, \mathbf{f}_{jiK}), \quad \forall (i, j) \in \mathcal{J}_N.$$

Introducing Lagrange multipliers λ_{ik} and μ_{ijk} and considering the partial derivative of the Lagrangian to \mathbf{f}_{ijk} leads to

$$\frac{\partial U_{ij}}{\partial \mathbf{f}_{ijk}}(\mathbf{f}_{ij1}, \dots, \mathbf{f}_{ijK}) = \lambda_{jk} - \lambda_{ik}, \quad \forall (i, j) \in \mathcal{J}_N, \quad \forall k \in \mathcal{I}_K. \quad (10.11)$$

If this set of equations can be solved analytically for the flows, then the solution to the corresponding minimum cost flow problem can be implemented

asymptotically by a dynamical system of the form

$$\dot{x}_{ik}(t) = b_{ik} + \sum_{\substack{j=1 \\ j \neq i}}^N f_{ijk}(x(t)), \quad \forall t \in \mathbb{R}, \quad \forall i \in \mathcal{I}_N, \quad \forall k \in \mathcal{I}_K. \quad (10.12)$$

Keeping in mind an application to compartmental systems such as a traffic problem, we suggest two possible forms for U_{ij} :

$$U_{ij}^A(\mathbf{f}_{ij1}, \dots, \mathbf{f}_{ijK}) = v_{ij}^A \left(\sum_{k=1}^K |\mathbf{f}_{ijk}|^\alpha \right)$$

and

$$U_{ij}^{\text{PN}}(\mathbf{f}_{ij1}, \dots, \mathbf{f}_{ijK}) = v_{ij}^{\text{P}} \left(\sum_{k=1}^K \max(\mathbf{f}_{ijk}, 0)^\alpha \right) + v_{ij}^{\text{N}} \left(\sum_{k=1}^K \max(-\mathbf{f}_{ijk}, 0)^\alpha \right),$$

with $\alpha > 1$, v_{ij}^A , v_{ij}^{P} , and v_{ij}^{N} strictly convex and increasing on \mathbb{R}^+ , and $v_{ij}^{\text{P}} = v_{ji}^{\text{N}}$, $\forall (i, j) \in \mathcal{I}_N$. The assumption $\alpha > 1$ is necessary to maintain strict convexity of the total cost, but the interesting case corresponds to $\alpha = 1$: for $U_{ij} = U_{ij}^A$ the cost then depends on the total flow through the arc, while for $U_{ij} = U_{ij}^{\text{PN}}$ the two directions are considered separately (this may physically correspond to the presence of an interconnection for each direction). For $\alpha = 1$ the cost is still convex, but not strictly convex (as in the case of linear cost functions) and similar behavior as in section 10.5.1 can be expected.

For $U_{ij} = U_{ij}^A$ ($(i, j) \in \mathcal{I}_N$), equation (10.11) results in

$$v_{ij}^{A'} \left(\sum_{l=1}^K |\mathbf{f}_{ijl}|^\alpha \right) |\mathbf{f}_{ijk}|^{\alpha-1} \operatorname{sgn}(\mathbf{f}_{ijk}) = \lambda_{jk} - \lambda_{ik}, \quad \forall (i, j) \in \mathcal{I}_N, \quad \forall k \in \mathcal{I}_K.$$

With

$$D_{ij} \triangleq v_{ij}^{A'} \left(\sum_{l=1}^K |\mathbf{f}_{ijl}|^\alpha \right),$$

and thus

$$\mathbf{f}_{ijk} = \operatorname{sgn}(\lambda_{jk} - \lambda_{ik}) \left| \frac{\lambda_{jk} - \lambda_{ik}}{D_{ij}} \right|^{\frac{1}{\alpha-1}},$$

D_{ij} satisfies

$$D_{ij} = v_{ij}^{A'} \left(\sum_{k=1}^K \left| \frac{\lambda_{jk} - \lambda_{ik}}{D_{ij}} \right|^{\frac{\alpha}{\alpha-1}} \right). \quad (10.13)$$

In some cases (e.g. if $v_{ij}^{A'}$ is a power function) this equation can be solved analytically for D_{ij} . The function f_{ijk} ($(i, j) \in \mathcal{I}_N$, $k \in \mathcal{I}_K$) from (10.12) satisfies

$$f_{ijk}(x) = \operatorname{sgn}(\tilde{\lambda}_{jk}(x) - \tilde{\lambda}_{ik}(x)) \left| \frac{\tilde{\lambda}_{jk}(x) - \tilde{\lambda}_{ik}(x)}{D_{ij}} \right|^{\frac{1}{\alpha-1}}, \quad \forall x \in \mathbb{R}^{N \times K},$$

for some function $\tilde{\lambda} : \mathbb{R}^{N \times K} \rightarrow \mathbb{R}^{N \times K}$, where D_{ij} is replaced by the solution of (10.13) (but in terms of $\tilde{\lambda}(x)$ instead of λ), or (if (10.13) cannot be solved analytically) can be modeled as a time-dependent variable with dynamics defined by

$$\tau \dot{D}_{ij}(t) = v_{ij}^{A'} \left(\sum_{k=1}^K \left| \frac{\tilde{\lambda}_{jk}(x(t)) - \tilde{\lambda}_{ik}(x(t))}{D_{ij}(t)} \right|^{\frac{\alpha}{\alpha-1}} \right) - D_{ij}(t),$$

$\forall t \in \mathbb{R}$, where τ is sufficiently small with respect to the transient time of (10.12). The right hand side is decreasing in $D_{ij}(t)$, and as a consequence $D_{ij}(t)$ will asymptotically follow the corresponding (unique) solution of (10.13) (with $\lambda = \tilde{\lambda}(x(t))$) with an error which decreases with decreasing τ and tends to zero if $x(t)$ approaches a constant value.

We again obtain a dynamical system which asymptotically implements the solution of the multi-commodity minimum cost flow problem. For $U_{ij} = U_{ij}^{\text{PN}}$ a similar approach can be followed, but possibly with twice as many extra variables.

10.7 Conclusion

For a separable and convex cost function, the solution to the minimum cost flow problem can be implemented as a dynamical system. The flow through an arc is a function of the state variables, which correspond to the accumulation of the commodity in the nodes. We can formulate conditions on these functions which guarantee convergence to an equilibrium state where the corresponding flows minimize the total cost under the constraints of flow conservation. When the cost is linear and hard bounds are introduced on the flows, the dynamical system is described by discontinuous differential equations.

The dynamical system implementation is related to the dual problem to the minimum cost flow problem. The flow determining functions are independent of the inflow or outflow at the nodes and therefore the dynamical system is robust with respect to the external in- or outflow. A decentralized implementation of the optimal flows is possible, with each flow variable dependent on local information only. The system equations can be chosen equal to those of the clustering model with a general network structure (and equal weighting factors), but with different interaction functions. The approach can be extended to the multi-commodity minimum cost flow problem.

Chapter 11

Conclusion

11.1 The Kuramoto-Sakaguchi model

Although the Kuramoto-Sakaguchi model is hard to analyze for a general configuration of the natural frequencies, it is possible to obtain mathematical results for a finite as well as an infinite number of oscillators.

For a finite number of oscillators we can formulate a sufficient condition for partial entrainment of a given subset of oscillators; the result remains non-trivial when the number of oscillators tends to infinity. The critical values for the coupling strength produced by this result are conservative, and for the estimation of the transition value defining the onset of partial entrainment for a given subset, we propose a procedure based on neglecting oscillators which do not belong to this subset. For the investigated distribution of natural frequencies, this analytical estimation agrees well with simulation results.

Entrainment may disappear with increasing coupling strength due to interaction with other oscillators, and a similar phenomenon can be observed in arrays of Josephson junctions, where it is also possible that a junction leaving one entrained subset joins another entrained subset.

The analytical results in this dissertation for the Kuramoto-Sakaguchi model with an infinite number of oscillators concern configurations of the natural frequencies corresponding to a perturbation of a unimodal distribution. We present equations which characterize the natural frequencies of the oscillators in the entrained subsets up to first order in the perturbation size.

Similarly as in the model with a finite number of oscillators, the presence of an entrained subset may obstruct entrainment in a neighboring subset when the coupling strength is increased; the length of the natural frequency interval corresponding to an entrained subset may decrease when the coupling strength is increased, and the entrained subset may even disappear.

Somewhat similar to the occurrence of orbital resonance in the solar system, entrainment of oscillators in intervals with a low frequency density may be induced by entrainment in other subsets, resulting from high frequency densities for the corresponding oscillators. The differences in average frequencies of the entrained subsets involved in the resonance phenomenon have rational ratios.

11.2 The clustering model

We have proposed a model which captures the partial entrainment behavior of the Kuramoto-Sakaguchi model, but also applies to other systems outside the field of coupled oscillators. The basic model can be generalized to include a general interaction structure, weighting factors associated with the agents, a relaxation of (one of) the conditions on the interaction functions, and time-dependence. The model and its generalizations allow for a thorough mathematical analysis; (for most versions) the clustering behavior can be characterized by necessary and/or sufficient conditions, the cluster structure can be investigated for varying coupling strength, and the internal behavior of a cluster can be shown to approach an equilibrium situation. When the model is periodic in time, the results of the time-invariant model can be applied by considering the time averages of the parameters and interaction functions.

The results of the basic model can also be extended to the case of an infinite number of agents. Based on the necessary and sufficient conditions pertaining to the basic clustering model with a finite number of agents we can derive conditions which can be shown to be necessary (under some conditions on the interaction function) for the clustering behavior of the model with an infinite number of agents. The cluster structure can also be investigated for varying coupling strength.

Besides the similarities of clustering behavior with partial entrainment in the Kuramoto-Sakaguchi model, we have described applications to compartmental systems, opinion formation and cycling races.

- A system of interconnected water tanks, where the interconnections have a maximal throughput, leads to the same system equations as the clustering model with a general network structure and weighting factors. The results for the clustering model are useful for determining whether or not the system of water tanks is prone to flooding.
- For the application to opinion formation we have identified opinions with the time derivatives of the state variables of the clustering model. The clustering behavior corresponds to the outcome of the opinion formation process: there may be several smaller groups, each with their own opinion, there may be a polarization between two opposite opinions, or a global consensus may be reached.

- For platoon forming in cycling races, we have proposed a procedure to choose the parameters of the model, such that simulation results agree with results from databases, both qualitatively and quantitatively. We were also able to characterize the difficulty level of the races by a single parameter, which corresponds to the coupling strength of the clustering model.

Inspired by the relation with compartmental systems, we have been able to implement the solution to the minimum cost flow problem as a dynamical system. The flows are functions of the state variables, which correspond to the accumulation of the commodity in the nodes. Under some conditions on these functions, it can be shown that there is always convergence to an equilibrium state where the corresponding flows solve the minimum cost flow problem. The flow determining functions are independent of the inflow or outflow at the nodes and therefore the dynamical system is robust with respect to the external in- or outflow. A decentralized implementation of the optimal flows is possible, with each flow variable dependent on local information only. The system equations can be chosen equal to those of the clustering model with a general network structure (and equal weighting factors), but with different interaction functions. The approach remains useful when the cost is linear and hard bounds are introduced on the flows f_{ij} , and can be extended to the multi-commodity minimum cost flow problem.

Appendix A

Proofs

A.1 Proofs for chapter 2

A.1.1 Proof of proposition 2.3

For $a \in (0, \pi - 2|\alpha|)$, yet to be determined, consider the region

$$R_a \triangleq \{\theta \in \mathbb{R}^N : |\theta_i - \theta_j| \leq a, \forall i, j \in S_1\}.$$

We will determine conditions such that a well-chosen initial condition leads to a solution θ of (1.2) which remains in R_a . The initial value $\theta(0)$ belongs to $R_{a'}$, with $a' \in (0, a)$ also yet to be determined, and is such that $\theta_m(0) \leq \theta_i(0) \leq \theta_M(0)$, $\forall i \in S_1$. This implies that $\theta_m(t) \leq \theta_i(t) \leq \theta_M(t)$, $\forall t \in \mathbb{R}^+$, by remark 2.1. Consequently we only need to derive conditions guaranteeing that $\theta_M(t) - \theta_m(t)$ will remain smaller than or equal to a . With $\varphi \triangleq \frac{\theta_M - \theta_m}{2}$ and $\omega \triangleq \frac{\omega_M - \omega_m}{2}$ it follows that

$$\dot{\varphi}(t) = \omega - \frac{K}{N} \sin \varphi(t) \sum_{j=1}^N \cos \left(\theta_j(t) - \frac{\theta_m(t) + \theta_M(t)}{2} - \alpha \right).$$

We will investigate the behavior of φ for t -values for which $\varphi(t) \in [\frac{a'}{2}, \frac{a}{2}]$. In this situation the terms in the summation for which $j \in S_1$ are bounded by

$$\cos \left(\theta_j(t) - \frac{\theta_m(t) + \theta_M(t)}{2} - \alpha \right) \geq \cos \left(\frac{a}{2} + |\alpha| \right),$$

since $\theta_m(t) \leq \theta_j(t) \leq \theta_M(t)$ and thus $\left| \theta_j(t) - \frac{\theta_m(t) + \theta_M(t)}{2} \right| \leq \frac{a}{2}$. For $j \in S_3$, we apply that $\cos\left(\theta_j(t) - \frac{\theta_m(t) + \theta_M(t)}{2} - \alpha\right) \geq -1$, and thus

$$\begin{aligned} \dot{\varphi}(t) \leq \omega - \frac{K}{N} \sin \varphi(t) & \left(|S_1| \cos\left(\frac{a}{2} + |\alpha|\right) - |S_3| \right. \\ & \left. + \sum_{j \in S_2} \cos\left(\theta_j(t) - \frac{\theta_m(t) + \theta_M(t)}{2} - \alpha\right) \right). \end{aligned}$$

With $\varphi(t) \in [\frac{a'}{2}, \frac{a}{2}]$, and $\gamma_i = \frac{|S_i|}{N}$ ($i \in \{1, 3\}$), we obtain

$$\begin{aligned} \frac{\dot{\varphi}(t)}{K \sin \varphi(t)} \leq \frac{\omega}{K \sin \frac{a'}{2}} - \gamma_1 \cos\left(\frac{a}{2} + |\alpha|\right) + \gamma_3 \\ - \frac{1}{N} \sum_{j \in S_2} \cos\left(\theta_j(t) - \frac{\theta_m(t) + \theta_M(t)}{2} - \alpha\right), \end{aligned}$$

or

$$\begin{aligned} \frac{1}{K} \ln \left(\frac{\tan \frac{\varphi(t)}{2}}{\tan \frac{\varphi(t_0)}{2}} \right) \leq & \left(\frac{\omega}{K \sin \frac{a'}{2}} - \gamma_1 \cos\left(\frac{a}{2} + |\alpha|\right) + \gamma_3 \right) (t - t_0) \\ & - \frac{1}{N} \sum_{j \in S_2} \int_{t_0}^t \cos\left(\theta_j(t') - \frac{\theta_m(t') + \theta_M(t')}{2} - \alpha\right) dt', \end{aligned}$$

for all $t_0, t \geq 0$, such that $\varphi(t') \in [\frac{a'}{2}, \frac{a}{2}]$, $\forall t' \in [t_0, t]$. We want to establish conditions for which the right hand side is upper bounded by $\frac{1}{K} \ln \left(\frac{\tan \frac{a}{4}}{\tan \frac{a'}{4}} \right)$, guaranteeing that for increasing t , $\varphi(t)$, with $\varphi(t_0) = \frac{a'}{2}$, can only leave the interval $[\frac{a'}{2}, \frac{a}{2}]$ by becoming smaller than $\frac{a'}{2}$.

Consider a $j \in S_2$. Then

$$\begin{aligned} 0 < \left| \omega_j - \frac{\omega_m + \omega_M}{2} \right| - 2K \leq & \left| \dot{\theta}_j(t) - \frac{\dot{\theta}_m(t) + \dot{\theta}_M(t)}{2} \right| \\ & \leq \left| \omega_j - \frac{\omega_m + \omega_M}{2} \right| + 2K. \end{aligned}$$

Lemma A.1. *Assume the continuously differentiable function $\theta_0 : \mathbb{R} \rightarrow \mathbb{R} : t \mapsto \theta_0(t)$ satisfies $0 < \Omega_1 \leq |\dot{\theta}_0(t)| \leq \Omega_2$, $\forall t \in \mathbb{R}^+$. Then*

$$\int_0^t \cos \theta_0(t') dt' \leq t \cos \Theta_0 + \frac{2}{\Omega_1} (\sin \Theta_0 - \Theta_0 \cos \Theta_0),$$

for all $\Theta_0 \in [0, \frac{\pi}{2}]$ satisfying

$$2 \left(\frac{1}{\Omega_1} - \frac{1}{\Omega_2} \right) (\sin \Theta_0 - \Theta_0 \cos \Theta_0) - \frac{2\pi}{\Omega_2} \cos \Theta_0 \leq 0.$$

Proof. The result is invariant under the substitution $\theta_0 \leftrightarrow -\theta_0$, and therefore we will only consider the case for which $\dot{\theta}(t) > 0, \forall t \in \mathbb{R}^+$. Since θ_0 is strictly increasing we can perform the substitution $t' = \theta_0^{-1}(\theta'_0)$ in the following integral.

$$\int_0^t (\cos \theta_0(t') - \cos \Theta_0) dt' = \int_{\theta_0(0)}^{\theta_0(t)} \frac{(\cos \theta'_0 - \cos \Theta_0) d\theta'_0}{\dot{\theta}_0(\theta_0^{-1}(\theta'_0))}.$$

Setting $I_+(t) \triangleq \{\theta'_0 \in [\theta_0(0), \theta_0(t)] : \cos(\theta'_0) \geq \cos \Theta_0\}$ and $I_-(t) \triangleq \{\theta'_0 \in [0, \theta_0(t)] : \cos(\theta'_0) \leq \cos \Theta_0\}$, we obtain the inequality

$$\begin{aligned} \int_0^t \cos \theta_0(t') dt' - t \cos \Theta_0 &\leq \int_{I_+(t)} \frac{(\cos \theta'_0 - \cos \Theta_0) d\theta'_0}{\Omega_1} \\ &\quad + \int_{I_-(t)} \frac{(\cos \theta'_0 - \cos \Theta_0) d\theta'_0}{\Omega_2}. \end{aligned}$$

Denoting by $n(t)$ the number of connected components of $I_+(t)$ if $I_+(t) \neq \emptyset$ and setting $n(t) \triangleq 1$ for $I_+(t) = \emptyset$ we obtain

$$\begin{aligned} &\int_0^t \cos \theta_0(t') dt' - t \cos \Theta_0 \\ &\leq \frac{n(t)}{\Omega_1} \int_{-\Theta_0}^{\Theta_0} (\cos \theta'_0 - \cos \Theta_0) d\theta'_0 \\ &\quad + \frac{n(t) - 1}{\Omega_2} \int_{\Theta_0}^{2\pi - \Theta_0} (\cos \theta'_0 - \cos \Theta_0) d\theta'_0 \\ &= \frac{2n(t)}{\Omega_1} (\sin \Theta_0 - \Theta_0 \cos \Theta_0) \\ &\quad - \frac{2(n(t) - 1)}{\Omega_2} (\sin \Theta_0 + (\pi - \Theta_0) \cos \Theta_0) \\ &= \frac{2}{\Omega_1} (\sin \Theta_0 - \Theta_0 \cos \Theta_0) + (n(t) - 1) \\ &\quad \times \left(2 \left(\frac{1}{\Omega_1} - \frac{1}{\Omega_2} \right) (\sin \Theta_0 - \Theta_0 \cos \Theta_0) - \frac{2\pi}{\Omega_2} \cos \Theta_0 \right), \end{aligned}$$

proving the lemma. \square

With the same notation as in the lemma, define $\mathcal{F}_{\Omega_1, \Omega_2} : [0, 1] \rightarrow \mathbb{R}$ by

$$\mathcal{F}_{\Omega_1, \Omega_2}(\cos \Theta_0) \triangleq 2 \left(\frac{1}{\Omega_1} - \frac{1}{\Omega_2} \right) (\sin \Theta_0 - \Theta_0 \cos \Theta_0) - \frac{2\pi}{\Omega_2} \cos \Theta_0,$$

$\forall \Theta_0 \in [0, \frac{\pi}{2}]$. Notice that $\mathcal{F}_{\Omega_1, \Omega_2}(0) \geq 0$, $\mathcal{F}_{\Omega_1, \Omega_2}(1) < 0$, and one can also verify that $\mathcal{F}_{\Omega_1, \Omega_2}''(\Theta_0) \geq 0$, $\forall \Theta_0 \in [0, \frac{\pi}{2}]$. It follows that the function $\mathcal{F}_{\Omega_1, \Omega_2}$ is convex and that the value $\cos \tilde{\Theta}_0 \in [0, 1]$ for which

$$(\cos \tilde{\Theta}_0, 0) = \lambda(0, \mathcal{F}_{\Omega_1, \Omega_2}(0)) + (1 - \lambda)(1, \mathcal{F}_{\Omega_1, \Omega_2}(1)),$$

for some $\lambda \in [0, 1]$, will satisfy $\mathcal{F}_{\Omega_1, \Omega_2}(\cos \tilde{\Theta}_0) \leq 0$. It can be calculated as

$$\cos \tilde{\Theta}_0 = \frac{\Omega_2 - \Omega_1}{\Omega_2 - \Omega_1 + \pi \Omega_1}.$$

In order to apply this result to θ_j , with $j \in S_2$, Ω_1 and Ω_2 have to be replaced by $|\omega_j - \frac{\omega_m + \omega_M}{2}| - 2K$ and $|\omega_j - \frac{\omega_m + \omega_M}{2}| + 2K$ respectively. Considering the definition of $\tilde{\Theta}_j$ ($j \in S_2$) and applying lemma A.1, we obtain the following inequality:

$$\begin{aligned} \frac{1}{K} \ln \left(\frac{\tan \frac{\varphi(t)}{2}}{\tan \frac{\varphi(t_0)}{2}} \right) &\leq \frac{1}{N} \sum_{j \in S_2} \frac{2 \left(\sin \tilde{\Theta}_j - \tilde{\Theta}_j \cos \tilde{\Theta}_j \right)}{|\omega_j - \frac{\omega_m + \omega_M}{2}| - 2K} \\ &+ \left(\frac{\omega}{K \sin \frac{a'}{2}} - \gamma_1 \cos \left(\frac{a}{2} + |\alpha| \right) + \gamma_3 + \frac{1}{N} \sum_{j \in S_2} \cos \tilde{\Theta}_j \right) (t - t_0), \quad (\text{A.1}) \end{aligned}$$

$\forall t_0, t \in \mathbb{R}^+ : \varphi([t_0, t]) \subset [\frac{a'}{2}, \frac{a}{2}]$. By requiring that

$$\frac{\omega}{K \sin \frac{a'}{2}} - \gamma_1 \cos \left(\frac{a}{2} + |\alpha| \right) + \gamma_3 + \tilde{\gamma}_2 \leq 0, \quad (\text{A.2a})$$

and

$$\frac{1}{K} \ln T \leq \frac{1}{K} \ln \left(\frac{\tan \frac{a}{4}}{\tan \frac{a'}{4}} \right), \quad (\text{A.2b})$$

it is guaranteed that $\varphi(t) \leq \frac{a}{2}$, $\forall t \in \mathbb{R}^+$, (keeping in mind the choice of the initial condition) and that there is partial entrainment with respect to S_1 . Set $a' \triangleq 4 \arctan \left(\frac{\tan \frac{a}{4}}{T} \right)$, automatically satisfying (A.2b). Inequality (A.2a) then

becomes

$$\frac{\omega(T^2 + \tan^2 \frac{a}{4})}{2KT \tan \frac{a}{4}} - \gamma_1 \cos(\frac{a}{2} + |\alpha|) + \tilde{\gamma}_2 + \gamma_3 \leq 0,$$

or, setting $s \triangleq \tan \frac{a}{4} \in (0, \tan(\frac{\pi}{4} - \frac{|\alpha|}{2}))$,

$$\frac{\omega(T^2 + s^2)(1 + s^2)}{2KT} - \gamma_1 s ((1 - s^2) \cos \alpha - 2s |\sin \alpha|) + s(1 + s^2) (\tilde{\gamma}_2 + \gamma_3) \leq 0.$$

Each value of $s \in (0, \tan(\frac{\pi}{4} - \frac{|\alpha|}{2}))$ leads to a sufficient condition for the existence of a solution which is partially entrained with respect to S_1 . To obtain the least stringent condition, one could take the derivative of the left hand side with respect to s and then substitute the appropriate root of the resulting third order polynomial in the inequality. To avoid obtaining too complicated expressions we take the derivative of the left hand side and we calculate the (non-negative) root \tilde{s} under the assumption that ω is equal to zero and in the absence of the term in $|\sin \alpha|$:

$$\tilde{s} = \sqrt{\frac{\gamma_1 \cos \alpha - \tilde{\gamma}_2 - \gamma_3}{3(\gamma_1 \cos \alpha + \tilde{\gamma}_2 + \gamma_3)}}.$$

Substitution of \tilde{s} in the previous inequality leads to proposition 2.3.

A.1.2 Proof of proposition 2.4

We can repeat the derivation of the previous section with $\omega = 0$, leading to the inequality (A.1) which now holds for all $t_0, t \in \mathbb{R}^+$ for which $\varphi(t_0, t) \subset (0, \frac{a}{2}]$. By demanding K to be sufficiently small the summation $\sum_{j \in S_2} \cos \tilde{\Theta}_j$ in (A.1) can be made as small as needed and again setting $a' \triangleq 4 \arctan\left(\frac{\tan \frac{a}{4}}{T}\right)$, (A.2b) can be satisfied. Since $\gamma_1 \cos \alpha > \gamma_3$, (A.2a) holds with strict inequality if a and K are chosen sufficiently small. From (A.1) we then obtain that $\lim_{t \rightarrow \infty} \varphi(t) = 0$, leading to proposition 2.4.

A.2 Proofs for chapter 3

A.2.1 The expression for $\tilde{\rho}'_d(\omega, t_0, t)$ concerning the drifting oscillators

The differential equation

$$\frac{\partial \tilde{\rho}'_d}{\partial t}(\omega, t_0, t) = \tilde{\rho}'_d(\omega, t_0, t) K r_0 \cos \tilde{\theta}_\omega(t - t_0)$$

$$+ \frac{\partial}{\partial \theta} (\rho_{d,0}(\theta, \omega)(K r'_R(t) \sin \theta - K r'_I(t) \cos \theta)) \Big|_{\theta=\tilde{\theta}_\omega(t-t_0)}$$

can be solved for $\tilde{\rho}'_d$ as

$$\begin{aligned} \tilde{\rho}'_d(\omega, t_0, t) = & e^{\int_0^t K r_0 \cos \tilde{\theta}_\omega(t'-t_0) dt'} \left(\tilde{\rho}'_d(\omega, t_0, 0) + \int_0^t dt' e^{-\int_0^{t'} K r_0 \cos \tilde{\theta}_\omega(t''-t_0) dt''} \right. \\ & \left. \times \frac{\partial}{\partial \theta} (\rho_{d,0}(\theta, \omega)(K r'_R(t') \sin \theta - K r'_I(t') \cos \theta)) \Big|_{\theta=\tilde{\theta}_\omega(t'-t_0)} \right), \end{aligned}$$

with

$$\int_0^t K r_0 \cos \tilde{\theta}_\omega(t' - t_0) dt' = K r_0 \int_0^t \frac{1 - \tan^2 \left(\frac{\tilde{\theta}_\omega(t'-t_0)}{2} \right)}{1 + \tan^2 \left(\frac{\tilde{\theta}_\omega(t'-t_0)}{2} \right)} dt'$$

(using the substitution $u = \tan \left(\frac{\tilde{\theta}_\omega(t'-t_0)}{2} \right) = \sin \varphi_\omega + \cos \varphi_\omega \tan \left(\frac{\gamma_\omega(t'-t_0)}{2} \right)$)

$$\begin{aligned} &= K r_0 \int_{\tan \left(\frac{\tilde{\theta}_\omega(-t_0)}{2} \right)}^{\tan \left(\frac{\tilde{\theta}_\omega(t-t_0)}{2} \right)} \frac{(1-u^2) 2 du}{(1+u^2) \omega (1+u^2 - 2u \sin \varphi_\omega)} \\ &= \int_{\tan \left(\frac{\tilde{\theta}_\omega(-t_0)}{2} \right)}^{\tan \left(\frac{\tilde{\theta}_\omega(t-t_0)}{2} \right)} \left(\frac{2u}{1+u^2} + \frac{2 \sin \varphi_\omega - 2u}{1+u^2 - 2u \sin \varphi_\omega} \right) du \\ &= \left[\ln \left(\frac{1+u^2}{1+u^2 - 2u \sin \varphi_\omega} \right) \right]_{\tan \left(\frac{\tilde{\theta}_\omega(-t_0)}{2} \right)}^{\tan \left(\frac{\tilde{\theta}_\omega(t-t_0)}{2} \right)} \\ &= \ln \left(\frac{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(-t_0)}{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t-t_0)} \right). \end{aligned}$$

It follows that

$$\begin{aligned}
& \tilde{\rho}'_d(\omega, t_0, t) \\
&= \frac{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(-t_0)}{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t - t_0)} \left(\tilde{\rho}'_d(\omega, t_0, 0) + \int_0^t dt' \frac{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t' - t_0)}{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(-t_0)} \right. \\
&\quad \left. \times \frac{\partial}{\partial \theta} \left(\frac{K \cos \varphi_\omega (r'_R(t') \sin \theta - r'_I(t') \cos \theta)}{2\pi (1 - \sin \varphi_\omega \sin \theta)} \right) \Big|_{\theta = \tilde{\theta}_\omega(t' - t_0)} \right) \\
&= \tilde{\rho}'_d(\omega, t_0, 0) \frac{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(-t_0)}{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t - t_0)} \\
&\quad + \frac{K \cos \varphi_\omega}{2\pi (1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t - t_0))} \int_0^t dt' \left(r'_R(t') \cos \theta + r'_I(t') \sin \theta \right. \\
&\quad \left. + \frac{\sin \varphi_\omega \cos \theta (r'_R(t') \sin \theta - r'_I(t') \cos \theta)}{1 - \sin \varphi_\omega \sin \theta} \right) \Big|_{\theta = \tilde{\theta}_\omega(t' - t_0)}.
\end{aligned}$$

The integrand can be rewritten as

$$\begin{aligned}
& \frac{r'_R(t') \cos \theta + r'_I(t') \sin \theta - r'_I(t') \sin \varphi_\omega}{1 - \sin \varphi_\omega \sin \theta} \Big|_{\theta = \tilde{\theta}_\omega(t' - t_0)} \\
&= \frac{r'_R(t')(1 - u^2) + r'_I(t')(2u - \sin \varphi_\omega(1 + u^2))}{1 + u^2 - \sin \varphi_\omega 2u} \Big|_{u = \sin \varphi_\omega + \cos \varphi_\omega \tan\left(\frac{\gamma_\omega(t' - t_0)}{2}\right)} \\
&= \frac{1}{1 + \sin^2 \varphi_\omega + \cos^2 \varphi_\omega v^2 + 2 \sin \varphi_\omega \cos \varphi_\omega v - \sin \varphi_\omega 2(\sin \varphi_\omega + \cos \varphi_\omega v)} \\
&\quad \times \left(r'_R(t')(1 - \sin^2 \varphi_\omega - \cos^2 \varphi_\omega v^2 - 2 \sin \varphi_\omega \cos \varphi_\omega v) \right. \\
&\quad \left. + r'_I(t')(2(\sin \varphi_\omega + \cos \varphi_\omega v) - \sin \varphi_\omega(1 + \sin^2 \varphi_\omega + \cos^2 \varphi_\omega v^2 \right. \\
&\quad \left. + 2 \sin \varphi_\omega \cos \varphi_\omega v)) \right) \Big|_{v = \tan\left(\frac{\gamma_\omega(t' - t_0)}{2}\right)} \\
&= \frac{1}{\cos^2 \varphi_\omega(1 + v^2)} \left(r'_R(t')(\cos^2 \varphi_\omega(1 - v^2) - \sin \varphi_\omega \cos \varphi_\omega 2v) \right. \\
&\quad \left. + r'_I(t')(\cos^3 \varphi_\omega 2v + \sin \varphi_\omega \cos^2 \varphi_\omega(1 - v^2)) \right) \Big|_{v = \tan\left(\frac{\gamma_\omega(t' - t_0)}{2}\right)} \\
&= r'_R(t')(\cos(\gamma_\omega(t' - t_0)) - \tan \varphi_\omega \sin(\gamma_\omega(t' - t_0))) \\
&\quad + r'_I(t')(\cos \varphi_\omega \sin(\gamma_\omega(t' - t_0)) + \sin \varphi_\omega \cos(\gamma_\omega(t' - t_0))) \\
&= \frac{r'_R(t')}{\cos \varphi_\omega} \cos(\gamma_\omega(t' - t_0) + \varphi_\omega) + r'_I(t') \sin(\gamma_\omega(t' - t_0) + \varphi_\omega).
\end{aligned}$$

By considering the representation of r'_R and r'_I in terms of complex exponentials, we obtain

$$\begin{aligned}
& \tilde{\rho}'_d(\omega, t_0, t) \\
&= \tilde{\rho}'_d(\omega, t_0, 0) \frac{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(-t_0)}{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t - t_0)} + K \rho_{d,0}(\tilde{\theta}_\omega(t - t_0), \omega) \sum_{\gamma \in \Gamma} \int_0^t dt' \\
&\quad \times e^{i\gamma t'} \left(\frac{R'_R(\gamma)}{\cos \varphi_\omega} \cos(\gamma_\omega(t' - t_0) + \varphi_\omega) + R'_I(\gamma) \sin(\gamma_\omega(t' - t_0) + \varphi_\omega) \right) \\
&= \tilde{\rho}'_d(\omega, t_0, 0) \frac{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(-t_0)}{1 - \sin \varphi_\omega \sin \tilde{\theta}_\omega(t - t_0)} + K \rho_{d,0}(\tilde{\theta}_\omega(t - t_0), \omega) \\
&\quad \times \sum_{\gamma \in \Gamma} \left(\frac{R'_R(\gamma)}{2 \cos \varphi_\omega} \left(\frac{e^{i((\gamma_\omega + \gamma)t - \gamma_\omega t_0 + \varphi_\omega)}}{i(\gamma_\omega + \gamma)} - \frac{e^{i(-\gamma_\omega t_0 + \varphi_\omega)}}{i(\gamma_\omega + \gamma)} \right. \right. \\
&\quad + \frac{e^{i(-\gamma_\omega + \gamma)t + \gamma_\omega t_0 - \varphi_\omega}}{i(-\gamma_\omega + \gamma)} - \frac{e^{i(\gamma_\omega t_0 - \varphi_\omega)}}{i(-\gamma_\omega + \gamma)} \Big) + \frac{R'_I(\gamma)}{2i} \left(\frac{e^{i((\gamma_\omega + \gamma)t - \gamma_\omega t_0 + \varphi_\omega)}}{i(\gamma_\omega + \gamma)} \right. \\
&\quad \left. \left. - \frac{e^{i(-\gamma_\omega t_0 + \varphi_\omega)}}{i(\gamma_\omega + \gamma)} - \frac{e^{i(-\gamma_\omega + \gamma)t + \gamma_\omega t_0 - \varphi_\omega}}{i(-\gamma_\omega + \gamma)} + \frac{e^{i(\gamma_\omega t_0 - \varphi_\omega)}}{i(-\gamma_\omega + \gamma)} \right) \right).
\end{aligned}$$

We determine the initial density $\tilde{\rho}'_d(\omega, t_0, 0)$ (and thus also $\rho'_d(\theta, \omega, 0)$) by imposing that

$$\begin{aligned}
\tilde{\rho}'_d(\omega, t_0, t) &= K \rho_{d,0}(\tilde{\theta}_\omega(t - t_0), \omega) \sum_{\gamma \in \Gamma} e^{i\gamma t} \\
&\quad \times \left(\frac{R'_R(\gamma)}{2 \cos \varphi_\omega} \left(\frac{e^{i(\gamma_\omega(t-t_0) + \varphi_\omega)}}{i(\gamma_\omega + \gamma)} + \frac{e^{i(-\gamma_\omega(t-t_0) - \varphi_\omega)}}{i(-\gamma_\omega + \gamma)} \right) \right. \\
&\quad \left. + \frac{R'_I(\gamma)}{2i} \left(\frac{e^{i(\gamma_\omega(t-t_0) + \varphi_\omega)}}{i(\gamma_\omega + \gamma)} - \frac{e^{i(-\gamma_\omega(t-t_0) - \varphi_\omega)}}{i(-\gamma_\omega + \gamma)} \right) \right).
\end{aligned}$$

Since

$$\begin{aligned}
e^{\pm i(\gamma_\omega(t-t_0) + \varphi_\omega)} &= e^{\pm i\varphi_\omega} \frac{(1 \pm iv)^2}{1 + v^2} \Big|_{v=\tan\left(\frac{\gamma_\omega(t-t_0)}{2}\right)} \\
&= e^{\pm i\varphi_\omega} \frac{\left(1 \pm i \left(\frac{u - \sin \varphi_\omega}{\cos \varphi_\omega}\right)\right)^2}{1 + \left(\frac{u - \sin \varphi_\omega}{\cos \varphi_\omega}\right)^2} \Big|_{u=\tan\left(\frac{\tilde{\theta}_\omega(t-t_0)}{2}\right)} \\
&= e^{\pm i\varphi_\omega} \frac{(e^{\mp i\varphi_\omega} \pm iu)^2}{1 + u^2 - 2u \sin \varphi_\omega} \Big|_{u=\tan\left(\frac{\tilde{\theta}_\omega(t-t_0)}{2}\right)} \\
&= \frac{e^{\mp i\varphi_\omega} - u^2 e^{\pm i\varphi_\omega} \pm 2iu}{1 + u^2 - 2u \sin \varphi_\omega} \Big|_{u=\tan\left(\frac{\tilde{\theta}_\omega(t-t_0)}{2}\right)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mp 2i \sin \varphi_\omega \frac{1+\cos \theta}{2} + \cos \theta e^{\pm i \varphi_\omega} \pm i \sin \theta}{1 - \sin \theta \sin \varphi_\omega} \Bigg|_{\theta=\tilde{\theta}_\omega(t-t_0)} \\
&= \frac{\cos \theta \cos \varphi_\omega \pm i (\sin \theta - \sin \varphi_\omega)}{1 - \sin \theta \sin \varphi_\omega} \Bigg|_{\theta=\tilde{\theta}_\omega(t-t_0)},
\end{aligned}$$

we can write $\tilde{\rho}'_d(\omega, t_0, t)$ as

$$\begin{aligned}
\tilde{\rho}'_d(\omega, t_0, t) &= \frac{K \rho_{d,0}(\theta, \omega)}{1 - \sin \theta \sin \varphi_\omega} \sum_{\gamma \in \Gamma} \frac{e^{i \gamma t}}{\gamma_\omega^2 - \gamma^2} \\
&\quad \times \left(\frac{R'_R(\gamma)}{\cos \varphi_\omega} (i \gamma \cos \theta \cos \varphi_\omega + \gamma_\omega (\sin \theta - \sin \varphi_\omega)) \right. \\
&\quad \left. + \frac{R'_I(\gamma)}{i} (-\gamma (\sin \theta - \sin \varphi_\omega) - i \gamma_\omega \cos \theta \cos \varphi_\omega) \right) \Bigg|_{\theta=\tilde{\theta}_\omega(t-t_0)}.
\end{aligned}$$

A.2.2 The expression for $s(t)$ concerning the entrained oscillators

The differential equation for s can be rewritten as

$$\begin{aligned}
\dot{s}(t) &= \frac{v(\tilde{\theta}_{\omega'}(t+s(t)), \omega, t) - \dot{\tilde{\theta}}_{\omega'}(t+s(t))}{\tilde{\theta}_{\omega'}(t+s(t))} \\
&= \frac{\omega - \omega' - K r'_R(t) \sin \tilde{\theta}_{\omega'}(t+s(t)) + K r'_I(t) \cos \tilde{\theta}_{\omega'}(t+s(t))}{\omega' - K r_0 \sin \tilde{\theta}_{\omega'}(t+s(t))} \\
&= \frac{\frac{\omega - \omega'}{\omega'} (1 + u^2) - \frac{K r'_R(t)}{\omega'} 2u + \frac{K r'_I(t)}{\omega'} (1 - u^2)}{1 + u^2 - 2u \sin \varphi_{\omega'}} \Bigg|_{u=\tan\left(\frac{\tilde{\theta}_{\omega'}(t+s(t))}{2}\right)} \\
&= \frac{1}{1 + (\sin \varphi_{\omega'} + \cos \varphi_{\omega'} v)^2 - 2(\sin \varphi_{\omega'} + \cos \varphi_{\omega'} v) \sin \varphi_{\omega'}} \left(\frac{\omega - \omega'}{\omega'} \right. \\
&\quad \times \left(1 + (\sin \varphi_{\omega'} + \cos \varphi_{\omega'} v)^2 \right) - 2 \frac{K r'_R(t)}{\omega'} (\sin \varphi_{\omega'} + \cos \varphi_{\omega'} v) \\
&\quad \left. + \frac{K r'_I(t)}{\omega'} (1 - (\sin \varphi_{\omega'} + \cos \varphi_{\omega'} v)^2) \right) \Bigg|_{v=\tan\left(\frac{\gamma_{\omega'}(t+s(t))}{2}\right)} \\
&= \frac{1}{\cos^2 \varphi_{\omega'} (1 + v^2)} \left(\frac{\omega - \omega'}{\omega'} (2 + \cos^2 \varphi_{\omega'} (v^2 - 1) + 2v \sin \varphi_{\omega'} \cos \varphi_{\omega'}) \right. \\
&\quad \left. - 2 \frac{K r'_R(t)}{\omega'} (\sin \varphi_{\omega'} + \cos \varphi_{\omega'} v) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{Kr'_1(t)}{\omega'} (\cos^2 \varphi_{\omega'} (1 - v^2) - 2v \sin \varphi_{\omega'} \cos \varphi_{\omega'}) \Big|_{v=\tan(\frac{\gamma_{\omega'}(t+s(t))}{2})} \\
& = \frac{\omega - \omega'}{\omega'} \left(\frac{1 + \cos(\gamma_{\omega'}(t+s(t)))}{\cos^2 \varphi_{\omega'}} - \cos(\gamma_{\omega'}(t+s(t))) + \tan \varphi_{\omega'} \sin(\gamma_{\omega'}(t+s(t))) \right) \\
& \quad - \frac{Kr'_R(t)}{\omega' \cos \varphi_{\omega'}} (\tan \varphi_{\omega'} (1 + \cos(\gamma_{\omega'}(t+s(t)))) + \sin(\gamma_{\omega'}(t+s(t)))) \\
& \quad + \frac{Kr'_1(t)}{\omega'} (\cos(\gamma_{\omega'}(t+s(t))) - \tan \varphi_{\omega'} \sin(\gamma_{\omega'}(t+s(t)))) \\
& = \frac{\omega - \omega'}{\omega' \cos^2 \varphi_{\omega'}} + \frac{\omega - \omega'}{2\omega'} \left(e^{i\gamma_{\omega'}(t+s(t))} (\tan^2 \varphi_{\omega'} - i \tan \varphi_{\omega'}) \right. \\
& \quad \left. + e^{-i\gamma_{\omega'}(t+s(t))} (\tan^2 \varphi_{\omega'} + i \tan \varphi_{\omega'}) \right) - \sum_{\gamma \in \Gamma} \left(e^{i\gamma t} \frac{K}{\gamma_{\omega'}} R'_R(\gamma) \tan \varphi_{\omega'} \right. \\
& \quad \left. + e^{i(\gamma t + \gamma_{\omega'}(t+s(t)) + \varphi_{\omega'})} \frac{K}{2i\gamma_{\omega'}} \left(\frac{R'_R(\gamma)}{\cos \varphi_{\omega'}} - iR'_I(\gamma) \right) \right. \\
& \quad \left. - e^{i(\gamma t - \gamma_{\omega'}(t+s(t)) - \varphi_{\omega'})} \frac{K}{2i\gamma_{\omega'}} \left(\frac{R'_R(\gamma)}{\cos \varphi_{\omega'}} + iR'_I(\gamma) \right) \right).
\end{aligned}$$

The dynamics on the slow time scale are found by setting $\tau \triangleq Kr'_{\text{RMS}}t$ and $s_s(\tau) \triangleq s(t)$:

$$\begin{aligned}
\dot{s}_s(\tau) & = \frac{\omega - \omega'}{Kr'_{\text{RMS}}\omega' \cos^2 \varphi_{\omega'}} + \frac{\omega - \omega'}{2Kr'_{\text{RMS}}\omega'} \\
& \quad \times \left(e^{i\gamma_{\omega'}(\frac{\tau}{Kr'_{\text{RMS}}} + s_s(\tau))} (\tan^2 \varphi_{\omega'} - i \tan \varphi_{\omega'}) \right. \\
& \quad \left. + e^{-i\gamma_{\omega'}(\frac{\tau}{Kr'_{\text{RMS}}} + s_s(\tau))} (\tan^2 \varphi_{\omega'} + i \tan \varphi_{\omega'}) \right) \\
& \quad - \sum_{\gamma \in \Gamma} \left(e^{\frac{i\gamma\tau}{Kr'_{\text{RMS}}}} \frac{R'_R(\gamma)}{\gamma_{\omega'} r'_{\text{RMS}}} \tan \varphi_{\omega'} \right. \\
& \quad \left. + e^{i(\frac{(\gamma + \gamma_{\omega'})\tau}{Kr'_{\text{RMS}}} + \gamma_{\omega'} s_s(\tau) + \varphi_{\omega'})} \frac{1}{2i\gamma_{\omega'} r'_{\text{RMS}}} \left(\frac{R'_R(\gamma)}{\cos \varphi_{\omega'}} - iR'_I(\gamma) \right) \right. \\
& \quad \left. - e^{i(\frac{(\gamma - \gamma_{\omega'})\tau}{Kr'_{\text{RMS}}} - \gamma_{\omega'} s_s(\tau) - \varphi_{\omega'})} \frac{1}{2i\gamma_{\omega'} r'_{\text{RMS}}} \left(\frac{R'_R(\gamma)}{\cos \varphi_{\omega'}} + iR'_I(\gamma) \right) \right).
\end{aligned}$$

Since the oscillator is moving at the average velocity $\gamma_{\omega'}$, s and s_s are bounded, and thus the difference $\omega - \omega'$ is of the same order of magnitude as Kr'_{RMS} . For the zero order approximation of the expression for $s(t)$ (and $s_s(\tau)$), the fast exponentials can be replaced by their average, which is zero.

If $r'_{\text{RMS}} \ll |\gamma - \gamma_{\omega'}|$, $\forall \gamma \in \Gamma$, then this holds for all exponentials, and the

dynamics reduce to

$$\dot{s}_s(\tau) = \frac{\omega - \omega'}{Kr'_{\text{RMS}}\omega' \cos^2 \varphi_{\omega'}},$$

with a bounded solution for s_s if and only if $\omega = \omega'$, implying that oscillators with a slightly different value of ω will move at a different long term average frequency, and therefore $\omega \notin \mathcal{E}$.

Assume that $|\gamma - \gamma_{\omega'}| = \mathcal{O}(r'_{\text{RMS}})$ for some $\gamma \in \Gamma$, and that γ is the only element from Γ with this property. Substituting the fast exponentials by zero, leads to (keeping in mind that $R'_R(-\gamma_{\omega'}) = \overline{R'_R(\gamma_{\omega'})}$ and $R'_I(-\gamma_{\omega'}) = \overline{R'_I(\gamma_{\omega'})}$, and that also $-\gamma \in \Gamma$)

$$\begin{aligned} \dot{s}_s(\tau) &= \frac{\omega - \omega'}{Kr'_{\text{RMS}}\omega' \cos^2 \varphi_{\omega'}} + \frac{|R'_R(\gamma) + i \cos \varphi_{\omega'} R'_I(\gamma)|}{\gamma_{\omega'} r'_{\text{RMS}} \cos \varphi_{\omega'}} \\ &\times \sin \left(\frac{(\gamma - \gamma_{\omega'})\tau}{Kr'_{\text{RMS}}} - \gamma_{\omega'} s_s(\tau) - \varphi_{\omega'} + \arg(R'_R(\gamma) + i \cos \varphi_{\omega'} R'_I(\gamma)) \right). \quad (\text{A.3}) \end{aligned}$$

First assume that $\gamma \neq \gamma_{\omega'}$. We will show that $\omega \notin \mathcal{E}$. Extending the system with the equation $\dot{\tau} = 1$, then results in a vector field in the two dimensional (τ, s_s) -plane which is invariant under translations in the $(\gamma_{\omega'}, \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}})$ -direction. Pick a particular solution, parametrized by $(\tau, s_{s,\omega}^p(\tau))$, $\tau \in \mathbb{R}$. From the translation invariance and the fact that $s_{s,\omega}^p$ is bounded, it can be derived that any point in the (τ, s_s) -plane can be written in a *unique* way as the sum of a point from the trajectory of the particular solution $s_{s,\omega}^p$ and a scalar multiple of the vector $(\gamma_{\omega'}, \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}})$, i.e.

$$\forall (\tau, s_s^0) \in \mathbb{R}^2, \exists! (\sigma, \lambda) \in \mathbb{R}^2 : (\tau, s_s^0) = (\sigma, s_{s,\omega}^p(\sigma)) + \lambda(\gamma_{\omega'}, \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}}).$$

For a slightly different value $\tilde{\omega}$ of ω we obtain a new vector field $(1, \dot{s}_{s,\tilde{\omega}})$, which, in the point characterized by (σ, λ) can be written as

$$\begin{aligned} (1, \dot{s}_{s,\omega}^p(\sigma) + \xi) &= \left(1 + \frac{\xi \gamma_{\omega'}}{\gamma_{\omega'} \dot{s}_{s,\omega}^p(\sigma) - \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}}} \right) (1, \dot{s}_{s,\omega}^p(\sigma)) \\ &\quad - \frac{\xi}{\gamma_{\omega'} \dot{s}_{s,\omega}^p(\sigma) - \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}}} (\gamma_{\omega'}, \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}}), \end{aligned}$$

with $\xi = \frac{\tilde{\omega} - \omega}{Kr'_{\text{RMS}}\omega' \cos^2 \varphi_{\omega'}}$. It follows that, along the flow of the new vector field, the parameter λ varies in (τ) -time as

$$\frac{d\lambda}{d\tau}(\tau) = - \frac{\xi}{\gamma_{\omega'} \dot{s}_{s,\omega}^p(\sigma(\tau)) - \frac{\gamma - \gamma_{\omega'}}{Kr'_{\text{RMS}}}},$$

with $|\frac{d\lambda}{d\tau}(\tau)|$ bounded from below since $|\dot{s}_{s,\omega}^p|$ is upper bounded. Furthermore a possible singularity by the denominator becoming zero cannot pose any problems, since λ cannot become infinite in a finite time interval, while the denominator cannot change sign, as can be derived from the translation invariance of the vector field. It follows that λ will grow unbounded along the new vector field $(1, \dot{s}_{s,\tilde{\omega}})$, and since $s_{s,\omega}^p$ is bounded and $\gamma \neq \gamma_{\omega'}$, $s_{s,\tilde{\omega}}$ cannot be bounded if $\tilde{\omega} \neq \omega$, and therefore $\omega \notin \mathcal{E}$.

Consider the case $\gamma_{\omega'} \in \Gamma$. From (A.3) it can be derived that $\omega \in \mathcal{E}$ and that the corresponding entrained subset will consist of all oscillators for which

$$|\omega - \omega'| \leq K |R'_C(\gamma_{\omega'})|,$$

where

$$R'_C(\gamma_{\omega}) \triangleq R'_R(\gamma_{\omega}) + i \cos \varphi_{\omega} R'_I(\gamma_{\omega}), \quad \forall \omega \in \mathbb{R} \setminus [-Kr_0, Kr_0],$$

with the zeroth order approximation of $s_s(\tau)$ and $s(t)$ settling into a constant value ($\sim \mathcal{O}(1)$), which (for a generic initial condition) can be calculated as

$$s(t) = \frac{1}{\gamma_{\omega'}} \left(\arcsin \left(\frac{\omega - \omega'}{K |R'_C(\gamma_{\omega'})|} \right) + \arg(R'_C(\gamma_{\omega'})) - \varphi_{\omega'} \right).$$

If there is more than one $\gamma \in \Gamma$ with $|\gamma - \gamma_{\omega'}| = \mathcal{O}(r'_{\text{RMS}})$, then, for simplicity, we assume that the fraction of oscillators involved in the entrainment is small enough to neglect its contribution to the equation (3.7).

A.2.3 The self-consistency equation

With the results of sections 3.3 and 3.4, (the first order approximation of) equation (3.7) can be written as

$$\begin{aligned} & (r_0 + r'_R(t) + ir'_I(t))e^{i\alpha} \\ &= \int_{-\infty}^{\infty} d\omega g_R(\omega) \int_{-\pi}^{\pi} d\theta e^{i\theta} (\rho_e(\theta, \omega, t) + \rho_{d,0}(\theta, \omega) + \rho'_d(\theta, \omega, t)) \\ &= \int_{-Kr_0}^{Kr_0} d\omega g_R(\omega) \exp(i\theta_0) \left(1 + i \operatorname{Im} \sum_{\gamma \in \Gamma} \frac{KR'(\gamma)e^{i(\gamma t - \theta_0)}}{Kr_0 \cos \theta_0 + i\gamma} \right) \Bigg|_{\theta_0 = \arcsin(\frac{\omega}{Kr_0})} \\ &+ \sum_{\gamma_{\omega'} \in \Gamma} \int_{\omega' - K|R'_C(\gamma_{\omega'})|}^{\omega' + K|R'_C(\gamma_{\omega'})|} d\omega g_R(\omega) \exp \left(i\tilde{\theta}_{\omega'} \left(t + \frac{1}{\gamma_{\omega'}} \left(\arcsin \left(\frac{\omega - \omega'}{K|R'_C(\gamma_{\omega'})|} \right) \right. \right. \right. \\ &\quad \left. \left. \left. + \arg(R'_C(\gamma_{\omega'})) - \varphi_{\omega'} \right) \right) \right) + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \int_{-\pi}^{\pi} d\theta \frac{\cos \varphi_{\omega} e^{i\theta}}{2\pi(1 - \sin \varphi_{\omega} \sin \theta)} \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \int_{-\pi}^{\pi} d\theta \frac{K \cos \varphi_\omega e^{i\theta}}{2\pi(1 - \sin \varphi_\omega \sin \theta)^2} \sum_{\gamma \in \Gamma} \frac{e^{i\gamma t}}{\gamma_\omega^2 - \gamma^2} \\
& \times \left(\left(\frac{\gamma R'_R(\gamma)}{\cos \varphi_\omega} + i\gamma_\omega R'_1(\gamma) \right) i \cos \theta \cos \varphi_\omega \right. \\
& \left. + \left(\frac{\gamma_\omega R'_R(\gamma)}{\cos \varphi_\omega} + i\gamma R'_1(\gamma) \right) (\sin \theta - \sin \varphi_\omega) \right).
\end{aligned}$$

Denote by \mathcal{C} the curve around the unit circle in the complex plane in counterclockwise direction, and set $z_1 \triangleq i \tan\left(\frac{\varphi_\omega}{2}\right)$ and $z_2 \triangleq i \cot\left(\frac{\varphi_\omega}{2}\right)$. (Notice that z_1 and z_2 are the roots of $z^2 - \frac{2iz}{\sin \varphi_\omega} - 1 = 0$.) We can then calculate the integrals over θ as follows.

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{e^{i\theta} d\theta}{1 - \sin \varphi_\omega \sin \theta} &= -\frac{2}{\sin \varphi_\omega} \int_{\mathcal{C}} \frac{z dz}{z^2 - \frac{2i}{\sin \varphi_\omega} z - 1} \\
&= -\frac{2\pi i}{\sin \varphi_\omega} \left(\frac{z_1}{z_1 - \frac{i}{\sin \varphi_\omega}} \right) \\
&= \frac{2\pi i}{\sin \varphi_\omega} \left(\frac{1}{\cos \varphi_\omega} - 1 \right),
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\cos \theta e^{i\theta} d\theta}{(1 - \sin \varphi_\omega \sin \theta)^2} &= -\frac{2}{i \sin^2 \varphi_\omega} \int_{\mathcal{C}} \frac{(z^3 + z) dz}{\left(z^2 - \frac{2i}{\sin \varphi_\omega} z - 1\right)^2} \\
&= -\frac{4\pi i}{i \sin^2 \varphi_\omega} \frac{d}{dz} \left(\frac{z^3 + z}{(z - z_2)^2} \right) \Bigg|_{z=z_1} \\
&= -\frac{4\pi}{\sin^2 \varphi_\omega} \left(\frac{z_1^3 - 3z_1^2 z_2 - z_2 - z_1}{(z_1 - z_2)^3} \right) \\
&= -\frac{4\pi}{\sin^2 \varphi_\omega} \left(\frac{\cos \varphi_\omega - 1}{2 \cos \varphi_\omega} \right) \\
&= \frac{2\pi}{\sin^2 \varphi_\omega} \left(\frac{1}{\cos \varphi_\omega} - 1 \right),
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{(\sin \theta - \sin \varphi_\omega) e^{i\theta} d\theta}{(1 - \sin \varphi_\omega \sin \theta)^2} &= \frac{2}{\sin^2 \varphi_\omega} \int_{\mathcal{C}} \frac{(z^3 - z - 2i \sin \varphi_\omega z^2) dz}{\left(z^2 - \frac{2i}{\sin \varphi_\omega} z - 1\right)^2} \\
&= \frac{4\pi i}{\sin^2 \varphi_\omega} \frac{d}{dz} \left(\frac{(z^3 - z - 2i \sin \varphi_\omega z^2)}{(z - z_2)^2} \right) \Bigg|_{z=z_1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4\pi i}{\sin^2 \varphi_\omega} \left(\frac{z_1^3 - 3z_1^2 z_2 + z_2 + z_1 + 4i \sin \varphi_\omega z_1 z_2}{(z_1 - z_2)^3} \right) \\
&= \frac{2\pi i}{\sin^2 \varphi_\omega} (1 - \cos \varphi_\omega).
\end{aligned}$$

It follows that

$$\begin{aligned}
&(r_0 + r'_R(t) + ir'_I(t))e^{i\alpha} \\
&= \int_{-Kr_0}^{Kr_0} d\omega g_R(\omega) \exp(i\theta_0) \left(1 + i \operatorname{Im} \sum_{\gamma_{\omega'} \in \Gamma} \frac{KR'(\gamma_{\omega'})e^{i(\gamma_{\omega'}t - \theta_0)}}{Kr_0 \cos \theta_0 + i\gamma_{\omega'}} \right) \Big|_{\theta_0 = \arcsin\left(\frac{\omega}{Kr_0}\right)} \\
&\quad + \sum_{\gamma_{\omega'} \in \Gamma} \int_{\omega' - K|R'_C(\gamma_{\omega'})|}^{\omega' + K|R'_C(\gamma_{\omega'})|} d\omega g_R(\omega) \exp\left(i\tilde{\theta}_{\omega'} \left(t + \frac{1}{\gamma_{\omega'}} \left(\arcsin\left(\frac{\omega - \omega'}{K|R'_C(\gamma_{\omega'})|} \right) \right. \right. \right. \\
&\quad \left. \left. \left. + \arg(R'_C(\gamma_{\omega'})) - \varphi_{\omega'} \right) \right) \right) + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{i(1 - \cos \varphi_\omega)}{\sin \varphi_\omega} \\
&\quad + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{iK(1 - \cos \varphi_\omega)}{\sin^2 \varphi_\omega} \sum_{\gamma_{\omega'} \in \Gamma} \frac{e^{i\gamma_{\omega'}t}}{\gamma_{\omega} - \gamma_{\omega'}} (R'_R(\gamma_{\omega'}) + i \cos \varphi_\omega R'_I(\gamma_{\omega'})) \\
&= Kr_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta_0 d\theta_0 g_R(Kr_0 \sin \theta_0) \exp(i\theta_0) \\
&\quad \times \left(1 + i \operatorname{Im} \sum_{\gamma_{\omega'} \in \Gamma} \frac{KR'(\gamma_{\omega'})e^{i(\gamma_{\omega'}t - \theta_0)}}{Kr_0 \cos \theta_0 + i\gamma_{\omega'}} \right) \\
&\quad + \sum_{\gamma_{\omega'} \in \Gamma} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K|R'_C(\gamma_{\omega'})| \cos \theta_0 d\theta_0 g_R(\omega' + K|R'_C(\gamma_{\omega'})| \sin \theta_0) \\
&\quad \times \exp\left(i\tilde{\theta}_{\omega'} \left(t + \frac{1}{\gamma_{\omega'}} (\theta_0 + \arg(R'_C(\gamma_{\omega'})) - \varphi_{\omega'}) \right) \right) \\
&\quad + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{i(1 - \cos \varphi_\omega)}{\sin \varphi_\omega} \\
&\quad + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{iK(1 - \cos \varphi_\omega)}{\sin^2 \varphi_\omega} \sum_{\gamma_{\omega'} \in \Gamma} \frac{e^{i\gamma_{\omega'}t}}{\gamma_{\omega} - \gamma_{\omega'}} (R'_R(\gamma_{\omega'}) + i \cos \varphi_\omega R'_I(\gamma_{\omega'})).
\end{aligned}$$

With

$$\begin{aligned}
e^{i\tilde{\theta}_{\omega'}(t)} &= \frac{1 - u^2 + 2iu}{1 + u^2} \Big|_{u = \sin \varphi_{\omega'} + \cos \varphi_{\omega'} \tan\left(\frac{\gamma_{\omega'}t}{2}\right)} \\
&= \frac{1 + i \left(\sin \varphi_{\omega'} + \cos \varphi_{\omega'} \tan\left(\frac{\gamma_{\omega'}t}{2}\right) \right)}{1 - i \left(\sin \varphi_{\omega'} + \cos \varphi_{\omega'} \tan\left(\frac{\gamma_{\omega'}t}{2}\right) \right)}
\end{aligned}$$

$$\begin{aligned}
(\tan \left(\frac{\gamma_{\omega'} t}{2} \right)) &= i \frac{1 - e^{i\gamma_{\omega'} t}}{1 + e^{i\gamma_{\omega'} t}} \\
&= \frac{e^{i\gamma_{\omega'} t} (1 + \cos \varphi_{\omega'} + i \sin \varphi_{\omega'}) + 1 - \cos \varphi_{\omega'} + i \sin \varphi_{\omega'}}{e^{i\gamma_{\omega'} t} (1 - \cos \varphi_{\omega'} - i \sin \varphi_{\omega'}) + 1 + \cos \varphi_{\omega'} - i \sin \varphi_{\omega'}} \\
&= \frac{e^{i\gamma_{\omega'} t} + \frac{1 - e^{-i\varphi_{\omega'}}}{1 + e^{i\varphi_{\omega'}}}}{e^{i\gamma_{\omega'} t} \frac{1 - e^{i\varphi_{\omega'}}}{1 + e^{i\varphi_{\omega'}}} + e^{-i\varphi_{\omega'}}} \\
&= \frac{e^{i(\gamma_{\omega'} t + \varphi_{\omega'})} + i \tan \left(\frac{\varphi_{\omega'}}{2} \right)}{1 - i e^{i(\gamma_{\omega'} t + \varphi_{\omega'})} \tan \left(\frac{\varphi_{\omega'}}{2} \right)} \\
&= i \tan \left(\frac{\varphi_{\omega'}}{2} \right) - 2i \cot \varphi_{\omega'} \sum_{k=1}^{\infty} \left(i e^{i(\gamma_{\omega'} t + \varphi_{\omega'})} \tan \left(\frac{\varphi_{\omega'}}{2} \right) \right)^k,
\end{aligned}$$

and (applying the fact that $\text{Im} \beta = \frac{\beta - \bar{\beta}}{2i}$, $\forall \beta \in \mathbb{C}$)

$$\begin{aligned}
\text{Im} \sum_{\gamma_{\omega'} \in \Gamma} K R'(\gamma_{\omega'}) \frac{e^{i(\gamma_{\omega'} t - \theta_0)}}{K r_0 \cos \theta_0 + i \gamma_{\omega'}} \\
&= \sum_{\gamma_{\omega'} \in \Gamma} \frac{K}{2(K^2 r_0^2 \cos^2 \theta_0 + \gamma_{\omega'}^2)} \\
&\quad \times \left(-e^{i(\gamma_{\omega'} t - \theta_0)} (R'_R(\gamma_{\omega'}) + i R'_I(\gamma_{\omega'})) (i K r_0 \cos \theta_0 + \gamma_{\omega'}) \right. \\
&\quad \left. + e^{-i(\gamma_{\omega'} t - \theta_0)} (R'_R(-\gamma_{\omega'}) - i R'_I(-\gamma_{\omega'})) (i K r_0 \cos \theta_0 - \gamma_{\omega'}) \right),
\end{aligned}$$

we can obtain a decomposition in complex exponentials and identify the coefficients (recall that $0 \notin \Gamma$ and that $\gamma \in \Gamma \Leftrightarrow -\gamma \in \Gamma$):

$$\begin{aligned}
r_0 e^{i\alpha} &= K r_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta_0 d\theta_0 g_R(K r_0 \sin \theta_0) e^{i\theta_0} \\
&\quad + \sum_{\gamma_{\omega'} \in \Gamma} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K |R'_C(\gamma_{\omega'})| \cos \theta_0 d\theta_0 g_R(\omega' + K |R'_C(\gamma_{\omega'})| \sin \theta_0) \\
&\quad \times i \tan \left(\frac{\varphi_{\omega'}}{2} \right) + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{i(1 - \cos \varphi_{\omega})}{\sin \varphi_{\omega}}, \\
R'(\gamma_{\omega'}) e^{i\alpha} &= K r_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta_0 d\theta_0 g_R(K r_0 \sin \theta_0) e^{i\theta_0} \frac{K}{2(K^2 r_0^2 \cos^2 \theta_0 + \gamma_{\omega'}^2)} \\
&\quad \times \left(e^{-i\theta_0} (R'_R(\gamma_{\omega'}) + i R'_I(\gamma_{\omega'})) (K r_0 \cos \theta_0 - i \gamma_{\omega'}) \right. \\
&\quad \left. - e^{i\theta_0} (R'_R(\gamma_{\omega'}) - i R'_I(\gamma_{\omega'})) (K r_0 \cos \theta_0 - i \gamma_{\omega'}) \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\gamma_{\omega''} \in \Gamma \cap \{\frac{\gamma_{\omega'}'}{k} : k \in \mathbb{N}_0\}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K |R'_C(\gamma_{\omega''})| \cos \theta_0 d\theta_0 \\
& \quad \times g_R(\omega'' + K |R'_C(\gamma_{\omega''})| \sin \theta_0) (-2i) \cot \varphi_{\omega''} \\
& \quad \times \left(i e^{i(\theta_0 + \arg(R'_C(\gamma_{\omega''})))} \tan \left(\frac{\varphi_{\omega''}}{2} \right) \right)^{\frac{\gamma_{\omega'}'}{\gamma_{\omega''}}} \\
& + \int_{\mathbb{R} \setminus \mathcal{E}} d\omega g_R(\omega) \frac{iK(1 - \cos \varphi_{\omega})}{\sin^2 \varphi_{\omega}(\gamma_{\omega} - \gamma_{\omega'})} (R'_R(\gamma_{\omega'}) + i \cos \varphi_{\omega} R'_I(\gamma_{\omega'})),
\end{aligned}$$

$\forall \gamma_{\omega'} \in \Gamma$.

A.3 Proofs for chapter 5

A.3.1 Proof of lemma 5.1

We will treat both directions of the proof at the same time.

Each inequality from (5.9a) with $k \notin \{k' - 1, k'\}$ is equivalent to an equality from (5.10a) with $k \notin \{k' - 1, k', k' + 1\}$. Each inequality from (5.9b) with $k \neq k'$ is equivalent to an equality from (5.10b) with $k \notin \{k', k' + 1\}$. The equalities (5.9c) and (5.10c) are equivalent and imply (5.10a) for $k = k'$.

Now we will show that (5.9) implies (5.10b) for $k \in \{k', k' + 1\}$ and that (5.10) implies (5.9b) for $k = k'$. Pick $(G_{k',1}^2, G_{k',2}^2) \in \mathcal{P}_2(G_{k'}^2)$ and $(G_{k'+1,1}^2, G_{k'+1,2}^2) \in \mathcal{P}_2(G_{k'+1}^2)$. Then $G_{k',1}^1 \triangleq G_{k',1}^2 \cup G_{k'+1,1}^2$ and $G_{k',2}^1 \triangleq G_{k',2}^2 \cup G_{k'+1,2}^2$ satisfy $(G_{k',1}^1, G_{k',2}^1) \in \mathcal{P}_2(G_{k'}^1)$.

Extend the domain of \tilde{v} by setting

$$\tilde{v}(G_-, \emptyset, G_+) = 0,$$

for arbitrary subsets $G_-, G_+ \subset \mathcal{I}_N$. (The actual value of $\tilde{v}(G_-, \emptyset, G_+)$ is irrelevant, but it needs to be finite.) For notational convenience, set

$$\begin{aligned}
A_1 &\triangleq G_{k'}^{2<}, & A_2 &\triangleq G_{k',1}^2, & A_3 &\triangleq G_{k',2}^2, \\
A_4 &\triangleq G_{k'+1,1}^2, & A_5 &\triangleq G_{k'+1,2}^2, & A_6 &\triangleq G_{k'+1}^{2>},
\end{aligned}$$

$$\begin{aligned}
A_{i_1, \dots, i_n} &\triangleq A_{i_1} \cup \dots \cup A_{i_n}, \\
\Gamma_{i_1, \dots, i_n} &\triangleq \sum_{i \in A_{i_1, \dots, i_n}} \gamma_i.
\end{aligned}$$

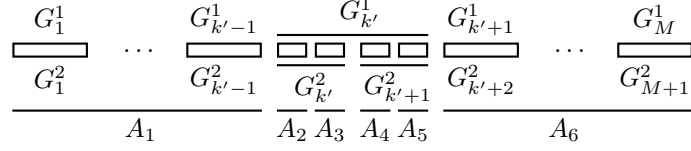


Figure A.1: Schematic representation of the clusters and the notation.

First notice that $\tilde{v}(A_1, A_{2,3,4,5}, A_6)$ is a weighted average of $\tilde{v}(A_1, A_{2,3}, A_{4,5,6})$ and $\tilde{v}(A_{1,2,3}, A_{4,5}, A_6)$:

$$\tilde{v}(A_1, A_{2,3,4,5}, A_6) = \frac{\Gamma_{2,3}\tilde{v}(A_1, A_{2,3}, A_{4,5,6}) + \Gamma_{4,5}\tilde{v}(A_{1,2,3}, A_{4,5}, A_6)}{\Gamma_{2,3,4,5}},$$

and from (5.9c) or (5.10c) it then follows that

$$\tilde{v}(A_1, A_{2,3,4,5}, A_6) = \tilde{v}(A_1, A_{2,3}, A_{4,5,6}) = \tilde{v}(A_{1,2,3}, A_{4,5}, A_6),$$

and thus that (5.9) implies (5.10a) for $k \in \{k' - 1, k' + 1\}$ and (5.10) implies (5.9a) for $k \in \{k' - 1, k'\}$.

Furthermore, one can verify that

$$\begin{aligned} & \tilde{v}(A_{1,2,4}, A_{3,5}, A_6) - \tilde{v}(A_1, A_{2,4}, A_{3,5,6}) \\ &= \frac{\Gamma_3\tilde{v}(A_{1,2,4}, A_3, A_6) + \Gamma_5\tilde{v}(A_{1,2,4}, A_5, A_6)}{\Gamma_{3,5}} \\ & \quad - \frac{\Gamma_2\tilde{v}(A_1, A_2, A_{3,5,6}) + \Gamma_4\tilde{v}(A_1, A_4, A_{3,5,6})}{\Gamma_{2,4}} \\ &= \frac{\Gamma_3\tilde{v}(A_{1,2}, A_3, A_{4,5,6}) + \Gamma_5\tilde{v}(A_{1,2,3,4}, A_5, A_6)}{\Gamma_{3,5}} \\ & \quad - \frac{\Gamma_2\tilde{v}(A_1, A_2, A_{3,4,5,6}) + \Gamma_4\tilde{v}(A_{1,2,3}, A_4, A_{5,6})}{\Gamma_{2,4}} \\ & \quad - 2 \left(\frac{1}{\Gamma_{2,4}} + \frac{1}{\Gamma_{3,5}} \right) \sum_{i \in A_3} \sum_{j \in A_4} K \gamma_i \gamma_j F_{ij} \\ &= \left(\frac{\Gamma_2\Gamma_3\Gamma_{2,3,4,5}}{\Gamma_{2,3}\Gamma_{2,4}\Gamma_{3,5}} - \frac{\Gamma_3(\Gamma_2\Gamma_5 - \Gamma_3\Gamma_4)}{\Gamma_{2,3}\Gamma_{2,4}\Gamma_{3,5}} \right) \tilde{v}(A_{1,2}, A_3, A_{4,5,6}) \\ & \quad + \left(\frac{\Gamma_4\Gamma_5\Gamma_{2,3,4,5}}{\Gamma_{4,5}\Gamma_{2,4}\Gamma_{3,5}} + \frac{\Gamma_5(\Gamma_2\Gamma_5 - \Gamma_3\Gamma_4)}{\Gamma_{4,5}\Gamma_{2,4}\Gamma_{3,5}} \right) \tilde{v}(A_{1,2,3,4}, A_5, A_6) \\ & \quad - \left(\frac{\Gamma_2\Gamma_3\Gamma_{2,3,4,5}}{\Gamma_{2,3}\Gamma_{2,4}\Gamma_{3,5}} + \frac{\Gamma_2(\Gamma_2\Gamma_5 - \Gamma_3\Gamma_4)}{\Gamma_{2,3}\Gamma_{2,4}\Gamma_{3,5}} \right) \tilde{v}(A_1, A_2, A_{3,4,5,6}) \\ & \quad - \left(\frac{\Gamma_4\Gamma_5\Gamma_{2,3,4,5}}{\Gamma_{4,5}\Gamma_{2,4}\Gamma_{3,5}} - \frac{\Gamma_4(\Gamma_2\Gamma_5 - \Gamma_3\Gamma_4)}{\Gamma_{4,5}\Gamma_{2,4}\Gamma_{3,5}} \right) \tilde{v}(A_{1,2,3}, A_4, A_{5,6}) \end{aligned}$$

$$\begin{aligned}
& -2 \left(\frac{1}{\Gamma_{2,4}} + \frac{1}{\Gamma_{3,5}} \right) \sum_{i \in A_3} \sum_{j \in A_4} K \gamma_i \gamma_j F_{ij} \\
&= \frac{\Gamma_2 \Gamma_3 \Gamma_{2,3,4,5}}{\Gamma_{2,3} \Gamma_{2,4} \Gamma_{3,5}} (\tilde{v}(A_{1,2}, A_3, A_{4,5,6}) - \tilde{v}(A_1, A_2, A_{3,4,5,6})) \\
&\quad + \frac{\Gamma_4 \Gamma_5 \Gamma_{2,3,4,5}}{\Gamma_{4,5} \Gamma_{2,4} \Gamma_{3,5}} (\tilde{v}(A_{1,2,3,4}, A_5, A_6) - \tilde{v}(A_{1,2,3}, A_4, A_{5,6})) \\
&\quad + \frac{\Gamma_2 \Gamma_5 - \Gamma_3 \Gamma_4}{\Gamma_{2,4} \Gamma_{3,5}} (\tilde{v}(A_{1,2,3}, A_{4,5}, A_6) - \tilde{v}(A_1, A_{2,3}, A_{4,5,6})) \\
&\quad - 2 \left(\frac{1}{\Gamma_{2,4}} + \frac{1}{\Gamma_{3,5}} \right) \sum_{i \in A_3} \sum_{j \in A_4} K \gamma_i \gamma_j F_{ij}.
\end{aligned}$$

The third term is zero, because of either (5.9c) or (5.10c). It follows that (5.10) implies (5.9b) for $k = k'$. If $A_3 = \emptyset$ or $A_4 = \emptyset$, then $\Gamma_3 = \Gamma_4 = 0$ and one of the first two terms and the last term will equal zero, and one can derive that (5.9) implies (5.10b) for $k \in \{k', k' + 1\}$.

A.3.2 Systems with a convex gradient function

Consider the state space \mathbb{R}^N , equipped with a constant metric described by the (symmetric and positive definite) matrix $G \in \mathbb{R}^{N \times N}$. Let $V : \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuously differentiable function with a Lipschitz continuous gradient $G^{-1} \nabla V$, and consider the associated gradient system:

$$\dot{x}(t) = -G^{-1} \nabla V(x(t)), \quad \forall t \in \mathbb{R}. \quad (\text{A.4})$$

For a trajectory $x : \mathbb{R} \rightarrow \mathbb{R}^N$, the positive limit set Γ^+ is defined by

$$\Gamma^+ \triangleq \left\{ y \in \mathbb{R}^N : \liminf_{t \rightarrow +\infty} \|x(t) - y\| = 0 \right\}.$$

For a solution x of (A.4) with $x|_{\mathbb{R}_+}$ bounded, the positive limit set is non-empty and, using V as a Lyapunov function and applying LaSalle's invariance theorem (see e.g. [28]), one can show that Γ^+ consists of equilibrium points of (A.4). If the equilibrium points of (A.4) are isolated, then it easily follows that Γ^+ is a singleton and x converges to a single equilibrium point. If they are not isolated, and no further restrictions are imposed on the function V , then Γ^+ may contain more than one equilibrium point as is illustrated below. Under some extra conditions on the function V , it can be shown that Γ^+ is a singleton. Most of these results, such as those of [2], are based on (extensions of) Lojasiewicz's inequality (see e.g. [32]) which requires real analyticity of the function V . In the following theorem we only require Lipschitz continuity of the first derivatives of V , but we will impose a convexity assumption to obtain the desired result.

Theorem A.1. *Let $V : \mathbb{R}^N \rightarrow \mathbb{R}$ be differentiable with ∇V Lipschitz continuous and assume that V is convex. Let $G \in \mathbb{R}^{N \times N}$ be symmetric and positive definite, and consider a solution x of the system (A.4) for which $x|_{\mathbb{R}^+}$ is bounded. Then x converges to an equilibrium point of (A.4).*

Proof. Denote by Γ^+ the positive limit set of x and pick $y \in \Gamma^+$; then y is an equilibrium point of (A.4) by the classical Lyapunov theory. Since V is non-increasing along x it follows that $V(x(t)) \geq V(y)$, $\forall t \in \mathbb{R}$ (by continuity of V). Pick a $t \in \mathbb{R}$ for which $x(t) \neq y$. (If no such t exists then the conclusion of the theorem immediately follows.) Then the function

$$v : \mathbb{R} \rightarrow \mathbb{R} : s \mapsto v(s) = V(y + s(x(t) - y))$$

is differentiable and convex with $v(1) \geq v(0)$. It follows that $\frac{dv}{ds}(1) \geq 0$ and thus

$$\nabla V(x(t))^T(x(t) - y) \geq 0,$$

implying that $\dot{x}(t)^T G(x(t) - y) \leq 0$, and therefore $(x(t) - y)^T G(x(t) - y)$ is non-increasing in t . As a consequence

$$\lim_{t \rightarrow +\infty} (x(t) - y)^T G(x(t) - y) = \liminf_{t \rightarrow +\infty} (x(t) - y)^T G(x(t) - y) = 0,$$

or

$$\lim_{t \rightarrow +\infty} x(t) = y.$$

□

A gradient system with bounded solutions not converging to an equilibrium point

We present an example of a gradient system (with a gradient function which is non-convex and not real analytic) where the conclusion of the previous theorem does not hold. The expression for the gradient function is inspired by an example in [2].

Consider the plane \mathbb{R}^2 , equipped with the standard metric (i.e. $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$), and the function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$V(r \cos \theta, r \sin \theta) = V_P(r, \theta), \quad \forall r \in \mathbb{R}^+, \forall \theta \in [0, 2\pi],$$

where V_P is the representation of V in polar coordinates, and is defined by

$$V_P(r, \theta) = 0, \quad \forall r \in [0, 1], \forall \theta \in [0, 2\pi],$$

and

$$V_P(r, \theta) = \exp\left(-\frac{1}{(r-1)^2}\right) \left((r-1)^2 + 1 + \sin\left(\frac{1}{(r-1)^2} - \theta\right) \right),$$

$\forall r > 1, \forall \theta \in [0, 2\pi]$. The function V is radially unbounded, implying that each trajectory x of the gradient system (A.4) remains bounded for positive time (i.e. $x|_{\mathbb{R}^+}$ is bounded) and has a non-empty positive limit set Γ^+ consisting of equilibrium points. The set of equilibrium points of the system (A.4) is the closed unit disk in \mathbb{R}^2 ; it follows that if $\|x(0)\| > 1$, then $\lim_{t \rightarrow +\infty} \|x(t)\| = 1$. Consider the curves

$$\mathcal{C}_C \triangleq \left\{ (r \cos \theta, r \sin \theta) : r > 1, \theta > -C; r = 1 + \frac{1}{\sqrt{C+\theta}} \right\},$$

with $C \in \mathbb{R}$, representing spirals in \mathbb{R}^2 around the unit circle. On a curve \mathcal{C}_C , represented in polar coordinates by r and θ , the derivative of $r - 1 - \frac{1}{\sqrt{C+\theta}}$ along a solution of (A.4) equals

$$\begin{aligned} -\frac{\partial V_P}{\partial r}(r, \theta) - \frac{1}{2(C+\theta)^{3/2}} \frac{1}{r} \frac{\partial V_P}{\partial \theta}(r, \theta) &= \frac{2}{(r-1)^3} \exp\left(-\frac{1}{(r-1)^2}\right) \\ &\times \left(-(r-1)^2 - 1 - \sin C - (r-1)^4 + \cos C + \frac{(r-1)^6}{4r} \cos C \right). \end{aligned}$$

For $C = C_1 \triangleq -\frac{\pi}{4}$ and r close to one this derivative is positive, while for $C = C_2 \triangleq \frac{\pi}{2}$ and r close to one this derivative is negative. Consider a solution with $\|x(0)\| > 1$. Since $\lim_{t \rightarrow +\infty} \|x(t)\| = 1$, there exists a $T \in \mathbb{R}$ for which the trajectory of $x|_{(T, +\infty)}$ lies between \mathcal{C}_{C_1} and \mathcal{C}_{C_2} , while $\|x(t)\|$ approaches 1. It follows that the positive limit set of x is the unit circle.

A.4 Proofs for chapter 8

A.4.1 Proof of theorem 8.1

Taking the partial derivative to b in (8.1) results in

$$\frac{\partial^2 x}{\partial t \partial b}(b, t) = 1 - \phi(b, t) \frac{\partial x}{\partial b}(b, t), \quad \forall b, t \in \mathbb{R}, \quad (\text{A.5})$$

with

$$\phi(b, t) \triangleq K \int_{-\infty}^{+\infty} g(b') \frac{df}{dx}(x(b', t) - x(b, t)) db', \quad \forall b, t \in \mathbb{R}.$$

The differential equation (A.5) can be solved for $\frac{\partial x}{\partial b}(b, t)$ as

$$\frac{\partial x}{\partial b}(b, t) = \exp\left(-\int_0^t \phi(b, t') dt'\right) \left(\int_0^t \exp\left(\int_0^{t'} \phi(b, t'') dt''\right) dt' + \frac{\partial x}{\partial b}(b, 0)\right), \quad (\text{A.6})$$

and since $\frac{\partial x}{\partial b}(b, 0) \geq 0$, $\forall b \in \mathbb{R}$, $\frac{\partial x}{\partial b}(b, t)$ is non-negative for all $t \geq 0$, $\forall b \in \mathbb{R}$.

Non-negativeness of $\frac{\partial^2 x}{\partial t \partial b}(b, t)$

We will show that $\frac{\partial^2 x}{\partial t \partial b}(b, t)$ is also non-negative for all $b \in \mathbb{R}$ and $t \in \mathbb{R}^+$. By the choice of the initial condition it follows from (A.5) that $\frac{\partial^2 x}{\partial t \partial b}(b, 0) \geq \epsilon$, $\forall b \in \mathbb{R}$, for some $\epsilon > 0$, since $\phi(b, t) \leq K \frac{df}{dx}(0)$. Assume that $\frac{\partial^2 x}{\partial t \partial b}(b, t)$ is negative for some $b \in \mathbb{R}$ and some $t > 0$ and define t_1 as

$$t_1 \triangleq \inf\{t : \exists b \in \mathbb{R} : \frac{\partial^2 x}{\partial t \partial b}(b, t) < 0\}.$$

For $t \in [0, t_1]$ the inequality $\frac{\partial^2 x}{\partial t \partial b}(b, t) \geq 0$ holds for all $b \in \mathbb{R}$, and thus $b' - b$ and

$$\frac{\partial}{\partial t}(x(b', t) - x(b, t)) = \int_b^{b'} \frac{\partial^2 x}{\partial t \partial b}(b'', t) db''$$

have the same sign. Since $\frac{\partial x}{\partial b}(b, 0) \geq 0$, $\forall b \in \mathbb{R}$, it follows that for fixed b and b' , $|x(b', t) - x(b, t)|$ is non-decreasing in t in the interval $[0, t_1]$, and thus $\frac{df}{dx}(x(b', t) - x(b, t))$ and therefore also $\phi(b, t)$ are non-increasing in t in the interval $[0, t_1]$.

From (A.5) and (A.6) one can derive that

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\exp\left(\int_0^t \phi(b, t') dt'\right) \frac{\partial^2 x}{\partial t \partial b}(b, t) \right) \\ &= -\frac{\partial \phi}{\partial t}(b, t) \exp\left(\int_0^t \phi(b, t') dt'\right) \frac{\partial x}{\partial b}(b, t), \quad \forall b, t \in \mathbb{R}. \quad (\text{A.7}) \end{aligned}$$

Since $\exp\left(\int_0^t \phi(b, t') dt'\right) \frac{\partial^2 x}{\partial t \partial b}(b, t) \Big|_{t=0} \geq \epsilon$, $\forall b \in \mathbb{R}$, and the right hand side of (A.7) is non-negative for $t \in [0, t_1]$ it follows that

$$\exp\left(\int_0^t \phi(b, t') dt'\right) \frac{\partial^2 x}{\partial t \partial b}(b, t) \Big|_{t=t_1} \geq \epsilon.$$

In the expression

$$\frac{\partial \phi}{\partial t}(b, t) = K \int_{-\infty}^{+\infty} g(b') \frac{d^2 f}{dx^2}(x(b', t) - x(b, t)) \left(\frac{\partial x}{\partial t}(b', t) - \frac{\partial x}{\partial t}(b, t) \right) db'$$

for $t \in \mathbb{R}^+$, the factor $\frac{d^2 f}{dx^2}(x(b', t) - x(b, t))$ has the opposite sign of $x(b', t) - x(b, t)$, which has the same sign as $b' - b$ (since $\frac{\partial x}{\partial t}(b, t) \geq 0, \forall b \in \mathbb{R}, \forall t \in \mathbb{R}^+$), and therefore the integrand is non-negative if and only if $\frac{\partial x}{\partial t}(b', t) - \frac{\partial x}{\partial t}(b, t)$ and $b' - b$ have opposite signs. Since $|\frac{\partial x}{\partial t}(b', t) - \frac{\partial x}{\partial t}(b, t) - (b' - b)| \leq 2KF$ it follows that $|\frac{\partial x}{\partial t}(b', t) - \frac{\partial x}{\partial t}(b, t)| \leq 2KF$ whenever $\frac{\partial x}{\partial t}(b', t) - \frac{\partial x}{\partial t}(b, t)$ and $b' - b$ have different signs. As a consequence $\frac{\partial \phi}{\partial t}(b, t)$ is upper bounded by $2K^2F \sup(\frac{d^2 f}{dx^2})$, and for any $\Delta t > 0$ the right hand side of (A.7) is lower bounded for $t \in [t_1, t_1 + \Delta t]$ (where the bound is independent of b). This implies the existence of a $\Delta t_1 \in (0, \Delta t)$ for which

$$\exp\left(\int_0^t \phi(b, t') dt'\right) \frac{\partial^2 x}{\partial t \partial b}(b, t) \geq 0,$$

for $t \in [t_1, t_1 + \Delta t_1]$, for all $b \in \mathbb{R}$, and thus $\frac{\partial^2 x}{\partial t \partial b}(b, t)$ is also non-negative for $t \in [0, t_1 + \Delta t_1], \forall b \in \mathbb{R}$, contradicting the definition of t_1 . It follows that $\frac{\partial^2 x}{\partial t \partial b}(b, t) \geq 0, \forall t \in \mathbb{R}^+, \forall b \in \mathbb{R}$.

The intervals I_k and J

Since $\frac{\partial^2 x}{\partial t \partial b}(b, t) \geq 0, \forall (b, t) \in \mathbb{R} \times \mathbb{R}^+$, the expression $x(b_2, t) - x(b_1, t)$, for $b_2 > b_1$, is non-decreasing in t . If it is bounded then the same holds for $x(b, t) - x(b_1, t)$ and $x(b_2, t) - x(b, t)$ with $b \in (b_1, b_2)$ arbitrary. It follows that we can find intervals I_k ($k \in N_I \subset \mathbb{Z}$) of non-zero length such that for any $k \in N_I$ and $b_1, b_2 \in I_k$ the value of $x(b_2, t) - x(b_1, t)$ is bounded in t , and (since it is non-decreasing) tends to a constant for $t \rightarrow \infty$. Let the intervals I_k be maximal, i.e. such that they cannot be extended and let N_I also be maximal, i.e. such that, whenever $b_1, b_2 \in \mathbb{R}$ with $b_1 \neq b_2$ and $x(b_2, t) - x(b_1, t)$ bounded in t , $b_1, b_2 \in I_k$ for some $k \in N_I$. Denote by c_k , resp. d_k , the lower, resp. upper, endpoint of the interval I_k . Then for any $\epsilon > 0$ and $b \in I_k$, $x(d_k + \epsilon, t) - x(b, t)$ and $x(b, t) - x(c_k - \epsilon, t)$ are unbounded (and non-decreasing), and it is easy to verify that for $t \rightarrow \infty$ the expression

$$\begin{aligned} \left\langle \frac{\partial x}{\partial t}(b, t) \right\rangle_{I_k} &= \frac{1}{\alpha_{I_k}} \int_{I_k} \frac{\partial x}{\partial t}(b, t) g(b) db \\ &= \frac{K}{\alpha_{I_k}} \int_{c_k}^{d_k} b g(b) db + \frac{1}{\alpha_{I_k}} \int_{c_k}^{d_k} \int_{-\infty}^{+\infty} g(b') f(x(b', t) - x(b, t)) g(b) db \\ &= \langle b \rangle_{I_k} + \frac{K}{\alpha_{I_k}} \int_{c_k}^{d_k} \int_{-\infty}^{c_k} g(b') f(x(b', t) - x(b, t)) g(b) db \end{aligned}$$

$$+ \frac{K}{\alpha_{I_k}} \int_{c_k}^{d_k} \int_{d_k}^{+\infty} g(b') f(x(b'), t) - x(b, t) g(b) db$$

will approach

$$\langle b \rangle_{I_k} - KF\alpha_{(-\infty, c_k)} + KF\alpha_{(d_k, +\infty)},$$

while similarly, for any $b \in J \triangleq \mathbb{R} \setminus \bigcup_{k \in N_I} I_k$,

$$\lim_{t \rightarrow \infty} \frac{\partial x}{\partial t}(b, t) = b - KF\alpha_{(-\infty, b)} + KF\alpha_{(b, +\infty)}.$$

Since

$$\begin{aligned} \frac{\partial x}{\partial t}(b, t) &= b + K \int_b^{\infty} g(b') f(x(b'), t) - x(b, t) db' \\ &\quad + K \int_{-\infty}^b g(b') f(x(b'), t) - x(b, t) db', \quad \forall b, t \in \mathbb{R}, \end{aligned}$$

with the first integral non-decreasing in t and upper bounded, and the second integral non-increasing in t and lower bounded, it follows that $\lim_{t \rightarrow +\infty} \frac{\partial x}{\partial t}(b, t)$ exists for all $b \in \mathbb{R}$.

Defining v by

$$v(b) = \begin{cases} b - KF\alpha_{(-\infty, b)} + KF\alpha_{(b, +\infty)}, & b \in J; \\ \langle b \rangle_{I_k} - KF\alpha_{(-\infty, c_k)} + KF\alpha_{(d_k, +\infty)}, & b \in I_k \text{ for some } k \in N_I, \end{cases} \quad (\text{A.8})$$

we obtain that $\lim_{t \rightarrow \infty} \frac{\partial x}{\partial t}(b, t) = v(b), \forall b \in \mathbb{R}$. For any $p \in (c_k, d_k)$ it also follows that

$$\begin{aligned} &\left\langle \frac{\partial x}{\partial t}(b, t) \right\rangle_{(p, d_k)} - \left\langle \frac{\partial x}{\partial t}(b, t) \right\rangle_{(c_k, p)} \\ &= \langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} \\ &\quad - \left(\frac{K}{\alpha_{(p, d_k)}} + \frac{K}{\alpha_{(c_k, p)}} \right) \int_{c_k}^p db g(b) \int_p^{d_k} db' g(b') f(x(b'), t) - x(b, t) \\ &\geq \langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} - (\alpha_{(c_k, p)} + \alpha_{(p, d_k)}) KF \\ &= \langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} - \alpha_{I_k} KF, \end{aligned}$$

and thus

$$\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} - \alpha_{I_k} KF \leq \inf_{t \in \mathbb{R}} \left(\left\langle \frac{\partial x}{\partial t}(b, t) \right\rangle_{(p, d_k)} - \left\langle \frac{\partial x}{\partial t}(b, t) \right\rangle_{(c_k, p)} \right).$$

The right hand side cannot be negative since $\frac{\partial^2 x}{\partial t \partial b}(b, t) \geq 0, \forall (b, t) \in \mathbb{R} \times \mathbb{R}^+$. It can be verified not to be positive by the fact that $x(b_2, t) - x(b_1, t)$ is bounded for any $b_2 \in (p, d_k), b_1 \in (c_k, p)$, and $\frac{\partial x}{\partial t}(b, t)$ is bounded by $|b| + KF$. It follows that $\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)} - \alpha_{I_k} KF \leq 0$ (i.e. (8.3d)). Taking the limits $p \xrightarrow{\geq} c_k$ and $p \xrightarrow{\leq} d_k$ then results in $\langle b \rangle_{(c_k, d_k)} \leq c_k + \alpha_{I_k} KF$ and $\langle b \rangle_{(c_k, d_k)} \geq d_k - \alpha_{I_k} KF$. From the fact that v must be non-decreasing we obtain that $\lim_{b \xrightarrow{<} c_k} v(b) \leq \lim_{b \xrightarrow{>} c_k} v(b) = \langle v \rangle_{I_k} = \lim_{b \xrightarrow{<} d_k} v(b) \leq \lim_{b \xrightarrow{>} d_k} v(b)$. Combining these results leads to $\langle b \rangle_{(c_k, d_k)} = c_k + \alpha_{I_k} KF$ (i.e. (8.3b)), $\langle b \rangle_{(c_k, d_k)} = d_k - \alpha_{I_k} KF$ (i.e. (8.3c)) and $\lim_{b \xrightarrow{<} c_k} v(b) = \lim_{b \xrightarrow{>} c_k} v(b) = \langle v \rangle_{I_k} = \lim_{b \xrightarrow{<} d_k} v(b) = \lim_{b \xrightarrow{>} d_k} v(b)$, proving that v is continuous. Furthermore, since v is non-decreasing in J , it follows that $\frac{dv}{db}(b) = 1 - 2KFg(b) \geq 0, \forall b \in J$, and the sets I_k and J satisfy the equations (8.3).

A.4.2 Cluster structure with increasing coupling strength

In this section we will provide the mathematical justification of the explanation at the end of section 8.1. We will show that at a local maximum a cluster will arise, and we will investigate the growth of this cluster while the coupling strength K increases.

For this section we will assume that g satisfies the property that for every $b \in \mathbb{R}$ there exists an $\epsilon > 0$ such that in each of the intervals $(b - \epsilon, b)$ and $(b, b + \epsilon)$ g is either non-increasing or non-decreasing.

A solution satisfying (8.3a), (8.3b) and (8.3c)

We will investigate the second equation in (8.4), since it characterizes the clusters I_k . After solving this equation one obtains a family of intervals for which the first equation can be used to calculate the corresponding values of the coupling strength.

For $m \in \mathbb{R}$ and $h \in \mathbb{R}^+$, define \mathcal{F} by

$$\mathcal{F}(m, h) \triangleq \langle b \rangle_{(m-h, m+h)} - m.$$

The equation $\mathcal{F}(m, h) = 0$ relates the midpoint m of an interval I_k to its half length h . If $g(m) > 0$ then the function \mathcal{F} is continuously differentiable for $h > 0$ and along a solution of $\mathcal{F}(m, h) = 0$ we obtain

$$\frac{\partial \mathcal{F}}{\partial m} = \frac{\partial}{\partial m} \left(\frac{\int_{m-h}^{m+h} bg(b)db}{\int_{m-h}^{m+h} g(b)db} - m \right)$$

$$= \frac{1}{\left(\int_{m-h}^{m+h} g(b)db\right)^2} \left(((m+h)g(m+h) - (m-h)g(m-h)) \int_{m-h}^{m+h} g(b)db \right. \\ \left. - (g(m+h) - g(m-h)) \int_{m-h}^{m+h} bg(b)db \right) - 1$$

$$(\mathcal{F}(m, h) = 0 \Leftrightarrow \int_{m-h}^{m+h} bg(b)db = m \int_{m-h}^{m+h} g(b)db)$$

$$= \frac{h}{\int_{m-h}^{m+h} g(b)db} (g(m+h) + g(m-h)) - 1,$$

and similarly

$$\frac{\partial \mathcal{F}}{\partial h} = \frac{h}{\int_{m-h}^{m+h} g(b)db} (g(m+h) - g(m-h)),$$

and thus

$$\frac{dm}{dh} = -\frac{\frac{\partial \mathcal{F}}{\partial h}}{\frac{\partial \mathcal{F}}{\partial m}} \\ = -\frac{g(m+h) - g(m-h)}{g(m+h) + g(m-h) - \frac{1}{h} \int_{m-h}^{m+h} g(b)db}.$$

At the origination of a cluster (with increasing K), when $h = 0$, the derivative $\frac{\partial \mathcal{F}}{\partial m}$ equals zero, preventing the application of the implicit function theorem. Instead we will look for a point where $\frac{\partial \mathcal{F}}{\partial m} \neq 0$. First we consider a slightly more general setting, allowing the starting point (of the curve we are looking for, described by $\mathcal{F}(m, h) = 0$) to be either a local maximum, a plateau — i.e. an interval where g is constant — or a cluster that had already been formed and satisfies a particular condition.

Starting point for the curve satisfying $\mathcal{F}(m, h) = 0$. We consider an $m_0 \in \mathbb{R}$ and an $h_0 \in \mathbb{R}^+$ satisfying $\mathcal{F}(m_0, h_0) = 0$ and

$$\frac{1}{2h_0} \int_{m_0-h_0}^{m_0+h_0} g(b)db = g(m_0 - h_0) = g(m_0 + h_0), \quad (\text{A.9})$$

for which g is increasing in $(m_0 - h_0 - \epsilon_0, m_0 - h_0)$ and decreasing in $(m_0 + h_0, m_0 + h_0 + \epsilon_0)$ for some $\epsilon_0 > 0$.

Then, with $0 < \epsilon_1 < \frac{\epsilon_0}{2}$,

$$\begin{aligned} \mathcal{F}(m_0 + \epsilon_1, h_0 + \epsilon_1) &= \int_{m_0 - h_0}^{m_0 + h_0 + 2\epsilon_1} g(b) db \\ &= \int_{m_0 - h_0}^{m_0 + h_0 + 2\epsilon_1} (b - m_0) g(b) db - \epsilon_1 \int_{m_0 - h_0}^{m_0 + h_0 + 2\epsilon_1} g(b) db \end{aligned}$$

$$(\langle b \rangle_{(m_0 - h_0, m_0 + h_0)}) = m_0 \Leftrightarrow \int_{m_0 - h_0}^{m_0 + h_0} (b - m_0) g(b) db = 0$$

$$\begin{aligned} &= \int_{m_0 + h_0}^{m_0 + h_0 + 2\epsilon_1} (b - m_0) g(b) db - \epsilon_1 \int_{m_0 - h_0}^{m_0 + h_0 + 2\epsilon_1} g(b) db \\ &= \int_0^{2\epsilon_1} (h_0 + u) g(m_0 + h_0 + u) du - \epsilon_1 \int_{m_0 - h_0}^{m_0 + h_0} g(b) db \\ &\quad - \epsilon_1 \int_0^{2\epsilon_1} g(m_0 + h_0 + u) du \\ &= h_0 \int_0^{2\epsilon_1} g(m_0 + h_0 + u) du - \epsilon_1 \int_{m_0 - h_0}^{m_0 + h_0} g(b) db \\ &\quad + \int_0^{2\epsilon_1} (u - \epsilon_1) g(m_0 + h_0 + u) du \\ &= h_0 \int_0^{2\epsilon_1} g(m_0 + h_0 + u) du - \epsilon_1 \int_{m_0 - h_0}^{m_0 + h_0} g(b) db \\ &\quad + \int_0^{\epsilon_1} u' (g(m_0 + h_0 + \epsilon_1 + u') - g(m_0 + h_0 + \epsilon_1 - u')) du' \end{aligned}$$

(g is decreasing in $(m_0 + h_0, m_0 + h_0 + 2\epsilon_1)$)

$$< 2\epsilon_1 h_0 g(m_0 + h_0) - \epsilon_1 \int_{m_0 - h_0}^{m_0 + h_0} g(b) db = 0, \quad \text{by (A.9).}$$

Analogously, $\mathcal{F}(m_0 - \epsilon_1, h_0 + \epsilon_1) > 0$, and thus there exists an $m' \in (m_0 - \epsilon_1, m_0 + \epsilon_1)$ satisfying $\mathcal{F}(m', h_0 + \epsilon_1) = 0$. We will show that

$$g(m' - h_0 - \epsilon_1) < \frac{1}{2(h_0 + \epsilon_1)} \int_{m' - h_0 - \epsilon_1}^{m' + h_0 + \epsilon_1} g(b) db, \quad (\text{A.10a})$$

$$g(m' + h_0 + \epsilon_1) < \frac{1}{2(h_0 + \epsilon_1)} \int_{m' - h_0 - \epsilon_1}^{m' + h_0 + \epsilon_1} g(b) db. \quad (\text{A.10b})$$

At least one of the inequalities holds, since $\frac{1}{2(h_0 + \epsilon_1)} \int_{m' - h_0 - \epsilon_1}^{m' + h_0 + \epsilon_1} g(b) db$

is a convex combination of $\frac{1}{2h_0} \int_{m_0-h_0}^{m_0+h_0} g(b)db$, $\frac{1}{2(m_0+\epsilon_1-m')} \int_{m'-h_0-\epsilon_1}^{m_0-h_0} g(b)db$ and $\frac{1}{2(m'+\epsilon_1-m_0)} \int_{m_0+h_0}^{m'+h_0+\epsilon_1} g(b)db$ and is thus smaller than $\frac{1}{2h_0} \int_{m_0-h_0}^{m_0+h_0} g(b)db$, which could not be true if both of the above inequalities were not satisfied. We will show that both inequalities hold. Assuming that

$$g(m' - h_0 - \epsilon_1) \geq \frac{1}{2(h_0 + \epsilon_1)} \int_{m'-h_0-\epsilon_1}^{m'+h_0+\epsilon_1} g(b)db,$$

we will find a contradiction. (The other case is analogous.) Introducing the following shorthand notations

$$\begin{aligned} a_1 &= m' - h_0 - \epsilon_1, & a_2 &= m_0 - h_0, \\ a_3 &= m_0 + h_0, & a_4 &= m' + h_0 + \epsilon_1, \\ \bar{g} &= \frac{1}{a_4 - a_1} \int_{a_1}^{a_4} g(b)db, \end{aligned}$$

we can derive

$$\begin{aligned} m' &= \langle b \rangle_{(a_1, a_4)} \\ &= \frac{\int_{a_1}^{a_4} b(g(b) - \bar{g} + \bar{g}) db}{\alpha_{(a_1, a_4)}} \end{aligned}$$

$$\left(\frac{a_1 + a_4}{2} = m' \right)$$

$$\begin{aligned} &= \frac{1}{\alpha_{(a_1, a_4)}} \left(\int_{a_1}^{a_4} b(g(b) - \bar{g}) db + \bar{g}m'(a_4 - a_1) \right) \\ &= \frac{1}{\alpha_{(a_1, a_4)}} \int_{a_1}^{a_4} (b - b_0)(g(b) - \bar{g}) db + m' \end{aligned}$$

where we let $b_0 \in (a_3, a_4)$ be the unique value satisfying $g(b_0) = \bar{g}$ (b_0 is unique since g is decreasing in this interval)

$$\begin{aligned} &= \frac{1}{\alpha_{(a_1, a_4)}} \left(\int_{a_1}^{a_2} (b - b_0)(g(b) - \bar{g}) db + \int_{a_2}^{a_3} (b - m_0)(g(b) - \bar{g}) db \right. \\ &\quad \left. + \int_{a_2}^{a_3} (m_0 - b_0)(g(b) - \bar{g}) db + \int_{a_3}^{a_4} (b - b_0)(g(b) - \bar{g}) db \right) + m' \end{aligned}$$

(considering that $g(b) \geq \bar{g}$, $\forall b \in (a_1, a_3)$, the first integral is negative, the second integral is zero, the third integral is non-positive and the fourth integral is negative)

$$< m'.$$

Because of this contradiction, it follows that (A.10) holds.

Continuation of the curve satisfying $\mathcal{F}(m, h) = 0$. From (A.10) we obtain that $\frac{\partial \mathcal{F}}{\partial m}(m', h_0 + \epsilon_1) < 0$. This implies that m' is the unique solution in the interval $(m_0 - \epsilon_1, m_0 + \epsilon_1)$ of the equation $\mathcal{F}(m, h_0 + \epsilon_1) = 0$ in m . (If there were more than one solution, $\frac{\partial \mathcal{F}}{\partial m}$ could not be negative in all of them.) It also guarantees that we can apply the implicit function theorem, leading to a function $\tilde{m} : (h_i, h_f) \rightarrow \mathbb{R}$ for some $h_i, h_f \in \mathbb{R}$, satisfying $h_i < h_0 + \epsilon_1 < h_f$ with $\tilde{m}(h_0 + \epsilon_1) = m'$ and $\mathcal{F}(\tilde{m}(h), h) = 0$, $\forall h \in (h_i, h_f)$. We can repeat this for all ϵ_1 in the interval $(0, \frac{\epsilon_0}{2})$, and, together with the uniqueness result for m' , we can conclude that all obtained functions \tilde{m} can be seen as equal to (or restrictions of) one function \tilde{m} defined on (h_i, h_f) where we can take $h_i = h_0$ and $h_f \geq h_0 + \frac{\epsilon_0}{2}$, such that for all $h \in (h_i, h_f)$ the inequalities (A.10) hold:

$$g(\tilde{m}(h) - h) < \frac{1}{2h} \int_{\tilde{m}(h)-h}^{\tilde{m}(h)+h} g(b) db, \quad (\text{A.11a})$$

$$g(\tilde{m}(h) + h) < \frac{1}{2h} \int_{\tilde{m}(h)-h}^{\tilde{m}(h)+h} g(b) db. \quad (\text{A.11b})$$

From (A.11) one derives that $|\frac{dm}{dh}(\tilde{m}(h), h)| < 1$, $\forall h \in (h_i, h_f)$, which in turn implies that, as h increases, $\tilde{m}(h) - h$ will decrease and $\tilde{m}(h) + h$ will increase. As a result the coupling strength K (calculated by the first equation from (8.4)) will increase with increasing h .

We already know that when K is increased, a cluster arises at a local maximum (or possibly at a plateau). Since this will happen at all local maximums (at the appropriate values for K) we are guaranteed that outside the clusters (the intervals I_k) the value of g will be smaller than (or equal to in some special cases) $1/(2KF)$ (as follows from the first equation of (8.4)), implying equation (8.3a). The inequalities (A.11) may hold in a larger interval for h when replacing the strict inequalities by non-strict inequalities. (We will refer to those as (A.11').) If, for a cluster corresponding to the interval $[\tilde{m}(h) - h, \tilde{m}(h) + h]$, the value of $g(b)$ would become larger than $1/(2KF)$ for $b = \tilde{m}(h) + h$ or $b = \tilde{m}(h) - h$ when h is increased, then this value of b must belong to another interval I_k for which the corresponding cluster has already 'collided' with the former cluster. At the moment of their 'collision' they will have merged into a new cluster, which will inherit (8.4) (cfr. infra) and the property (A.11') from the constituting clusters. This allows us to apply the implicit function theorem

again, unless both inequalities in (A.11') are equalities, and then the conditions (A.9) are satisfied and we can apply the entire reasoning of this section again. The conditions concerning the increase and decrease of g at the edges of the interval will also be satisfied if there are no neighboring clusters to merge with. (In case (A.9) holds and g is constant in $(m_0 - h_0 - \epsilon_0, m_0 - h_0)$ or $(m_0 + h_0, m_0 + h_0 + \epsilon_0)$ for some $\epsilon_0 > 0$, then the cluster corresponding to $(m_0 - h_0, m_0 + h_0)$ can be extended to include this interval. For the associated value of K , there will be a discontinuity in the cluster growth; the/an endpoint(s) of the cluster interval will change discontinuously when K is increased. In case both endpoints change discontinuously with increasing K , there is no unique \tilde{m} describing the transition in the corresponding interval for h .)

Of course we still have to check whether the domain of the function \tilde{m} can always be extended far enough to be able to merge with the other clusters or to go on for $K \rightarrow \infty$ if there are no other clusters.

Consider a function \tilde{m} associated to a cluster and let (h_i, h_f) now be the maximal interval for which \tilde{m} is well-defined. If the inequalities (A.11') are not satisfied anymore for some h in (h_i, h_f) , then the cluster has already been able to merge with another cluster. As long as (A.11') is satisfied we know that $|\frac{dm}{dh}| \leq 1$. From this property it follows that $m_f \triangleq \lim_{h \rightarrow h_f} \tilde{m}(h)$ exists and by the continuity of \mathcal{F} it satisfies $\mathcal{F}(m_f, h_f) = 0$. Since (h_i, h_f) is maximal it follows that $\frac{\partial \mathcal{F}}{\partial m}(m_f, h_f) = 0$. Together with (A.11') the conditions (A.9) follow and if there are no neighboring clusters to merge with we can apply the results from this section again.

In other words, we have proven that for every $K > 0$ there will be a division of \mathbb{R} in intervals I_k and J satisfying the first three conditions of (8.3). If g is never constant in intervals of non-zero length, then the endpoints of the intervals I_k are continuous functions of K (from the origination of a cluster until it merges with another cluster). We will now prove that the intervals I_k also satisfy (8.3d).

The condition (8.3d)

Given a function \tilde{m} associated to a cluster (i.e. with $\mathcal{F}(\tilde{m}(h), h) = 0$ in $[h_i, h_f]$), define the functions \tilde{c} , \tilde{d} and \tilde{K} as follows:

$$\begin{aligned}\tilde{c}(h) &\triangleq \tilde{m}(h) - h, \\ \tilde{d}(h) &\triangleq \tilde{m}(h) + h, \\ \tilde{K}(h) &\triangleq \frac{\tilde{d}(h) - \tilde{c}(h)}{2\alpha_{(\tilde{c}(h), \tilde{d}(h))} F},\end{aligned}$$

$\forall h \in [h_i, h_f]$, in which we restrict h_f in such a way that (A.11') is satisfied for all $h \in [h_i, h_f]$. For simplicity we assume that \tilde{m} is differentiable in $[h_i, h_f]$. (If \tilde{m} is differentiable in $[h_i, h_f]$ except in isolated points, then the following reasoning still holds. If \tilde{m} is not differentiable in an interval (h_1, h_2) then $g(\tilde{m}(h) + h)$ and $g(\tilde{m}(h) - h)$ are constant for all $h \in (h_1, h_2)$, and one can set $\tilde{m}(h) \triangleq \frac{h-h_1}{h_2-h_1}\tilde{m}(h_2) + \frac{h_2-h}{h_2-h_1}\tilde{m}(h_1)$, $\forall h \in (h_1, h_2)$.) This guarantees that $|\frac{d\tilde{m}}{dh}(h)| \leq 1$ and thus $\frac{d\tilde{c}}{dh}(h) \leq 0$, $\frac{d\tilde{d}}{dh}(h) \geq 0$ and (as can be verified) $\frac{d\tilde{K}}{dh}(h) \geq 0$, $\forall h \in [h_i, h_f]$. Pick a $p \in [\tilde{c}(h_1), \tilde{d}(h_1)]$, for some $h_1 \in [h_i, h_f]$. Now define the function C by

$$C(h) \triangleq \langle b \rangle_{(p, \tilde{d}(h))} - \langle b \rangle_{(\tilde{c}(h), p)} - \alpha_{(\tilde{c}(h), \tilde{d}(h))} \tilde{K}(h)F,$$

$\forall h \in [h_1, h_f]$. Then

$$\begin{aligned} \frac{dC}{dh}(h) &= \frac{\tilde{d}(h)g(\tilde{d}(h))\frac{d\tilde{d}}{dh}(h)}{\alpha_{(p, \tilde{d}(h))}} - \frac{\langle b \rangle_{(p, \tilde{d}(h))}g(\tilde{d}(h))\frac{d\tilde{d}}{dh}(h)}{\alpha_{(p, \tilde{d}(h))}} + \frac{\tilde{c}(h)g(\tilde{c}(h))\frac{d\tilde{c}}{dh}(h)}{\alpha_{(\tilde{c}(h), p)}} \\ &\quad - \frac{\langle b \rangle_{(\tilde{c}(h), p)}g(\tilde{c}(h))\frac{d\tilde{c}}{dh}(h)}{\alpha_{(\tilde{c}(h), p)}} \\ &\quad - \left(g(\tilde{d}(h))\frac{d\tilde{d}}{dh}(h) - g(\tilde{c}(h))\frac{d\tilde{c}}{dh}(h) \right) \tilde{K}(h)F - \alpha_{(\tilde{c}(h), \tilde{d}(h))} \frac{d\tilde{K}}{dh}(h)F \\ &\leq \frac{g(\tilde{d}(h))\frac{d\tilde{d}}{dh}(h)}{\alpha_{(p, \tilde{d}(h))}} \left(\tilde{d}(h) - \langle b \rangle_{(p, \tilde{d}(h))} - \alpha_{(p, \tilde{d}(h))} \tilde{K}(h)F \right) \\ &\quad - \frac{g(\tilde{c}(h))\frac{d\tilde{c}}{dh}(h)}{\alpha_{(\tilde{c}(h), p)}} \left(-\tilde{c}(h) + \langle b \rangle_{(\tilde{c}(h), p)} - \alpha_{(\tilde{c}(h), p)} \tilde{K}(h)F \right) \end{aligned}$$

(from $\mathcal{F}(\tilde{m}(h), h) = 0$ and the definition of \tilde{K} follow equivalent relations as in (8.3b) and (8.3c))

$$\begin{aligned} &= \frac{g(\tilde{d}(h))\frac{d\tilde{d}}{dh}(h)}{\alpha_{(p, \tilde{d}(h))}} \left(\langle b \rangle_{(\tilde{c}(h), \tilde{d}(h))} + \alpha_{(\tilde{c}(h), \tilde{d}(h))} \tilde{K}(h)F - \langle b \rangle_{(p, \tilde{d}(h))} \right. \\ &\quad \left. - \alpha_{(p, \tilde{d}(h))} \tilde{K}(h)F \right) - \frac{g(\tilde{c}(h))\frac{d\tilde{c}}{dh}(h)}{\alpha_{(\tilde{c}(h), p)}} \left(\alpha_{(\tilde{c}(h), \tilde{d}(h))} \tilde{K}(h)F \right. \\ &\quad \left. - \langle b \rangle_{(\tilde{c}(h), \tilde{d}(h))} + \langle b \rangle_{(\tilde{c}(h), p)} - \alpha_{(\tilde{c}(h), p)} \tilde{K}(h)F \right). \end{aligned}$$

Using the fact that $\alpha_{(\tilde{c}(h), p)} + \alpha_{(p, \tilde{d}(h))} = \alpha_{(\tilde{c}(h), \tilde{d}(h))}$ and

$$\langle b \rangle_{(\tilde{c}(h), \tilde{d}(h))} = \frac{\alpha_{(\tilde{c}(h), p)} \langle b \rangle_{(\tilde{c}(h), p)} + \alpha_{(p, \tilde{d}(h))} \langle b \rangle_{(p, \tilde{d}(h))}}{\alpha_{(\tilde{c}(h), \tilde{d}(h))}}$$

we can rewrite this in terms of $C(h)$:

$$\frac{dC}{dh}(h) \leq - \left(\frac{g(\tilde{d}(h)) \frac{d\tilde{d}}{dh}(h)}{\alpha_{(p,\tilde{d}(h))}} \frac{\alpha_{(\tilde{c}(h),p)}}{\alpha_{(\tilde{c}(h),\tilde{d}(h))}} - \frac{g(\tilde{c}(h)) \frac{d\tilde{c}}{dh}(h)}{\alpha_{(\tilde{c}(h),p)}} \frac{\alpha_{(p,\tilde{d}(h))}}{\alpha_{(\tilde{c}(h),\tilde{d}(h))}} \right) C(h).$$

The factor in brackets is nonnegative and thus $\frac{dC}{dh}(h) \leq 0$ whenever $C(h) \geq 0$. It follows that if $C(h_1) \leq 0$, then $C(h) \leq 0$, $\forall h \in [h_1, h_f]$.

As already mentioned at the end of section 8.1 the first occurrence of (8.3d) for a given p when increasing K is automatically satisfied since then either $p = c_k$ or $p = d_k$ for some k and thus (8.3d) follows from either (8.3b) or (8.3c). As we have just shown (8.3d) will remain satisfied when the clusters continue to grow with further increases of K .

Two clusters merging

In this section we will verify that two clusters, represented by the intervals I_k and I_{k+1} , with $d_k - c_{k+1} \rightarrow 0$ (as a result of a limiting process $K \xrightarrow{\leq} K_1$ for some K_1) and satisfying the equations (8.3b), (8.3c) and (8.3d), will result in a new cluster, represented by (c_k, d_{k+1}) (except for a possible inclusion of the/an endpoint(s)), satisfying the same conditions. Since this merging process leaves J unchanged, the inequalities (8.3a) remain satisfied. For equations (8.3b) and (8.3c) we will consider the equivalent form (8.4).

It follows that, with $c_{k+1} = d_k$,

$$\begin{aligned} \frac{\int_{c_k}^{d_{k+1}} g(b') db'}{d_{k+1} - c_k} &= \frac{1}{d_{k+1} - c_k} \left(\int_{c_k}^{d_k} g(b') db' + \int_{c_{k+1}}^{d_{k+1}} g(b') db' \right) \\ &= \frac{1}{d_{k+1} - c_k} \left(\frac{d_k - c_k}{2KF} + \frac{d_{k+1} - c_{k+1}}{2KF} \right) \\ &= \frac{1}{2KF}, \end{aligned}$$

and

$$\begin{aligned} \langle b \rangle_{(c_k, d_{k+1})} &= \frac{1}{\alpha_{(c_k, d_{k+1})}} \left(\alpha_{(c_k, d_k)} \langle b \rangle_{(c_k, d_k)} + \alpha_{(c_{k+1}, d_{k+1})} \langle b \rangle_{(c_{k+1}, d_{k+1})} \right) \\ &= \frac{1}{\frac{d_{k+1} - c_k}{2KF}} \left(\frac{(d_k - c_k)}{2KF} \frac{(c_k + d_k)}{2} + \frac{(d_{k+1} - c_{k+1})}{2KF} \frac{(c_{k+1} + d_{k+1})}{2} \right) \\ &= \frac{d_{k+1} + c_k}{2}. \end{aligned}$$

Concerning (8.3d) we will assume that $p \in I_k$. The case $p \in I_{k+1}$ is analogous.

$$\begin{aligned} & \langle b \rangle_{(p, d_{k+1})} - \langle b \rangle_{(c_k, p)} \\ &= \frac{\alpha_{(p, d_k)}}{\alpha_{(p, d_{k+1})}} (\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)}) + \frac{\alpha_{(c_{k+1}, d_{k+1})}}{\alpha_{(p, d_{k+1})}} (\langle b \rangle_{(c_{k+1}, d_{k+1})} - \langle b \rangle_{(c_k, p)}), \end{aligned}$$

and with

$$\begin{aligned} & \langle b \rangle_{(c_{k+1}, d_{k+1})} - \langle b \rangle_{(c_k, p)} \\ &= \langle b \rangle_{(c_k, d_k)} + \frac{d_{k+1} - c_k}{2} - \langle b \rangle_{(c_k, p)} \\ &= \frac{\alpha_{(c_k, p)} \langle b \rangle_{(c_k, p)} + \alpha_{(p, d_k)} \langle b \rangle_{(p, d_k)}}{\alpha_{(c_k, d_k)}} + \alpha_{(c_k, d_{k+1})} KF - \langle b \rangle_{(c_k, p)} \\ &= \frac{\alpha_{(p, d_k)}}{\alpha_{(c_k, d_k)}} (\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)}) + \alpha_{(c_k, d_{k+1})} KF, \end{aligned}$$

it follows that

$$\begin{aligned} & \langle b \rangle_{(p, d_{k+1})} - \langle b \rangle_{(c_k, p)} \\ &= \frac{\alpha_{(p, d_k)}}{\alpha_{(p, d_{k+1})}} \frac{\alpha_{(c_k, d_{k+1})}}{\alpha_{(c_k, d_k)}} (\langle b \rangle_{(p, d_k)} - \langle b \rangle_{(c_k, p)}) + \frac{\alpha_{(c_k, d_{k+1})}}{\alpha_{(p, d_{k+1})}} \alpha_{(c_{k+1}, d_{k+1})} KF \\ &\leq \alpha_{(c_k, d_{k+1})} KF. \end{aligned}$$

A.5 Proofs for chapter 10

A.5.1 Proof of proposition 10.1

Since the function $U_{ij}'^{-1}$ is increasing with $\lim_{\sigma \rightarrow \pm\infty} U_{ij}'^{-1}(\sigma) = \pm\infty$, the function mapping $\lambda_j - \lambda_i \in \mathbb{R}$ to $\int_0^{\lambda_j - \lambda_i} U_i'^{-1}(\sigma) d\sigma - B_0 |\lambda_j - \lambda_i|$, with $B_0 > 0$ arbitrary, grows unbounded for $\lambda_j - \lambda_i \rightarrow \pm\infty$, for each $i, j \in \mathcal{I}_N$, with $i \neq j$. It follows that for any $B > 0$, there exists a $D > 0$ for which

$$\frac{1}{2} \sum_{(i, j) \in \mathcal{I}_N} \int_0^{\tilde{\lambda}_j(x) - \tilde{\lambda}_i(x)} U_{ij}'^{-1}(\sigma) d\sigma \geq B \Delta \tilde{\lambda}(x), \quad \forall x \in \mathbb{R}^N, \text{ with } \Delta \tilde{\lambda}(x) \geq D.$$

For

$$B = \sum_{i=1}^N |b_i| + 1,$$

there exists a $D > 0$, such that, $\forall x \in \mathbb{R}^N$, with $\Delta\tilde{\lambda}(x) \geq D$ (taking into account that $\sum_{i=1}^N b_i = 0$),

$$\begin{aligned} V(\tilde{\lambda}(x)) &= -\sum_{i=1}^N b_i \left(\tilde{\lambda}_i(x) - \tilde{\lambda}_1(x) \right) + \frac{1}{2} \sum_{(i,j) \in \mathcal{I}_N} \int_0^{\tilde{\lambda}_j(x) - \tilde{\lambda}_i(x)} U'_{ij}(\sigma) d\sigma \\ &\geq -\sum_{i=1}^N |b_i| \Delta\tilde{\lambda}(x) + \left(\sum_{i=1}^N |b_i| + 1 \right) \Delta\tilde{\lambda}(x) \\ &= \Delta\tilde{\lambda}(x). \end{aligned}$$

For each $C \in \mathbb{R}$, $\|x\| \rightarrow \infty$, with $x \in L_C$, implies $\Delta\tilde{\lambda}(x) \rightarrow \infty$, and thus also $V(\tilde{\lambda}(x)) \rightarrow \infty$; it follows that $V \circ \tilde{\lambda}$ is radially unbounded in L_C .

A.5.2 Proof of the equivalence of the conditions (C3) and (D)

Under the assumption that $\frac{d\bar{\lambda}_i}{dx_i}(x_i) > 0$, $\forall x_i \in \mathbb{R}$, $\forall i \in \mathcal{I}_N$, we will show that the function $\Delta\tilde{\lambda}$ is radially unbounded in L_C , for all $C \in \mathbb{R}$, if and only if one of the three conditions (D) is satisfied.

- First assume that, for each $C \in \mathbb{R}$, $\Delta\tilde{\lambda}$ is radially unbounded in L_C . Pick $i_1, i_2 \in \mathcal{I}_N$ arbitrary, with $i_1 \neq i_2$, set $x_i = 0$, $\forall i \in \mathcal{I}_N \setminus \{i_1, i_2\}$, and set $x_{i_1} = -x_{i_2}$, such that $x \in L_0$. Consider the limit $x_{i_1} = -x_{i_2} \rightarrow +\infty$. Then $\Delta\tilde{\lambda}(x) \rightarrow \bar{\lambda}_{i_1}(x_{i_1}) - \bar{\lambda}_{i_2}(x_{i_2}) \rightarrow +\infty$, implying that

$$\lim_{x_{i_1} \rightarrow +\infty} \bar{\lambda}_{i_1}(x_{i_1}) = +\infty \quad \text{and/or} \quad \lim_{x_{i_2} \rightarrow -\infty} \bar{\lambda}_{i_2}(x_{i_2}) = -\infty. \quad (\text{A.12})$$

If the condition (D1) does not hold, then there exists an $i_B \in \mathcal{I}_N$ for which $\bar{\lambda}_{i_B}(x_{i_B})$ remains bounded as $x_{i_B} \rightarrow +\infty$, and thus, setting $i_1 = i_B$ and $i_2 \in \mathcal{I}_N \setminus \{i_B\}$ in (A.12), it follows that

$$\lim_{x_i \rightarrow -\infty} \bar{\lambda}_i(x_i) = -\infty, \quad \forall i \in \mathcal{I}_N \setminus \{i_B\}. \quad (\text{A.13})$$

If neither conditions (D1) and (D2) hold, then it follows from (A.13) that $\bar{\lambda}_{i_B}(x_{i_B})$ also remains bounded as $x_{i_B} \rightarrow -\infty$, implying that

$$\lim_{x_i \rightarrow +\infty} \bar{\lambda}_i(x_i) = +\infty, \quad \forall i \in \mathcal{I}_N \setminus \{i_B\}, \quad (\text{A.14})$$

and (A.13) and (A.14) then imply (D3).

- Assume that one of the three conditions (\mathcal{D}) holds. For each $i, j \in \mathcal{I}_N$, with $i \neq j$, it follows that if $x_i \rightarrow +\infty$ and $x_j \rightarrow -\infty$, or $x_i \rightarrow -\infty$ and $x_j \rightarrow +\infty$, then

$$\Delta\tilde{\lambda}(x) \geq |\bar{\lambda}_i(x_i) - \bar{\lambda}_j(x_j)| \rightarrow +\infty.$$

Therefore, for each $Q > 0$, there exists a $D_1 > 0$ such that, if $x_k \geq D_1$ and $x_l \leq -D_1$ for some $k, l \in \mathcal{I}_N$, with $k \neq l$, then $\Delta\tilde{\lambda}(x) \geq Q$. On the other hand, for any $C \in \mathbb{R}$ there exists a $D_0 > 0$ such that, if $\|x\| \geq D_0$ and $x \in L_C$, there are $k, l \in \mathcal{I}_N$, with $k \neq l$, satisfying $x_k \geq D_1$ and $x_l \leq -D_1$, and therefore $\Delta\tilde{\lambda}(x) \geq Q$. Consequently, for each $C \in \mathbb{R}$, $\Delta\tilde{\lambda}$ is radially unbounded in L_C .

A.5.3 Lemma A.2

Lemma A.2. *If the continuously differentiable functions $\tilde{\lambda}_i : \mathbb{R}^N \rightarrow \mathbb{R}$, with $N > 2$, are such that for any $i, j, k \in \mathcal{I}_N$ with $i \neq j$, $i \neq k$, and $j \neq k$, $\tilde{\lambda}_j(x) - \tilde{\lambda}_i(x)$ is independent of x_k , for all $x \in \mathbb{R}^N$, then there exist functions $\bar{\lambda}_i : \mathbb{R} \rightarrow \mathbb{R}$, with the property that*

$$\bar{\lambda}_j(x_j) - \bar{\lambda}_i(x_i) = \tilde{\lambda}_j(x) - \tilde{\lambda}_i(x), \quad \forall x \in \mathbb{R}^N, \quad \forall i, j \in \mathcal{I}_N \text{ with } i \neq j.$$

Proof. Pick an $x^o \in \mathbb{R}^N$, and set

$$\bar{\lambda}_1(x_1) \triangleq \left(\tilde{\lambda}_1(x) - \tilde{\lambda}_2(x) \right) \Big|_{x_2=x_2^o}.$$

Since

$$\begin{aligned} \tilde{\lambda}_2(x) - \tilde{\lambda}_1(x) + \bar{\lambda}_1(x_1) &= \tilde{\lambda}_2(x) - \tilde{\lambda}_1(x) - \left(\tilde{\lambda}_2(x) - \tilde{\lambda}_1(x) \right) \Big|_{x_2=x_2^o} \\ &= \tilde{\lambda}_2(x) - \tilde{\lambda}_3(x) - \left(\tilde{\lambda}_2(x) - \tilde{\lambda}_3(x) \right) \Big|_{x_2=x_2^o} \\ &\quad - \tilde{\lambda}_1(x) + \tilde{\lambda}_3(x) + \left(\tilde{\lambda}_1(x) - \tilde{\lambda}_3(x) \right) \Big|_{x_2=x_2^o} \\ &= \tilde{\lambda}_2(x) - \tilde{\lambda}_3(x) - \left(\tilde{\lambda}_2(x) - \tilde{\lambda}_3(x) \right) \Big|_{x_2=x_2^o}, \end{aligned}$$

with $\tilde{\lambda}_2(x) - \tilde{\lambda}_3(x)$ independent of x_1 , we can set

$$\bar{\lambda}_2(x_2) \triangleq \tilde{\lambda}_2(x) - \tilde{\lambda}_1(x) + \bar{\lambda}_1(x_1).$$

For $j > 2$, set

$$\bar{\lambda}_j(x_j) \triangleq \tilde{\lambda}_j(x) - \tilde{\lambda}_1(x) + \bar{\lambda}_1(x_1)$$

$$= \tilde{\lambda}_j(x) - \tilde{\lambda}_2(x) + \bar{\lambda}_2(x_2).$$

It follows that this expression is independent of x_2 and x_1 and therefore a function of x_j only. The equalities

$$\bar{\lambda}_j(x_j) - \bar{\lambda}_i(x_i) = \tilde{\lambda}_j(x) - \tilde{\lambda}_i(x), \quad \forall x \in \mathbb{R}^N, \quad \forall i, j \in \mathcal{I}_N \text{ with } i \neq j,$$

immediately follow. □

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