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Supersymmetric Schur functions and Lie superalgebra representations

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Els Moens,
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Preface

> *"Fermionen und Bosonen sind fundamental verschiedene Typen von Quantenteilchen. So wenig ein gewöhnlicher Spiegel einen Apfel wie eine Birne aussehen läst, so wenig vermag eine der üblichen Symmetrien Fermionen in Bosonen zu verwandeln. Das schafft erst der Zauberspiegel der Supersymmetrie."* *(Prof. Dr. J. Jolie [30])*

Lie superalgebras and their representations continue to play an important role in the understanding and exploitation of supersymmetry in physical systems. The Lie superalgebras under consideration here, namely $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$ (sometimes denoted by $U(m|n)$ or $SU(m|n)$), have applications in quantum mechanics [1, 42], nuclear physics [6, 14, 27], string theory [19, 23], conformal field theory [21], supergravity [2, 35], M-theory [20], lattice QCD [7, 9, 15], solvable lattice models [62], spin systems [24] and quantum systems [58]. Also their affine extensions [21, 24] or q-deformations [1, 58] play an important role. In most of the applications, it are the irreducible representations or "multiplets" of $\mathfrak{gl}(m|n)$ (sometimes referred to as simple $\mathfrak{gl}(m|n)$ modules) that play a role. But, despite the fact that Lie superalgebras and their representations have been the subject of much attention, there is no complete description of the finite-dimensional complex irreducible representations of even the simplest family of basic classical Lie superalgebras, $\mathfrak{sl}(m|n)$.

Representation theory of Lie superalgebras, and in particular of $\mathfrak{gl}(m|n)$ or its simple counterpart $\mathfrak{sl}(m|n)$, is not a straightforward copy of the corresponding theory for simple Lie algebras. The development of $\mathfrak{gl}(m|n)$ representation theory is quite remarkable. Shortly after the classification of finite-dimensional simple Lie superalgebras [31, 59], Kac considered the problem of classifying all finite-dimensional irreducible representations of the basic classical Lie superalgebras [32]. For a subclass of these irreducible representations, known as "typical" representations, Kac derived a character formula closely analogous to the Weyl character formula for irreducible representations of simple Lie algebras [32]. The problem of obtaining a character formula for the remaining "atypical" irreducible representations has been the subject of intensive investigation.

From a computational and practical point of view, it is useful to identify characters with supersymmetric S-functions, since it is easy to work with S-functions, for which many properties are known. Although this identification holds for covariant and contravariant irreducible representations [10, 22], where the corresponding S-function is labelled by a single partition λ, it fails for mixed tensor irreducible representations, where the corresponding S-function is labelled by a composite partition $\bar{\nu};\mu$. The problem is well described and analysed in [70], where furthermore a character formula for atypical $\mathfrak{gl}(m|n)$ irreducible representations is conjectured. Since then, some partial solutions to this problem were given, e.g. for so-called generic representations [53], for singly atypical representations [12, 68, 71], or for tame representations [34]. More recently, the character problem for $\mathfrak{gl}(m|n)$ was principally solved by Serganova [60], who gave an algorithm to compute composition factor multiplicities of so-called Kac-modules, and thus indirectly the character. In [73], a substantially simpler method was conjectured to compute these composition factor multiplicities; this conjecture was proved by Brundan [13]. Still, the method using composition factor multiplicities of Kac-modules remains a rather indirect way of computing characters. Recently, there was a further breakthrough for this problem. Developing on the work of Brundan, Yucai Su and Zhang [66] managed to compute the generalized Kazhdan-Lusztig polynomials of $\mathfrak{gl}(m|n)$ irreducible representations, leading to a relatively explicit character formula for all these irreducible representations, and thus proving that the character formula conjectured in [70] holds.

The main idea of Chapter 1 is to fix some notation and terminology, and to discuss briefly the ring of symmetric functions. As already mentioned, symmetric and supersymmetric functions will be parametrized by either partitions or composite partitions. Those objects play an important role in this thesis. So, we will give their definition, their representation by means of Young diagrams and we will explain the notion of (composite) tableaux. In the second section we consider the ring of symmetric functions. The bases, known for this ring, are discussed briefly. Next to those bases, there still exists another and even more important basis: the symmetric Schur functions. As several bases of the ring of symmetric functions are defined, the relation between those bases is given by means of transition matrices. Due to the relation between symmetric functions and characters of the irreducible representations, the remainder of the chapter is dedicated to the symmetric Schur functions. Here we make a difference beween the symmetric functions parametrized by either a partition or a composite partition. For both families, we give the definition and a lot of properties and formulas. Some properties, known for symmetric functions parametrized by a partition, are generalized to symmetric functions parametrized by composite partitions. To end this chapter, and for further reference, we put together all the definitions and formulas of this chapter.

The supersymmetric functions and in particular the supersymmetric Schur functions are the main objects of this thesis. They are discussed in Chapter 2. This chapter, which is essentially a supersymmetric analogue of the previous chapter, is composed of three parts. Broadly speaking, the first part is dedicated to supersymmetric functions indexed by a partition, the second part summarizes the different bases of the ring of supersymmetric functions and the third part deals with supersymmetric functions indexed by a composite partition. The three most important bases namely the elementary, the complete and the Schur supersymmetric functions are defined, as well as their mutual relations. Next to those definitions, we also introduce the notion of a supertableau, and the formula for supersymmetric Schur functions by means of supertableaux. In the second part, we extend the number of bases by defining the remaining bases, analogous to the symmetric case. We prove that the supersymmetric power sums, the monomial and the so-called 'forgotten' supersymmetric functions are indeed supersymmetric. The relations between all those bases is proved to be given by the same transition matrices as those given in the first chapter. As the supersymmetric bases and the symmetric bases are defined in a similar way, their generating functions turn out to be similar as well. The third part dedicated to the supersymmetric Schur functions indexed by a composite partition, generalizes the definition and properties of composite symmetric functions. To make the similarity complete, we introduce the notion of composite supertableaux. Those composite supertableaux give rise to a new formula for composite supersymmetric Schur functions. This formula looks more complicated than the other formulas by means of (super)tableaux, but we illustrate that the extra conditions in the definiton of composite supertableaux just as in the formula for supersymmetric Schur functions are necessary. Again, we put together all the definitions and formulas of this chapter for further reference at the end of the chapter.

Until this point we did not need the notion of Lie superalgebras, but they are essential for the remaining chapters. As already mentioned, a superalgebra is the keystone in understanding supersymmetry. Although we will only need the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ for subsequent use, we give a short introduction defining superalgebras, Lie superalgebras and the enveloping algebra in general. The Cartan subalgebra and the root systems are discussed in particular for $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$. Since we want to link the first two chapters with the remaining of this thesis, we need to introduce representations, irreducible representations with highest weight (or highest weight modules) and their characters. The connections between the previous chapters and the representations is given by the bijection between $\mathfrak{gl}(m|n)$ highest weights and composite partitions. The covariant and contravariant modules are well known; the mixed tensor modules give rise to characters which do not allways coincide with supersymmetric functions. For the typical representations, King already proved that the characters coincide with supersymmetric functions, so we will also distinguish between typical and atypical representations. Seeing that the atypicality will play an important role, we define atypicality and the atypicality matrix in terms of highest weights and in terms of (composite) partitions.

It was already known [10, 22] that the characters of irreducible covariant and contravariant tensor representations in $\mathfrak{gl}(m|n)$ yield the supersymmetric S-functions. Comparing the different formulas for symmetric and supersymmetric S-functions, one formula is missing with regard to supersymmetric Schur functions. In Chapter 4, we derive a new, determinantal formula for the supersymmetric Schur polynomial $s_\lambda(x/y)$. The origin of this formula goes back to representation theory of the Lie superalgebra $\mathfrak{gl}(m|n)$. In particular, we show that all covariant representations are 'tame' by performing a sequence of simple odd reflections. This sequence is determined by a well-considered sequence of simple odd roots. This way, we can apply a character formula due to Kac and Wakimoto, leading to our new expression for $s_\lambda(x/y)$. The special choice of λ being the zero partition, gives rise to the denominator identity for $\mathfrak{gl}(m|n)$. This identity corresponds to a determinantal identity combining Cauchy's double alternant with Vandermonde's determinant. Next to the representation theoretic proof, we provide two other independent proofs for this determinantal formula. The first proof makes use of Macdonald's four characterizing properties of supersymmetric Schur functions; we prove that the determinantal formula is homogeneous and that the cancellation, factorization and restriction properties are fulfilled. A last and a more direct proof ties up our formula with that of Sergeev-Pragacz. This chapter is based on [47, 49].

In Chapter 5 we intend to show that there is still another family of atypical representations for which the character is given by a composite S-function. We distinguish between normally, critically and quasicritically related roots. If all the roots are critically related, the representation and its highest weight are also called critical. The relation between a highest weight and a composite partition is connected to the notion of being critical, and the notion of a critical composite partition is introduced. Next to the covariant and contravariant representations which are critical, we indicate that the class of critical representations is much larger. It is shown, with the same techniques as in Chapter 4, that the representations corresponding to a critical composite partition (subject to a technical restriction) are tame, and its character formula is computed using the character formula of Kac and Wakimoto. Again, this new expression gives rise to a determinantal expression for the characters of irreducible critical representations.

The main goal of this formula is to link this determinantal formula to composite supersymmetric S-functions. This last equality however is conjectured. The proof of this conjecture is outlined and the difficulties encountered are discussed. In fact, the proof of the conjecture is reduced to the proof of an identity for composite supersymmetric Schur functions presented as a lemma. Altough we are convinced that for critical composite partitions with no zero in the overlap (if presented in an $m \times n$-rectangle) the composite supersymmetric Schur functions are the characters of the irreducible representations with corresponding highest weight, the remaining lemma is not proved yet. The results of this chapter can be found in [50, 51].

In the previous chapters we have given different formulas to define supersymmetric functions. The formulas by means of the complete supersymmetric functions given in Chapter 2 and the new determinantal formulas deduced in Chapter 4 respectively Chapter 5 gives rise to new formulas for the t-dimension of covariant respectively mixed tensor representations V of the Lie superalgebra $\mathfrak{gl}(m|n)$ by means of a determinant. The parameter t keeps track of the $\mathbb{Z}$-grading of V. If we compare both formulas for covariant representations, the equality gives rise to a Hankel determinant identity for a special choice of λ. For the mixed tensor representations, the equality between the two formulas gives also rise to a determinant identity for a special choice of $\bar{\nu};\mu$. In this last case, the analysis of the determinant requires a lot of advanced determinant calculus. The first part of this chapter with respect to dimension formulas of covariant representations is based on [48, 49].

Chapter 1

Symmetric Schur Functions

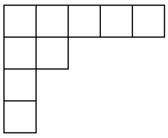
The purpose of the first chapter is on the one hand to fix some notation and terminology, on the other hand, to discuss briefly the ring of symmetric functions. A lot of these notions, notation and terminology will be taken from [46, Part I]. As most of the objects considered turn out to be parametrized by partitions and composite partitions, we will introduce them in the first section, as well as their presentation by means of Young diagrams and the notion of (composite) tableaux. The second section is dedicated to the ring of symmetric functions and the bases known for this ring. A special class of symmetric functions, the Schur functions, is defined in the third section as are the most important properties related to this thesis. In the next section, we consider the generating functions and in the fifth section we give a lot of properties and formulas for the symmetric functions parametrized by either a partition or a composite partition. To end this chapter, and for further reference, we put together all the definitions and formulas of this chapter.

1.1 Partitions and composite partitions

1.1.1 Partitions and diagrams

A PARTITION λ of a non-negative number N, is any sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers with $\lambda_1 \geq \lambda_2 \geq \ldots$, containing only finitely many non-zero terms and $\sum_i \lambda_i = |\lambda| = N$. The number of non-zero parts is $\ell(\lambda)$ and is called the LENGTH of λ and the sum of the parts $|\lambda|$ is the WEIGHT of λ (see Figure 1.1).

The YOUNG DIAGRAM F^{λ} of shape λ is the set of left-adjusted rows of squares with λ_i squares (or boxes) in the ith row reading from top to bottom. We shall usually denote the diagram of a partition λ by the same symbol λ. The boxes are often referred to by means of their row and column number; e.g. (1,4) refers to the fourth box on the first row. For example, the Young diagram of $(5, 2, 1, 1)$ is given in Figure 1.1.

$F^{(5,2,1,1)} =$ 

- $\ell(\lambda) = 4$
- $|\lambda| = 9$

Figure 1.1: Young Diagram of $\lambda = (5, 2, 1, 1)$

The CONJUGATE of a partition λ is the partition λ' whose diagram is the transpose of the diagram λ, the diagram obtained by reflection in the main diagonal. Hence, λ'_i is the number of boxes in the ith column of λ. In Figure 1.2, this is illustrated for $\lambda = (5, 2, 1, 1)$ and its conjugate $\lambda' = (4, 2, 1, 1, 1)$.

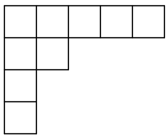
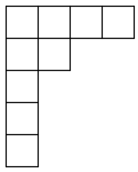
$F^{(5,2,1,1)} =$  and $F^{(4,2,1,1,1)} =$ 

Figure 1.2: A partition and its conjugate partition

Given two partitions λ, μ, we can define several operations. The SUM $\lambda + \mu$ of the partitions λ and μ is defined by $(\lambda + \mu)_i = \lambda_i + \mu_i$. The UNION $\lambda \cup \mu$ of the partitions λ and μ is given by the partition whose parts are of those of λ and μ, arranged in descending order.

For example, if $\lambda = (4, 2)$ and $\mu = (3, 3, 1)$, then $\lambda + \mu = (7, 5, 1)$, $\lambda \cup \mu = (4, 3, 3, 2, 1)$, see Figure 1.3 where the shaded boxes correspond to the parts of μ.
The operations $+$ and $\cup$ are dual to each other: $(\lambda + \mu)' = \lambda' \cup \mu'$. E.g. $\lambda' \cup \mu' = (2, 2, 1, 1) \cup (3, 2, 2) = (3, 2, 2, 2, 2, 1, 1) = (\lambda + \mu)'$.

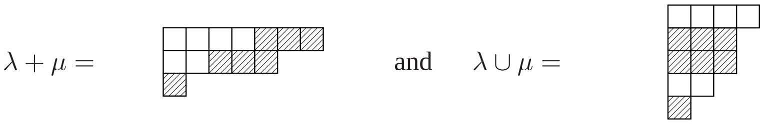


Figure 1.3: Sum and union of $\lambda = (4, 2)$ and $\mu = (3, 3, 1)$

The set $\mathcal{P}_N$ of partitions of N can be ordered in different ways. The only orderings that we will use are the lexicographic ordering L_N and the reverse lexicographical ordering L'_N. The ordering L_N (resp. L'_N) is the subset of $\mathcal{P}_N \times \mathcal{P}_N$ consisting of all (λ, μ) such that either $\lambda = \mu$, or else the first non-vanishing difference $\lambda_i - \mu_i$ is positive (resp. negative). Notice that both orderings are total orderings. For example, when $N = 5$, the lexicographic ordering on $\mathcal{P}_5$ is given by

$$(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3), (1^5);$$

whereas the reverse lexicographical ordering on $\mathcal{P}_5$ is given by

$$(1^5), (2, 1^3), (2^2, 1), (3, 1^2), (3, 2), (4, 1), (5).$$

Notice that in this notation the exponent indicates the number of times each integer occurs as a part.

1.1.2 Skew diagrams and tableaux

If λ, μ are partitions, we shall write $\lambda \supset \mu$ to mean that the diagram of μ is embodied in the diagram of λ, i.e. $\mu_i \leq \lambda_i$ for all $i \geq 1$. The set-theoretic difference $\theta = \lambda - \mu$ is called a skew diagram. For $\lambda = (5, 4, 2, 1)$ and $\mu = (3, 3, 1)$ the skew diagram is the shaded region in Figure 1.4. The conjugate of a skew diagram $\theta = \lambda - \mu$ is $\theta' = \lambda' - \mu'$ and $|\theta| = \sum \theta_i = |\lambda| - |\mu|$. A PATH in a skew diagram θ is a sequence $s_0, s_1, \ldots, s_r$

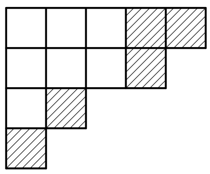
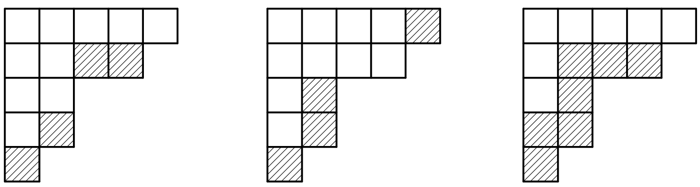


Figure 1.4: A skew diagram θ

of squares in θ such that s_{i-1} and s_i have a common side, for $1 \le i \le m$. A subset of θ is CONNECTED if any two boxes in that subset are connected by a path. The maximal connected components of θ are themselves skew diagrams, and they are called the CONNECTED COMPONENTS of θ. In Figure 1.4 there are three connected components.

A skew diagram is a HORIZONTAL m-STRIP (resp. a VERTICAL m-STRIP) if $|\theta| = m$ and $\theta'_i \le 1$ (resp. $\theta_i \le 1$) for each $i \ge 1$. In other words, a horizontal (resp. vertical) strip has at most one square in each column (resp. row). A skew diagram θ is a BORDER STRIP, also called a RIBBON, if θ is connected and contains no 2×2 block of squares, so that successive rows (or columns) of θ overlap by exactly one square. In Figure 1.5 there is an example of a horizontal, a vertical strip and a border strip.



(a) Horizontal strip $\theta = (0, 2, 0, 1, 1)$

(b) Vertical strip $\theta = (1, 0, 1, 1, 1)$

(c) Border strip $\theta = (0, 3, 1, 2, 1)$

Figure 1.5: A horizontal strip, a vertical strip and a border strip.

A (COLUMN STRICT) TABLEAU T, often referred to as SEMISTANDARD YOUNG TABLEAUX or SSYT, is a sequence of partitions

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(r)} = \lambda$$

such that each skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ $(1 \le i \le r)$ is a horizontal strip. Graphically, a tableau is a numbered skew diagram obtained by numbering each square of the skew diagram $\theta^{(i)}$ with the number i. The numbers inserted in $\lambda - \mu$ must increase strictly down each column and weakly from left to right along each row. The skew diagram $\lambda - \mu$ is called the SHAPE of the tableau T, and the sequence $(|\theta^{(1)}|, \ldots, |\theta^{(r)}|)$ is the WEIGHT of T. Hence, the weight can easily be obtained by counting the absolute frequency of the numbers i in the tableau $(1 \le i \le r)$, i.e. $|\theta^{(i)}|$ is the number of i's in the tableau (see Figure 1.6).

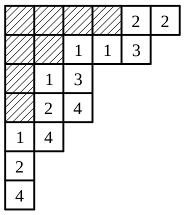


Figure 1.6: Tableau with shape $\lambda - \mu = (6,5,3,3,2,1,1) - (4,2,1,1)$ and weight $(4,4,2,3)$

1.1.3 Composite partitions

The COMPOSITE YOUNG DIAGRAM $F^{\overline{\nu};\mu} = F(\ldots, -\nu_2, -\nu_1; \mu_1, \mu_2, \ldots)$, specified by the pair of partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$, consists of two conventional Young diagrams F^μ and F^ν. The former is composed of boxes arranged in left-adjusted rows of lengths $\mu_1, \mu_2, \ldots$ (from top to bottom), and the latter of boxes arranged in right-adjusted rows of lengths $\nu_1, \nu_2, \ldots$ (from bottom to top).

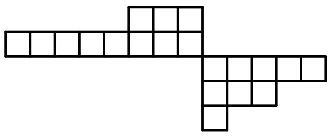


Figure 1.7: Composite Young Diagram $F^{\bar{\nu};\mu} = F^{(\overline{3},\overline{8});(5,3,1)}$

A manner of juxtaposition of F^μ and F^ν to form $F^{\overline{\nu};\mu}$ was given in [18]. To some extent this is a refining of the back-to-back notation of [3] and [36]. By way of illustration, for $\bar{\nu};\mu = (\overline{3},\overline{8});(5,3,1)$ the composite Young diagram is displayed in Figure 1.7. Note that in $(\overline{3},\overline{8})$ we have used the convention of putting the minus-signs on top of the integers; so in the given example $\mu = (5,3,1)$ and $\nu = (8,3)$. We shall refer to $\bar{\nu};\mu$ as being a COMPOSITE PARTITION. A composite partition is called an m-STANDARD COMPOSITE PARTITION if and only if $\ell(\mu) + \ell(\nu) \leq m$.

1.2 The ring of symmetric functions

In [46, §I.2] the ring of symmetric functions is described for infinitely many variables x. In this section we will give a survey of it but only for a finite set of variables $x_1, \ldots, x_m$.

Consider the ring $\mathbb{Z}[x_1, \ldots, x_m]$ of polynomials in m independent variables $x = (x_1, \ldots, x_m)$ with integer coefficients. The symmetric group S_m acts on this ring by permuting the variables. The symmetric polynomials, i.e. polynomials invariant under the action of S_m, form a subring $\Lambda_m = \mathbb{Z}[x_1, \ldots, x_m]^{S_m}$, which is a graded ring. For

$$\Lambda_m = \bigoplus_{k \geq 0} \Lambda_m^k$$

where Λ_m^k consists of the homogeneous symmetric polynomials of degree k with Λ_m^0 spanned by the zero degree polynomial 1.

Several different bases are defined on Λ_m.

1. Let λ be any partition of length $\ell(\lambda) \leq m$. For each m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m$ we denote by x^α the monomial $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$. The polynomial

$$m_\lambda(x) = m_\lambda(x_1, \ldots, x_m) = \sum_\alpha x^\alpha \tag{1.1}$$

   summed over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_m)$, is clearly symmetric, and the m_λ, as λ runs through all partitions with $\ell(\lambda) \leq m$, form a $\mathbb{Z}$-basis of Λ_m. The m_λ are called the MONOMIAL SYMMETRIC FUNCTIONS.

2. For each integer $r > 0$ the rth ELEMENTARY SYMMETRIC FUNCTION e_r is the sum of all products of r distinct variables x_i, so that

$$e_0 = 1 \quad \text{and} \quad e_r = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1} x_{i_2} \ldots x_{i_r} = m_{(1^r)}, \text{ for } r \geq 1. \tag{1.2}$$

   In particular e_r is defined to be zero for $r < 0$ and $e_r = 0$ for $r > m$. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, e_λ is defined as $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$. Given that the functions e_r are algebraically independent over $\mathbb{Z}$, the e_λ with $\ell(\lambda') \leq m$ form a $\mathbb{Z}$-basis for Λ_m.

3. For each $r \geq 0$ the rth COMPLETE SYMMETRIC FUNCTION h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \ldots$, such that

$$h_r = \sum_{|\lambda| = r} m_\lambda. \tag{1.3}$$

   In particular $h_0 = 1$ and $h_1 = e_1$. It is convenient to define h_r to be zero if $r < 0$. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, h_λ is defined as $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$.

$\Lambda_m = \mathbb{Z}[h_1, \ldots, h_m]$ where the h_r are algebraically independent. Note that $h_{m+1}, h_{m+2}, \ldots,$ are non-zero polynomials in $h_1, \ldots, h_m$. So, the h_λ, with $\ell(\lambda') \leq m$, form a $\mathbb{Z}$-basis of Λ_m; for example, if $m = 2$, $h_{(3^2)} = 4h_{(2^2,1^2)} - 4h_{(2,1^4)} + h_{(1^6)}$.

4. For each $r \geq 1$ the rth POWER SUM is

$$p_r = \sum_i x_i^r = m_r(x). \tag{1.4}$$

In the same way as e_λ and h_λ, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$. It is shown [46] that $nh_n = \sum_{r=1}^n p_r h_{n-r}$. So it is clear that $\mathbb{Q}[p_1, \ldots, p_n] = \mathbb{Q}[h_1, \ldots, h_n]$. Since the complete symmetric functions h_r are algebraically independent over $\mathbb{Z}$, and hence also over $\mathbb{Q}$, the p_r are also algebraically independent over $\mathbb{Q}$. So, the p_λ form a $\mathbb{Q}$-basis of Λ_m. But they do not form a $\mathbb{Z}$-basis of Λ_m; for example, $h_2 = \frac{1}{2}(p_1^2 + p_2)$ does not have integer coefficients when expressed in terms of the p_λ.

On the ring of symmetric polynomials Λ_m, a ring homomorphism ω is defined as:

$$\omega : \Lambda_m \to \Lambda_m : e_r \to h_r \qquad \text{For all } r \geq 0. \tag{1.5}$$

This homomorphism is an involution [46], i.e. ω^2 is the identity map.

The involution ω maps a power sum onto itself:

$$\omega(p_\lambda) = \varepsilon_\lambda p_\lambda \quad \text{with } \varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}. \tag{1.6}$$

Using this involution, a fifth $\mathbb{Z}$-basis of Λ_m can be defined for any partition λ, namely

$$f_\lambda = \omega(m_\lambda). \tag{1.7}$$

These elements are called the FORGOTTEN SYMMETRIC FUNCTIONS, as there is no simple direct description.

1.3 Symmetric Schur functions

1.3.1 Symmetric Schur functions indexed by a partition λ

Let $x = (x_1, \ldots, x_m)$ be a set of variables, Λ_m the ring of symmetric functions in x and let $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\ell(\lambda) \leq m$ be a partition. One can associate with any partition an alternant, which is the $m \times m$ determinant

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + m - j})_{1 \leq i,j \leq m} \tag{1.8}$$

where

$$\delta = (m-1, m-2, \ldots, 1, 0). \tag{1.9}$$

In particular, for $\lambda = ()$, the zero partition, we have Vandermonde's determinant $a_\delta = \prod_{1\le i<j\le m}(x_i - x_j)$.

The SCHUR FUNCTIONS, often called S-FUNCTIONS, are defined as

$$s_\lambda(x) = s_\lambda(x_1, \ldots, x_m) = \frac{a_{\lambda+\delta}}{a_\delta}. \tag{1.10}$$

It is clear from the definition that s_λ is symmetric and homogeneous of degree $|\lambda|$. Note that for any positive integer c

$$s_{(\lambda_1+c,\ldots,\lambda_m+c)}(x) = \left(\prod_{i=1}^{m} x_i^c\right) s_{(\lambda_1,\ldots,\lambda_m)}(x). \tag{1.11}$$

There is also a more combinatorial definition of a Schur function in terms of column strict tableaux of shape λ. Each tableau determines a monomial

$$x^T = \prod_{i=1}^{m} x_i^{|\theta^{(i)}|}$$

where $(|\theta^{(1)}|, \ldots, |\theta^{(m)}|)$ is the weight of the tableau. For $m = 5$ and $\lambda = (4,3,1)$ a possible tableau and corresponding monomial are given in Figure 1.8.

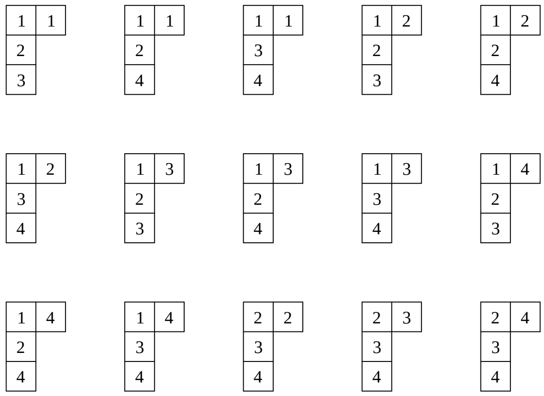
$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 5 & \\ \cline{1-3} 4 & & & \\ \cline{1-1}\end{array} \qquad \longleftrightarrow \qquad x^T = x_1^2 x_2^3 x_4^2 x_5.$$

Figure 1.8: Tableau of shape $\lambda = (4,3,1)$ and weight $(2,3,0,2,1)$.

Then, the symmetric Schur function can be expressed by means of tableaux:

$$s_\lambda(x) = \sum_T x^T \tag{1.12}$$

where the sum is taken over all column strict tableaux of shape λ and with entries from 1 to m. E.g. suppose $m = 4$ and $\lambda = (2,1,1)$. Then all the tableaux and the corresponding terms are given in Figure 1.9.



$$s_{(2,1,1)}(x_1,x_2,x_3,x_4) = {x_1}^2x_2x_3 + {x_1}^2x_2x_4 + {x_1}^2x_3x_4 + x_1{x_2}^2x_3 + x_1{x_2}^2x_4$$
$$+ x_1x_2x_3x_4 + x_1x_2{x_3}^2 + x_1x_2x_3x_4 + x_1{x_3}^2x_4 + x_1x_2x_3x_4$$
$$+ x_1x_2{x_4}^2 + x_1x_3{x_4}^2 + {x_2}^2x_3x_4 + x_2{x_3}^2x_4 + x_2x_3{x_4}^2$$

Figure 1.9: All tableaux and corresponding terms for $\lambda = (2,1,1)$.

If $\mu \subset \lambda$, then one defines the SKEW SYMMETRIC SCHUR FUNCTIONS $s_{\lambda/\mu}(x)$ as

$$s_{\lambda/\mu}(x) = \sum_T x^T, \tag{1.13}$$

where the sum is taken over all column strict tableaux of shape $\lambda - \mu$, otherwise $s_{\lambda/\mu}(x) = 0$.

The Jacobi-Trudi formula and the Nägelsbach-Kostka formula give s_λ in terms of the elementary and complete symmetric functions [46] :

$$s_\lambda(x) = \det\Big(h_{\lambda_i - i + j}(x)\Big)_{1 \le i,j \le \ell(\lambda)} \tag{1.14}$$

$$s_\lambda(x) = \det\Big(e_{\lambda'_i - i + j}(x)\Big)_{1 \le i,j \le \ell(\lambda')}. \tag{1.15}$$

So, if $\lambda = (n)$ and $\lambda = (1^n)$, we have that

$$s_{(n)}(x) = h_n(x) \qquad \text{and} \qquad s_{(1^n)}(x) = e_n(x). \tag{1.16}$$

These formulas can be extended to skew symmetric Schur functions [46, §I.5]:

$$s_{\lambda/\mu}(x) = \det\Big(h_{\lambda_i - \mu_j - i + j}(x)\Big)_{1 \le i,j \le \ell(\lambda)} \tag{1.17}$$

$$s_{\lambda/\mu}(x) = \det\Big(e_{\lambda'_i-\mu'_j-i+j}(x)\Big)_{1\le i,j\le \ell(\lambda')}. \tag{1.18}$$

The symmetric Schur functions can also be expressed in terms of the symmetric monomial functions, namely:

$$s_\lambda(x) = \sum_{\mu\le\lambda} K_{\lambda\mu} m_\mu(x) \tag{1.19}$$

where the sum is taken over all partitions $\mu \le \lambda$ in the lexicographical ordering. The numbers $K_{\lambda\mu}$ are called KOSTKA NUMBERS and it is shown that

$$K_{\lambda\mu} = \text{the number of (column strict) tableaux } T \text{ of shape } \lambda \text{ and weight } \mu. \tag{1.20}$$

Therefore, $K_{\lambda\mu} \ge 0$ and $K_{\lambda\lambda} = 1$. For example, if $\lambda = (2,2,1)$ then $K_{(2,2,1)(2,1^3)} = 2$ and $K_{(2,2,1)(1^5)} = 5$ (see Figure 1.10). Thus,

$$s_{(2,2,1)}(x) = m_{(2^2,1)}(x) + 2\, m_{(2,1^3)}(x) + 5\, m_{(1^5)}(x).$$

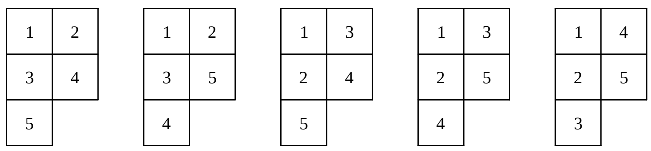


Figure 1.10: The different tableaux T of shape $(2,2,1)$ and weight (1^5).

Other determinantal formulas for s_λ are Giambelli's formula [46] and the ribbon formula [45]. Giambelli's formula depends on the Frobenius notation for partitions. Suppose that the main diagonal of the Young diagram of λ consists of r-boxes (i,i), $(1 \le i \le r)$. Let $\alpha_i = \lambda_i - i$ be the number of boxes in the ith row of λ to the right of (i,i), for $1 \le i \le r$, and $\beta_i = \lambda'_i - i$ be the number of boxes in the ith column of λ below (i,i), for $1 \le i \le r$. We have $\alpha_1 > \alpha_2 > \ldots > \alpha_r \ge 0$ and $\beta_1 > \beta_2 > \ldots > \beta_r \ge 0$, and we denote the partition λ by

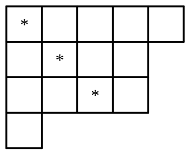
$$\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r) = (\alpha|\beta).$$

Clearly, the conjugate of $(\alpha|\beta)$ is $(\beta|\alpha)$.

Then, Giambelli's formula is given by

$$s_{(\alpha|\beta)}(x) = \det(s_{(\alpha_i|\beta_j)}(x))_{1\le i,j\le r},$$

where, for $a, b \ge 0$, $s_{(a|b)} = h_{a+1}e_b - h_{a+2}e_{b-1} + \ldots + (-1)^b h_{a+b+1}e_0$; if a or b is negative, $s_{(a|b)} = 0$ except when $a + b = -1$, in which case $s_{(a|b)} = (-1)^b$.



$$\lambda = (5,4,4,1) = (4,2,1|3,1,0)$$

Figure 1.11: Frobenius notation for partitions.

1.3.2 Symmetric Schur functions indexed by a composite partition $\bar{\nu};\mu$

Let $x = (x_1, \ldots, x_m)$ be the set of variables. Let $\bar{\nu};\mu$ be a composite partition with $\ell(\mu) = p$, $\ell(\nu) = q$ and $p + q \leq m$. Thus, $\bar{\nu};\mu$ is a m-standard composite partition. Then, we can associate an m-tuple $(\mu_1, \ldots, \mu_p, 0, \ldots, 0, -\nu_q, \ldots, -\nu_1)$ with the composite partition $\bar{\nu};\mu$. Extending (1.11), the symmetric Schur function indexed by this composite partition is defined by:

$$s_{\bar{\nu};\mu}(x) = \left(\prod_{i=1}^{m} x_i^{-\nu_1}\right) s_\lambda(x) \quad \text{with } \lambda = (\mu_1+\nu_1, \mu_2+\nu_1, \ldots, -\nu_2+\nu_1, 0). \tag{1.21}$$

The formula analogous to (1.15) for symmetric Schur functions indexed by a composite partition was postulated by Balantekin and Bars [5] in terms of characters (see Chapter 3), and proved in [18], namely

$$s_{\overline{\nu};\mu}(x) = \det \left(\begin{array}{c|c} \dot{e}_{\nu'_l+k-l}(x) & e_{\mu'_j-k+j-1}(x) \\ \hline \dot{e}_{\nu'_l-i+l-1}(x) & e_{\mu'_j+i-j}(x) \end{array} \right) \tag{1.22}$$

where the indices i, j, k resp. l run from top to bottom, from left to right, from bottom to top resp. from right to left. The dotted function $\dot{e}_r(x) = e_r(\overline{x}) = e_r(\frac{1}{x_1}, \ldots, \frac{1}{x_m})$. In the same way, we also have that

$$s_{\overline{\nu};\mu}(x) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l+k-l}(x) & h_{\mu_j-k+j-1}(x) \\ \hline \dot{h}_{\nu_l-i+l-1}(x) & h_{\mu_j+i-j}(x) \end{array} \right) \tag{1.23}$$

where the indices i, j, k resp. l run from top to bottom, from left to right, from bottom to top resp. from right to left and with the function $\dot{h}_r(x) = h_r(\overline{x})$.

Example 1.1 *Let $\nu = (1,1)$ and $\mu = (3,1)$,*

$$\begin{aligned} s_{(\overline{1},\overline{1});(3,1)}(x) &= \det \left(\begin{array}{c|ccc} \dot{e}_2(x) & e_1(x) & 0 & 0 \\ \hline \dot{e}_1(x) & e_2(x) & e_0(x) & 0 \\ \dot{e}_0(x) & e_3(x) & e_1(x) & e_0(x) \\ 0 & e_4(x) & e_2(x) & e_1(x) \end{array} \right) \\ &= \det \left(\begin{array}{cc|cc} \dot{h}_1(x) & \dot{h}_2(x) & h_1(x) & 0 \\ \dot{h}_0(x) & \dot{h}_1(x) & h_2(x) & 0 \\ \hline 0 & \dot{h}_0(x) & h_3(x) & h_0(x) \\ 0 & 0 & h_4(x) & h_1(x) \end{array} \right) \end{aligned}$$

The formulas (1.22) and (1.23) are defined independent of the number of variables x. So, these formulas can easily be generalized for composite partitions which are not necessarily m-standard. Suppose $\bar{\nu};\mu$ is a not a m-standard composite partition, i.e. $\ell(\mu) + \ell(\nu) > m$ then the $s_{\bar{\nu};\mu}(x)$ is related to a Schur function indexed by a standard composite partition:

$$s_{\bar{\nu};\mu}(x) = (-1)^{c+\bar{c}+1} s_{\overline{\nu-h};\mu-h}(x) \tag{1.24}$$

where $h = \ell(\mu) + \ell(\nu) - m - 1$ and $\mu - h$ and $\nu - h$ are the partitions obtained from the diagrams F^{μ} and F^{ν} respectively, by the removal of continuous boundary strips each of length h starting at the foot of the first columns of F^{μ} resp. F^{ν} and extending over c and $\bar{c}$ columns respectively. This MODIFICATION RULE was introduced in [37]. The boundary strip is said to be removable if $h \geq 0$ and if the juxtaposition of $F^{\mu-h}$ and $F^{\nu-h}$ yields a regular composite Young diagram $F^{\overline{\alpha};\beta}$ i.e. the resulting diagrams F^{β} and $F^{\overline{\alpha}}$ correspond to the partitions β and α. If α or β is not partition, then $F^{\overline{\alpha};\beta}$ is said to be irregular. If $F^{\overline{\alpha};\beta}$ is irregular then the strip h is not removable and $s_{\overline{\alpha};\beta}$ is to be interpreted as being identically zero. This modification rule can be illustrated using Formula (1.22) and Example 1.1.

Example 1.2 *Suppose $m = 3$. Given (1.24), $h = 2 + 2 - 3 - 1 = 0$ and hence $c = \bar{c} = 0$. So,*

$$s_{(\overline{1},\overline{1});(3,1)}(x) = -s_{(\overline{1},\overline{1});(3,1)}(x) \Leftrightarrow s_{(\overline{1},\overline{1});(3,1)}(x) = 0.$$

The same result follows from (1.22):

$$s_{(\overline{1},\overline{1});(3,1)}(x) = \det \left(\begin{array}{c|ccc} \dot{e}_2(x) & e_1(x) & 0 & 0 \\ \hline \dot{e}_1(x) & e_2(x) & e_0(x) & 0 \\ \dot{e}_0(x) & e_3(x) & e_1(x) & e_0(x) \\ 0 & 0 & e_2(x) & e_1(x) \end{array} \right) \equiv 0$$

as the second column is proportional to the first column since $\dot{e}_2(x) = \frac{e_1(x)}{x_1x_2x_3}$, $\dot{e}_1(x) = \frac{e_2(x)}{x_1x_2x_3}$ *and* $\dot{e}_0(x) = \frac{e_3(x)}{x_1x_2x_3}$.

Suppose $m = 2$*. Applying (1.24), with* $h = 1$ *and hence* $c = \overline{c} = 1$*, gives*

$$s_{(\overline{1},\overline{1});(3,1)}(x) = -s_{(\overline{1});(3)}(x),$$

which also follows from (1.22):

$$\begin{aligned} s_{(\overline{1},\overline{1});(3,1)}(x) &= \det \left(\begin{array}{c|ccc} \dot{e}_2(x) & e_1(x) & 0 & 0 \\ \hline \dot{e}_1(x) & e_2(x) & e_0(x) & 0 \\ \dot{e}_0(x) & 0 & e_1(x) & e_0(x) \\ 0 & 0 & e_2(x) & e_1(x) \end{array} \right) \\ &= -\det \left(\begin{array}{c|ccc} \dot{e}_1(x) & e_0(x) & 0 & 0 \\ \hline \dot{e}_0(x) & e_1(x) & e_0(x) & 0 \\ 0 & e_2(x) & e_1(x) & e_0(x) \\ 0 & 0 & e_2(x) & e_1(x) \end{array} \right) \\ &= -s_{(\overline{1});(3)}(x) \end{aligned}$$

where the second determinant is derived from the first by multiplying the first column and dividing the second column with the factor x_1x_2 *and swapping over the first two columns.*

If necessary, the modification rule (1.24) should be applied more than once until either $h < 0$ i.e. the resulting composite partition is m-standard, or else the resulting symmetric Schur function is shown to be zero. This is illustrated in Figure 1.12 for $m = 4$ and $\bar{\nu}; \mu = (\overline{1}, \overline{1}, \overline{1}, \overline{2}, \overline{3}, \overline{3}, \overline{3}); (3, 3, 2, 2, 1, 1)$ with

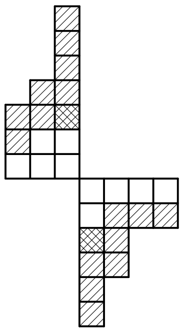
$$s_{\bar{\nu};\mu}(x) = (-1)^{4+3+1} s_{\overline{\beta_1};\alpha_1} = (-1)^{1+1+1} s_{\overline{\beta_2};\alpha_2} = -s_{\overline{\beta_2};\alpha_2}$$

1.4 Generating functions

We briefly recall some generating function expansions in terms of symmetric functions, taken from [46].

The generating functions for the e_r and h_r are given by

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i=1}^{n} (1 + x_i t) \tag{1.25}$$



$h_1 = 8$	$h_2 = 1$
$c_1 = 4$	$c_2 = 1$
$\overline{c}_1 = 3$	$\overline{c}_2 = 1$
$\alpha_1 = (4,1,1)$	$\alpha_2 = (4,1)$
$\beta_1 = (3,2,1)$	$\beta_2 = (3,2)$

Figure 1.12: Modification rules for $m = 4$ and $\nu = (3,3,3,2,1,1,1)$ and $\mu = (4,4,2,2,1,1)$.

and

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i=1}^{n} \frac{1}{(1 - x_i t)} \tag{1.26}$$

From (1.25) and (1.26) we have that

$$H(t)E(-t) = 1 \tag{1.27}$$

or, equivalently,

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0, \qquad \text{for all } n \geq 1. \tag{1.28}$$

The generating function of the p_r is

$$P(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)} \quad \text{or} \quad P(-t) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}. \tag{1.29}$$

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two finite or infinite sequences of independent variables. The product $\prod_{i,j}(1 - x_i y_j)^{-1}$ has three expansions in terms of the different bases.

The first of these expansions is, with the sum taken over all partitions λ,

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} m_\lambda(x) h_\lambda(y) = \sum_{\lambda} h_\lambda(x) m_\lambda(y). \tag{1.30}$$

The expansion in terms of the symmetric Schur functions is given by

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda s_\lambda(x)s_\lambda(y) \tag{1.31}$$

summed over all partition λ. The last expansion is one in terms of the power sums:

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda z_\lambda^{-1}p_\lambda(x)p_\lambda(y) \tag{1.32}$$

summed over all partition λ and with

$$z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.

By applying the involution ω to the symmetric functions of the x variables we obtain from (1.30), (1.31) and (1.32):

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda m_\lambda(x)e_\lambda(y) = \sum_\lambda e_\lambda(x)m_\lambda(y), \tag{1.30'}$$

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda s_\lambda(x)s_{\lambda'}(y), \tag{1.31'}$$

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda \varepsilon_\lambda z_\lambda^{-1}p_\lambda(x)p_\lambda(y) \tag{1.32'}$$

with ε defined in (1.6).

1.5 Properties of symmetric Schur functions

1.5.1 Transition matrices

Several $\mathbb{Z}$-bases of the space Λ_m were constructed above. In this section we will give a survey of the matrices of transitions between these bases [46, 67]. It is clear that rows and columns of these matrices are labeled by the partitions of N. We will arrange them in reverse lexicographical order (see §1.1).

Let J be the matrix such that $J_{\lambda\mu} = 1$ for $\lambda' = \mu$ and $J_{\lambda\mu} = 0$ for $\lambda' \neq \mu$ and let $K = (K_{\lambda\mu})$ be the matrix composed by the Kostka numbers.

Given two bases, the transition matrix $M(u,v) = (M_{\lambda,\mu})$ such that

$$u_\lambda = \sum_\mu M_{\lambda\mu} v_\mu$$

is given in Table 1.1 where M' is the transposed matrix of M and $M^* = (M^{-1})'$.

	e	h	m	f	s
e	1	$K'JK^*$	$K'JK$	$K'K$	$K'J$
h	$K'JK^*$	1	$K'K$	$K'JK$	K'
m	$K^{-1}JK^*$	$K^{-1}K^*$	1	$K^{-1}JK$	K^{-1}
f	$K^{-1}K^*$	$K^{-1}JK^*$	$K^{-1}JK$	1	$K^{-1}J$
s	JK^*	K^*	K	JK	1

Table 1.1: Transition Matrices

1.5.2 Operations on symmetric Schur functions

On the symmetric Schur functions, the S-functions, there are several important operations defined [46]. The (OUTER) PRODUCT of S-functions, arises from ordinary multiplication of polynomials:

$$s_\mu(x) s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x). \tag{1.33}$$

Remark that this operation is clearly commutative, associative and distributive over addition. As a short notation we will also use $s_{\mu\nu}(x) = s_{\mu\cdot\nu}(x) \equiv s_\mu(x)s_\nu(x)$.

The coefficients $c^\lambda_{\mu\nu}$, known as THE LITTLEWOOD-RICHARDSON COEFFICIENTS, may be calculated by a simple combinatorial rule [44] involving the partitions ν and μ. The proof of this combinatorial rule can be found in [46, §I.9]. We have

$$c^\lambda_{\mu\nu} = 0 \qquad \text{unless} \qquad |\lambda| = |\mu| + |\nu| \quad \text{and} \quad \mu, \nu \subset \lambda.$$

Let T be a tableau. From T we derive a WORD or sequence $w(T)$ by reading the symbols in T from right to left in successive rows, starting with the top row.

A word $w = a_1 a_2 \ldots a_N$ in the symbols $1, 2, \ldots, n$ is said to be a LATTICE PERMUTATION if for $1 \leq r \leq N$ and $1 \leq i \leq n-1$, the number of occurrences of the symbol i in $a_1 a_2 \ldots a_r$ is not less than the number of occurrences of $i+1$.

For example, given the tableau T with shape $\lambda - \mu = (5,3,2) - (1,1)$ in Figure 1.13, the word is $w(T) = 32113241$, which is not a lattice permutation.

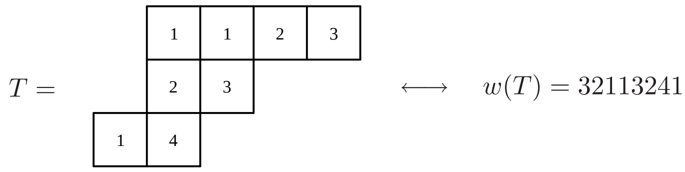


Figure 1.13: Tableau T and corresponding word $w(T)$.

Proposition 1.3 (Littlewood-Richardson rule) *Let λ, μ, ν be partitions. Then $c^{\lambda}_{\mu\nu}$ is equal to the number of tableaux T of shape $\lambda - \mu$ and weight ν such that $w(T)$ is a lattice permutation.*

For example $c^{\lambda}_{\mu\nu} = 3$ for $\lambda = (4,3,2,1,1)$, $\mu = (3,2,1)$ and $\nu = (2,2,1)$, as illustrated in Figure 1.14; in Figure 1.14(a) the word is $w(T) = (1,1,2,2,3)$, in Figure 1.14(b) the word is $w(T) = (1,2,1,2,3)$ and in Figure 1.14(c) the word is $w(T) = (1,2,3,1,2)$.

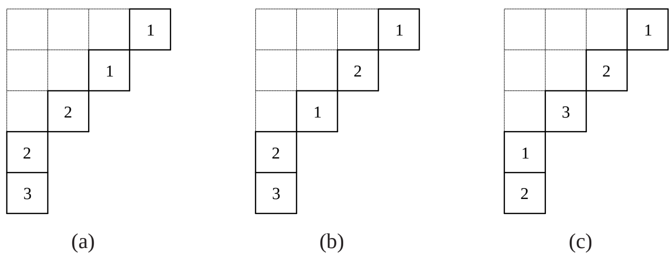


Figure 1.14: Different lattice permutations for T of shape $\lambda - \mu$ and weight ν.

Several properties can be derived for the outer product.
Together with (1.16), we have

$$s_{\mu}(x)h_r(x) = s_{\mu}(x)s_{(r)}(x) = \sum_{\lambda} s_{\lambda}(x) \tag{1.34}$$

with $\lambda - \mu$ a horizontal r-strip (Pieri's formula) and

$$s_{\mu}(x)e_r(x) = s_{\mu}(x)s_{(1^r)}(x) = \sum_{\lambda} s_{\lambda}(x) \tag{1.35}$$

with $\lambda - \mu$ a vertical r-strip. An extension of Pieri's formula is also proved [46]

$$s_\mu(x)h_\nu(x) = \sum_\lambda K_{\lambda-\mu,\nu} s_\lambda(x) \tag{1.36}$$

with $|\nu| = |\lambda - \mu|$ and $K_{\lambda-\mu,\nu}$ the Kostka number as defined in (1.20).

As a simple consequence of $s_{\mu\nu\rho} = s_{\mu\nu}s_\rho = s_\mu s_{\nu\rho} = s_\nu s_{\mu\rho}$ and (1.33), we have

$$\sum_\sigma c^\lambda_{\rho\sigma} c^\sigma_{\mu\nu} = \sum_\eta c^\lambda_{\mu\eta} c^\eta_{\nu\rho} = \sum_\tau c^\lambda_{\nu\tau} c^\tau_{\mu\rho}. \tag{1.37}$$

For the skew symmetric Schur functions, one can show that

$$s_{\lambda/\mu}(x) = \sum_\eta c^\lambda_{\mu\eta} s_\eta(x). \tag{1.38}$$

This allows one to define

$$s_{(\lambda/\mu)/\nu}(x) = \sum_\eta c^\lambda_{\mu\eta} s_{\eta/\nu}(x). \tag{1.39}$$

Expanding the last function and using (1.37) implies

$$s_{(\lambda/\mu)/\nu}(x) \quad = \quad \sum_{\eta,\rho} c^\lambda_{\mu\eta} c^\eta_{\rho\nu} s_\eta(x) = \sum_{\sigma,\rho} c^\lambda_{\nu\sigma} c^\sigma_{\rho\mu} s_\sigma(x) = s_{(\lambda/\nu)/\mu)}(x). \tag{1.40}$$

This means that the ordering of μ and ν can been changed. So, one can write this as

$$s_{(\lambda/\mu)/\nu}(x) = s_{(\lambda/\nu)/\mu}(x) = s_{\lambda/(\mu\nu)}(x). \tag{1.41}$$

Using (1.37) in the expansion of (1.39), we can easily see that

$$s_{\lambda/(\mu\nu)}(x) = \sum_{\eta,\sigma} c^\lambda_{\mu\eta} c^\eta_{\nu\sigma} s_\sigma(x) = \sum_{\rho,\sigma} c^\lambda_{\rho\sigma} c^\rho_{\mu\nu} s_\sigma(x) = \sum_\rho c^\rho_{\mu\nu} s_{\lambda/\rho}(x). \tag{1.42}$$

Now let $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$, $z = (z_1, z_2, \ldots)$ be three sets of independent variables. Then, by (1.33) and (1.38) we have

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(x) s_\lambda(z) s_\mu(y) = \sum_\nu s_\nu(x) s_\nu(z) \sum_\mu s_\mu(z) s_\mu(y).$$

Using (1.31) twice

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(x)s_\lambda(z)s_\mu(y) = \prod_{i,k}(1-x_iz_k)^{-1}\prod_{j,k}(1-y_jz_k)^{-1} = \sum_\lambda s_\lambda(x,y)s_\lambda(z)$$

where $s_\lambda(x,y)$ denotes the Schur function indexed by λ in $(x_1,x_2,\ldots,y_1,y_2,\ldots)$. From this equality, we can conclude that

$$s_\lambda(x,y) = \sum_\mu s_{\lambda/\mu}(x)s_\mu(y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}s_\mu(y)s_\nu(x) \tag{1.43}$$

and more generally [46], we have

$$s_{\lambda/\mu}(x,y) = \sum_\nu s_{\lambda/\nu}(x)s_{\nu/\mu}(y) \tag{1.44}$$

summed over all partitions ν such that $\lambda \supset \nu \supset \mu$. Together with (1.38) this becomes,

$$s_{\lambda/\mu}(x,y) = \sum_\eta s_{\lambda/\mu\eta}(x)s_\eta(y). \tag{1.44'}$$

1.5.3 Generalizations to the case of composite partitions

As a first generalization we can extend (1.12) to symmetric Schur functions indexed by a composite Young diagam. Hence we have to introduce the concept of a composite Young tableau [39].

A COMPOSITE YOUNG TABLEAU is a numbered composite Young diagram formed by inserting positive and negative entries chosen from the sets $M = \{1,2,\ldots,m\}$ and $\overline{M} = \{\overline{1},\overline{2},\ldots,\overline{m}\}$ into each box of F^μ, resp. F^ν, in such a way that the entries are

- non-decreasing across rows and strictly increasing down columns
- $r(j) + \overline{r}(\overline{j}) \leq j$ for $j \in \{1,2,\ldots,m\}$, where $r(j)$ and $\overline{r}(\overline{j})$ are the lowest row numbers in F^μ resp. F^ν containing j resp. $\overline{j}$.

An entry $\overline{i}$ is to be interpreted as $-i$, and the corresponding weight w has components $w_i = n_i - n_{\overline{i}}$, for $i = 1,\ldots,m$, where n_i and $n_{\overline{i}}$ are the number of entries i and $\overline{i}$ in $T^{\bar{\nu};\mu}$.

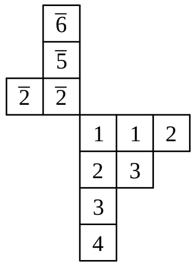


Figure 1.15: A standard composite tableau

Given $m = 7$, this is illustrated in Figure 1.15 for the composite partition $\bar{\nu};\mu = (\overline{1},\overline{1},\overline{2});(3,2,1,1)$ and weight $w = (2,0,2,1,-1,-1,0)$.

Using those standard composite Young tableaux, the symmetric Schur functions indexed by a composite diagram $\bar{\nu};\mu$ can be computed as:

$$s_{\bar{\nu};\mu}(x) = \sum_{T^{\bar{\nu};\mu}} x^{T^{\bar{\nu};\mu}}. \tag{1.45}$$

The extra condition $r(j) + r(\overline{j}) \leq j$ on standard composite Young tableau is a simple consequence of the connection between the symmetric Schur functions indexed by a partition and those indexed by a m-standard composite partition,

$$s_{\bar{\nu};\mu}(x) = \left(\prod_{i=1}^{m} x_i^{-\nu_1}\right) s_\lambda(x) \quad \text{with } \lambda = (\mu_1+\nu_1, \mu_2+\nu_1, \ldots, -\nu_2+\nu_1, 0). \tag{1.46}$$

With each composite tableau $T^{\bar{\nu};\mu}$ of shape $\bar{\nu};\mu$ corresponds a tableau T of shape λ. In order to fill the boxes of $F^{\bar{\nu}}$ we construct F^λ by appending a rectangle with m rows and ν_1 columns and removing boxes at the bottom of this rectangle corresponding to the boxes of $F^{\bar{\nu}}$ (see Figure 1.16 for $\bar{\nu};\mu = (\overline{2},\overline{3});(3,2,1,1)$). Given a standard tableau F^λ, a column of $F^{\bar{\nu}}$ contains $\overline{i}$ if and only if there is no i in the corresponding column of F^λ, if we put $F^{\bar{\nu}}$ on top of F^λ as shown in the Figure 1.16. The numbers $\overline{i}$ are ordered in $F^{\bar{\nu}}$ according to the other conditions of a composite tableau.
According to the condition that the entries have to increase along a column, it follows that $r_\mu(p) \leq p$. Suppose

$$K_p^\lambda = \min(\{c \mid p \text{ does not occur in column } c \text{ of } F^\lambda\} \cup \{\mu_1 + \nu_1 + 1\}).$$

If $K_p^\lambda \in \{1, \ldots, \nu_1\}$, let $q = \max(\{t \mid t < p, t \text{ occurs in column } K_p^\lambda\})$ and ρ_q its row number along the column K_p^λ. Then it is easy to check that $\overline{r}_\nu(\overline{p}) = p - \rho_q$ and

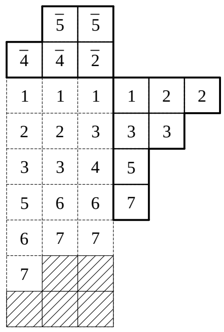


Figure 1.16: Extra condition: $r(j) + \overline{r}(\overline{j}) \leq j$

$r_\mu(p) \leq \rho_q$. If $K_p^\lambda \in \{\nu_1 + 1, \ldots, \nu_1 + \mu_1 + 1\}$, there is a p in the first ν_1 columns, and thus $\overline{r}_\nu(\overline{p}) = 0$. This implies the extra condition.
In Figure 1.16, $K_5^\lambda = 2 \in \{1, 2, 3\}$, $q = 3$ and $\rho_q = 3$ gives $\overline{r}_\nu(\overline{5}) = 2$ and $r_\mu(5) \leq 3$; $K_6^\lambda = 4 \in \{4, 5, 6, 7\}$, $q = 5$ and $\rho_q = 3$ gives $\overline{r}_\nu(\overline{6}) = 0$ and $r_\mu(6) = 0 \leq 6$.

Although the determinantal formula (1.22) does not seem to be a practical computational tool, King showed [36, 41] that its Laplace expansion yields the formula

$$s_{\bar{\nu};\mu}(x) = \sum_\zeta (-1)^{|\zeta|} s_{\mu/\zeta}(x) s_{\nu/\zeta'}(\overline{x}). \tag{1.47}$$

Conversely [3, 37]

$$s_\nu(\overline{x}) s_\mu(x) = \sum_\zeta s_{\overline{\nu/\zeta};\mu/\zeta}(x), \tag{1.48}$$

where the symmetric functions in the right hand side are defined by

$$s_{\overline{\lambda/\mu};\nu/\eta}(x) = \sum_{\zeta,\xi} c^\lambda_{\zeta\mu} c^\nu_{\xi\eta} s_{\overline{\zeta};\xi}(x). \tag{1.49}$$

Often, we will use the notation by means of composite partitions to denote $s_\nu(\overline{x})$:

$$s_\nu(\overline{x}) = s_{\overline{\nu}}(x), \quad \text{with } \overline{\nu} = (-\nu_1, -\nu_2, \ldots) \text{ and } \overline{x_i} = \frac{1}{x_i}. \tag{1.50}$$

Analogous to (1.42), this definition can easily be extended to

$$s_{\overline{\lambda/\mu\nu};\eta/\kappa\tau}(x) = \sum_{\zeta,\xi} c^\zeta_{\mu\nu} c^\xi_{\kappa\tau} s_{\overline{\lambda/\zeta};\eta/\xi}(x). \tag{1.51}$$

Formula (1.48) can be generalized to a product of two Schur functions indexed by skew partitions.

$$s_{\lambda/\mu}(x)s_{\eta/\nu}(\overline{x}) = \sum_{\rho} s_{\overline{\eta/\nu\rho};\lambda/\mu\rho}(x) \tag{1.52}$$

Proof.

$$\begin{aligned}
s_{\lambda/\mu}(x)s_{\eta/\nu}(\overline{x}) &\overset{(1.38)}{=} \left(\sum_{\sigma} c^{\lambda}_{\sigma\mu}s_{\sigma}(x)\right)\left(\sum_{\tau} c^{\eta}_{\tau\nu}s_{\tau}(\overline{x})\right)\\
&\overset{(1.48)}{=} \sum_{\sigma,\tau} c^{\lambda}_{\sigma\mu}c^{\eta}_{\tau\nu}\left(\sum_{\rho} s_{\overline{\tau/\rho};\sigma/\rho}(x)\right)\\
&\overset{(1.49)}{=} \sum_{\rho}\sum_{\sigma,\tau}\sum_{\gamma,\delta} c^{\lambda}_{\sigma\mu}c^{\eta}_{\tau\nu}c^{\tau}_{\rho\gamma}c^{\sigma}_{\rho\delta}s_{\overline{\gamma};\delta}(x)\\
&= \sum_{\rho}\sum_{\gamma,\delta}\left(\sum_{\tau} c^{\eta}_{\tau\nu}c^{\tau}_{\rho\gamma}\right)\left(\sum_{\sigma} c^{\lambda}_{\sigma\mu}c^{\sigma}_{\rho\delta}\right)s_{\overline{\gamma};\delta}(x)\\
&\overset{(1.37)}{=} \sum_{\rho}\sum_{\gamma,\delta}\left(\sum_{\alpha} c^{\eta}_{\alpha\gamma}c^{\alpha}_{\nu\rho}\right)\left(\sum_{\beta} c^{\lambda}_{\beta\delta}c^{\beta}_{\mu\rho}\right)s_{\overline{\gamma};\delta}(x)\\
&\overset{(1.49)}{=} \sum_{\rho}\sum_{\alpha,\beta} c^{\alpha}_{\nu\rho}c^{\beta}_{\mu\rho}s_{\overline{\eta/\alpha};\lambda/\beta}(x)\\
&\overset{(1.51)}{=} \sum_{\rho} s_{\overline{\eta/\nu\rho};\lambda/\mu\rho}(x)
\end{aligned}$$

□

The combination of (1.47) and (1.48) yields

$$s_{\bar{\nu};\mu}(x) = \sum_{\zeta,\rho}(-1)^{|\zeta|}s_{\overline{\nu/\zeta'\rho};\mu/\zeta\rho}(x) \tag{1.53}$$

Further generalizations of (1.47) and (1.53) are,

$$s_{\overline{\nu/\mu};\lambda/\tau}(x) = \sum_{\zeta}(-1)^{|\zeta|}s_{\nu/\mu\zeta'}(\overline{x})s_{\lambda/\tau\zeta}(x) \tag{1.54a}$$

$$s_{\overline{\nu/\mu};\lambda/\tau}(x) = \sum_{\zeta,\rho}(-1)^{|\zeta|}s_{\overline{\nu/\mu\zeta'\rho};\lambda/\tau\zeta\rho}(x) \tag{1.54b}$$

Proof.

$$s_{\overline{\nu/\mu};\lambda/\tau}(x) \overset{(1.38)}{=} \sum_{\sigma,\eta} c^{\nu}_{\sigma\mu}c^{\lambda}_{\eta\tau}s_{\overline{\sigma};\eta}(x)$$

$$\stackrel{(1.47)}{=} \sum_{\sigma,\eta} c^{\nu}_{\sigma\mu} c^{\lambda}_{\eta\tau} \left(\sum_{\zeta} (-1)^{|\zeta|} s_{\sigma/\zeta'}(\overline{x}) s_{\eta/\zeta}(x) \right)$$

Applying (1.40) provides (1.54a); (1.54a) together with (1.52) gives (1.54b). □

Finally, to end this section we will generalize (1.44) and (1.44'). Let $(x, y) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ be a set of variables, then:

$$s_{\overline{\nu/\eta};\lambda/\mu}(x,y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{\sigma/\rho};\tau/\rho}(y) \tag{1.55a}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x,y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\lambda/\tau}(x) s_{\overline{\sigma/\eta\rho};\tau/\eta\rho}(y) \tag{1.55b}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x,y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma\rho};\lambda/\tau\rho}(x) s_{\overline{\sigma/\eta};\tau/\mu}(y) \tag{1.55c}$$

Proof.

$$\begin{aligned}
s_{\overline{\nu/\eta};\lambda/\mu}(x,y) &\stackrel{(1.54a)}{=} \sum_{\zeta} (-1)^{|\zeta|} s_{\nu/\eta\zeta'}(\overline{x},\overline{y}) s_{\lambda/\mu\zeta}(x,y) \\
&\stackrel{(1.38)}{=} \sum_{\zeta,\varphi,\psi} (-1)^{|\zeta|} c^{\varphi}_{\eta\zeta'} c^{\psi}_{\mu\zeta} s_{\nu/\varphi}(\overline{x},\overline{y}) s_{\lambda/\psi}(x,y) \qquad (*) \\
&\stackrel{(1.44')}{=} \sum_{\zeta,\varphi,\psi} \sum_{\sigma,\tau} (-1)^{|\zeta|} c^{\varphi}_{\eta\zeta'} c^{\psi}_{\mu\zeta} s_{\nu/\varphi\sigma}(\overline{x}) s_{\lambda/\psi\tau}(x) s_{\sigma}(\overline{y}) s_{\tau}(y) \\
&\stackrel{(1.38)}{=} \sum_{\sigma,\tau} \left(\sum_{\zeta} (-1)^{|\zeta|} s_{\nu/\eta\zeta'\sigma}(\overline{x}) s_{\lambda/\mu\zeta\tau}(x) \right) s_{\sigma}(\overline{y}) s_{\tau}(y)
\end{aligned}$$

Applying (1.54a) and (1.48) proves (1.55a). Replacing $\sigma\eta$ resp. $\mu\tau$ by φ resp. ψ provides (1.55b). Starting from $(*)$ and applying (1.44) we have that

$$\begin{aligned}
s_{\overline{\nu/\eta};\lambda/\mu}(x,y) &= \sum_{\zeta,\varphi,\psi} \sum_{\sigma,\tau} (-1)^{|\zeta|} c^{\varphi}_{\eta\zeta'} c^{\psi}_{\mu\zeta} s_{\nu/\sigma}(\overline{x}) s_{\lambda/\tau}(x) s_{\sigma/\varphi}(\overline{y}) s_{\tau/\psi}(y) \\
&\stackrel{(1.38)}{=} \sum_{\sigma,\tau} s_{\nu/\sigma}(\overline{x}) s_{\lambda/\tau}(x) \left(\sum_{\zeta} (-1)^{|\zeta|} s_{\sigma/\eta\zeta'}(\overline{y}) s_{\tau/\mu\zeta}(y) \right)
\end{aligned}$$

Finally, applying (1.54a) and (1.48) proves (1.55c). □

1.6 Appendix: Formulary

In this appendix we want to gather all the definitions and formulas derived in this chapter. That way, we provide an overview of the different formulas for symmetric functions indexed by a partition or a composite partition. The numbering used in this section is the same numbering as used in the text. That way, the numbers will not increase as the formulas are rearranged. On the other hand, the context or proof can easily be found in the text and we maintain an overview.

$$\sum_{\sigma} c^{\lambda}_{\rho\sigma} c^{\sigma}_{\mu\nu} = \sum_{\eta} c^{\lambda}_{\mu\eta} c^{\eta}_{\nu\rho} = \sum_{\tau} c^{\lambda}_{\nu\tau} c^{\tau}_{\mu\rho}. \tag{1.37}$$

1.6.1 Symmetric Schur functions indexed by a partition λ

$$s_\lambda(x) = s_\lambda(x_1,\dots,x_m) = \frac{a_{\lambda+\delta}}{a_\delta}. \tag{1.10}$$

$$s_\lambda(x) = \sum_T x^T \qquad T \text{ a (column strict) tableau of shape } \lambda. \tag{1.12}$$

$$s_{\lambda/\mu}(x) = \sum_T x^T \qquad T \text{ a (column strict) tableau of shape } \lambda - \mu. \tag{1.13}$$

$$s_\lambda(x) = \det\Big(h_{\lambda_i - i + j}(x)\Big)_{1\le i,j\le \ell(\lambda)} \tag{1.14}$$

$$s_\lambda(x) = \det\Big(e_{\lambda'_i - i + j}(x)\Big)_{1\le i,j\le \ell(\lambda')}. \tag{1.15}$$

$$s_{\lambda/\mu}(x) = \det\Big(h_{\lambda_i - \mu_j - i + j}(x)\Big)_{1\le i,j\le \ell(\lambda)} \tag{1.17}$$

$$s_{\lambda/\mu}(x) = \det\Big(e_{\lambda'_i - \mu'_j - i + j}(x)\Big)_{1\le i,j\le \ell(\lambda')}. \tag{1.18}$$

$$s_{(n)}(x) = h_n(x) \qquad \text{and} \qquad s_{(1^n)}(x) = e_n(x). \tag{1.16}$$

$$s_\lambda(x) = \sum_{\mu\le\lambda} K_{\lambda\mu} m_\mu(x) \qquad \text{with } K_{\lambda\mu} \text{ the Kostka numbers (see (1.20))} \tag{1.19}$$

$$E(t) = \sum_{r\ge 0} e_r t^r = \prod_{i=1}^{n} (1 + x_i t) \tag{1.25}$$

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i=1}^{n} \frac{1}{(1 - x_i t)} \tag{1.26}$$

$$P(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)} \quad \text{or} \quad P(-t) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)} \tag{1.29}$$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} m_\lambda(x) h_\lambda(y) = \sum_{\lambda} h_\lambda(x) m_\lambda(y) \tag{1.30}$$

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} m_\lambda(x) e_\lambda(y) = \sum_{\lambda} e_\lambda(x) m_\lambda(y) \tag{1.30'}$$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y) \tag{1.31}$$

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y) \tag{1.31'}$$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} z_\lambda^{-1} p_\lambda(x) p_\lambda(y) \tag{1.32}$$

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} \varepsilon_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) \tag{1.32'}$$

with $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.

$$s_\mu(x) s_\nu(x) = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_\lambda(x). \tag{1.33}$$

$$s_\mu(x) h_r(x) = \sum_{\lambda} s_\lambda(x) \qquad \lambda - \mu \text{ a horizontal } r\text{-strip.} \tag{1.34}$$

$$s_\mu(x) e_r(x) = \sum_{\lambda} s_\lambda(x) \qquad \lambda - \mu \text{ a vertical } r\text{-strip.} \tag{1.35}$$

$$s_\mu(x) h_\nu(x) = \sum_{\lambda} K_{\lambda-\mu,\nu} s_\lambda(x) \qquad \text{with } |\nu| = |\lambda - \mu|. \tag{1.36}$$

$$s_{\lambda/\mu}(x) = \sum_{\nu} c_{\mu\nu}^{\lambda} s_\nu(x). \tag{1.38}$$

$$s_{(\lambda/\mu)/\nu}(x) = \sum_\eta c^\lambda_{\mu\eta} s_{\eta/\nu}(x). \tag{1.39}$$

$$s_{\lambda/(\mu\nu)}(x) = \sum_\eta c^\eta_{\mu\nu} s_{\lambda/\eta}(x). \tag{1.42}$$

$$s_{(\lambda/\mu)/\nu}(x) = s_{\lambda/(\mu\nu)}(x) = s_{(\lambda/\nu)/\mu}(x). \tag{1.41}$$

$$s_\lambda(x,y) = \sum_\mu s_{\lambda/\mu}(x) s_\mu(y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(y) s_\nu(x) \tag{1.43}$$

$$s_{\lambda/\mu}(x,y) = \sum_\nu s_{\lambda/\nu}(x) s_{\nu/\mu}(y) \qquad \text{with } \lambda \supset \nu \supset \mu. \tag{1.44}$$

$$s_{\lambda/\mu}(x,y) = \sum_\eta s_{\lambda/\mu\eta}(x) s_\eta(y) \tag{1.44'}$$

1.6.2 Symmetric Schur functions indexed by a composite partition $\bar{\nu};\mu$

$$s_{\bar{\nu};\mu}(x) = \left(\prod_{i=1}^m x_i^{-\nu_1}\right) s_\lambda(x) \quad \text{with } \lambda = (\mu_1+\nu_1, \mu_2+\nu_1, \ldots, -\nu_2+\nu_1, 0). \tag{1.56}$$

$$s_{\overline{\nu};\mu}(x) = \det\left(\begin{array}{c|c} \dot{e}_{\nu'_l+k-l}(x) & e_{\mu'_j-k+j-1}(x) \\ \hline \dot{e}_{\nu'_l-i+l-1}(x) & e_{\mu'_j+i-j}(x) \end{array}\right), \tag{1.22}$$

$$s_{\overline{\nu};\mu}(x) = \det\left(\begin{array}{c|c} \dot{h}_{\nu_l+k-l}(x) & h_{\mu_j-k+j-1}(x) \\ \hline \dot{h}_{\nu_l-i+l-1}(x) & h_{\mu_j+i-j}(x) \end{array}\right) \tag{1.23}$$

where the indices i, j, k resp. l run from top to bottom, from left to right, from bottom to top resp. from right to left.

$$s_{\bar{\nu};\mu}(x) = (-1)^{c+\bar{c}+1} s_{\overline{\nu-h};\mu-h}(x) \qquad \text{with } h = \ell(\mu)+\ell(\nu)-m-1. \tag{1.24}$$

$$s_{\overline{\nu};\mu} = \sum_{T^{\overline{\nu};\mu}} x^{T^{\overline{\nu};\mu}} \qquad \text{with } T^{\overline{\nu};\mu} \text{ a standard composite Young tableau.} \tag{1.45}$$

$$s_{\bar{\nu};\mu}(x) = \sum_{\zeta}(-1)^{|\zeta|} s_{\mu/\zeta}(x) s_{\nu/\zeta'}(\overline{x}) \tag{1.47}$$

$$s_{\overline{\lambda/\mu};\nu/\eta}(x) = \sum_{\zeta,\xi} c^{\lambda}_{\zeta\mu} c^{\nu}_{\xi\eta} s_{\bar{\zeta};\xi}. \tag{1.49}$$

$$s_{\overline{\lambda/\mu\nu};\eta/\kappa\tau}(x) = \sum_{\zeta,\xi} c^{\zeta}_{\mu\nu} c^{\xi}_{\kappa\tau} s_{\overline{\lambda/\zeta};\eta/\xi}. \tag{1.51}$$

$$s_{\nu}(\overline{x}) s_{\mu}(x) = \sum_{\zeta} s_{\overline{\nu/\zeta};\mu/\zeta}(x) \tag{1.48}$$

$$s_{\lambda/\mu}(x) s_{\eta/\nu}(\overline{x}) = \sum_{\rho} s_{\overline{\eta/\nu\rho};\lambda/\mu\rho}(x) \tag{1.52}$$

$$s_{\bar{\nu};\mu}(x) = \sum_{\zeta,\rho}(-1)^{|\zeta|} s_{\overline{\nu/\zeta'\rho};\mu/\zeta\rho}(x) \tag{1.53}$$

$$s_{\overline{\nu/\mu};\lambda/\tau}(x) = \sum_{\zeta}(-1)^{|\zeta|} s_{\nu/\mu\zeta'}(\overline{x}) s_{\lambda/\tau\zeta}(x) \tag{1.54a}$$

$$s_{\overline{\nu/\mu};\lambda/\tau}(x) = \sum_{\zeta,\rho}(-1)^{|\zeta|} s_{\overline{\nu/\mu\zeta'\rho};\lambda/\tau\zeta\rho}(x) \tag{1.54b}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x,y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{\sigma/\rho};\tau/\rho}(y) \tag{1.55a}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x,y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\lambda/\tau}(x) s_{\overline{\sigma/\eta\rho};\tau/\eta\rho}(y) \tag{1.55b}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x,y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma\rho};\lambda/\tau\rho}(x) s_{\overline{\sigma/\eta};\tau/\mu}(y) \tag{1.55c}$$

Chapter 2

Supersymmetric Schur Functions

Where Chapter 1 was dedicated to symmetric functions, we will give a supersymmetric analogue in this second chapter. Essentially, this chapter contains three parts. In the first section, we define the supersymmetric functions indexed by a partition λ. We introduce the elementary and complete supersymmetric functions as well as the supersymmetric Schur functions. Apart from the relation between those functions we also introduce the notion of a supertableau. The second section gives a summary of all the bases for the ring of supersymmetric functions. Next to the known bases (the elementary, complete supersymmetric functions and the power sums), we define the monomial and the so-called forgotten supersymmetric functions. We prove that those functions are indeed supersymmetric and that the relations between the different bases, in terms of the transition matrices, are still valid and the same as in the symmetric case. To end this section, we prove that the bases have similar generation functions as the symmetric bases. The third part of this chapter deals with supersymmetric Schur functions indexed by a composite partition. After giving the definition and some properties equivalent with the properties of composite symmetric functions, we give a definition of composite supertableaux and introduce a new formula for the composite supersymmetric Schur functions by means of those composite supertableaux. The definitions and properties of the supersymmetric Schur functions deduced in this chapter are summarized in the last section.

2.1 Supersymmetric functions indexed by λ

Let $x = x^{(m)} = (x_1, x_2, \ldots, x_m)$ and $y = y^{(n)} = (y_1, y_2, \ldots, y_n)$ be two sets of independent variables. A polynomial $p(x/y)$ is said to be DOUBLY SYMMETRIC if it is symmetric in x and y separately. The ring of all doubly symmetric polynomials is $\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n]^{S_m \times S_n}$. A function $f(x/y)$ satisfies the cancellation property if the result of substituting $x_m = t$ and $y_n = -t$ in $f(x/y)$ is given by $f(x^{(m-1)}/y^{(n-1)})$. Then, a function $f(x/y)$ is said to be SUPERSYMMETRIC if it is doubly symmetric and if the cancellation property is fulfilled. So, the supersymmetric functions form a subring $\Lambda_{m|n}$ of $\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n]^{S_m \times S_n}$.

2.1.1 Elementary and complete supersymmetric functions

Let $x = x^{(m)} = (x_1, x_2, \ldots, x_m)$ and $y = y^{(n)} = (y_1, y_2, \ldots, y_n)$ be two sets of independent variables. The generating function $E(t)$ in formula (1.25) can be generalized [46, §I.3, Exercise 23] to a rational function[1] of t,

$$E_{x/y}(t) = \sum_{r \geq 0} e_r(x/y) t^r = \frac{\prod_{i=1}^{m}(1 + x_i t)}{\prod_{j=1}^{n}(1 - y_j t)}. \tag{2.1}$$

Thus, the ELEMENTARY SUPERSYMMETRIC FUNCTIONS can be written in terms of the elementary symmetric and the complete symmetric functions:

$$e_r(x/y) = \sum_{k=0}^{r} e_k(x) h_{r-k}(y). \tag{2.2}$$

Applying the involution ω (1.5) on (2.2) provides the COMPLETE SUPERSYMMETRIC FUNCTIONS

$$h_r(x/y) = \sum_{k=0}^{r} h_k(x) e_{r-k}(y), \tag{2.3}$$

which are generated by

$$H_{x/y}(t) = \sum_{r \geq 0} h_r(x/y) t^r = \frac{\prod_{j=1}^{n}(1 + y_j t)}{\prod_{i=1}^{m}(1 - x_i t)}. \tag{2.4}$$

From (2.1) and (2.4) follows that the symmetric equality (1.27) is still valid in the supersymmetric case, namely:

$$E_{x/y}(-t) H_{x/y}(t) = 1 \tag{2.5}$$

[1] Remark that our definition is slightly different from the definition of Macdonald's e_r^{Mac}: we have a minus sign in the denominator, such that $e_r(x/y) = e_r^{\mathrm{Mac}}(x/-y)$

Therefore, the supersymmetric e_r and h_r fulfill the same properties as the symmetric e_r and h_r. So, for any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$,

$$e_\lambda(x/y) = \prod_i e_{\lambda_i}(x/y) \qquad \text{and} \qquad h_\lambda(x/y) = \prod_i h_{\lambda_i}(x/y). \tag{2.6}$$

2.1.2 Supersymmetric Schur functions

Definition

Given the elementary supersymmetric functions $e_r(x/y)$, a generalization of Formula (1.15) gives the SUPERSYMMETRIC SCHUR FUNCTIONS $s_\lambda(x/y)$, namely:

$$s_\lambda(x/y) = \det\Big(e_{\lambda'_i - i + j}(x/y)\Big)_{1 \le i,j \le \ell(\lambda')}. \tag{2.7}$$

Macdonald shows that the supersymmetric Schur functions satisfy four properties which also characterize these functions.

- **Homogeneity:** $s_\lambda(x^{(m)}/y^{(n)})$ is a homogeneous function of degree $|\lambda|$.
- **Restriction:** Let $m \ge 1$ (resp. $n \ge 1$). The result of setting $x_m = 0$ (resp. $y_n = 0$) in $s_\lambda(x^{(m)}/y^{(n)})$ is the function $s_\lambda(x^{(m-1)}/y^{(n)})$ (resp. $s_\lambda(x^{(m)}/y^{(n-1)})$).
- **Cancellation:** Let $m, n \ge 1$. The result of substituting $x_m = t$ and $y_n = -t$ in $s_\lambda(x^{(m)}/y^{(n)})$ is the function $s_\lambda(x^{(m-1)}/y^{(n-1)})$.
- **Factorization:** If the partition λ satisfies $\lambda_m \ge n \ge \lambda_{m+1}$, so that λ can be written in the form $\lambda = ((n^m) + \tau) \cup \eta$ (see Figure 2.1), with τ (resp. η') a partition of length $\le m$ (resp. $\le n$), then

$$s_\lambda(x^{(m)}/y^{(n)}) = s_\tau(x^{(m)}) s_{\eta'}(y^{(n)}) \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j). \tag{2.8}$$

Formula (2.8) was first derived by Berele and Regev [10, Theorem 6.20] and will be referred to as the Berele-Regev formula. Stembridge [65] showed that the supersymmetric Schur functions are a $\mathbb{Z}$-basis of $\Lambda_{m|n}$.

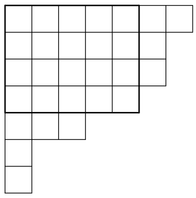
$F^\lambda =$  with $\lambda = ((n^m) + \tau) \cup \eta$, $\tau = (2,1,1)$, $\eta = (3,1,1)$

Figure 2.1: Factorization if $\lambda_m \geq n \geq \lambda_{m+1}$ with $(m|n) = (4|5)$.

Properties

It is also possible to prove that the determinantal expression (1.14) can be extended to supersymmetric Schur functions, giving us $s_\lambda(x/y)$ in terms of the complete supersymmetric functions:

$$s_\lambda(x/y) = \det\Big(h_{\lambda_i - i + j}(x/y)\Big)_{1 \leq i,j \leq \ell(\lambda)} \tag{2.9}$$

Obviously,

$$s_{(r)}(x/y) = h_{(r)}(x/y) \qquad \text{and} \qquad s_{(1^r)}(x/y) = e_{(r)}(x/y). \tag{2.10}$$

It is clear from the definition of $e_r(x/y)$ and $h_r(x/y)$ together with formulas (2.7) and (2.9) that

$$s_\lambda(y/x) = s_{\lambda'}(x/y). \tag{2.11}$$

In addition to (2.7) and (2.9), the supersymmetric Schur functions can be written in terms of the symmetric Schur functions [46, §I.5, Exercise 23], namely

$$s_\lambda(x/y) = \sum_\mu s_\mu(x) s_{(\lambda/\mu)'}(y) = \sum_\mu s_{\lambda/\nu}(x) s_{\nu'}(y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x) s_{\nu'}(y). \tag{2.12}$$

From this equation it is clear that $s_\lambda \equiv 0$ if $\lambda_{m+1} > n$, as $s_\mu(x) \equiv 0$ if $\ell(\mu) > |x| = m$.

It is shown [69] that the supersymmetric Schur functions obey the same outer product rules:

$$s_\mu(x/y) s_\nu(x/y) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x/y). \tag{2.13}$$

This allows us to define the supersymmetric Schur functions labeled by a skew partition:

$$s_{\lambda/\mu}(x/y) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(x/y). \tag{2.14}$$

Analogous to (1.17) and (1.18) these skew supersymmetric Schur functions can also be expressed by a determinantal formula in terms of the elementary and complete supersymmetric functions:

$$s_{\lambda/\mu}(x/y) = \det\Big(e_{\lambda'_i-\mu'_j-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda')}, \tag{2.15}$$

$$s_{\lambda/\mu}(x/y) = \det\Big(h_{\lambda_i-\mu_j-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda)}. \tag{2.16}$$

In combination with (2.12) and (2.14), $s_{\lambda/\mu}(x/y)$ can be written as an expansion of symmetric functions in different ways.

$$s_{\lambda/\mu}(x/y) = \sum_{\nu,\sigma,\tau} c^{\lambda}_{\mu\nu} c^{\nu}_{\sigma\tau} s_\sigma(x) s_{\tau'}(y)$$

Applying (2.14), (2.12) and (1.37) in various ways, this becomes

$$\begin{aligned}
s_{\lambda/\mu}(x/y) &= \sum_{\nu,\sigma} c^{\lambda}_{\mu\nu} s_\sigma(x) s_{(\nu/\sigma)'}(y) = \sum_{\sigma} s_\sigma(x) s_{(\lambda/\mu\sigma)'}(y) && \text{(2.17a)}\\
&= \sum_{\nu,\tau} c^{\lambda}_{\mu\nu} s_{\nu/\tau}(x) s_{\tau'}(y) = \sum_{\tau} s_{\lambda/\mu\tau}(x) s_{\tau'}(y) && \text{(2.17b)}\\
&= \sum_{\rho,\sigma,\tau} c^{\lambda}_{\rho\sigma} c^{\rho}_{\mu\tau} s_\sigma(x) s_{\tau'}(y) = \sum_{\rho} s_{\lambda/\rho}(x) s_{(\rho/\mu)'}(y) && \text{(2.17c)}\\
&= \sum_{\rho,\sigma,\tau} c^{\lambda}_{\rho\tau} c^{\rho}_{\mu\sigma} s_\sigma(x) s_{\tau'}(y) = \sum_{\rho} s_{\rho/\mu}(x) s_{(\lambda/\rho)'}(y) && \text{(2.17d)}
\end{aligned}$$

2.1.3 Supertableaux

A SUPERTABLEAU or BITABLEAU S of type $(m|n)$ and shape $\lambda - \mu$ (where $\mu \subset \lambda$) [46, §I.5, Exercise 23] is a sequence of partitions

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(m+n)} = \lambda$$

such that the skew diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ is a horizontal strip for $1 \le i \le m$ and a vertical strip for $m+1 \le i \le m+n$. Graphically, each square of $\theta^{(i)}$ can be filled by the symbol i, $1 \le i \le m$, and each square of $\theta^{(m+j)}$ by the symbol j', $1 \le j \le n$. With respect to the order $1 < \ldots < m < 1' < \ldots < n'$, there are two conditions on S:

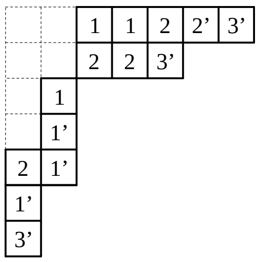
1. the symbols $i \in \{1, \ldots, m\}$ and $j' \in \{1', \ldots, n'\}$ increase in the weak sense from left to right along each row and from top to bottom in each column of S,

2. there is at most one symbol j' in each row, and at most one symbol i in each column.

With each such supertableau S we associate a monomial $(x/y)^S$ obtained by replacing each symbol i (resp. j') by x_i (resp. y_j) and forming the product of all x's and y's. Then,

$$s_{\lambda/\mu}(x/y) = \sum_S (x/y)^S \tag{2.18}$$

summed over all supertableaux S of type $(m|n)$ and shape $\lambda - \mu$. A supertableau and its associated monomial are given for $(m|n) = (2|3)$ and $\lambda - \mu = (7,5,2,2,2,1,1) - (2,2,1,1)$ in Figure 2.2.



$$\longleftrightarrow \quad (x/y)^S = x_1^3\, x_2^4\, y_1^3\, y_2\, y_3^3$$

Figure 2.2: A supertableau S and corresponding term $(x/y)^S$.

2.2 Supersymmetric bases

With reference to the similarity of the elementary and complete symmetric functions and supersymmetric functions, we will define the supersymmetric equivalent of the symmetric power functions, the monomial and the forgotten symmetric functions.

2.2.1 Supersymmetric power functions

In order to define these functions we generalize (1.29):

$$P_{x/y}(t) = \frac{d}{dt} \log H_{x/y}(t) = \frac{H'_{x/y}(t)}{H_{x/y}(t)} = \sum_r p_r(x/y) t^{r-1} \tag{2.19}$$

Then, we have the following property:

Proposition 2.1 *The supersymmetric power functions are given in terms of the symmetric power functions by:*

$$p_r(x/y) = p_r(x) + (-1)^{r-1} p_r(y). \tag{2.20}$$

Proof. Using (2.4) we have that

$$\begin{aligned}
H'_{x/y}(t) &= \frac{d}{dt}\left(\prod_{j=1}^{n}(1+y_jt)\prod_{i=1}^{m}(1-x_it)^{-1}\right) \\
&= \sum_{k=1}^{n} y_k \frac{\prod_{j=1,j\neq k}^{n}(1+y_jt)}{\prod_{i=1}^{m}(1-x_it)} + \sum_{l=1}^{m} \frac{x_l}{(1-x_lt)^2} \frac{\prod_{j=1}^{n}(1+y_jt)}{\prod_{i=1,i\neq l}^{m}(1-x_it)} \\
&= \frac{\prod_j (1+y_jt)}{\prod_i (1-x_it)}\left(\sum_{k=1}^{n}\frac{y_k}{(1+y_kt)} + \sum_{l=1}^{m}\frac{x_l}{(1-x_lt)}\right) \\
&= H_{x/y}(t)\left(\sum_{k=1}^{n}\frac{y_k}{(1+y_kt)} + \sum_{l=1}^{m}\frac{x_l}{(1-x_lt)}\right)
\end{aligned}$$

Expanding the generating function for the symmetric power sums, gives us that

$$P(t) = \sum_{r\geq 1} p_r t^{r-1} = \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} = \sum_{i\geq 1}\frac{x_i}{(1-x_it)}$$

Thus, $P_{x/y}(t) = \sum_r \Big(p_r(x) + (-1)^{r-1}p_r(y)\Big)t^{r-1} = \sum_r p_r(x/y)t^{r-1}$ □

2.2.2 The monomial supersymmetric functions and the forgotten supersymmetric functions

For any given partition λ we can define the MONOMIAL SUPERSYMMETRIC FUNCTION $m_\lambda(x/y)$ and the FORGOTTEN SUPERSYMMETRIC FUNCTION $f_\lambda(x/y)$ in terms of the monomial and forgotten symmetric functions:

$$m_\lambda(x/y) = \sum_{\mu\cup\nu=\lambda} m_\mu(x) f_\nu(y) \tag{2.21}$$

and

$$f_\lambda(x/y) = \omega(m_\lambda(x/y)) = \sum_{\mu\cup\nu=\lambda} f_\mu(x) m_\nu(y) \tag{2.22}$$

where ω is the involution defined in (1.5). We shall prove that these functions are indeed supersymmetric and that the connections between the different bases, the transition matrices, are the same matrices as for the symmetric bases.

Proposition 2.2 *For any $r \geq 0$,*

$$e_r(x/y) = m_{(1^r)}(x/y), \tag{2.23}$$

and

$$h_r(x/y) = \sum_{\lambda} m_\lambda(x/y), \quad \textit{with } |\lambda| = r \tag{2.24}$$

Proof. Both expressions follow from the definitions of the different supersymmetric functions.

$$\begin{aligned} m_{(1^r)}(x/y) &= \sum_{s+t=r} m_{(1^s)}(x) f_{(1^t)}(y) = \sum_{s+t=r} m_{(1^s)}(x)\omega(m_{(1^t)}(y)) \\ &= \sum_{s=0}^{r} e_{(s)}(x)\omega(e_{(r-s)}(y)) = \sum_{s=0}^{r} e_{(s)}(x) h_{(r-s)}(y) \\ &= e_r(x/y) \end{aligned}$$

The second expression (2.24) can be deduced in a similar way:

$$\begin{aligned} \sum_{\lambda,|\lambda|=r} m_\lambda(x/y) &= \sum_{\lambda,|\lambda|=r} \sum_{\mu\cup\nu=\lambda} m_\mu(x) f_\nu(y) \\ &= \sum_{\mu}\sum_{\nu} m_\mu(x) f_\nu(y) \qquad (\text{with } |\mu|+|\nu|=r) \\ &= \sum_{\mu} m_\mu(x)\omega\left(\sum_{\nu,|\nu|=r-|\mu|} m_\nu(y)\right) = \sum_{\mu} m_\mu(x)\omega(h_{r-|\mu|}(y)) \\ &= \sum_{\mu} m_\mu(x) e_{r-|\mu|}(y) = \sum_{k=0}^{r}\left(\sum_{\substack{\mu \\ |\mu|=k}} m_\mu(x)\right) e_{r-k}(y) \\ &= \sum_{k=0}^{r} h_k(x) e_{r-k}(y) = h_r(x/y) \end{aligned}$$

□

As $m_r(x/y) = m_{(r)}(x) + \omega(m_{(r)}(y)) = p_{(r)}(x) + (-1)^{r-1}p_{(r)}(y)$ the supersymmetric equivalence of (1.4) is valid, namely:

$$p_r(x/y) = m_r(x/y). \tag{2.25}$$

In order to prove the connection between the supersymmetric Schur functions and the monomial supersymmetric functions, we need the following lemma.

Lemma 2.3 *Let λ, φ, χ be partitions and $\xi = \varphi \cup \chi$ then*

$$K_{\lambda\xi} = \sum_{\mu,\nu} c^{\lambda}_{\mu\nu} K_{\mu\varphi} K_{\nu\chi} \tag{2.26}$$

where $K_{\lambda\mu}$ are the Kostka numbers and $c^{\lambda}_{\mu\nu}$ the Littlewood-Richardson coefficients.

Proof. As $h_\xi = h_\varphi h_\chi$ and taking into account Table 1.1 then, by (1.33),

$$\begin{aligned} \sum_{\lambda} K_{\lambda\xi} s_\lambda(x) &= \sum_{\mu,\nu} K_{\mu\varphi} K_{\nu\chi} s_\mu(x) s_\nu(x) = \sum_{\mu,\nu,\lambda} K_{\mu\varphi} K_{\nu\chi} c^{\lambda}_{\mu\nu} s_\lambda(x) \\ &= \sum_{\lambda} \left(\sum_{\mu,\nu} c^{\lambda}_{\mu\nu} K_{\mu\varphi} K_{\nu\chi} \right) s_\lambda(x) \end{aligned}$$

□

Relying on this lemma, we have the following proposition.

Proposition 2.4 *Let λ be a partition, then*

$$s_\lambda(x/y) = \sum_{\xi} K_{\lambda\xi} m_\xi(x/y). \tag{2.27}$$

Proof. The right hand side of (2.27) is equal to

$$\sum_{\xi} K_{\lambda\xi} \left(\sum_{\varphi\cup\chi=\xi} m_\varphi(x) f_\chi(y) \right) = \sum_{\varphi,\chi} K_{\lambda\xi} m_\varphi(x) f_\chi(y) \text{ with } \varphi \cup \chi = \xi$$

The left hand side of (2.27) is equal to (see §1.5.1):

$$\sum_{\mu,\nu} c^{\lambda}_{\mu\nu} s_\mu(x) s_{\nu'}(y) \quad = \quad \sum_{\mu,\nu} c^{\lambda}_{\mu\nu} \left(\sum_{\varphi} K_{\mu\varphi} m_\varphi(x) \right) \left(\sum_{\chi} (JK)_{\nu'\chi} f_\chi(y) \right)$$

As $(JK)_{\nu'\chi} = \sum_\psi J_{\nu'\psi}K_{\psi\chi} = K_{\nu\chi}$, this becomes

$$\begin{aligned}\sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x)s_{\nu'}(y) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}\Big(\sum_\varphi K_{\mu\varphi}m_\varphi(x)\Big)\Big(\sum_\chi K_{\nu\chi}f_\chi(y)\Big)\\ &= \sum_{\varphi,\chi}\Big(\sum_{\mu,\nu} c^\lambda_{\mu\nu}K_{\mu\varphi}K_{\nu\chi}\Big)m_\varphi(x)f_\chi(y)\end{aligned}$$

Using (2.26), the left hand side equals

$$\sum_{\varphi,\chi} K_{\lambda\xi}m_\varphi(x)f_\chi(y) \text{ with } \varphi\cup\chi = \xi.$$

□

In a similar way, we can prove that, given a partition λ,

$$s_\lambda(x/y) = \sum_\xi (JK)_{\lambda\xi}f_\xi(x/y).$$

Therefore, the relations, in terms of the transition matrices, are still valid for the supersymmetric monomial functions, the supersymmetric forgotten functions and the supersymmetric Schur functions.

This also implies that the supersymmetric monomial (resp. forgotten) functions as defined in (2.21) and (2.22) are indeed supersymmetric.

2.2.3 Generating functions

Suppose $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$, $z = (z_1, z_2, \ldots)$, $u = (u_1, u_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ are finite or infinite sequences of independent variables.

In [69] the generating function for the $s_\lambda(x/y)$ is given by

$$G(x, y, z) = \frac{\prod_{i,a}(1 + y_i z_a)}{\prod_{j,b}(1 - x_j z_b)} \tag{2.28}$$

Using (1.31) and (1.31'), Formula (2.28) is equal to

$$\begin{aligned}&\Big(\sum_\nu s_{\nu'}(y)s_\nu(z)\Big)\Big(\sum_\mu s_\mu(x)s_\mu(z)\Big) = \sum_{\nu,\mu} s_\mu(x)s_{\nu'}(y)\Big(s_\mu(z)s_\nu(z)\Big)\\ &\overset{(1.33)}{=} \sum_{\nu,\mu} s_\mu(x)s_{\nu'}(y)\sum_\lambda c^\lambda_{\nu\mu}s_\lambda(z) = \sum_\lambda\Big(\sum_{\nu,\mu} c^\lambda_{\nu\mu}s_\mu(x)s_{\nu'}(y)\Big)s_\lambda(z)\end{aligned}$$

Applying (2.12), we have that

$$G(x,y,z) = \frac{\prod_{i,a}(1+y_i z_a)}{\prod_{j,b}(1-x_j z_b)} = \sum_\lambda s_\lambda(x/y)s_\lambda(z) \tag{2.29}$$

Then, it is easy to prove that:

$$\sum_\lambda s_\lambda(x/y)s_\lambda(u/v) = \frac{\prod_{a,j}(1+u_a y_j)\prod_{b,i}(1+v_b x_i)}{\prod_{a,i}(1-u_a x_i)\prod_{b,j}(1-v_b y_j)} \tag{2.30}$$

Proof. Applying (2.29) twice and taking into account that $s_\mu(y/x) = s_{\mu'}(x/y)$, the right hand side of (2.30) equals:

$$\sum_\lambda s_\lambda(x/y)s_\lambda(u)\sum_\mu s_{\mu'}(y/x)s_{\mu'}(v) = \sum_\lambda s_\lambda(x/y)s_\lambda(u)\sum_\mu s_\mu(x/y)s_{\mu'}(v)$$

This gives rise to the left hand side of (2.30), using (2.12) and (2.13),

$$\sum_{\lambda,\mu,\nu} c^\nu_{\lambda\mu} s_\nu(x/y)s_\lambda(u)s_{\mu'}(v) = \sum_\nu s_\nu(x/y)s_\nu(u/v)$$

□

Using (2.30), we can also prove that

$$\sum_\lambda h_\lambda(x/y)m_\lambda(u/v) = \frac{\prod_{a,j}(1+u_a y_j)\prod_{b,i}(1+v_b x_i)}{\prod_{a,i}(1-u_a x_i)\prod_{b,j}(1-v_b y_j)} \tag{2.31}$$

Proof. Using (2.30) and the transition matrices, the left hand side becomes

$$\sum_\lambda s_\lambda(x/y)s_\lambda(u/v) = \sum_\lambda \left(\sum_\alpha K^*_{\lambda\alpha} h_\alpha(x/y)\right)\left(\sum_\beta K_{\lambda\beta} m_\beta(u/v)\right)$$

As $\sum_\lambda K^*_{\lambda\alpha}K_{\lambda\beta} = \sum_\lambda K^{-1}_{\alpha\lambda}K_{\lambda\beta} = \delta_{\alpha\beta}$, with $\delta_{\alpha\beta}$ the Kronecker delta, this yields

$$\sum_{\alpha,\beta} \delta_{\alpha\beta} h_\alpha(x/y)m_\beta(u/v) = \sum_\alpha h_\alpha(x/y)m_\alpha(u/v)$$ □

Similar to Section 1.4, there is a third expansion:

$$\sum_\lambda z_\lambda^{-1} p_\lambda(x/y)p_\lambda(u/v) = \frac{\prod_{a,j}(1+u_a y_j)\prod_{b,i}(1+v_b x_i)}{\prod_{a,i}(1-u_a x_i)\prod_{b,j}(1-v_b y_j)} \tag{2.32}$$

Proof. First of all we prove that

$$p_\lambda(x/y)p_\lambda(u/v) = p_\lambda(z/w) \tag{2.33}$$

where

$$z = \{u_a x_i | a = 1, 2, \ldots; i = 1, 2, \ldots\} \cup \{v_b y_j | b = 1, 2, \ldots; j = 1, 2, \ldots\},$$
$$w = \{u_a y_j | a = 1, 2, \ldots; j = 1, 2, \ldots\} \cup \{v_b x_i | b = 1, 2, \ldots; i = 1, 2, \ldots\}.$$

Since, $p_\lambda(x/y) = \prod_i p_{\lambda_i}(x/y)$, and as the outer product is commutative, it is sufficient to prove this statement for $\lambda = (r)$.

$$\begin{aligned} p_r(x/y)p_r(u/v) &= (p_r(x) + (-1)^{r-1}p_r(y))(p_r(u) + (-1)^{r-1}p_r(v)) \\ &= (p_r(ux) + p_r(vy)) + (-1)^{r-1}(p_r(vx) + p_r(uy)) \\ &= p_r(z) + (-1)^{r-1}p_r(w) = p_r(z/w) \end{aligned}$$

where $p_r(ux)$ is the power sum in the variables ux with

$$ux = \{u_a x_i | a = 1, \ldots, M; i = 1, \ldots, m\}.$$

On the other side, $H_{x/y}(t) = \sum_\lambda z_\lambda^{-1} p_\lambda(x/y) t^{|\lambda|} = \dfrac{\prod_j (1 + y_j t)}{\prod_i (1 - x_i t)}$. So, $H_{z/w}(1) = \sum_\lambda z_\lambda^{-1} p_\lambda(z/w) = \dfrac{\prod_j (1 + w_j)}{\prod_i (1 - z_i)}$, which proves formula (2.32), using (2.33). □

2.3 Supersymmetric Schur functions indexed by $\bar{\nu}; \mu$

2.3.1 Definition and properties

Given a composite partition $\bar{\nu}; \mu$ (§ 1.1.3), one can define the corresponding supersymmetric Schur function, also called supersymmetric S-function [4, 5]. Let $\dot{h}_r(x/y) = h_r(\overline{x}/\overline{y})$, where $\overline{x}_i = x_i^{-1}$ and $\overline{y}_j = y_j^{-1}$. Then:

$$s_{\bar{\nu};\mu}(x/y) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l + k - l}(x/y) & h_{\mu_j - k - j + 1}(x/y) \\ \hline \dot{h}_{\nu_l - i - l + 1}(x/y) & h_{\mu_j + i - j}(x/y) \end{array} \right) \tag{2.34}$$

where i, j, k resp. l runs from top to bottom, from left to right, from bottom to top, resp. from right to left. For $\nu = 0$, this supersymmetric S-function is the so-called supersymmetric Schur function as defined in (2.9). Obviously, Formula (2.34) is the supersymmetric equivalent of (1.23). For a genuine composite partition, the functions $s_{\bar{\nu};\mu}(x/y)$ have many properties similar to ordinary Schur functions [17, 18, 28, 29, 41]. For instance, it is shown [18] that from this determinantal definition the following identity can be derived:

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\zeta} (-1)^{|\zeta|} s_{\mu/\zeta}(x/y) s_{\nu/\zeta'}(\overline{x}/\overline{y}) \tag{2.35}$$

Using this formula we can prove the following identities:

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x) s_{\overline{\sigma'/\rho};\tau'/\rho}(y) \tag{2.36a}$$

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) \tag{2.36b}$$

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\rho,\varphi,\psi} s_{\overline{\nu/\varphi\rho};\mu/\psi\rho}(x) s_{\overline{\varphi'};\psi'}(y) \tag{2.36c}$$

Proof. As the supersymmetric functions occurring in the right hand side of (2.35) are indexed by ordinary skew partitions, we can apply the properties of Chapter 1.

$$\begin{aligned}
s_{\bar{\nu};\mu}(x/y) &\overset{(2.35)}{=} \sum_{\zeta} (-1)^{|\zeta|} s_{\mu/\zeta}(x/y) s_{\nu/\zeta'}(\overline{x}/\overline{y}) \\
&\overset{(2.17b)}{=} \sum_{\zeta,\tau,\sigma} (-1)^{|\zeta|} s_{\mu/\zeta\tau}(x) s_{\tau'}(y) s_{\nu/\zeta'\sigma}(\overline{x}) s_{\sigma'}(\overline{y}) \\
&= \sum_{\zeta,\tau,\sigma} (-1)^{|\zeta|} \big(s_{\mu/\zeta\tau}(x) s_{\nu/\zeta'\sigma}(\overline{x})\big)\big(s_{\tau'}(y) s_{\sigma'}(\overline{y})\big) \\
&\overset{(1.52)}{=} \sum_{\zeta,\tau,\sigma} (-1)^{|\zeta|} \Big(\sum_{\rho} s_{\overline{\nu/\zeta'\sigma\rho};\mu/\zeta\tau\rho}(x)\Big)\Big(\sum_{\rho} s_{\overline{\sigma'/\rho};\tau'/\rho}(y)\Big) \\
&= \sum_{\tau,\sigma,\rho} \left(\sum_{\zeta,\rho} (-1)^{|\zeta|} s_{\overline{\nu/\zeta'\sigma\rho};\mu/\zeta\tau\rho}(x)\right) s_{\overline{\sigma'/\rho};\tau'/\rho}(y)
\end{aligned}$$

Applying (1.54b) provides (2.36a). Replacing ρ by ρ' and observing that $\sigma'/\rho' = (\sigma/\rho)'$, Formula (2.36b) follows immediately. Replacing σ/ρ resp. τ/ρ by φ resp. ψ, proves the last expression (2.36c). □

By analogy with (1.49), we can define

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\zeta,\xi} c^{\lambda}_{\zeta\mu} c^{\nu}_{\xi\eta} s_{\bar{\xi};\zeta}(x/y). \tag{2.37}$$

The generalization of (2.36b) is then given by

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) \tag{2.38a}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\lambda/\tau}(x) s_{\overline{(\sigma/\eta\rho)'};(\tau/\mu\rho)'}(y) \tag{2.38b}$$

Proof. These formulas can easily be deduced from (2.36b) by applying (2.37), (1.49), (1.51) and (1.37). □

We can also prove that

$$s_{\bar{\nu};\mu}(x/y) = s_{\overline{\nu'};\mu'}(y/x). \tag{2.39}$$

Proof.

$$\begin{aligned} s_{\bar{\nu};\mu}(x/y) &\overset{(2.36b)}{=} \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) \\ &\overset{(1.49)}{=} \sum_{\rho,\sigma,\tau} \sum_{\alpha,\beta} \sum_{\gamma,\delta} c^{\nu}_{\alpha\sigma} c^{\mu}_{\beta\tau} c^{\sigma}_{\rho\gamma} c^{\tau}_{\rho\delta} s_{\overline{\alpha};\beta}(x) s_{\overline{\gamma'};\delta'}(y) \\ &\overset{(1.37)}{=} \sum_{\alpha,\beta} \sum_{\gamma,\delta} \sum_{\rho,\zeta,\xi} c^{\nu}_{\gamma\zeta} c^{\zeta}_{\alpha\rho} c^{\mu}_{\xi\delta} c^{\xi}_{\rho\beta} s_{\overline{\alpha};\beta}(x) s_{\overline{\gamma'};\delta'}(y) \\ &\overset{(1.49)}{=} \sum_{\rho,\zeta,\xi} s_{\overline{\zeta/\rho};\xi/\rho}(x) s_{\overline{(\nu/\zeta)'};(\mu/\xi)'}(y) \\ &= \sum_{\rho,\zeta,\xi} s_{\overline{\nu'/\zeta'};\mu'/\xi'}(y) s_{\overline{(\zeta'/\rho')'};(\xi'/\rho')'}(x) \end{aligned}$$

It is clear that in the last expression the sum can be taken over all ρ', ζ', ξ' instead of ρ, ζ, ξ without changing the expression. Applying (2.36b) provides (2.39). □

2.3.2 $(m|n)$-standard composite partitions and supertableaux

Definition 2.5 *A composite partition is said to be an* $(m|n)$-STANDARD COMPOSITE PARTITION *if and only if there exist* J *and* L *such that*

$$J = \min\{j \,|\, \mu'_{j+1} + \nu'_{n-j+1} \le m\} \quad \textit{with} \quad 0 \le J \le n, \tag{2.40}$$

$$L = \min\{l \,|\, \mu_{m-l+1} + \nu_{l+1} \le n\} \quad \textit{with} \quad 0 \le L \le m, \tag{2.41}$$

In such a case let $I = m - L$ *and* $K = n - J$.

Due to the definitions of J and L as minimum values of j and l satisfying the given conditions, it follows that $\mu'_J + \nu'_{n-J} \geq \mu'_{(J-1)+1} + \nu'_{n-(J-1)+1} > m$ and $\mu_{m-L} + \nu_L \geq \mu_{m-(L-1)+1} + \nu_{(L-1)+1} > n$, so that $\mu'_J + \nu'_K > m$ and $\mu_I + \nu_L > n$. Thus, for each $(m|n)$-standard composite partition there exists a composite Young diagram that fits inside a cross of arm width m and leg width n [17, 40] as shown in Figure 2.3. The corresponding composite diagram is also called an $(m|n)$-STANDARD COMPOSITE DIAGRAM.

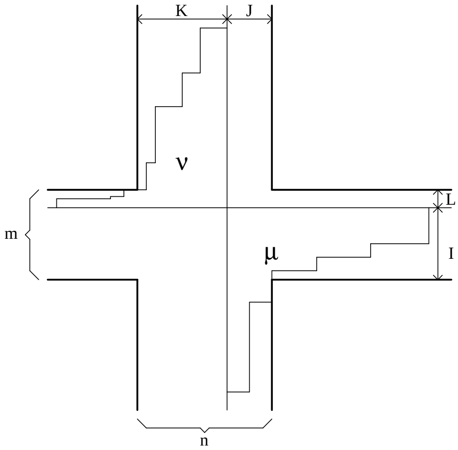


Figure 2.3: $(m|n)$-standard composite partition

In order to define $(m|n)$-standard composite supertableaux, we need another property of composite supersymmetric Schur functions.

Lemma 2.6 *Suppose $y = y^{(n)} = (y_1, \ldots, y_n)$. Let ν and μ be partitions, then*

$$s_{\bar{\nu};\mu}(x/y) = \sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)})y_n^{a-b}. \tag{2.42}$$

Proof. We prove this statement using the formulas given earlier in this chapter.

$$\begin{aligned}
s_{\bar\nu;\mu}(x/y) &\overset{(2.36c)}{=} \sum_{\varphi,\psi,\rho} s_{\overline{\nu/(\rho\varphi)};\mu/(\rho\psi)}(x) s_{\overline{\varphi'};\psi'}(y) \\
&\overset{(1.55a)}{=} \sum_{\varphi,\psi,\rho} s_{\overline{\nu/(\rho\varphi)};\mu/(\rho\psi)}(x) \left(\sum_{\eta,\kappa,\tau} s_{\overline{(\varphi/\kappa)'};(\psi/\tau)'}(y^{(n-1)}) s_{\overline{\kappa/\eta'};\tau'/\eta'}(y_n) \right) \\
&= \sum_{\kappa,\tau} \sum_{\varphi,\psi,\rho} s_{\overline{\nu/(\rho\varphi)};\mu/(\rho\psi)}(x) s_{\overline{(\varphi/\kappa)'};(\psi/\tau)'}(y^{(n-1)}) \sum_{\eta} s_{\overline{\kappa'/\eta'};\tau'/\eta'}(y_n) \\
&\overset{(1.48),(2.38b)}{=} \sum_{\kappa,\tau} s_{\overline{\nu/\kappa};\mu/\tau}(x/y^{(n-1)}) s_{\kappa'}(\overline{y_n}) s_{\tau'}(y_n).
\end{aligned}$$

As $s_\lambda(x^{(m)}) = 0$ if $\ell(\lambda) > \left|x^{(m)}\right| = m$, the right hand side equals

$$\sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)}) s_{(b)}(\overline{y_n}) s_{(a)}(y_n) = \sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)}) y_n^{a-b}. \quad \square$$

Thus, isolating y_n indicates that we can separate a vertical strip in both F^μ and F^ν, the diagrams of μ resp. ν. Repeating this construction for all y_j, $j = n, \ldots, 1$, successively, the remaining S-function in the right hand side of (2.42) becomes independent of the variables y. This symmetric Schur function is indexed by a composite partition which is not necessarily m-standard. So, we possibly need the modification rules (1.24) [37]. Let $t \geq 0$ be the number of times we need to apply the modification rules. So, we have three kind of boxes; those corresponding to the separation of the y_j, the boxes removed by the modification rules, and the remaining boxes. We can put j' resp. $\overline{j'}$ in the boxes of the vertical strips in F^μ resp. F^ν corresponding to the separation of y_j. The boxes corresponding to the removal due to the modification rules are filled with a j^* resp. $\overline{j^*}$, $j = 1, \ldots, t$. The remaining boxes are filled with i and $\bar{i}$, $i \in \{1, \ldots, m\}$, in such a way that it becomes a composite Young tableau as defined in §1.5.3. In order to define an $(m|n)$-standard composite supertableau $S^{\bar\nu;\mu}$, let us fix some notations. Let $n(i^*)$ resp. $\overline{n}(\overline{i^*})$ be the number of boxes of F^μ resp. F^ν containing i^* resp. $\overline{i^*}$. Let $r(i)$ resp. $R(i^*)$ ($\overline{r}(\bar{i})$ resp. $\overline{R}(\overline{i^*})$) be the lowest resp. greatest row number of F^μ (resp. F^ν) containing i^* (resp. $\overline{i^*}$). And finally, let $c(i^*)$ resp. $\overline{c}(\overline{i^*})$ be the greatest column number of F^μ resp. F^ν containing i^* resp. $\overline{i^*}$.

A $(m|n)$-STANDARD COMPOSITE SUPERTABLEAU $S^{\bar\nu;\mu}$ is a numbered composite Young diagram $F^{\bar\nu;\mu}$, not necessarily an $(m|n)$-standard composite diagram, formed by inserting positive and negative entries chosen from $M = \{1, \ldots, m, t^*, \ldots, 1^*, 1', \ldots, n'\}$ and $\overline{M} = \{\bar{1}, \ldots, \overline{m}, \overline{t^*}, \ldots, \overline{1^*}, \overline{1'}, \ldots, \overline{n'}\}$ $(t \geq 0)$ into each box of F^μ resp. $F^{\bar\nu}$ in such a way that the following conditions are fulfilled.

- The elements obey an ordering $1 < \ldots < m < t^* < \ldots < 1^* < 1' < \ldots < n'$.
- The entries $i \in \{1,2,\ldots,m\}$ and $\bar{i} \in \{\bar{1},\bar{2},\ldots,\overline{m}\}$ are non-decreasing across rows and strictly increasing down columns; the entries $j' \in \{1',2',\ldots,n'\}$, and $\overline{j'} \in \{\overline{1'},\overline{2'},\ldots,\overline{n'}\}$, are strictly increasing across rows and non-decreasing down columns; the boxes filled by an entry i^* resp. $\overline{i^*}$, $1 \leq i \leq t$, form a connected component starting in the first column of F^μ resp. $F^{\bar{\nu}}$.
- $r(i) + \overline{r}(\bar{i}) \leq i$ for all $i \in \{1,2,\ldots,m\}$.
- $n(i^*) = \overline{n}(\overline{i^*}) = R(i^*) + \overline{R}(\overline{i^*}) - m - 1$.
- $R(i^*) = m + 1 + \overline{c}(\overline{i^*}) - \overline{r}(\overline{i^*})$ and $\overline{R}(\overline{i^*}) = m + 1 + c(i^*) - r(i^*)$.

With each composite supertableau, we can link a rational term in the variables $x = (x_1,\ldots,x_m)$ and $y = (y_1,\ldots,y_n)$. The entries i and $\bar{i}$ are the positive and negative powers of x_i, the entries with a dash j' and $\overline{j'}$ are the positive and negative powers of y_j, and we will not take into consideration the entries with a star as there are as many entries i^* in F^μ as there are $\overline{i^*}$ in F^ν.

This is illustrated in Figure 2.4 for $\bar{\nu};\mu = (\bar{1}^2,\bar{3}^2,\bar{5},\bar{6}^3,\bar{7});(8,6,5^2,4,3^3,2,1^2)$. The term corresponding with the given $(3|2)$-standard composite supertableau is $\frac{x_1^2 y_1^4}{x_3 y_2^2}$. Remark that $\bar{\nu};\mu$ itself is not a $(3|2)$-standard composite partition.

Then, the supersymmetric Schur functions are given by:

$$s_{\bar{\nu};\mu}(x^{(m)}/y^{(n)}) = \sum_{S^{\bar{\nu};\mu}} (-1)^{(\sum_{i=1}^{t}(c(i^*)+\overline{c}(\overline{i^*})+1))(1-\delta_{0t})} (x/y)^{S^{\bar{\nu};\mu}}, \tag{2.43}$$

where the sum is taken over all $(m|n)$-standard composite supertableaux of shape $\bar{\nu};\mu$ and with δ_{0t} the Kronecker delta.

Formula (2.43) is illustrated in Figure 2.5 and Figure 2.7. In those figures the standard composite supertableaux are given together with the corresponding terms of $s_{\bar{\nu};\mu}(x/y)$.

From this formula, we get that

$$s_{(\bar{1},\bar{1});(1,1)}(x^{(2)}/y^{(1)}) = \frac{x_1 x_2}{y_1^2} + \frac{y_1^2}{x_1 x_2} + \frac{x_1}{y_1} + \frac{x_2}{y_1} + \frac{y_1}{x_1} + \frac{y_1}{x_2} + 1.$$

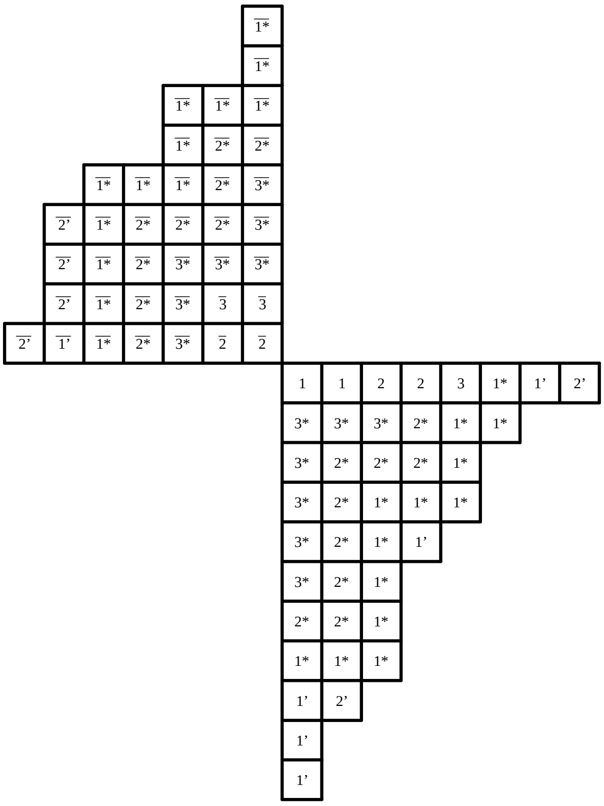


Figure 2.4: $(3|2)$-Standard composite supertableau

Let us check this expansion using Formula (2.36a).

$$
\begin{aligned}
s_{(\overline{1}^2);(1^2)}(x^{(2)}/y^{(1)}) &= s_{(\overline{1},\overline{1});(1,1)}(x)s_{();()}(y) + s_{(\overline{1},\overline{1});(1)}(x)s_{();(1)}(y) \\
&+ s_{(\overline{1},\overline{1});()}(x)s_{();(2)}(y) + s_{(\overline{1});(1,1)}(x)s_{(\overline{1});()}(y) + s_{(\overline{1});(1)}(x)\left(s_{(\overline{1});(1)}(y) + s_{();()}(y)\right) \\
&+ s_{(\overline{1});()}(x)\left(s_{(\overline{1});(2)}(y) + s_{();(1)}(y)\right) + s_{();(1,1)}(x)s_{(\overline{2});()}(y) \\
&+ s_{();(1)}(x)\left(s_{(\overline{2});(1)}(y) + s_{(\overline{1});()}(y)\right) + s_{();()}(x)\left(s_{(\overline{2});(2)}(y) + s_{(\overline{1});(1)}(y) + s_{();()}(y)\right)
\end{aligned}
$$

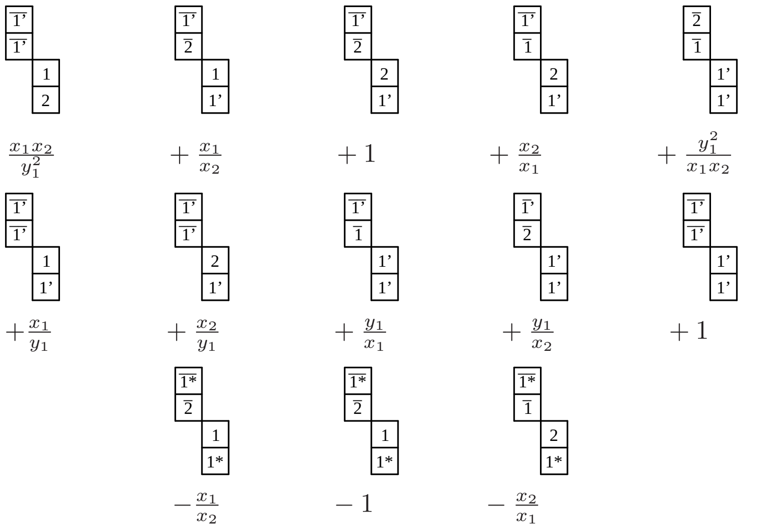


Figure 2.5: $(2|1)$-standard composite supertableaux $S^{\bar{\nu};\mu}$ for $\bar{\nu};\mu = (\bar{1},\bar{1});(1,1)$.

Using the modification rules and substituting $s_{();()} \equiv 1$, this becomes

$$\begin{aligned} s_{(\bar{1}^2);(1^2)}(x/y) =& - s_{(\bar{1});(1)}(x) + 0 + s_{(\bar{1},\bar{1});()}(x)s_{();(2)}(y) + 0 + s_{(\bar{1});(1)}(x) \\ &+ s_{(\bar{1});()}(x)\big(s_{();(1)}(y)\big) + s_{();(1,1)}(x)s_{(\bar{2});()}(y) \\ &+ s_{();(1)}(x)\big(s_{(\bar{1});()}(y)\big) + 1 \end{aligned}$$

thus,

$$\begin{aligned} s_{(\bar{1}^2);(1^2)}(x/y) =& s_{(\bar{1},\bar{1});()}(x)s_{();(2)}(y) + s_{(\bar{1});()}(x)s_{();(1)}(y) + s_{();(1,1)}(x)s_{(\bar{2});()}(y) \\ &+ s_{();(1)}(x)s_{(\bar{1});()}(y) + 1 \end{aligned}$$

which is exactly the same as the expansion gathered from Formula (2.43).

Observe that it is necessary to make the requirements on the order of the entries. For, suppose $j' < i^*$ for all j' and i^*, the following tableau (Figure 2.6) would have been standard, giving an extra term for $s_{\bar{\nu};\mu}(x/y)$.

In the expression (2.43) a minus sign appears in contrast with (1.12), (1.45) and (2.18). The minus sign is needed in applying the modification rules and is necessary as illustrated for $s_{(\bar{1},\bar{1},\bar{1});(2,1)}(x/y)$.

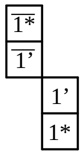


Figure 2.6: Extra tableau under the assumption $j' < i^*$ for all j' and i^*

The composite partition $(\overline{1},\overline{1},\overline{1});(2,1)$ is $(2|1)$-standard but the expansion of $s_{(\overline{1},\overline{1},\overline{1});(2,1)}(x/y)$ contains negative terms.

$$s_{(\overline{1}^3);(2,1)}(x/y) = \frac{x_1^2x_2}{y_1^3} + \frac{x_1x_2^2}{y_1^3} + \frac{x_1^2}{y_1^2} + 2\frac{x_1x_2}{y_1^2} + \frac{x_2^2}{y_1^2} + \frac{x_1}{y_1} + \frac{x_2}{y_1} - \frac{y_1^2}{x_1x_2} - \frac{y_1}{x_1} - \frac{y_1}{x_2}. \tag{2.44}$$

Those terms and their sign are given in Figure 2.7.

We obtain (2.44) using (2.36a):

$$\begin{aligned}
s_{(\overline{1}^3);(2,1)}(x^{(2)}/y^{(1)}) &= s_{(\overline{1},\overline{1},\overline{1});(2,1)}(x)s_{();()}(y) + s_{(\overline{1},\overline{1},\overline{1});(2)}(x)s_{();(1)}(y) \\
&+s_{(\overline{1},\overline{1},\overline{1});(1,1)}(x)s_{();(1)}(y) + s_{(\overline{1},\overline{1},\overline{1});(1)}(x)s_{();(1,1)}(y) + s_{(\overline{1},\overline{1},\overline{1});(1)}(x)s_{();(2)}(y) \\
&+s_{(\overline{1},\overline{1},\overline{1});()}(x)s_{();(2,1)}(y) + s_{(\overline{1},\overline{1});(2,1)}(x)s_{(\overline{1});()}(y) \\
&+s_{(\overline{1},\overline{1});(2)}(x)\big(s_{(\overline{1});(1)}(y) + s_{();()}(y)\big) + s_{(\overline{1},\overline{1});(1,1)}(x)\big(s_{(\overline{1});(1)}(y) + s_{();()}(y)\big) \\
&+s_{(\overline{1},\overline{1});(1)}(x)\big(s_{(\overline{1});(2)}(y) + s_{();(1)}(y)\big) + s_{(\overline{1},\overline{1});(1)}(x)\big(s_{(\overline{1});(1,1)}(y) + s_{();(1)}(y)\big) \\
&+s_{(\overline{1},\overline{1});()}(x)\big(s_{(\overline{1});(2,1)}(y) + s_{();(2)}(y) + s_{();(1,1)}(y)\big) \\
&+s_{(\overline{1});(2,1)}(x)s_{(\overline{2});()}(y) + s_{(\overline{1});(2)}(x)\big(s_{(\overline{2});(1)}(y) + s_{(\overline{1});()}(y)\big) \\
&+s_{(\overline{1});(1,1)}(x)\big(s_{(\overline{2});(1)}(y) + s_{(\overline{1});()}(y)\big) + s_{(\overline{1});(1)}(x)\big(s_{(\overline{2});(1,1)}(y) + s_{(\overline{1});(1)}(y)\big) \\
&+s_{(\overline{1});(1)}(x)\big(s_{(\overline{2});(2)}(y) + s_{(\overline{1});(1)}(y) + s_{();()}(y)\big) \\
&+s_{(\overline{1});()}(x)\big(s_{(\overline{2});(2,1)}(y) + s_{(\overline{1});(2)}(y) + s_{(\overline{1});(1,1)}(y) + s_{();(1)}(y)\big) \\
&+s_{();(2,1)}(x)s_{(\overline{3});()}(y) + s_{();(2)}(x)\big(s_{(\overline{3});(1)}(y) + s_{(\overline{2});()}(y)\big) \\
&+s_{();(1,1)}(x)\big(s_{(\overline{3});(1)}(y) + s_{(\overline{2});()}(y)\big) \\
&+s_{();(1)}(x)\big(s_{(\overline{3});(2)}(y) + s_{(\overline{2});(1)}(y) + s_{(\overline{1});()}(y)\big) \\
&+s_{();(1)}(x)\big(s_{(\overline{3});(1,1)}(y) + s_{(\overline{2});(1)}(y)\big) \\
&+s_{();()}(x)\big(s_{(\overline{3});(2,1)}(y) + s_{(\overline{2});(2)}(y) + s_{(\overline{2});(1,1)}(y) + s_{(\overline{1});(1)}(y)\big)
\end{aligned}$$

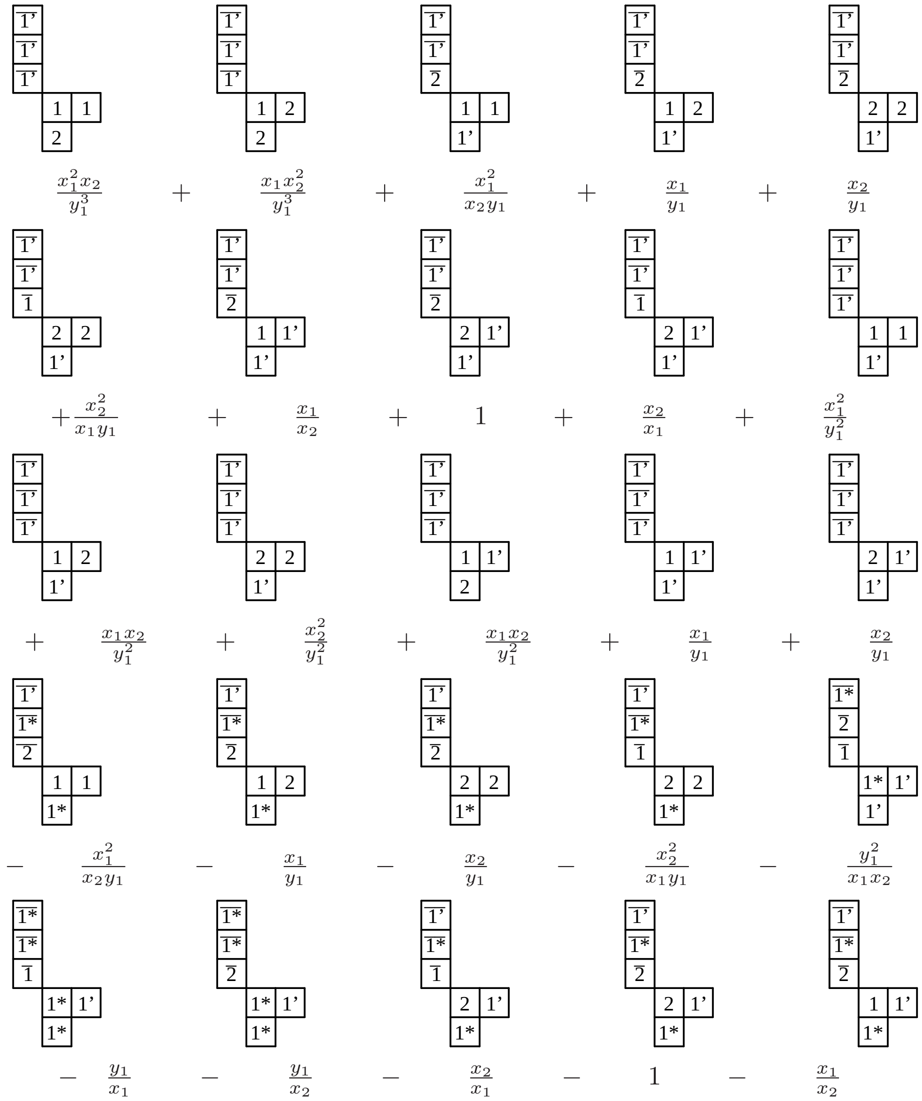


Figure 2.7: Supertableaux for $\bar{\nu};\mu = (\bar{1},\bar{1},\bar{1});(2,1)$ and corresponding terms

After simplification using the modification rules this becomes:

$$\begin{aligned} s_{(\bar{1}^3);(2,1)}(x^{(2)}/y^{(1)}) = &-s_{(\bar{1});()}(x)s_{();(1)}(y) - s_{(\bar{1},\bar{1});()}(x)s_{();(2)}(y) - s_{(\bar{1});(2)}(x)s_{(\bar{1});()}(y) \\ &-s_{(\bar{1});(1)}(x) + s_{(\bar{1});(2)}(x)s_{(\bar{1});()}(y) + s_{(\bar{1});(1)}(x) + s_{();(2,1)}(x)s_{(\bar{3});()}(y) \\ &+s_{();(2)}(x)s_{(\bar{2});()}(y) + s_{();(1,1)}(x)s_{(\bar{2});()}(y) + s_{();(1)}(x)s_{(\bar{1});()}(y) \\ &= -\, s_{(\bar{1});()}(x)s_{();(1)}(y) - s_{(\bar{1},\bar{1});()}(x)s_{();(2)}(y) + s_{();(2,1)}(x)s_{(\bar{3});()}(y) \\ &+s_{();(2)}(x)s_{(\bar{2});()}(y) + s_{();(1,1)}(x)s_{(\bar{2});()}(y) + s_{();(1)}(x)s_{(\bar{1});()}(y) \end{aligned}$$

This expression is equal to (2.44).

2.4 Appendix: Formulary

2.4.1 Supersymmetric functions indexed by a (skew) partition

$$e_r(x/y) = \sum_{k=0}^{r} e_k(x)h_{r-k}(y) \tag{2.2}$$

$$h_r(x/y) = \sum_{k=0}^{r} h_k(x)e_{r-k}(y) \tag{2.3}$$

$$p_r(x/y) = p_r(x) + (-1)^{r-1}p_r(y) \tag{2.20}$$

$$m_\lambda(x/y) = \sum_{\mu\cup\nu=\lambda} m_\mu(x)f_\nu(y) \tag{2.21}$$

$$f_\lambda(x/y) = \omega(m_\lambda(x/y)) = \sum_{\mu\cup\nu=\lambda} f_\mu(x)m_\nu(y) \tag{2.22}$$

$$s_{\lambda/\mu}(x/y) = \sum_\nu c^\lambda_{\mu,\nu} s_\nu(x/y) = \sum_\nu s_{\lambda/\mu\nu}(x)s_{\nu'}(y) \tag{2.14}$$

$$s_{\lambda/\mu}(x/y) = \sum_S (x/y)^S, \quad S \text{ a supertableaux of shape } \lambda-\mu. \tag{2.18}$$

$$s_\lambda(x/y) = \det\Big(e_{\lambda'_i-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda')} \tag{2.7}$$

$$s_\lambda(x/y) = \det\Big(h_{\lambda_i-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda')} \tag{2.9}$$

$$s_{\lambda/\mu}(x/y) = \det\Big(e_{\lambda'_i-\mu'_j-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda')} \tag{2.15}$$

$$s_{\lambda/\mu}(x/y) = \det\Big(h_{\lambda_i-\mu_j-i+j}(x/y)\Big)_{1\le i,j\le \ell(\lambda)} \tag{2.16}$$

$$E_{x/y}(t) = \sum_{r\ge 0} e_r(x/y)t^r = \frac{\prod_{i=1}^m (1+x_i t)}{\prod_{j=1}^n (1-y_j t)} \tag{2.1}$$

$$H_{x/y}(t) = \sum_{r\ge 0} h_r(x/y)t^r = \frac{\prod_{j=1}^m (1+y_j t)}{\prod_{i=1}^n (1-x_i t)} \tag{2.4}$$

$$P_{x/y}(t) = \frac{d}{dt}\log H_{x/y}(t) = \frac{H'_{x/y}(t)}{H_{x/y}(t)} = \sum_r p_r(x/y)t^{r-1} \tag{2.19}$$

$$\sum_\lambda s_\lambda(x/y)s_\lambda(u/v) = \frac{\prod_{a,j}(1+u_a y_j)\prod_{b,i}(1+v_b x_i)}{\prod_{a,i}(1-u_a x_i)\prod_{b,j}(1-v_b y_j)} \tag{2.30}$$

$$\sum_\lambda h_\lambda(x/y)m_\lambda(u/v) = \frac{\prod_{a,j}(1+u_a y_j)\prod_{b,i}(1+v_b x_i)}{\prod_{a,i}(1-u_a x_i)\prod_{b,j}(1-v_b y_j)} \tag{2.31}$$

$$\sum_\lambda z_\lambda^{-1} p_\lambda(x/y)p_\lambda(u/v) = \frac{\prod_{a,j}(1+u_a y_j)\prod_{b,i}(1+v_b x_i)}{\prod_{a,i}(1-u_a x_i)\prod_{b,j}(1-v_b y_j)} \tag{2.32}$$

$$s_{(r)}(x/y) = h_{(r)}(x/y) \quad \text{and} \quad s_{(1^r)}(x/y) = e_{(r)}(x/y) \tag{2.10}$$

$$s_\lambda(y/x) = s_{\lambda'}(x/y) \tag{2.11}$$

$$s_\lambda(x/y) = s_\tau(x)s_{\eta'}(y)\prod_{i=1}^m\prod_{j=1}^n (x_i+y_j), \quad \text{if } \lambda_m \ge n \text{ and } \lambda = (\kappa+\tau)\cup\eta \tag{2.8}$$

$$s_\lambda(x/y) = \sum_\mu s_\mu(x)s_{(\lambda/\mu)'}(y) = \sum_\mu s_{\lambda/\nu}(x)s_{\nu'}(y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}s_\mu(x)s_{\nu'}(y) \tag{2.12}$$

$$s_\mu(x/y)s_\nu(x/y) = s_{\mu\nu}(x/y) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x/y) \tag{2.13}$$

$$s_{\lambda/\mu}(x/y) = \sum_\sigma s_\sigma(x) s_{(\lambda/\mu\sigma)'}(y) \tag{2.17a}$$

$$s_{\lambda/\mu}(x/y) = \sum_\tau s_{\lambda/\mu\tau}(x) s_{\tau'}(y) \tag{2.17b}$$

$$s_{\lambda/\mu}(x/y) = \sum_\rho s_{\lambda/\rho}(x) s_{(\rho/\mu)'}(y) \tag{2.17c}$$

$$s_{\lambda/\mu}(x/y) = \sum_\rho s_{\rho/\mu}(x) s_{(\lambda/\rho)'}(y) \tag{2.17d}$$

2.4.2 Supersymmetric Schur functions indexed by a composite partition $\bar{\nu};\mu$

$$s_{\bar{\nu};\mu}(x/y) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l+k-l}(x/y) & h_{\mu_j-k-j+1}(x/y) \\ \hline \dot{h}_{\nu_l-i-l+1}(x/y) & h_{\mu_j+i-j}(x/y) \end{array} \right) \tag{2.34}$$

where i, j, k resp. l runs from top to bottom, from left to right, from bottom to top, resp. from right to left.

$$s_{\bar{\nu};\mu}(x/y) = \sum_\zeta (-1)^{|\zeta|} s_{\mu/\zeta}(x/y) s_{\nu/\zeta'}(\overline{x}/\overline{y}) \tag{2.35}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\zeta,\xi} c^\lambda_{\zeta\mu} c^\nu_{\xi\eta} s_{\bar{\xi};\zeta}(x/y). \tag{2.37}$$

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x) s_{\overline{\sigma'/\rho};\tau'/\rho}(y) \tag{2.36a}$$

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) \tag{2.36b}$$

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\rho,\varphi,\psi} s_{\overline{\nu/\varphi\rho};\mu/\psi\rho}(x) s_{\overline{\varphi'};\psi'}(y) \tag{2.36c}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) \tag{2.38a}$$

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\lambda/\tau}(x) s_{\overline{(\sigma/\eta\rho)'};(\tau/\mu\rho)'}(y) \tag{2.38b}$$

$$s_{\bar{\nu};\mu}(x/y) = s_{\overline{\nu'};\mu'}(y/x). \tag{2.39}$$

$$s_{\bar{\nu};\mu}(x/y) = \sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)}) y_n^{a-b}. \tag{2.42}$$

$$s_{\bar{\nu};\mu}(x^{(m)}/y^{(n)}) = \sum_{S^{\bar{\nu};\mu}} (-1)^{(\sum_{i=1}^{t}(c(i^*)+\bar{c}(\overline{i^*})+1))(1-\delta_{0t})}(x/y)^{S^{\bar{\nu};\mu}} \tag{2.43}$$

where the sum is taken over all $(m|n)$-standard composite supertableaux.

Chapter 3

Lie superalgebras and their representations

As Lie superalgebras play an important role in understanding supersymmetry, we will discuss those superalgebras here. After a short introduction, we will define the Lie superalgebras, the enveloping algebra, and the Cartan subalgebra and root systems. This description is based on [59]. Next, within the scope of the earlier chapters, we consider the irreducible representations of the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ and their characters. We consider the highest weight representations and discuss the typical and atypical representations. We prove that there is a unique correspondence between a highest weight and an $(m|n)$-standard composite partition in $\mathfrak{gl}(m|n)$. The connection between those two notions makes it possible to link covariant, contravariant and mixed tensor modules to ordinary partitions and composite partitions and to transfer representation theoretic notions as atypicality and atypicality matrix to the context of partitions.

3.1 Introduction

Many attempts have been made to make general relativity consistent with quantum field theory. In all the most successful attempts a new symmetry is required. So the "super"-symmetry, which successfully combines the interactions between gravity and the gauge symmetries, had to move beyond Lie algebras to "graded" Lie algebras, namely to Lie superalgebras.

In this chapter we will describe Lie superalgebras based on [59, 71, 72]. Lie superalgebras and their representations continue to play an important role in the understanding and exploitation of supersymmetry in physical systems. The Lie superalgebras under consideration here, namely $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$ (sometimes denoted by $U(m|n)$ or $SU(m|n)$), have applications in quantum mechanics [1, 42], nuclear physics [6, 27, 14], string theory [19, 23], conformal field theory [21], supergravity [35, 2], M-theory [20], lattice QCD [7, 9, 15], solvable lattice models [62], spin systems [24] and quantum systems [58]. Also their affine extensions [21, 24] or q-deformations [1, 58] play an important role. In most of the applications, the irreducible representations or "multiplets" of $\mathfrak{gl}(m|n)$ play a role.

A first review was given by Corwin, Ne'eman and Sternberg [16], presenting the subject as it was known in 1974. A complete classification of the finite dimensional simple Lie superalgebras over $\mathbb{C}$ has been given by Kac [31] and Scheunert [59].

Representation theory of Lie superalgebras, and in particular of $\mathfrak{gl}(m|n)$ or its simple counterpart $\mathfrak{sl}(m|n)$, is not a straightforward copy of the corresponding theory for simple Lie algebras.

3.2 Definition of Lie superalgebras

In this chapter, denote by K the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let Γ be one of the rings $\mathbb{Z}$ or $\mathbb{Z}_2$. The two elements of $\mathbb{Z}_2$ will be denoted by $\bar{0}$ and $\bar{1}$.

Definition 3.1 *Let V be a vector space over the field K. A Γ-GRADING OF THE VECTOR SPACE V is a family $(V_\gamma)_{\gamma\in\Gamma}$ of subspaces of V such that*

$$V = \bigoplus_{\gamma\in\Gamma} V_\gamma.$$

A vector space V is said to be Γ-GRADED if it is equipped with a Γ-grading.

An element of V is called HOMOGENEOUS of degree γ, $\gamma \in \Gamma$, if it is an element of V_γ. In the case of $\Gamma = \mathbb{Z}_2$, the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are called even (resp. odd). Every element $y \in V$ has a unique decomposition of the form

$$y = \sum_{\gamma\in\Gamma} y_\gamma \qquad \text{with} \qquad y_\gamma \in V_\gamma, \gamma \in \Gamma$$

where only finite many y_γ are different from zero. The element y_γ is called the homogeneous component of y of degree γ.

A subspace U of V is called Γ-graded if it contains the homogeneous components of all of its elements, i.e. if

$$U = \bigoplus_{\gamma\in\Gamma} (U \cap V_\gamma).$$

On any $\mathbb{Z}$-graded vector space $V = \bigoplus_{j\in\mathbb{Z}} V_j$, there exists a natural $\mathbb{Z}_2$-grading which is said to be induced by the $\mathbb{Z}$-grading and which is defined by

$$V_{\bar{0}} = \bigoplus_{j\in\mathbb{Z}} V_{2j} \qquad \text{and} \qquad V_{\bar{1}} = \bigoplus_{j\in\mathbb{Z}} V_{2j+1}.$$

Definition 3.2 *An algebra A over a field K is Γ-graded if the underlying vector space of A is Γ-graded,*

$$A = \bigoplus_{\gamma\in\Gamma} A_\gamma,$$

and if

$$A_\alpha A_\beta \subset A_{\alpha+\beta}, \qquad \text{for all } \alpha, \beta \in \Gamma.$$

Evidently, A_0 is a subalgebra of A.

Definition 3.3 *A $\mathbb{Z}_2$-graded algebra is called a* SUPERALGEBRA.

Definition 3.4 *A superalgebra A is said to be $\mathbb{Z}$-graded if there exists a family $(A_j)_{j\in\mathbb{Z}}$ of $\mathbb{Z}_2$-graded subspaces of A such that*

$$A = \bigoplus_{j\in\mathbb{Z}} A_j \quad \text{and} \quad A_i A_j \subset A_{i+j}, \text{ for all } i, j \in \mathbb{Z}.$$

The $\mathbb{Z}$-grading $(A_j)_{j\in\mathbb{Z}}$ is said to be consistent with the $\mathbb{Z}_2$-grading of A if

$$A_{\bar{0}} = \bigoplus_{j\in\mathbb{Z}} A_{2j} \quad \text{and} \quad A_{\bar{1}} = \bigoplus_{j\in\mathbb{Z}} A_{2j+1}.$$

Definition 3.5 *A* LIE SUPERALGEBRA $\mathfrak{g}$ *is a $\mathbb{Z}_2$-graded algebra over a field of characteristic 0 (typically $\mathbb{R}$ or $\mathbb{C}$) together with a bilinear operation $[.,.]$ from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$, called the* LIE SUPERBRACKET *or* SUPERCOMMUTATOR*, such that $\forall a \in \mathfrak{g}_\alpha, \forall b \in \mathfrak{g}_\beta, \forall c \in \mathfrak{g}$, and $\forall \alpha, \beta \in \mathbb{Z}_2$:*

$$[a, b] \quad \in \quad \mathfrak{g}_{\alpha+\beta} \tag{3.1a}$$

$$
\begin{aligned}
[a,b] &= -(-1)^{\alpha\beta}[b,a] && (3.1b)\\
[a,[b,c]] &= [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]. && (3.1c)
\end{aligned}
$$

Equation (3.1b) indicates that this bracket is "supersymmetric" and (3.1c) is called the "super Jacobi identity".

It is obvious that the subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is a Lie algebra.

Example 3.6 The simplest example of a Lie superalgebra is $\mathfrak{gl}(m|n)$, $m, n \in \mathbb{Z}_{\geq 0}$. This algebra is, analogous to $\mathfrak{gl}(m)$, the algebra of all matrices of order $m + n$.

$$
\mathfrak{gl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = x \mid A \in M_{m\times m}, B \in M_{m\times n}, C \in M_{n\times m}, D \in M_{n\times n} \right\},
$$

with $M_{p\times q}$ the vector space of all $p \times q$ complex matrices. The even subspace $\mathfrak{gl}_{\bar{0}}(m|n)$ and the odd subspace $\mathfrak{gl}_{\bar{1}}(m|n)$ are defined by:

$$
\begin{aligned}
\mathfrak{gl}_{\bar{0}}(m|n) &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = x \mid A \in M_{m\times m}, D \in M_{n\times n} \right\},\\
\mathfrak{gl}_{\bar{1}}(m|n) &= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = x \mid B \in M_{m\times n}, C \in M_{n\times m} \right\},\\
&= \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = x \mid B \in M_{m\times n} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = x \mid C \in M_{n\times m} \right\}\\
&= \mathfrak{gl}(m|n)_{+1} \oplus \mathfrak{gl}(m|n)_{-1}
\end{aligned}
$$

So, $\mathfrak{gl}(m|n)$ has a $\mathbb{Z}$-grading consistent with the $\mathbb{Z}_2$-grading:

$$
\begin{aligned}
\mathfrak{gl}(m|n) &= \mathfrak{gl}_{\bar{0}}(m|n) \oplus \mathfrak{gl}_{\bar{1}}(m|n) = \mathfrak{gl}_{-1}(m|n) \oplus \mathfrak{gl}_{0}(m|n) \oplus \mathfrak{gl}_{+1}(m|n)\\
\mathfrak{gl}_{\bar{0}}(m|n) &= \mathfrak{gl}(m) \oplus \mathfrak{gl}(n).
\end{aligned}
$$

The bracket is determined in the natural matrix representation by

$$
[a,b] = ab - (-1)^{\alpha\beta} ba, \qquad \forall a \in \mathfrak{g}_\alpha \text{ and } \forall b \in \mathfrak{g}_\beta,
$$

where the juxtaposition in the right hand side denotes matrix multiplication.

For a matrix in $\mathfrak{gl}(m|n)$ the supertrace [16, 31, 59] is defined as $str(x) = tr(A) - tr(D)$. Observe that the supertrace does not depend upon the choice of the homogeneous basis for $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

Definition 3.7 *A bilinear form $f(a,b)$ on a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called*

consistent	$\Leftrightarrow$	$f(a,b) = 0$ *for all* $a \in \mathfrak{g}_{\bar{0}}$ *and for all* $b \in \mathfrak{g}_{\bar{1}}$,
supersymmetric	$\Leftrightarrow$	$f(a,b) = (-1)^{\alpha\beta} f(b,a)$, *for all* $a \in \mathfrak{g}_{\bar{\alpha}}$ *and for all* $b \in \mathfrak{g}_{\bar{\beta}}$,
invariant	$\Leftrightarrow$	$f([a,b],c) = f(a,[b,c])$.

In [31], V. Kac proved the following lemma.

Lemma 3.8 *The bilinear form $f(a,b) = str(ab)$ is consistent, supersymmetric and invariant.*

As a corollary of the supersymmetry, $str([a,b]) = str(ab - (-1)^{\alpha\beta} ba) = 0$. So, one can define the subalgebra $\mathfrak{sl}(m|n)$:

$$\mathfrak{sl}(m|n) = \{x \in \mathfrak{gl}(m|n) : str(x) = 0\}. \tag{3.2}$$

Definition 3.9 *A Lie superalgebra is called* SIMPLE *if it does not have any non-trivial graded ideals and if $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$.*
A Lie superalgebra is called SEMI-SIMPLE *if it does not contain any non-trivial solvable ideals.*

A (two-sided) ideal I of a Lie superalgebra $\mathfrak{g}$ is a GRADED IDEAL of $\mathfrak{g}$ if I is a graded vector space.

This definition of simple Lie superalgebras implies that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ as $[\mathfrak{g},\mathfrak{g}]$ is a graded ideal. It can be shown [59] that a left or right graded ideal of $\mathfrak{g}$ is automatically a two-sided ideal. According to the definition, a simple Lie superalgebra might still contain a non-trivial non-graded ideal. Actually, however, this is not the case (see [59]).

Proposition 3.10 *A simple Lie superalgebra $\mathfrak{g}$ does not have any left or right ideals (graded or not) except for $\{0\}$ and $\mathfrak{g}$.*

The Lie superalgebra $\mathfrak{sl}(m|n)$ is simple if $m \neq n$. In the case $m = n$, the algebra $\mathfrak{sl}(m|n)$ contains the $2m \times 2m$ identity matrix I_{2m}. Then $\mathbb{C}.I_{2m}$ is a graded ideal of $\mathfrak{sl}(m|m)$. The quotient algebra $\mathfrak{sl}(m|n)/\mathbb{C}.I_{2m}$ is again a simple Lie superalgebra.

In [31] V. Kac gave a classification of the different families of simple Lie superalgebras over an arbitrary field of characteristic 0.

3.3 The enveloping algebra of a Lie superalgebra

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be an associative Lie superalgebra over K ($K = \mathbb{R}$ or $K = \mathbb{C}$) and let $\mathcal{T}(\mathfrak{g})$ be the tensor algebra of the vector space $\mathfrak{g}$. So,

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{n=0}^{+\infty} \mathcal{T}_n(\mathfrak{g}) \text{ with } \begin{cases} \mathcal{T}_0(\mathfrak{g}) = K, \\ \mathcal{T}_n(\mathfrak{g}) = \overbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}^{n \text{ times}} \end{cases}. \tag{3.3}$$

The multiplication on a tensor algebra is given by the usual tensor product.

The $\mathbb{Z}_2$-grading of $\mathfrak{g}$ induces a $\mathbb{Z}_2$-grading of $\mathcal{T}(\mathfrak{g})$ such that the canonical injection $\mathfrak{g} \to \mathcal{T}(\mathfrak{g})$ is an even linear mapping and that $\mathcal{T}(\mathfrak{g})$ is a superalgebra. We consider the two-sided ideal I of $\mathcal{T}(\mathfrak{g})$ which is generated by the elements

$$a \otimes b - (-1)^{\alpha\beta} b \otimes a - [a, b] \text{ with } a \in \mathfrak{g}_\alpha,\ b \in \mathfrak{g}_\beta; \alpha, \beta \in \mathbb{Z}_2. \tag{3.4}$$

These elements are of course homogeneous and hence, I is a graded ideal. Therefore, if we define

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I, \tag{3.5}$$

it follows that $\mathcal{U}(\mathfrak{g})$ is an associative superalgebra. This algebra is called the (UNIVERSAL) ENVELOPING SUPERALGEBRA OF $\mathfrak{g}$.

3.4 A Cartan subalgebra and simple root systems

A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a CARTAN SUBALGEBRA if it is nilpotent and equal to its normalizer, which is the set of those elements x in $\mathfrak{g}$ such that $[x, \mathfrak{h}] \subset \mathfrak{h}$. The Cartan subalgebra is the maximal Abelian subalgebra of $\mathfrak{g}_{\bar{0}}$. The Cartan subalgebra of the subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is also called the Cartan subalgebra of the Lie superalgebra $\mathfrak{g}$. All Cartan subalgebras of a classical superalgebra $\mathfrak{g}$ have the same dimension. By definition, the dimension of a Cartan subalgebra $\mathfrak{h}$ is the rank of $\mathfrak{g}$.

From now on, we will only consider the Lie superalgebra $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$. The Cartan subalgebra of $\mathfrak{gl}(m|n)$ is given by the vector space $\mathfrak{h}$ of diagonal matrices and has dimension $m+n$. The restriction to $\mathfrak{sl}(m|n)$ requires the supertrace condition to be satisfied. Hence the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(m|n)$ has dimension $m+n-1$ and is spanned by

$$\begin{aligned} h_i &= E_{ii} - E_{i+1,i+1} \ (1 \le i \le m-1 \text{ or } m+1 \le i \le m+n-1), \\ h_m &= E_{mm} + E_{m+1,m+1}, \end{aligned} \tag{3.6}$$

where E_{ij} is the matrix with entry 1 at position (i,j) and 0 elsewhere.

Consider, the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$. The $\epsilon\delta$-basis of $\mathfrak{h}^*$ is given by $\{\epsilon_i \mid 1 \leq i \leq m\} \cup \{\delta_j \mid 1 \leq j \leq n\}$. This dual space $\mathfrak{h}^*$ contains special elements α such that

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x,\ h \in \mathfrak{h}\}. \tag{3.7}$$

The set Δ of those eigenvalues α is by definition the ROOT SYSTEM of $\mathfrak{g}$, namely:

$$\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\} \tag{3.8}$$

A root α is called EVEN (resp. ODD) if $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq \emptyset$ (resp. $g_\alpha \cap \mathfrak{g}_{\bar{1}} \neq \emptyset$). The set of all the even resp. odd roots is indicated by $\Delta_{\bar{0}}$ resp. $\Delta_{\bar{1}}$.

Let now $\mathfrak{b}_{\bar{0}}$ be a Borel subalgebra of $\mathfrak{g}_{\bar{0}}$, i.e. $\mathfrak{b}_{\bar{0}}$ is a maximum solvable subalgebra of $\mathfrak{g}_{\bar{0}}$. Obviously, the Cartan subalgebra $\mathfrak{h}$ is a subalgebra of $\mathfrak{b}_{\bar{0}}$. We can extend $\mathfrak{b}_{\bar{0}}$ to a Borel subalgebra $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ of $\mathfrak{g}$ (usually, this extension is not unique!). Then

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+ \qquad \text{and} \qquad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \tag{3.9}$$

where $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are subalgebras of $\mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{n}^+] \subset \mathfrak{n}^+$, $[\mathfrak{h}, \mathfrak{n}^-] \subset \mathfrak{n}^-$ and $\dim \mathfrak{n}^+ = \dim \mathfrak{n}^-$. A root α is called POSITIVE (resp. NEGATIVE) if $\mathfrak{g}^\alpha \cap \mathfrak{n}^+ \neq 0$ (resp. $\mathfrak{g}^\alpha \cap \mathfrak{n}^- \neq 0$). Denote by $\Delta^+, \Delta^+_{\bar{0}}, \Delta^+_{\bar{1}}$ (resp. $\Delta^-, \Delta^-_{\bar{0}}, \Delta^-_{\bar{1}}$) the subsets of the positive (resp. negative) roots in $\Delta, \Delta_{\bar{0}}, \Delta_{\bar{1}}$.

A positive root is called SIMPLE if it cannot be decomposed into a sum of positive roots. The set of all simple roots is called a SIMPLE ROOT SYSTEM of $\mathfrak{g}$ and is denoted here by Π.

The so-called DISTINGUISHED CHOICE [31] for a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is such that $\mathfrak{g}_{+1} \subset \mathfrak{n}^+$ and $\mathfrak{g}_{-1} \subset \mathfrak{n}^-$. Note that contrary to the case of simple Lie algebras not all the choices of a set of simple roots are equivalent.

In this $\epsilon\delta$-basis the EVEN ROOTS of $\mathfrak{g} = \mathfrak{gl}(m|n)$ are of the form $\epsilon_i - \epsilon_j$ or $\delta_i - \delta_j$, and the ODD ROOTS are of the form $\pm(\epsilon_i - \delta_j)$. As a system of SIMPLE ROOTS one takes the so-called DISTINGUISHED SET [16]

$$\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}. \tag{3.10}$$

For this choice we have

$$\Delta^+_{\bar{0}} \quad = \quad \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq m\} \cup \{\delta_i - \delta_j | 1 \leq i < j \leq n\},$$

$$\Delta_{\bar{1}}^{+} \quad = \quad \{\beta_{ij} = \epsilon_i - \delta_j | 1 \leq i \leq m,\ 1 \leq j \leq n\}. \tag{3.11}$$

Thus, in the distinguished basis there is only one simple root which is odd.

As usual, we put

$$\rho_0 = \frac{1}{2}(\sum_{\alpha \in \Delta_{0,+}} \alpha), \qquad \rho_1 = \frac{1}{2}(\sum_{\alpha \in \Delta_{1,+}} \alpha), \qquad \rho = \rho_0 - \rho_1. \tag{3.12}$$

Or, explicitly in the $\epsilon\delta$-basis

$$\rho = \frac{1}{2}\sum_{i=1}^{m}(m - n - 2i + 1)\epsilon_i + \frac{1}{2}\sum_{j=1}^{n}(m + n - 2j + 1)\delta_j. \tag{3.13}$$

There is a symmetric form $(\ ,\ )$ on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$, and in the natural basis it takes the form $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{ij}$, where δ_{ij} is the usual Kronecker symbol.

It is easy to check that the odd roots are isotropic : $(\alpha, \alpha) = 0$ if $\alpha \in \Delta_1$.

The WEYL GROUP of a root system is generated by reflections with respect to the simple roots for any choice of a set of positive roots. The Weyl group of a Lie superalgebra $\mathfrak{g}$ is the Weyl group W of $\mathfrak{g}_{\bar{0}}$ [31]. The Weyl group of $\mathfrak{gl}(m|n)$ is hence the direct product of symmetric groups $S_m \times S_n$. For $w \in W$, we denote by $\varepsilon(w)$ its signature.

The WEIGHT SPACE $\mathfrak{h}^*$ is spanned by ϵ_i ($i = 1, \ldots, m$) and δ_j ($j = 1, \ldots, n$). So, a weight $\Lambda \in \mathfrak{h}^*$ can be written as

$$\Lambda = \lambda_1\epsilon_1 + \cdots + \lambda_m\epsilon_m + \mu_1\delta_1 + \cdots + \mu_n\delta_n, \tag{3.14}$$

which we will usually write as

$$\Lambda = (\lambda_1, \ldots, \lambda_m; \mu_1, \ldots, \mu_n). \tag{3.15}$$

A weight $\Lambda \in \mathfrak{h}^*$ with $\Lambda = \sum_{i=1}^{m} \lambda_i\epsilon_i + \sum_{j=1}^{n} \mu_j\delta_j$ is called INTEGRAL if and only if $\lambda_i \in \mathbb{Z},\ \mu_j \in \mathbb{Z}$; it is called INTEGRAL DOMINANT if and only if it is integral and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$.

To every weight $\Lambda \in \mathfrak{h}^*$, with $\Lambda = \sum_{i=1}^{m} \lambda_i\epsilon_i + \sum_{j=1}^{n} \mu_j\delta_j$, we can associate its so-called KAC-DYNKIN LABEL namely:

$$\Lambda = [a_1, a_2, \ldots, a_{m-1}; a_m; a_{m+1}, \ldots, a_{m+n-1}] \tag{3.16}$$

where $a_i = \lambda_i - \lambda_{i+1}$, for $i < m$, $a_m = \lambda_m + \mu_1$, and $a_{m+j} = \mu_j - \mu_{j+1}$, for $j < n$.

In terms of the Kac-Dynkin labels, a weight $\Lambda \in \mathfrak{h}^*$ is said to be integral dominant if and only if its Kac-Dynkin label is such that $a_i \in \mathbb{Z}_{\geq 0}$ for $i \neq m$, whereas a_m can be any complex number. For our purpose it is sufficient to consider only those Λ for which $a_m \in \mathbb{Z}$.

Note that Λ expressed in terms of the $\epsilon\delta$-basis represents a unique weight of $\mathfrak{gl}(m|n)$ whereas the Kac-Dynkin labels of Λ represent a unique weight of $\mathfrak{sl}(m|n)$ rather than $\mathfrak{gl}(m|n)$. For example, for $(m|n) = (4|5)$, $\Lambda_1 = (5,4,4,1;0,\bar{2},\bar{2},\bar{4},\bar{4})$ and $\Lambda_2 = (4,3,3,0;1,\bar{1},\bar{1},\bar{3},\bar{3})$ have the same Kac-Dynkin label $[1,0,3;1;2,0,2,0]$.

3.5 Representations

3.5.1 Irreducible and faithful representations

Definition 3.11 *Let $\mathfrak{g}$ be a $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and consider the superalgebra $\mathrm{End}(V) = \mathrm{End}_{\bar{0}}(V) \oplus \mathrm{End}_{\bar{1}}(V)$ of endomorphisms of V. A linear representation φ of $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ into $\mathrm{End}(V)$, that is*

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y),$$
$$\varphi([x,y]) = [\varphi(x), \varphi(y)],$$
$$\varphi(\mathfrak{g}_{\bar{0}}) \subset \mathrm{End}_{\bar{0}}(V) \text{ and } \varphi(\mathfrak{g}_{\bar{1}}) \subset \mathrm{End}_{\bar{1}}(V),$$

for all $x, y \in \mathfrak{g}$ and $\alpha \in \mathbb{C}$.

The vector space V is the representation space. It has the structure of a $\mathfrak{g}$-module by $x(v) = \varphi(x)v$ for $x \in \mathfrak{g}$ and $v \in V$.

For example, the ADJOINT REPRESENTATION $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(V)$ is defined as

$$\mathrm{ad}_a : \mathfrak{g} \to \mathrm{End}(V) : x \to \mathrm{ad}_a(x) = [a, x] \tag{3.17}$$

The dimension (resp. the superdimension) of the representation φ is the dimension (resp. graded dimension) of the vector space V:

$$\begin{aligned} \dim\varphi &= \dim V_{\bar{0}} + \dim V_{\bar{1}} \\ \mathrm{sdim}\,\varphi &= \dim V_{\bar{0}} - \dim V_{\bar{1}}. \end{aligned}$$

A representation is said to be FAITHFUL if $\varphi(x) = \varphi(y)$ implies that $x = y$ for $x, y \in \mathfrak{g}$ and TRIVIAL if $\varphi(x) = 0$ for all $x \in \mathfrak{g}$.

A representation $\varphi : \mathfrak{g} \to \mathrm{End}(V)$ is called IRREDUCIBLE if the $\mathfrak{g}$-module V contains no non-trivial $\mathfrak{g}$-submodules. The $\mathfrak{g}$-module V is then called a SIMPLE MODULE. Otherwise, the representation φ is said to be REDUCIBLE. In that case, one has $V = V' + V''$, as a sum of vector spaces, where the $\mathfrak{g}$-module V' is an invariant subspace under φ. Note that considered as modules, the sum $V = V' + V''$ might be either a direct or semidirect sum. If the representation is equivalent to a direct sum of irreducible components, it is said to be COMPLETELY REDUCIBLE. The $\mathfrak{g}$-module V is then called a SEMI-SIMPLE module. A representation which is not equivalent to a direct sum of two or more non-zero representations is called INDECOMPOSABLE.

Let φ and ψ be two representations of $\mathfrak{g}$ with representation spaces V and V'. We can define the direct sum $\varphi \oplus \psi$ with representation space $V \oplus V'$ and the tensor product $\varphi \otimes \psi$ with representation space $V \otimes V'$ of those two representations. The action of the representations $\varphi \oplus \psi$ and $\varphi \otimes \psi$ on the corresponding representation spaces is given by, $x \in \mathfrak{g},\ v \in V$ and $v' \in V'$:

$$(\varphi \oplus \psi)(x)v \oplus v' = \varphi(x)v \oplus \psi(x)v' \tag{3.18}$$

$$(\varphi \otimes \psi)(x)v \otimes v' = \varphi(x)v \otimes v' + v \otimes \psi(x)v'. \tag{3.19}$$

If φ and ψ are both irreducible representations, the tensor product $\varphi \otimes \psi$ will be in general reducible but, contrary to the Lie algebra case, the tensor product of two irreducible representations is not necessarily completely reducible.

Any representation of a basic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ can be decomposed into a direct sum of irreducible representations of the even subspace $\mathfrak{g}_{\bar{0}}$

3.5.2 Highest weight representations

Let $\mathfrak{g}$ be a Lie superalgebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping superalgebra, $\mathfrak{g}'$ a subalgebra of $\mathfrak{g}$ and V a $\mathfrak{g}'$-module. V can be extended to a $\mathcal{U}(\mathfrak{g}')$-module. We consider the $\mathbb{Z}_2$-graded space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} V$. This space is defined as the factor space of $\mathcal{U}(\mathfrak{g}) \otimes V$ by the linear span of the elements of the form $gh \otimes v - g \otimes h(v),\ g \in \mathcal{U}(\mathfrak{g}),\ h \in \mathcal{U}(\mathfrak{g}')$ and $v \in V$. The so constructed $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}')} V$ is said to be INDUCED FROM THE $\mathfrak{g}'$-MODULE V and is denoted $\mathrm{Ind}_{\mathfrak{g}'}^{\mathfrak{g}}(V)$.

Let $\mathfrak{g}$ be a basic classical Lie superalgebra and $\mathfrak{h}$ its Cartan subalgebra. We fix the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, containing $\mathfrak{h}$ as defined in (3.9):

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let $\Lambda \in \mathfrak{h}^*$ be a linear function on $\mathfrak{h}$. We define a one-dimensional even $\mathfrak{b}$-module $\langle v_\Lambda \rangle = \mathrm{span}(v_\lambda)$ by

$$\mathfrak{n}^+(v_\Lambda) = 0 \qquad \text{and} \qquad h(v_\Lambda) = \Lambda(h)v_\Lambda,\ (h \in \mathfrak{h}).$$

The equality $\mathfrak{n}^+ v_\Lambda = 0$ means that the action of the positive root elements upon v vanishes, and the condition $h(v_\Lambda) = \Lambda(h) v_\Lambda$ implies that Λ is a weight with corresponding weight vector v_Λ.

The induced module $\bar{V}(\Lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\langle v_\Lambda \rangle)$ for any integral dominant weight $\Lambda \in \mathfrak{h}^*$ is called the KAC MODULE. This $\mathfrak{g}$-module contains a unique maximal submodule $I(\Lambda)$, and we set

$$V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda).$$

The $\mathfrak{g}$-module $V(\Lambda)$ is called an IRREDUCIBLE REPRESENTATION WITH HIGHEST WEIGHT Λ or a HIGHEST WEIGHT MODULE.

Kac [31] proved the following proposition:

Proposition 3.12 *(a) v_Λ is a unique vector in $V(\Lambda)$, up to a constant factor, for which $\mathfrak{n}^+(v_\Lambda) = 0$.*

(b) Any finite dimensional irreducible representation of $\mathfrak{g}$ is of the form $V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda)$ where Λ is an integral dominant weight.

(c) Any finite dimensional simple $\mathfrak{g}$-module is uniquely characterized by its integral dominant weight Λ: two $\mathfrak{g}$-modules $V(\Lambda)$ and $V(\Lambda')$ are isomorphic if and only if $\Lambda = \Lambda'$.

(d) The finite dimensional simple $\mathfrak{g}$-module $V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda)$ has the weight decomposition

$$V(\Lambda) = \bigoplus_{\lambda \leq \Lambda} V_\lambda, \quad \textit{with } V_\Lambda = \{v \in V \mid h(v) = \lambda(h)v, \ h \in \mathfrak{h}\}, \tag{3.20}$$

where λ is called a weight of the representation V_λ and V_λ is the corresponding weight space.

3.5.3 Typical and atypical representations and character formulas

Every finite dimensional $\mathfrak{g}$-module of a (semi)-simple Lie algebra $\mathfrak{g}$ is completely reducible. Unfortunately, this property is not valid for Lie superalgebras, since there exists the famous Djoković-Hochschild theorem:

Theorem 3.13 *Let $\mathfrak{g}$ be a Lie superalgebra over an algebraically closed field. All the finite dimensional representations of $\mathfrak{g}$ are completely reducible if and only if $\mathfrak{g}$ is isomorphic to the direct sum of a semi-simple Lie algebra with finitely many Lie superalgebras of the type $B(0, n)$.*

The root system of $B(0,n)$, one of the families of orthosymplectic superalgebras, is given by

$$\Delta_{\bar{0}} = \{\pm\delta_i \pm \delta_j, \pm 2\delta_i\} \qquad \text{and} \qquad \Delta_{\bar{1}} = \{\pm\delta_j\}. \tag{3.21}$$

So, all simple Lie superalgebras, except the $B(0,n)$ algebras, have so-called INDECOMPOSABLE, i.e. not completely reducible, representations.

This leads to the definition of two types of irreducible representations for a Lie superalgebra $\mathfrak{g}$. Let Λ be a highest weight for a finite dimensional irreducible representation $V(\Lambda)$ of $\mathfrak{g}$:

(i) either the representation $V(\Lambda)$ can not be extended to an indecomposable representation of $\mathfrak{g}$. In this case $V(\Lambda)$ is called a TYPICAL REPRESENTATION.

(ii) or else the representation $V(\Lambda)$ can be extended with another $\mathfrak{g}$-module in such a way that the new representation is an indecomposable representation of $\mathfrak{g}$. In this case $V(\Lambda)$ is an ATYPICAL REPRESENTATION of $\mathfrak{g}$.

The typical representations are the ones which satisfy all the "nice" properties of the irreducible representations of Lie algebras; the atypical representations usually cause a great deal of worries. But we cannot avoid the atypical ones, since, for instance, they appear in the decomposition of the tensor product of two typical representations.

Kac proved the following useful theorem to distinguish typical representations from atypical representations by means of their highest weight.

Theorem 3.14 *Let $\Lambda \in \mathfrak{h}^*$ be a highest weight of a finite dimensional irreducible module $V(\Lambda)$ of a classical Lie superalgebra $\mathfrak{g}$. Then the following statements are equivalent:*

(i) $V(\Lambda)$ *is typical;*

(ii) $(\Lambda+\rho,\alpha) \neq 0$, *for all* $\alpha \in \bar{\Delta}_{\bar{1}}^+$, *where* $\bar{\Delta}_{\bar{1}} = \{\alpha \in \Delta_{\bar{1}} \mid 2\alpha \notin \Delta_{\bar{0}}\}$;

(iii)

$$\mathrm{ch}(V(\Lambda)) = L^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)}, \tag{3.22}$$

with $L = \dfrac{\prod_{\alpha\in\Delta_{\bar{0}}^+}(e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha\in\Delta_{\bar{1}}^+}(e^{\alpha/2} + e^{-\alpha/2})}$, W *the Weyl group and* ρ *defined in (3.12).* e^λ *is the formal exponential function on* $\mathfrak{h}^*$ *such that* $e^\lambda(\mu) = \delta_{\lambda\mu}$ *for* $\lambda, \mu \in \mathfrak{h}^*$, *which satisfies* $e^\lambda e^\mu = e^{\lambda+\mu}$.

Remark that for $\mathfrak{g} = \mathfrak{sl}(m|n)$, $\bar{\Delta}_{\bar{1}}^{+} = \Delta_{\bar{1}}^{+}$.

Expression (3.22) gives the character formula for a typical representation of a simple Lie superalgebra. The problem of obtaining a character formula for the remaining "atypical" irreducible representations has been the subject of intensive investigation, both in the mathematics and physics literature.

At this point it is convenient to say something about the connection between representations of $\mathfrak{gl}(m|n)$ and of $\mathfrak{sl}(m|n)$, which is similar to that between $\mathfrak{gl}(m)$ and $\mathfrak{sl}(m)$. Recall that $\mathfrak{sl}(m|n)$ consists of those elements of $\mathfrak{gl}(m|n)$ with zero supertrace. Define the element σ in the standard ϵ-δ-basis by

$$\sigma = \sum_{i=1}^{m} \epsilon_i - \sum_{j=1}^{n} \delta_j, \tag{3.23}$$

or in coordinates $\sigma = (1, 1, \ldots, 1;\ -1, -1, \ldots, -1)$. Then $\sigma = 0$ in the weight space of $\mathfrak{sl}(m|n)$ (but not in the weight space of $\mathfrak{gl}(m|n)$). So two highest weights Λ and $\Lambda + j\sigma$ of $\mathfrak{gl}(m|n)$ stand for the same highest weight in $\mathfrak{sl}(m|n)$. This implies that the corresponding highest weight representations V_Λ and $V_{\Lambda+j\sigma}$ must have the same character as $\mathfrak{sl}(m|n)$ representations. Then their $\mathfrak{gl}(m|n)$ characters are also the same, up to a factor. More explicitly,

$$\mathrm{ch}\, V_{\Lambda+j\sigma} = (e^\sigma)^j \ \mathrm{ch}\, V_\Lambda, \tag{3.24}$$

with e^σ the formal exponential.

3.6 Highest weight and composite partitions

Often it is useful to represent a highest weight Λ by a composite Young diagram. There exists a bijection between $\mathfrak{gl}(m|n)$ integral dominant weights Λ and an $(m|n)$-standard composite partition $\bar{\nu}; \mu$.

Lemma 3.15 *Let Λ be a $\mathfrak{gl}(m|n)$ integral dominant weight and let JL be the set of all (j, l) with $0 \leq j \leq n$ and $0 \leq l \leq m$ such that*

$$\begin{array}{rlcl} \textit{either} & j = 0 & \textit{or} & \Lambda_{m+j} > -l, \\ \textit{and either} & j = n & \textit{or} & \Lambda_{m+j+1} \leq -l, \\ \textit{and either} & l = m & \textit{or} & \Lambda_{m-l} \geq j, \\ \textit{and either} & l = 0 & \textit{or} & \Lambda_{m-l+1} < j. \end{array} \tag{3.25}$$

Then let $J = \min\{j \mid (j,l) \in JL\}$ *and* $L = \min\{l \mid (J,l) \in JL\}$*. Then the corresponding composite partition* $\bar{\nu};\mu$ *is* $(m|n)$*-standard, where*

$$\begin{array}{llll} \mu_i = \Lambda_i & \text{for } i = 1,2,\ldots,I = m-L & \text{and } \mu_i \le J \text{ for } i > I, \\ \mu'_j = m + \Lambda_{m+j} & \text{for } j = 1,2,\ldots,J & \text{and } \mu'_j \le I \text{ for } j > J, \\ \nu'_k = -\Lambda_{m+n-k+1} & \text{for } k = 1,2,\ldots,K = n-J & \text{and } \nu'_k \le L \text{ for } k > K, \\ \nu_l = n - \Lambda_{m-l+1} & \text{for } l = 1,2,\ldots,L & \text{and } \nu_l \le K \text{ for } l > L. \end{array} \tag{3.26}$$

Conversely, if the composite partition $\bar{\nu};\mu$ *is* $(m|n)$*-standard with* I,J,K,L *as in Definition 2.5 then the corresponding* $\mathfrak{gl}(m|n)$ *integral dominant weight* Λ *is given by*

$$\begin{aligned} \Lambda = (\mu_1,\mu_2,\ldots,\mu_I, n-\nu_L,\ldots,n-\nu_2,n-\nu_1 \mid \\ \mu'_1 - m, \mu'_2 - m, \ldots, \mu'_J - m, -\nu'_K,\ldots,-\nu'_2,-\nu'_1). \end{aligned} \tag{3.27}$$

Proof. Suppose we have a weight Λ and the corresponding $J = \min\{j \mid (j,l) \in JL\}$ and $L = \min\{l \mid (J,l) \in JL\}$ according to (3.25). Let, by (3.26),

$$\begin{aligned} \mu &= (\Lambda_1,\ldots,\Lambda_{m-L}) \cup (\Lambda_{m+1}+L,\ldots,\Lambda_{m+J}+L)' \\ \nu &= (n-\Lambda_m,\ldots,n-\Lambda_{m-L+1}) \cup (-\Lambda_{m+n}-L,\ldots,-\Lambda_{m+J+1}-L)' \end{aligned}$$

$$\begin{aligned} \mu' &= (\Lambda_{m+1}+m,\ldots,\Lambda_{m+J}+m) \cup (\Lambda_1 - J,\ldots,\Lambda_{m-L}-J)' \\ \nu' &= (-\Lambda_{m+n},\ldots,-\Lambda_{m+J+1}) \cup (J-\Lambda_m,\ldots,J-\Lambda_{m-L+1})' \end{aligned}$$

From this construction we see that

$$\begin{aligned} \mu_{m-L+1} + \nu_{L+1} &\le J + (n-J) = n \\ \mu'_{J+1} + \nu'_{n-J+1} &\le (m-L) + L = m \end{aligned}$$

Thus $\bar{\nu};\mu$ is $(m|n)$-standard by construction, since there exist (see Definition 2.5) numbers $J^* = \min\{j \mid \mu'_{j+1} + \nu'_{n-j+1} \le m\}$ and $L^* = \min\{l \mid \mu_{m-l+1} + \nu_{l+1} \le n\}$ with $J^* \le J$ and $L^* \le L$.

As J^* and L^* are minimal, we have the following relations:

$$\begin{array}{lcl} \mu_{m-L^*+1} = J^* & \text{and} & \mu'_{J^*} > m - L^*, \\ \nu'_{n-J^*+1} = L^* & \text{and} & \nu_{L^*} > n - J^*. \end{array}$$

Taking into account that $J^* \le J$ and $L^* \le L$, we have:

$$\begin{array}{lrcl} 1. & \mu_{m-L^*+1} + \nu_{L^*+1} \le n & \Leftrightarrow & J^* + (n - \Lambda_{m-L^*}) \le n, \\ 2. & \mu'_{J^*+1} + \nu'_{n-J^*+1} \le m & \Leftrightarrow & (\Lambda_{m+J^*+1} + m) + L^* \le m, \\ 3. & \mu'_{J^*} > m - L^* & \Leftrightarrow & \Lambda_{m+J^*} + m > m - L^*, \\ 4. & \nu_{L^*} > n - J^* & \Leftrightarrow & n - \Lambda_{n-L^*+1} > n - J^*. \end{array}$$

Simplification proves that the four equations (3.25) are fulfilled for $(j,l) = (J^*, L^*)$, that is:

$$\Lambda_{m-L^*} \geq J^*, \quad \Lambda_{m+J^*+1} \leq -L^*, \quad \Lambda_{m+J^*} > -L^*, \text{ and } \Lambda_{n-L^*+1} < J^*.$$

Thus, $J^* \geq J$ and $L^* \geq L$ and so, $J^* = J$ and $L^* = L$. □

Remark that it is not important whether J or L is minimized first. Indeed, suppose

$$J = \min\{j \mid (j,l) \in JL\} \qquad \text{and} \qquad L = \min\{l \mid (J,l) \in JL\}$$

and

$$L' = \min\{l \mid (j,l) \in JL\} \qquad \text{and} \qquad J' = \min\{j \mid (j,L) \in JL\}.$$

If $L' = L$ then it is clear that also $J = J'$. So, suppose $L' < L$ ($L' > L$ is similar). If $J' < J$ then there exist $(J', L') \in JL$ with $J' < J$ which is a contradiction with J being minimal. If $J' = J$, then $(J, L') \in JL$ which is a contradiction with L being minimal. So, if $L' < L$ then $J' > J$. Then we have the following relations:

$$\Lambda_{m+J} \geq \Lambda_{m+J'} > -L' > -L \geq \Lambda_{m+J+1} \geq \Lambda_{m+J'+1},$$
$$\Lambda_{m-L} \geq \Lambda_{m-L'} \geq J' > J > \Lambda_{m-L+1} \geq \Lambda_{m-L'+1}.$$

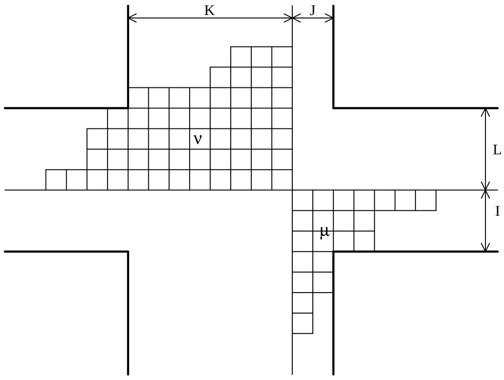
Thus, $(J, L') \in JL$ which is a contradiction with L being minimal.

Let us illustrate this Lemma in $\mathfrak{gl}(7|10)$ for

$$\Lambda = (7,4,4,1,0,0,\bar{2}; 0,\bar{2},\bar{5},\bar{5},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7}).$$

For $j = 0$, it is impossible to find an l such that $\Lambda_{m+1} = 0 \leq -l$. For $j = 1$, we have to find an l such that $\Lambda_{m+1} = 0 > -l$ and $\Lambda_{m+2} = -2 \leq -l$; so, $l \in \{1, 2\}$. For neither $(j,l) = (1,1)$ nor $(j,l) = (1,2)$ is the third condition $\Lambda_{m-l} = 0 > j = 1$ fulfilled. For $j = 2$, we find that $-\Lambda_{m+2} = 2 < l \leq -\Lambda_{m+3} = 5$. As $\Lambda_{m-4} \geq 2$ and $\Lambda_{m-4+1} < 2$ we find that $l = 4$. Therefore we find $(J, L) = (2, 4)$.
So, $\mu = (7,4,4) \cup (4,2)' = (7,4,4,2,2,1,1)$, $\nu' = (7^3, 6, 5^4) \cup (4, 2^2, 1)' = (7^3, 6, 5^4, 4, 3, 1^2)$ and $\nu = (12, 10, 10, 9, 8, 4, 3)$.
Conversely, starting from $\bar{\nu}; \mu = (\bar{3},\bar{4},\bar{8},\bar{9},\bar{1}0,\bar{1}0,\bar{1}2); (7,4,4,2,2,1,1)$ we find, that $(J, L) = (2, 4)$ using Definition 2.5. This is illustrated in Figure 3.1.

As every integral dominant weight Λ defines a composite partition $\bar{\nu}; \mu$, we will indicate this correspondence by $\Lambda_{\bar{\nu};\mu}$. In the special case where $\nu = ()$ we will write Λ_μ.



$$\Updownarrow$$

$$\Lambda = (\underbrace{7,4,4}_{I=3},\underbrace{1,0,0,\bar{2}}_{L=4};\underbrace{0,\bar{2}}_{J=2},\underbrace{\bar{5},\bar{5},\bar{5},\bar{5},\bar{6},\bar{7},\bar{7},\bar{7}}_{K=8}).$$

Figure 3.1: Highest weight and corresponding composite partition $\bar{\nu};\mu$.

3.7 Covariant, contravariant and mixed tensor modules

Berele and Regev [10], and Sergeev [61], showed that the tensor product of N copies of the natural $(m+n)$-dimensional representation $V = \mathbb{C}^{m+n}$ of $\mathfrak{g} = \mathfrak{gl}(m|n)$ is completely reducible, and that the irreducible components V_λ can be labeled by a partition λ of N such that λ is inside the (m,n)-hook, denoted by $\lambda \in \mathcal{H}_{m,n}$, i.e. such that $\lambda_{m+1} \leq n$. These representations V_λ are known as COVARIANT MODULES.

Analogously, they showed that the tensor product of N copies of the dual V^* of the natural $(m+n)$-dimensional representation V of $\mathfrak{g} = \mathfrak{gl}(m|n)$ is also completely reducible. The irreducible components $V_{\bar{\lambda}}$ are again labeled by a partition λ of N such that $\lambda \in \mathcal{H}_{m,n}$. These representations $V_{\bar{\lambda}}$ are known as the CONTRAVARIANT MODULES.

In the early days of Lie superalgebra representation theory, the notion of graded tensors was introduced [22]. The MIXED TENSOR MODULES, the mixed tensor irreducible representations, occur as irreducible components in the tensor product of M copies of the natural $(m+n)$-dimensional representation V and N copies of the dual V^* of V. This tensor product however is in general not completely reducible.

It was believed [4, 5] that the standard methods of covariant, contravariant and mixed tensor representations with the corresponding Young techniques yield the characters of $\mathfrak{gl}(m|n)$ irreducible representations in terms of supersymmetric S-functions. Although this is certainly true for the covariant and contravariant tensor representations [10, 22], it is not so for the mixed tensor representations, as already observed in [38, 57].

From a computational and practical point of view, it is useful to identify characters with supersymmetric S-functions, since it is easy to work with S-functions, for which many properties are known (see Chapter 2). As just mentioned, this identification holds for covariant and contravariant irreducible representations [22, 10], where the corresponding S-function is labeled by a single partition λ, but in general fails for mixed tensor irreducible representations, where the corresponding S-function is labeled by a composite partition $\bar{\nu};\mu$.

In Chapter 5 we show that there is still another family of atypical representations for which the character is given by a (composite) S-function, namely the so-called critical $\mathfrak{gl}(m|n)$ representations.

3.8 Atypicality and the atypicality matrix of Λ

Up to here, we studied V or $V(\Lambda)$ within the distinguished basis. For this basis we found a unique highest weight and we gave a definition of typical and atypical representations. However, there are different choices for a basis which are not necessarily isomorphic. In another basis we will have evidently another highest weight. In this section we will point out that the atypicality does not depend upon the choice of the simple roots. In order to do so, we start with Theorem 3.14. From this theorem we know that $V(\Lambda)$ being typical is equivalent to $(\Lambda+\rho,\alpha)\neq 0$, for all $\alpha\in\Delta_{\bar{1}}^{+}$. In the same way, we will say that the highest weight $\Lambda=\Lambda_{\bar{\nu};\mu}$ and the corresponding composite partition $\bar{\nu};\mu$ are typical. The module $V(\Lambda_\lambda)$ resp. $V(\Lambda_{\bar{\nu};\mu})$ will be denoted by V_λ resp. $V_{\bar{\nu};\mu}$.

If $(\Lambda+\rho,\alpha)=0$, for some $\alpha\in\Delta_{\bar{1}}^{+}$, the representation V_Λ is called atypical. If there is only one such positive root α, the representation is called singly atypical. Let $\Lambda\in\mathfrak{h}^*$; the ATYPICALITY OF Λ, denoted by atyp$(\Lambda+\rho)$, is the maximal number of linearly independent roots β_i such that $(\beta_i,\beta_j)=0$ and $(\Lambda+\rho,\beta_i)=0$ for all i and j [34]. Such a set $\{\beta_i\}$ is called a Λ-MAXIMAL ISOTROPIC SUBSET of Δ.

Given a set of positive roots Δ^+ of Δ, and a simple odd root α, one may construct a new set of positive roots [34, 52] by

$$\Delta^{+\prime} = (\Delta^+ \cup \{-\alpha\}) \setminus \{\alpha\}. \tag{3.28}$$

The set $\Delta^{+\prime}$ is called a SIMPLE REFLECTION of Δ^+. Since we use only simple reflections with respect to simple odd roots, $\Delta_{\bar{0}}^+$ remains invariant, but $\Delta_{\bar{1}}^+$ will change and the new ρ is given by :

$$\rho' = \rho + \alpha. \tag{3.29}$$

Let V be a finite-dimensional irreducible $\mathfrak{g}$-module and Λ the highest weight of V in the distinguished basis. If $\Delta^{+\prime}$ is obtained from Δ^+ by a simple α-reflection, where α is odd, and Λ' denotes the highest weight of V with respect to $\Delta^{+\prime}$, then [34]

$$\Lambda' = \Lambda - \alpha \text{ if } (\Lambda, \alpha) \neq 0; \qquad \Lambda' = \Lambda \text{ if } (\Lambda, \alpha) = 0. \tag{3.30}$$

If α is a *simple* odd root from Δ^+ then $(\rho, \alpha) = \frac{1}{2}(\alpha, \alpha) = 0$ [34, p. 421], and therefore, following (3.29) and (3.30) :

$$\begin{aligned} \Lambda' + \rho' &= \Lambda + \rho \ \text{ if } \ (\Lambda + \rho, \alpha) \neq 0, \\ \Lambda' + \rho' &= \Lambda + \rho + \alpha \ \text{ if } \ (\Lambda + \rho, \alpha) = 0. \end{aligned} \tag{3.31}$$

From this, one deduces that for the $\mathfrak{g}$-module V, atyp$(\Lambda + \rho)$ is independent of the choice of Δ^+; then atyp$(\Lambda+\rho)$ is referred to as the atypicality of V. If atyp$(\Lambda+\rho) = 0$, V is TYPICAL, otherwise it is ATYPICAL.

Let us first consider the covariant case with $\lambda \in \mathcal{H}_{m,n}$ where we compute the atypicality of V_λ in the distinguished basis. For this purpose, it is sufficient to compute the numbers $(\Lambda_\lambda + \rho, \beta_{ij})$, with $\beta_{ij} = \epsilon_i - \delta_j$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, and count the number of zeros. It is convenient to put the numbers $(\Lambda_\lambda + \rho, \beta_{ij})$ in a $(m \times n)$-matrix, the ATYPICALITY MATRIX $A(\Lambda)$ [70, 71], and give the matrix entries in the (m, n)-rectangle together with the Young frame of λ. This is illustrated here in Figure 3.2 for $\lambda = (11, 9, 4, 3, 2, 2, 2, 1)$ in $\mathfrak{gl}(8|5)$. So for this example, atyp$(\Lambda_\lambda + \rho) = 3$, since the atypicality matrix contains three zeros. In the following, we will sometimes refer to (i, j) as the position of β_{ij}; the position is also identified by a box in the (m, n)-rectangle.

In the more general case, Λ corresponds to a composite partition $\bar{\nu}; \mu$. Let us now consider the atypicality of $V_{\bar{\nu};\mu}$, in the distinguished basis. The entries of the atypicality

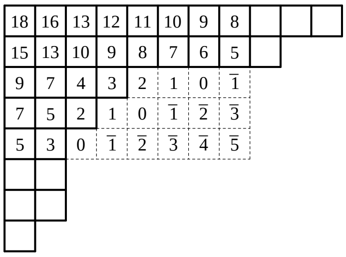


Figure 3.2: Atypicality matrix $A(\Lambda)$ of $\Lambda_\lambda = (11, 9, 4, 3, 2; 3, 2, 0, 0, 0, 0, 0, 0)$.

matrix $A(\Lambda_{\bar{\nu};\mu})$ in terms of the composite partition $\bar{\nu};\mu$ in $\mathfrak{gl}(m|n)$ are given by:

$$\begin{aligned} A(\Lambda)_{ij} &= \mu_i + \mu'_j - i - j + 1 && \text{for } 1 \le i \le I \text{ and } 1 \le j \le J; \\ A(\Lambda)_{i,n-k+1} &= \mu_i - \nu'_k - i + k + m - n && \text{for } 1 \le i \le I \text{ and } 1 \le k \le K; \\ A(\Lambda)_{m-l+1,j} &= \mu'_j - \nu_l - j + l - m + n && \text{for } 1 \le l \le L \text{ and } 1 \le j \le J; \\ A(\Lambda)_{m-l+1,n-k+1} &= -\nu_l - \nu'_k + l + k - 1 && \text{for } 1 \le l \le L \text{ and } 1 \le k \le K. \end{aligned} \tag{3.32}$$

This is illustrated in general in Figure 3.3.

It might be noted that $A(\Lambda)_{ij} = h_{ij}$ resp. $A(\Lambda)_{m-l+1,n-k+1} = -h_{lk}$, where h_{ij} is the hook length of the box in the ith row and jth column of the Young diagram F^μ resp. F^ν. These entries of the atypicality matrix are non-zero, since h_{ij} and h_{kl} are necessarily positive. If $V_{\bar{\nu};\mu}$ is atypical, the zero's occur in part ② or ③ or in both of them, see Figure 3.3.

The properties of the atypicality matrix $A(\Lambda)$ have been studied in detail in [26]. We summarize here some of these properties.

Lemma 3.16 *1. Let $\Lambda = [a_1, a_2, \ldots, a_{m-1}; a_m; a_{m+1}, \ldots, a_{m+n-1}]$ be the Kac-Dynkin label of Λ in $\mathfrak{gl}(m|n)$; then*

$$\begin{aligned} A(\Lambda)_{ij} - A(\Lambda)_{i+1,j} &= a_i + 1, && 1 \le i \le m-1, 1 \le j \le n, \\ A(\Lambda)_{m,1} &= a_0, && \\ A(\Lambda)_{ij} - A(\Lambda)_{i,j+1} &= a_{m+j} + 1, && 1 \le i \le m, 1 \le j \le n-1. \end{aligned} \tag{3.33}$$

2. An atypicality matrix $A(\Lambda)$ satisfies $A(\Lambda)_{ij} + A(\Lambda)_{kl} = A(\Lambda)_{il} + A(\Lambda)_{kj}$. Conversely, any $(m \times n)$ matrix satisfying this condition for all pairs (i, j) and (k, l) with $1 \le i, k \le m$ and $1 \le j, l \le n$ is the atypicality matrix of a unique element $\Lambda \in \mathfrak{h}^*$.

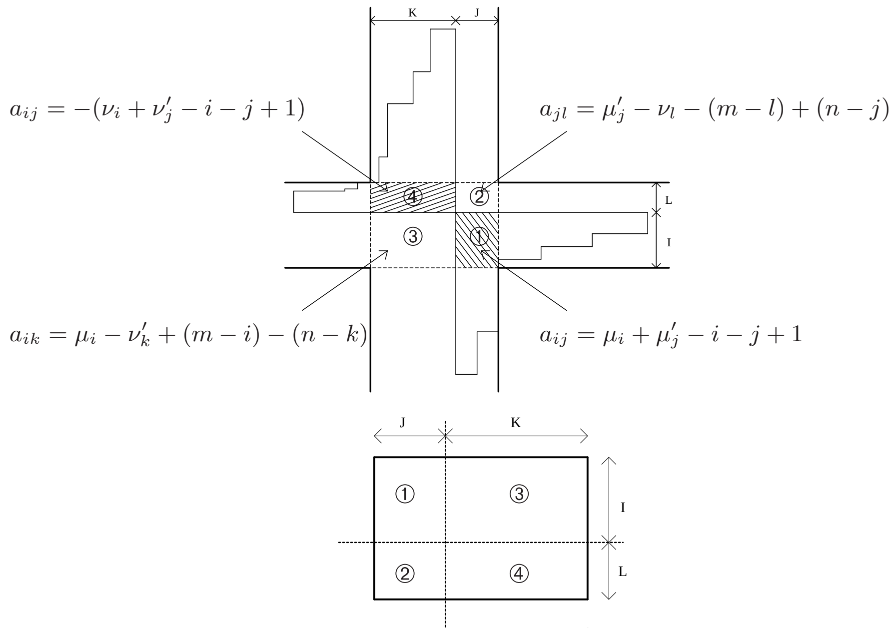


Figure 3.3: Atypicality matrix of $\Lambda_{\bar{\nu};\mu}$.

3. *Λ is dominant if and only if*

$$
\begin{aligned}
&A(\Lambda)_{ij} - A(\Lambda)_{i+1,j} - 1 \geq 0, \quad 1 \leq i < m, 1 \leq j \leq n,\\
&A(\Lambda)_{ij} - A(\Lambda)_{i,j+1} - 1 \geq 0, \quad 1 \leq i \leq m, 1 \leq j < n.
\end{aligned} \tag{3.34}
$$

Moreover, Λ is integral dominant if the expressions on the left-hand side of (3.34) are all integers.

As for finite dimensional modules the highest weight Λ is necessarily integral dominant, the zeros of $A(\Lambda)$ lie in distinct rows and columns. Furthermore, a zero lies to the right of another zero if and only if it lies above it.

Chapter 4

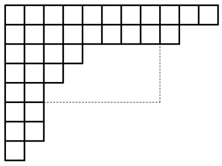
A determinantal formula for supersymmetric Schur polynomials

In this chapter we derive a new formula for the supersymmetric Schur polynomial $s_\lambda(x/y)$. The origin of this formula goes back to representation theory of the Lie superalgebra $\mathfrak{gl}(m|n)$. In particular, we show how a character formula due to Kac and Wakimoto can be applied to covariant representations, leading to a new expression for $s_\lambda(x/y)$. This new expression gives rise to a determinantal formula for $s_\lambda(x/y)$. In particular, the denominator identity for $\mathfrak{gl}(m|n)$ corresponds to a determinantal identity combining Cauchy's double alternant with Vandermonde's determinant. We provide a second and independent proof of the new determinantal formula by showing that it satisfies the four characteristic properties of supersymmetric Schur polynomials. A third and more direct proof ties up our formula with that of Sergeev-Pragacz.

4.1 Introduction

Just as the functions $s_\lambda(x)$ are characters of simple modules of the Lie algebra $\mathfrak{gl}(m)$, the supersymmetric S-functions are characters of (a class of) simple modules of the Lie superalgebra $\mathfrak{gl}(m|n)$ [10]. In this context, a different formula for $s_\lambda(x/y)$ was found by Sergeev (see [55]) and in [70]; the first proof of this formula was given by

Pragacz [55]. To describe the so-called Sergeev-Pragacz formula, let λ be a partition with $\lambda_{m+1} \leq n$. Consider the Young diagram F^λ, let F^κ be the part of F^λ that falls within the $m \times n$ rectangle, and let F^τ, resp. F^η, be the remaining part to the right, resp. underneath this rectangle; i.e. $\lambda = (\kappa + \tau) \cup \eta$. This is illustrated, for $m = 5$, $n = 8$ and $\lambda = (11, 9, 4, 3, 2, 2, 2, 1)$, as follows :

$F^\lambda =$  hence

$$\begin{aligned} \kappa &= (8,8,4,3,2) \\ \tau &= (3,1) \\ \eta &= (2,2,1) \end{aligned} \tag{4.1}$$

Then, the Sergeev-Pragacz formula for $s_\lambda(x/y)$ is given by

$$s_\lambda(x/y) = D_0^{-1} \sum_{w \in S_m \times S_n} \varepsilon(w)\, w\Big(x^{\tau+\delta_m} y^{\eta'+\delta_n} \prod_{(i,j)\in F^\kappa} (x_i + y_j) \Big), \tag{4.2}$$

where $(i, j) \in F^\kappa$ if and only if the box with row-index i (read from left to right) and column-index j (read from top to bottom) belongs to F^κ, and

$$D_0 = \prod_{1\leq i<j\leq m} (x_i - x_j) \prod_{1\leq i<j\leq n} (y_i - y_j). \tag{4.3}$$

This formula is useful for the computation of $s_\lambda(x/y)$, and even for the computation of Littlewood-Richardson coefficients [11, 69]. Note that for the special case that $\lambda_m \geq n$, (4.2) becomes the Berele-Regev formula (2.8).

Observe that D_0 in (4.2) is just Weyl's denominator for $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. So when $\lambda = 0$, (4.2) does not yield a new denominator identity related to $\mathfrak{gl}(m|n)$; it only gives the denominator identity for $\mathfrak{gl}(m)$ and $\mathfrak{gl}(n)$.

A simple comparison between symmetric and supersymmetric functions reveals that most of the formulas of Chapter 1 have a supersymmetric form. Nevertheless, a determinantal formula for supersymmetric functions similar to (1.10) did not exist yet. In this chapter, we shall give a new formula for $s_\lambda(x/y)$. In its simplest form, this yields a new determinantal formula for supersymmetric S-functions. Furthermore, this formula yields a genuine denominator identity related to $\mathfrak{gl}(m|n)$.

First, we introduce some new notations. Let

$$D(x) = \prod_{1\leq i<j\leq m} (x_i - x_j) \qquad \text{and} \qquad E(x,y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i - y_j) \tag{4.4}$$

and define D by

$$D = \frac{\prod_{1\le i<j\le m}(x_i - x_j)\prod_{1\le i<j\le n}(y_i - y_j)}{\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j)} = \frac{D(x)D(y)}{E(x,-y)}. \tag{4.5}$$

Let λ be a partition with $\lambda_{m+1} \le n$ and put

$$k = \min\{j|\lambda_j + m + 1 - j \le n\}; \tag{4.6}$$

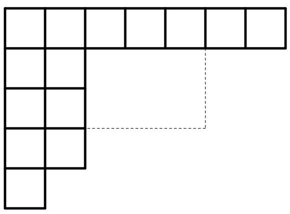
since $\lambda_{m+1} \le n$, we have that $1 \le k \le m+1$. Then the new formula reads

$$s_\lambda(x/y) = (-1)^{mn-m+k-1} D^{-1} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \tag{4.7}$$

where the (rectangular) blocks of the determinant are given by

$$\begin{aligned} R &= \left(\frac{1}{x_i + y_j}\right)_{\substack{1\le i\le m,\\ 1\le j\le n}}, \\ X_\lambda &= \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m,\\ 1\le j\le k-1}}, \qquad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1,\\ 1\le j\le n}}. \end{aligned}$$

For example, let $m = 3$, $n = 5$ and $\lambda = (7,2,2,2,1)$. Then $\kappa = (5,2,2)$, $\tau = (2)$, $\eta = (2,1)$ and $k = 2$ (see Figure 4.1).



$k = 2$ since:

$$\lambda_1 + 3 + 1 - 1 = 10 > 5,$$
$$\lambda_2 + 3 + 1 - 2 = 4 \le 5.$$

Figure 4.1: The partition $\lambda = (7,2,2,2,1)$ in $\mathfrak{gl}(3|5)$.

Thus, according to formula (4.7),

$$s_{(7,2,2,2,1)} = -D^{-1} \det \begin{pmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \frac{1}{x_1+y_3} & \frac{1}{x_1+y_4} & \frac{1}{x_1+y_5} & x_1^4 \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3} & \frac{1}{x_2+y_4} & \frac{1}{x_2+y_5} & x_2^4 \\ \frac{1}{x_3+y_1} & \frac{1}{x_3+y_2} & \frac{1}{x_3+y_3} & \frac{1}{x_3+y_4} & \frac{1}{x_3+y_5} & x_3^4 \\ y_1^6 & y_2^6 & y_3^6 & y_4^6 & y_5^6 & 0 \\ y_1^4 & y_2^4 & y_3^4 & y_4^4 & y_5^4 & 0 \\ y_1^0 & y_2^0 & y_3^0 & y_4^0 & y_5^0 & 0 \end{pmatrix}. \tag{4.8}$$

When $\lambda = 0$ it follows from (2.9) or (4.2) that $s_\lambda(x/y) = 1$. The new formula (4.7) gives rise to a denominator identity for $\mathfrak{gl}(m|n)$. Suppose $m \leq n$ ($m \geq n$ is similar); when $\lambda = 0$, it follows from (4.6) that $k = 1$. So the X_λ-block and 0-block disappear in (4.7). Changing the order of the R-block and Y_λ-block, implies

$$\det \begin{pmatrix} y_1^{n-m-1} & \cdots & y_n^{n-m-1} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_n^0 \\ \frac{1}{x_1+y_1} & \cdots & \frac{1}{x_1+y_n} \\ \vdots & & \vdots \\ \frac{1}{x_m+y_1} & \cdots & \frac{1}{x_m+y_n} \end{pmatrix} = \frac{\prod_{1\leq i<j\leq m}(x_i - x_j)\prod_{1\leq i<j\leq n}(y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j)}. \tag{4.9}$$

Clearly, when $m = n$, this is simply Cauchy's double alternant; when $m = 0$, it is just Vandermonde's determinant. When $0 < m < n$, it is a combination of the two. These type of determinants have already been encountered in a different context [8] (we found this reference in [43]); here they are for the first time related to a denominator identity.

4.2 Covariant modules for $\mathfrak{gl}(m|n)$

The first step to find a determinantal formula for the supersymmetric functions labeled by a partition λ is showing that the covariant modules are tame. The covariant modules can be labeled by a partition $\lambda \in \mathcal{H}_{m,n}$ (see § 3.7). This means that λ is inside the (m,n)-hook, i.e. such that $\lambda_{m+1} \leq n$.

Let us first consider V_λ in the distinguished basis fixed by (3.10), namely

$$\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}.$$

Then, the highest weight Λ_λ of V_λ in the standard ϵ-δ-basis is given by [70]

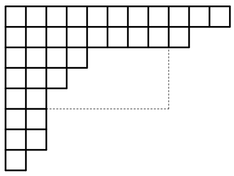
$$\Lambda_\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_m\epsilon_m + \nu'_1\delta_1 + \cdots + \nu'_n\delta_n, \tag{4.10}$$

where $\nu'_j = \max\{0, \lambda'_j - m\}$ for $1 \leq j \leq n$. See Figure 4.2.

From the combinatorics of atypicality matrices [71], it follows that the Λ_λ-maximal isotropic subset (see § 3.8) is given by:

$$S_{\Lambda_\lambda} = \{\beta_{i,\lambda_i+m+1-i} | 1 \leq i \leq m,\ 1 \leq \lambda_i + m + 1 - i \leq n\}.$$

That is to say, one finds the zeros in the atypicality matrix as follows : on row m in column $\lambda_m + 1$; on row $m-1$ in column $\lambda_{m-1} + 2$; in general one has, see also Figure 4.3 :



$$\longleftrightarrow \qquad \Lambda_\lambda = (11, 9, 4, 3, 2, ; 3, 2, 0, 0, 0, 0, 0, 0)$$

Figure 4.2: The Young diagram of a partition λ and the corresponding highest weight Λ_λ.

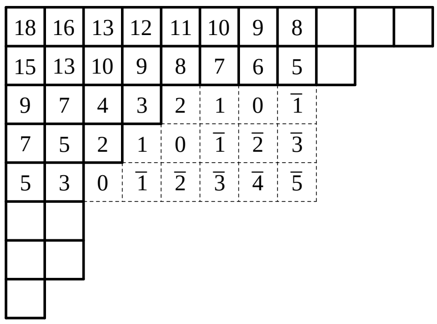


Figure 4.3: Atypicality matrix $A(\Lambda_\lambda)$ together with λ

Proposition 4.1 *The atypicality matrix has its zeros on row i and in column $\lambda_i + m - i + 1$ ($i = m, m-1, \ldots$) as long as these column indices are not exceeding n.*

If one can choose a $(\Lambda + \rho)$-maximal isotropic subset S_Λ in Δ^+ such that $S_\Lambda \subset \Pi \subset \Delta^+$ (Π is the set of simple roots with respect to Δ^+), then the $\mathfrak{g}$-module V is called TAME, and a character formula is known due to Kac and Wakimoto [34]. It reads :

$$\operatorname{ch} V = j_\Lambda^{-1} e^{-\rho} R^{-1} \sum_{w \in W} \varepsilon(w) w \Big(e^{\Lambda+\rho} \prod_{\beta \in S_\Lambda} (1 + e^{-\beta})^{-1} \Big), \tag{4.11}$$

where

$$R = \prod_{\alpha \in \Delta_{\bar 0}^+} (1 - e^{-\alpha}) \Big/ \prod_{\alpha \in \Delta_{\bar 1}^+} (1 + e^{-\alpha}) \tag{4.12}$$

and j_Λ is a normalization coefficient to make sure that the coefficient of e^Λ on the right hand side of (4.11) is 1.

Clearly, S_{Λ_λ} is in general not a subset of the set of simple roots Π (since Π contains only one odd root, $\epsilon_m - \delta_1$). So formula (4.11) cannot be applied. The purpose is to show that there exists a sequence of simple odd α-reflections such that for the new $\Delta^{+\prime}$, where the module V_λ has highest weight Λ', there exists a $(\Lambda' + \rho')$-maximal isotropic subset $S_{\Lambda'}$ with $S_{\Lambda'} \subset \Pi' \subset \Delta^{+\prime}$.

In § 3.8 we already introduced the notion of a simple odd reflection. We will repeat these notions here. Given a set of positive roots Δ^+ of Δ, and a simple odd root α, the new set of positive roots [34, 52] Δ', the simple reflection of Δ, is given by

$$\Delta^{+\prime} = (\Delta^+ \cup \{-\alpha\}) \setminus \{\alpha\}.$$

Since we use only simple reflections with respect to simple odd roots, $\Delta_{\bar{0}}^+$ remains invariant, but $\Delta_{\bar{1}}^+$ will change and the new ρ is given by $\rho' = \rho + \alpha$.

Let V be a finite-dimensional irreducible $\mathfrak{g}$-module and Λ the highest weight of V in the distinguished basis. If $\Delta^{+\prime}$ is obtained from Δ^+ by a simple α-reflection, where α is odd, and Λ' denotes the highest weight of V with respect to $\Delta^{+\prime}$, then [34]

$$\Lambda' = \Lambda - \alpha \text{ if } (\Lambda, \alpha) \neq 0; \qquad \Lambda' = \Lambda \text{ if } (\Lambda, \alpha) = 0.$$

If α is a *simple* odd root from Δ^+ then $(\rho, \alpha) = \frac{1}{2}(\alpha, \alpha) = 0$ [34, p. 421], and therefore, following (3.29) and (3.30) :

$$\begin{aligned} \Lambda' + \rho' &= \Lambda + \rho \text{ if } (\Lambda + \rho, \alpha) \neq 0, \\ \Lambda' + \rho' &= \Lambda + \rho + \alpha \text{ if } (\Lambda + \rho, \alpha) = 0. \end{aligned} \tag{4.13}$$

Definition 4.2 *For $\lambda \in \mathcal{H}_{m,n}$, the (m, n)-index of λ is the number*

$$k = \min\{i | \lambda_i + m + 1 - i \leq n\}, \qquad (1 \leq k \leq m + 1). \tag{4.14}$$

In what follows, k will always denote this number; it will be a crucial entity for our developments. In the atypicality matrix, k corresponds to the smallest row number in which there occurs a zero; see Figure 4.3. By Proposition 4.1, $m - k + 1$ is the atypicality of V_λ. If $k = m + 1$, then $S_{\Lambda_\lambda} = \emptyset$ and V_λ is typical and trivially tame. Thus in the following, we shall assume that $k \leq m$. To begin with, Δ^+ corresponds to the distinguished choice, and Π is the distinguished set of simple roots (3.10). The highest weight of V_λ is given by Λ_λ. Denote $\Lambda^{(1)} = \Lambda_\lambda$, $\rho^{(1)} = \rho$ and $\Pi^{(1)} = \Pi$. Now we perform a sequence of simple odd $\alpha^{(i)}$-reflections; each of these reflections preserve $\Delta_{\bar{0}}^+$ but may change $\Lambda^{(i)} + \rho^{(i)}$ and $\Pi^{(i)}$. Denote the sequence of reflections by :

$$\Lambda^{(1)} + \rho^{(1)},\, \Pi^{(1)} \quad \stackrel{\alpha^{(1)}}{\longrightarrow} \quad \Lambda^{(2)} + \rho^{(2)},\, \Pi^{(2)} \quad \stackrel{\alpha^{(2)}}{\longrightarrow} \quad \cdots \quad \stackrel{\alpha^{(f)}}{\longrightarrow} \quad \Lambda' + \rho',\, \Pi' \tag{4.15}$$

where, at each stage, $\alpha^{(i)}$ is an odd root from $\Pi^{(i)}$. For given λ, consider the following sequence of odd roots (with positions on row m, row $m-1$, ..., row k) :

$$\begin{array}{ll} \text{row } m\ : & \beta_{m,1}, \beta_{m,2}, \ldots, \beta_{m,\lambda_k-k+m} \\ \text{row } m-1\ : & \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,\lambda_k-k+m-1} \\ \vdots & \vdots \\ \text{row } k\ : & \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,\lambda_k} \end{array} \tag{4.16}$$

in this particular order (i.e. starting with $\beta_{m,1}$ and ending with β_{k,λ_k}). Then we have :

Lemma 4.3 *The sequence (4.16) is a proper sequence of simple odd reflections for Λ_λ, i.e. $\alpha^{(i)}$ is a simple odd root from $\Pi^{(i)}$. At the end of the sequence, one finds :*

$$\begin{aligned} \Pi' =& \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{k-1} - \delta_1, \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{\lambda_k - 1} - \delta_{\lambda_k}, \\ & \delta_{\lambda_k} - \epsilon_k, \epsilon_k - \delta_{\lambda_k+1}, \delta_{\lambda_k+1} - \epsilon_{k+1}, \epsilon_{k+1} - \delta_{\lambda_k+2}, \ldots, \delta_{\lambda_k+m-k} - \epsilon_m, \\ & \epsilon_m - \delta_{\lambda_k+m+1-k}, \delta_{\lambda_k+m+1-k} - \delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1} - \delta_n\}. \end{aligned} \tag{4.17}$$

Furthermore,

$$\Lambda' + \rho' = \Lambda_\lambda + \rho + \sum_{i=k+1}^{m} \sum_{j=\lambda_i+1}^{\lambda_k-k+i} \beta_{i,j}. \tag{4.18}$$

Proof. Let us consider, in the first stage, the reflections with respect to the roots in row m. Clearly, $\alpha^{(1)} = \beta_{m,1} = \epsilon_m - \delta_1$ is an odd root from $\Pi^{(1)} = \Pi$. Performing the reflection with respect to $\beta_{m,1}$ implies that $\Pi^{(2)}$ contains $\epsilon_{m-1} - \delta_1, \delta_1 - \epsilon_m, \epsilon_m - \delta_2$ as simple odd roots. Thus $\Pi^{(2)}$ contains $\alpha^{(2)} = \beta_{m,2}$. A reflection with respect to $\beta_{m,2}$ implies that $\Pi^{(3)}$ contains $\epsilon_{m-1} - \delta_1, \delta_2 - \epsilon_m, \epsilon_m - \delta_3$ as simple odd roots. So this process continues, and after $\lambda_k - k + m$ such reflections (i.e. at the end of row m), we have

$$\begin{aligned} \Pi^{(\lambda_k-k+m+1)} =& \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-2} - \epsilon_{m-1}, \epsilon_{m-1} - \delta_1, \\ & \delta_1 - \delta_2, \ldots, \delta_{\lambda_k-k+m-1} - \delta_{\lambda_k-k+m}, \delta_{\lambda_k-k+m} - \epsilon_m, \\ & \epsilon_m - \delta_{\lambda_k-k+m+1}, \delta_{\lambda_k+m+1-k} - \delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1} - \delta_n\}. \end{aligned} \tag{4.19}$$

Observe that this process can continue since $\lambda_k - k + m < n$ by definition of the (m,n)-index k of λ. So after the first stage (i.e. after the reflections with respect to odd roots of row m) there are three odd roots in $\Pi^{(\lambda_k-k+m+1)}$, and the set is ready to continue the reflections with respect to the elements of row $m-1$, since $\beta_{m-1,1}$ belongs to $\Pi^{(\lambda_k-k+m+1)}$. Since at each stage $\alpha^{(i)}$ is a simple odd root, (3.31) implies

$$\begin{aligned} \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} \ \text{ if } \ (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) \neq 0, \\ \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} + \alpha^{(i)} \ \text{ if } \ (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) = 0. \end{aligned}$$

Examining this explicitly for the elements of row m yields

$$\Lambda^{(\lambda_k-k+m+1)} + \rho^{(\lambda_k-k+m+1)} = \Lambda_\lambda + \rho + \sum_{j=\lambda_m+1}^{\lambda_k-k+m} \beta_{m,j}. \tag{4.20}$$

If $k=m$ the lemma follows. If $k<m$ the process continues; suppose this is the case. But now we are in a situation where the elements of row $m-1$ play completely the same role as those of row m in the first stage. This means that at the end of the second stage, the new set of simple roots is given by

$$\begin{aligned} \Pi^{(i)} =& \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-3} - \epsilon_{m-2}, \epsilon_{m-2} - \delta_1, \\ & \delta_1 - \delta_2, \ldots, \delta_{\lambda_k-k+m-2} - \delta_{\lambda_k-k+m-1}, \delta_{\lambda_k-k+m-1} - \epsilon_{m-1}, \\ & \epsilon_{m-1} - \delta_{\lambda_k-k+m}, \delta_{\lambda_k-k+m} - \epsilon_m, \epsilon_m - \delta_{\lambda_k-k+m+1}, \\ & \delta_{\lambda_k+m+1-k} - \delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1} - \delta_n\}, \end{aligned} \tag{4.21}$$

and the new $\Lambda^{(i)} + \rho^{(i)}$ by

$$\Lambda^{(i)} + \rho^{(i)} = \Lambda_\lambda + \rho + \sum_{j=\lambda_m+1}^{\lambda_k-k+m} \beta_{m,j} + \sum_{j=\lambda_{m-1}+1}^{\lambda_k-k+m-1} \beta_{m-1,j} \tag{4.22}$$

(the last addition follows by inspecting the atypicality matrix). Continuing with the remaining stages (i.e. rows in (4.16)) leads to (4.17) and (4.18). □

Corollary 4.4 *The covariant module V_λ is tame.*

Proof. Having performed the simple odd reflections (4.16), one can compute the atypicality matrix for $\Lambda' + \rho'$ using (4.18). This gives :

$$(\Lambda' + \rho', \beta_{ij}) = 0 \text{ for all } (i,j) \text{ with } k \leq i \leq m, \lambda_k + 1 \leq j \leq \lambda_k + m + 1 - k. \tag{4.23}$$

Therefore the set

$$S_{\Lambda'} = \{\epsilon_k - \delta_{\lambda_k+1}, \epsilon_{k+1} - \delta_{\lambda_k+2}, \ldots, \epsilon_m - \delta_{\lambda_k+m+1-k}\} \tag{4.24}$$

is a $(\Lambda' + \rho')$-maximal isotropic subset. Furthermore, $S_{\Lambda'} \subset \Pi'$, see (4.17). □

Let us illustrate some of these notions for the partition λ of Figure 4.3 in Figure 4.4 and in Figure 4.5.

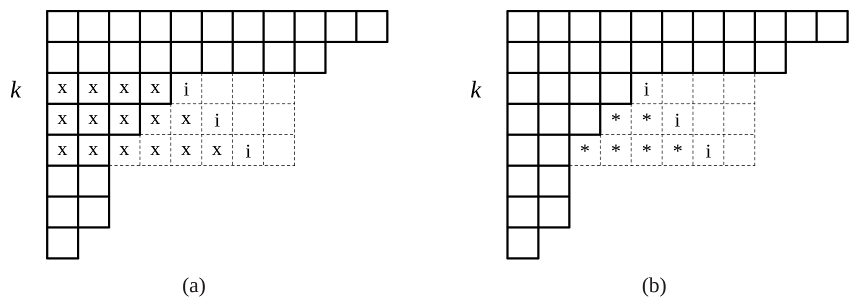


Figure 4.4: Notions concerning simple odd reflections.

In Figure 4.4(a) the positions marked with "i" refer to the $(\Lambda' + \rho')$-maximal isotropic set (4.24). The first element, with row index k, is simply the position of the box just to the right of the Young diagram F^{λ} in row k. For the remaining positions one continues in the direction of the diagonal until one reaches row m. For convenience, let us refer to these positions as "the isotropic diagonal." The positions of the odd roots that have been used for the sequence of reflections to go from Λ_{λ} and Π to Λ' and Π' are marked by "x" in Figure 4.4(a). So, they are simply all positions to the left of the isotropic diagonal. Finally, Figure 4.4(b) shows the positions of those β_{ij} that appear on the right hand side of (4.18); they are marked by "*". These are all positions to the left of the isotropic diagonal that are not inside F^{λ}.

This is illustrated in more details in Figure 4.5 for the same λ.

In this figure, the boxes indexed by "X" refer to the simple odd roots in Π' at that stage. The boxes marked with "0" refer to the positions of the zero's in the atypicality matrix. The odd root $\beta_{i,j}$ above every arrow, is the simple odd root used for the simple reflection at that stage. For the given partition λ, the corresponding set of simple roots Π contains only one odd root, more specifically $\beta_{5,1}$. Using $\beta_{5,1}$ in the first step to perform a simple odd reflection, provides us three simple odd roots; $\beta_{5,1}$, $\beta_{5,2}$ and $\beta_{4,1}$. So we can use $\beta_{5,2}$ in the next step and the new set of simple roots contains the odd roots $\beta_{4,2}$, $\beta_{5,2}$ and $\beta_{5,3}$. Looking at the position of the zeros in the atypicality matrix in the first two steps in this sequence of odd reflections, we can see that for neither $\beta_{5,1}$

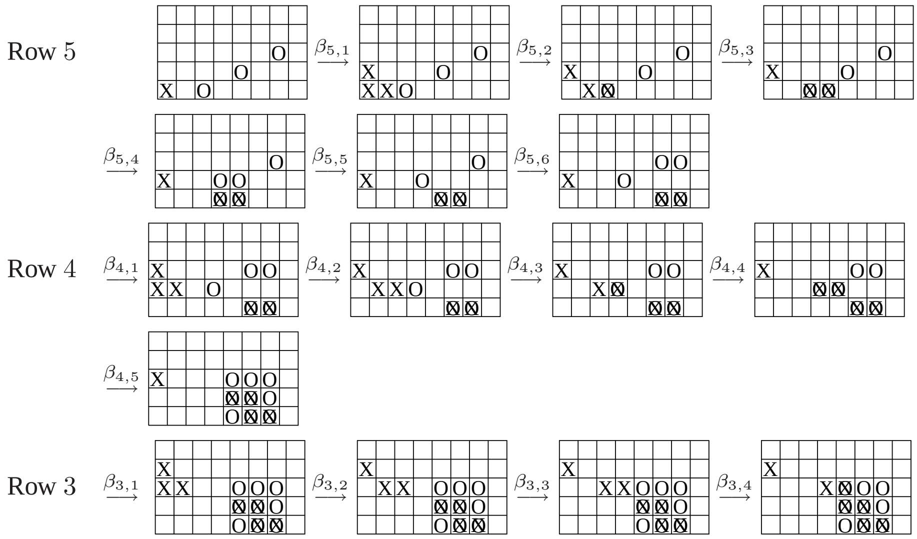


Figure 4.5: The simple odd roots of Π' and the positions of the zero's in the atypicality matrix

or $\beta_{5,2}$ the atypicality matrix contains a zero on the position corresponding to these odd roots. Following (4.13), the atypicality matrix will not change in the first two steps. To accomplish the simple odd reflection with respect to $\beta_{5,3}$, the atypicality matrix will change as on position $(5,3)$ the atypicality matrix contains a zero. According to (4.13), elements in the fifth row of the atypicality matrix will increase by one, the elements in the third column will decrease by one. Given the atypicality matrix (see Figure 4.3), we can see that a zero appears on position $(5,4)$. In the next step, the atypicality matrix will change again by increasing the last row and decreasing the fourth column by one. This will make the zeros to move along the last row as shown in the next rectangle. After finishing the simple reflections with respect to the odd roots in the last row, we can continue the process in the fourth row, as $\beta_{4,2}$ became a simple odd root in the first step. In the same way, the process will continue with respect to the odd roots in the remaining rows corresponding to (4.16). At the end of the simple reflections with respect to the fourth row, there will be nine zeros in the atypicality matrix placed together in a square. We still need the reflections with respect to the roots in the third row, with row index k, in order to obtain a set of three independent odd roots β_1, β_2 and β_3 such that $(\Lambda' + \rho', \beta_i) = 0$, for $i = 1,\ 2$ or 3.

One can see from these examples that the (m,n)-index k determines all other necessary ingredients.

We are now in a position to evaluate the character formula (4.11),

$$\operatorname{ch} V_\lambda = j_{\Lambda'}^{-1} e^{-\rho'} R'^{-1} \sum_{w\in W} \varepsilon(w) w\Big(e^{\Lambda'+\rho'} \prod_{\beta\in S_{\Lambda'}} (1+e^{-\beta})^{-1}\Big), \tag{4.25}$$

where R' is given by (4.12) with $\Delta_{\bar{1}}^+$ replaced by $\Delta'_{1,+}$ ($\Delta_{\bar{0}}^+$ remains unchanged). The mn elements of $\Delta_{\bar{1}}^{+\prime}$ are $\pm\beta_{ij}$, where one must take the minus-sign if β_{ij} appears in the list (4.16) (i.e. if its position is marked by "x" in Figure 4.4(a) and the plus-sign otherwise. However, all this information is not necessary here, since by definition of ρ and R

$$e^{-\rho'} R'^{-1} = e^{-\rho} R^{-1}.$$

Putting, as usual in this context,

$$x_i = e^{\epsilon_i}, \qquad y_j = e^{\delta_j} \qquad (1 \le i \le m, 1 \le j \le n), \tag{4.26}$$

one has

$$e^{-\rho} R^{-1} = D^{-1} \prod_{i=1}^{m} x_i^{(m-n-1)/2} \prod_{j=1}^{n} y_j^{(n-m-1)/2},$$

with D given in (4.5). Using (4.18) and (4.24), one finds

$$\begin{aligned} e^{\Lambda'+\rho'} \prod_{\beta\in S_{\Lambda'}} (1+e^{-\beta})^{-1} &= \prod_{i=1}^{m} x_i^{(1-m+n)/2} \prod_{j=1}^{n} y_j^{(1+m-n)/2} \prod_{i=1}^{k-1} x_i^{\lambda_i+m-i-n} \\ &\times \prod_{i=k}^{m} x_i^{\lambda_k+m+1-k-n} \prod_{j=1}^{\lambda_k} y_j^{\lambda'_j+n-j-m} \prod_{j=\lambda_k+1}^{\lambda_k+m+1-k} y_j^{n-\lambda_k-m-1+k} \\ &\times \prod_{j=\lambda_k+m+2-k}^{n} y_j^{n-j} \Big/ \prod_{i=k}^{m} (x_i + y_{\lambda_k+i+1-k}). \end{aligned}$$

In order to rewrite this in a more appropriate form, let us introduce some further notation related to the partition $\lambda \in \mathcal{H}_{m,n}$. Clearly, the (m,n)-index k defined in (4.14) plays again an essential role. Related to this, let us also put

$$l = \lambda_k + 1, \qquad r = n - m + k - l. \tag{4.27}$$

The numbers $k,\ l,\ r$ are illustrated in Figure 4.6.

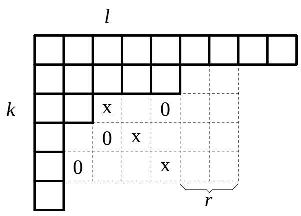


Figure 4.6: The numbers $k,\ l,\ r$ for a given partition λ in $\mathfrak{gl}(m|n)$.

Now we have

$$\operatorname{ch} V_\lambda = j_{\Lambda'}^{-1} D^{-1} \sum_{w\in W} \varepsilon(w) w(t_\lambda),$$

with

$$t_\lambda = \prod_{i=1}^{k-1} x_i^{\lambda_i+m-i-n} \prod_{j=1}^{l-1} y_j^{\lambda'_j+n-j-m} \prod_{i=k}^{m} \frac{y_{l+i-k}^r}{x_i^r(x_i+y_{l+i-k})} \prod_{j=l+m+1-k}^{n} y_j^{n-j}. \tag{4.28}$$

This form also allows us to deduce $j_{\Lambda'} = j_{\Lambda_\lambda}$. Indeed, consider in $W = S_m \times S_n$ the subgroup H of elements $w = \sigma_x \times \sigma_y$, where σ_x is a permutation of $(x_k, x_{k+1}, \ldots, x_m)$ and σ_y is the same permutation of $(y_l, y_{l+1}, \ldots, y_{l+m-k})$. Each element of H leaves t_λ invariant. Furthermore, $\varepsilon(w) = 1$ for $w \in H$. Since H is isomorphic to S_{m-k+1},

$$\sum_{w\in H} \varepsilon(w) w(t_\lambda) = (m-k+1)!\, t_\lambda.$$

So, in order to have multiplicity one for the highest weight term, one must take

$$j_{\Lambda'} = (m-k+1)!.$$

We can now conclude this section by the following two alternative formulas for the computation of $\operatorname{ch} V_\lambda$ (i.e. of $s_\lambda(x/y)$) :

$$\operatorname{ch} V_\lambda = \frac{D^{-1}}{(m-k+1)!} \sum_{w\in S_m\times S_n} \varepsilon(w) w(t_\lambda) \tag{4.29}$$

$$= D^{-1} \sum_{w\in (S_m\times S_n)/S_{m-k+1}} \varepsilon(w) w(t_\lambda), \tag{4.30}$$

where the second sum is over the cosets of $H = S_{m-k+1}$ in $S_m \times S_n$.

4.3 A determinantal formula for $s_\lambda(x/y)$

Let $\lambda \in \mathcal{H}_{m,n}$; the important quantities related to λ are given by the (m,n)-index k (see (4.14)) and the related numbers l and r (see (4.27)). Since $\mathrm{ch}\, V_\lambda = s_\lambda(x/y)$, (4.29) or (4.30), together with the expression (4.28) for t_λ, yield a (new) expression for the supersymmetric S-function. In this section we shall rewrite (4.29) in a nicer form; in particular we shall show that it is equivalent to a determinantal form for $s_\lambda(x/y)$.

The first step in this process is the following :

Lemma 4.5 *Let t_λ be given by (4.28). Then*

$$\frac{1}{(m-k+1)!} \sum_{w \in S_m \times S_n} \varepsilon(w) w(t_\lambda) = (-1)^{(k-1)(l-1)} \det(C), \tag{4.31}$$

where C is the following square matrix of order $n+k-1$:

$$C = \begin{pmatrix} 0 & Y_\lambda^{(1)} \\ X_\lambda & R^{(r)} \\ 0 & Y^{(r)} \end{pmatrix} \tag{4.32}$$

with

$$X_\lambda = \left(x_i^{\lambda_j+m-n-j} \right)_{\substack{1\le i\le m,\\ 1\le j\le k-1}}, \qquad R^{(r)} = \left(\frac{y_j^r}{x_i^r(x_i+y_j)} \right)_{\substack{1\le i\le m,\\ 1\le j\le n}} \tag{4.33}$$

$$Y_\lambda^{(1)} = \left(y_j^{\lambda'_i+n-m-i} \right)_{\substack{1\le i\le l-1,\\ 1\le j\le n}}, \qquad Y^{(r)} = \left(y_j^{r-i} \right)_{\substack{1\le i\le r,\\ 1\le j\le n}} \tag{4.34}$$

Proof. Applying Laplace's theorem [63, Section 1.8] for the expansion of $\det(C)$ with respect to columns $1, 2, \ldots, k-1$, one finds

$$\det(C) = (-1)^{\frac{k(k-1)}{2}} \sum_{1\le i'_1<\cdots<i'_{k-1}\le n+k-1} (-1)^{i'_1+\cdots+i'_{k-1}} C^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1} \bar{C}^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}, \tag{4.35}$$

where $C^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}$ is a minor of C of order $k-1$ (i.e. the determinant of the submatrix of C obtained by keeping only rows $i'_1,\ldots,i'_{k-1}$ and columns $1,\ldots,k-1$), and $\bar{C}^{i'_1,\ldots,i'_{k-1}}_{1,\ldots,k-1}$ is the complementary minor (i.e. the determinant of the submatrix of C obtained by deleting rows $i'_1,\ldots,i'_{k-1}$ and columns $1,\ldots,k-1$). Because of the zero blocks in C, the only row indices that yield a nonzero minor are those with

$l \leq i'_1 < \cdots < i'_{k-1} \leq l + m - 1$. So it is convenient to put $i_\kappa = i'_\kappa - l + 1$ $(1 \leq \kappa \leq k-1)$. Then,

$$\det(C) = (-1)^{\frac{k(k-1)}{2}} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq m} (-1)^{i_1 + \cdots + i_{k-1} + (k-1)(l-1)} C^{i_1,\ldots,i_{k-1}}_{1,\ldots,k-1} \bar{C}^{i_1,\ldots,i_{k-1}}_{1,\ldots,k-1}, \tag{4.36}$$

where

$$C^{i_1,\ldots,i_{k-1}}_{1,\ldots,k-1} = \det \begin{pmatrix} x_{i_1}^{\lambda_1+(m-1)-n} & \cdots & x_{i_1}^{\lambda_{k-1}+(m-k+1)-n} \\ \vdots & & \vdots \\ x_{i_{k-1}}^{\lambda_1+(m-1)-n} & \cdots & x_{i_{k-1}}^{\lambda_{k-1}+(m-k+1)-n} \end{pmatrix},$$

$$\bar{C}^{i_1,\ldots,i_{k-1}}_{1,\ldots,k-1} = \det \begin{pmatrix} y_1^{\lambda'_1+(n-1)-m} & \cdots & y_n^{\lambda'_1+(n-1)-m} \\ \vdots & & \vdots \\ y_1^{\lambda'_{l-1}+(n-l+1)-m} & \cdots & y_n^{\lambda'_{l-1}+(n-l+1)-m} \\ \frac{y_1^r}{x_{i_k}^r(x_{i_k}+y_1)} & \cdots & \frac{y_n^r}{x_{i_k}^r(x_{i_k}+y_n)} \\ \vdots & & \vdots \\ \frac{y_1^r}{x_{i_m}^r(x_{i_m}+y_1)} & \cdots & \frac{y_n^r}{x_{i_m}^r(x_{i_m}+y_n)} \\ y_1^{r-1} & \cdots & y_n^{r-1} \\ \vdots & & \vdots \\ y_1^1 & \cdots & y_n^1 \\ y_1^0 & \cdots & y_n^0 \end{pmatrix}.$$

The number of terms on the right hand side of (4.36) is $\binom{m}{k-1}(k-1)!n! = m!n!/(m-k+1)!$, so this is the same as the number of distinct terms on the left hand side of (4.31). For $(i_1,\ldots,i_{k-1}) = (1,\ldots,k-1)$, and the diagonal term in the minor and in the complementary minor, the contribution on the right hand side of (4.36) is now easily seen to be $(-1)^{k(k-1)+(k-1)(l-1)}t_\lambda = (-1)^{(k-1)(l-1)}t_\lambda$. But by definition of the determinant, every term on the right hand side of (4.36) is (up to the overall sign factor $(-1)^{(k-1)(l-1)}$) of the form $\varepsilon(w)w(t_\lambda)$ with $w \in S_m \times S_n$. Conversely, every term of the form $\varepsilon(w)w(t_\lambda)$ appears as a term on the right hand side of (4.36). It follows that (4.31) holds. □

Lemma 4.6 *With k, l and r defined as in (4.14) and (4.27), one has*

(i) $\lambda_{k-1} - \lambda_k \geq r \geq 0$,

(ii) $\lambda'_{l-1+i} = k-1$ *for* $1 \le i \le r$.

Proof. By (4.27), $r = n-m+k-l = n-m+k-\lambda_k-1$, so by definition of k it follows that $r \ge 0$. Suppose that $\lambda_{k-1}-\lambda_k < r$; this would imply that $\lambda_{k-1}+m-(k-1)+1 \le n$, contradicting the definition of k as the minimal index for which $\lambda_j+m-j+1 \le n$ holds. So (i) holds. To prove (ii), observe that by definition of the conjugate partition,

$$\lambda'_j = k-1 \text{ for } \lambda_k+1 \le j \le \lambda_{k-1}.$$

Using (i), $\lambda'_j = k-1$ for $\lambda_k+1 \le j \le \lambda_k+r$, i.e. for $l \le j \le l+r-1$. □

Lemma 4.7 *Let C be defined as in Lemma 4.5. Then*

$$\det(C) = (-1)^{m(n-m)+l(k-1)} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \tag{4.37}$$

where X_λ has been defined in (4.33), and

$$R = \left(\frac{1}{x_i+y_j}\right)_{1\le i\le m,\ 1\le j\le n}, \qquad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i}\right)_{1\le i\le n-m+k-1,\ 1\le j\le n}.$$

Proof. If $r=0$ the result is easy; so suppose that $r>0$. Consider C, and let R_i denote its row i. For $1 \le i \le m$, multiply R_{l-1+i} by x_i^r yielding the matrix C' with rows R'_i. Clearly, $\det(C) = \prod_{i=1}^m x_i^{-r} \det(C')$, and the element on position $(l-1+i, k-1+j)$ $(1 \le i \le m;\ 1 \le j \le n)$ in C' is $\frac{y_j^r}{x_i+y_j}$. Now consider the following row operations on C' (for all i with $1 \le i \le m$) :

$$R'_{l-1+i} \longrightarrow R'_{l-1+i} - \sum_{s=0}^{r-1} (-1)^s x_i^s R'_{l+m+s}.$$

This means the matrix element $\frac{y_j^r}{x_i+y_j}$ at position $(l-1+i, k-1+j)$ becomes

$$\frac{y_j^r}{x_i+y_j} - \sum_{s=0}^{r-1} (-1)^s x_i^s y_j^{r-s-1} = \frac{y_j^r}{x_i+y_j} - \frac{y_j^r + (-1)^{r-1} x_i^r}{x_i+y_j} = \frac{(-1)^r x_i^r}{x_i+y_j}.$$

Dividing these rows by $(-1)^r x_i^r$ (and then the first $k-1$ columns by $(-1)^r$) finally leads to

$$\det(C) = (-1)^{mr+r(k-1)} \det \begin{pmatrix} 0 & Y_\lambda^{(1)} \\ X_\lambda & R \\ 0 & Y^{(r)} \end{pmatrix}$$
$$= (-1)^{r(m+k-1)+m(l-1)+n(k-1)} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda^{(1)} & 0 \\ Y^{(r)} & 0 \end{pmatrix}.$$

Using lemma 4.6 (ii), it follows that $r - i = \lambda'_{l-1+i} + n - (l-1+i) - m$, thus

$$\begin{pmatrix} Y_\lambda^{(1)} \\ Y^{(r)} \end{pmatrix} = Y_\lambda. \tag{4.38}$$

Simplifying the sign leads to (4.37). □

Combining lemmas 4.5 and 4.7 leads to the main result :

Theorem 4.8 *Let $\lambda \in \mathcal{H}_{m,n}$ and k be the (m,n)-index of λ. Then*

$$s_\lambda(x/y) = (-1)^{mn-m+k-1} D^{-1}$$
$$\times \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\le i\le m \\ 1\le j\le n}} & \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m \\ 1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1 \\ 1\le j\le n}} & 0 \end{pmatrix}. \tag{4.39}$$

In section 4.1 we have already pointed out that for $\lambda = 0$ and $m \le n$, (4.39) gives rise to the determinantal identity (4.9) combining Cauchy's double alternant with Vandermonde's determinant. It is easy to verify that for $\lambda = 0$ and $m > n$ we have that $k = n - m + 1$, and then (4.39) gives rise to an identity equivalent to (4.9).

Finally, let us consider the case with $k = m + 1$ (corresponding to a typical representation V_λ in terms of the previous section). Then the blocks X_λ and Y_λ are square matrices, and one finds :

$$\begin{aligned} s_\lambda(x/y) &= D^{-1} \det(x_i^{\lambda_j+m-n-j})_{1\le i,j\le m} \det(y_j^{\lambda'_i+n-m-i})_{1\le i,j\le n} \\ &= s_\tau(x) s_{\eta'}(y) \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j) = s_\tau(x)s_{\eta'}(y)E(x,-y), \end{aligned} \tag{4.40}$$

where τ and η are the parts of λ defined in (4.1). This is in agreement with (2.8).

4.4 Four characterizing properties

As mentioned in § 2.1.2, supersymmetric S-polynomials satisfy four properties which also characterize these polynomials. Here we shall show that the polynomials defined by means of the right hand side of (4.39) do indeed satisfy these four properties. This gives an independent proof of the determinantal expression (4.39).

So let $x^{(m)} = (x_1, \ldots, x_m)$, $y^{(n)} = (y_1, \ldots, y_n)$, $\lambda \in \mathcal{H}_{m,n}$, denote by k the (m,n)-index of λ, and define :

$$s_\lambda(x^{(m)}/y^{(n)}) = s_\lambda(x/y) = (-1)^{mn-m+k-1} D_{m,n}^{-1} \det \begin{pmatrix} R & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}, \qquad (4.41)$$

where

$$R = \left(\frac{1}{x_i + y_j} \right)_{\substack{1 \le i \le m \\ 1 \le j \le n}},$$

$$X_\lambda = \left(x_i^{\lambda_j + m - n - j} \right)_{\substack{1 \le i \le m \\ 1 \le j \le k-1}}, \qquad Y_\lambda = \left(y_j^{\lambda'_i + n - m - i} \right)_{\substack{1 \le i \le n-m+k-1 \\ 1 \le j \le n}},$$

and $D_{m,n}$ is an obvious notation for (4.5). If $\lambda \notin \mathcal{H}_{m,n}$, then $s_\lambda(x^{(m)}/y^{(n)})$ is defined to be zero.

Before showing that (4.41) satisfies the four characteristic properties, we need some preliminary results. First of all, it will sometimes be convenient to use an alternative formula to (4.41) :

Lemma 4.9 *Let $\lambda \in \mathcal{H}_{m,n}$, k the (m,n)-index of λ, $l = \lambda_k + 1$, $r = n - m + k - l$, and*

$$R^{(r)} = \left(\frac{y_j^r}{x_i^r (x_i + y_j)} \right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

then

$$s_\lambda(x^{(m)}/y^{(n)}) = (-1)^{(m-k+1)(l-1)+n(k-1)} D_{m,n}^{-1} \det \begin{pmatrix} R^{(r)} & X_\lambda \\ Y_\lambda & 0 \end{pmatrix}. \qquad (4.42)$$

Proof. The equivalence of the two determinants follows from (4.37). □

Lemma 4.10 *Let $\lambda \in \mathcal{H}_{m,n}$ and k the (m,n)-index of λ. Then $\lambda' \in \mathcal{H}_{n,m}$ and the (n,m)-index of λ' is given by $n - m + k$.*

Proof. For any partition λ and any two positive integers u and v, we have that

$$\lambda_u \le v \Rightarrow \lambda'_{v+1} \le u-1. \tag{4.43}$$

Applying this to $\lambda \in \mathcal{H}_{m,n}$ for $(u,v)=(m+1,n)$ implies that $\lambda' \in \mathcal{H}_{n,m}$. Put $k'=n-m+k$. Since k is the (m,n)-index of λ, one has by definition that $\lambda_k \le k'-1$. Applying (4.43) gives $\lambda'_{k'} \le k-1 = m-n+k'-1$, implying that the (n,m)-index of λ' is less than or equal to k'. On the other hand, suppose that the (n,m)-index of λ' is strictly less than k'. Then $\lambda'_{k'-1} \le m-n+k'-2=k-2$. But then (4.43) implies $\lambda_{k-1} \le k'-2 = n-m+k-2$, contradicting the fact that k is (m,n)-index of λ. □

Corollary 4.11

$$s_{\lambda'}(y^{(n)}/x^{(m)}) = s_\lambda(x^{(m)}/y^{(n)}). \tag{4.44}$$

Proof. Using (4.41), Lemma 4.10, and the fact that transposing a matrix leaves the determinant invariant, the result follows. □

Now we are in a position to prove the validity of the four characteristic properties.

Proposition 4.12 (Homogeneity) *$s_\lambda(x^{(m)}/y^{(n)})$ is a homogeneous polynomial of degree $|\lambda|$.*

Proof. The factor $D_{m,n}^{-1}$ on the right hand side of (4.41) stands for the multiplication by all (x_i+y_j) and for the division by all (x_i-x_j) and (y_i-y_j) ($i<j$). Clearly, the determinant on the right hand side of (4.41) is divisible by all (x_i-x_j) and (y_i-y_j) ($i<j$). Hence $s_\lambda(x^{(m)}/y^{(n)})$ is a polynomial, and by definition of the determinant it is also homogeneous. From this determinant, one finds

$$\begin{aligned}\deg s_\lambda(x^{(m)}/y^{(n)}) \quad &= \quad mn - m(m-1)/2 - n(n-1)/2 + \Big(\sum_{j=1}^{k-1}(\lambda_j+m-n-j) \\ &+ \sum_{i=1}^{n-m+k-1}(\lambda'_i+n-m-i) - (m-k+1)\Big) = |\lambda|,\end{aligned}$$

since $\sum_{j=1}^{k-1}\lambda_j + \sum_{i=1}^{n-m+k-1}\lambda'_i = |\lambda| + (n-m+k-1)(k-1)$. □

Proposition 4.13 (Factorization) *If the partition λ satisfies $\lambda_m \ge n \ge \lambda_{m+1}$, so that λ can be written in the form $\lambda = ((n^m)+\tau)\cup\eta$, with τ (resp. η) a partition of length $\le m$ (resp. $\le n$), then*

$$s_\lambda(x^{(m)}/y^{(n)}) = s_\tau(x^{(m)})s_{\eta'}(y^{(n)})\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j).$$

Proof. If $n \geq \lambda_{m+1}$, then $k = m+1$, and then the right hand side of (4.41) reduces to the right hand side of (4.40). □

Proposition 4.14 (Cancellation) *Let $m, n \geq 1$. The result of substituting $x_m = t$ and $y_n = -t$ in $s_\lambda(x^{(m)}/y^{(n)})$ is the polynomial $s_\lambda(x^{(m-1)}/y^{(n-1)})$.*

Proof. If k, the (m,n)-index of λ, is equal to $m+1$, then it follows from Proposition 4.13 that the result of this substitution is 0; but also $s_\lambda(x^{(m-1)}/y^{(n-1)}) = 0$ since then $\lambda \notin \mathcal{H}_{m-1,n-1}$.
So let λ be such that $k \leq m$. Then $\lambda \in \mathcal{H}_{m-1,n-1}$ and the $(m-1,n-1)$-index of λ is also k. Consider the right hand side of (4.41); divide $D^{-1}_{m,n}$ by $(x_m + y_n)$ and multiply row m of the matrix by $(x_m + y_n)$. Now make the substitution $x_m = t$ and $y_n = -t$; one obtains $D^{-1}_{m-1,n-1}$. In the determinant, all entries of row m are zero except the entry in column n, which is 1. So, apart from introducing an extra factor $(-1)^{m+n}$, we can delete row m and column n. But the remaining determinant is exactly the one in the expression of $s_\lambda(x^{(m-1)}/y^{(n-1)})$, and because of the factor $(-1)^{m+n}$ also the sign works out correctly. □

The only property that is more tedious to prove is the last one :

Proposition 4.15 (Restriction) *Let $m \geq 1$ (respectively $n \geq 1$). The result of setting $x_m = 0$ (resp. $y_n = 0$) in $s_\lambda(x^{(m)}/y^{(n)})$ is the polynomial $s_\lambda(x^{(m-1)}/y^{(n)})$ (resp. $s_\lambda(x^{(m)}/y^{(n-1)})$).*

Proof. By (4.44), it is sufficient to prove only the case $y_n = 0$. For $D_{m,n}$, we have

$$D^{-1}_{m,n}|_{y_n=0} = \frac{\prod_{i=1}^{m} x_i}{\prod_{j=1}^{n-1} y_j} D^{-1}_{m,n-1}. \tag{4.45}$$

There are now two cases to consider : the $(m,n-1)$-index of λ is k, or the $(m,n-1)$-index of λ is $k+1$ (no other values are possible).
First case. We shall use expression (4.42). Suppose that the $(m,n-1)$-index of λ is also k. This means that $\lambda_k + m + 1 - k < n$, or $r > 0$. By (4.38), this also means that the last row of the matrix in (4.42) consists of n ones followed by $k-1$ zeros. But substituting $y_n = 0$ in the matrix of (4.42), means that in column n all entries except the last one (which is 1) become zero. Now expand the determinant with respect to column n. The result is a new determinant of order $(n-1)+k-1$. But when going from $(x^{(m)}, y^{(n)})$ to $(x^{(m)}, y^{(n-1)})$, k remains the same, n reduces to $n-1$ and r reduces to $r-1$. So if we multiply row i ($1 \leq i \leq m$) of this new determinant by x_i, and divide column j ($1 \leq j \leq n-1$) of this new determinant by y_j, the resulting determinant is exactly that of $s_\lambda(x^{(x)}/y^{(n-1)})$ in the form (4.42). Furthermore, the

factors appearing in (4.45) have been taken care of, and one can verify that also the sign is correct. So in this case the lemma holds.

Second case. Suppose that the $(m, n-1)$-index of λ is $k+1$. This means that $\lambda_k + m + 1 - k = n$, or $r = 0$. Consider the matrix in (4.41). It is easy to see that all powers of y_j are strictly positive (on the last row of the matrix, the power of y_j is $\lambda'_{n-m+k-1} - k + 1 = \lambda'_{\lambda_k} - k + 1 \geq 1$). So substituting $y_n = 0$ in (4.41) yields a matrix of which column n consists of $(\frac{1}{x_1}, \ldots, \frac{1}{x_m}, 0, \ldots, 0)$. Let us push this column to the end, so that it becomes column $n+k-1$. Up to a sign, the determinant remains the same. Next, we use the factors in (4.45) to multiply in the determinant row i by x_i $(1 \leq i \leq m)$ and divide column j by y_j $(1 \leq j \leq n-1)$. The resulting matrix is of the following type :

$$\begin{pmatrix} \frac{x_1}{y_1(x_1+y_1)} & \cdots & \frac{x_1}{y_{n-1}(x_1+y_{n-1})} & x_1^{\lambda_1+m-n} & \cdots & x_1^{\lambda_{k-1}+m-n-k} & 1 \\ \vdots & & \vdots & \vdots & & & \vdots \\ \frac{x_m}{y_1(x_m+y_1)} & \cdots & \frac{x_m}{y_{n-1}(x_m+y_{n-1})} & x_m^{\lambda_1+m-n} & \cdots & x_m^{\lambda_{k-1}+m-n-k} & 1 \\ y_1^{\lambda'_1+n-m-2} & \cdots & y_{n-1}^{\lambda'_1+n-m-2} & 0 & \cdots & & 0 \\ \vdots & & \vdots & \vdots & & & \vdots \\ y_1^{\lambda'_{n-m+k-1}-k} & \cdots & y_{n-1}^{\lambda'_{n-m+k-1}-k} & 0 & \cdots & & 0 \end{pmatrix} \tag{4.46}$$

Now we can do the following : for every column j $(1 \leq j \leq n-1)$, subtract from this column $1/y_j$ times the last column of the matrix, i.e.

$$C_j \longrightarrow C_j - \frac{1}{y_j} C_{n+k-1}.$$

This implies that the matrix element on position (i, j), where $1 \leq i \leq m$ and $1 \leq j \leq n-1$, becomes

$$\frac{x_i}{y_j(x_i+y_j)} - \frac{1}{y_j} = \frac{-1}{x_i+y_j}.$$

After dealing with sign changes in rows and columns, we get, up to a sign :

$$D_{m,n-1}^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n-1}} & \left(x_i^{\lambda_j+m-n+1-j}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq k}} \\ \left(y_j^{\lambda'_i+n+1-m-i}\right)_{\substack{1\leq i\leq n-m+k-1 \\ 1\leq j\leq n-1}} & 0 \end{pmatrix}, \tag{4.47}$$

where we have used again that $\lambda_k + m - n - 1 - k = 0$ to see that the powers of x_i agree in the last column. But (4.47) is simply the same as the right hand side of (4.41) with n replaced by $n-1$ and k by $k+1$, so it is (up to a sign) $s_\lambda(x^{(m)}/y^{(n-1)})$. To verify that also the sign agrees is left as an exercise. □

Finally, we have

Theorem 4.16 *The polynomials $s_\lambda(x^{(m)}/y^{(n)})$, defined by (4.41), are the supersymmetric Schur polynomials.*

Proof. This follows immediately from Propositions 4.12-4.15, and the result that these four properties characterize the supersymmetric Schur polynomials [56, 46]. □

4.5 A proof of coincidence with the Sergeev-Pragacz formula

In this final section, we provide a direct proof [75], based on the Berele-Regev formula (2.8) and Laplace's theorem [63, Section 1.8], that our determinantal formula coincides with the Sergeev-Pragacz formula.

Let us write $x = x' + x''$ for a decomposition of $x = (x_1, x_2, \dots, x_m)$ into two disjoint subsets of fixed size, say $|x'| = p$ and $|x''| = q$ with $p + q = m$. Recall the definition of $E(x', x'')$ in (4.4).

Lemma 4.17 *For $m = p+q$, let $\mu = (\mu_1, \dots, \mu_p)$, $\nu = (\nu_1, \dots, \nu_q)$ be two partitions and $\lambda = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$. Then*

$$\sum_{x'+x''} \frac{s_{\mu+(q^p)}(x')s_\nu(x'')}{E(x', x'')} = s_\lambda(x) \tag{4.48}$$

where the sum is over all possible decompositions $x = x' + x''$ with the size of x' equal to p and the size of x'' equal to q.

Proof. We can rewrite the lhs of expression (4.48) using the determinantal formula for S-functions and the equality $D(x) = (-1)^{\frac{p(p+1)}{2}+r_1+\cdots+r_p} D(x')D(x'')E(x', x'')$ with the elements of x' denoted by $x_{r_1}, \dots, x_{r_p}$ and those of x'' by $x_{s_1}, \dots, x_{s_q}$:

$$\sum_{x'+x''} \frac{s_{\mu+(q^p)}(x')s_\nu(x'')}{E(x', x'')} = \frac{(-1)^{\frac{p(p+1)}{2}}}{D(x)} \sum_{x'+x''} (-1)^{r_1+\cdots+r_p} |x_{r_i}^{\mu_j+q+p-j}||x_{s_i}^{\nu_j+q-j}|. \tag{4.49}$$

The numerator of this sum is the Laplace expansion of the following determinant with respect to columns $1, 2, \dots, p$:

$$\left| \; x^{\mu+(q^p)+\delta_p} \quad x^{\nu+\delta_q} \; \right| = \left| x^{\lambda+\delta_m} \right| \tag{4.50}$$

with $\lambda = (\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q)$ and δ_l given in (1.9). The result follows. □

Remark 4.18 The previous lemma and its proof can easily be extended for an arbitrary p-tuple μ over $\mathbb{Z}$ and an arbitrary q-tuple ν over $\mathbb{Z}$. In this case, $s_\alpha(x)$ (α an arbitrary m-tuple) is defined according to formula (1.14). We can extend this generalization to the supersymmetric case where $s_\alpha(x/y)$ is defined as

$$s_\alpha(x/y) = \det\Big(h_{\alpha_i - i + j}(x/y)\Big)_{1 \le i,j \le \ell(\alpha)} \qquad (\alpha \text{ a } t\text{-tuple with } \alpha_i \in \mathbb{Z}). \quad (4.51)$$

Note that for a t-tuple over $\mathbb{Z}$, $\alpha + \delta_t$ must consist of nonnegative distinct integers for s_α to be nonzero.

Lemma 4.19 *Suppose* $|y| = n$, $y' = (y_1, \ldots, y_{n-1})$ *and* η *an arbitrary* t*-tuple, then*

$$s_\eta(x/y) = \sum_\zeta s_\zeta(x/y')(y_n)^d, \quad (4.52)$$

where the sum is taken over all 2^t t*-tuples* ζ *such that* $(\eta - \zeta)_i \in \{0, 1\}$ *and* $d = |\eta - \zeta|$.

Proof. First suppose that η is a partition. If we sum only over all ζ such that $(\eta - \zeta)$ is a vertical strip, the summation follows from the expression of $s_\eta(x/y)$ by means of supertableaux [10]. All the other terms vanish as there exists an index i such that $\zeta_{i+1} = \zeta_i + 1$.
Suppose now that η is not a partition. Consider $\eta + \delta_t$. If $\eta + \delta_t$ has a negative component, then $s_\eta(x/y) = 0$; but then also each $s_\zeta(x/y')$ is zero since every $\zeta + \delta_t$ has a negative component. If $\eta + \delta_t$ has two equal parts, then $s_\eta(x/y) = 0$; then the right hand side of (4.52) consists of zero terms and terms cancelling each other two by two. Finally, if $\eta + \delta_t$ consists of distinct nonnegative parts, then $s_\eta(x/y) = \pm s_\lambda(x/y)$, with λ a partition and $\sigma(\eta + \delta_t) = \lambda + \delta_t$, for some permutation σ. Now apply (4.52) to $s_\lambda(x/y)$; applying σ^{-1} to each t-tuple in both sides of this equation yields the result □

Lemma 4.20 *Let* $m = p + q$, $\mu = (\mu_1, \ldots, \mu_p)$ *an arbitrary* p*-tuple and* $\nu = (\nu_1, \nu_2, \ldots)$ *an arbitrary* t*-tuple. Denote* $\lambda = (\mu_1, \ldots, \mu_p, \nu_1, \nu_2, \ldots)$. *Then,*

$$s_\lambda(x/-y) = \sum_{x'+x''} \frac{s_{\mu+(q^p)}(x'/-y)s_\nu(x''/-y)}{E(x', x'')} \quad (4.53)$$

where the sum is over all possible decompositions $x = x' + x''$ *with the size of* x' *equal to* p *and the size of* x'' *equal to* q.

Proof. One can use, e.g., induction on n, i.e. the number of variables $y = (y_1, \ldots, y_n)$. If $n = 0$ then we are reduced to the symmetric case and the result follows from Lemma 4.17 and Remark 4.18. Otherwise, one separates the variable y_n. We can apply Lemma 4.19 twice on the right hand side of (4.53); with $(\mu - \alpha)_i \in \{0, 1\}$, $a = |\mu - \alpha|$ and $(\nu - \beta)_i \in \{0, 1\}$, $b = |\nu - \beta|$:

$$\sum_{x'+x''} \frac{s_{\mu+(q^p)}(x'/-y)s_\nu(x''/-y)}{E(x',x'')} =$$
$$= \sum_{x'+x''} \frac{1}{E(x',x'')} \left(\sum_\alpha s_{\alpha+(q^p)}(x'/-y')(-y_n)^a \right) \left(\sum_\beta s_\beta(x''/-y')(-y_n)^b \right)$$
$$= \sum_\alpha \sum_\beta \left(\sum_{x'+x''} \frac{s_{\alpha+(q^p)}(x'/-y')s_\beta(x''/-y')}{E(x',x'')} \right) (-y_n)^{a+b}$$

Using induction, this expression reduces to $\sum_\gamma s_\gamma(x'/-y')(-y_n)^c$ with $c = a + b$, $\gamma = (\alpha_1, \ldots, \alpha_p, \beta_1, \beta_2, \ldots)$ and $(\lambda - \gamma)_i \in \{0, 1\}$. Now it is easy to see that this expression is equal to $s_\lambda(x/-y)$ applying (4.52). □

We can now give a direct proof of the determinantal formula.

Theorem 4.21 *Let $\lambda \in \mathcal{H}_{m,n}$ and k be the (m, n)-index of λ. Then*

$$\frac{E(x,y)}{D(x)D(y)} \det \begin{pmatrix} \frac{1}{x-y} & X_\lambda \\ Y_\lambda & 0 \end{pmatrix} = \pm s_\lambda(x/-y), \tag{4.54}$$

where the (rectangular) blocks of the determinant are given by

$$\frac{1}{x-y} = \left(\frac{1}{x_i - y_j} \right)_{1\le i\le m,\ 1\le j\le n},$$
$$X_\lambda = \left(x_i^{\lambda_j+m-n-j} \right)_{\substack{1\le i\le m, \\ 1\le j\le k-1}}, \qquad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i} \right)_{\substack{1\le i\le n-m+k-1, \\ 1\le j\le n}}.$$

Proof. Suppose $|x'| = k - 1 \equiv p$ and $|y'| = n - m + k - 1 \equiv q$. We will indicate the indices of the elements of x' by i_t ($t = 1, \ldots, p$) and those of y' by j_t ($t = 1, \ldots, q$). The determinant in (4.54) has a double Laplace expansion, with partitions $\alpha = (\lambda_1, \ldots, \lambda_p) - (q^p)$ and $\beta = (\lambda'_1, \ldots, \lambda'_q) - (p^q)$ determined by X_λ and Y_λ, so the left hand side of (4.54) equals :

$$\frac{E(x,y)}{D(x)D(y)} \sum_{x=x'+x''} \sum_{y=y'+y''} (-1)^P D(x')D(y')s_\alpha(x')s_\beta(y') \det\left(\frac{1}{x''-y''} \right)$$

$$
\begin{aligned}
&(\text{ with } P = \frac{p(p+1)}{2} + \frac{q(q+1)}{2} + \sum_{t=1}^{p} i_t + \sum_{t=1}^{q} j_t) \\
= \; & \frac{E(x,y)}{D(x)D(y)} \sum_{x'+x''} \sum_{y'+y''} (-1)^P D(x')D(y')s_\alpha(x')s_\beta(y')D(x'')D(y'')/E(x'',y'') \\
= \; & \sum_{x'+x''} \sum_{y'+y''} s_\alpha(x')s_\beta(y') \frac{E(x,y')E(x',y'')}{E(x',x'')E(y',y'')} \\
= \; & \sum_{y'+y''} \left(\sum_{x'+x''} \frac{s_\alpha(x')E(x',y'')}{E(x',x'')} \right) \frac{s_\beta(y')(-1)^{mq}E(y',x)}{E(y',y'')}.
\end{aligned}
$$

Now we can apply the Berele-Regev formula (2.8) twice. Putting $\eta = \alpha + ((m-k+1)^p)$ and $\chi = \beta + (m^q) = (\lambda'_1, \ldots, \lambda'_q) + ((m-k+1)^q)$, there comes

$$
\begin{aligned}
\pm \sum_{y'+y''} \left(\sum_{x'+x''} \frac{s_\eta(x'/-y'')}{E(x',x'')} \right) \frac{s_\chi(y'/-x)}{E(y',y'')} \\
= \pm \sum_{y'+y''} \left(\sum_{x'+x''} \frac{s_\eta(x'/-y'')}{E(x',x'')} \right) \frac{s_\chi(-y'/x)}{E(y',y'')}.
\end{aligned}
$$

Finally, we use Lemma 4.20, and duality (2.11); the last expression becomes :

$$
\pm \sum_{y'+y''} s_\alpha(x/-y'')s_\chi(-y'/x) \frac{1}{E(y',y'')} = \pm(-1)^{|\alpha|} \sum_{y'+y''} \frac{s_{\alpha'}(-y''/x)s_\chi(-y'/x)}{E(y',y'')}.
$$

As $\alpha' = (\lambda'_{q+1}, \lambda'_{q+2}, \ldots)$ and $\chi = (\lambda'_1, \ldots, \lambda'_q) + ((m-k+1)^q)$, this is equal to $\pm s_{\lambda'}(-y/x) = \pm s_\lambda(x/-y)$, using once more Lemma 4.20 and duality. □

Thus, the proof uses essentially a double Laplace expansion, twice the application of the Berele-Regev formula, and twice Lemma 4.20. Finally, observe that the sign in Theorem 4.21 depends on the partition λ, to be precise it is equal to

$$
(-1)^{\sum_{i=1}^{n-m+k-1} \lambda'_i + \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - 1}.
$$

To end this chapter, we will summarise the main results. We gave a determinantal formula for the supersymmetric Schur polynomials indexed by a partition λ:

$$
s_\lambda(x/y) = \pm D^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right) & \left(x_i^{\lambda_j+m-n-j}\right) \\ \left(y_j^{\lambda'_i+n-m-i}\right) & 0 \end{pmatrix}.
$$

The equivalence between the determinant and the supersymmetric Schur functions is proved in three different ways. The first proof is a representation theoretic proof (Section 4.3), followed by a proof by means of the four characterizing properties given by Macdonald (Section 4.4). A more straightforward proof, using a double Laplace expansion and a special case of the formula of Sergeev-Pragacz, is given in Section 4.5.

Chapter 5

Determinantal formula for composite supersymmetric Schur functions

It was already known that the characters of covariant and contravariant representations correspond to supersymmetric Schur functions. In this chapter we intend to show that there is still another family of atypical representations for which the character is given by a composite S-function. Based upon the definition of critical representations, the notion of “critical composite partition” is introduced. It is shown that for critical composite partitions (subject to a technical restriction) the corresponding $\mathfrak{gl}(m|n)$ representation V_Λ is tame, so its character formula can be computed. This expression gives rise to a determinantal formula for $\mathrm{ch}(V_\Lambda)$. Analogous to the previous chapter, the main goal of this formula is to link this determinantal formula to composite supersymmetric S-functions. This last equality however is conjectured. We give the outlines of a proof and point out the difficulties we encountered. Essentially, the proof of the conjecture reduces to an identity for composite supersymmetric Schur functions.

5.1 Introduction

This chapter is presenting some new results for irreducible representations of the Lie superalgebra $\mathfrak{gl}(m|n)$. For a subclass of these irreducible representations, known as "typical" representations, Kac derived a character formula closely analogous to the Weyl character formula for irreducible representations of simple Lie algebras [32]. The problem of obtaining a character formula for the remaining "atypical" irreducible representations has been the subject of intensive investigation, both in the mathematics and physics literature. In the early days of Lie superalgebra representation theory, the notion of graded tensors was introduced [22], and it was believed [4, 5] that the standard methods of covariant, contravariant and mixed tensor representations with the corresponding Young techniques yield the characters of $\mathfrak{gl}(m|n)$ irreducible representations in terms of supersymmetric S-functions. Although this is certainly true for the covariant and contravariant tensor representations [22, 10], it is not so for the mixed tensor representations, as already observed in [38, 57]. The problem is well described and analysed in [70], where furthermore a character formula for atypical $\mathfrak{gl}(m|n)$ irreducible representations is conjectured. Since then, some partial solutions to this problem were given, e.g. for so-called generic representations [53], for singly atypical representations [12, 71, 68], or for tame representations [34]. More recently, the character problem for $\mathfrak{gl}(m|n)$ was solved in principle by Serganova [60], who gave an algorithm to compute composition factor multiplicities of so-called Kac-modules, and thus indirectly the character. In [73], a substantially simpler method was conjectured to compute these composition factor multiplicities; this conjecture was proved by Brundan [13]. Still, the method using composition factor multiplicities of Kac-modules remains a rather indirect way of computing characters. Recently, there was a further breakthrough for this problem. Building on the work of Brundan, Yucai Su and Zhang [66] managed to compute the generalized Kazhdan-Lusztig polynomials of $\mathfrak{gl}(m|n)$ irreducible representations, leading to a relatively explicit character formula for all these irreducible representations, and thus proving that the character formula conjectured in [70] holds.

From a computational and practical point of view, it is useful to identify characters with supersymmetric S-functions, since it is easy to work with S-functions, for which many properties are known (see Chapter 2). As just mentioned, this identification holds for covariant and contravariant irreducible representations [22, 10], where the corresponding S-function is labeled by a single partition λ, but fails for mixed tensor irreducible representations, where the corresponding S-function is labeled by a composite partition $\bar{\nu};\mu$. In the present chapter, we intend to show that there is still another family of atypical representations for which the character is given by a (composite) S-function, namely the so-called critical $\mathfrak{gl}(m|n)$ representations.

5.2 Normally, critically and quasicritically related roots

Let $\Lambda \in \mathfrak{h}^*$ be a highest weight and $\bar{\nu};\mu$ the corresponding composite partition as defined in § 3.6; we will denote this correspondence by $\Lambda = \Lambda_{\bar{\nu};\mu}$. With the notation of [26], we distinguish between *normally, critically and quasicritically* related roots of the $(\Lambda+\rho)$-isotropic set. Consider the set of odd roots $\{\gamma_1,\ldots,\gamma_a\}$ with $\gamma_s = \beta_{i_s,j_s}$ such that $(\Lambda_{\bar{\nu};\mu}+\rho,\beta_{i_s,j_s}) = 0$ where $j_1 < j_2 < \cdots < j_a$. Notice that $a =$atyp($\Lambda_{\bar{\nu};\mu}+\rho$). If we put all the numbers $(\Lambda_{\bar{\nu};\mu}+\rho,\beta_{i_s,j_s})$ in the atypicality matrix $A(\Lambda_{\bar{\nu};\mu})$ (see § 3.8), then it should be noticed that the $\gamma_1,\ldots,\gamma_a$ are ordered from the bottom left-hand corner to the top right-hand corner of this matrix. Let x_{pq} with $1 \le p < q \le a$ be the entry in $A(\Lambda_{\bar{\nu};\mu})$ at the intersection of the column containing the γ_p zero with the row containing the γ_q zero and x_{qp} the entry at the intersection of the row containing the γ_p zero and the column containing the γ_q zero. As shown in [26], $x_{pq} = -x_{qp}$ and therefore $A(\Lambda_{\bar{\nu};\mu})$ has the following form:

$$A(\Lambda_{\bar{\nu};\mu}) = \begin{pmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & x_{1a} & \ldots & x_{2a} & \ldots & & \ldots & 0 & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & x_{13} & \ldots & x_{23} & \ldots & 0 & \ldots & -x_{3a} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & x_{12} & \ldots & 0 & \ldots & -x_{23} & \ldots & -x_{2a} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & 0 & \ldots & -x_{12} & \ldots & -x_{13} & \ldots & -x_{1a} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{pmatrix} \tag{5.1}$$

Let h_{pq} be the hook length between the zeros corresponding to γ_p and γ_q, i.e. the number of steps needed to go from the γ_p zero of $A(\Lambda_{\bar{\nu};\mu})$ via x_{pq} to the γ_q zero, where the zeros themselves are included in the count.

In Figure 5.1 the atypicality matrix is given for $\bar{\nu};\mu = (\bar{4},\bar{6},\bar{6},\bar{6});(3,3,2,2)$ in $\mathfrak{gl}(5|7)$. The weight Λ is threefold atypical with $\gamma_1 = \beta_{51}, \gamma_2 = \beta_{32}$ and $\gamma_3 = \beta_{14}$. The hook lengths are $h_{12} = 4, h_{13} = 8$ and $h_{23} = 5$.

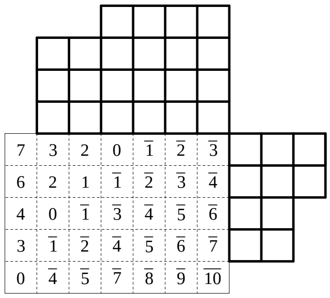


Figure 5.1: $\bar{\nu};\mu = (\bar{4},\bar{6},\bar{6},\bar{6});(3,3,2,2)$ and the atypicality matrix of $\Lambda_{\bar{\nu};\mu}$ in $\mathfrak{gl}(5|7)$.

Definition 5.1 *Let Λ be a highest weight of $\mathfrak{gl}(m|n)$ with atyp$(\Lambda+\rho) = a$ and atypical roots $\{\gamma_1,\dots,\gamma_a\}$. Then for every $1 \le p < q \le a$: γ_p and γ_q are* NORMALLY RELATED *if and only if $x_{pq}+1 > h_{pq}$; γ_p and γ_q are* QUASICRITICALLY RELATED *if and only if $x_{pq}+1 = h_{pq}$; γ_p and γ_q are* CRITICALLY RELATED *if and only if $x_{pq}+1 < h_{pq}$.*

In Figure 5.1, $x_{12}+1 = 5$, $x_{13}+1 = 8$ and $x_{23}+1 = 4$. Thus, γ_1 and γ_2 are normally related, γ_1 and γ_3 are quasicritically related and γ_2 and γ_3 are critically related.

If each pair (γ_i,γ_{i+1}) $(i = 1,2,\dots,a-1)$ is critically related, then all elements of $\{\gamma_1,\dots,\gamma_a\}$ are critically related. Then the composite partition $\bar{\nu};\mu$, the highest weight $\Lambda_{\bar{\nu};\mu}$ and the representation $V_{\Lambda_{\bar{\nu};\mu}} \equiv V_{\bar{\nu};\mu}$ are called *critical*. This coincides with the notion of *totally connected*, as described in [73, 66]. There is a simple combinatorial way to check criticality:

Proposition 5.2 *Suppose $\bar{\nu};\mu$ is standard in $\mathfrak{gl}(m|n)$ with atyp$(\Lambda_{\bar{\nu};\mu}+\rho) = a$. Let $\gamma_s = \beta_{i_s,j_s}$ so that $(\Lambda_{\bar{\nu};\mu}+\rho,\gamma_s) = 0$ ($s = 1,\dots,a$) and*

$$\begin{aligned}\mathcal{M} &= \{\mu_{i_1}+m-i_1, \mu_{i_1-1}+m-i_1+1, \dots, \mu_{i_a}+m-i_a\},\\ \mathcal{N} &= \{\nu'_{j_1}+n-j_1, \nu'_{j_1-1}+n-j_1+1, \dots, \nu'_{j_a}+n-j_a\}.\end{aligned}$$

Then the composite partition $\bar{\nu};\mu$ is critical for $\mathfrak{gl}(m|n)$ if and only if

$$\begin{aligned}\mathcal{M}\cup\mathcal{N} &= \{\mu_{i_1}+m-i_1, \mu_{i_1}+m-i_1+1, \dots, \mu_{i_1}+m-i_a+j_a-j_1-a+1\},\\ &= \{\mu_{i_1}+m-i_1, \mu_{i_1}+m-i_1+1, \dots, \mu_{i_1}+m-i_1+h_{1a}-a\},\end{aligned}$$

i.e. if and only if $\mathcal{M}\cup\mathcal{N}$ is a set of consecutive integers.

Proof. Suppose $\mathcal{M} \cup \mathcal{N} \neq \{\mu_{i_1}+m-i_1, \mu_{i_1}+m-i_1+1, \ldots, \mu_{i_1}+m-i_a+j_a-j_1-a+1\}$. This means that at least one integer is missing between $\mu_{i_1}+m-i_1$ and $\mu_{i_a}+m-i_a$. So, there exists a p such that

$$x_{p,(p+1)} > i_p - i_{p+1} + j_{p+1} - j_p - 1 = h_{p,(p+1)} - 2 \Leftrightarrow x_{p,(p+1)} + 1 \geq h_{p,(p+1)},$$

a contradiction. Conversely, suppose $\mathcal{M} \cup \mathcal{N}$ is a set of consecutive numbers. Define, with $p < q$, the sets $\mathcal{M}^{(pq)}$ and $\mathcal{N}^{(pq)}$ as:

$$\begin{aligned} \mathcal{M}^{(pq)} &= \{\mu_{i_p}+m-i_p, \mu_{i_p-1}+m-i_p+1, \ldots, \mu_{i_q}+m-i_q\}, \\ \mathcal{N}^{(pq)} &= \{\nu'_{j_p}+n-j_p, \nu'_{j_p-1}+n-j_p+1, \ldots, \nu'_{j_q}+n-j_q\}. \end{aligned}$$

The set $\mathcal{M}^{(pq)} \cup \mathcal{N}^{(pq)} = \{\mu_{i_p}+m-i_p, \mu_{i_p}+m-i_p+1, \ldots, \mu_{i_q}+m-i_q+j_q-j_p-q+p\}$ is also a set of consecutive numbers. This implies that

$$\mu_{i_q}+m-i_q = \mu_{i_p}+m-i_q+j_q-j_p-q+p \Leftrightarrow \mu_{i_q}-\mu_{i_p} = j_q-j_p-q+p$$

for every $p < q$. So, with $h_{pq} = j_q - j_p - i_q + i_p + 1$,

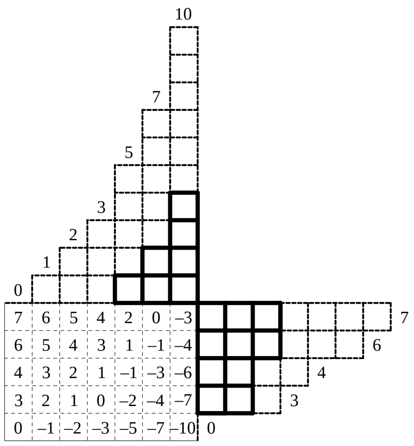
$$\begin{aligned} x_{pq}+1 &= \mu_{i_q}+m-i_q-(\nu'_{j_p}+n-j_p)+1 \\ &= \mu_{i_q}+m-i_q-(\mu_{i_p}+m-i_p)+1 \\ &= h_{pq}+p-q < h_{pq}, \end{aligned}$$

meaning that γ_p and γ_q are critically related for every p, q ($p < q$). □

This property is illustrated for $\mathfrak{gl}(5|7)$ in Figure 5.2 of a critical composite partition. Note how the Young diagrams, together with the $(m \times n)$-rectangle, determine the numbers attached to these diagrams; how the differences of these numbers determine the entries in the $(m \times n)$-rectangle and hence also the zeros; and how criticality can be read off from these numbers.

It is easy to verify that covariant or contravariant representations are always critical. The class of critical representations is however much larger. To understand this, let us concentrate again on the above example. In particular, consider $\mu = (3,3,2,2)$ fixed in $\mathfrak{gl}(5|7)$, and let us determine all possible $\nu \neq 0$ such that $\bar{\nu};\mu$ is critical. Using Proposition 5.2, one finds, listed according to the length of ν':

- if $\ell(\nu') = 1$, all $\bar{\nu};\mu$ are critical;
- if $\ell(\nu') = 2$, all $\bar{\nu};\mu$ are critical as long as $\nu'_2 \notin \{1,2\}$;
- if $\ell(\nu') = 3$, all $\bar{\nu};\mu$ are critical as long as $\nu'_3 \notin \{2,3\}$.



$$\bar{\nu};\mu = (\bar{1},\bar{1},\bar{2},\bar{3});(3,3,2,2)$$

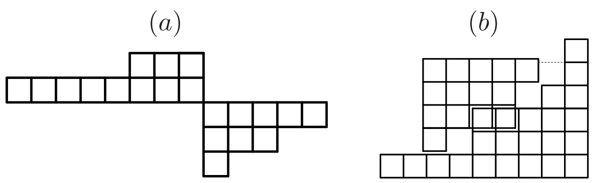
$$\begin{aligned}(\gamma_1,\gamma_2,\gamma_3) &= (\beta_{5,1},\beta_{4,4},\beta_{1,6})\\ \mathcal{M} &= \{0,3,4,6,7\}\\ \mathcal{N} &= \{0,1,2,3,5,7\}\\ \mathcal{M}\cup\mathcal{N} &= \{0,1,2,3,4,5,6,7\}\end{aligned}$$

$$\begin{aligned}x_{12} &= 3, & h_{12} &= 5\\ x_{13} &= 7, & h_{13} &= 10\\ x_{23} &= 4, & h_{23} &= 6\end{aligned}$$

Figure 5.2: Illustration of Proposition 5.2

One can continue with this description for $\ell(\nu') \geq 4$, but it becomes slightly more intricate (some of the representations are no longer multiply atypical, and thus also not critical). In any case, this illustrates that for given m, n and μ, one can describe the corresponding critical representations $\bar{\nu};\mu$ using Proposition 5.2, and that the class of critical representations is indeed much larger than just the covariant representations (those with $\nu = 0$) and the contravariant representations (those with $\mu = 0$). In general, it illustrates that $\bar{\nu};\mu$ is critical if the size of μ and ν is sufficiently small compared to m and n.

In §1.1.3 we introduced composite partitions together with the visualization by means of juxtaposition. Let m and n be fixed. There is also another way to visualize $\bar{\nu};\mu$ by putting them together in a $(m \times n)$-rectangle. The partition μ is now composed of boxes arranged in left-adjusted rows of lengths $\mu_1, \mu_2, \ldots$ starting at the top left-hand corner of this rectangle, and the partition ν of boxes arranged in right-adjusted rows of lengths $\nu_1, \nu_2, \ldots$ starting at the bottom right-hand corner of the rectangle. For $\bar{\nu};\mu = (\bar{1},\bar{1},\bar{2},\bar{5},\bar{5},\bar{9});(5,4,4,1)$ and $(m|n) = (5|7)$ this is illustrated in (5.2)(b). Observe that in this second visualization, there can be overlap between the two diagrams.



(5.2)

5.3 Some examples

In this section, we shall give some examples of composite partitions, their Young diagrams (both in the $(m \times n)$-cross and in the $(m \times n)$-rectangle), their atypicality matrix, and some related composite partitions. To do so, we will use the notions given in §3.6 and the definition of σ:

$$\sigma = \sum_{i=1}^{m} \epsilon_i - \sum_{j=1}^{n} \delta_j, \tag{3.23}$$

or in coordinates $\sigma = (1, 1, \ldots, 1;\ -1, -1, \ldots, -1)$.

Let us take $\mathfrak{gl}(m|n) = \mathfrak{gl}(5|7)$, and consider as first example the composite partition

$$\bar{\nu}; \mu = (\overline{3}, \overline{3}, \overline{4}, \overline{6}, \overline{7}); (5, 5, 5, 4, 2, 1, 1, 1). \tag{5.3}$$

The Young diagram of $\bar{\nu}; \mu$ – in its proper corner position in the $(m \times n)$-cross – is given in Figure 5.3(a). So in this case, $(I, J, K, L) = (0, 5, 2, 5)$, and hence we find

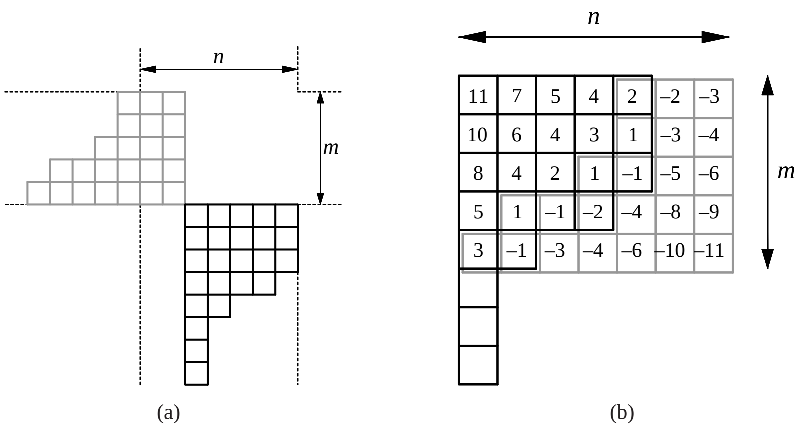


Figure 5.3: Young diagram of a composite partition $\bar{\nu}; \mu$ in (a) the $(m \times n)$-cross and (b) the $(m \times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (5, 5, 5, 4, 2, 1, 1, 1)$ and $\nu = (7, 6, 4, 3, 3)$.

from (3.27) that the corresponding weight $\Lambda = \Lambda_{\bar{\nu};\mu}$ is given by

$$\Lambda = (4, 4, 3, 1, 0;\ 3, 0, -1, -1, -2, -5, -5). \tag{5.4}$$

The Young diagram of $\bar{\nu};\mu$ is also given in Figure 5.3(b), where it is represented in the $(m \times n)$-rectangle. Notice that in this case, there is overlap between the two diagrams (that of μ given in black and that of $\bar{\nu}$ given in gray). Furthermore, in the last figure we also give the atypicality matrix $A(\Lambda)$, in the appropriate positions of the $(m \times n)$-rectangle. Notice that there are no zeros in this matrix, so Λ is typical.

We can now consider the closely related weight

$$\widetilde{\Lambda} = \Lambda + \sigma = (5,5,4,2,1;\ 2,-1,-2,-2,-3,-6,-6). \tag{5.5}$$

Using (3.27), it is easy to work out the composite partition corresponding to $\widetilde{\Lambda}$. One finds

$$\overline{\widetilde{\nu}};\widetilde{\mu} = (\overline{2},\overline{2},\overline{2},\overline{3},\overline{5},\overline{6});(5,5,4,2,1,1,1), \tag{5.6}$$

with $(I,J,K,L) = (5,1,6,0)$. Now we can consider the Young diagram of $\overline{\widetilde{\nu}};\widetilde{\mu}$, once in its corner position in the $(m \times n)$-cross – given here in Figure 5.4(a); and once represented in the $(m \times n)$-rectangle – given in Figure 5.4(b). Also the atypicality

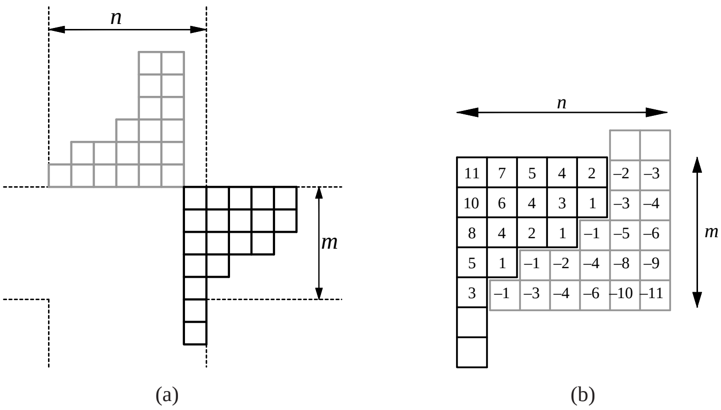


Figure 5.4: Young diagram of a composite partition $\bar{\nu};\mu$ in (a) the $(m \times n)$-cross and (b) the $(m \times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (5,5,4,2,1,1,1)$ and $\nu = (6,5,3,2,2,2)$.

matrix is once again given, and obviously $A(\Lambda) = A(\widetilde{\Lambda})$, since $(\sigma,\beta_{ij}) = 0$ for all odd roots β_{ij}. Notice that in the $(m \times n)$-rectangle (Figure 5.4(b)), the Young diagrams of $\widetilde{\mu}$ and $\overline{\widetilde{\nu}}$ have no overlap and just "touch" each other along their boundaries. All positive entries in the atypicality matrix are inside the diagram of $\widetilde{\mu}$, whereas all negative entries of $A(\widetilde{\Lambda})$ are inside $\overline{\widetilde{\nu}}$. This is no coincidence. One can show that this is a general feature of typical weights. More explicitly, let $\bar{\nu};\mu$ be a composite partition with corresponding weight $\Lambda = \Lambda_{\bar{\nu};\mu}$ and suppose Λ is typical. Then there is a unique integer j such that $\widetilde{\Lambda} = \Lambda + j\sigma$, for which the corresponding composite partition is $\overline{\widetilde{\nu}};\widetilde{\mu}$, satisfies the following properties:

- the Young diagrams of $\widetilde{\mu}$ and $\overline{\widetilde{\nu}}$ have no overlap (no intersection) in the $(m \times n)$-rectangle;
- each box in the $(m \times n)$-rectangle is either part of the Young diagram of $\widetilde{\mu}$ or else of the Young diagram of $\overline{\widetilde{\nu}}$;
- all positive entries in the atypicality matrix are inside the Young diagram of $\widetilde{\mu}$, and all negative entries are inside the Young diagram of $\overline{\widetilde{\nu}}$.

As a second example in $\mathfrak{gl}(5|7)$, let us take the composite partition

$$\bar{\nu};\mu = (\overline{2},\overline{2},\overline{3},\overline{6},\overline{7});(4,3,3,1,1). \tag{5.7}$$

The Young diagram of $\bar{\nu};\mu$, properly situated in the $(m \times n)$-cross, is given in Figure 5.5(a). Note that $(I,J,K,L) = (5,0,7,0)$, and we find from (3.27) that the corre-

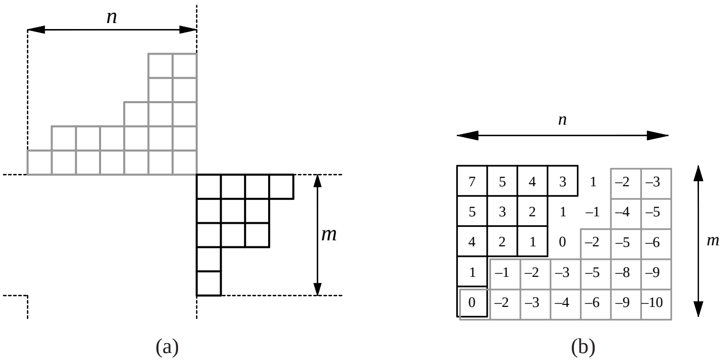


Figure 5.5: Young diagram of a composite partition $\bar{\nu};\mu$ in (a) the $(m \times n)$-cross and (b) the $(m \times n)$-rectangle, together with its atypicality matrix. Here, $\mu = (4,3,3,1,1)$ and $\nu = (7,6,3,2,2)$.

sponding weight $\Lambda = \Lambda_{\bar{\nu};\mu}$ is given by

$$\Lambda = (4,3,3,1,1;\ -1,-2,-2,-2,-3,-5,-5). \tag{5.8}$$

The Young diagram of $\bar{\nu};\mu$ is also given in Figure 5.5(b), represented in the $(m\times n)$-rectangle. Notice the overlap between the two diagrams. As for the previous example, we also give the atypicality matrix $A(\Lambda)$, in the appropriate positions of the $(m\times n)$-rectangle in Figure 5.5(b). Notice that there are two zeros in this matrix, so Λ is atypical. By the entry "4" in the hook connecting the two zeros (in the terminology of Definition 5.1, $x_{12}=4$ and $h_{12}=6$), it follows that Λ is critical.

Let us consider the closely related weight

$$\widetilde{\Lambda}=\Lambda-\sigma=(3,2,2,0,0;\ 0,-1,-1,-1,-2,-4,-4). \tag{5.9}$$

Using (3.27), the composite partition corresponding to $\widetilde{\Lambda}$ is

$$\overline{\widetilde{\nu}};\widetilde{\mu}=(\overline{2},\overline{2},\overline{3},\overline{6});(3,2,2), \tag{5.10}$$

with again $(I,J,K,L)=(5,0,7,0)$. The Young diagram of $\overline{\widetilde{\nu}};\widetilde{\mu}$, properly positioned in the $(m\times n)$-cross, is given in Figure 5.6(a); and in Figure 5.6(b) it is once again given but now positioned in the $(m\times n)$-rectangle, together with the atypicality matrix (again $A(\Lambda)=A(\widetilde{\Lambda})$). Notice that in the $(m\times n)$-rectangle (Figure 5.6(b)), the

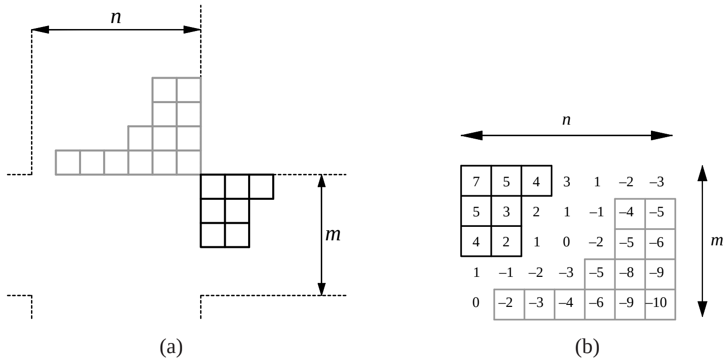


Figure 5.6: Young diagram of a composite partition $\bar{\nu};\mu$ in (a) the $(m\times n)$-cross and (b) the $(m\times n)$-rectangle, together with its atypicality matrix. Here, $\mu=(3,2,2)$ and $\nu=(6,3,2,2)$.

Young diagrams of $\widetilde{\mu}$ and $\overline{\widetilde{\nu}}$ have no overlap, and the zeros of the atypicality matrix are positioned in the "gap" between the two diagrams. This is a general property of critical atypical weights. More explicitly, let $\bar{\nu};\mu$ be a composite partition with corresponding weight $\Lambda=\Lambda_{\bar{\nu};\mu}$ and suppose Λ is atypical and critical. Then there is a unique integer j such that $\widetilde{\Lambda}=\Lambda+j\sigma$, for which the corresponding composite partition $\overline{\widetilde{\nu}};\widetilde{\mu}$ satisfies the following properties

- the Young diagrams of $\widetilde{\mu}$ and $\overline{\overline{\nu}}$, positioned in the $(m \times n)$-rectangle, do not cover the complete rectangle but leave a connected "gap";
- all the zeros of the atypicality matrix appear in this connected gap.

It is possible that the Young diagrams of $\widetilde{\mu}$ and $\overline{\overline{\nu}}$ overlap in the $(m \times n)$-rectangle, but there will be no zeros in that area. In what follows, the composite partition $\bar{\nu};\mu$ is assumed to be critical with no zeros in the overlap.

5.4 A character formula for a special class of composite partitions

We shall construct a formula for $\mathrm{ch}(\Lambda_{\bar{\nu};\mu})$ where $\bar{\nu};\mu$ is standard, critical and such that $\bar{\nu};\mu$ do not overlap if represented in a $(m \times n)$-rectangle (see (5.2)(b)). In what follows, we will only consider such composite partitions $\bar{\nu};\mu$. Let $\Lambda_{\bar{\nu};\mu}$ be the highest weight corresponding to $\bar{\nu};\mu$. We can generalize the definition of the (m,n)-index of an ordinary partition λ (cf. [47]) to composite partitions $\bar{\nu};\mu$ in $\mathfrak{gl}(m|n)$:

Definition 5.3 *For $\bar{\nu};\mu$ a standard composite partition, the (m,n)-index of $\bar{\nu};\mu$ is the number*

$$k = \min\Bigg(\Bigg\{ i \in \{1,\ldots,m\} | \exists j \in \{1,\ldots,n\} : \mu_i + \left\langle \mu'_{n-j+1} - m \right\rangle + (m-i) = \nu'_j + \langle \nu_{m-i+1} - n \rangle + (n-j) \Bigg\} \cup \{m+1\}\Bigg) \tag{5.11}$$

where $\langle a \rangle = \max(0,a)$. In what follows, k will always denote this number. In the special case where $\nu = 0$, this definition coincides with the one given in (4.14). Since, if $\nu = 0$ we have that

$$\begin{aligned} \mu_i + \left\langle \mu'_{n-j+1} - m \right\rangle + (m-i) &= n - j \\ &\Updownarrow \\ \mu_i + (m-i) - n + 1 &= -j - \left\langle \mu'_{n-j+1} - m \right\rangle + 1 \leq 0 \\ &\Updownarrow \\ \mu_i + m - i + 1 &\leq n \end{aligned}$$

When the representation is typical k will be equal to $m+1$; otherwise k corresponds to the smallest row number in the atypicality matrix in which there occurs a zero. Thus in the following we shall assume that $k \leq m$.

Recall that Δ_+ corresponds to the distinguished choice, and Π is the distinguished set of simple roots (3.10). The highest weight of $V_{\bar{\nu};\mu}$ is given by $\Lambda_{\bar{\nu};\mu}$. With respect to another set of simple roots Π' (with the corresponding ρ'), $V_{\bar{\nu};\mu}$ has a different highest weight Λ'. We shall follow the technique of simple odd reflections, described in §4.2. Denote $\Lambda^{(1)} = \Lambda_{\bar{\nu};\mu}$, $\rho^{(1)} = \rho$ and $\Pi^{(1)} = \Pi$. Now we perform a sequence of simple odd $\alpha^{(i)}$-reflections [47]; each of these reflections preserve $\Delta_{0,+}$ but may change $\Lambda^{(i)} + \rho^{(i)}$ and $\Pi^{(i)}$. Denote the sequence of reflections by:

$$\Lambda^{(1)} + \rho^{(1)},\, \Pi^{(1)} \quad \stackrel{\alpha^{(1)}}{\longrightarrow} \quad \Lambda^{(2)} + \rho^{(2)},\, \Pi^{(2)} \quad \stackrel{\alpha^{(2)}}{\longrightarrow} \quad \cdots \quad \stackrel{\alpha^{(f)}}{\longrightarrow} \quad \Lambda' + \rho',\, \Pi' \tag{5.12}$$

where, at each stage, $\alpha^{(i)}$ is an odd root from $\Pi^{(i)}$. For given $\bar{\nu};\mu$, consider the following sequence of odd roots (with positions on row m, row $m-1$, ..., row k):

$$\begin{array}{ll} \text{row } m: & \beta_{m,1}, \beta_{m,2}, \ldots, \beta_{m,\min\{n,\mu_k-k+m\}} \\ \text{row } m-1: & \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,\min\{n,\mu_k-k+m-1\}} \\ \vdots & \vdots \\ \text{row } k: & \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,\mu_k} \end{array} \tag{5.13}$$

in this particular order (i.e. starting with $\beta_{m,1}$ and ending with β_{k,μ_k}). Then we have:

Lemma 5.4 *Let $\bar{\nu};\mu$ be standard and critical in $\mathfrak{gl}(m|n)$ and suppose ν and μ do not overlap in the $(m \times n)$-rectangle. Then the sequence (5.13) is a proper sequence of simple odd reflections for $\Lambda_{\bar{\nu};\mu}$, i.e. $\alpha^{(i)}$ is a simple odd root from $\Pi^{(i)}$. At the end of the sequence, one finds:*

$$\begin{aligned} \Pi' = \{&\epsilon_1 - \epsilon_2, \ldots, \epsilon_{k-2} - \epsilon_{k-1}, \epsilon_{k-1} - \delta_1, \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{\mu_k - 1} - \delta_{\mu_k}, \\ &\delta_{\mu_k} - \epsilon_k, \epsilon_k - \delta_{\mu_k+1}, \delta_{\mu_k+1} - \epsilon_{k+1}, \epsilon_{k+1} - \delta_{\mu_k+2}, \ldots, \delta_{\mu_k+m-k} - \epsilon_m, \\ &\epsilon_m - \delta_{\mu_k+m+1-k}, \delta_{\mu_k+m+1-k} - \delta_{\mu_k+m+2-k}, \ldots, \delta_{n-1} - \delta_n\}. \end{aligned} \tag{5.14}$$

Furthermore,

$$\Lambda' + \rho' = \Lambda_\lambda + \rho + \sum_{i=k+1}^{k+a-1} \sum_{j=\mu_i+1}^{\mu_k-k+i} \beta_{i,j} + \sum_{i=k+a}^{m} \sum_{j=\mu_i+1}^{\max\{0,n-\nu_{m-i+1}\}} \beta_{i,j}. \tag{5.15}$$

Proof. This proof is similar to the proof of Lemma 4.3 (see also [47]). But observe that in the first stage (i.e. the reflections with respect to odd roots of row m), $\mu_k - k + m < n$ is not necessarily true. So the sequence of odd reflections will end either with β_{m,μ_k-k+m} or with $\beta_{m,n}$. In the first case, $\Pi^{(\mu_k-k+m+1)}$ has three odd roots; in the second case, $\Pi^{(n+1)}$ contains only two odd roots. However, in both cases the set is ready to continue the reflections with respect to the elements of row $m-1$, since $\beta_{m-1,1}$ belongs to $\Pi^{(\lambda_k-k+m+1)}$ as well as to $\Pi^{(n+1)}$. Continuing with the other stages of (5.13) leads to (5.14). Remark that this sequence of simple odd reflections can always be performed, independent of whether $\bar{\nu};\mu$ is critical or not.

Criticality does, however, play an important role in (5.15), since the changes of the atypicality matrix at each step of the sequence are governed by

$$\begin{aligned} \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} \text{ if } (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) \neq 0, \\ \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} + \alpha^{(i)} \text{ if } (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) = 0. \end{aligned} \tag{5.16}$$

Suppose that in the original situation the first zero is in the last row, so $\gamma_1 = \beta_{m,\mu_m+1}$. This assumption does not harm the generality, as $\Lambda_{\bar{\nu};\mu} + \rho$ will not change anyway until the first zero is reached in the atypicality matrix, according to (5.16). Examining the sequence of odd reflections explicitly for the elements of row m yields

$$\Lambda^{(\min\{n,\mu_k-k+m\}+1)} + \rho^{(\min\{n,\mu_k-k+m\}+1)} = \Lambda_{\bar{\nu};\mu} + \rho + \sum_{j=\mu_m+1}^{\min\{n-\nu_1,\mu_k-k+m\}} \beta_{m,j}. \tag{5.17}$$

More generally, we have that $\mu_k - k + i \leq n - \nu_{m-i+1}$ if $k+1 \leq i \leq k+a-1$, and $\mu_k - k + i > n - \nu_{m-i+1}$ if $k + a \leq i \leq m$. This explains the two different contributions in (5.15). To see that criticality is necessary, consider two consecutive roots γ_p and γ_{p+1}. For simplicity, consider again γ_1 and γ_2 with γ_1 on row m. If γ_1 and γ_2 are critically related, then there appears a zero on row $m-1$ of the atypicality matrix after finishing the first stage, and one can continue until there appears an extra zero in the row of γ_2. If they are not critically related, then no extra zero can be obtained in this row. To follow the argument, this is illustrated in the following examples, where the atypicality matrix is given in the initial situation, after finishing the first stage, and after finishing the second (and in this case final) stage:

Critically related roots:

4	3	1	0
2	1	–1	–2
0	–1	–3	–4

$\longrightarrow$

3	2	1	0
1	0	–1	–2
1	0	–1	–2

$\longrightarrow$

3	1	0	0
3	1	0	0
1	–1	–2	–2

Non critically related roots:

5	4	2	0
2	1	–1	–3
0	–1	–3	–5

$\longrightarrow$

4	3	2	0
1	0	–1	–3
1	0	–1	–3

$\longrightarrow$

4	2	1	0
3	1	0	–1
1	–1	–2	–3

Thus, if $\bar{\nu};\mu$ is critical, there is at least one zero at row $m-1$ after the first stage. This zero corresponds to $\beta_{m-1,\mu_{m-1}+1}$. At the second stage, the elements of row $m-1$ play the same role as the elements of row m in the first stage, and one continues the process. Schematically, the zeros in the atypicality matrix move up along those positions corresponding to boxes that are not covered by the Young diagrams $F(\mu)$ and $F(\nu)$. Continuing with the remaining stages leads to (5.15). □

Corollary 5.5 *The critical representation $V_{\bar{\nu};\mu}$ is tame.*

Proof. Having performed the simple odd reflections (5.13), one can compute the atypicality matrix for $\Lambda' + \rho'$ using (5.15). This gives:

$$(\Lambda' + \rho', \beta_{ij}) = 0 \text{ for all } (i,j) \text{ with } k \leq i \leq k+a-1, \mu_k+1 \leq j \leq \mu_k + a. \quad (5.18)$$

Therefore the set

$$S_{\Lambda'} = \{\epsilon_k - \delta_{\mu_k+1}, \epsilon_{k+1} - \delta_{\mu_k+2}, \ldots, \epsilon_{k+a-1} - \delta_{\mu_k+a}\} \quad (5.19)$$

is a $(\Lambda' + \rho')$-maximal isotropic subset. Furthermore, $S_{\Lambda'} \subset \Pi'$, see (5.14). This implies that $V_{\bar{\nu};\mu}$ is tame [47]. If $\bar{\nu};\mu$ is not critical, (5.18) does not hold, as explained in the proof of Lemma 5.4. □

Let us illustrate some of these notions for $\bar{\nu};\mu = (\bar{3},\bar{3});(9,5,3,3,2,2,1)$ in $\mathfrak{gl}(5|7)$: In (5.7)(a), the atypicality matrix associated with $\bar{\nu};\mu$ is given. In (5.7(b)) the positions

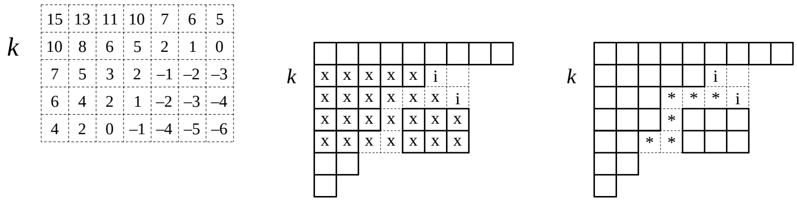


Figure 5.7: Some of the notion illustrated for $\bar{\nu};\mu = (\bar{3},\bar{3});(9,5,3,3,2,2,1)$ in $\mathfrak{gl}(5|7)$.

marked with “i” refer to the $(\Lambda' + \rho')$-maximal isotropic set (5.19). For convenience, let us refer to these positions as “the isotropic diagonal.” The positions of the odd roots that have been used for the sequence of reflections to go from $\Lambda_{\bar{\nu};\mu}$ and Π to Λ' and Π' are marked by “x” in (5.7(b)). So, they are simply all positions to the left of the isotropic diagonal. Finally, (5.7(c)) shows the positions of those β_{ij} that appear on the right hand side of (5.15); they are marked by “*”. These are all positions to the left of the isotropic diagonal that are not inside $F(\bar{\nu};\mu)$. One can see from this example and others that the (m,n)-index k determines all other necessary ingredients.

5.5 A determinantal formula for $\mathrm{ch}(V_{\bar{\nu};\mu})$

Let $\bar{\nu};\mu$ be a standard and critical composite partition without overlap in the $(m \times n)$-rectangle. As the $\mathfrak{g}$-module $V_{\bar{\nu};\mu}$ is tame, a character formula is known due to Kac and Wakimoto [34] (see also § 4.2). It reads, in terms of Λ':

$$\mathrm{ch}\, V_{\bar{\nu};\mu} = j_{\Lambda'}^{-1} e^{-\rho'} R'^{-1} \sum_{w\in W} \varepsilon(w) w\Big(e^{\Lambda'+\rho'} \prod_{\beta\in S_{\Lambda'}} (1+e^{-\beta})^{-1}\Big), \tag{5.20}$$

where

$$R' = \prod_{\alpha\in\Delta_{0,+}} (1-e^{-\alpha}) \Big/ \prod_{\alpha\in\Delta'_{1,+}} (1+e^{-\alpha}) \tag{5.21}$$

and $j_{\Lambda'}$ is a normalization coefficient to make sure that the coefficient of $e^{\Lambda'}$ on the right hand side of (5.20) is 1. By definition of ρ and R

$$e^{-\rho'} R'^{-1} = e^{-\rho} R^{-1} = D^{-1} \prod_{i=1}^{m} x_i^{(m-n-1)/2} \prod_{j=1}^{n} y_j^{(n-m-1)/2}$$

with D given in (5.23). As usual in this context we put

$$x_i = e^{\epsilon_i}, \qquad y_j = e^{\delta_j} \qquad (1 \le i \le m, 1 \le j \le n). \tag{5.22}$$

Now we have

$$\operatorname{ch} V_{\bar{\nu};\mu} = j^{-1}_{\Lambda'_{\bar{\nu};\mu}} D^{-1} \sum_{w \in W} \varepsilon(w) w(t_{\bar{\nu};\mu}),$$

with

$$D = \frac{\prod_{1 \le i < j \le m}(x_i - x_j) \prod_{1 \le i < j \le n}(y_i - y_j)}{\prod_{i=1}^{m} \prod_{j=1}^{n}(x_i + y_j)} \tag{5.23}$$

and, using (5.15) and (5.19),

$$t_{\bar{\nu};\mu} = \prod_{i=1}^{k-1} x_i^{\mu_i + m - i - n} \prod_{j=1}^{l-1} y_j^{\mu'_j + n - j - m} \prod_{i=k}^{k+a-1} \frac{y^r_{i-k+l}}{x^r_i (x_i + y_{i-k+l})} \times \prod_{i=k+a}^{n} x_i^{m-i-\nu_{m-i+1}} \prod_{j=l+a}^{n} y_j^{n-j-\nu'_{n-j+1}} \tag{5.24}$$

where $l = \mu_k + 1$ and $r = n - m + k - l$ and $j_{\Lambda'_{\bar{\nu};\mu}} = a!$ (due to symmetry there are $a!$ elements of $S_m \times S_n$ that leave $t_{\bar{\nu};\mu}$ invariant).

This expression can be written in a nicer form:

Theorem 5.6 *Let $t_{\bar{\nu};\mu}$ be given by (5.24) and $r = n - m + k - \mu_k - 1$. Then*

$$\frac{1}{a!} \sum_{w \in S_m \times S_n} \varepsilon(w) w(t_{\bar{\nu};\mu}) = (-1)^{(m-a)(l-1)+n(m-a-k+1)} \det(C), \tag{5.25}$$

where C is the following square matrix of order $n + m - a$:

$$C = \begin{pmatrix} 0 & Y_{\mu'} & 0 \\ X_\mu & R^{(r)} & X_\nu \\ 0 & Y_{\nu'} & 0 \end{pmatrix} \qquad \textit{with} \quad R^{(r)} = \left(\frac{y^r_j}{x^r_i (x_i + y_j)} \right)_{1 \le i \le m,\, 1 \le j \le n} \tag{5.26}$$

and with

$$X_\mu = \left(x_i^{\mu_j + m - n - j} \right)_{\substack{1 \le i \le m, \\ 1 \le j \le k-1}}, \qquad X_\nu = \left(x_i^{m-j-\nu_{m-j+1}} \right)_{\substack{1 \le i \le m, \\ k+a \le j \le m}},$$

$$Y_{\mu'} = \left(y_j^{\mu'_i + n - m - i} \right)_{\substack{1 \le i \le l-1, \\ 1 \le j \le n}}, \qquad Y_{\nu'} = \left(y_j^{n-i-\nu'_{n-i+1}} \right)_{\substack{l+a \le i \le n, \\ 1 \le j \le n}}.$$

Proof. The proof is similar to that of Lemma 4.5 (see also [47, Lemma 3.1]). Apply Laplace's theorem for the expansion of $\det(C)$ with respect to columns $1, 2, \ldots, k-1, k+n, k+n+1, \ldots, n+m-a$. Keeping track of the zero blocks, one finds

$$\det(C) = (-1)^P \sum_{1\le i_1<\cdots<i_{m-a}\le m} (-1)^{i_1+\cdots+i_{m-a}+(m-a)(l-1)} \det(C_x)\det(C_y), \tag{5.27}$$

where $P = \frac{(m-a)(m-a+1)}{2}$, C_x is the $(m-a)\times(m-a)$-matrix consisting of rows $i_1, i_2, \ldots, i_{m-a}$ of the matrix $\begin{pmatrix} X_\mu & X_\nu \end{pmatrix}$, and C_y is the $n\times n$-matrix

$$\begin{pmatrix} Y_{\mu'} \\ \tilde{R}^{(r)} \\ Y_{\nu'} \end{pmatrix},$$

where $\tilde{R}^{(r)}$ is obtained by removing rows $i_1, i_2, \ldots, i_{m-a}$ in $R^{(r)}$. The number of terms on the rhs of (5.27) is $\binom{m}{m-a}(m-a)!n! = m!n!/a!$; due to symmetry considerations this is the same as the number of distinct terms on the lhs of (5.25). For $(i_1, \ldots, i_{m-a}) = (1, \ldots, k-1, k+n, ..., n+m-a)$, and the diagonal term in $\det C_x$ and $\det C_y$, the contribution on the rhs of (5.27) is now easily seen to be

$$(-1)^{(m-a)(l-1)+n(m-a-k+1)} t_{\bar{\nu};\mu}.$$

But by definition of the determinant, every term on the rhs of (5.27) is (up to the overall sign factor $(-1)^{(m-a)(l-1)}$) of the form $\varepsilon(w)w(t_{\bar{\nu};\mu})$ with $w \in S_m \times S_n$. Conversely, every term of the form $\varepsilon(w)w(t_{\bar{\nu};\mu})$ appears as a term on the rhs of (5.27). It follows that (5.25) holds. □

With the same notation, one finds

Corollary 5.7 *The character of a critical representation labeled by a standard composite partition $\bar{\nu};\mu$ (without zeros in the overlap) has the following determinantal form:*

$$\mathrm{ch}\, V_{\bar{\nu};\mu} = (-1)^{(m-a)(l-1)+n(m-a-k+1)} D^{-1} \det(C) \tag{5.28}$$

with C defined in (5.26).

As an example, let $m = 4$, $n = 5$ and $\bar{\nu};\mu = (\bar{1},\bar{1},\bar{4});(3,1)$. One finds

6	4	3	2	–1
3	1	0	–1	–4
1	–1	–2	–3	–6
0	–2	–3	–4	–7

$$\begin{aligned} k &= 2 \\ l &= \mu_k + 1 = 2 \\ a &= 2 \\ &\Rightarrow\ r = n - m + k - l = 1 \\ &\Rightarrow\ n + m - a = 7 \end{aligned}$$

Thus, according to formula (5.26),

$$\mathrm{ch}V_{(\bar{1},\bar{1},\bar{4});(3,1)} = D^{-1}\ \det C,$$

with

$$C = \begin{pmatrix} 0 & y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & 0 \\ x_1 & \frac{y_1}{x_1(x_1+y_1)} & \frac{y_2}{x_1(x_1+y_2)} & \frac{y_3}{x_1(x_1+y_3)} & \frac{y_4}{x_1(x_1+y_4)} & \frac{y_5}{x_1(x_1+y_5)} & x_1^{-4} \\ x_2 & \frac{y_1}{x_2(x_2+y_1)} & \frac{y_2}{x_2(x_2+y_2)} & \frac{y_3}{x_2(x_2+y_3)} & \frac{y_4}{x_2(x_2+y_4)} & \frac{y_5}{x_2(x_2+y_5)} & x_2^{-4} \\ x_3 & \frac{y_1}{x_3(x_3+y_1)} & \frac{y_2}{x_3(x_3+y_2)} & \frac{y_3}{x_3(x_3+y_3)} & \frac{y_4}{x_3(x_3+y_4)} & \frac{y_5}{x_3(x_3+y_5)} & x_3^{-4} \\ x_4 & \frac{y_1}{x_4(x_4+y_1)} & \frac{y_2}{x_4(x_4+y_2)} & \frac{y_3}{x_4(x_4+y_3)} & \frac{y_4}{x_4(x_4+y_4)} & \frac{y_5}{x_4(x_4+y_5)} & x_4^{-4} \\ 0 & y_1^0 & y_2^0 & y_3^0 & y_4^0 & y_5^0 & 0 \\ 0 & y_1^{-3} & y_2^{-3} & y_3^{-3} & y_4^{-3} & y_5^{-3} & 0 \end{pmatrix}.$$

5.6 The character formula and $s_{\bar{\nu};\mu}(x/y)$

The determinantal formula is very explicit. The main goal of this determinantal formula however is that it allows us to make the link with another explicit formula that is even more useful, namely:

Conjecture 5.8 *Let $\bar{\nu};\mu$ be a standard and critical composite partition with no overlap. The character $\mathrm{ch}\,V_{\bar{\nu};\mu}$ is equal to $s_{\bar{\nu};\mu}(x/y)$ as defined in (2.34).*

The purpose of this section is to investigate whether the techniques of § 4.5 can be extended to the current case of critical composite partitions.

Recall that

$$D(x) = \prod_{1\le i<j\le m} (x_i - x_j) \qquad \text{and} \qquad E(x,y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j).$$

Suppose $|x| = m$ and $|y| = n$. From the definition of $D(x)$ and $E(x,y)$ we derive, for $\overline{x} = \frac{1}{x}$ and $\overline{y} = \frac{1}{y}$, that:

$$D(\overline{x}) = (-1)^{\frac{m(m-1)}{2}} D(x) \left(\prod_{i=1}^{m} x_i^{-m+1} \right), \tag{5.29}$$

$$E(\overline{x},\overline{y}) = (-1)^{mn} \left(\prod_{i=1}^{m} x_i^{-n} \right) \left(\prod_{j=1}^{n} y_j^{-m} \right) E(x,y), \tag{5.30}$$

$$\frac{D(\overline{x})D(\overline{y})}{E(\overline{x},\overline{y})} = (-1)^{P} \left(\prod_{i=1}^{m} x_i^{n-m+1} \right) \left(\prod_{j=1}^{n} y_j^{m-n+1} \right) \frac{D(x)D(y)}{E(x,y)} \tag{5.31}$$

$$\text{with } P = \frac{m(m-1)}{2} + \frac{n(n-1)}{2} + mn$$

Lemma 5.9 *Suppose ν is a partition, then*

$$s_{\overline{\nu}}(x) = \frac{(-1)^{\frac{m(m-1)}{2}}}{D(x)} \left| x_j^{-\nu_i+i-1} \right| = \frac{\left| x_j^{-\nu_{m-i+1}+m-i} \right|}{D(x)}.$$

Proof. This formula is derived from the determinantal formula (1.10) for S-functions [46], applying properties of determinants:

$$\begin{aligned} s_{\overline{\nu}}(x) &= s_{\nu}(\overline{x}) = \frac{\left|\overline{x}_j^{\nu_i+m-i}\right|}{D(\overline{x})} = (-1)^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} x_i^{m-1} \frac{\left|x_j^{-\nu_i-m+i}\right|}{D(x)} \\ &= (-1)^{\frac{m(m-1)}{2}} \frac{\left|x_j^{-\nu_i+i-1}\right|}{D(x)} = \frac{\left|x_j^{-\nu_{m-i+1}+m-i}\right|}{D(x)}. \end{aligned}$$

□

If ν is an arbitrary t-tuple over $\mathbb{Z}$ and μ an arbitrary s-tuple over $\mathbb{Z}$, we can still define $s_{\bar{\nu};\mu}(x|y)$ through formula (2.34). Note that $\nu+\delta_t$ and $\mu+\delta_s$ must be nonnegative distinct integers for $s_{\bar{\nu};\mu}(x|y)$ to be nonzero. We need the following generalization of Lemma 2.6.

Lemma 5.10 *Suppose $y = y^{(n)} = (y_1,\ldots,y_n)$. Let ν be an arbitrary t-tuple over $\mathbb{Z}$ and μ an arbitrary s-tuple over $\mathbb{Z}$, then*

$$s_{\bar{\nu};\mu}(x/y) = \sum_{\alpha,\beta} s_{\overline{\beta};\alpha}(x/y^{(n-1)}) y_n^{a-b}, \tag{5.32}$$

where $a = |\mu - \alpha|$, $b = |\nu - \beta|$, and the sum is taken over all α and β such that $(\nu - \beta)_i \in \{0, 1\}$.

Proof. (Compare with the proof of Lemma 4.19.) This follows from the previous lemma and the determinant (2.34) for $s_{\bar{\nu};\mu}(x/y)$. If there are two identical columns in this determinant, then $s_{\bar{\nu};\mu}(x/y) \equiv 0$. But then also in the right hand side of (5.32), the terms will either vanish or else cancel each other two by two. If all columns in the determinant are different, they can be permuted such that $s_{\bar{\nu};\mu}(x/y) = \pm s_{\overline{\phi};\psi}(x/y)$ where ϕ and ψ are partitions. Applying Lemma 2.6 and performing the inverse permutation for the S-functions in the right hand side yields the result. □

Lemma 5.11 *For $m = p + q$, let $\varphi = (\varphi_1, \ldots, \varphi_p)$ and $\sigma = (\sigma_1, \ldots, \sigma_q)$ be two partitions and $\lambda = (\varphi_1 + g - q, \ldots, \varphi_p + g - q, -\sigma_q + h, \ldots, -\sigma_1 + h)$. Suppose $g, h \in \mathbb{Z}$, then*

$$\sum_{x'+x''} \frac{(\prod x')^g (\prod x'')^h s_\varphi(x') s_{\overline{\sigma}}(x'')}{E(x', x'')} = s_\lambda(x), \tag{5.33}$$

where the sum is over all possible decompositions $x = x' + x''$ with the size of x' equal to p and the size of x'' equal to q.

Proof. We can rewrite the left hand side of (5.33) using the determinantal formula (1.10) for S-functions and the equality

$$D(x) = (-1)^{\frac{p(p+1)}{2} + r_1 + \cdots + r_p} D(x') D(x'') E(x', x''), \tag{5.34}$$

with the elements of x' denoted by $x_{r_1}, \ldots, x_{r_p}$ ($r_1 < \cdots < r_p$) and those of x'' by $x_{s_1}, \ldots, x_{s_q}$ ($s_1 < \cdots < s_q$):

$$\begin{aligned}
&\sum_{x'+x''} \frac{(\prod x')^g (\prod x'')^h s_\varphi(x') s_{\overline{\sigma}}(x'')}{E(x', x'')} \\
&= \sum_{x'+x''} \frac{(\prod x')^g (\prod x'')^h}{E(x', x'')} \cdot \frac{\left| \left({x_{r_i}}^{\varphi_j + p - j} \right)_{\substack{i=1\ldots p \\ j=1\ldots p}} \right|}{D(x')} \cdot \frac{\left| \left({x_{s_i}}^{-\sigma_{q-j+1} + q - j} \right)_{\substack{i=1\ldots q \\ j=1\ldots q}} \right|}{D(x'')} \\
&= \frac{1}{D(x)} \sum_{x'+x''} (-1)^P \left| \left({x_{r_i}}^{(\varphi_j + g - q) + (p + q - j)} \right)_{\substack{i=1\ldots p \\ j=1\ldots p}} \right| \\
&\qquad \times \left| \left({x_{s_i}}^{(-\sigma_{m-j+1} + h) + (m - j)} \right)_{\substack{i=1\ldots q \\ j=p+1\ldots m}} \right|.
\end{aligned}$$

with $P = \frac{p(p+1)}{2} + r_1 + \cdots + r_p$. The numerator of this sum is the Laplace expansion of the following determinant with respect to columns $1, \ldots, p$:

$$\left| x_i^{(\varphi_j+g-q)+(m-j)} \quad x_i^{(-\sigma_{m-j+1}+h)+(m-j)} \right| = \left| x^{\lambda+\delta_m} \right|$$

with $\lambda = (\varphi_1 + g - q, \ldots, \varphi_p + g - q, -\sigma_q + h, \ldots, -\sigma_1 + h)$ and $\delta_m = (m-1, m-2, \ldots, 0)$, so the result follows. □

Observe that in this result, λ is not necessarily a partition, but it could be an arbitrary integer m-tuple. In such a case, $s_\lambda(x)$ is still well defined by $|x^{\lambda+\delta_m}|/|x^{\delta_m}|$.

Now we are almost in a position to prove Conjecture 5.8. However, as we shall see, there remains a complication. First we will give the outlines of a proof and the problem occurring for a special case (Conjecture 5.12), and then finally for the general case (Conjecture 5.13).

The special case consists of a subclass of all standard and critical composite partitions without overlap. This subclass is characterized by $n = l + a - 1$; in other words, we will consider standard composite partitions where the first zero in the atypicality matrix (the zero in the row with index k) is *in the last column*. An example is given in Figure 5.8, with $\bar{\nu}; \mu = (\overline{2}, \overline{6}); (6, 4, 2, 2, 1)$ in $\mathfrak{gl}(8|5)$. In this case $k = 4$, $l = 3$, $a = 3$ and $r = -2$. Remark that we can also consider the conjugate case where $m = k+a-1$,

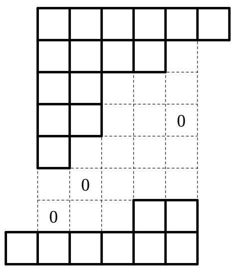


Figure 5.8: $\bar{\nu}; \mu = (\overline{2}, \overline{6}); (6, 4, 2, 2, 1)$

meaning that all the zeros are in the last a rows.

Conjecture 5.12 *Suppose $\bar{\nu}; \mu$ is a standard and critical composite partition with no overlap in $\mathfrak{gl}(m|n)$ with $n = l + a - 1$ or $m = k + a - 1$. Then*

$$\mathrm{ch}(V_{\bar{\nu};\mu}) = \pm s_{\bar{\nu};\mu}(x/y). \tag{5.35}$$

We only have to prove the equality (5.35) if $n = l + a - 1$; when $m = k + a - 1$ the equality follows from the case $n = l + a - 1$ and the fact that

$$s_{\bar{\nu};\mu}(x/y) = s_{\overline{\nu'};\mu'}(y/x). \tag{5.36}$$

We shall give the outlines of a proof, and show that it finally reduces to an identity for composite supersymmetric Schur functions.

Let $p = k + a - 1$ and $q = m - k - a + 1$. Note that in this special case $r = -q$. First substitute y_j by $-y_j$ in the determinantal formula (Corollary 5.7) of $\mathrm{ch}(V_{\bar{\nu};\mu})$. Next, take the Laplace expansion of this determinant with respect to columns $1, 2, \ldots, n+k-1$. So, with the elements of x' denoted by $x_{r_1}, \ldots, x_{r_p}$ and those of x'' by $x_{s_1}, \ldots, x_{s_q}$, we have the following expression for the character:

$$\begin{aligned}
&\frac{E(x,y)}{D(x)D(y)} \det \begin{pmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right) & 0 \\ \left(x_i^{\mu_j+m-n-j}\right) & \left(\frac{y_j^r}{x_i^r(x_i-y_j)}\right) & \left(x_i^{-\nu'_{m-j+1}+m-j}\right) \end{pmatrix} \\
&= \frac{E(x,y)}{D(x)D(y)} \sum_{x'+x''} (-1)^P \begin{vmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right)_{\substack{1\le i\le n-a \\ 1\le j\le n}} \\ \left(x_{r_i}^{\mu_j+m-n-j}\right)_{\substack{1\le i\le p \\ 1\le j\le k-1}} & \left(\frac{y_j^r}{x_{r_i}^r(x_{r_i}-y_j)}\right)_{\substack{1\le i\le n-a \\ 1\le j\le k-1}} \end{vmatrix} \\
&\quad \times \left| \left(x_{s_i}^{-\nu'_j+j-1}\right)_{\substack{1\le i\le, q \\ 1\le j\le q}} \right|,
\end{aligned}$$

with $P = \frac{(n+k-1)(n+k)}{2} + r_1 + \cdots + r_p$.

In the right hand side of this expression we can rewrite the first determinant, using Theorem 4.8:

$$\begin{aligned}
&\frac{E(x',y)}{D(x')D(y)} \begin{vmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right)_{\substack{i=1,\ldots,n-a \\ j=1,\ldots,n}} \\ \left(x_{r_i}^{\mu_j+m-n-j}\right)_{\substack{i=1,\ldots,p \\ j=1,\ldots,k-1}} & \left(\frac{y_j^r}{x_{r_i}^r(x_{r_i}-y_j)}\right)_{\substack{i=1,\ldots,p \\ j=1,\ldots,n}} \end{vmatrix} \\
&= \frac{E(x',y)}{D(x')D(y)} \frac{\prod_{i=1}^p x_{r_i}^q}{\prod_{j=1}^n y_j^q} \begin{vmatrix} 0 & \left(y_j^{\mu'_i+n-p-i}\right)_{\substack{i=1,\ldots,n-a \\ j=1,\ldots,n}} \\ \left(x_{r_i}^{\mu_j+p-n-j}\right)_{\substack{i=1,\ldots,p \\ j=1,\ldots,k-1}} & \left(\frac{1}{x_{r_i}-y_j}\right)_{\substack{i=1,\ldots,p \\ j=1,\ldots,n}} \end{vmatrix} \\
&= \pm \left(\frac{\prod_{i=1}^p x_{r_i}}{\prod_{j=1}^n y_j}\right)^q s_\mu(x'/-y),
\end{aligned}$$

where the minus sign depends on the partition μ only. So, the Laplace expansion equals

$$\pm \sum_{x'+x''} (-1)^P \frac{E(x,y)}{D(x)D(y)} \frac{D(x')D(y)}{E(x',y)} \left(\frac{\prod_{i=1}^p x_{r_i}}{\prod_{j=1}^n y_j} \right)^q s_\mu(x'/-y) D(x'') s_\nu(\overline{x}'')$$

$$\stackrel{(5.34)}{=} \pm \sum_{x'+x''} \frac{E(x'',y)}{E(x',x'')} \left(\frac{\prod_{i=1}^p x_{r_i}}{\prod_{j=1}^n y_j} \right)^q s_\mu(x'/-y) s_\nu(\overline{x}'')$$

$$\stackrel{(5.30)}{=} \pm \sum_{x'+x''} \frac{(-1)^{qn} (\prod_{i=1}^q x_{s_i}^n)(\prod_{j=1}^n y_j^q) E(\overline{x}'',\overline{y})}{E(x',x'')} \frac{\prod_{i=1}^p x_{r_i}^q}{\prod_{j=1}^n y_j^q} s_\mu(x'/-y) s_\nu(\overline{x}'').$$

Applying a special case of the Sergeev-Pragacz formula (2.8) yields

$$\pm \sum_{x'+x''} \frac{(\prod_{i=1}^p x_{r_i})^q (\prod_{i=1}^q x_{s_i})^n}{E(x',x'')} s_\mu(x'/-y) s_{\nu+(n^q)}(\overline{x}''/-\overline{y}) \qquad (5.37)$$

The only step remaining to prove the conjecture, is to prove that this sum equals $\pm s_{\bar{\nu};\mu}(x/-y)$. Substituting every y_j by $-y_j$, will yield the theorem. □

So, the important fact to note is that the proof finally reduces to verifying identity (5.37). Next, we move our attention to the general case. Also here, the proof reduces to an identity of the form (5.37):

Conjecture 5.13 *Let $\bar{\nu};\mu$ be a standard and critical composite partition with no overlap in $\mathfrak{gl}(m|n)$. Then*

$$\mathrm{ch}(V_{\bar{\nu}:\mu}) = \pm s_{\bar{\nu};\mu}(x/y).$$

Let $p = l+a-1$ and $q = n-l-a+1$. First substitute y_j by $-y_j$ in the determinantal formula of $\mathrm{ch}(V_{\bar{\nu}:\mu})$. Next, take the Laplace expansion of the determinant with respect to rows $1, 2, \ldots, m+l-1$. So, with the elements of y' denoted by $y_{r_1}, \ldots, y_{r_p}$ and those of y'' by $y_{s_1}, \ldots, y_{s_q}$, we have the following expression

$$\frac{E(x,-y)}{D(x)D(y)} \det \begin{pmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right) & 0 \\ \left(x_i^{\mu_j+m-n-j}\right) & \left(\frac{y_j^r}{x_i^r(x_i-y_j)}\right) & \left(x_i^{-\nu_{m-j+1}+m-j}\right) \\ 0 & \left(y_j^{-\nu'_{n-i+1}+n-i}\right) & 0 \end{pmatrix}$$

$$= \frac{E(x,-y)}{D(x)D(y)} \sum_{y'+y''} (-1)^P C_1 \left| \left(y_{s_j}^{-\nu'_{q-i+1}+q-i} \right)_{\substack{i=1,\ldots,q \\ j=1,\ldots,q}} \right|,$$

with $P = \frac{(m+l-1)(m+l)}{2} + r_1 + \cdots + r_p$. The determinant C_1 equals

$$\begin{vmatrix} 0 & \left(y_{r_j}^{\mu'_i+n-m-i}\right)_{\substack{1\le i\le l-1\\1\le j\le p}} & 0 \\ \left(x_i^{\mu_j+m-n-j}\right)_{\substack{1\le i\le m\\1\le j\le k-1}} & \left(\frac{y_{r_j}^r}{x_i^r(x_i-y_{r_j})}\right)_{\substack{1\le i\le m\\1\le j\le p}} & \left(x_i^{-\nu_{m-j+1}+m-j}\right)_{\substack{1\le i\le m\\k+a\le j\le m}} \end{vmatrix}$$

$$= \left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q \begin{vmatrix} 0 & \left(y_{r_j}^{\mu'_i+p-m-i}\right) & 0 \\ \left(x_i^{\mu_j+m-p-j}\right) & \left(\frac{y_{r_j}^{r'}}{x_i^{r'}(x_i-y_{r_j})}\right) & \left(x_i^{-\nu_{m-j+1}+q+m-j}\right) \end{vmatrix},$$

where $r' = r-q = p-m+k-l$. Thus, this determinantal expression coincides with the determinantal formula (5.26) of $\mathrm{ch}(V_{\overline{\eta};\mu})$ in $\mathfrak{gl}(m|p)$ with $\eta = (\nu_1, \ldots, \nu_{m-k-a+1}) - (q^{m-k-a+1})$. According to Conjecture 5.12, the determinant C_1 equals

$$\pm\frac{D(x)D(y')}{E(x,y')}\left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y').$$

So, $\eta' = (\nu'_{q+1}, \nu'_{q+2}, \ldots)$ and let $\beta' = (\nu'_1, \ldots, \nu'_q)$. As the minus sign depends on the composite partition $\overline{\eta};\mu$ only, the Laplace expansion equals

$$\begin{aligned} &\pm\sum_{y'+y''}(-1)^P\frac{E(x,y)}{D(x)D(y)}\frac{D(x)D(y')}{E(x,y')}\left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y')D(y'')s_{\beta'}(\overline{y}'') \\ &= \pm\sum_{y'+y''}\frac{E(x,y'')}{E(y',y'')}\left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y')s_{\beta'}(\overline{y}'') \\ &= \pm\sum_{y'+y''}\frac{\left(\prod_{i=1}^m x_i\right)^q\left(\prod_{j=1}^q y_{s_j}\right)^m}{E(y',y'')}\frac{\prod_{j=1}^p y_{r_j}^q}{\prod_{i=1}^m x_i^q}s_{\overline{\eta};\mu}(x/-y')s_{\beta'}(\overline{y}'')E(\overline{y}'',\overline{x}), \end{aligned}$$

where we have used (5.30). Simplifying and applying again the special case of the Sergeev-Pragacz formula and (5.36), this becomes

$$\pm\sum_{y'+y''}\frac{\left(\prod_{j=1}^p y_{r_j}\right)^q\left(\prod_{j=1}^q y_{s_j}\right)^m}{E(y',y'')}s_{\overline{\eta'};\mu'}(-y'/x)s_{\beta'+(m^q)}(-\overline{y}''/\overline{x}). \qquad (5.38)$$

If we can prove that (5.38) equals $\pm s_{\overline{\nu'};\mu'}(-y/x)$, the conjecture is established by applying (5.36) and substituting every y_j by $-y_j$. □

Essentially, those two conjectures are proved if we would be able to prove next lemma.

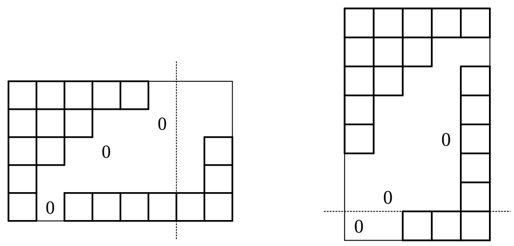
Lemma 5.14 *Suppose that $|x| = m$, $|y| = n$ and that p and q are positive integers with $m = p + q$. Let $\bar{\nu};\mu$ be a critical composite partition with no zeros in the overlap when presented in the $(m \times n)$-rectangle. With $\nu = (\kappa_1, \ldots, \kappa_q, \eta_1, \eta_2, \ldots)$, we find that:*

$$\sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n s_{\overline{\eta};\mu}(x'/y) s_{\kappa+(n^q)}(\overline{x}''/\overline{y})}{E(x', x'')} = s_{\overline{\nu};\mu}(x/y) \tag{5.39}$$

where the sum is over all possible decompositions $x = x' + x''$ with the size of x' equal to p and the size of x'' equal to q.

Originally we believed that (5.39) was generally valid for any composite partition $\bar{\nu};\mu$, since it seemed to be the supersymmetric version of (5.33). However, this is unfortunately not the case. Since we do not need (5.39) in general, but only in special cases ($\bar{\nu};\mu$ critical, no zeros in overlap, p and q special values), we believe it is still valid in those special cases where needed. But we have not been able to prove this. In the following, we shall describe some of the problems we encountered, indicating that (5.39) is certainly not valid in general but only for the special cases needed.

First of all, we will need an extra restriction on the choice of p and q. In the first application (5.37) of Lemma 5.14 $p = k + a - 1$; the second application (5.38) should be translated to conjugate partitions first. But, by taking the conjugate, it is possible that $l + a - 1$ is not turned over in $k + a - 1$ as illustrated in Figure 5.9.



(a) $p = l + a - 1 = 6$ (b) $p = k+a-1 = 7$

Figure 5.9: In (a) $\bar{\nu};\mu$ in $\mathfrak{gl}(5|8)$ and in (b) $\bar{\nu};\mu$ in $\mathfrak{gl}(8|5)$.

Nevertheless, the choice of p and q can not be arbitrary either. Let us illustrate this for the composite partition $\bar{\nu};\mu = (\overline{3});(1)$ in $\mathfrak{gl}(3|3)$ where $k = 1$ and $a = 2$. If

$p = k + a - 1 = 2$, then we have that $q = 1$, $\kappa = (3)$ and $\eta = ()$. In the assumption that Lemma 5.14 holds, the lhs of (5.39) is in this example given by

$$\begin{aligned} \text{lhs} &\overset{(5.32)}{=} \sum_{x'+x''} \frac{(\prod x')(\prod x'')^3 (s_{(1)}(x'/y') + y_3)(s_{(6)}(\overline{x}''/\overline{y}') + s_{(5)}(\overline{x}''/\overline{y}')y_3^{-1})}{E(x', x'')} \\ &\overset{(5.39)}{=} s_{(\overline{3});(1)}(x/y') + s_{(\overline{2});(1)}(x/y')\overline{y_3} + s_{(\overline{3});()}(x/y')y_3 + s_{(\overline{2});()}(x/y') \\ &\overset{(2.36a)}{=} s_{(\overline{3});(1)}(x'/y). \end{aligned}$$

Here we use the notation $y' = (y_1, \ldots, y_{n-1}) = (y_1, y_2)$. Writing out every term in both sides of (5.39), tells us that the expression is true. On the other hand, if we make the same computation but with the assumption that $p = 1$, then $q = 2$, $\kappa = (3, 0)$ and $\eta = ()$ and thus:

$$\begin{aligned} \text{lhs} \overset{(5.32)}{=} \sum_{x'+x''} \frac{(\prod x')^2(\prod x'')^3}{E(x', x'')} &\big(s_{(1)}(x'/y') + y_3\big)\big(s_{(6,3)}(\overline{x}''/\overline{y}') \\ &+ s_{(5,3)}(\overline{x}''/\overline{y}')y_3^{-1} + s_{(6,2)}(\overline{x}''/\overline{y}')y_3^{-1} + s_{(5,2)}(\overline{x}''/\overline{y}')y_3^{-2}\big) \end{aligned}$$

In this expression there are several terms which are not of the proper type of the lhs of (5.39) e.g.:

$$\sum_{x'+x''} \frac{(\prod x')^2(\prod x'')^3}{E(x', x'')} s_{(1)}(x'/y')s_{(6,2)}(\overline{x}''/\overline{y}')y_3^{-1}$$

is not equal to a supersymmetric Schur function as the power of $(\prod x'')$ is different from $|y'| = 2$. Indeed, by writing out every term in both sides of (5.39), we find that the expression are different.

On the other hand, observe that the restrictions on the composite partition $\bar{\nu};\mu$ are necessarily. If we consider a composite partition which is not critical, say $\bar{\nu};\mu = (\overline{2}, \overline{2}); (2, 1)$ in $\mathfrak{gl}(3|3)$, or a composite partition $\bar{\nu};\mu$ which is critical but with zeros in the overlap, say $\bar{\nu};\mu = (\overline{3}, \overline{3}); (2, 2, 1)$ in $\mathfrak{gl}(3|3)$, then (5.39) is not true. These inequalities were found by writing out every single term in both sides of (5.39).

As Lemma 5.14 looks like a supersymmetric extension of Lemma 5.11, one could hope to reduce it to Lemma 5.11 by induction on $|y|$, starting from $|y| = 0$ (where it is certainly true, since this is Lemma 5.11). In the following, we indicate the problems of such an attempt by induction.

The case where $n=0$, coincides with Lemma 5.11, since $s_{\overline{\eta};\mu}(x) = \left(\prod_i x_i\right)^{-\eta_1} s_\xi(x)$ with $\xi = (\mu_1+\eta_1, \mu_2+\eta_1, \ldots, \eta_1-\eta_2, 0)$ [41] and with $g = q-\eta_1,\ \varphi = \xi$ and $\sigma = \kappa + (n^p)$. Suppose $n>0$ and denote by $y^{(n)} = (y_1, \ldots, y_n)$. We can use Lemma 2.6 to isolate y_n, giving:

$$\sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n s_{\overline{\eta};\mu}(x'/y) s_{\kappa+(n^q)}(\overline{x}''/\overline{y})}{E(x',x'')}$$
$$= \sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n}{E(x',x'')} \left(\sum_{\alpha,\beta} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) y_n^{a-b} \right) \left(\sum_{\gamma} s_{\gamma+(n^q)}(\overline{x}''/\overline{y}^{(n-1)}) y_n^{-c} \right) \tag{5.40}$$

where μ/α, η/β and κ/γ are vertical strips of length a, b and c respectively. Rearranging terms, this sum equals

$$\sum_{\alpha,\beta,\gamma} \left(\sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n}{E(x',x'')} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) s_{\gamma+(n^q)}(\overline{x}''/\overline{y}^{(n-1)}) \right) y_n^{a-b-c} \tag{5.41}$$

In order to be able to apply induction, the power of x'' has to be $n-1 = |y^{(n-1}|$. This is easily solved by applying the formula of Berele and Regev (2.8) twice:

$$\sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n}{E(x',x'')} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) s_{\gamma+(n^q)}(\overline{x}''/\overline{y}^{(n-1)})$$
$$= \sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n}{E(x',x'')} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) s_{\gamma+(1^q)}(\overline{x}'') \prod_{i,j}(x''_i + y'_j)$$
$$= \sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^{n-1}}{E(x',x'')} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) s_{\gamma}(\overline{x}'') \prod_{i,j}(x''_i + y'_j)$$
$$= \sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^{n-1}}{E(x',x'')} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) s_{\gamma+((n-1)^q)}(\overline{x}''/\overline{y}^{(n-1)}) \tag{5.42}$$

If we would be able to use induction, the sum (5.41) will reduce to

$$\sum_{\alpha,\tau} s_{\overline{\tau};\alpha}(x/y^{(n-1)}) y_n^{a-(b+c)} \tag{5.43}$$

where μ/α is a vertical strip of length a and $(\nu-\tau)_i \in \{0,1\}$ with $|\nu-\tau| = b+c$ and $\tau = (\gamma_1, \ldots, \gamma_q, \beta_1, \beta_2, \ldots)$. Applying Lemma 5.10, this is equal to $s_{\overline{\nu};\mu}(x/y)$, which would provide us a proof of Lemma 5.14.

So what went wrong? The problem is in fact the identification of the factor

$$\sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^{n-1}}{E(x',x'')} s_{\overline{\beta};\alpha}(x'/y^{(n-1)}) s_{\gamma+((n-1)^q)}(\overline{x}''/\overline{y}^{(n-1)})$$

in (5.42) with $s_{\overline{\tau};\alpha}(x/y^{(n-1)})$ in (5.43). It turns out that $\overline{\tau};\alpha$ not necessarily satisfies the special conditions of Lemma 5.14, so induction fails.

Let us illustrate this by means of an example. Writing out (5.42) for $\bar{\nu};\mu = (\overline{1},\overline{1});(1,1,1)$ in $\mathfrak{gl}(4|3)$ gives the following result when $p = q = 2$:

$$\begin{aligned}\sum_{x'+x''} \frac{(\prod x')^2 (\prod x'')^2}{E(x',x'')} \Big(& s_{(1,1,1)}(x'/y') s_{(\overline{3},\overline{3})}(x''/y') + s_{(1,1,1)}(x'/y') s_{(\overline{3},\overline{2})}(x''/y') y_3^{-1} \\ & + s_{(1,1,1)}(x'/y') s_{(\overline{2},\overline{2})}(x''/y') y_3^{-2} + s_{(1,1)}(x'/y') s_{(\overline{3},\overline{3})}(x''/y') y_3 \\ & + s_{(1,1)}(x'/y') s_{(\overline{3},\overline{2})}(x''/y') + s_{(1,1)}(x'/y') s_{(\overline{2},\overline{2})}(x''/y') y_3^{-1} \\ & + s_{(1)}(x'/y') s_{(\overline{3},\overline{3})}(x''/y') y_3^2 + s_{(1)}(x'/y') s_{(\overline{3},\overline{2})}(x''/y') y_3 \\ & + s_{(1)}(x'/y') s_{(\overline{2},\overline{2})}(x''/y') + s_{()}(x'/y') s_{(\overline{3},\overline{3})}(x''/y') y_3^3 \\ & + s_{()}(x'/y') s_{(\overline{3},\overline{2})}(x''/y') y_3^2 + s_{()}(x'/y') s_{(\overline{2},\overline{2})}(x''/y') y_3 \Big)\end{aligned} \tag{5.44}$$

and (5.43) gives rise to

$$\begin{aligned} & s_{(\overline{1},);(1,1,1)}(x/y') + s_{(\overline{1});(1,1,1)}(x/y') y_3^{-1} + s_{();(1,1,1)}(x/y') y_3^{-2} \\ & + s_{(\overline{1},\overline{1});(1,1)}(x/y') y_3 + s_{(\overline{1});(1,1)}(x/y') + s_{();(1,1)}(x/y') y_3^{-1} \\ & + s_{(\overline{1},\overline{1});(1)}(x/y') y_3^2 + s_{(\overline{1});(1)}(x/y') y_3 + s_{();(1)}(x/y') \\ & + s_{(\overline{1},\overline{1});()}(x/y') y_3^3 + s_{(\overline{1});()}(x/y') y_3^2 + s_{();()}(x/y') y_3 \end{aligned} \tag{5.45}$$

Remark that in both expressions $y' = y^{(n-1)} = (y_1, y_2)$.

There occur some problems, although it is possible to check by computing every term, that the sum (5.44) indeed equals $s_{\bar{\nu};\mu}(x/y)$. The problems occur as, in some terms, the composite partitions we should expect using induction are not critical anymore; for example in the first term $(\overline{1},\overline{1},\overline{1});(1,1)$ is not critical in $\mathfrak{gl}(4|2)$. In the same manner, the second and fourth term do not lead to a critical composite partition either. In most of the other terms, the expected composite partition $\overline{\tau};\alpha$ would lead to a different value of p . So, most of the terms are not exactly equal to $s_{\overline{\tau};\alpha}(x/y^{(n-1)})$. But, extra terms in every single computation of a term in (5.44) will cancel out. For example, the first term T_1 is equal to:

$$\begin{aligned} T_1 &= \sum_{x'+x''} \frac{(\prod x')^2(\prod x'')^2}{E(x',x'')} s_{(1,1,1)}(x'/y')s_{(\overline{3},\overline{3})}(x''/y') \\ &= s_{(\overline{1},\overline{1});(1,1,1)}(x/y') + s_{();(1,1,1)}(x)s_{(\overline{1},\overline{1});()}(y') + s_{();(1,1,1,1)}(x)s_{(\overline{1},\overline{2});()}(y'), \end{aligned}$$

whereas the ninth term T_9 equals:

$$\begin{aligned} T_9 &= \sum_{x'+x''} \frac{(\prod x')^2(\prod x'')^2}{E(x',x'')} s_{(1)}(x'/y')s_{(\overline{2},\overline{2})}(x''/y') \\ &= s_{();(1)}(x/y') - s_{();(1,1,1)}(x)s_{(\overline{1},\overline{1});()}(y') - s_{();(1,1,1,1)}(x)s_{(\overline{1},\overline{2});()}(y'). \end{aligned}$$

The sum gives $T_1 + T_9 = s_{(\overline{1},\overline{1});(1,1,1)}(x/y') + s_{();(1)}(x/y')$.

So, to conclude this section, we notice that the two conjectures are proved if we can prove Lemma 5.14. A proof by induction is so far the best hope. However, in such a proof it remains to be argued why extra terms in the identification of (5.42) and (5.43) always cancel.

Chapter 6

Dimension formulas for $\mathfrak{gl}(m|n)$ representations

In this chapter we investigate new formulas for the dimension and superdimension of the covariant and mixed tensor representations V of the Lie superalgebra $\mathfrak{gl}(m|n)$. The notion of t-dimension is introduced, where the parameter t keeps track of the $\mathbb{Z}$-grading of V. A formula for the t-dimension is derived from the determinantal formula for the supersymmetric Schur polynomials $s_\lambda(x/y)$ and the determinantal formula for $s_{\bar{\nu};\mu}(x/y)$. It expresses the t-dimension as a simple determinant. For a special choice of λ, the new t-dimension formula for V_λ gives rise to a Hankel determinant identity.

6.1 Introduction

In this chapter we consider again the Lie superalgebra $\mathfrak{gl}(m|n)$, denoted by $\mathfrak{g}$. Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. Let Λ be the highest weight of V. We shall consider the specialization of $\operatorname{ch} V$ determined by

$$\begin{aligned} F(e^{\epsilon_i}) &= 1 \quad (i = 1, \ldots, m) \\ F(e^{\delta_j}) &= t \quad (j = 1, \ldots, n). \end{aligned} \tag{6.1}$$

This specialization is consistent with the $\mathbb{Z}$-grading of $\mathfrak{g}$, and the corresponding $\mathbb{Z}$-grading of V. The specialization of the character of V under F is referred to as the

t-DIMENSION of V and denoted by $\dim_t(V)$:

$$\dim_t(V) = F(\mathrm{ch}\, V) = \sum_{\mu} \dim V(\mu) F(e^{\mu}). \tag{6.2}$$

Often, the t-dimension would be defined [33, §10] as $F(e^{-\Lambda}\,\mathrm{ch}\, V)$, with Λ the highest weight of V; but here (6.2) is more convenient. The t-dimension of V stands for the polynomial

$$F(e^{\Lambda}) \bigoplus_{j\in\mathbb{Z}_+} \dim V_{-j}\, t^j, \tag{6.3}$$

where $V = V_0 \oplus V_{-1} \oplus V_{-2} \oplus \cdots$ is the $\mathbb{Z}$-grading of V. Note that for the $\mathbb{Z}_2$-grading $V = V_{\bar 0} \oplus V_{\bar 1}$ we have $V_{\bar 0} = V_0 \oplus V_{-2} \oplus \cdots$ and $V_{\bar 1} = V_{-1} \oplus V_{-3} \oplus \cdots$. Therefore, the dimension of V is found by putting $t = 1$ in the expression for the t-dimension, whereas the superdimension of V is found by putting $t = -1$. So the t-dimension can also be seen as an extension of the notion of dimension and superdimension. For example, let us consider the adjoint representation. The weights of the adjoint representation are the roots. So,

$$\mathrm{ch}(V) = \sum_{i,j} e^{\epsilon_i-\epsilon_j} + \sum_{i,j} e^{\delta_i-\delta_j} + \sum_{i,j} e^{\epsilon_i-\delta_j} + \sum_{i,j} e^{-\epsilon_i+\delta_j}$$

Under the specialization (6.1), we have that

$$\dim_t(V) = m^2 + n^2 + m\, n\, \frac{1}{t} + m\, n\, t$$

The first part of this chapter is dealing with the computation of the t-dimension of a particular class of finite-dimensional irreducible representations of $\mathfrak{gl}(m|n)$, namely the covariant representations. In the second part, we consider the t-dimension of the mixed tensor representations.

6.2 t-dimension formula for covariant representations

There exist a number of expressions for $s_\lambda(x/y)$ (see Chapter 2). In order to compute the t-dimension, we will use two formulas. The first is the classical formula relating the supersymmetric Schur function $s_\lambda(x/y)$ to the determinant of complete supersymmetric polynomials (2.3). More precisely,

$$s_\lambda(x/y) = \det_{1\le i,j\le \ell(\lambda)} \Big(h_{\lambda_i - i + j}(x/y) \Big),$$

where $\ell \equiv \ell(\lambda)$ is the length of the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

The second is a our determinantal formula for supersymmetric Schur polynomials (4.39).

Since $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$, the specialization (6.1) corresponds to putting each $x_i = 1$ and $y_j = t$ in $s_\lambda(x/y)$. For the elementary and complete symmetric functions, such specializations are well-known:

$$h_r(x_1, \ldots, x_m)\Big|_{x_i=1} = \binom{m+r-1}{r} = \binom{m+r-1}{m-1}, \tag{6.4}$$

$$e_r(x_1, \ldots, x_m)\Big|_{x_i=1} = \binom{m}{r}. \tag{6.5}$$

Thus it follows from (2.3) and (2.9) that

Proposition 6.1 *The t-dimension of V_λ is given by the determinant*

$$\dim_t V_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(\sum_{k=0}^{\lambda_i - i + j} \binom{m + \lambda_i - i + j - k - 1}{\lambda_i - i + j - k} \binom{n}{k} t^k \right). \tag{6.6}$$

Although this formula is simple to derive, it should be observed that in general the matrix elements in the right hand side of (6.6) do not have a “closed form” expression [54]: they remain polynomials in t. Even for $t = 1$, the expression

$$\sum_{k=1}^{r} \binom{m+r-k-1}{r-k} \binom{n}{k}$$

cannot be simplified in general. Only for $t = -1$ we have

$$\sum_{k=0}^{r} \binom{m+r-k-1}{r-k} \binom{n}{k} (-1)^k = \binom{m-n-1+r}{r}. \tag{6.7}$$

This is related to the fact that

$$\sum_{k=0}^{r} \binom{m+r-k-1}{r-k} \binom{n}{k} t^k = \binom{m+r-1}{r} {}_2F_1 \left({-r, -n \atop -m-r+1} ; -t \right), \tag{6.8}$$

in terms of the ${}_2F_1$ hypergeometric function [54, 64], and the terminating ${}_2F_1$ series – with general parameters – is summable only with argument 1.

This implies that for $t = -1$, i.e. the superdimension formula $\mathrm{sdim}\, V_\lambda$, the expression (6.6) can be simplified, leading to the following corollary.

Corollary 6.2 *The superdimension of a covariant module V_λ is given by:*

$$\operatorname{sdim} V_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(\binom{m-n-1+\lambda_i-i+j}{\lambda_i-i+j} \right) \tag{6.9}$$

$$= \frac{\prod_{i<j}(\lambda_i - i - \lambda_j + j)}{\prod_i (\lambda_i - i + l(\lambda))!} \prod_i (m-n+1-i)_{\lambda_i}. \tag{6.10}$$

Herein, $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol [64], and the determinant in (6.9) can be written in closed form using [43, (3.11)]. So in general the superdimension has a closed form expression (6.10), whereas the dimension has not.

Observe that (6.10) yields: if $m \le n$ then $\operatorname{sdim} V_\lambda = 0$ when $\lambda_1 + m > n$ and $\operatorname{sdim} V_\lambda \neq 0$ when $\lambda_1 + m \le n$; if $m > n$ then $\operatorname{sdim} V_\lambda = 0$ when $\lambda'_1 + n > m$ and $\operatorname{sdim} V_\lambda \neq 0$ when $\lambda'_1 + n \le m$ (where λ' is the conjugate of λ).

Remark. Formula (6.10) has the structure of a dimension formula of $\mathfrak{gl}(m-n)$. Indeed, the dimension of the covariant module V_λ in $\mathfrak{gl}(m)$ is given by

$$\dim V_\lambda = \prod_{(i,j)\in\lambda} \frac{(m+j-i)}{h(i,j)} \tag{6.11}$$

where $h(i,j)$ is the hook length given by $h(i,j) = \lambda_i - i + \lambda'_j - j + 1$ for $(i,j) \in \lambda$. If $\ell(\lambda) \le m$, this formula can be written in an alternative expression [74]:

$$\dim V_\lambda = \prod_{1\le i<j\le m} \frac{(\lambda_i - i - \lambda_j + j)}{(j-i)}. \tag{6.12}$$

By rearranging factors, (6.12) can be rewritten as:

$$\dim V_\lambda = \prod_{1\le i<j\le \ell(\lambda)} \frac{(\lambda_i - i - \lambda_j + j)}{(j-i)} \prod_{i=1}^{\ell(\lambda)} \frac{\binom{\lambda_i+m-i}{m-i}}{\binom{\lambda_i+l(\lambda)-i}{\ell(\lambda)-i}}. \tag{6.13}$$

This formula is valid for every partition λ, even when $\ell(\lambda) > m$, in which case it gives 0. It is easy to see that (6.10) is the same expression as (6.13), replacing m by $m-n$. In other words $\operatorname{sdim} V_\lambda$ for $\mathfrak{gl}(m|n)$ coincides with $\dim V_\lambda$ for $\mathfrak{gl}(m-n)$. This equality between superdimenisons in $\mathfrak{gl}(m|n)$ and dimensions in $\mathfrak{gl}(m)$ has been considered before [10, 25, 38].

6.2.1 A formula for the t-dimension

In this section we shall consider the new determinantal formula for supersymmetric Schur polynomials (4.39), and use it to compute the t-dimension. This time, the expression for $\dim_t(V_\lambda)$ is quite different from (6.6): it reduces again to a determinant, but now the matrix elements are closed forms in t instead of hypergeometric series in t. We shall simplify this expression and discuss some applications.

The starting point is the following determinantal formula for the supersymmetric Schur function $s_\lambda(x/y)$. Let $x = x^{(m)} = (x_1, \ldots, x_m)$ and $y = y^{(n)} = (y_1, \ldots, y_n)$; let λ be a partition with $\lambda \in \mathcal{H}_{m,n}$ and let k be the (m, n)-index of λ. Let us recall formula (4.39):

$$s_\lambda(x/y) = \pm D^{-1} \det \begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\le i\le m\\1\le j\le n}} & \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m\\1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1\\1\le j\le n}} & 0 \end{pmatrix} \tag{6.14}$$

with

$$D = \frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{i,j}(x_i + y_j)}.$$

Observe that the sign in (4.41) is $(-1)^{mn-m+k-1}$; since its role is not essential here, we shall usually just write $\pm$.

In order to deduce a t-dimension formula from this, we will need some simple properties of symmetric polynomials and a careful analysis of the determinant in (6.14) using row and column operations.

We have already mentioned the complete and elementary symmetric functions. Another class that we need are the monomial symmetric functions $m_\lambda(x)$ (see (1.1)). The number of terms in $m_\lambda(x)$ is easy to count, so that we have the following counterpart of (6.4) and (6.5):

$$m_{(0^{r_0}1^{r_1}\ldots k^{r_k})}(x_1, \ldots, x_m)\Big|_{x_i=1} = \frac{m!}{r_0!r_1!\ldots r_k!} \text{ where } \sum_{i=0}^{k} r_i = m. \tag{6.15}$$

The following lemma gives some simple decomposition properties of symmetric functions:

Lemma 6.3 *Let $x = x' + x''$ be a decomposition of $x = (x_1, \ldots, x_m)$ in two disjoint subsets. Then*

$$h_r(x) = \sum_{k=0}^{r} h_k(x')h_{r-k}(x''), \quad \textit{and} \quad m_\lambda(x) = \sum_{\mu\cup\nu=\lambda} m_\mu(x')m_\nu(x'').$$

Proof. The proof for the $h_r(x)$ polynomials follows immediately from the generating function for these polynomials (1.26). For the m_λ, we use induction on $|x''|$. First, let $x'' = (x_m)$. By the definition of $m_\lambda(x)$, with $\lambda = (\lambda_1, \lambda_2, \ldots)$, if follows that

$$m_\lambda(x) = m_\lambda(x') + \sum_{\lambda_i\cup\mu=\lambda} m_\mu(x')m_{\lambda_i}(x_m) = \sum_{\mu\cup\nu=\lambda} m_\mu(x')m_\nu(x_m).$$

Now assume that the property holds for $|x''| \leq q$. Let $\bar{x}' = x' \backslash \{x_i\}$ and $\bar{x}'' = x'' \cup \{x_i\}$ for a certain $x_i \in x'$. Then, using the induction hypothesis:

$$\begin{aligned} m_\lambda(x) &= \sum_{\tau\cup\kappa=\lambda} m_\tau(x')m_\kappa(x'') = \sum_{\tau\cup\kappa=\lambda}\left(\sum_{\mu\cup\eta=\tau} m_\mu(\bar{x}')m_\eta(x_i)\right)m_\kappa(x'') \\ &= \sum_{\mu\cup\nu=\lambda} m_\mu(\bar{x}')\left(\sum_{\eta\cup\kappa=\nu} m_\eta(x_i)m_\kappa(x'')\right) = \sum_{\mu\cup\nu=\lambda} m_\mu(\bar{x}')m_\nu(\bar{x}''). \end{aligned}$$

□

Next, we shall use a number of times the same sequence of elementary row or column operations in matrices. So it is convenient to fix these in an algorithm:

Algorithm 1 Given a matrix with at least m rows, with R_i denoting row i. The algorithm consists of the following row operations:

Step 1:	$R_i \longrightarrow \dfrac{R_i - R_1}{x_i - x_1},$	for $1 < i \leq m$;
Step 2:	$R_i \longrightarrow \dfrac{R_i - R_2}{x_i - x_2},$	for $2 < i \leq m$;
$\vdots$		
Step $m-1$:	$R_m \longrightarrow \dfrac{R_m - R_{m-1}}{x_m - x_{m-1}}.$	

So the total number of row operations is $m(m-1)/2$.

Algorithm 2 Given a matrix with at least n columns, with C_j denoting column j. This algorithm consists of the following $n(n-1)/2$ column operations:

$$
\begin{array}{lll}
\text{Step 1:} & C_j \longrightarrow \dfrac{C_j - C_1}{y_j - y_1}, & \text{for } 1 < j \leq n; \\
\text{Step 2:} & C_k \longrightarrow \dfrac{C_j - C_2}{y_j - y_2}, & \text{for } 2 < j \leq n; \\
\quad\vdots & & \\
\text{Step } n-1: & C_n \longrightarrow \dfrac{C_n - C_{n-1}}{y_n - y_{n-1}}. &
\end{array}
$$

Lemma 6.4 *Let $(r_1, r_2, \ldots)$ be a sequence of (non-negative) integers, and consider matrices*

$$A = \Big(h_{r_j}(x_i) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}, \qquad B = \Big(h_{r_i}(y_j) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}.$$

Then Algorithm 1 transforms A into $A^\star$, and Algorithm 2 transforms B into $B^\star$, with

$$A^\star = \Big(h_{r_j-i+1}(x_1, \ldots, x_i) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}, \qquad B^\star = \Big(h_{r_i-j+1}(y_1, \ldots, y_j) \Big)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}.$$

Proof. It is sufficient to give the proof for A only (so we assume $p \geq m$). Denote by $A^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that the (i,j)-element of $A^{(s)}$ is given by $A^{(s)}_{i,j} = h_{r_j-s}(x_1, \ldots, x_s, x_i)$, by induction on s. Clearly, in the first step the elements $h_{r_j}(x_i)$ are replaced by

$$\frac{x_i^{r_j} - x_1^{r_j}}{x_i - x_1} = x_i^{r_j-1} + x_i^{r_j-2}x_1 + \ldots + x_i x_1^{r_j-2} + x_1^{r_j-1} = h_{r_j-1}(x_1, x_i).$$

Now we can assume that after step s we have $A^{(s)}_{i,j} = h_{r_j-s}(x_1, \ldots, x_s, x_i)$ for all $i > s$. Step $s+1$ consist of the operations $R_i \longrightarrow (R_i - R_{s+1})/(x_i - x_{s+1})$ for all $i > s+1$. Thus the element $A^{(s+1)}_{i,j}$ becomes, using Lemma 6.3 a number of times:

$$
\begin{aligned}
&\frac{h_{r_j-s}(x_1, \ldots, x_s, x_i) - h_{r_j-s}(x_1, \ldots, x_s, x_{s+1})}{x_i - x_{s+1}} \\
&\quad = \sum_{l=0}^{r_j-s-1} h_l(x_1, \ldots, x_s) \frac{x_i^{r_j-s-l} - x_{s+1}^{r_j-s-l}}{x_i - x_{s+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{r_j-s-1} h_l(x_1,\ldots,x_s)(x_i^{r_j-s-l-1} + x_i^{r_j-s-l-2}x_{s+1} + \ldots + x_{s+1}^{r_j-s-l-1}) \\
&= \sum_{l=0}^{r_j-s-1} h_l(x_1,\ldots,x_s)h_{r_j-s-l-1}(x_{s+1},x_i) = h_{r_j-s-1}(x_1,\ldots,x_{s+1},x_i).
\end{aligned}
$$

Since the algorithm applies in total $i-1$ row transformations on row i, it follows that $A^\star_{i,j} = h_{r_j-i+1}(x_1,\ldots,x_i)$. □

Lemma 6.5 *Algorithm 1 transforms*

$$R = \left(\frac{1}{x_i+y_j}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \textit{ into } R^\star = \left(\frac{(-1)^{i-1}}{\prod_{l=1}^{i}(x_l+y_j)}\right)_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}.$$

Proof. Denote by $R^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that the (i,j)-element of $R^{(s)}$ is given by $R^{(s)}_{i,j} = \dfrac{(-1)^s}{\prod_{l=1}^{s}(x_l+y_j)(x_i+y_j)}$. In the first step, the operations are $R_i \longrightarrow \frac{R_i-R_1}{x_i-x_1}$ for $i>1$, so

$$R^{(1)}_{i,j} = \left(\frac{1}{x_i+y_j} - \frac{1}{x_1+y_j}\right)\frac{1}{x_i-x_1} = \frac{-1}{(x_1+y_j)(x_i+y_j)}.$$

Next we use induction on s. One finds:

$$
\begin{aligned}
R^{(s+1)}_{i,j} &= \left(\frac{(-1)^s}{\prod_{l=1}^{s}(x_l+y_j)(x_i+y_j)} - \frac{(-1)^s}{\prod_{l=1}^{s}(x_l+y_j)(x_{s+1}+y_j)}\right)\frac{1}{x_i-x_{s+1}} \\
&= \frac{(-1)^s}{\prod_{l=1}^{s}(x_l+y_j)} \cdot \frac{(-1)}{(x_{s+1}+y_j)(x_i+y_j)}.
\end{aligned}
$$

Since the algorithm applies in total $i-1$ row transformations on row i, the result follows. □

The following is a technical lemma on partitions, using the reverse lexicographic ordering (see Section 1.1.1) for partitions of the same integer. So when we write $\lambda \leq \mu$, this means that λ and μ are partitions of the same integer (i.e. $|\lambda| = |\mu|$) with either $\lambda = \mu$ or else the first non-vanishing difference $\lambda_i - \mu_i$ negative.

Lemma 6.6 *Assume that α,β,ν,μ are partitions with $\ell(\alpha) = s+1$, $\ell(\beta) = s+2$ and $\ell(\nu) = 2$. Then, for $i,s,t \in \mathbb{Z}_{\geq 0}$:*

$$\alpha \leq (i,1^s), \quad \mu\cup(t) = \alpha, \quad \nu \leq (t,1) \qquad \Leftrightarrow \qquad \beta = \mu\cup\nu \leq (i,1^{s+1}).$$

Proof. Assume that $\alpha \leq (i, 1^s)$, $\mu \cup (t) = \alpha$ and $\nu \leq (t, 1)$, then $|\beta| = |\mu| + |\nu| = (i + s - t) + (t + 1) = |(i, 1^{s+1})|$. Furthermore $\beta_1 = \max(\mu_1, \nu_1) \leq \max(\mu_1, t) = \alpha_1 \leq i$, so $\beta \leq (i, 1^{s+1})$.
Conversely, assume that $\beta = \mu \cup \nu \leq (i, 1^{s+1})$, then ν is of the form $\nu = (\beta_k, \beta_l)$ ($\beta_l > 0$), so $\nu \leq (\beta_k + \beta_l - 1, 1)$. Put $t = \beta_k + \beta_l - 1$ and $\alpha = \mu \cup (t)$. Then $|\alpha| = |\mu| + |(t)| = (i + s + 1 - \beta_k - \beta_l) + (\beta_k + \beta_l - 1) = i + s$. Since $\ell(\mu) = s$ we have that $|\mu| \geq s$, and $|(t)| \leq i$. So $\alpha_1 = \max(\mu_1, t) \leq \max(\mu_1, i) \leq i$, thus $\alpha \leq (i, 1^s)$. □

This technical lemma is needed in the following:

Lemma 6.7 *Let* $Y_j = \frac{1}{1+y_j}$ *and consider the matrix*

$$R = \left(\frac{(-1)^{i+1}}{(1+y_j)^i} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} = \left((-1)^{i+1} Y_j^i \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}.$$

Algorithm 2 transforms R *into*

$$R^\star = \left((-1)^{i+j} \sum_{\alpha \leq (i, 1^{j-1})} m_\alpha(Y_1, \ldots, Y_j) \right)_{\substack{1 \leq i \leq p \\ 1 \leq j \leq n}}.$$

Proof. Observe that $1/(y_i - y_j) = Y_i Y_j / (Y_j - Y_i)$. Denote, as usual, by $R^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that

$$R_{i,j}^{(s)} = (-1)^{i+s+1} \sum_{\alpha \leq (i, 1^s)} m_\alpha(Y_1, \ldots, Y_s, Y_j), \qquad \text{for all } j > s.$$

Step 1 consist of the column operations $C_j \longrightarrow \dfrac{C_j - C_1}{Y_1 - Y_j} Y_1 Y_j$, so $R_{i,j}^{(1)}$ is given by

$$\begin{aligned} &\left((-1)^{i+1} Y_j^i - (-1)^{i+1} Y_1^i \right) \frac{Y_1 Y_j}{Y_1 - Y_j} \\ &= (-1)^{i+2} (Y_j^i Y_1 + Y_j^{i-1} Y_1^2 + \ldots + Y_j^2 Y_1^{i-1} + Y_j Y_1^i) \\ &= (-1)^{i+2} \sum_{\alpha \leq (i,1)} m_\alpha(Y_1, Y_j). \end{aligned}$$

Next we use induction on s. This yields, using Lemma 6.3:

$R_{i,j}^{(s+1)}$

$$
\begin{aligned}
&= (-1)^{i+s+1} \sum_{\alpha \le (i,1^s)} \Big(m_\alpha(Y_1,\ldots,Y_s,Y_j) - m_\alpha(Y_1,\ldots,Y_s,Y_{s+1}) \Big) \frac{Y_j Y_{s+1}}{Y_{s+1}-Y_j} \\
&= (-1)^{i+s+1} \sum_{\alpha \le (i,1^s)} \sum_{\mu \cup (t) = \alpha} m_\mu(Y_1,\ldots,Y_s)(Y_j^t - Y_{s+1}^t) \frac{Y_j Y_{s+1}}{Y_{s+1}-Y_j} \\
&= (-1)^{i+s+2} \sum_{\alpha \le (i,1^s)} \sum_{\mu \cup (t) = \alpha} m_\mu(Y_1,\ldots,Y_s) \sum_{\nu \le (t,1)} m_\nu(Y_{s+1},Y_j).
\end{aligned}
$$

Next, we use Lemma 6.6 and finally Lemma 6.3 again:

$$
\begin{aligned}
R_{i,j}^{(s+1)} &= (-1)^{i+s+2} \sum_{\beta \le (i,1^{s+1})} \Big(\sum_{\mu \cup \nu = \beta} m_\mu(Y_1,\ldots,Y_s) m_\nu(Y_{s+1},Y_j) \Big) \\
&= (-1)^{i+s+2} \sum_{\beta \le (i,1^{s+1})} m_\beta(Y_1,\ldots,Y_{s+1},Y_j).
\end{aligned}
$$

Since the algorithm applies in total $j-1$ column transformations on column j, the result follows. □

The next lemma is about the specialization of such matrix elements. By $y=1$ we mean the substitution $(y_1 = 1, \ldots, y_j = 1)$.

Lemma 6.8

$$
R_{i,j} = \sum_{\alpha \le (i,1^{j-1})} m_\alpha(y_1,\ldots,y_j) \Big|_{y=1} = \binom{i+j-2}{j-1}.
$$

Proof. It is easy to verify (e.g. using (6.15)) that $R_{1,j} = 1$ and $R_{i,1} = 1$. Now,

$$
\begin{aligned}
R_{i,j} &= \sum_{\alpha \le (i,1^{s+1})} m_\alpha(y_1,\ldots,y_j) \Big|_{y=1} \\
&= \Big(\Big(\sum_{\mu \le (i,1^{j-2})} m_\mu(y_1,\ldots,y_{j-1}) \Big) y_j + \Big(\sum_{\nu \le (i-1,1^{j-1})} m_\nu(y_1,\ldots,y_j) \Big) y_j \Big) \Big|_{y=1} \\
&= R_{i,j-1} + R_{i-1,j}.
\end{aligned}
$$

Now the result follows. □

Now we have all ingredients to determine the specialization of (4.41).

Theorem 6.9 *The t-dimension of V_λ is given by* $\dim_t(V_\lambda) = \pm(1+t)^{mn}R(\lambda)$ *with*

$$R(\lambda) = \det \begin{pmatrix} \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}}\binom{i+j-2}{j-1}\right)_{\substack{1\le i\le m\\1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1}\right)_{\substack{1\le i\le m\\1\le j\le k-1}} \\ \left(t^{\lambda'_i+n-m-i-j+1}\binom{\lambda'_i+n-m-i}{j-1}\right)_{\substack{1\le i\le n-m+k-1\\1\le j\le n}} & 0 \end{pmatrix}. \tag{6.16}$$

Proof. Consider the determinant in (4.41) and apply Algorithm 1 on the corresponding matrix. From Lemmas 6.4 and 6.5 it follows that the first m rows of this matrix become

$$\begin{pmatrix} \left(\frac{(-1)^{i-1}}{\prod_{l=1}^{i}(x_l+y_j)}\right)_{\substack{1\le i\le m\\1\le j\le n}} & \left(h_{\lambda_j+m-n-i-j+1}(x_1,\ldots,x_i)\right)_{\substack{1\le i\le m\\1\le j\le k-1}} \end{pmatrix},$$

while the determinant has been multiplied by a factor $\prod_{i>j}(x_i-x_j)$. Now we can make the substitution $x_i = 1$; then (4.41) becomes

$$\pm\frac{\prod_j (1+y_j)^m}{\prod_{i<j}(y_i-y_j)} \det \begin{pmatrix} \left(\frac{(-1)^{i-1}}{(1+y_j)^i}\right)_{\substack{1\le i\le m\\1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1}\right)_{\substack{1\le i\le m\\1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1\\1\le j\le n}} & 0 \end{pmatrix}.$$

Next apply Algorithm 2 on the first n columns of this matrix. Using Lemmas 6.4 and 6.7, this becomes

$$\prod_j (1+y_j)^m \det(C).$$

with

$$C = \begin{pmatrix} \left((-1)^{i+j}\sum_{\alpha\le(i,1^{j-1})} m_\alpha(Y_1,\ldots,Y_j)\right)_{\substack{1\le i\le m\\1\le j\le n}} & \left(\binom{\lambda_j+m-n-j}{i-1}\right)_{\substack{1\le i\le m\\1\le j\le k-1}} \\ \left(h_{\lambda'_i+n-m-i-j+1}(y_1,\ldots,y_j)\right)_{\substack{1\le i\le n-m+k-1\\1\le j\le n}} & 0 \end{pmatrix}.$$

Finally, substituting $y_j = t$, using Lemma 6.8, and the fact that we are dealing with homogeneous symmetric polynomials, leads to the result. □

Compared to (6.6), (6.16) has the advantage that each matrix element is a simple binomial coefficient multiplied by a power of t or $(1+t)$, and no longer a finite series of type ${}_2F_1(-t)$. So in general (6.16) is easier to compute. Furthermore, its simple form is more appropriate to deduce certain properties of the t-dimension for particular V_λ, as we shall demonstrate in the following section.

6.2.2 Further simplifications, examples and applications

Let λ be a partition, $\lambda \in \mathcal{H}_{m,n}$, and λ' its conjugate. Recall the definition of the (m,n)-index k of λ in (4.14) and also the definition of the related integer r in (4.27):

$$k = \min\{i|\lambda_i + m + 1 - i \leq n\}, \qquad (1 \leq k \leq m+1);$$
$$r = n - m + k - \lambda_k - 1.$$

Since λ is in the (m,n)-hook, λ' is in the (n,m)-hook, and we can define its (n,m)-index k' and the corresponding number r':

$$k' = \min\{i|\lambda'_i + n + 1 - i \leq m\}, \qquad (1 \leq k' \leq n+1);$$
$$r' = m - n + k' - \lambda'_{k'} - 1.$$

Applying the determinant formula (4.41) for $s_\lambda(x/y)$ and for $s_{\lambda'}(y/x)$ yields the same, with determinants of transposed matrices. Comparing the orders of the matrices implies that $n + k - 1 = m + k' - 1$, so we have

$$n + k = m + k', \qquad r = k' - \lambda_k - 1, \qquad r' = k - \lambda'_{k'} - 1. \tag{6.17}$$

Furthermore, from Lemma 4.6 we know that $\lambda'_{\lambda_k + l} = k - 1$ for all $1 \leq l \leq r$. So the binomials on the last r rows of the matrix in (6.16) take the values

$$\binom{\lambda'_i + n - m - i}{j-1} = \binom{r-l}{j-1} \qquad \text{for } 1 \leq l \leq r, \text{ and } i = \lambda_k + l.$$

By the triangularity of the matrix with such binomial coefficients as entries, the determinant in (6.16) can thus be reduced according to the last r rows.

Completely analogous, the remaining determinant can be reduced according to the last r' columns. What remains is the determinant of a matrix of order $n+k-1-r-r'$, and we have

Corollary 6.10 *The t-dimension of V_λ is given by $\dim_t(V_\lambda) = \pm(1+t)^{mn} R'(\lambda)$ with*

$R'(\lambda) =$

$$\det \begin{pmatrix} \left(\frac{(-1)^{i+j+r+r'}}{(1+t)^{i+j+r+r'-1}} \binom{i+j+r+r'-2}{j-1} \right)_{\substack{1\leq i\leq m-r' \\ 1\leq j\leq n-r}} & \left(\binom{\lambda_j+m-n-j}{i+r'-1} \right)_{\substack{1\leq i\leq m-r' \\ 1\leq j\leq \lambda'_{k'}}} \\ \left(t^{\lambda'_i+n-m-i-j-r+1} \binom{\lambda'_i+n-m-i}{j+r-1} \right)_{\substack{1\leq i\leq \lambda_k \\ 1\leq j\leq n-r}} & 0 \end{pmatrix}. \tag{6.18}$$

An interesting application follows from this formula for the special case of $\lambda = \Big((n-a)^{(m-a)}\Big)$, where $a = 0, 1, \ldots, \min(m,n)$. For such a rectangular λ, we have

$$k = m - a + 1,\ k' = n - a + 1,\ r = n - a,\ r' = m - a,\ \lambda_k = 0,\ \lambda'_{k'} = 0,$$

and so the determinant in (6.18) reduces:

$$\begin{aligned}
&\dim_t(V_\lambda)\\
&\quad = \pm\,(1+t)^{mn} \det_{1\le i,j\le a}\left(\frac{(-1)^{i+j+r+r'}}{(1+t)^{i+j+r+r'-1}} \binom{i+j+r+r'-2}{j-1} \right)\\
&\quad = \pm\,(1+t)^{mn-a(r+r'-1)} \det_{1\le i,j\le a}\left(\frac{(-1)^{i+j}}{(1+t)^{i+j}} \binom{i+j+m+n-2a-2}{j-1} \right).
\end{aligned}$$

The resulting determinant can be further simplified: in the corresponding matrix, multiply row i by $(-1)^{i+1}(1+t)^{i+1}$ for all $1 \le i \le a$, and then multiply column j by $(-1)^{1+j}(1+t)^{j-1}$ for all $1 \le j \le a$. This yields:

$$\dim_t(V_\lambda) = (1+t)^{(m-a)(n-a)} \det_{1\le i,j\le a}\left(\binom{i+j+m+n-2a-2}{j-1} \right).$$

Now the matrix elements have no longer a power of $(1+t)$, but only a binomial coefficient. The remaining determinant can easily be computed. Taking out common factors in rows and columns, it becomes

$$\prod_{i=1}^{a} \frac{(i+m+n-2a-1)!}{(i+n-a-1)!(i+m-a-1)!} \det_{1\le i,j\le a}\Big((m+n-2a+i)_{j-1}\Big).$$

The last determinant is of the form

$$\det_{1\le i,j\le a}\Big((x_i)_{j-1}\Big) = \det_{1\le i,j\le a}\Big(x_i^{j-1}\Big) = \prod_{1\le i<j\le a}(x_j - x_i),$$

see [43, (2.2)].

So we finally obtain, for $\lambda = \Big((n-a)^{(m-a)}\Big)$, that

$$\begin{aligned}
\dim_t(V_\lambda) &= (1+t)^{(m-a)(n-a)} \prod_{i=0}^{a-1} \frac{(m+n-2a+i)!\ i!}{(n-a+i)!(m-a+i)!}\\
&= (1+t)^{(m-a)(n-a)} \prod_{i=0}^{a-1} \frac{\binom{m+n-2a+i}{n-a+i}}{\binom{m-a+i}{i}}
\end{aligned}$$

Using (6.13) we find that

$$\dim_t(V_\lambda) = (1+t)^{(m-a)(n-a)} \dim(V_\mu) \quad \text{with } \mu = (m-a)^a \text{ in } \mathfrak{gl}(n).$$

Comparing this with (6.6), we obtain a closed form expression for determinants of the type (6.6) where $\lambda = \left((n-a)^{(m-a)}\right)$. Replacing m by $m+1$, $m-a$ by s, and reversing the order of the rows of the corresponding matrix, this yields, using the ${}_2F_1$ notation:

$$\begin{aligned} &\det_{0\le i,j\le s}\left(\binom{n+i+j}{m}\, {}_2F_1\left({m-n-i-j, -n \atop -n-i-j}; -t\right)\right) \\ &= \quad (-1)^{s(s+1)/2}(1+t)^{(s+1)(s+n-m)} \prod_{i=1}^{m-s} \frac{\binom{2s+n-m+i}{s+1}}{\binom{s+i}{s+1}} \qquad (s \le m). \end{aligned}$$

The change of order of the rows implies we are dealing with a Hankel determinant, and for such determinants the row and column indices are usually starting from 0. This determinant identity can be written in a number of alternative ways. E.g. applying a transformation on the ${}_2F_1$, and denoting $t/(t+1)$ by z, one can write

$$\det_{0\le i,j\le s}(A_{i+j}) = (-1)^{s(s+1)/2}(1-z)^{(s+1)(m-s)} \prod_{i=1}^{m-s} \frac{\binom{2s+n-m+i}{s+1}}{\binom{s+i}{s+1}}, \tag{6.19}$$

where

$$A_k = \binom{n+k}{m}\, {}_2F_1\left({-m, -n \atop -n-k}; z\right). \tag{6.20}$$

Since this is a polynomial identity in n, the condition that n must be an integer can be dropped. Replacing n by u and z by $-v$, one can write this in the following form:

Corollary 6.11 *Let m and s be positive integers with $s \le m$, u and v arbitrary variables, and*

$$A_k = \sum_{l=0}^{m} \binom{u+k-l}{m-l}\binom{u}{l} v^l.$$

Then the Hankel determinant is given by

$$\det_{0\le i,j\le s}(A_{i+j}) = (-1)^{s(s+1)/2}(1+v)^{(s+1)(m-s)} \prod_{i=1}^{m-s} \frac{\binom{2s+u-m+i}{s+1}}{\binom{s+i}{s+1}}.$$

It seems to be difficult to find an independent proof of this corollary, even with the methods of [43, §2.6]. Here, it is a simple consequence of the two different t-dimension formulas for a particular V_λ.

6.3 t-dimension formula for mixed tensor representations

6.3.1 A formula for the t-dimension

In this section we shall consider the new determinantal formula for the characters of mixed tensor representations (5.28), and use it to compute the t-dimension. We will also compute the t-dimension of these representations by means of (2.34). Both expressions for $\dim_t(V_{\bar{\nu};\mu})$ are quite different: the first formula reduces again to a determinant, but now the matrix elements are closed forms in t instead of hypergeometric series in t. We shall give some examples and discuss a special case.

Since the specializations (6.1) correspond to putting each $x_i = 1$ and $y_j = t$, we already described this specialization for the elementary and complete symmetric functions (6.4) and (6.5) and the monomial symmetric functions (6.15). For the symmetric Schur functions this specialization can be found in [46, §3, Ex. 4] and is equal to the dimension formula (6.11), more exactly:

$$s_\lambda(x_1, \ldots, x_m)\Big|_{x_i=1} = \prod_{(i,j)\in\lambda} \frac{m + c(i,j)}{h(i,j)}, \tag{6.21}$$

where the content $c(i,j)$ is given by $c(i,j) = j - i$ and the hook length $h(i,j)$ is given by $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ for $(i,j) \in \lambda$.
For a special class of symmetric Schur functions we can rewrite (6.21) as a binomial coefficient. Suppose, $\lambda = (a, 1^{b-1})$. As the numerator and denominator of (6.21) are given by

$$\begin{aligned} \prod_{(i,j)\in\lambda} (m + c(i,j)) &= (m+1-1)\ldots(m+a-1).(m+1-2)\ldots(m+1-b) \\ &= \frac{(m+a-1)!}{(m-b)!} \end{aligned} \tag{6.22}$$

and

$$\prod_{(i,j)\in\lambda} h(i,j) = (a+b-1)(a-1)!(b-1)!, \tag{6.23}$$

the number of monomials in $s_\lambda(x)$ is equal to:

$$s_{(a,1^{b-1})}(x_1, \ldots, x_m)\Big|_{x_i=1} = \frac{(m+a-1)!}{(m-b)!(a+b-1)(a-1)!(b-1)!}. \tag{6.24}$$

In particular for $\lambda = (a, 1^{m-1})$, we have that

$$s_{(a,1^{m-1})}(x_1, \ldots, x_m)\Big|_{x_i=1} = \frac{(m+a-2)!}{(a-1)!(m-1)!} = \binom{m+a-2}{m-1}. \tag{6.25}$$

In Section 6.2.1 we introduced two algorithms. We will use the same sequence of elementary row and column operations here.

Lemma 6.12 *Let $(r_1, r_2, \ldots)$ be a sequence of (non-negative) integers. We denote by $\overline{x_i} = \frac{1}{x_i}$ for all i and $\overline{y_j} = \frac{1}{y_j}$ for all j. Consider the matrices*

$$G = \Big(s_{r_j}(\overline{x_i}) \Big)_{\substack{1\le i\le p\\ 1\le j\le q}}, \qquad H = \Big(s_{r_i}(\overline{y_j}) \Big)_{\substack{1\le i\le p\\ 1\le j\le q}}.$$

Then Algorithm 1 transforms G into $G^\star$, and Algorithm 2 transforms H into $H^\star$, with

$$G^\star = \Big((-1)^{i+1} s_{(r_j,1^{i-1})}(\overline{x_1}, \ldots, \overline{x_i}) \Big),\ H^\star = \Big((-1)^{j+1} s_{(r_i,1^{j-1})}(\overline{y_1}, \ldots, \overline{y_j}) \Big),$$

in both matrices $1 \le i \le p$ and $1 \le j \le q$.

Proof. It is sufficient to give the proof for G only (so we assume $p \ge m$). Denote by $G^{(s)}$ the matrix obtained after step s of the algorithm. We shall prove that the (i,j)-element of $G^{(s)}$ is given by $G^{(s)}_{i,j} = (-1)^s s_{(r_j,1^s)}(\overline{x_1}, \ldots, \overline{x_s}, \overline{x_i})$, by induction on s. In the first step, the elements $s_{(r_j)}(\overline{x_i})$ are replaced by

$$\begin{aligned} \frac{s_{(r_j)}(\overline{x_i}) - s_{(r_j)}(\overline{x_1})}{x_i - x_1} &= \frac{-\overline{x_1}\,\overline{x_i}(s_{(r_j)}(\overline{x_i}) - s_{(r_j)}(\overline{x_1}))}{\overline{x_i} - \overline{x_1}} \\ &= \frac{-s_{(1)}(\overline{x_1}) s_{(1)}(\overline{x_i})(s_{(r_j)}(\overline{x_i}) - s_{(r_j)}(\overline{x_1}))}{\overline{x_i} - \overline{x_1}} \end{aligned}$$

Using (1.33) and the fact that $s_\lambda(x^{(m)}) = 0$ if $\ell(\lambda) > m$, we have that $s_{(1)}(\overline{x_i}) s_{(r_j)}(\overline{x_i}) = s_{(r_j,1)}(\overline{x_i}) + s_{(r_j+1)}(\overline{x_i}) = s_{(r_j+1)}(\overline{x_i})$. Thus,

$$G^{(1)}_{i,j} = -\left(\sum_{x'+x''} \frac{s_{(r_j+1)}(\overline{x'}) s_{(1)}(\overline{x''})}{E(\overline{x'}, \overline{x''})} \right) \overset{(4.48)}{=} -s_{(r_j,1)}(\overline{x_1}, \overline{x_i}),$$

with $|x'| = |x''| = 1$. Now we can assume that after step s we have that $G^{(s)}_{i,j} = (-1)^s s_{(r_j,1^s)}(\overline{x_1} \ldots, \overline{x_s}, \overline{x_i})$ for all $i > s$. Step $s+1$ consist of the operations $R_i \longrightarrow (R_i - R_{s+1})/(x_i - x_{s+1})$ for all $i > s+1$. Thus the element $G^{(s+1)}_{i,j}$ becomes:

$$\frac{(-1)^s s_{(r_j,1^s)}(\overline{x_1}, \ldots, \overline{x_s}, \overline{x_i}) - (-1)^s s_{(r_j,1^s)}(\overline{x_1}, \ldots, \overline{x_s}, \overline{x_{s+1}})}{x_i - x_{s+1}} =$$

$$\overset{(1.33)}{=} \quad (-1)^{s+1}\sum_{\mu,\nu} c_{\mu\nu}^{(r_j,1^s)} s_\mu(\overline{x_1},\ldots,\overline{x_s})(s_\nu(\overline{x_i}) - s_\nu(\overline{x_{s+1}}))\frac{\overline{x_i}\,\overline{x_{s+1}}}{\overline{x_i}-\overline{x_{s+1}}}$$

$$= \quad (-1)^{s+1}\sum_{\mu,\nu} c_{\mu\nu}^{(r_j,1^s)} s_\mu(\overline{x_1},\ldots,\overline{x_s})(s_\nu(\overline{x_i}) - s_\nu(\overline{x_{s+1}}))\frac{s_{(1)}(\overline{x_i})s_{(1)}(\overline{x_{s+1}})}{\overline{x_i}-\overline{x_{s+1}}}$$

$$\overset{(1.33)}{=} \quad (-1)^{s+1}\sum_{\mu,\nu} c_{\mu\nu}^{(r_j,1^s)} s_\mu(\overline{x_1},\ldots,\overline{x_s}) s_{(\nu,1)}(\overline{x_i}\,\overline{x_{s+1}}))$$

This implies that $\nu = (a)$ and $\mu = (r_j - a + 1, 1^{s-1})$. The sum can be rewritten and is equal to:

$$(-1)^{s+1}\sum_{1\le a\le r_j} c^{(r_j,1^s)}_{(r_j-a+1,1^{s-1}),(a,1)} s_{(r_j-a+1,1^{s-1})}(\overline{x_1},\ldots,\overline{x_s}) s_{(a,1)}(\overline{x_i}\,\overline{x_{s+1}}))$$
$$= (-1)^{s+1} s_{(r_j,1^{s+1})}(\overline{x_1},\ldots,\overline{x_{s+1}},\overline{x_i})$$

In the last step, we used (1.33) and the fact that $s_\lambda(x^{(m)}) = 0$ if $\ell(\lambda) > m$. □

Theorem 6.13 *The t-dimension of $V_{\bar{\nu};\mu}$ is given by*

$$\dim_t(V_{\bar{\nu};\mu}) = \pm(1+t)^{mn} t^{rn} \det\begin{pmatrix} 0 & R_{12} & 0 \\ R_{21} & R_{22} & R_{23} \\ 0 & R_{32} & 0 \end{pmatrix} \tag{6.26}$$

with

$$R_{12} = \left(t^{\mu_i'+n-m-i-r-j+1}\binom{\mu_i'+n-m-i-r}{j-1}\right)_{\substack{1\le i\le l-1\\ 1\le j\le n}},$$

$$R_{21} = \left(\binom{\mu_j+m-n+r-j}{i-1}\right)_{\substack{1\le i\le m\\ 1\le j\le k-1}},$$

$$R_{22} = \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}}\binom{i+j-2}{j-1}\right)_{\substack{1\le i\le m\\ 1\le j\le n}},$$

$$R_{23} = \left((-1)^{i-1}\binom{\nu_{m-j+1}-m-r+i+j-2}{i-1}\right)_{\substack{1\le i\le m\\ 1\le j\le m-k-a+1}},$$

$$R_{32} = \left(t^{\nu'_{n-i+1}-n+r+i+j-1}\binom{\nu'_{n-i+1}-n+r+i+j-2}{j-1}\right)_{\substack{1\le i\le n-l-a+1\\ 1\le j\le n}}$$

and with k the (m,n)-index defined in (5.3), $l = \mu_k + 1$ and $r = n - m + k - l$.

Proof. Consider the character formula $\operatorname{ch} V_{\bar{\nu};\mu} = \pm D^{-1}\det(C)$ from (5.28) where $\det(C)$ is defined in (5.26). First we multiply the rows $l-1+i$ for $i=1,\ldots,m$ with x_i^r and divide the columns $k-1+j$ for $j=1,\ldots,n$, with y_j^r. The determinant of C becomes

$$\det(C) = \left(\frac{\prod_{j=1}^n y_j}{\prod_{i=1}^m x_i}\right)^r \det\begin{pmatrix} 0 & Y^{\mu'} & 0 \\ X^\mu & R & X^\nu \\ 0 & Y^{\nu'} & 0 \end{pmatrix}$$

with

$$\begin{aligned}
R &= \left(\tfrac{1}{x_i+y_j}\right)_{1\le i\le m,\,1\le j\le n} \\
X^\mu &= \left(x_i^{\mu_j+m-n+r-j}\right)_{\substack{1\le i\le m,\\ 1\le j\le k-1}} &&= \left(h_{\mu_j+m-n+r-j}(x_i)\right)_{\substack{1\le i\le m,\\ 1\le j\le k-1}}, \\
X^\nu &= \left(x_i^{-\nu_{m-j+1}m+r-j}\right)_{\substack{1\le i\le m,\\ k+a\le j\le m}} &&= \left(s_{(\nu_{m-j+1}-m-r+j)}(\overline{x_i})\right)_{\substack{1\le i\le m,\\ k+a\le j\le m}}, \\
Y^{\mu'} &= \left(y_j^{\mu'_i+n-m-r-i}\right)_{\substack{1\le i\le l-1,\\ 1\le j\le n}} &&= \left(h_{\mu'_i+n-m-r-i}(y_j)\right)_{\substack{1\le i\le l-1,\\ 1\le j\le n}}, \\
Y^{\nu'} &= \left(y_j^{-\nu'_{n-i+1}+n-r-i}\right)_{\substack{l+a\le i\le n,\\ 1\le j\le n}} &&= \left(s_{(\nu'_{n-i+1}-n+r+i)}(\overline{y_j})\right)_{\substack{l+a\le i\le n,\\ 1\le j\le n}}.
\end{aligned}$$

Apply Algorithm 1 on rows i for $i=l,\ldots,m+l-1$; denote the block matrix of those rows by $R^{(l\ldots m+l-1)}$. From Lemma 6.4, Lemma 6.5 and Lemma 6.12 it follows that

$$R^{(l\ldots m+l-1)} = \left(\widetilde{X^\mu} \mid \widetilde{R} \mid \widetilde{X^\nu}\right)$$

where

$$\begin{aligned}
\widetilde{R} &= \left(\frac{(-1)^{i-1}}{\prod_{p=1}^i (x_p+y_j)}\right)_{1\le i\le m,\,1\le j\le n}, \\
\widetilde{X^\mu} &= \left(h_{\mu_j+m-n+r-j-i+1}(x_1,\ldots,x_i)\right)_{1\le i\le m,\,1\le j\le k-1}, \\
\widetilde{X^\nu} &= \left((-1)^{i-1} s_{(\nu_{m-j+1}-m-r+j,1^{i-1})}(\overline{x_1},\ldots,\overline{x_i})\right)_{1\le i\le m,\,k+a\le j\le m},
\end{aligned}$$

while the determinant has been divided by a factor $\prod_{i>j}(x_i-x_j)$. Now we can make the substitution $x_i = 1$; then, using (6.4) and (6.25), $\pm D^{-1}\det(C)$ becomes

$$\pm\frac{\prod_{j=1}^{n} y_j^r \prod_{j=1}^{n}(1+y_j)^m}{\prod_{i<j}(y_i-y_j)} \det\begin{pmatrix} 0 & Y^{\mu'} & 0 \\ \overline{X^{\mu}} & R_1 & \overline{X^{\nu}} \\ 0 & Y^{\nu'} & 0 \end{pmatrix} = \pm\frac{\prod_{j=1}^{n} y_j^r \prod_{j=1}^{n}(1+y_j)^m}{\prod_{i<j}(y_i-y_j)} \det(\widetilde{C})$$

where

$$\begin{aligned} R_1 &= \left(\tfrac{(-1)^{i-1}}{(1+y_j)^i}\right)_{1\le i\le m,\, 1\le j\le n}, \\ \overline{X^{\mu}} &= \left(\tbinom{\mu_j+m-n+r-j}{i-1}\right)_{1\le i\le m,\, 1\le j\le k-1}, \\ \overline{X^{\nu}} &= \left((-1)^{i-1}\tbinom{\nu_{m-j+1}-m-r+i+j-2}{i-1}\right)_{1\le i\le m,\, k+a\le j\le m}. \end{aligned}$$

Next we apply Algorithm 2 on columns j, for $j = k, \dots, k+n-1$ of matrix $\widetilde{C}$. Using Lemmas 6.4, 6.7 and 6.12, this becomes

$$\det(\widetilde{C}) = \prod_{i>j}(y_i-y_j)\det\begin{pmatrix} 0 & \widetilde{Y^{\mu'}} & 0 \\ \overline{X^{\mu}} & \widetilde{R_1} & \overline{X^{\nu}} \\ 0 & \widetilde{Y^{\nu'}} & 0 \end{pmatrix}$$

where

$$\begin{aligned} \widetilde{R_1} &= \left((-1)^{i+j}\sum_{\alpha\le(i,1^{j-1})} m_\alpha(Y_1,\dots,Y_j)\right)_{1\le i\le m,\, 1\le j\le n}, \\ \widetilde{Y^{\mu'}} &= \left(h_{\mu'_i+n-m-r-i-j+1}(y_1,\dots,y_j)\right)_{1\le i\le l-1,\, 1\le j\le n}, \\ \widetilde{Y^{\nu'}} &= \left((-1)^{j-1}s_{(\nu'_{n-i+1}-n+r+i,1^{j-1})}(\overline{y_1},\dots,\overline{y_j})\right)_{l+a\le i\le n,\, 1\le j\le n}. \end{aligned}$$

In this expression $Y_j = \frac{1}{1+y_j}$ for all $j = 1, \dots, n$ according to Lemma 6.7. Substituting $y_j = t$ and using Lemma 6.8, and Formulas (6.15) and (6.25), finally leads to the result. □

Similar to Section 6.2.1 we can also derive a t-dimension formula starting from the definition (2.34) of composite supersymmetric Schur functions:

$$s_{\bar{\nu};\mu}(x/y) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l+k-l}(x/y) & h_{\mu_j-k-j+1}(x/y) \\ \hline \dot{h}_{\nu_l-i-l+1}(x/y) & h_{\mu_j+i-j}(x/y) \end{array} \right)$$

where i, j, k resp. l runs from top to bottom, from left to right, from bottom to top, resp. from right to left. This formula can be rewritten

$$s_{\bar{\nu};\mu}(x/y) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l+\nu_1'-l-i+1}(x/y) & h_{\mu_j-\nu_1'-j+i}(x/y) \end{array} \right)$$

where i, j resp. l runs from top to bottom, from left to right, resp. from right to left.

It follows from (6.4) and (6.5) that

Proposition 6.14 *The t-dimension of $V_{\bar{\nu};\mu}$ is given by the determinant*

$$\dim_t V_{\bar{\nu};\mu} = \det (A \mid B)\,, \tag{6.27}$$

with

$$\begin{aligned} A &= \left(\sum_{k=0}^{\nu_l+\nu_1'-l-i+1} \binom{m+\nu_l+\nu_1'-l-i-k}{\nu_l+\nu_1'-l-i-k+1} \binom{n}{k} t^{-k} \right)_{1\le i\le \nu_1'+\mu_1',\, 1\le l\le \nu_1'}, \\ B &= \left(\sum_{k=0}^{\mu_j-\nu_1'-j+i} \binom{m+\mu_j-\nu_1'-j+i-k-1}{\mu_j-\nu_1'-j+i-k} \binom{n}{k} t^{k} \right)_{1\le i\le \nu_1'+\mu_1',\, 1\le j\le \mu_1'} \end{aligned}$$

where i, j resp. l runs from top to bottom, from left to right, resp. from right to left.

Compared to (6.27), (6.26) has the advantage that in general the dimension of the matrix is smaller and each matrix element is a simple binomial coefficient multiplied by a power of t or $(1+t)$.

Let us illustrate both formulas for $\bar{\nu};\mu = (\bar{1},\bar{1},\bar{3});(2,1)$ in $\mathfrak{gl}(m|n) = \mathfrak{gl}(3|3)$. It is easy to check that $k = 2$, $l = 2$ and $r = 0$. Using (6.27), we have that

$$\dim_t V_{\bar{\nu};\mu} = \det \begin{pmatrix} 3+\frac{3}{t} & \frac{9}{t}+\frac{3}{t^2}+6 & 21+\frac{45}{t}+\frac{30}{t^2}+\frac{6}{t^3} & 0 & 0 \\ 1 & 3+\frac{3}{t} & 15+\frac{30}{t}+\frac{18}{t^2}+\frac{3}{t^3} & 1 & 0 \\ 0 & 1 & 10+\frac{18}{t}+\frac{9}{t^2}+\frac{1}{t^3} & 3+3\,t & 0 \\ 0 & 0 & 6+\frac{9}{t}+\frac{3}{t^2} & 6+9\,t+3\,t^2 & 1 \\ 0 & 0 & 3+\frac{3}{t} & 10+18\,t+9\,t^2+t^3 & 3+3\,t \end{pmatrix} = 48\,\frac{\left(t^4+5\,t^3+10\,t^2+5\,t+1\right)(1+t)^4}{t^5}$$

Using (6.26), we find that

$$\begin{aligned} \dim_t V_{\bar{\nu};\mu} &= \pm(1+t)^9\,t^0\,\det \begin{pmatrix} 0 & t & 1 & 0 & 0 \\ 1 & \frac{1}{1+t} & -\frac{1}{(1+t)^2} & \frac{1}{(1+t)^3} & 1 \\ 1 & -\frac{1}{(1+t)^2} & \frac{2}{(1+t)^3} & -\frac{3}{(1+t)^4} & -3 \\ 0 & \frac{1}{(1+t)^3} & -\frac{3}{(1+t)^4} & \frac{6}{(1+t)^5} & 6 \\ 0 & \frac{1}{t^3} & -\frac{3}{t^4} & \frac{6}{t^5} & 0 \end{pmatrix} \\ &= 48\,\frac{\left(t^4+5\,t^3+10\,t^2+5\,t+1\right)(1+t)^4}{t^5} \end{aligned}$$

6.3.2 The t-dimension in the special case $\bar{\nu};\mu = (\overline{b^c});(b^c)$

Let us consider the Lie superalgebra $\mathfrak{gl}(m|n)$. When we have the composite partition $\bar{\nu};\mu = (\overline{b}^c);(b^c)$ with either $n = m+b-2c$ or $n = m+2b-c$, the t-dimension turns out to be proportional to a power of t times a power of $1+t$. As both cases are equivalent, we will only consider the first case where $n = m+b-2c$. The other case is identical to $\overline{\nu'};\mu'$ in $\mathfrak{gl}(n|m)$, which coincides with the first one. An example of $\bar{\nu};\mu = (\overline{b}^c);(b^c)$ is given in Figure 6.1 where $n = m+b-2c$.
It is easy to verify that $k = 2c+1$, $l = 1$, $r = b$ and $a = n-b$. In order to have a critical composite partition with no zeros in the overlap, there is an extra condition $2c \leq m$. If $2c = m$ then the composite partition will be typical as the two partitions will fill the whole $(m \times n)$-rectangle.

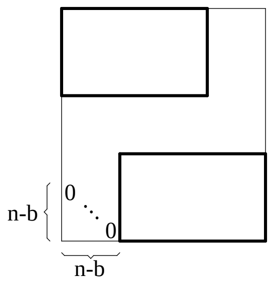


Figure 6.1: $\bar{\nu}; \mu = (\overline{b}^c); (b^c)$ in $\mathfrak{gl}(m|n)$.

The t-dimension of a mixed tensor representation is given by a determinantal formula (6.26). Translated to the composite partition $\bar{\nu}; \mu = (\overline{b}^c); (b^c)$ this formula reads

$$\dim_t V_{(\overline{b}^c);(b^c)} = (1+t)^{mn} t^{bn} \det(C) \tag{6.28}$$

with

$$C = \begin{pmatrix} \left(\binom{b+2c-j}{i-1}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,c}} & \left(\binom{c-j}{i-1}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,c}} & \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}}\binom{i+j-2}{j-1}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \\ 0 & 0 & \left(\frac{(-1)^{j-1}}{t^{i+j+c-1}}\binom{i+j+c-2}{j-1}\right)_{\substack{i=1,\dots,b \\ j=1,\dots,n}} \end{pmatrix} \tag{6.29}$$

We will prove that

Theorem 6.15 *Let $\bar{\nu}; \mu = (\overline{b}^c); (b^c)$ be a composite partition with $n = m + b - 2c$ and $n - b \geq 0$; the t-dimension of $V_{\bar{\nu};\mu}$ is then given by:*

$$\dim_t V_{(\overline{b}^c);(b^c)} = \pm((b+c)!)^c t^{-bc}(1+t)^{2cb} \times \frac{\prod_{j=1}^{n+c}(j+c-1)! \prod_{j=1}^{c-1}(c+b+j)^{c-j} \prod_{j=1}^{c}(j-1)! \prod_{j=1}^{b}(j-1)! \prod_{j=1}^{n-b}(j-1)!}{\prod_{i=1}^{m-c}(i+c-1)! \prod_{i=1}^{b}(i+c-1)! \prod_{j=1}^{n}(j-1)! \prod_{j=1}^{2c}(b+j-1)!}. \tag{6.30}$$

First simplifications

By the triangularity of columns $c+i$, where $i=1,\ldots,c$, the determinant can be reduced according to those columns; $\det(C)$ becomes

$$\det(C) = \pm\det\begin{pmatrix} \left(\binom{b+2c-j}{i+c-1}\right)_{\substack{i=1,\ldots,m-c\\ j=1,\ldots,c}} & \left(\frac{(-1)^{i+c+j}}{(1+t)^{i+j+c-1}}\binom{i+j+c-2}{j-1}\right)_{\substack{i=1,\ldots,m-c\\ j=1,\ldots,n}} \\ 0 & \left(\frac{(-1)^{j-1}}{t^{i+j+c-1}}\binom{i+j+c-2}{j-1}\right)_{\substack{i=1,\ldots,b\\ j=1,\ldots,n}} \end{pmatrix}. \tag{6.31}$$

In the last b rows, a factor $t^{i+n+c-1}$ with $i=1,\ldots,b$, can be put in front. Next, we apply the following operations on the last b rows R_i:

for j from 2 to b do
for i from b to j by -1 do
$R_{i+m-c} \to R_{i+m-c} - R_{i+m-c-1}$.

After s steps the entries in the last b rows become $(-1)^{j-1}t^{n-j}\binom{i+j+c-2-s}{j-1-s}$. Since the algorithm applies in total $i-1$ row transformations on row $i+m-c$, the entries in those last b rows become

$$(-1)^{j-1}t^{n-j}\binom{j+c-1}{j-i}. \tag{6.32}$$

On the first c columns K_j we apply the following operations:

for i from c to 2 by -1 do
for j from 1 to $i-1$ do (6.33)
$K_j \to K_j - K_{j+1}$.

After s steps the elements on column K_j become $\binom{b+2c-j-s}{i+c-1-s}$. At the end of this sequence (6.33), we applied $(c-j)$ transformations on column K_j. This way, the entries become

$$\left(\binom{b+c}{i+j-1}\right)_{i=1,\ldots,m-c,\, j=1,\ldots,c}.$$

Changing the order of those c columns the entries in the first c columns are converted into

$$\left(\binom{b+c}{i-j+c}\right)_{i=1,\ldots,m-c,\, j=1,\ldots,c}. \tag{6.34}$$

So, $\det(C)$ can be replaced by:

$$\det(C) = \pm F \det \begin{pmatrix} \left(\binom{b+c}{i-j+c}\right)_{\substack{i=1,\dots,m-c\\ j=1,\dots,c}} & \left(\frac{(-1)^{i+c+j}}{(1+t)^{i+j+c-1}}\binom{i+j+c-2}{j-1}\right)_{\substack{i=1,\dots,m-c\\ j=1,\dots,n}} \\ 0 & \left((-1)^{j-1}t^{n-j}\binom{j+c-1}{j-i}\right)_{\substack{i=1,\dots,b\\ j=1,\dots,n}} \end{pmatrix} \tag{6.35}$$

and

$$F = t^{-\frac{b(b+1)}{2}-b(n+c-1)}.$$

Eliminating the denominators and common factors

First of all, we substitute $(t+1)$ by T. In the next step we eliminate denominators and common factors by multiplying

- the first $m-c$ rows by $(-1)^{i+c-1}(i+c-1)!\,T^{n+i+c-1}$,
- the last b rows by $(i+c-1)!$, where $i = 1, \dots, b$,
- the first c columns by $\dfrac{(-1)^c(b+j-1)!}{(b+c)!\,T^{n+c}}$.
- finally, the last n columns by $\dfrac{(-1)^{j-1}(j-1)!}{(j+c-1)!}$, $j = 1, \dots, n$.

The resulting determinant is given in the next formula:

$$\begin{aligned} \det(C) &= \pm t^{-\frac{b(b+1)}{2}-b(n+c-1)} T^{-(n+c)(n-b)-\frac{(m-c)(m-c-1)}{2}} \\ &\times \frac{((b+c)!)^c \prod_{j=1}^{n}(j+c-1)!}{\prod_{i=1}^{m-c}(i+c-1)! \prod_{j=1}^{n}(j-1)! \prod_{j=1}^{c}(b+j-1)! \prod_{i=1}^{b}(i+c-1)!} \\ &\times \det \begin{pmatrix} \left((-1)^{i-1}(i+c-1)_{\underline{j-1}}(b+j-1)_{\underline{i-1}}\,T^{i-1}\right) & \left((c+j)_{i-1}T^{n-j}\right) \\ 0 & \left((j-1)_{\underline{i-1}}t^{n-j}\right) \end{pmatrix} \end{aligned} \tag{6.36}$$

Herein, $(a)_r = a(a+1)\cdots(a+r-1)$ is the Pochhammer symbol or rising factorial, and by $(a)_{\underline{r}} = a(a-1)\cdots(a-r+1)$ we denote the falling factorial.

Generalisations by means of variables

In order to compute the remaining determinant, we introduce some variables in such a way that the expansion of the determinant is proportional to a factorization in these variables. More exactly:

$$\begin{aligned} A(b,c,n) \\ &= \begin{pmatrix} \left((-1)^{i-1}(R_j+i-1)_{\underline{j-1}}(b+j-1)_{\underline{i-1}}\,T^{i-1}\right) & \left((C+j)_{i-1}T^{n-j}\right) \\ 0 & \left((A_i+j-1)_{\underline{i-1}}t^{n-j}\right) \end{pmatrix} \\ &= \begin{pmatrix} (A_{11})_{i=1,\dots,m-c,\ j=1,\dots,c} & (A_{12})_{i=1,\dots,m-c,\ j=1,\dots,n} \\ 0 & (A_{22})_{i=1,\dots,b,\ j=1,\dots,n} \end{pmatrix} \end{aligned} \tag{6.37}$$

From now on, we can even assume that the dimension variables $b,\ c,\ n$ are independent. The variables R_j, for $j=1,\dots,c$, A_i for $i=1,\dots,b$ and C are parameters.

The determinant is independent of the variables A_i

To prove that $\det(A(b,c,n))$ is independent of the variables A_i we will only consider the last b rows and more exactly A_{22}. The entries of A_{22} are:

$$(A_{22})_{ij} = t^{n-j}(A_i+j-1)_{\underline{i-1}}. \tag{6.38}$$

It is clear that the first row of A_{22} is independent of A_1 as the elements are equal to $t^{n-j}(A_1+j-1)_{\underline{0}} = t^{n-j}$. So, we can replace A_1 by no matter what variable, for example A_2. Next, let us consider the row operation where we replace the second row R_2 by R_2-R_1. The entries become

$$t^{n-j}(A_2+j-1)_{\underline{1}}-t^{n-j} = t^{n-j}(A_2-1+j-1) = t^{n-j}(A_2^*+j-1)_{\underline{1}} \text{ with } A_2^* = A_2-1.$$

This implies that $\det(A(b,c,n)) = \det(A(b,c,n))|_{A_2=A_2^*}$. Since we can perform this transformation several times, we can replace A_2 by any value. Suppose that we already considered the first k rows, and that we can replace the variables A_j, $j=1,\dots,k$ by arbitrary variables. Then we can choose to put $A_k = A_{k+1}-1$. Next, replace the $(k+1)$-th row R_{k+1} by $R_{k+1}-kR_k$:

$$\begin{aligned} &t^{n-j}(A_{k+1}+j-1)_{\underline{k}} - k\,t^{n-j}(A_{k+1}-1+j-1)_{\underline{k-1}} \\ &= t^{n-j}(A_{k+1}+j-2)_{\underline{k-1}}(A_{k+1}+j-1-k) \\ &= t^{n-j}(A_{k+1}-1+j-1)_{\underline{k}} \end{aligned} \tag{6.39}$$

So, we may replace A_{k+1} by $A_{k+1} - 1$, and by repetition by any value. We can conclude that the determinant is independent of the choices of the value of every A_i.

Without a loss of generality, we put $A(b,c,n) = A(b,c,n)\big|_{A_i=0, i=1,\dots,b}$:

$$A(b,c,n) = \begin{pmatrix} \left((-1)^{i-1}(R_j+i-1)_{\underline{j-1}}(b+j-1)_{\underline{i-1}}\, T^{i-1}\right) & \left((C+j)_{i-1}T^{n-j}\right) \\ 0 & \left((j-1)_{\underline{i-1}}t^{n-j}\right) \end{pmatrix} \tag{6.40}$$

The determinant $\det(A(b,c,n))$ is divisible by $\prod_{p=2}^{c}\prod_{q=1}^{p-1}(R_p+b+q)$

The entries of $(A_{11})_{ij}$ have the following form:

$$(A_{11})_{ij} = (-1)^{i-1}T^{i-1}(R_j+i-1)_{\underline{j-1}}(b+j-1)_{\underline{i-1}} \tag{6.41}$$

We will prove the existence of every factor $(R_j + b + 1)$ and illustrate the other factors and the algorithm by an example. First of all, we replace column K_j by $K_j - (R_j)_{\underline{j-1}}K_1$; the entries transform into:

$$(-1)^{i-1}T^{i-1}\left[(R_j+i-1)_{\underline{j-1}}(+j-1)_{\underline{i-1}} - (R_j)_{\underline{j-1}}(b)_{\underline{i-1}}\right] = (-1)^{i-1}T^{i-1}X$$

If $i < j$ then

$$\begin{aligned} X &= (R_j)_{\underline{j-i}}\Big[(R_j+1)_{i-1}(b+(j-1))(b-1+(j-1))\dots(b-i+2+(j-1)) \\ &\quad - (R_j+i-1-(j-1))(R_j+i-2-(j-1))\dots(R_j+1-(j-1))(b)_{\underline{i-1}}\Big] \\ &= (R_j)_{\underline{j-i}}\Big[(R_j+1)_{i-1}\sum_{k=0}^{i-1}(j-1)^k e_{i-1+k}(b,b-1,\dots,b-i+2) \\ &\quad - (b)_{\underline{i-1}}\sum_{k=0}^{i-1}(-1)^k(j-1)^k e_{i-1+k}(R_j+1,R_j+2,\dots,R_j+i-1)\Big] \end{aligned}$$

If $i \geq j$ then

$$\begin{aligned} X &= (b)_{\underline{i-j}}\Big[(R_j+i-1)_{\underline{j-1}}(b+1)_{j-1} - (R_j)_{\underline{j-1}}(b+j-i)_{\underline{j-1}}\Big] \\ &= (b)_{\underline{i-j}}\Big[(b+1)_{j-1}\sum_{k=0}^{j-1}(i-1)^k e_{j-1-k}(R_j, R_j-1, \ldots, R_j-j+2) \\ &\quad - (R_j)_{\underline{j-1}}\sum_{k=0}^{j-1}(i-1)^k e_{j-1-k}(b+1, b+2, \ldots, b+j-1)\Big] \end{aligned}$$

As both expressions become zero when putting $R_j = -b-1$, column K_j can be divided by a factor $R_j + b + 1$, with $j = 2, \ldots, c$.

The same process can be repeated for column K_j, with $j = 3, \ldots, c$, using K_2 in the operations and yielding a factor $R_j + b + 2$; next we can repeat this process for K_j, with $j = 4, \ldots, c$, using K_3 in the operations and yielding a factor $R_j + b + 3$, etc. The computation is always the same, so we will not give it explicitly. But let us illustrate the whole process for $b = 2$, $c = 4$ and $n = 3$. The matrix is then given by a (7×7)-matrix. As the column operations only affect the first c columns, we will consider only those columns. The last b rows have a zero in columns 1 to c; therefore they will not been mentioned either.

$$\begin{pmatrix} 1 & R_2 & R_3\,(R_3-1) & R_4\,(R_4-1)\,(R_4-2) \\ -2-2\,t & -3\,(R_2+1)\,(1+t) & -4\,(R_3+1)\,R_3\,(1+t) & -5\,(R_4+1)\,R_4\,(R_4-1)\,(1+t) \\ 2\,(1+t)^2 & 6\,(R_2+2)\,(1+t)^2 & 12\,(R_3+2)\,(R_3+1)\,(1+t)^2 & 20\,(R_4+2)\,(R_4+1)\,R_4\,(1+t)^2 \\ 0 & -6\,(R_2+3)\,(1+t)^3 & -24\,(R_3+3)\,(R_3+2)\,(1+t)^3 & -60\,(R_4+3)\,(R_4+2)\,(R_4+1)\,(1+t)^3 \\ 0 & 0 & 24\,(R_3+4)\,(R_3+3)\,(1+t)^4 & 120\,(R_4+4)\,(R_4+3)\,(R_4+2)\,(1+t)^4 \end{pmatrix}$$

$$\text{for } j \text{ from 2 to } c \text{ do } K_j \rightarrow K_j - (R_j)_{\underline{j-1}} K_1.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2-2\,t & -\,(R_2+3)\,(1+t) & -2\,R_3\,(R_3+3)\,(1+t) & -3\,R_4\,(R_4+3)\,(R_4-1)\,(1+t) \\ 2\,(1+t)^2 & 4\,(1+t)^2\,(R_2+3) & 2\,(R_3+3)\,(5\,R_3+4)\,(1+t)^2 & 6\,R_4\,(R_4+3)\,(3\,R_4+2)\,(1+t)^2 \\ 0 & -6\,(R_2+3)\,(1+t)^3 & -24\,(R_3+3)\,(R_3+2)\,(1+t)^3 & -60\,(R_4+3)\,(R_4+2)\,(R_4+1)\,(1+t)^3 \\ 0 & 0 & 24\,(R_3+4)\,(R_3+3)\,(1+t)^4 & 120\,(R_4+4)\,(R_4+3)\,(R_4+2)\,(1+t)^4 \end{pmatrix}$$

$$\text{for } j \text{ from 2 to } c \text{ do } K_j \rightarrow \frac{1}{(j-1)(R_j+b+1)} K_j.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2-2t & -1-t & -(1+t)R_3 & -(1+t)R_4(R_4-1) \\ 2(1+t)^2 & 4(1+t)^2 & (5R_3+4)(1+t)^2 & 2(3R_4+2)R_4(1+t)^2 \\ 0 & -6(1+t)^3 & -12(R_3+2)(1+t)^3 & -20(R_4+2)(R_4+1)(1+t)^3 \\ 0 & 0 & 12(R_3+4)(1+t)^4 & 40(R_4+4)(R_4+2)(1+t)^4 \end{pmatrix}$$

for j from 3 to c do $j \to K_j - (R_j)_{\underline{j-2}}K_2$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2-2t & -1-t & 0 & 0 \\ 2(1+t)^2 & 4(1+t)^2 & (1+t)^2(R_3+4) & 2(1+t)^2R_4(R_4+4) \\ 0 & -6(1+t)^3 & -6(1+t)^3(R_3+4) & -2(R_4+4)(7R_4+5)(1+t)^3 \\ 0 & 0 & 12(R_3+4)(1+t)^4 & 40(R_4+4)(R_4+2)(1+t)^4 \end{pmatrix}$$

for j from 3 to c do $K_j \to \dfrac{1}{(j-2)(R_j+b+2)}K_j$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2-2t & -1-t & 0 & 0 \\ 2(1+t)^2 & 4(1+t)^2 & (1+t)^2 & R_4(1+t)^2 \\ 0 & -6(1+t)^3 & -6(1+t)^3 & -(7R_4+5)(1+t)^3 \\ 0 & 0 & 12(1+t)^4 & 20(R_4+2)(1+t)^4 \end{pmatrix}$$

for j from 4 to c do $K_j \to K_j - (R_j)_{\underline{j-3}}K_3$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2-2t & -1-t & 0 & 0 \\ 2(1+t)^2 & 4(1+t)^2 & (1+t)^2 & 0 \\ 0 & -6(1+t)^3 & -6(1+t)^3 & -(1+t)^3(R_4+5) \\ 0 & 0 & 12(1+t)^4 & 8(1+t)^4(R_4+5) \end{pmatrix}$$

for j from 4 to c do $K_j \to \dfrac{1}{(j-3)(R_j+b+3)}K_j$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2-2t & -1-t & 0 & 0 \\ 2(1+t)^2 & 4(1+t)^2 & (1+t)^2 & 0 \\ 0 & -6(1+t)^3 & -6(1+t)^3 & -(1+t)^3 \\ 0 & 0 & 12(1+t)^4 & 8(1+t)^4 \end{pmatrix}$$

In the end, the entries of A_{12} are independent of R_j. The product

$$\prod_{p=2}^{c}\prod_{q=1}^{p-1}(R_p+b+q)$$

is a polynomial in $R_1, R_2, \ldots, R_c$ with highest degree term $R_2R_3^2R_4^3\cdots R_c^{c-1}$. This polynomial divides $\det(C)$, therefore $\det(C)$ is also a polynomial in $R_2, \ldots, R_c$. Looking at (6.40), it is easy to see that the highest degree possible in the computation of the determinant is also $R_2R_3^2R_4^3\cdots R_c^{c-1}$. Therefore, we found every factor in the variables R_j. To compute the coefficient, it is sufficient to look at the leading coefficients in R_j in every term of A_{11}. So,

Lemma 6.16 *Let $A(b,c,n)$ be the matrix defined in (6.40).*

$$\det(A(b,c,n)) = \pm\prod_{p=2}^{c}\prod_{q=1}^{p-1}(R_p+b+q)\det(\overline{A}(b,c,n)) \tag{6.42}$$

where

$$\overline{A}(b,c,n) = \begin{pmatrix} \left((b+j-1)_{\underline{i-1}}T^{i-1}\right)_{\substack{i=1,\ldots,m-c\\ j=1,\ldots,c}} & \left((-C-j)_{\underline{i-1}}T^{n-j}\right)_{\substack{i=1,\ldots,m-c\\ j=1,\ldots,n}} \\ 0 & \left((j-1)_{\underline{i-1}}t^{n-j}\right)_{\substack{i=1,\ldots,b\\ j=1,\ldots,n}} \end{pmatrix}. \tag{6.43}$$

Remark that we first eliminated the minus sign in the first c columns.

$\det(\overline{A})$ is divisible by $\prod_{p=1}^{n-b}\prod_{q=1}^{c}(C+b+p+q-1)$

One of the methods given in [43] to compute determinants is the method by identification of factors. To prove that a determinant is divisible by a factor $C+b+p+q-1$, we have to find a vector in the kernel of the matrix.

For every (p,q) we construct a vector $v_{(p,q)} \in M_{(n+c)\times 1}$:

$$v_{(p,q)} = \begin{pmatrix} \left(-\binom{q-1}{i-1} T^{n-b-q} t^{q-i}\right)_{i=1,\ldots,c} \\ (0)_{i=1,\ldots,p-1} \\ \left((-1)^{i-1}\binom{b}{i-1} T^{p-1} t^{i-1}\right)_{i=1,\ldots,b+1} \\ (0)_{i=1,\ldots,n-b-p} \end{pmatrix} \qquad (6.44)$$

Lemma 6.17 *For $\overline{A}(b,c,n)$ given in (6.43), and $v_{(p,q)}$ defined in (6.44), we have that*

$$\textit{if} \quad C = -b-p-q+1 \quad \Rightarrow \quad \overline{A}(b,c,n)\cdot v_{(p,q)} = 0. \qquad (6.45)$$

Proof. *Part 1: the first $n+b-c$ rows in the product $\overline{A}(b,c,n)\cdot v_{(p,q)}$.*
As $C = -b-p-q+1$ it follows that $(-C-j)_{\underline{i-1}} = (b+p+q-j-1)_{\underline{i-1}}$.

$$\begin{aligned}
&(\overline{A}(b,c,n)\cdot v_{(p,q)})_i \\
&= \sum_{j=1}^{c} -(b+j-1)_{\underline{i-1}} \binom{q-1}{j-1} T^{i-1+n-b-q} t^{q-j} \\
&\quad + \sum_{j=1}^{n} (b+p+q-j-1)_{\underline{i-1}} T^{n-j} (v_{(p,q)})_{c+j} \\
&= \sum_{j} -(b+j-1)_{\underline{i-1}} \binom{q-1}{j-1} T^{i-1+n-b-q} t^{q-j} \\
&\quad + \sum_{j} (b+p+q-j-1)_{\underline{i-1}} T^{n-j} (-1)^{j-p} \binom{b}{j-p} t^{j-p} T^{p-1} \\
&= \sum_{j} (-1)^{i} (1-b-j)_{i-1} \binom{q-1}{j-1} T^{i-q-1+n-b} t^{q-j} \\
&\quad + \sum_{j} (-1)^{i-1} (-b-q+j+1)_{i-1} T^{n-j-1} (-1)^{j} \binom{b}{j} t^{j} \\
&= \left(\sum_{j} (-1)^{i} (1-b-j)_{i-1} \binom{q-1}{j-1} t^{q-j} \right.
\end{aligned}$$

$$+\sum_j (-1)^{i+j-1}(-b-q+j+1)_{i-1}T^{b+q-i-j}\binom{b}{j}t^j\Bigg)\, T^{n-b+i-q-1}$$

In order to have a product equal to zero, we will prove the following identity:

$$\sum_j \binom{q-1}{j-1}(1-b-j)_{i-1}t^{q-j} = \sum_j (1+j-b-q)_{i-1}(-1)^j\binom{b}{j}t^jT^{b+q-i-j}.$$

$$\text{lhs} \quad = \quad \sum_J \binom{q-1}{J}(1-b-q+J)_{i-1}t^J$$

In the rhs we expand the power of $(1+t)$, change summation indices, and perform the Chu-Vandermonde (binomial) summation theorem:

$$\begin{aligned}
\text{rhs} \quad &= \quad \sum_j (1+j-b-q)_{i-1}(-1)^j\binom{b}{j}t^j(1+t)^{b+q-i-j} \\
&= \quad \sum_j (1+j-b-q)_{i-1}(-1)^j\binom{b}{j}t^j\left(\sum_k \binom{b+q-i-j}{k}t^k\right) \\
&= \quad \sum_{k,j} (-1)^j(1+j-b-q)_{i-1}\binom{b}{j}\binom{b+q-i-j}{k}t^{k+j} \\
&= \quad \sum_{K,j} (-1)^j(1+j-b-q)_{i-1}\binom{b}{j}\binom{b+q-j-i}{K-j}t^K \\
&= \quad \sum_{K,j} (-1)^j(1+j-b-q)_{i-1}\binom{b}{j}\binom{b+q-j-1}{K-j}\frac{(-b-q+K+1)_{i-1}}{(-b-q+j+1)_{i-1}}t^K \\
&= \quad \sum_K (1+K-b-q)_{i-1}\left(\sum_j (-1)^j\binom{b}{j}\binom{b+q-j-1}{K-j}\right)t^K \\
&\overset{(6.7)}{=} \quad \sum_K (1+K-b-q)_{i-1}\binom{q-1}{K}t^K = \text{lhs}
\end{aligned}$$

Part 2: the last b rows in the product $\overline{A}(b,c,n)\cdot v_{(p,q)}$.
For the entries in the last b rows, we first simplify the expression, put common factors in front and perform the Chu-Vandemonde (binomial) summation theorem:

$$0+\sum_{j=1}^{n}(j-1)_{\underline{i-1}}t^{n-j}(v(p,q))_{c+j}$$

$$\begin{aligned}
&= \sum_j (j-1)_{\underline{i-1}} t^{n-j} (-1)^{j-p} \binom{b}{j-p} t^{j-p} T^{p-1} \\
&= \sum_j (j-p-1)_{\underline{i-1}} t^{n-p} (-1)^{j} \binom{b}{j} T^{p-1} \\
&= (-1)^{i-1} t^{n-p} T^{p-1} \sum_j (-1)^j \binom{b}{j} (p+1-j)_{i-1} \\
&= (-1)^{i-1} t^{n-p} T^{p-1} \sum_j (-1)^j \binom{b}{j} \binom{p-j+i-1}{p-j} (i-1)! \\
&= (-1)^{i-1} t^{n-p} T^{p-1} \sum_j (-1)^j \binom{b}{j} \binom{p-j+i-1}{p-j} (i-1)! \\
&\overset{(6.7)}{=} (-1)^{i-1} t^{n-p} T^{p-1} (i-1)! \binom{i-1-b+p}{p}.
\end{aligned}$$

As $1 \leq i \leq b$, we find that $p-b \leq p+i-1-b \leq p-1$. Therefore, $\binom{i-1-b+p}{p} = 0$. The result follows. □

As the determinant can be divided by

$$\prod_{p=1}^{n-b} \prod_{q=1}^{c} (C+b+p+q-1), \tag{6.46}$$

$\det(\overline{A}(b,c,n))$ is a polynomial in C of degree at least $(n-b)c$.

$\det(\overline{A}(b,c,n))$ is a polynomial of degree $(n-b)c$ in C and $\frac{b(b-1)}{2}$ in t

In order to put an upper bound to the degree of $\det(\overline{A}(b,c,n))$ in C and to determine the degree of $\det(\overline{A}(b,c,n))$ in t, we apply some new column and row operations. First of all, we put $t = T-1$ in $\overline{A}(b,c,n)$. The first sequence of operations affects the columns K_j, with $j = c+1, \ldots, c+n-1$:

for i from 1 to $n-1$ do
for j from $c+1$ to $c+n-i$ do $K_j \rightarrow K_j - TK_{j+1}$, (6.47)

whereas the second sequence of operations changes rows R_i, with $i = 2, \dots, b$:

$$\begin{aligned} &\text{for } j \text{ from } 2 \text{ to } b \text{ do} \\ &\quad \text{for } i \text{ from } b \text{ to } j \text{ by } -1 \text{ do} \\ &\qquad R_{n+c-b+i} \to R_{n+c-b+i} - (n-i+1)TR_{n+c-b+i-1}. \end{aligned} \tag{6.48}$$

The operations (6.47) only affect the last n columns. We will first look at the elements in the first $n-b$ rows of those columns.

Lemma 6.18 *The operations (6.47) transform*

$$(-C-j)_{\underline{i-1}}T^{n-j} \qquad \textit{into} \qquad (-1)^{n-i-j-1}(i-1)_{\underline{n-j}}T^{n-j}. \tag{6.49}$$

Proof. We prove this lemma by induction on the number of transformations applied on every row. To do so, we will prove that after s steps the entries become

$$T^{n-j}(i-1)_{\underline{s}}(-C-j-s)_{\underline{i-s-1}}. \tag{6.50}$$

After one step we have that:

$$\begin{aligned} &(-C-j)_{\underline{i-1}}T^{n-j} - T(-C-j-1)_{\underline{i-1}}T^{n-j-1} \\ &\quad = \big((-C-j)-(-C-j-i+1)\big)(-C-j-1)_{\underline{i-2}}T^{n-j} \\ &\quad = (i-1)(-C-j-1)_{\underline{i-2}}T^{n-j}. \end{aligned}$$

Suppose that after $s-1$ steps the elements became $T^{n-j}(i-1)_{s-1}(-C-j-s+1)_{\underline{i-s}}$, then we will prove that after another operation those elements become (6.50). Indeed,

$$\begin{aligned} &T^{n-j}(i-1)_{\underline{s-1}}(-C-j-s+1)_{\underline{i-s}} - T^{n-j}(i-1)_{\underline{s-1}}(-C-j-s)_{\underline{i-s}} \\ &\quad = T^{n-j}(i-1)_{\underline{s-1}}(-C-j-s)_{\underline{i-s-1}}\big((-C-j-s+1)-(-C-j-i+1)\big) \\ &\quad = T^{n-j}(i-1)_{\underline{s-1}}(-C-j-s)_{\underline{i-s-1}}(i-s) \end{aligned}$$

which is equal to (6.50). As we make $n-j$ transformations on column j, this yields the lemma. □

The row operations (6.48) affect the last b rows. The elements in the last n columns of those rows however, are first changed by (6.47), the other entries are zero.

Lemma 6.19 *The operations (6.47) transform the elements* $(j-1)_{\underline{i-1}}t^{n-j}$ *in the last* b *rows into:*

$$(-1)^{n-j}\sum_{q=0}^{n-j}\binom{n-j}{q}(i-1)_{\underline{q}}(n-i+1)_{i-q-1}t^q. \tag{6.51}$$

Proof. We will prove this Lemma also by induction on the number of transformations; after s transformations the entries are

$$(-1)^s (j-1)_{\underline{i-s-1}} t^{n-j-s} \sum_{q=0}^{s} \binom{s}{q} (i-1)_{\underline{q}} (j)_{s-q} t^q. \tag{6.52}$$

After one step the elements are changed into:

$$\begin{aligned}
&(j-1)_{\underline{i-1}} t^{n-j} - (1+t)(j)_{\underline{i-1}} t^{n-j-1} \\
&\quad = (j-1)_{\underline{i-2}} t^{n-j-1} \big((j-i+1)t + (1+t)j\big) \\
&\quad = -(j-1)_{\underline{i-2}} t^{n-j-1} (i-1)t + j)
\end{aligned}$$

Suppose that we have already applied $s-1$ transformations on a row. In the next step the entries will become:

$$(-1)^{s-1} (j-1)_{\underline{i-s-1}} t^{n-j-s} \times \text{term}$$

with

$$\begin{aligned}
\text{term} &= \sum_{q=0}^{s-1} \binom{s-1}{q} (i-1)_{\underline{q}} \Big(t(j-i+s)(j)_{s-q-1} - (1+t)j(j+1)_{s-q-1} \Big) t^q \\
&= \sum_{q=0}^{s-1} \binom{s-1}{q} (i-1)_{\underline{q}} \Big(t(j-i+s)(j)_{s-q-1} - (1+t)(j)_{s-q} \Big) t^q \\
&= \sum_{q=0}^{s-1} \binom{s-1}{q} (i-1)_{\underline{q}} \Big(t(j)_{s-q-1}(-i+q+1) - (j)_{s-q} \Big) t^q \\
&= -\binom{s-1}{0} (i-1)_{\underline{0}} (j)_s t^0 + \binom{s-1}{s-1} (i-1)_{\underline{s-1}} (j)_0 (s-i) t^s \\
&\quad + \sum_{q=1}^{s-1} \left(\binom{s-1}{q-1} (i-1)_{\underline{q-1}} (j)_{s-q} (-i+q) - \binom{s-1}{q} (i-1)_{\underline{q}} (j)_{s-q} \right) t^q \\
&= -\binom{s}{0} (i-1)_{\underline{0}} (j)_s t^0 - \binom{s}{s} (i-1)_{\underline{s}} (j)_0 t^s \\
&\quad - \sum_{q=1}^{s-1} (i-1)_{\underline{q}} (j)_{s-q} \left(\binom{s-1}{q-1} + \binom{s-1}{q} \right) t^q \\
&= -\sum_{q=0}^{s} \binom{s}{q} (i-1)_{\underline{q}} (j)_{s-q} t^q.
\end{aligned}$$

This proves the induction. Since the algorithm applies in total $n-j$ transformations in column j, the entries become

$$(-1)^{n-j}(j-1)_{\underline{i-n+j-1}}\sum_{q=0}^{n-j}\binom{n-j}{q}(i-1)_{\underline{q}}(j)_{n-j-q}t^q.$$

(6.51) follows from the equality $(j-1)_{\underline{i-1-n+j}}(j)_{n-j-q}=(n-i+1)_{i-q-1}$. □

Lemma 6.20 *The operations (6.48) transform the elements (6.51) in the last b rows into:*

$$(-1)^{n-i-j+1}(j-1)_{i-1}t^{i-1}. \tag{6.53}$$

Proof. It is easy to prove by means of induction that after s transformations the entries in the last b rows are given by

$$(-1)^{n-j}\sum_{q=0}^{n-j}\binom{n-j}{q}(n-i+1)_{i-q-1}t^q\sum_{p=0}^{s}\binom{s}{p}(i-1-p)_{\underline{q}}(-1)^pT^p.$$

As we need $i-1$ operations in row i, we get:

$$(-1)^{n-j}\sum_{q=0}^{n-j}\binom{n-j}{q}(n-i+1)_{i-q-1}t^q\sum_{p=0}^{i-1}\binom{i-1}{p}(i-1-p)_{\underline{q}}(-1)^pT^p. \tag{6.54}$$

In this expression:

$$\binom{i-1}{p}(i-1-p)_{\underline{q}}=(i-1)_{\underline{q}}\binom{i-1-q}{p}. \tag{6.55}$$

Consequently, (6.54) can be replaced by:

$$\begin{aligned}
&(-1)^{n-j}\sum_{q=0}^{n-j}\binom{n-j}{q}(n-i+1)_{i-q-1}t^q(i-1)_{\underline{q}}\sum_{p=0}^{i-1}\binom{i-1-q}{p}(-1)^pT^p\\
=\quad&(-1)^{n-j}\sum_{q=0}^{n-j}\binom{n-j}{q}(n-i+1)_{i-q-1}t^q(i-1)_{\underline{q}}(1-T)^{i-q-1}\\
=\quad&(-1)^{n-j-i+1}t^{i-1}\sum_{q=0}^{n-j}\binom{n-j}{q}(n-i+1)_{i-q-1}(i-1)_{\underline{q}}(-1)^q
\end{aligned}$$

In this expression the sum can be rewritten:

$$\begin{aligned}
&\sum_{q=0}^{n-j} \binom{n-j}{q} (n-i+1)_{i-q-1} (i-1)_{\underline{q}} (-1)^q \\
&= \frac{(j-1)!(n-j)!}{(n-i)!} \sum_q \frac{(n-q-1)!}{(n-j-q)!(j-1)!} \frac{(i-1)!}{q!(i-q-1)!} (-1)^q \\
&= \frac{(j-1)!(n-j)!}{(n-i)!} \sum_q \binom{n-q-1}{n-j-q} \binom{i-1}{q} (-1)^q \\
&\overset{(6.7)}{=} \frac{(j-1)!(n-j)!}{(n-i)!} \binom{n-i}{n-j} = (j-1)_{\underline{i-1}}.
\end{aligned}$$

This proves the lemma. □

As a result of Lemma 6.20, we can divide the last b rows by a factor t^{i-1}, $i = 1, \ldots, b$.

Lemma 6.21 *The row and column operations described, transform $\overline{A}(b, c, n)$ in*

$$\overline{A}'(b, c, n) = \begin{pmatrix} (b+j-1)_{\underline{i-1}} T^{i-1} & (i-1)_{\underline{n-j}} (-C-n)_{\underline{i-n+j-1}} T^{n-j} \\ 0 & (-1)^{n+1-i-j} (j-1)_{\underline{i-1}} \end{pmatrix} \tag{6.56}$$

where

$$\det(\overline{A}'(b, c, n)) = t^{\frac{b(b-1)}{2}} \det(\overline{A}(b, c, n)). \tag{6.57}$$

For example, with $b = 3$, $c = 2$ and $n = 5$,

$$\overline{A}'(3,2,5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 3T & 4T & 0 & 0 & 0 & T & -C-5 \\ 6T^2 & 12T^2 & 0 & 0 & 2T^2 & -2T(C+5) & (C+5)_2 \\ 6T^3 & 24T^3 & 0 & 6T^3 & -6T^2(C+5) & 3T(C+5)_2 & -(C+5)_3 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 0 & 0 & 2 & -6 & 12 \end{pmatrix}$$

It is clear that the maximal degree in C is determined by the determinant of the sub-matrix consisting of rows $c+1,\ c+2, \ldots, c+n-b$ and columns $c+b+1,\ c+b+$

$2, \ldots, c+n$; more exactly

$$\begin{pmatrix} (-C-n)_{c-n+b+1} & \cdots & (-C-n)_{c-1} & (-C-n)_{c} \\ (-C-n)_{c-n+b+2} & \cdots & (-C-n)_{c} & (-C-n)_{c+1} \\ \vdots & \ddots & \vdots & \vdots \\ (-C-n)_{c} & \cdots & (-C-n)_{c+n-b-2} & (-C-n)_{c+n-b-1} \end{pmatrix} \tag{6.58}$$

So, the maximal degree of $\overline{A}'(b,c,n)$ in C is $(n-b)c$. This proves that we have found every factor in C.

The coefficient of $C^{(n-b)c}$ in the computation of $\det(\overline{A}'(b,c,n))$ equals $\det(\tilde{(}A)(b,c,n)$ with $\tilde{A}(b,c,n)$ defined in the next lemma.

Lemma 6.22

$$\begin{aligned} \tilde{A}(b,c,n) &= \begin{pmatrix} (b+j-1)_{\underline{i-1}}T^{i-1} & (-1)^{n+1-i-j}(i-1)_{\underline{n-j}}T^{n-j} \\ 0 & (-1)^{n+1-i-j}(j-1)_{\underline{i-1}} \end{pmatrix} \\ &= \begin{pmatrix} (\tilde{A}_{11})_{i=1,\ldots,m-c,\ j=1,\ldots,c} & (\tilde{A}_{12})_{i=1,\ldots,m-c,\ j=1,\ldots,n} \\ 0 & (\tilde{A}_{22})_{i=1,\ldots,b,\ j=1,\ldots,n} \end{pmatrix} \end{aligned} \tag{6.59}$$

with

$$\det(\overline{A}(b,c,n)) = \prod_{p=1}^{n-b}\prod_{q=1}^{c}(C+b+p+q-1)t^{\frac{b(b-1)}{2}}\det(\tilde{A}(b,c,n)). \tag{6.60}$$

In the example mentioned above, $\tilde{A}(b,c,n)$ becomes

$$\tilde{A}(3,2,5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 3T & 4T & 0 & 0 & 0 & T & -1 \\ 6T^2 & 12T^2 & 0 & 0 & 2T^2 & -2T & 1 \\ 6T^3 & 24T^3 & 0 & 6T^3 & -6T^2 & 3T & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 0 & 0 & 2 & -6 & 12 \end{pmatrix} \tag{6.61}$$

The determinant of $\tilde{A}(b,c,n)$

The determinant can be divided into nine blocks.

$$\tilde{A}(b,c,n) = \begin{pmatrix} (A_{11})_{\substack{i=1,\dots,c\\ j=1,\dots,c}} & (A_{12})_{\substack{i=1,\dots,c\\ j=1,\dots,b}} & (A_{13})_{\substack{i=1,\dots,c\\ j=1,\dots,n-b}} \\ (A_{21})_{\substack{i=1,\dots,n-b\\ j=1,\dots,c}} & (A_{22})_{\substack{i=1,\dots,n-b\\ j=1,\dots,b}} & (A_{23})_{\substack{i=1,\dots,n-b\\ j=1,\dots,n-b}} \\ (A_{31})_{\substack{i=1,\dots,b\\ j=1,\dots,c}} & (A_{32})_{\substack{i=1,\dots,b\\ j=1,\dots,b}} & (A_{33})_{\substack{i=1,\dots,b\\ j=1,\dots,n-b}} \end{pmatrix} \tag{6.62}$$

In this section we will prove that:

Lemma 6.23 *With $\tilde{A}(b,c,n)$ defined in (6.59) and $A_{11},\ A_{23}$ and A_{32} given in (6.62), we find that*

$$\det(\tilde{A}(b,c,n)) = \det(A_{11})\det(A_{23})\det(A_{32}) \tag{6.63}$$

We will first make A_{11} lower triangular using column operations:

for p from 1 to $c-1$ do

for j from $p+1$ to c do $K_j \to K_j - \binom{j-1}{p-1} K_p$.

Using the equality

$$\sum_{i=0}^{k} \binom{k}{i}\binom{a}{b-i} = \binom{a+k}{b} \tag{6.64}$$

it is easy to prove that the operations transform

$$(b+j-1)_{\underline{i-1}}T^{i-1} = (i-1)!\binom{b+j-1}{i-1}T^{i-1} \quad \text{into} \quad (i-1)!\binom{b}{i-j}T^{i-1}.$$

Next, we divide the first c columns by $(j-1)!T^{j-1}$. Finally, the elements $(b+j-1)_{\underline{i-1}}T^{i-1}$ become:

$$(i-1)_{\underline{i-j}}\binom{b}{i-j}T^{i-j}. \tag{6.65}$$

More specific, the triangularity of the first c columns with only 1s along the diagonal, implies:

$$\det(A_{11}) = \det\left((b+j-1)_{\underline{i-1}}\, T^{i-1}\right) = T^{\frac{c(c-1)}{2}} \prod_{j=1}^{c}(j-1)!. \tag{6.66}$$

In the example, $\det(\tilde{A}(3,2,5))$ becomes:

$$\det(\tilde{A}(3,2,5)) = T\ \det\left(\begin{array}{cc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3T & 1 & 0 & 0 & 0 & T & -1 \\ \hline 6T^2 & 6T & 0 & 0 & 2T^2 & -2T & 1 \\ 6T^3 & 18T^2 & 0 & 6T^3 & -6T^2 & 3T & -1 \\ \hline 0 & 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 0 & 0 & 2 & -6 & 12 \end{array}\right)$$

Using the first c columns, we transform $\begin{pmatrix} A_{12} & A_{13} \end{pmatrix}$ into a zero block. The column operations are:

for j from $c+1$ to $c+n$ do

for i from $n+c+1-j$ to c do

$K_j \to K_j - (b+j-1)_{\underline{i-1}} K_i.$

This way, a Laplace expansion reduces the dimension of the matrix to a $n \times n$-matrix. In general, this leads to:

$$\det(\tilde{A}) = T^{\frac{c(c-1)}{2}} \prod_{j=1}^{c} (j-1)!\ \det\begin{pmatrix} A'_{22} & A'_{23} \\ A'_{32} & A'_{33} \end{pmatrix} \tag{6.67}$$

For $b=3$, $c=2$ and $n=5$, we get for our example:

$$\det(\tilde{A}(3,2,5)) = T\ \det\left(\begin{array}{ccc|cc} 0 & 0 & 2T^2 & -2T-6T^2 & 1+12T^2+6T \\ 0 & 6T^3 & -6T^2 & 3T-18T^3 & -1+48T^3+18T^2 \\ \hline 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 2 & -6 & 12 \end{array}\right)$$

The $b \times b$ block A'_{32} is upper triangular. First, we divide the last b rows by $(-1)^{n-1}(i-1)!$, changing the entries in those rows into $(-1)^{i+j}\binom{j-1}{i-1}$. This way, $\det(\tilde{A}(b,c,n))$ is divided by

$$\det(A'_{32}) = \det(A_{32}) = (-1)^{b(n-1)} \prod_{i=1}^{b} (i-1)!. \tag{6.68}$$

The following column operations turn A'_{33} into a zero block:

for j from 1 to b do

for i from 1 to $n-b$ do

$$K_{b+i} \to K_{b+i} + (-1)^{i-1}\binom{i+j-2}{i-1}\binom{b+i-1}{i+j-1}K_{b-j+1}.$$

The Laplace expansion with respect to the last b rows gives that

$$\det(\tilde{A}) = (-1)^{b(n-1)}T^{\frac{c(c-1)}{2}}\prod_{j=1}^{c}(j-1)!\prod_{i=1}^{b}(i-1)!\ \det\left(A''_{23}\right) \tag{6.69}$$

In the example, we have that:

$$\det(\tilde{A}(3,2,5)) = 2\,T\ \det\begin{pmatrix} -2\,T & 1+8\,T \\ 3\,T-24\,T^2 & -1+96\,T^2 \end{pmatrix}$$

While making zero blocks of A_{13} and A_{33} in $\tilde{A}$ the entries of A_{23} also changed. To undo those changes, we apply some new row and column operations, turning the matrix A''_{23} back into the original matrix A_{23}. These operations are given by:

for i from $n-b-1$ to 1 by -1 do

for j from 1 to i do

$$K_{i+1} \to K_{i+1} + \binom{b}{i-j+1}K_j.$$

and

for i from $n-b-1$ to 1 by -1 do for j from 1 to i do

$$R_{i+1} \to R_{i+1} + (-1)^{i+j-1}\binom{b+i-j}{i-j+1}\binom{c+i}{i-j+1}(i-j+1)!T^{i-j+1}R_{i-j+1}.$$

To conclude the computation of the determinant, we still need to compute $\det(A_{23})$, which is equal to

$$\begin{aligned} \det(A_{23}) &= \det\left(\left((-1)^{n-b-c+1-i-j}T^{n-b-j}(i-c-1)_{\underline{n-b-j}}\right)_{1\le i,j\le n-b}\right) \\ &= (-1)^{-c(n-b)}\ T^{\frac{(n-b)(n-b-1)}{2}}\ \det\left((i-c-1)_{\underline{n-b-j}}\right) \end{aligned}$$

$$= \quad (-1)^{-c(n-b)}\, T^{\frac{(n-b)(n-b-1)}{2}} \prod_{j=1}^{n-b} (j-1)!$$

In the last step we made use of the computation of (6.66). So,

$$\det(\tilde{A}) = (-1)^{b(n-1)-c(n-b)}\, T^{\frac{c(c-1)}{2}+\frac{(n-b)(n-b-1)}{2}} \prod_{j=1}^{c}(j-1)! \prod_{j=1}^{b}(j-1)! \prod_{j=1}^{n-b}(j-1)!. \tag{6.70}$$

Conclusion

To conclude this section, we combine all the results (6.36) and (6.37), giving:

$$\begin{aligned}
&\dim_t V_{(\bar{b}^c);(b^c)} \overset{(6.28)}{=} (1+t)^{mn} t^{rn} \det(C) \\
&= \pm t^{-\frac{b(b+1)}{2}-b(n+c-1)} T^{-(n+c)(n-b)-\frac{(m-c)(m-c-1)}{2}} \\
&\quad \times \frac{((b+c)!)^c \prod_{j=1}^{n}(j+c-1)!}{\prod_{i=1}^{m-c}(i+c-1)! \prod_{j=1}^{n}(j-1)! \prod_{j=1}^{c}(b+j-1)! \prod_{i=1}^{b}(i+c-1)!} \\
&\quad \times \det(A(b,c,n))\Big|_{\substack{R_j = c,\, j=1,\ldots,c,\ C=c \\ A_i = 0,\, i=1,\ldots,b}} .
\end{aligned}$$

Herein,

$$\begin{aligned}
\det(A(b,c,n)) &\overset{(6.42)}{=} \prod_{p=2}^{c}\prod_{q=1}^{p-1}(R_p+b+q)\det(\overline{A}(b,c,n)) \\
&\overset{(6.60)}{=} \prod_{p=2}^{c}\prod_{q=1}^{p-1}(R_p+b+q) \prod_{p=1}^{n-b}\prod_{q=1}^{c}(C+b+p+q-1) t^{\frac{b(b-1)}{2}} \det(\tilde{A}(b,c,n) \\
&\overset{(6.70)}{=} \prod_{p=2}^{c}\prod_{q=1}^{p-1}(R_p+b+q) \prod_{p=1}^{n-b}\prod_{q=1}^{c}(C+b+p+q-1) t^{\frac{b(b-1)}{2}} \\
&\quad \times \prod_{j=1}^{c}(j-1)! \prod_{j=1}^{b}(j-1)! \prod_{j=1}^{n-b}(j-1)!\ (1+t)^{\frac{c(c-1)}{2}+\frac{(n-b)(n-b-1)}{2}}
\end{aligned}$$

The power of t is

$$\frac{b(b-1)}{2} - \frac{b(b+1)}{2} - b(n+c-1) + bn = -b\,c$$

Taking into account that $m = n - b + 2c$, the power of $(1+t)$ is given by:

$$mn + \frac{c(c-1)}{2} + \frac{(n-b)(n-b-1)}{2} - (n-b)(n+c) - \frac{(m-c)(m-c-1)}{2} = 2cb$$

The remaining coefficient is equal to

$$\frac{((b+c)!)^c \prod_{j=1}^{n}(j+c-1)! \prod_{q=1}^{c}(c+b+q)_{n-b} \prod_{q=1}^{c-1}(c+b+q)^{c-q}}{\prod_{i=1}^{m-c}(i+c-1)! \prod_{j=1}^{n}(j-1)! \prod_{j=1}^{c}(b+j-1)! \prod_{i=1}^{b}(i+c-1)!}$$
$$\times \prod_{j=1}^{c}(j-1)! \prod_{j=1}^{b}(j-1)! \prod_{j=1}^{n-b}(j-1)!$$
$$= \quad ((b+c)!)^c \frac{\prod_{j=0}^{n+c-1}(j+c)! \prod_{j=1}^{c-1}(c+b+j)^{c-j} \prod_{j=0}^{c-1} j! \prod_{j=0}^{b-1} j! \prod_{j=0}^{n-b-1} j!}{\prod_{i=0}^{m-c-1}(i+c)! \prod_{i=0}^{b-1}(i+c)! \prod_{j=0}^{n-1} j! \prod_{j=0}^{2c-1}(b+j)!}$$

This proves Theorem 6.15.

Comparing this with Proposition 6.14, we obtain a closed form expression for determinants of the type (6.27) where $\bar{\nu};\mu = (\overline{b}^c);(b^c)$. This yields, using the ${}_2F_1$ notation and Theorem 6.15, that:

Proposition 6.24 *Let b, c and n be positive numbers, with $m = n - b + 2c$. Let*

$$A = \left(\binom{m+b+c-l-i}{b+c-l-i+1} {}_2F_1 \left(\begin{matrix} l+i-b-c-1, -n \\ i+l-m-b-c \end{matrix} ; -\frac{1}{t} \right) \right)_{\substack{1\le i\le 2c,\\ 1\le l\le c}},$$

$$B = \left(\binom{m+b-c-j+i-1}{b-c-j+i} {}_2F_1 \left(\begin{matrix} c+j-b-i, -n \\ c+j-m-b-i+1 \end{matrix} ; -t \right) \right)_{\substack{1\le i\le 2c,\\ 1\le j\le c}},$$

where i, j resp. l runs from top to bottom, from left to right, resp. from right to left. Then the determinantal identity is given by

$$\begin{aligned}
&\det(A \mid B) \\
&= \pm((b+c)!)^c \frac{\prod_{j=0}^{n+c-1}(j+c)! \prod_{j=1}^{c-1}(c+b+j)^{c-j} \prod_{j=0}^{c-1} j! \prod_{j=0}^{b-1} j! \prod_{j=0}^{n-b-1} j!}{\prod_{i=0}^{m-c-1}(i+c)! \prod_{i=0}^{b-1}(i+c)! \prod_{j=0}^{n-1} j! \prod_{j=0}^{2c-1}(b+j)!} \frac{(1+t)^{2cb}}{t^{bc}}.
\end{aligned}$$

Appendix A

Nederlandstalige samenvatting

In deze appendix is het ons doel om in het Nederlands een overzicht te geven van de in dit proefschrift gegeven resultaten. Het is zeker niet de bedoeling een Nederlandstalige vertaling te geven. Wel willen we hier de belangrijkste resultaten schetsen in hun context. Hiertoe zullen we de hoofdstukken doorlopen in chronologische volgorde.

A.1 Symmetrische Schur-functies

De objecten, de symmetrische en supersymmetrische functies, die we behandelen in dit proefschrift worden gekarakteriseerd door partities en samengestelde partities. Het eerste hoofdstuk vormt bijgevolg de basis waarop de andere hoofdstukken steunen; hier worden de notaties en terminologie vastgelegd.

A.1.1 Partities en samengestelde partities

Een PARTITIE λ van een niet-negatief geheel getal N, is een rij $\lambda = (\lambda_1, \lambda_2, \ldots)$ van niet-negatieve gehele getallen die voldoen aan $\lambda_1 \geq \lambda_2 \geq \ldots$ en met enkel een eindig aantal λ_i's verschillend van nul. Het aantal niet-nul delen van de partitie, noemt men de LENGTE van de partitie nl. $\ell(\lambda)$. De som van alle delen $|\lambda|$ is het GEWICHT van λ en is gelijk aan N. Partities worden voorgesteld door middel van hun YOUNG DIAGRAM F^λ.

Veronderstel dat λ, μ twee partities zijn, dan bedoelen we met $\lambda \supset \mu$ dat het diagram van μ bevat is in het diagram van λ, nl. $\mu_i \leq \lambda_i$ for all $i \geq 1$. Het verschil $\theta = \lambda - \mu$ noemen we een SCHEEF DIAGRAM.

Een (kolom strikt) TABLEAU T is een rij van partities

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(r)} = \lambda$$

zodat elk scheef diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ $(1 \leq i \leq r)$ een horizontale strip is; dit betekent dat elke kolom van het scheef diagram hoogstens één box bevat. We kunnen een tableau ook grafisch voorstellen als een genummerd diagram waarbij elk vierkant uit het scheef diagram $\theta^{(i)}$ het nummer i krijgt.

Een SAMENGESTELD YOUNG DIAGRAM $F^{\overline{\nu};\mu} = F(\ldots, -\nu_2, -\nu_1; \mu_1, \mu_2, \ldots)$, gekarakteriseerd door twee partities $\mu = (\mu_1, \mu_2, \ldots)$ en $\nu = (\nu_1, \nu_2, \ldots)$, bestaat uit de Young diagrammen F^{μ} en F^{ν}. Deze diagrammen worden naast elkaar geplaatst [18] zoals weergegeven in Figuur A.1. Merk op dat we in $F^{\overline{\nu};\mu} = F^{(\overline{3},\overline{8});(5,3,1)}$ gebruik maken van de conventie om het minteken boven het getal te plaatsen.

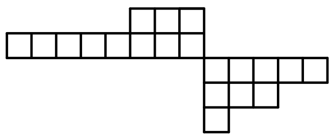


Figure A.1: Composite Young Diagram $F^{\overline{\nu};\mu} = F^{(\overline{3},\overline{8});(5,3,1)}$

Een SAMENGESTELD YOUNG TABLEAU [39] is een genummerd samengesteld Young diagram. Dit tableau wordt verkregen door de boxen van F^{μ} te vullen met de positieve getallen $M = \{1, 2, \ldots, m\}$ en de boxen van F^{ν} met de negatieve getallen $\overline{M} = \{\overline{1}, \overline{2}, \ldots, \overline{m}\}$, zodanig dat de getallen niet dalen volgens de rijen, en strikt stijgen volgens de kolommen. Bovendien is $r(j) + \overline{r}(\overline{j}) \leq j$ voor $j \in \{1, 2, \ldots, m\}$, met $r(j)$ en $\overline{r}(\overline{j})$ de kleinste rijnummers in F^{μ} resp. F^{ν} die een j resp. $\overline{j}$ bevatten.

A.1.2 De ring van symmetrische functies

We beschouwen de ring $\mathbb{Z}[x_1, \ldots, x_m]$ van veeltermen in m onafhankelijke veranderlijken $x = (x_1, \ldots, x_m)$ met gehele coëfficiënten. De symmetrische groep S_m werkt in op deze ring door de variabelen onderling te permuteren. We kunnen nu verschillende basissen definiëren voor $\Lambda_m = \mathbb{Z}[x_1, \ldots, x_m]^{S_m}$, de deelring van $\mathbb{Z}[x_1, \ldots, x_m]$

invariant onder S_m. Voor een partitie λ geven we hier een opsomming van deze basissen. Voor meer details verwijzen we naar de Engelstalige tekst.

De MONOMIALE symmetrische functies: $m_\lambda(x) = \sum_\alpha x^\alpha$.
De ELEMENTAIRE symmetrische functies: $e_r = \sum_{i_1<\ldots<i_r} x_{i_1}x_{i_2}\ldots x_{i_r}$.
De COMPLETE symmetrische functies: $h_r = \sum_{|\lambda|=r} m_\lambda$.
De MACHTSOMMEN: $p_r = \sum_i x_i^r = m_r(x)$.

A.1.3 Symmetrische Schur-functies

Symmetrische Schur-functies geïndexeerd door een partitie $\boldsymbol{\lambda}$

We nemen opnieuw een set van variabelen $x = (x_1, \ldots, x_m)$ en een partitie $\lambda = (\lambda_1, \lambda_2, \ldots)$ met $\ell(\lambda) \leq m$. De SCHUR-FUNCTIES, vaak ook S-FUNCTIES genoemd, worden gedefinieerd als

$$s_\lambda(x) = s_\lambda(x_1, \ldots, x_m) = \frac{\det(x^{\lambda+\delta})}{\det(x^\delta)},$$

waarbij $\delta = (m-1, m-2, \ldots, 2, 1, 0)$.

Verschillende formules zijn gekend in verband met symmetrische Schur-functies. Onder andere kunnen we deze Schur-functies ook definiëren door middel van tableaus van de vorm λ:

$$s_\lambda(x) = \sum_T x^T \quad \text{met } x^T = \prod_{i=1}^m x_i^{|\theta^{(i)}|},$$

waarbij de som genomen wordt over alle mogelijke tableaus.

Andere belangrijke formules zijn de Jacobi-Trudi-formule en de Nägelsbach-Kostka-formule in termen van de elementaire en complete symmetrische functies [46] :

$$s_\lambda(x) = \det\Big(h_{\lambda_i-i+j}(x)\Big)_{1\leq i,j\leq \ell(\lambda)} = \det\Big(e_{\lambda'_i-i+j}(x)\Big)_{1\leq i,j\leq \ell(\lambda')}.$$

Symmetrische Schur-functies geïndexeerd door een samengestelde partitie $\bar{\nu};\mu$

We vertrekken opnieuw van een set van variabelen $x = (x_1, \ldots, x_m)$. Veronderstel dat $\bar{\nu};\mu$ een samengestelde partitie is met $\ell(\mu) = p$, $\ell(\nu) = q$ en $p+q \leq m$; we noemen $\bar{\nu};\mu$ dan een m-STANDAARD SAMENGESTELDE PARTITIE. De symmetrische Schur-functie geïndexeerd door een samengestelde partitie kan gekoppeld worden aan de definitie van een symmetrische functie gekarakteriseerd door een partitie λ nl.:

$$s_{\bar{\nu};\mu}(x) = \left(\prod_{i=1}^{m} x_i^{-\nu_1}\right) s_\lambda(x) \quad \text{met } \lambda = (\mu_1+\nu_1, \mu_2+\nu_1, \ldots, -\nu_2+\nu_1, 0).$$

Volgende formules werden gepostuleerd door Balantekin and Bars [5] en bewezen in [18], namelijk:

$$s_{\overline{\nu};\mu}(x) = \det \left(\begin{array}{c|c} \dot{e}_{\nu_l'+k-l}(x) & e_{\mu_j'-k+j-1}(x) \\ \hline \dot{e}_{\nu_l'-i+l-1}(x) & e_{\mu_j'+i-j}(x) \end{array} \right) \tag{A.1}$$

$$s_{\overline{\nu};\mu}(x) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l+k-l}(x) & h_{\mu_j-k+j-1}(x) \\ \hline \dot{h}_{\nu_l-i+l-1}(x) & h_{\mu_j+i-j}(x) \end{array} \right) \tag{A.2}$$

waarbij de indices i, j, k respectievelijk l lopen van boven naar onder, van links naar rechts, van onder naar boven en van rechts naar links. Verder definiëren we de functies met een punt als volgt: $\dot{e}_r(x) = e_r(\overline{x}) = e_r(\frac{1}{x_1}, \ldots, \frac{1}{x_m})$ en $\dot{h}_r(x) = h_r(\overline{x})$.

Gebruikmakend van de samengestelde Young tableaus, kunnen we de symmetrische Schur-functies, geïndexeerd door een samengestelde partitie, berekenen als:

$$s_{\overline{\nu};\mu} = \sum_{T^{\overline{\nu};\mu}} x^{T^{\overline{\nu};\mu}},$$

waarbij de som genomen wordt over alle mogelijke samengestelde Young-tableaus.

A.2 Supersymmetrische Schur-functies

In Hoofdstuk 2 willen we de theorie van de symmetrische functies vertalen naar de supersymmetrische functies; functies en formules uit Hoofdstuk 1 krijgen een supersymmetrische tegenhanger.

A.2.1 Supersymmetrische functies geïndexeerd door λ

Neem twee verzamelingen $x = (x_1, x_2, \ldots, x_m)$ en $y = (y_1, y_2, \ldots, y_n)$ van onafhankelijke veranderlijken. Een veelterm $p(x/y)$, symmetrisch in x en y, noemen we SUPERSYMMETRISCH als na substitutie van $x_m = t$ en $y_n = -t$ in $p(x/y)$ het polynoom onafhankelijk wordt van t. Deze supersymmetrische functies vormen een deelring $\Lambda_{m|n}$ van $\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n]^{S_m \times S_n}$.

De ELEMENTAIRE SUPERSYMMETRISCHE FUNCTIES en de COMPLETE SUPERSYMMETRISCHE FUNCTIES worden gedefinieerd in termen van de elementaire en complete symmetrische functies [46]:

$$e_r(x/y) = \sum_{k=0}^{r} e_k(x) h_{r-k}(y) \quad \text{en} \quad h_r(x/y) = \sum_{k=0}^{r} h_k(x) e_{r-k}(y).$$

Gegeven de elementaire en complete supersymmetrische functies kunnen we de SUPERSYMMETRISCHE SCHUR-FUNCTIES $s_\lambda(x/y)$ definiëren als:

$$s_\lambda(x/y) = \det\Big(e_{\lambda'_i - i + j}(x/y)\Big)_{1 \le i,j \le \ell(\lambda')}. \tag{A.3}$$

Bovendien kunnen we ook aantonen dat:

$$s_\lambda(x/y) = \det\Big(h_{\lambda_i - i + j}(x/y)\Big)_{1 \le i,j \le \ell(\lambda')}. \tag{A.4}$$

Macdonald toont aan dat de supersymmetrische Schur-functies voldoen aan de volgende vier eigenschappen.

- **Homogeniteit:** $s_\lambda(x^{(m)}/y^{(n)})$ is een homogene functie van graad $|\lambda|$.
- **Restrictie:** Als we $x_m = 0$ (resp. $y_n = 0$) stellen in $s_\lambda(x^{(m)}/y^{(n)})$ dan krijgen we de functie $s_\lambda(x^{(m-1)}/y^{(n)})$ (resp. $s_\lambda(x^{(m)}/y^{(n-1)})$).
- **Annulering:** Na substitutie van $x_m = t$ en $y_n = -t$ wordt $s_\lambda(x^{(m)}/y^{(n)})$ de functie $s_\lambda(x^{(m-1)}/y^{(n-1)})$.
- **Ontbinden in factoren:** Indien de partitie λ zodanig is dat $\lambda_m \ge n \ge \lambda_{m+1}$, dan kunnen we λ schrijven als $\lambda = ((n^m) + \tau) \cup \eta$ met τ (resp. η) een partitie met lengte $\le m$ (resp. $\le n$). De bijhorende S-functie kan dan geschreven worden als

$$s_\lambda(x^{(m)}/y^{(n)}) = s_\tau(x^{(m)}) s_{\eta'}(y^{(n)}) \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j). \tag{A.5}$$

Formule (A.5) werd afgeleid door Berele and Regev [10, Theorem 6.20].

Een SUPERTABLEAU of BITABLEAU S van het type $(m|n)$ en vorm $\lambda - \mu$ [46, §I.5, Exercise 23] is een rij van partities

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(m+n)} = \lambda$$

zodanig dat elke scheef diagram $\theta^{(i)} = \lambda^{(i)} - \lambda^{(i-1)}$ een horizontale strip is voor $1 \leq i \leq m$ en een verticale strip voor $m+1 \leq i \leq m+n$. Grafisch betekent dit dat elk vierkant van $\theta^{(i)}$ kan opgevuld worden met een symbool i, $1 \leq i \leq m$, en elk vierkant van $\theta^{(m+j)}$ met een symbool j', $1 \leq j \leq n$ met $1 < \ldots < m < 1' < \ldots < n'$; de symbolen $i \in \{1, \ldots, m\}$ (respectievelijk $j' \in \{1, \ldots, n'\}$) zijn niet-dalend volgens elke rij (resp. kolom); elke rij bevat hoogstens één symbool j' en elke kolom hoogstens één symbool i.

De supersymmetrische Schur-functies kunnen vervolgens bepaald worden aan de hand van deze supertableaus:

$$s_{\lambda/\mu}(x/y) = \sum_{S} (x/y)^S$$

waarbij gesommeerd wordt over alle supertableaus S van het type $(m|n)$ en vorm $\lambda - \mu$.

A.2.2 Supersymmetrische basissen

Naar analogie met de overeenkomsten tussen elementaire en complete symmetrische functies enerzijds en hun supersymmetrische varianten anderzijds, willen we hier eveneens het supersymmetrische equivalent van de symmetrische machtsfuncties, de monomiale en de 'vergeten' symmetrische functies definiëren.

De supersymmetrische machtsfuncties kunnen eenvoudig bepaald worden als:

$$p_r(x/y) = p_r(x) + (-1)^{r-1} p_r(y).$$

Stel λ is een willekeurige partitie. Dan hebben we de MONOMIALE SUPERSYMMETRISCHE FUNCTIES $m_\lambda(x/y)$ en de VERGETEN SUPERSYMMETRISCHE FUNCTIES $f_\lambda(x/y)$ als volgt gedefinieerd:

$$m_\lambda(x/y) = \sum_{\mu \cup \nu = \lambda} m_\mu(x) f_\nu(y)$$

and

$$f_\lambda(x/y) = \omega(m_\lambda(x/y)) = \sum_{\mu \cup \nu = \lambda} f_\mu(x) m_\nu(y).$$

Voor meer informatie over ω en de vergeten symmetrische functies verwijzen we naar de Engelstalige tekst. Verder, hebben we ook aangetoond dat al deze functies inderdaad supersymmetrisch zijn en dat de transitiematrices, de matrices die de basisovergangen weergeven, onveranderd blijven ten opzichte van het symmetrische equivalent.

A.2.3 Supersymmetrische Schur-functies geïndexeerd door $\bar{\nu};\mu$

Stel $\bar{\nu};\mu$ is een samengestelde partitie. De corresponderende supersymmetrische Schur-functie, ook supersymmetrische S-functie [4, 5], wordt gedefinieerd als

$$s_{\bar{\nu};\mu}(x/y) = \det \left(\begin{array}{c|c} \dot{h}_{\nu_l+k-l}(x/y) & h_{\mu_j-k-j+1}(x/y) \\ \hline \dot{h}_{\nu_l-i-l+1}(x/y) & h_{\mu_j+i-j}(x/y) \end{array} \right) \tag{A.6}$$

waarbij i, j, k resp. l van boven naar onder, van links naar rechts, van onder naar boven, respectievelijk van rechts naar links lopen. Als $\nu = 0$ vinden we de gewone supersymmetrische S-functies terug. Door de analogie met (A.2) is het evident dat vele eigenschappen en formules gelijkaardig zullen zijn aan de eigenschappen en formules gevonden voor gewone symmetrische Schur-functies [17, 18, 28, 29, 41].

Definitie A.1 *Een samengestelde partitie noemen we een* $(m|n)$-STANDAARD SAMENGESTELDE PARTITIE *als en slechts als er een* J *en* L *bestaan zodat*

$$\begin{aligned} J &= \min\{j \,|\, \mu'_{j+1} + \nu'_{n-j+1} \leq m\} \quad \textit{met} \quad 0 \leq J \leq n, \\ L &= \min\{l \,|\, \mu_{m-l+1} + \nu_{l+1} \leq n\} \quad \textit{met} \quad 0 \leq L \leq m, \end{aligned}$$

We stellen dan $I = m - L$ *en* $K = n - J$.

De grafische weergave van deze definitie bepaalt dat een samengestelde partitie $(m|n)$-standaard is als en slechts haar samengesteld Young diagram in een kruis past met armbreedte m en beenbreedte n [17, 40].

Ook voor samengestelde supersymmetrische Schur-functies hebben we een formule door middel van $(m|n)$-standaard samengestelde supertableaus. Vooraleer deze formule hier aan te halen, leggen we nog enkele notaties vast. Stel $n(i^*)$ resp. $\overline{n}(\overline{i^*})$ is het aantal vierkantjes in F^μ resp. F^ν dat i^* resp. $\overline{i^*}$ bevat; $r(i)$ resp. $R(i^*)$ ($\overline{r}(\overline{i}$ resp. $\overline{R}(\overline{i^*})$) is het kleinste resp. grootste rijnummer van de rijen in F^μ (resp. F^ν) die een i^* (resp. $\overline{i^*}$) bevatten; $c(i^*)$ resp. $\overline{c}(\overline{i^*})$ is het grootste kolomnummer van de kolommen van F^μ resp. F^ν die een i^* resp. $\overline{i^*}$ bevatten.

Een $(m|n)$-STANDAARD SAMENGESTELD SUPERTABLEAU $S^{\bar{\nu};\mu}$ is een genummerd samengesteld Young diagram $F^{\bar{\nu};\mu}$, niet noodzakelijk een $(m|n)$-standaard samengesteld diagram, verkregen door positieve en negatieve symbolen gekozen uit $M = \{1, \ldots, m, t^*, \ldots, 1^*, 1', \ldots, n'\}$ en $\overline{M} = \{\overline{1}, \ldots, \overline{m}, \overline{t^*}, \ldots, \overline{1^*}, \overline{1'}, \ldots, \overline{n'}\}$ $(t \geq 0)$ te plaatsen in elk vierkant van F^μ resp. $F^{\bar{\nu}}$ zodanig dat volgende voorwaarden voldaan zijn voor de geordende symbolen $1 < \ldots < m < t^* < \ldots < 1^* < 1' < \ldots < n'$:

- De symbolen $i \in \{1, 2, \ldots, m\}$ en $\bar{i} \in \{\bar{1}, \bar{2}, \ldots, \overline{m}\}$ zijn niet-dalend volgens de rijen en stijgen strikt volgens de kolommmen; de symbolen $j' \in \{1', 2', \ldots, m'\}$ en $\overline{j'} \in \{\overline{1'}, \overline{2'}, \ldots, \overline{n'}\}$, zijn strikt stijgend volgens de rijen en niet-dalend volgens de kolommen; boxen gevuld met een symbool i^* resp. $\overline{i^*}$, $1 \leq i \leq t$, vormen een samenhangende component startend in de eerste kolom van F^μ resp. $F^{\bar{\nu}}$.
- $r(i) + \overline{r}(\bar{i}) \leq i$ voor alle $i \in \{1, 2, \ldots, m\}$.
- $n(i^*) = \overline{n}(\overline{i^*}) = R(i^*) + \overline{R}(\overline{i^*}) - m - 1$ en $\overline{R}(\overline{i^*}) = m + 1 + c(i^*) - r(i^*)$.

Met elk samengesteld supertableau correspondeert een rationale term in de variabelen $x = (x_1, \ldots, x_m)$ en $y = (y_1, \ldots, y_n)$. De symbolen i resp. $\bar{i}$ zijn de positieve resp. negatieve machten van x_i, de symbolen met een accent zijn de machten van y_j, de overige symbolen worden niet in rekening gebracht, immers $|i^*| = |\overline{i^*}|$.

De supersymmetrische Schur-functie wordt dan gegeven door:

$$s_{\bar{\nu};\mu}(x^{(m)}/y^{(n)}) = \sum_{S^{\bar{\nu};\mu}} (-1)^{(\sum_{i=1}^{t}(c(i^*)+\overline{c}(\overline{i^*})+1))(1-\delta_{0t})}(x/y)^{S^{\bar{\nu};\mu}}.$$

De som wordt genomen over alle $(m|n)$-standaard samengestelde supertableaus van vorm $\bar{\nu};\mu$ en met δ_{0t} de Kronecker-delta.

Voor alle andere eigenschappen, rekenregels en formules in verband met symmetrische en supersymmetrische functies verwijzen we naar de Engelstalige tekst.

A.3 Lie superalgebra's en representaties

Vele pogingen zijn reeds ondernomen om relativiteitstheorie consistent te maken met quantumveldentheorie. In de meeste succesvolle pogingen is men moeten overstappen van gewone symmetrie naar modellen die gebruik maken van supersymmetrie. In de beschrijving van deze 'super'-symmetrie deed de theorie van Lie-superalgebra's haar intrede. Het grote knelpunt in de studie van Lie-superalgebra's is het feit dat de representatietheorie van Lie superalgebra's geen rechtstreekse vertaling is van de corresponderende theorie, gekend voor de simpele Lie-algebra's.

A.3.1 Definities

Een Lie-superalgebra wordt als volgt gedefinieerd:

Definitie A.2 *Een* LIE-SUPERALGEBRA $\mathfrak{g}$ *is een* $\mathbb{Z}_2$*-gegradeerde algebra over een veld van karakteristiek 0 samen met een bilineaire bewerking* $[.,.]$ *van* $\mathfrak{g}\times\mathfrak{g}$ *in* $\mathfrak{g}$*, het* LIE SUPERHAAKJE *of* SUPERCOMMUTATOR *genoemd, zodat* $\forall a \in \mathfrak{g}_\alpha, \forall b \in \mathfrak{g}_\beta, \forall c \in \mathfrak{g}$*, en* $\forall \alpha, \beta \in \mathbb{Z}_2$*:*

$$[a,b] \in \mathfrak{g}_{\alpha+\beta} \tag{A.7a}$$

$$[a,b] = -(-1)^{\alpha\beta}[b,a] \tag{A.7b}$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]. \tag{A.7c}$$

Vergelijking (A.7b) geeft aan dat het Lie-superhaakje "supersymmetrisch" is; (A.7c) noemen we de "super Jacobi-identiteit".

In dit proefschrift beperken we ons tot de Lie-superalgebra's $\mathfrak{gl}(m|n)$ en $\mathfrak{sl}(m|n)$.

Een deelalgebra $\mathfrak{h}$ van een Lie-algebra $\mathfrak{g}$ noemen we een CARTAN-DEELALGEBRA als de deelalgebra nilpotent is en gelijk is aan zijn normalisator. De Cartan-deelalgebra van een Lie-superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is de Cartan-deelalgebra, de maximale abelse deelalgebra, van $\mathfrak{g}_{\bar{0}}$.

Voor $\mathfrak{g} = \mathfrak{gl}(m|n)$ kan elk element van de duale ruimte $\mathfrak{h}^*$ van de Cartan subalgebra $\mathfrak{h}$ kan beschreven worden aan de hand van de $\epsilon\delta$-basis van $\mathfrak{h}^*$, gegeven door $\{\epsilon_i \mid 1 \leq i \leq m\} \cup \{\delta_j \mid 1 \leq j \leq n\}$. Een gewicht $\Lambda \in \mathfrak{h}^*$ kunnen we zodoende schrijven als

$$\Lambda = \lambda_1\epsilon_1 + \cdots + \lambda_m\epsilon_m + \mu_1\delta_1 + \cdots + \mu_n\delta_n, \text{ of nog } \Lambda = (\lambda_1, \ldots, \lambda_m; \mu_1, \ldots, \mu_n).$$

Zo een gewicht Λ wordt integraal dominant genoemd als en slechts als $\lambda_i \in \mathbb{Z},\ \mu_j \in \mathbb{Z}$ en als $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ en $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$.

De verzameling $\Delta = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$ met $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x,\ h \in \mathfrak{h}\}$, noemen we een WORTELSYSTEEM. In de $\epsilon\delta$-basis worden de EVEN WORTELS van $\mathfrak{g} = \mathfrak{gl}(m|n)$ gegeven door $\epsilon_i - \epsilon_j$ or $\delta_i - \delta_j$, en de ONEVEN WORTELS door $\pm(\epsilon_i - \delta_j)$. De verzamelingen van de POSITIEVE even en oneven wortels worden aangeduid met

$$\begin{aligned}\Delta_{\bar{0}}^+ &= \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq m\} \cup \{\delta_i - \delta_j | 1 \leq i < j \leq n\},\\ \Delta_{\bar{1}}^+ &= \{\beta_{ij} = \epsilon_i - \delta_j | 1 \leq i \leq m,\ 1 \leq j \leq n\}.\end{aligned}$$

Een positieve wortel wordt SIMPEL genoemd als hij niet kan geschreven worden als som van andere positieve wortels. Er zijn verschillende keuzemogelijkheden voor een set van simpele wortels. Als systeem van SIMPELE WORTELS nemen we altijd de standaard keuze [16]

$$\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}.$$

In dit systeem van simpele wortels is er slechts één oneven wortel. Verder stellen we, zoals gebruikelijk,

$$\rho_0 = \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{0,+}} \alpha\Big), \qquad \rho_1 = \frac{1}{2}\Big(\sum_{\alpha\in\Delta_{1,+}} \alpha\Big), \qquad \rho = \rho_0 - \rho_1.$$

Tot slot van deze paragraaf, voeren we nog het begrip representatie en hoogste gewicht in.

Definitie A.3 *Stel* $\mathfrak{g}$ *is* $\mathfrak{gl}(m|n)$ *of* $\mathfrak{sl}(m|n)$*. Veronderstel verder dat* $V = V_{\bar 0} \oplus V_{\bar 1}$ *een* $\mathbb{Z}_2$*-gegradeerde vectorruimte is en beschouw de superalgebra van endomorfismen* $\mathrm{End}(V) = \mathrm{End}_{\bar 0}(V) \oplus \mathrm{End}_{\bar 1}(V)$*. Een lineaire* REPRESENTATIE φ *van* $\mathfrak{g}$ *is een homomorfisme van* $\mathfrak{g}$ *in* $\mathrm{End}(V)$*.*

De dimensie (resp. de superdimensie) van een representation φ is de dimensie (resp. gegradeerde dimensie) van de vectorruimte V:

$$\begin{aligned} \dim\varphi &= \dim V_{\bar 0} + \dim V_{\bar 1} \\ \mathrm{sdim}\,\varphi &= \dim V_{\bar 0} - \dim V_{\bar 1}. \end{aligned}$$

We fixeren de Borel-deelalgebra $\mathfrak{b}$ van $\mathfrak{g}$ die $\mathfrak{h}$ bevat. Het geïnduceerde moduul $\bar V(\Lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\langle v_\Lambda\rangle)$, voor een willekeurig integraal dominant gewicht $\Lambda \in \mathfrak{h}^*$ noemt men een KAC MODUUL. Een dergelijk $\mathfrak{g}$-moduul bevat een uniek maximaal deelmoduul $I(\Lambda)$. Het quotiënt van beiden, genoteerd als

$$V(\Lambda) = \bar V(\Lambda)/I(\Lambda),$$

noemt men een ENKELVOUDIG HOOGSTE GEWICHT MODUUL of een IRREDUCIEBELE REPRESENTATIE MET HOOGSTE GEWICHT Λ.

A.3.2 Hoogste gewichten versus samengestelde partities

Vaak hebben we de voorstelling van een hoogste gewicht nodig door middel van een samengesteld Young diagram. De bijectie tussen een integraal dominant gewicht Λ en een $(m|n)$-standaard samengestelde partitie $\bar\nu;\mu$ wordt gegeven door volgend lemma.

Lemma A.4 *Stel Λ is een $\mathfrak{gl}(m|n)$ integraal dominant gewicht. Stel JL is de verzameling van alle (j,l) met $0 \leq j \leq n$ en $0 \leq l \leq m$ zodat*

$$\begin{array}{rlll} \textit{ofwel} & j=0 & \textit{ofwel} & \Lambda_{m+j} > -l, \\ \textit{en ofwel} & j=n & \textit{ofwel} & \Lambda_{m+j+1} \leq -l, \\ \textit{en ofwel} & l=m & \textit{ofwel} & \Lambda_{m-l} \geq j, \\ \textit{en ofwel} & l=0 & \textit{ofwel} & \Lambda_{m-l+1} < j. \end{array}$$

Veronderstel vervolgens dat $J = \min\{j \,|\, (j,l) \in JL\}$ en $L = \min\{l \,|\, (J,l) \in JL\}$. De corresponderende samengestelde partitie $\bar{\nu};\mu$ is $(m|n)$-standaard, met

$$\begin{array}{llll} \mu_i = \Lambda_i & \textit{voor } i = 1,2,\ldots,I = m-L & \textit{en } \mu_i \leq J & \textit{voor } i > I, \\ \mu'_j = m + \Lambda_{m+j} & \textit{voor } j = 1,2,\ldots,J & \textit{en } \mu'_j \leq I & \textit{voor } j > J, \\ \nu'_k = -\Lambda_{m+n-k+1} & \textit{voor } k = 1,2,\ldots,K = n-J & \textit{en } \nu'_k \leq L & \textit{voor } k > K, \\ \nu_l = n - \Lambda_{m-l+1} & \textit{voor } l = 1,2,\ldots,L & \textit{en } \nu_l \leq K & \textit{voor } l > L. \end{array}$$

Anderzijds, als $\bar{\nu};\mu$ een $(m|n)$-standaard samengestelde partitie is met I,J,K,L gedefinieerd in Definitie A.1, dan wordt het corresponderende $\mathfrak{gl}(m|n)$ integraal dominant gewicht Λ gegeven door

$$\Lambda = (\mu_1, \mu_2, \ldots, \mu_I, n-\nu_L, \ldots, n-\nu_2, n-\nu_1 \,| \\ \mu'_1 - m, \mu'_2 - m, \ldots, \mu'_J - m, -\nu'_K, \ldots, -\nu'_2, -\nu'_1).$$

Om wille van de connectie tussen $\bar{\nu};\mu$ en Λ noteren we $\Lambda = \Lambda_{\bar{\nu};\mu}$.

A.3.3 Covariante, contravariante en gemengde tensormodulen

Berele en Regev [10], en Sergeev [61], toonden aan dat het tensorproduct van N kopieen van de natuurlijke $(m+n)$-dimensionale representatie $V = \mathbb{C}^{m+n}$ van $\mathfrak{g} = \mathfrak{gl}(m|n)$ compleet reduceert. De irreduciebele componenten worden gelabeled door een partitie λ van N. Deze representaties noemt men de COVARIANTE MODULEN. Analoog toonden ze aan dat het tensorproduct van N kopieën van de duale V^* aanleiding geeft tot de CONTRAVARIANTE MODULEN. De GEMENGDE TENSORMODULEN verschijnen als irreduciebele componenten in het tensorproduct van M kopieën van V en N kopieën van de duale V^* van V. Merk op dat het tensorproduct in het algemeen niet volledig reduceerbaar is.

A.3.4 Atypicaliteit en de atypicaliteitsmatrix van Λ

De symmetrische vorm $(\ ,\ )$ in $\mathfrak{h}^*$, geïnduceerd door de invariante symmetrische vorm in $\mathfrak{g}$ neemt volgende vorm aan in termen van de $\epsilon\delta$-basis: $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ en $(\delta_i, \delta_j) = -\delta_{ij}$, met δ_{ij} het Kronecker symbool.

Om de atypicaliteit van een representatie met hoogste gewicht Λ te bepalen, berekenen we alle getallen $(\Lambda+\rho, \beta_{ij})$, met $\beta_{ij} = \epsilon_i - \delta_j$, voor $1 \leq i \leq m$ en $1 \leq j \leq n$ en we plaatsen ze in een $(m \times n)$-matrix, de ATYPICALITEITSMATRIX $A(\Lambda)$ [70, 71]. Indien $(\Lambda+\rho, \alpha) \neq 0$ dan wordt de representatie TYPISCH genoemd; bestaat er een $\alpha \in \Delta_{\bar{1}}^+$ waarvoor $(\Lambda+\rho, \alpha) = 0$ dan noemen we de representatie V_Λ atypisch. De ATYPICALITEIT VAN Λ, genoteerd als atyp$(\Lambda+\rho) = a$, is het maximaal aantal lineair onafhankelijke wortels β_i zodat $(\beta_i, \beta_j) = 0$ en $(\Lambda+\rho, \beta_i) = 0$ voor alle i en j [34]. Een dergelijke verzameling $\{\beta_i\}_{i=1,\ldots,a}$ noemen we een Λ-MAXIMAAL ISOTROPE DEELVERZAMELING S_Λ.

Eigenschappen van de atypicaliteitsmatrix $A(\Lambda)$ zijn in detail bestudeerd in [26].

A.4 Een determinantformule voor supersymmetrische Schur-veeltermen

In Hoofdstuk 4 hebben we een nieuwe formule afgeleid voor supersymmetrische Schur-veeltermen $s_\lambda(x/y)$. De representatietheorie van de Lie superalgebra $\mathfrak{gl}(m|n)$ en meer bepaald de karakterformule van Kac en Wakimoto ligt aan de oorsprong van deze formule. We gaan hier wat nader op in.

A.4.1 Covariante modulen in $\mathfrak{gl}(m|n)$ zijn tam

De eerste stap in onze zoektocht naar een nieuwe formule voor supersymmetrische functies (geïndexeerd door een partitie λ) is aantonen dat de covariante modulen tam zijn.

We beschouwen opnieuw de standaard keuze voor de verzameling van simpele wortels Π en het hoogste gewicht Λ_λ van V_λ in de $\epsilon\delta$-basis van $\mathfrak{h}^*$. Uit de studie van atypicaliteitsmatrices [71], volgt er dat de $(\Lambda_\lambda+\rho)$-maximale isotrope deelverzameling gegeven wordt door:

$$S_{\Lambda_\lambda} = \{\beta_{i,\lambda_i+m+1-i} | 1 \leq i \leq m,\ 1 \leq \lambda_i + m + 1 - i \leq n\}.$$

Indien we S_Λ zodanig kunnen kiezen dat $S_\Lambda \subset \Pi \subset \Delta^+$ dan wordt het $\mathfrak{g}$-moduul V TAM genoemd. Voor deze representaties is een karakterformule gekend, te danken aan Kac en Wakimoto [34]. Het is echter duidelijk dat in het algemene geval S_{Λ_λ} geen deelverzameling is van Π, aangezien Π slechts één oneven wortel bevat. Met behulp van een rij van simpele reflecties rond oneven wortels, kunnen we echter overgaan op een nieuw systeem van simpele wortels Π'. In de nieuwe $\Delta^{+\prime}$, waarbij het moduul V_λ

als hoogste gewicht nu Λ' heeft, is de $(\Lambda'+\rho')$-maximaal isotrope deelverzameling $S_{\Lambda'}$ nu wel zodanig dat $S_{\Lambda'} \subset \Pi' \subset \Delta^{+\prime}$. Bij elke stap in de rij van reflecties houden we rekening met volgende eigenschappen

$$\begin{aligned} \Lambda'+\rho' &= \Lambda+\rho \ \text{ als } \ (\Lambda+\rho,\alpha)\neq 0, \\ \Lambda'+\rho' &= \Lambda+\rho+\alpha \ \text{ als } \ (\Lambda+\rho,\alpha)=0. \end{aligned} \tag{A.8}$$

De rij van simpele wortels ligt volledig vast door wat we de (m,n)-index noemen van een partitie. Dit getal k speelt een cruciale rol in dit proefschrift.

Definitie A.5 *Stel $\lambda \in \mathcal{H}_{m,n}$, dan wordt de (m,n)-*INDEX VAN *λ gegeven door*

$$k = \min\{i|\lambda_i + m + 1 - i \leq n\}, \qquad (1 \leq k \leq m+1).$$

De rij van spiegelingen wordt dan gegeven door volgende oneven simpele wortels in de opgegeven volgorde:

$$\begin{array}{ll} \text{rij } m\ : & \beta_{m,1},\beta_{m,2},\ldots,\beta_{m,\lambda_k-k+m} \\ \text{rij } m-1\ : & \beta_{m-1,1},\beta_{m-1,2},\ldots,\beta_{m-1,\lambda_k-k+m-1} \\ \vdots & \vdots \\ \text{rij } k\ : & \beta_{k,1},\beta_{k,2},\ldots,\beta_{k,\lambda_k} \end{array} \tag{A.9}$$

Bij het voltooien van de rij van simpele reflecties, hebben we het volgende:

Lemma A.6 *De rij (A.9) is een rij van spiegelingen rond simpele oneven wortels voor Λ_λ. Op het einde van de rij hebben we:*

$$\begin{aligned} \Pi' = \{ & \epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \ldots, \epsilon_{k-1}-\delta_1, \delta_1-\delta_2, \delta_2-\delta_3, \ldots, \delta_{\lambda_k-1}-\delta_{\lambda_k}, \\ & \delta_{\lambda_k}-\epsilon_k, \epsilon_k-\delta_{\lambda_k+1}, \delta_{\lambda_k+1}-\epsilon_{k+1}, \epsilon_{k+1}-\delta_{\lambda_k+2}, \ldots, \delta_{\lambda_k+m-k}-\epsilon_m, \\ & \epsilon_m-\delta_{\lambda_k+m+1-k}, \delta_{\lambda_k+m+1-k}-\delta_{\lambda_k+m+2-k}, \ldots, \delta_{n-1}-\delta_n\}. \end{aligned}$$

Bovendien is er ook voldaan aan:

$$\Lambda'+\rho' = \Lambda_\lambda + \rho + \sum_{i=k+1}^{m} \sum_{j=\lambda_i+1}^{\lambda_k-k+i} \beta_{i,j}.$$

Als belangrijk gevolg van Lemma A.6 hebben we volgende eigenschap:

Gevolg A.7 *Elk covariant moduul V_λ is tam.*

Dit betekent dat we nu in staat zijn de karakterformule van Kac en Wakimoto te vertalen naar de covariante modulen. We voeren eerste enkele nieuwe notaties in:

$$D(x) = \prod_{1\le i<j\le m}(x_i - x_j),\; E(x,y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j) \text{ en } D = \frac{D(x)D(y)}{E(x,-y)}.$$

We krijgen uiteindelijk voor $\operatorname{ch} V_\lambda = s_\lambda(x/y)$:

$$\operatorname{ch} V_\lambda = \frac{D^{-1}}{(m-k+1)!}\sum_{w\in S_m\times S_n}\varepsilon(w)w(t_\lambda).$$

waarin

$$t_\lambda = \prod_{i=1}^{k-1} x_i^{\lambda_i+m-i-n}\prod_{j=1}^{l-1} y_j^{\lambda'_j+n-j-m}\prod_{i=k}^{m}\frac{y^r_{l+i-k}}{x_i^r(x_i+y_{l+i-k})}\prod_{j=l+m+1-k}^{n} y_j^{n-j}.$$

A.4.2 Een determinantformule voor $s_\lambda(x/y)$

Stel $\lambda \in \mathcal{H}_{m,n}$; de karakteristieke grootheden verbonden aan de partitie λ worden gegeven door de (m,n)-index k en de daarvan afgeleide grootheden $l = \lambda_k + 1$ en $r = n - m + k - l$. Aangezien $\operatorname{ch} V_\lambda = s_\lambda(x/y)$ kunnen we uit voorgaande paragraaf een nieuwe uitdrukking voor supersymmetrische S-functies afleiden. Deze formule kan in een praktische determinantformule gegoten worden zoals weergegeven in volgende formule:

Stelling A.8 *Stel $\lambda \in \mathcal{H}_{m,n}$ en k de (m,n)-index van λ. Dan is*

$$s_\lambda(x/y) = (-1)^{mn-m+k-1}D^{-1} \times \det\begin{pmatrix} \left(\frac{1}{x_i+y_j}\right)_{\substack{1\le i\le m\\1\le j\le n}} & \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m\\1\le j\le k-1}} \\ \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1\\1\le j\le n}} & 0 \end{pmatrix}. \quad \text{(A.10)}$$

Met behulp van bovenstaande formule hebben we een identiteit van determinanten bewezen die Cauchy's dubbele alternant combineert met de Vandermonde-determinant; dit resultaat krijgen we door λ de nulpartitie te stellen.

A.4.3 Vier karakteriserende eigenschappen

Zoals vermeld in §A.2.1 voldoen supersymmetrische S-veeltermen aan vier karakteriserende eigenschappen. We hebben aangetoond dat het rechterlid van (A.10) aan deze vier eigenschappen voldoet. Dit geeft ons onmiddellijk een onafhankelijk bewijs dat de determinant inderdaad een supersymmetrische Schur-veelterm is.

A.4.4 Een direct en onafhankelijk bewijs

Een derde bewijs is gebaseerd op de formule van Berele-Regev (A.5) en de Laplace ontwikkeling van een determinant.

Hiertoe voeren we een nieuwe notatie in: $x = x' + x''$ is een willekeurige decompositie van $x = (x_1, x_2, \ldots, x_m)$ in twee disjuncte deelverzamelingen met een vaste kardinaliteit; stel $|x'| = p$ en $|x''| = q$ zodat $p + q = m$. Het bewijs steunt onder andere op volgend lemma:

Lemma A.9 *Stel $m = p + q$ en veronderstel dat $\mu = (\mu_1, \ldots, \mu_p)$ een willekeurig p-tal is, en $\nu = (\nu_1, \nu_2, \ldots)$ een willekeurig t-tal (t een willekeurig positief geheel getal). Noem $\lambda = (\mu_1, \ldots, \mu_p, \nu_1, \nu_2, \ldots)$. Dan is,*

$$s_\lambda(x/-y) = \sum_{x'+x''} \frac{s_{\mu+(q^p)}(x'/-y)s_\nu(x''/-y)}{E(x', x'')} \tag{A.11}$$

waar we som wordt genomen over alle mogelijke decompostities $x = x' + x''$ met kardinaliteit van x' gelijk aan p en kardinaliteit van x'' gelijk aan q.

Gebruikmakend van Lemma A.9, een tweevoudige toepassing van (A.5) en een dubbele Laplace ontwikkeling, kunnen we nu volgende stelling bewijzen.

Stelling A.10 *Stel $\lambda \in \mathcal{H}_{m,n}$; k is de (m, n)-index van λ. Dan is*

$$\frac{E(x,y)}{D(x)D(y)} \det \begin{pmatrix} \frac{1}{x-y} & X_\lambda \\ Y_\lambda & 0 \end{pmatrix} = \pm s_\lambda(x/-y), \tag{A.12}$$

waar de (rechthoekige) blokken van de determinant gegeven worden door

$$\frac{1}{x-y} = \left(\frac{1}{x_i - y_j}\right)_{1\le i\le m,\ 1\le j\le n},$$

$$X_\lambda = \left(x_i^{\lambda_j+m-n-j}\right)_{\substack{1\le i\le m,\\ 1\le j\le k-1}}, \qquad Y_\lambda = \left(y_j^{\lambda'_i+n-m-i}\right)_{\substack{1\le i\le n-m+k-1,\\ 1\le j\le n}}.$$

Het minteken is enkel afhankelijk van de partitie λ en is gelijk aan

$$(-1)^{\sum_{i=1}^{n-m+k-1} \lambda'_i + \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - 1}.$$

Tot slot willen we nog even de resultaten van Hoofdstuk 4 op een rijtje zetten. We tonen aan dat covariante modulen tam zijn. De karakterformule van Kac en Wakimoto voor tamme representaties kan dan aangewend worden om een determinantformule op te stellen voor supersymmetrische Schur-functies. Naast deze rechtstreekse methode, geven we nog twee onafhankelijke bewijzen die de equivalentie tussen de gevonden determinant en de supersymmetrische Schur-veeltermen aantoont: een eerste bewijs steunt op de vier karakteriserende eigenschappen gegeven door Macdonald, een tweede onafhankelijk en meer direct bewijs steunt op een dubbele Laplace ontwikkeling en de formule van Berele en Regev.

A.5 Een determinantformule voor samengestelde supersymmetrische S-functies

A.5.1 Inleiding

In Hoofdstuk 5 beschrijven we nieuwe resultaten voor de irreduciebele representaties van de Lie superalgebra $\mathfrak{gl}(m|n)$. De karakters van "typische" representaties zijn gekend dankzij het werk van Kac [32]. Het vinden van een karakterformule voor de overige "atypische" irreduciebele representaties is het onderwerp van intensief onderzoek zowel in de wiskundige als fysische literatuur. In tegenstelling tot de karakters van covariante en contravariante modulen die samenvallen met de supersymmetrische S-functies gekarakteriseerd door een partitie λ, is dit niet meer het geval voor de gemengde tensorrepresentaties [38, 57]. Nochtans is het zowel uit rekenkundig als praktisch opzicht gemakkelijker indien we de karakters kunnen identificeren met deze supersymmetrische S-functies. Hiervoor zijn er immers heel wat rekenregels gekend (zie Hoofdstuk 2). In Hoofdstuk 5 tonen we aan dat er een andere familie van representaties bestaat, naast de covariante en contravariante, waarvoor het karakter samenvalt met een supersymmetrische S-functie, de zogenaamde kritische representaties in $\mathfrak{gl}(m|n)$.

A.5.2 Normaal, kritisch en quasikritisch gerelateerde wortels

Neem een hoogste gewicht $\Lambda_{\bar{\nu};\mu} \in \mathfrak{h}^*$ en de corresponderende samengestelde partitie $\bar{\nu};\mu$. In [26] wordt er onderscheid gemaakt tussen *normaal, kritisch en quasikritisch* gerelateerde wortels van de $(\Lambda_{\bar{\nu};\mu} + \rho)$-isotrope verzameling $S_{\Lambda_{\bar{\nu};\mu}}$. Beschouw de verzameling van oneven wortels $\{\gamma_1, \ldots, \gamma_a\}$ met $\gamma_s = \beta_{i_s,j_s}$ zodanig dat $(\Lambda_{\bar{\nu};\mu} + \rho, \beta_{i_s,j_s}) = 0$ met $j_1 < j_2 < \cdots < j_a$. Bermerk dat de kardinaliteit van $S_{\Lambda_{\bar{\nu};\mu}}$ gelijk is aan de atypicaliteit van $\Lambda_{\bar{\nu};\mu}$, i.e. a =atyp$(\Lambda_{\bar{\nu};\mu} + \rho)$. We noemen x_{pq}, $1 \leq p < q \leq a$, het element in de atypicaliteitsmatrix $A(\Lambda_{\bar{\nu};\mu})$ in de kolom die de nul bevat corresponderend met γ_p en in de rij die de nul bevat corresponderend met γ_q. Met h_{pq} duiden we de hoeklengte aan tussen de verschillende nullen, de positie van de nullen zelf inbegrepen.

Definitie A.11 *Stel Λ is een hoogste gewicht van $\mathfrak{gl}(m|n)$ met atyp$(\Lambda) = a$ en atypicaliteitswortels $\{\gamma_1, \ldots, \gamma_a\}$. Dan geldt er voor elke $1 \leq p < q \leq a$: γ_p en γ_q zijn* NORMAAL GERELATEERD *als en slechts als $x_{pq} + 1 > h_{pq}$; γ_p en γ_q zijn* QUASI-KRITISCH GERELATEERD *als en slechts als $x_{pq} + 1 = h_{pq}$; γ_p en γ_q zijn* KRITISCH GERELATEERD *als en slechts als $x_{pq} + 1 < h_{pq}$.*

Als alle koppel (γ_i, γ_{i+1}) $(i = 1, 2, \ldots, a-1)$ kritisch gerelateerd zijn dan noemen we de corresponderende samengestelde partitie $\bar{\nu}; \mu$, het hoogste gewicht $\Lambda_{\bar{\nu};\mu}$ en de representatie $V_{\Lambda_{\bar{\nu};\mu}} \equiv V_{\bar{\nu};\mu}$ kritisch. Er is een eenvoudige manier om combinatorisch na te gaan of een partitie al dan niet kritisch is:

Eigenschap A.12 *Neem een standaard samengestelde partitie $\bar{\nu}; \mu$ in $\mathfrak{gl}(m|n)$ met atyp$(\Lambda_{\bar{\nu};\mu} + \rho) = a$. Stel $\gamma_s = \beta_{i_s,j_s}$ zodanig dat $(\Lambda_{\bar{\nu};\mu} + \rho, \gamma_s) = 0$ $(s = 1, \ldots, a)$ en*

$$\begin{aligned} \mathcal{M} &= \{\mu_{i_1} + m - i_1, \mu_{i_1-1} + m - i_1 + 1, \ldots, \mu_{i_a} + m - i_a\}, \\ \mathcal{N} &= \{\nu'_{j_1} + n - j_1, \nu'_{j_1-1} + n - j_1 + 1, \ldots, \nu'_{j_a} + n - j_a\}. \end{aligned}$$

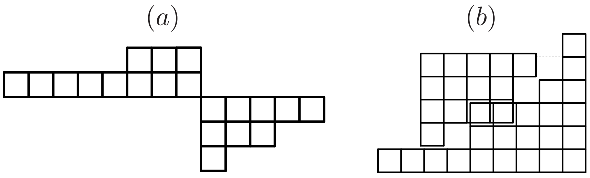
De samengestelde partitie $\bar{\nu}; \mu$ is kritisch in $\mathfrak{gl}(m|n)$ als en slechts als

$$\mathcal{M} \cup \mathcal{N} = \{\mu_{i_1} + m - i_1, \mu_{i_1} + m - i_1 + 1, \ldots, \mu_{i_1} + m - i_a + j_a - j_1 - a + 1\},$$

i.e. als en slechts als $\mathcal{M} \cup \mathcal{N}$ een verzameling is van opeenvolgende gehele getallen.

Op die manier is het eenvoudig na te gaan dat covariante en contravariante representaties altijd kritisch zijn. De klasse van kritische representaties is echter veel groter. In het algemeen zal $\bar{\nu}; \mu$ kritisch zijn als m en n groot genoeg zijn ten opzichte van μ en ν.

In §A.1.1 hebben we reeds een voorstelling ingevoerd voor de samengestelde partities. Soms is het echter gemakkelijker om de partities voor te stellen in een $(m \times n)$-rechthoek. De partitie μ wordt dan in de linkerbovenhoek geplaatst volgens links gealigneerde rijen, de partitie ν is rechts gealigneerd en wordt in de rechterbenedenhoek geplaatst. De twee voorstellingen worden geïllustreerd in (A.13) voor $\bar{\nu};\mu = (\bar{1},\bar{1},\bar{2},\bar{5},\bar{5},\bar{9});(5,4,4,1)$ en $(m|n) = (5|7)$



(A.13)

Algemeen kunnen we voor een willekeurig kritisch hoogste gewicht, een kritische samengestelde partitie $\bar{\nu};\mu$ vinden zodat er voldaan is aan de volgende voorwaarden:

- de Young-diagrammen van μ en ν, voorgesteld in de $(m \times n)$-rechthoek, bedekken de volledige rechthoek niet maar laten een samenhangende tussenruimte.
- alle nullen in de atypicaliteitsmatrix bevinden zich in deze tussenruimte.

Het is evenwel mogelijk dat de diagrammen van μ en ν elkaar overlappen in deze voorstelling, maar $\bar{\nu};\mu$ kan zodanig gekozen worden dat er zich geen nullen bevinden in dit gebied. In wat volgt, veronderstellen we altijd dat $\bar{\nu};\mu$ kritisch is met geen nullen in de overlap.

A.5.3 Een karakterformule voor kritische representaties

Net als in §A.4 kunnen we ook een (m,n)-index k toekennen aan een samengestelde partitie. Deze definitie voor k is een veralgemening van de (m,n)-index voor gewone partities en wordt gegeven door:

Definition A.1 *Voor een standaard samengestelde partitie $\bar{\nu};\mu$ wordt de (m,n)-index gegeven door*

$$k = \min\Bigg(\Bigg\{ i \in \{1,\ldots,m\} | \exists j \in \{1,\ldots,n\} : \mu_i + \left\langle \mu'_{n-j+1} - m \right\rangle + (m-i) = \nu'_j + \langle \nu_{m-i+1} - n \rangle + (n-j) \Bigg\} \cup \{m+1\}\Bigg) \tag{A.14}$$

waarbij $\langle a \rangle = \max(0,a)$.

Analoog als bij covariante en contravariante representaties, maken we opnieuw gebruik van een rij van spiegelingen met simpele oneven wortels. Voor een gegeven $\bar{\nu};\mu$, beschouwen we nu volgende rij van oneven wortels, in de gegeven volgorde:

$$\begin{array}{ll} \text{rij } m: & \beta_{m,1}, \beta_{m,2}, \ldots, \beta_{m,\min\{n,\mu_k-k+m\}} \\ \text{rij } m-1: & \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,\min\{n,\mu_k-k+m-1\}} \\ \vdots & \vdots \\ \text{rij } k: & \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,\mu_k} \end{array} \tag{A.15}$$

Door toepassing van deze rij van reflecties wordt de standaard keuze voor simpele wortels omgezet in

$$\begin{aligned} \Pi' = \{&\epsilon_1-\epsilon_2,\ldots,\epsilon_{k-2}-\epsilon_{k-1},\epsilon_{k-1}-\delta_1,\delta_1-\delta_2,\delta_2-\delta_3,\ldots,\delta_{\mu_k-1}-\delta_{\mu_k},\\ &\delta_{\mu_k}-\epsilon_k,\epsilon_k-\delta_{\mu_k+1},\delta_{\mu_k+1}-\epsilon_{k+1},\epsilon_{k+1}-\delta_{\mu_k+2},\ldots,\delta_{\mu_k+m-k}-\epsilon_m,\\ &\epsilon_m-\delta_{\mu_k+m+1-k},\delta_{\mu_k+m+1-k}-\delta_{\mu_k+m+2-k},\ldots,\delta_{n-1}-\delta_n\}. \end{aligned}$$

Hieruit kunnen we volgend resultaat afleiden:

Gevolg A.13 *Elke kritische representatie $V_{\bar{\nu};\mu}$ is tam.*

Gebruik makend van Gevolg A.13 kunnen we de karakterformule van Kac en Wakimoto [34] opnieuw toepassen. Het herschrijven van deze karakterformule in een meer geschikte vorm geeft volgende formule:

Stelling A.14 *Het karakter van een kritische representatie, gekarakteriseerd door een standaard samengestelde partitie $\bar{\nu};\mu$ zonder nullen in overlap heeft de volgende vorm:*

$$\mathrm{ch}\, V_{\bar{\nu};\mu} = (-1)^{(m-a)(l-1)+n(m-a-k+1)} D^{-1} \det(C) \tag{A.16}$$

met C de volgende vierkante matrix van orde $n+m-a$:

$$C = \begin{pmatrix} 0 & Y_{\mu'} & 0 \\ X_\mu & R^{(r)} & X_\nu \\ 0 & Y_{\nu'} & 0 \end{pmatrix} \qquad \textit{met} \quad R^{(r)} = \left(\frac{y_j^r}{x_i^r(x_i+y_j)} \right)_{1\le i\le m,\ 1\le j\le n}$$

en waarbij

$$\begin{aligned} X_\mu &= \left(x_i^{\mu_j+m-n-j}\right)_{\substack{1\le i\le m,\\ 1\le j\le k-1}}, & X_\nu &= \left(x_i^{m-j-\nu_{m-j+1}}\right)_{\substack{1\le i\le m,\\ k+a\le j\le m}}, \\ Y_{\mu'} &= \left(y_j^{\mu_i'+n-m-i}\right)_{\substack{1\le i\le l-1,\\ 1\le j\le n}}, & Y_{\nu'} &= \left(y_j^{n-i-\nu_{n-i+1}'}\right)_{\substack{l+a\le i\le n,\\ 1\le j\le n}}. \end{aligned}$$

A.5.4 De karakterformule en $s_{\bar{\nu};\mu}(x/y)$

Formule (A.16) voor $\mathrm{ch}(V_{\bar{\nu};\mu})$ is zeer expliciet. Het doel van deze formule is de identifacatie van deze formule met de supersymmetrische Schur-functies:

Conjectuur A.15 *Stel $\bar{\nu};\mu$ is een standaard samengestelde partitie die bovendien kritisch is en zonder nullen in de overlap. Dan is het karakter $\mathrm{ch}\, V_{\bar{\nu};\mu}$ gelijk aan $s_{\bar{\nu};\mu}(x/y)$ zoals die gedefinieerd werd in (A.6).*

Dit vermoeden is bewezen indien we volgend lemma kunnen bewijzen.

Lemma A.16 *Veronderstel dat $|x| = m$, $|y| = n$ en dat p en q positieve gehele getallen zijn met $m = p + q$. Verder veronderstellen we dat $\bar{\nu};\mu$ een kritische samengestelde partitie is met geen nullen in de overlap. Met $\nu = (\kappa_1, \ldots, \kappa_q, \eta_1, \eta_2, \ldots)$ vinden we dat:*

$$\sum_{x'+x''} \frac{(\prod x')^q (\prod x'')^n s_{\overline{\eta};\mu}(x'/y) s_{\kappa+(n^q)}(\overline{x}''/\overline{y})}{E(x', x'')} = s_{\overline{\nu};\mu}(x/y) \tag{A.17}$$

waarbij de som genomen wordt over alle mogelijke decomposities $x = x' + x''$ met $|x'| = p$ en $|x''| = q$.

Oorspronkelijk waren we ervan overtuigd dat Lemma A.16 geldig is voor een willekeurige samengestelde partitie $\bar{\nu};\mu$ omwille van het feit dat dit lemma de supersymmetrische versie lijkt van Lemma A.9. Helaas, dit blijkt niet het geval. Aangezien we Lemma A.16 niet algemeen nodig hebben, maar enkel in speciale gevallen, zijn we er wel van overtuigd dat het lemma geldig blijft in de speciale gevallen die we nodig hebben. Maar er steken een aantal problemen de kop op bij het bewijzen van Lemma A.16.

Eerst en vooral moet er nog een extra voorwaarde opgelegd worden aan de keuze van p en q. Door middel van een tegenvoorbeeld wordt de vrije keuze van p en q immers ontkracht; anderzijds laat de tweevoudige toepassing van Lemma A.16 niet toe een eenduidig verband op te leggen tussen p en q enerzijds en de (m, n)-index k en de daarvan afgeleide grootheid $l = \lambda_k + 1$ anderzijds.

Bij een bewijs door inductie op het aantal variabelen y maakt de eis dat $\bar{\nu};\mu$ kritisch is en zonder nullen in de overlap, de inductiestap niet mogelijk. Na het afzonderen van y_n blijven er immers sommen over analoog als in het linkerlid van (A.17), maar de identificatie met de door (A.17) verwachte samengestelde partities faalt. Naast de verwachte term verschijnen er extra termen bij de inductiestap. Gelukkig heffen deze extra termen elkaar op indien we vertrekken van een samengestelde partitie $\bar{\nu};\mu$ die kritisch is en geen nullen heeft in de overlap.

Een bewijs door inductie is tot nog toe onze beste hoop op succes. Een argument waarom de extra termen wegvallen ten opzichte van elkaar blijft echter de enige ontbrekende schakel in het bewijs van Conjectuur A.15

A.6 Dimensieformules voor representaties in $\mathfrak{gl}(m|n)$

Hoofdstuk 6 is in feite opgedeeld in twee grote delen. In het eerste deel bepalen we twee formules voor de t-dimensie van covariante representaties V_λ, het tweede deel besteed aandacht aan de t-dimensie van gemengde tensorrepresentaties $V_{\bar{\nu};\mu}$. In beide gevallen gaan we dieper in op een speciale keuze van λ enerzijds en $\bar{\nu};\mu$ anderzijds.

A.6.1 Inleiding

We beperken ons tot de Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ en de eindig-dimensionale irreduciebele representaties V. Voor elk van deze representaties zullen we aan de hand van de nieuwe determinantformules een dimensieformule bepalen. Hiertoe maken we gebruik van de afbeelding F die inwerkt op $\operatorname{ch} V$ en gedefinieerd wordt door

$$\begin{array}{ll} F(e^{\epsilon_i}) = 1 & (i = 1, \ldots, m) \\ F(e^{\delta_j}) = t & (j = 1, \ldots, n). \end{array} \tag{A.18}$$

Deze afbeelding is consistent met de $\mathbb{Z}$-gradatie van $\mathfrak{g}$, en de corresonderende $\mathbb{Z}$-gradatie van V. Het inwerken van F op het karakter van V geeft ons de t-DIMENSIE van V, genoteerd $\dim_t(V)$:

$$\dim_t(V) = F(\operatorname{ch} V) = \sum_\mu \dim V(\mu) F(e^\mu).$$

De t-dimensie is in feite een veralgemening van de dimensie van V, die we terugvinden door $t = 1$ te stellen, en de superdimensie van V, die we terugvinden door $t = -1$ te stellen.

A.6.2 *t*-dimensie van covariante representaties

De twee formules waarvan we starten om de t-dimensie te bepalen zijn enerzijds (A.4) en anderzijds (A.10).

De toepassing van F op (A.4) levert volgende formule:

Stelling A.17 *De t-dimensie van V_λ wordt gegeven door volgende determinant*

$$\dim_t V_\lambda = \det_{1\le i,j\le \ell(\lambda)} \left(\sum_{k=0}^{\lambda_i - i + j} \binom{m+\lambda_i - i + j - k - 1}{\lambda_i - i + j - k} \binom{n}{k} t^k \right). \qquad \text{(A.19)}$$

Aangezien $x_i = e^{\epsilon_i}$ en $y_j = e^{\delta_j}$, correspondeert (A.18) met de substitutie $x_i = 1$ en $y_j = t$ in de uitdrukking voor $s_\lambda(x/y)$. De toepassing van deze specializatie op (A.10) geeft na rij- en kolombewerkingen volgende formule voor de t-dimensie:

Stelling A.18 *De t-dimensie van V_λ wordt gegeven door* $\dim_t(V_\lambda) = \pm(1+t)^{mn}R(\lambda)$ *met*

$$R(\lambda) = \det \begin{pmatrix} \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}} \binom{i+j-2}{j-1} \right)_{\substack{1\le i\le m\\ 1\le j\le n}} & \left(\binom{\lambda_j + m - n - j}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}} \\ \left(t^{\lambda'_i + n - m - i - j + 1} \binom{\lambda'_i + n - m - i}{j-1} \right)_{\substack{1\le i\le n-m+k-1\\ 1\le j\le n}} & 0 \end{pmatrix}. \qquad \text{(A.20)}$$

Vergelijken we beide formules dan heeft (A.20) ten opzichte van (A.19) het grote voordeel dat elk matrixelement uit een enkele binomiaalcoëfficiënt bestaat vermenigvuldigd met een macht van t of $(1+t)$.

Een interessante toepassing volgt voor de speciale keuze $\lambda = \left((n-a)^{(m-a)}\right)$, met $a = 0, 1, \ldots, \min(m,n)$. Voor een dergelijke rechthoekige partitie λ is

$$\dim_t(V_\lambda) = \pm\,(1+t)^{mn-a(r+r'-1)} \det_{1\le i,j\le a} \left(\frac{(-1)^{i+j}}{(1+t)^{i+j}} \binom{i+j+m+n-2a-2}{j-1} \right)$$

waarbij $k = m-a+1,\ k' = n-a+1,\ r = n-a,\ r' = m-a,\ \lambda_k = 0,\ \lambda'_{k'} = 0$. Het minteken en de macht van $(1+t)$ kunnen voorop geplaatst worden en vervolgens delen we de noemers weg uit deze uitdrukking. Wat overblijft is een determinant waarvan de ontwikkeling terug te vinden is in [43, (2.2)]. Uiteindelijk krijgen we in het rechterlid een gesloten gedaante:

$$\dim_t(V_\lambda) = (1+t)^{(m-a)(n-a)} \prod_{i=0}^{a-1} \frac{\binom{m+n-2a+i}{n-a+i}}{\binom{m-a+i}{i}}.$$

Vergelijken we deze uitdrukking met (A.19), die kan herschreven worden in functie van ${}_2F_1$ functies, dan krijgen we een uitdrukking voor determinanten:

Eigenschap A.19

$$\begin{aligned}&\det_{0\le i,j\le s}\left(\binom{n+i+j}{m}\,{}_2F_1\left({m-n-i-j,-n \atop -n-i-j};-t\right)\right)\\ &= (-1)^{s(s+1)/2}(1+t)^{(s+1)(s+n-m)}\prod_{i=1}^{m-s}\frac{\binom{2s+n-m+i}{s+1}}{\binom{s+i}{s+1}} \qquad (s\le m).\end{aligned}$$

Deze identiteit kan op verschillende manieren geschreven worden. De uitdrukking die met rechtstreekse berekeningen moeilijk te vinden is, krijgen we hier als een eenvoudig gevolg van de gelijkheid van twee formules voor t-dimensie.

A.6.3 t-dimensie van gemengde tensorrepresentaties

Naar analogie met voorgaande paragraaf kunnen we ook hier twee formules opstellen voor de t-dimensie. We passen hierbij (A.18) toe op de formules (A.6) en (A.16). De t-dimensie kan door volgende stellingen berekend worden.

Stelling A.20 *De t-dimensie van $V_{\bar{\nu};\mu}$ wordt gegeven door volgende determinant*

$$\dim_t V_{\bar{\nu};\mu} = \det\left(A\mid B\right), \tag{A.21}$$

met

$$\begin{aligned} A &= \left(\sum_{k=0}^{\nu_l+\nu_1'-l-i+1}\binom{m+\nu_l+\nu_1'-l-i-k}{\nu_l+\nu_1'-l-i-k+1}\binom{n}{k}t^{-k}\right)_{1\le i\le\nu_1'+\mu_1',\,1\le l\le\nu_1'},\\ B &= \left(\sum_{k=0}^{\mu_j-\nu_1'-j+i}\binom{m+\mu_j-\nu_1'-j+i-k-1}{\mu_j-\nu_1'-j+i-k}\binom{n}{k}t^{k}\right)_{1\le i\le\nu_1'+\mu_1',\,1\le j\le\mu_1'}\end{aligned}$$

waar de indices i, j resp. l van boven naar onder, van links naar rechts, resp. van rechts naar links lopen.

De tweede formule leunt aan bij (A.20).

Stelling A.21 *De t-dimensie van $V_{\bar{\nu};\mu}$ wordt gegeven door*

$$\dim_t(V_{\bar{\nu};\mu}) = \pm(1+t)^{mn}t^{rn}\det\begin{pmatrix}0 & R_{12} & 0\\ R_{21} & R_{22} & R_{23}\\ 0 & R_{32} & 0\end{pmatrix} \tag{A.22}$$

met

$$R_{12} = \left(t^{\mu_i'+n-m-i-r-j+1} \binom{\mu_i'+n-m-i-r}{j-1} \right)_{\substack{1\le i\le l-1\\ 1\le j\le n}},$$

$$R_{21} = \left(\binom{\mu_j+m-n+r-j}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le k-1}},$$

$$R_{22} = \left(\frac{(-1)^{i+j}}{(1+t)^{i+j-1}} \binom{i+j-2}{j-1} \right)_{\substack{1\le i\le m\\ 1\le j\le n}},$$

$$R_{23} = \left((-1)^{i-1} \binom{\nu_{m-j+1}-m-r+i+j-2}{i-1} \right)_{\substack{1\le i\le m\\ 1\le j\le m-k-a+1}},$$

$$R_{32} = \left(t^{\nu'_{n-i+1}-n+r+i+j-1} \binom{\nu'_{n-i+1}-n+r+i+j-2}{j-1} \right)_{\substack{1\le i\le n-l-a+1\\ 1\le j\le n}}.$$

In het algemeen is de orde van de determinant in (A.22) kleiner dan de orde in (A.21). Bovendien hebben de matrixelementen een eenvoudigere gedaante: het product van een binomiaalcoëfficiënt met een macht van t of $(1+t)$.
Het speciale geval $\bar{\nu}; \mu = (\overline{b}^c); (b^c)$ met $n = m+b-2c$ leidt ook hier tot een identiteit.

Eigenschap A.22 *Stel b, c en n zijn positieve gehele getallen met $m = n-b+2c$, dan is*

$$\det(A \mid B) = \pm((b+c)!)^c \frac{\prod_{j=0}^{n+c-1}(j+c)! \prod_{j=1}^{c-1}(c+b+j)^{c-j} \prod_{j=0}^{c-1} j! \prod_{j=0}^{b-1} j! \prod_{j=0}^{n-b-1} j!}{\prod_{i=0}^{m-c-1}(i+c)! \prod_{i=0}^{b-1}(i+c)! \prod_{j=0}^{n-1} j! \prod_{j=0}^{2c-1}(b+j)!} t^{-bc}(1+t)^{2cb}$$

met

$$A = \left(\binom{m+b+c-l-i}{b+c-l-i+1} {}_2F_1\left(\begin{matrix} l+i-b-c-1, -n \\ i+l-m-b-c \end{matrix} ; -\frac{1}{t} \right) \right)_{\substack{1\le i\le 2c,\\ 1\le l\le c}},$$

$$B = \left(\binom{m+b-c-j+i-1}{b-c-j+i} {}_2F_1\left(\begin{matrix} c+j-b-i, -n \\ c+j-m-b-i+1 \end{matrix} ; -t \right) \right)_{\substack{1\le i\le 2c,\\ 1\le j\le c}},$$

waarbij i, j resp. l van boven naar onder, van links naar rechts, resp. van rechts naar links lopen.

Appendix B

Erratum

- P19 (L-5) The second item in the definition of a COMPOSITE YOUNG TABLEAU should be replaced by: $R(j) + \overline{R}(\bar{j}) \leq j$ for $j \in \{1, 2, \ldots, m\}$, where $R(j)$ and $\overline{R}(\bar{j})$ are the greatest row numbers in F^{μ} resp. F^{ν} containing j resp. $\bar{j}$.

- P20 Figure 1.15 should be replaced by Figure B.1.

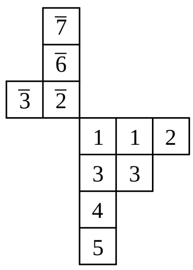


Figure B.1: A standard composite tableau

- P21 The caption of Figure 1.16 should be replaced by "Extra condition: $R(j) + \overline{R}(\bar{j}) \leq j$"

- P20(L-5)-21(L4) should be replaced by:
According to the condition that the entries have to increase along a column, it

follows that $R_\mu(p) \leq p$. Suppose

$$K_p^\lambda = \max(\{c \leq \nu_1 \mid p \text{ does not occur in column } c \text{ of } F^\lambda\} \cup \{0\}).$$

If $K_p^\lambda = 0$, there is a p in the first ν_1 columns, and thus $\overline{R}_\nu(\overline{p}) = 0$. If $K_p^\lambda \in \{1, \ldots, \nu_1\}$, let $q = \max(\{t \mid t < p, t \text{ occurs in column } K_p^\lambda\})$ and ρ_q its row number along the column K_p^λ. Then it is easy to check that $\overline{R}_\nu(\overline{p}) = p - \rho_q$ and $R_\mu(p) \leq \rho_q$. This implies the extra condition.
In Figure 1.16, $K_5^\lambda = 3 \in \{1, 2, 3\}$, $q = 4$ and $\rho_q = 3$ gives $\overline{R}_\nu(\overline{5}) = 2$ and $R_\mu(5) = 3 \leq 3$; $K_7^\lambda = 0$, thus $\overline{R}_\nu(\overline{7}) = 0$ and $R_\mu(7) = 4 \leq 7$.

- P44 (L-8) $r(i)$ should be replaced by $r(i^*)$, and $\overline{r}(\overline{i})$ should be replaced by $\overline{r}(\overline{i^*})$.
- P45 The third item in the definition of a $(m|n)$-STANDARD COMPOSITE SUPERTABLEAU $S^{\overline{\nu};\mu}$ must be replaced by $R(i)+\overline{R}(\overline{i}) \leq i$ for all $i \in \{1, 2, \ldots, m\}$.

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