Meerdimensionale distributies en veralgemeende hilberttransformaties in cliffordanalyse

Multidimensional distributions and generalized Hilbert transforms in Clifford analysis

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Woord vooraf

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Chapter 1

Introduction

In this doctoral thesis we study some specific families of multidimensional distributions in the framework of orthogonal Clifford analysis, meanwhile constructing several generalizations of the Clifford–Hilbert transform, their kernels belonging to one of those families of distributions. The generalized Hilbert transforms are developed within the framework of either orthogonal Clifford analysis (Part I), anisotropic Clifford analysis (Part II) or Hermitean Clifford analysis (Part III).

In the introductory Chapter 2 we take off with a presentation of the classical Hilbert transform on the real line. Although emphasis is on the theoretical point of view, we mention applications in the theoretical description of many devices and systems. In particular the notion of analytic signal, closely related to the Hilbert transform, is widely used in the theory of signals, circuits and systems. In the first section, the fundamental and characterizing properties of the Hilbert transform on the real line are recalled, since in the following sections and chapters, it is examined whether the multidimensional generalizations of that original one–dimensional Hilbert transform still submit to properly rephrased analogues of its most crucial properties. Special attention is paid to its relationship with the Cauchy integral in the complex plane. This naturally leads to the well–known Hardy space of holomorphic functions in the upper half of the complex plane and a second Hardy space on the real line, isomorphic with the first one, and consisting of the eigenfunctions of the Hilbert transform with eigenvalue 1. We then end the first section with the concept of analytic signal, involving the Hilbert transform as an indispensable tool for both global and local descriptions of a real valued signal. In the second section, some of the existing higher dimensional scalar valued generalizations of the one–dimensional Hilbert transform are exposed in order to have a clear view of the advantages and limitations of their associated analytic signals. At the same time, their properties are compared with those of the Hilbert transform on the real line. For more detailed information about the one–dimensional Hilbert transform, we refer the reader to a.o. [98, 79, 5, 60, 66]. Higher dimensional scalar valued Hilbert transforms are introduced in e.g. [99, 95, 55, 66].

The purpose of Part I is twofold. On the one hand we introduce in Chapter 4 some specific families of distributions in the framework of orthogonal Clifford analysis. Although certain of these distributions were already introduced, albeit dispersed, in the literature on harmonic analysis and on Clifford analysis, classifying those distributions in families offers structural clarity and completeness. Also some original results are obtained. On the other hand, Chapter 6 is devoted to an extensive study of new multidimensional Hilbert transforms, the convolution kernels of which are carefully selected from the distributions mentioned above.

In the introductory Chapter 3 the necessary language of orthogonal Clifford algebra and Clifford analysis is presented. The algebras under consideration found their origin in the paper [41] of W. K. Clifford who called them geometric algebras since they incorporate inside one single structure as well the inner product as the wedge product of vectors. They generalize a.o. Grassmann's exterior algebra and Hamilton's algebra of quaternions. One of the most simple nontrivial Clifford algebras is the algebra of complex numbers, obtained when constructing the universal Clifford algebra over the field of the real numbers. In complex analysis then, it is well–known that the two–dimensional Laplace operator may be decomposed as the multiplication of the Cauchy–Riemann operator with its complex conjugate. So, holomorphic functions, i.e. null solutions of the Cauchy–Riemann operator, are also harmonic. In the same sense orthogonal Clifford analysis arises as a higher dimensional function theory in a specific Clifford algebra setting, centred around the notion of so–called monogenic functions, i.e. null solutions of the Clifford–vector valued orthogonal Dirac operator, which is an elegant generalization to higher dimension of the Cauchy–Riemann operator. As this Dirac operator factorizes the higher dimensional Laplacian, monogenic functions are harmonic as well and moreover their properties constitute a refinement of those of harmonic functions. We end this chapter with the so–called spherical monogenics, since they play a fundamental role in the construction of our generalized Hilbert transforms. They are restrictions to the unit sphere of homogeneous, monogenic polynomials which we take Clifford–vector valued. Finally, we remark here that, over the years, Clifford analysis has gained more and more interest and has grown out to a proper branch of classical analysis. A profound study of Clifford analysis, drawing the parallels between the classical complex function theory on the one hand and this monogenic function theory on the other hand, can be found in the book [23] of Brackx, Delanghe and Sommen. For further study of this higher dimensional function theory and its applications we refer to e.g. [62, 60, 52, 77, 63, 88, 87, 8, 48, 82, 61].

In Chapter 4, four families of distributions in orthogonal Clifford analysis are introduced which were already considered by Brackx, Delanghe and Sommen in [26, 25]. The common and most striking feature of all those distributions is the way they act on scalar valued test functions defined in Euclidean space. By making use of the well–known spherical means, which arise naturally by introducing spherical co-ordinates, a simple, powerful and highly efficient technique was designed allowing to convert the action in the original setting of Euclidean space into an action on the real line by means of the distribution "finite parts". The above authors also saw that introducing generalized spherical means, involving spherical monogenics, would give rise to much more general Clifford distributions, encompassing those already constructed in the special case where the degree of the spherical monogenic considered is zero. In our turn we not only normalize those families of distributions, but we also closely study their properties, disclosing strong connections between them.

Next, the definition of the classical Clifford–Hilbert transform and its characterizing properties are introduced in Chapter 5. To our knowledge, Horváth was the first to define a vector valued Hilbert transform on Euclidean space $\mathbb{R}^m$ using Clifford algebra (see [69]). This multidimensional Hilbert transform in the orthogonal Clifford analysis setting was taken up again in the 1980's and further studied in e.g. [91, 60, 76, 50, 51]. It also plays a fundamental role in the study of Hardy spaces of monogenic functions, see e.g. [36, 77, 39, 4, 42, 49]. In the first section of this chapter, we present an alternative definition for the Clifford–vector valued Hilbert transform of Horváth, involving the multiplication with an extra basis vector. Its main properties are then examined; in particular its relationship with the Cauchy integral on $\mathbb{R}^{m+1}$ is disclosed, at the same time giving rise to a study of Hardy spaces of monogenic functions. Further, we also propose a higher dimensional generalization of the concept of analytic signal in the Clifford analysis context. Finally, to conclude this section, we deal with the interaction between the Clifford–Hilbert transform and the Clifford–Radon transform, both transforms being protagonists in multidimensional signal analysis theory. In the second section, the Clifford–Hilbert transform on closed surfaces in Euclidean space $\mathbb{R}^m$ is introduced. It is shown that, in general, its properties are weaker than the ones of the Clifford–Hilbert transform on $\mathbb{R}^m$, except for the case of the unit sphere to which we pay special attention.

We end Part I with our construction of two possible generalizations of the Clifford–Hilbert transform on $\mathbb{R}^m$ (Chapter 6). Their kernels are deliberately chosen from the distributions introduced in Chapter 4, in such a way that the corresponding convolution operators preserve as much traditional properties of the Clifford–Hilbert transform as possible. In the first approach for generalization it is shown that the kernels constitute a refinement of the generalized Hilbert kernels introduced by Horváth in [70]. Our resulting generalized Hilbert transforms are shown to be no longer unitary operators, yet they remain bounded singular operators on $L_2(\mathbb{R}^m)$. The second approach is based on the intimate relationship between the Hilbert transform and the Cauchy integral and starts with the construction of a generalized Cauchy integral on $\mathbb{R}^{m+1}$ involving a distribution from one of the aforementioned families as a generalized Cauchy kernel. A new generalized Hilbert transform in $\mathbb{R}^m$ is then defined as part of the L_2 or distributional boundary limits of the generalized Cauchy integral considered, and it is shown to be a bounded operator on the Sobolev spaces $W_2^n(\mathbb{R}^m)$. Also a connection is established between both generalizations through the action of a higher order Dirac derivative. Finally, a section is devoted to the action of the Radon transform on the two types of generalized Hilbert operators.

The (generalized) multidimensional Hilbert transforms on $\mathbb{R}^m$ considered so far and usually obtained as a part of the boundary limits of an associated Cauchy integral on $\mathbb{R}^{m+1}$, might be characterized as isotropic, since the metric in the underlying space is the standard Euclidean one. Part II now adopts the idea of an anisotropic (also called metric dependent or metrodynamical) Clifford setting, which offers the possibility of adjusting the co-ordinate system to preferential and not necessarily mutually orthogonal directions. In this new area of Clifford analysis (see e.g. [35, 53]), we construct the so–called anisotropic Clifford–Hilbert transform.

The basic language of anisotropic Clifford analysis is presented in the introductory Chapter 7. We first introduce the notion of metric tensor which gives rise to two bases in $\mathbb{R}^m$: a covariant one and a contravariant one. Then, a

Clifford algebra is constructed, depending on the metric tensor involved, and all necessary definitions and results of orthogonal Clifford analysis are adapted to this metric dependent setting. We introduce e.g. the concepts of Dirac operator, monogenicity and Laplace operator. We end this chapter with the definition and the study of the so–called anisotropic Fourier transform, the metric dependent analogue of the classical Fourier transform.

In Chapter 8, we first define a new anisotropic Clifford–Hilbert transform on $\mathbb{R}^m$, arising naturally as part of the non–tangential boundary limits of the anisotropic Cauchy integral on $\mathbb{R}^{m+1}$. Next, the former operator is shown to possess, formally, the same properties as the isotropic Clifford–Hilbert transform introduced in Chapter 5. Finally, the striking result is obtained that the associated anisotropic Cauchy integral on $\mathbb{R}^{m+1}$ is no longer uniquely determined, but may stem from a diversity of metric tensors of order $(m+1)$.

The central topic of Part III is the development of new higher dimensional Hilbert transforms in the framework of Hermitean Clifford analysis, a rather young branch of Clifford analysis (see e.g. [86, 85, 89, 43, 34, 6, 7, 54]).

The introductory Chapter 9 discusses the basic ingredients of Hermitean Clifford analysis, a new and successful branch of Clifford analysis, offering a refinement of the orthogonal case; it focusses on the simultaneous null solutions, called Hermitean monogenic functions, of two Hermitean Dirac operators which are invariant under the action of a realization of the unitary group. We first introduce in a natural way the elementary objects in Hermitean Clifford analysis by means of a so–called complex structure J, which is used to mould in some sense the orthogonal protagonists into their Hermitean counterparts. In the second section a splitting of the Hermitean monogenic system is considered, which has already been studied in [7], leading to the so–called homogeneous parts of complex spinor space.

While studying Clifford–Hermite wavelets in the context of Hermitean Clifford analysis, see [31, 32], the authors came across a new kind of operator, obtained as the composition of two orthogonal Clifford–Hilbert transforms on $\mathbb{R}^m$. The resulting operator, denoted $\mathcal{K}$, was shown to possess some typical properties of a classical Hilbert transform as well. In Chapter 10, we further extensively investigate this $\mathcal{K}$–transform, reobtaining it as the commutator of two new Hermitean Clifford–Hilbert transforms. In the first section of this chapter, we introduce, next to the Clifford–Hilbert transform already presented in Chapter 5, a second orthogonal Clifford–Hilbert transform on $\mathbb{R}^m$, its kernel being obtained by the action of the complex structure J on the orthogonal Clifford–Hilbert kernel. Through the action of the projection operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal Clifford–Hilbert kernel we get two new isotropic Hermitean Hilbert transforms. The commutator of the latter transforms then gives rise to the $\mathcal{K}$–transform, which is studied in the second section. Its connections and similarities with the standard Clifford–Hilbert transforms as well as with the newly introduced Hermitean Hilbert transforms are explicitly investigated, and in particular new Hardy spaces associated to this operator are defined and characterized. Some results also allow for a nice geometric interpretation. In the last section the concept of multidimensional analytic signal is revised.

In orthogonal Clifford analysis, the Clifford–Cauchy integral formula for monogenic functions has proven to be a corner stone of the function theory, as is the case for the traditional Cauchy formula for holomorphic functions in the complex plane. Naturally a Cauchy integral formula for Hermitean monogenic functions is essential in the further development of Hermitean Clifford analysis, but was not obtained before in a satisfactory way. In the first section of Chapter 11 we arrive at the desired result, by passing to the framework of circulant (2×2) matrix functions. As an additional result, the obtained Hermitean Clifford–Cauchy integral formula turns out to reduce to the traditional Martinelli–Bochner formula for holomorphic functions of several complex variables when considering the special case of functions taking values in the n–homogeneous part of complex spinor space. This means that the theory of Hermitean monogenic functions not only refines orthogonal Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, even encompassing some of its results. In the second section, we then arrive at the definition of a new Hermitean Clifford–Hilbert transform, arising naturally as part of the non–tangential boundary limits of the Hermitean Clifford–Cauchy integral. The resulting matrix Hilbert operator is shown to satisfy properly adapted analogues of the characteristic properties of the Hilbert transform in classical analysis and orthogonal Clifford analysis.

Chapter 2

Classical Hilbert transforms

The Hilbert transform is named after David Hilbert (1862–1943), one of the greatest mathematicians of the twentieth century, who, in his studies of integral equations, was the first to observe what is nowadays known as the Hilbert transform pair. However, the Hilbert transform and its properties were developed mainly by E. C. Titchmarsh and G. H. Hardy. It was Hardy who named it after Hilbert. The Hilbert transform is applied in the theoretical description of many devices and systems and directly implemented in the form of Hilbert analogue or digital filters. In particular the notion of analytic signal, closely related to the Hilbert transform, is widely used in the theory of signals, circuits and systems. In this work however, emphasis is laid on the theoretical point of view. For more detailed information about the Hilbert transform on the real line, we refer the reader to a.o. [98, 79, 5, 60, 66]. Higher dimensional scalar valued Hilbert transforms are introduced in e.g. [99, 95, 55, 66].

2.1 The Hilbert transform on the real line

2.1.1 Introduction

This first section may be seen as a basic framework for the whole of the underlying thesis, which is to be understood in the following sense. We recall the definition and the fundamental, characterizing properties of the Hilbert transform on the real line. In subsequent chapters, it is then examined whether the multidimensional generalizations of that original one–dimensional Hilbert transform may be defined, which still submit to properly rephrased analogues of its most crucial properties.

We consider signals $f(t)$, depending on a real (time) variable $t \in \mathbb{R}$; they may be real or complex valued. The signal f is mostly assumed to be a *finite energy signal*, i.e.

$$\|f\|_{L_2}^2 \equiv \int_{-\infty}^{+\infty} |f(t)|^2 \, dt < +\infty$$

in other words, $f \in L_2(\mathbb{R})$.

The Hilbert transform of a signal f then arises in a quite natural, but formal, way in the context of the Fourier transform. Consider a Fourier pair of functions f and F for which

$$\begin{aligned} F(\omega) &= \mathcal{F}_{t\to\omega}[f(t)](\omega) &&= \int_{-\infty}^{+\infty} f(t) \, \exp(-2\pi i \, t\omega) \, dt \\ f(t) &= \mathcal{F}^{-1}_{\omega\to t}[F(\omega)](t) &&= \int_{-\infty}^{+\infty} F(\omega) \, \exp(2\pi i \, \omega t) \, d\omega \end{aligned} \tag{2.1}$$

In a formal way the inverse Fourier formula (2.1) may be written as

$$f(t) = \int_0^{+\infty} \left[(F(\omega) + F(-\omega)) \cos(2\pi\omega t) + i \, (F(\omega) - F(-\omega)) \sin(2\pi\omega t) \right] \, d\omega$$

or

$$f(t) = \int_0^{+\infty} \left(a(\omega) \cos(2\pi\omega t) + b(\omega) \sin(2\pi\omega t) \right) \, d\omega$$

where we have put

$$\begin{aligned} a(\omega) &= F(\omega) + F(-\omega) &&= 2 \int_{-\infty}^{+\infty} f(u) \, \cos(2\pi u\omega) \, du \\ b(\omega) &= i \, (F(\omega) - F(-\omega)) &&= 2 \int_{-\infty}^{+\infty} f(u) \, \sin(2\pi u\omega) \, du \end{aligned}$$

Now define the so–called *conjugate integral* g of f by

$$g(t) = \int_0^{+\infty} \left(-b(\omega) \cos(2\pi\omega t) + a(\omega) \sin(2\pi\omega t) \right) \, d\omega$$

Then it holds that

$$\begin{aligned} f(t) + i\,g(t) &= \int_0^{+\infty} (a(\omega) - ib(\omega))\, \exp(2\pi i\,\omega t)\, d\omega \\ &= 2 \int_0^{+\infty} F(\omega)\, \exp(2\pi i\,\omega t)\, d\omega \end{aligned}$$

Let us first have a look at the Fourier transform of the function g. One has

$$\begin{aligned} & g(t) \\ &= \int_0^{+\infty} \left[-i\,(F(\omega) - F(-\omega))\cos(2\pi\omega t) + (F(\omega) + F(-\omega))\sin(2\pi\omega t)\right]\, d\omega \\ &= \int_0^{+\infty} \left(-i\,F(\omega)\exp(2\pi i\,\omega t) + i\,F(-\omega)\exp(-2\pi i\,\omega t)\right)\, d\omega \\ &= \int_{-\infty}^{+\infty} (-i\,\mathrm{sgn}(\omega)\,F(\omega))\exp(2\pi i\,\omega t)\, d\omega \end{aligned}$$

where we introduced the signum function

$$\mathrm{sgn}(\omega) = \frac{\omega}{|\omega|} = \begin{cases} +1\,, & \omega > 0 \\ -1\,, & \omega < 0 \end{cases}$$

So the Fourier transform G of g is given by

$$G(\omega) = -i\,\mathrm{sgn}(\omega)\,F(\omega)$$

Next, in a formal way, the functions f, g and $f + ig$, are defined as the following limits for $y \to 0+$:

$$\begin{aligned} f(x) &= \lim_{y\to 0+} U(x,y) \\ &= \lim_{y\to 0+} \int_0^{+\infty} (a(\omega)\cos(2\pi\omega x) + b(\omega)\sin(2\pi\omega x))\, \exp(-2\pi\omega y)\, d\omega \\ g(x) &= \lim_{y\to 0+} V(x,y) \\ &= \lim_{y\to 0+} \int_0^{+\infty} (-b(\omega)\cos(2\pi\omega x) + a(\omega)\sin(2\pi\omega x))\, \exp(-2\pi\omega y)\, d\omega \end{aligned}$$

and

$$\begin{aligned} f(x) + i\, g(x) &= \lim_{y \to 0+} \left(U(x,y) + i\, V(x,y) \right) \\ &= \lim_{y \to 0+} 2 \int_0^{+\infty} F(\omega) \exp(2\pi i\, \omega x)\, \exp(-2\pi \omega y)\, d\omega \end{aligned}$$

or still

$$\begin{aligned} f(x) + i\, g(x) &= \lim_{y \to 0+} W(z) \\ &= \lim_{y \to 0+} 2 \int_0^{+\infty} F(\omega) \exp(2\pi i\, \omega z)\, d\omega \; , \quad z = x + iy \end{aligned}$$

Note that the functions U and V are *harmonic* in the upper half plane

$$\mathbb{C}^+ = \{ z = x + iy \in \mathbb{C} : \mathrm{Im}(z) = y > 0 \}$$

i.e.

$$\begin{aligned} \Delta U(x,y) &= \left(\partial_x^2 + \partial_y^2 \right) U(x,y) = 0 \; , \quad y > 0 \\ \Delta V(x,y) &= \left(\partial_x^2 + \partial_y^2 \right) V(x,y) = 0 \; , \quad y > 0 \end{aligned}$$

while W is *holomorphic* there, i.e.

$$\partial_{z^c} W(z) = \frac{1}{2} \left(\partial_x + i\, \partial_y \right) W(x,y) = 0 \; , \quad y > 0$$

In the particular case where f is real valued, the functions a, b and g are real valued as well and so are the functions U and V. As they are then the real and imaginary parts of the holomorphic function W in $\mathbb{C}^+$, the functions U and V are *conjugate harmonic functions* in the upper half of the complex plane. Moreover, still for a real valued function f,

$$U(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-t)^2 + y^2}\, f(t)\, dt \tag{2.2}$$

and

$$V(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(x-t)}{(x-t)^2 + y^2}\, f(t)\, dt \tag{2.3}$$

Note that this holomorphic function W in the upper half plane may be written as

$$W(z) \;=\; 2 \int_{-\infty}^{+\infty} F(\omega)\, H(\omega)\, d\omega$$

where we have put $H(\omega) = \exp(2\pi i\, \omega z) Y(\omega)$ with Y the Heaviside function on the real axis. This function H is then the Fourier transform of

$$h(t) \;=\; -\frac{1}{2\pi i}\, \frac{1}{x + iy + t} = -\frac{1}{2\pi i}\, \frac{1}{z+t}\;, \quad y > 0$$

and hence, by Parseval's identity one has

$$\int_{-\infty}^{+\infty} F(\omega)\, H(\omega)\, d\omega \;=\; \int_{-\infty}^{+\infty} f(t)\, h(-t)\, dt$$

or

$$2 \int_{0}^{+\infty} F(\omega)\, \exp(2\pi i\, \omega z)\, d\omega \;=\; 2 \int_{-\infty}^{+\infty} f(t) \left(-\frac{1}{2\pi i}\right) \frac{1}{z-t}\, dt\;, \quad \mathrm{Im}(z) > 0$$

or still

$$W(z) \;=\; -\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{z-t}\, dt$$

which is in accordance with the relations (2.2) and (2.3) for the functions U and V. Now we search for a direct relationship between the functions f and g. One has

$$\begin{aligned}
g(x) \;&=\; 2 \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} f(u) \sin(2\pi\omega(x-u))\, du \\
&=\; \lim_{\lambda \to +\infty} 2 \int_0^{\lambda} d\omega \int_{-\infty}^{+\infty} f(u) \sin(2\pi\omega(x-u))\, du \\
&=\; \lim_{\lambda \to +\infty} \frac{1}{\pi} \int_{-\infty}^{+\infty} f(u)\, \frac{1 - \cos(2\pi(x-u)\lambda)}{x-u}\, du \\
&=\; \lim_{\lambda \to +\infty} \frac{1}{\pi} \int_{0}^{+\infty} \left(f(x-u) - f(x+u)\right) \frac{1 - \cos(2\pi u\lambda)}{u}\, du
\end{aligned}$$

If f is a sufficiently smooth function, the part involving $\cos(2\pi u\lambda)$ will tend to 0 as $\lambda \to +\infty$ and we obtain

$$g(x) \;=\; \frac{1}{\pi} \int_0^{+\infty} \frac{f(x-t) - f(x+t)}{t}\, dt \;=\; \frac{1}{\pi}\, \mathrm{Pv} \int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\, dt$$

where Pv denotes the so–called *Cauchy principal value*, which means that in the integral the singularity at $t = x$ will be approached in a symmetrical way, i.e.

$$\mathrm{Pv} \int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\, dt \;=\; \lim_{\varepsilon \to 0+} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, dt$$

In a similar way, starting from

$$f(x) \;=\; -2 \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} g(u) \sin(2\pi\omega(x-u))\, du$$

we obtain

$$f(x) \;=\; -\frac{1}{\pi} \int_0^{+\infty} \frac{g(x-t) - g(x+t)}{t}\, dt \;=\; -\frac{1}{\pi}\, \mathrm{Pv} \int_{-\infty}^{+\infty} \frac{g(t)}{x-t}\, dt$$

It was Hilbert who first noticed the reciprocity between the functions f and g; they form a so–called *Hilbert pair* and g is called the *Hilbert transform* of f.

2.1.2 Definition and first properties

Let $f \in L_2(\mathbb{R})$; the Hilbert transform $\mathcal{H}[f]$ of f is defined by

$$\begin{aligned}
\mathcal{H}[f](x) \;&=\; \frac{1}{\pi}\, \mathrm{Pv} \int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\, dt \;=\; \frac{1}{\pi}\, \mathrm{Pv} \int_{-\infty}^{+\infty} \frac{f(x-t)}{t}\, dt \\
&=\; \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{-\varepsilon} \frac{f(x-t)}{t}\, dt + \int_{+\varepsilon}^{+\infty} \frac{f(x-t)}{t}\, dt \right) \\
&=\; \frac{1}{\pi} \int_0^{+\infty} \frac{f(x-t) - f(x+t)}{t}\, dt
\end{aligned}$$

or, by means of a convolution

$$\mathcal{H}[f](x) \;=\; \left(\mathrm{Pv}\, \frac{1}{\pi t} * f(t) \right)(x) \tag{2.4}$$

where we call $H(t) = \mathrm{Pv}\, \frac{1}{\pi t}$ the *Hilbert kernel.*

The Hilbert transform is then directly seen to satisfy the following properties.

Property 2.1. *The Hilbert transform is a convolution operator, which is equivalent with saying that the Hilbert transform commutes with translations, i.e.*

$$\tau_a[\mathcal{H}[f]] \;=\; \mathcal{H}[\tau_a[f]]$$

with $\tau_a[f](t) = f(t-a)$, $a \in \mathbb{R}$.

Property 2.2. *The Hilbert kernel is a homogeneous distribution of degree* (-1), *which, for a convolution operator, is equivalent with saying that the Hilbert transform commutes with dilations, i.e.*

$$d_a[\mathcal{H}[f]] \;=\; \mathcal{H}[d_a[f]]$$

with $d_a[f](t) = \frac{1}{\sqrt{a}} f(\frac{t}{a})$, $a > 0$.

Property 2.3. *The Hilbert and Fourier transforms are interrelated in the following way:*

$$\begin{aligned} \mathcal{F}[\mathcal{H}[f]](\omega) &= -i\,\mathrm{sgn}(\omega)\,\mathcal{F}[f](\omega) \\ \mathcal{H}[\mathcal{F}[f]](\omega) &= \mathcal{F}[i\,\mathrm{sgn}(t)\,f(t)](\omega) \end{aligned}$$

The results contained in Property 2.3 are used to calculate Hilbert transforms in practice. They lead to the diagram in Figure 2.1, where the operator $i\mathcal{H}$ and not $\mathcal{H}$ plays a crucial role.

$$\begin{tikzcd}
f(t) \arrow[d, "\mathcal{F}"] & -\mathrm{sgn}(t) f(t) \arrow[l, "\cdot(-\mathrm{sgn}(t))"'] \arrow[d, "\mathcal{F}"] & f(t) \arrow[l, "\cdot(-\mathrm{sgn}(t))"'] \arrow[r, "i\mathcal{H}"] \arrow[d, "\mathcal{F}"] & i\mathcal{H}[f](t) \arrow[r, "i\mathcal{H}"] \arrow[d, "\mathcal{F}"] & f(t) \arrow[d, "\mathcal{F}"] \\
F(\omega) & i\mathcal{H}[F](\omega) \arrow[l, "i\mathcal{H}"] & F(\omega) \arrow[l, "i\mathcal{H}"] \arrow[r, "\cdot\mathrm{sgn}(\omega)"'] & \mathrm{sgn}(\omega)F(\omega) \arrow[r, "\cdot\mathrm{sgn}(\omega)"'] & F(\omega)
\end{tikzcd}$$

Figure 2.1: Relationship of the Hilbert transform with the Fourier transform

The Fourier image of the Hilbert kernel, i.e. $(-i)\,\mathrm{sgn}(\omega)$, is called the *Fourier symbol* of the Hilbert transform. That Fourier symbol clearly being a bounded function is equivalent with saying that the Hilbert transform is a bounded linear operator on $L_2(\mathbb{R})$. More precisely one has:

Property 2.4. *The Hilbert transform is a bounded linear operator on* $L_2(\mathbb{R})$, *and is a fortiori norm preserving, i.e.*

$$\|\mathcal{H}[f]\|^2_{L_2} \;=\; \int_{-\infty}^{+\infty} |\mathcal{H}[f](x)|^2 \, dx \;=\; \int_{-\infty}^{+\infty} |f(t)|^2 \, dt \;=\; \|f\|^2_{L_2}$$

More generally, it also preserves the inner product

$$(f,g)_{L_2} \;\equiv\; \int_{-\infty}^{+\infty} f^c(t)\, g(t)\, dt \;=\; (\mathcal{H}[f], \mathcal{H}[g])_{L_2}$$

where f^c stands for the complex conjugate function of f.

For the following property we introduce the identity operator $\mathbf{1}$ on $L_2(\mathbb{R})$.

Property 2.5. *The Hilbert transform* $\mathcal{H} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ *satisfies*

$$\mathcal{H}^2 \;=\; -\mathbf{1}$$

Corollary 2.1. *The Hilbert transform* $\mathcal{H} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ *is invertible with*

$$\mathcal{H}^{-1} \;=\; -\mathcal{H}$$

Corollary 2.2. *The Hilbert transform* $\mathcal{H} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ *is unitary, its adjoint being given by* $-\mathcal{H}$, *i.e.*

$$(\mathcal{H}[f], g)_{L_2} \;=\; (f, -\mathcal{H}[g])_{L_2} \;, \quad f, g \in L_2(\mathbb{R})$$

Corollary 2.3. *The Hilbert transform satisfies the orthogonality relation*

$$(f^c, \mathcal{H}[f])_{L_2} \;=\; 0$$

Remark 2.1. *In engineering sciences the L_2 norm of a signal f is considered, up to a possible constant, as its (finite) energy. This is why the above corollary is paraphrased as "the mutual energy of a signal and its Hilbert transform is zero".*

Property 2.6.

(i) If f is real, then $\mathcal{H}[f]$ is real.

(ii) If f is imaginary, then $\mathcal{H}[f]$ is imaginary.

(iii) If f is even, then $\mathcal{H}[f]$ is odd.

(iv) If f is odd, then $\mathcal{H}[f]$ is even.

We then end this subsection with some additional properties. First of all, the relationship between the Hilbert transform and the derivative operator is revealed. Secondly, the effect of the multiplication of the signal with the monomial t is examined. And last but not least, a remarkable result is given for the Hilbert transform of the product of two signals with non–overlapping spectra.

Property 2.7. *The Hilbert transform commutes with differentiation, i.e. if f and $\frac{df}{dt}$ are in $L_2(\mathbb{R})$, then*

$$\mathcal{H}\left[\frac{d}{dt}f(t)\right](x) \;=\; \frac{d}{dx}\left(\mathcal{H}[f](x)\right)$$

Property 2.8. *If f and tf are in $L_2(\mathbb{R})$, then*

$$\mathcal{H}[tf(t)](x) \;=\; x\,\mathcal{H}[f](x) - \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\,dt$$

Property 2.9 (Bedrosian [3])**.** *Let f and g be signals such that for some positive real number a*

$$|F(\omega)| \;=\; 0\,, \quad \text{if } |\omega| > a$$

and

$$|G(\omega)| \;=\; 0\,, \quad \text{if } |\omega| < a$$

then

$$\mathcal{H}[fg] \;=\; f\,\mathcal{H}[g]$$

Remark 2.2. *In engineering sciences, the above result is paraphrased as "only the high-pass signal in the product of low-pass and high-pass signals gets Hilbert transformed".*

2.1.3 Relationship with the Cauchy integral

We assume the real axis to be embedded in the complex plane $\mathbb{C}$. A function $F(z)$, depending on a complex variable $z = x + iy$, is called *holomorphic* in an open region $\Omega \subset \mathbb{C}$ if it is continuously differentiable and satisfies

$$\partial_{z^c} F \;=\; 0 \quad \text{in } \Omega$$

with

$$\partial_{z^c} \;=\; \frac{1}{2}\,(\partial_x + i\,\partial_y)$$

being the so–called Cauchy–Riemann operator. The fundamental solution of ∂_{z^c} is given by

$$E(z) \;=\; \frac{1}{2\pi}\frac{1}{z}$$

It satisfies the following properties:

(i) $E(z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$

(ii) $\displaystyle\lim_{|z|\to+\infty} E(z) = 0$

(iii) $\partial_{z^c} E(z) = \delta(z)$ in distributional sense

This fundamental solution $E(z)$, up to the complex factor i, now serves as the convolution kernel, usually called *Cauchy kernel*, for the Cauchy integral.

Definition 2.1. *For a signal $f(t)$ on the real t–axis, its* Cauchy integral *is defined in $\mathbb{C} \setminus \mathbb{R}$ by*

$$\mathcal{C}[f](z) \;=\; i \int_{-\infty}^{+\infty} E(z-t)\, f(t)\, dt \;=\; -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{(x-t)+iy}\, dt$$

It may be clear that the Cauchy integral is holomorphic in both the upper half plane $\mathbb{C}^+$ and the lower half plane $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. Let $f \in L_2(\mathbb{R})$; taking the supremum either in $\mathbb{C}^+$ or in $\mathbb{C}^-$, we also have that

$$\sup_{y \lesseqgtr 0} \int_{-\infty}^{+\infty} |\mathcal{C}[f](x+iy)|^2 \, dx < +\infty$$

The Hilbert transform comes into play now, when considering the non–tangential boundary limits of the Cauchy integral for $y \to \pm 0$. They lead to the so–called *Plemelj–Sokhotzki formulae.*

Property 2.10. *Let $f \in L_2(\mathbb{R})$, then the non–tangential boundary limits of the Cauchy integral are given by*

$$\begin{aligned}
\mathcal{C}^+[f](x) &\equiv \lim_{y\to 0+} \mathcal{C}[f](z) &= \frac{1}{2}\, f(x) + \frac{1}{2}\, i\,\mathcal{H}[f](x) \\
\mathcal{C}^-[f](x) &\equiv \lim_{y\to 0-} \mathcal{C}[f](z) &= -\frac{1}{2}\, f(x) + \frac{1}{2}\, i\,\mathcal{H}[f](x)
\end{aligned}$$

For a signal $f \in L_2(\mathbb{R})$, we call $\mathcal{C}^+[f]$ and $\mathcal{C}^-[f]$ its *Cauchy transforms.* They satisfy the following properties.

Corollary 2.4. *Let $f \in L_2(\mathbb{R})$, then*

(i) *$\mathcal{C}^+$ and $\mathcal{C}^-$ are bounded linear operators on $L_2(\mathbb{R})$*

(ii) *$f = \mathcal{C}^+[f] - \mathcal{C}^-[f]$ and $i\mathcal{H}[f] = \mathcal{C}^+[f] + \mathcal{C}^-[f]$*

(iii) *$\mathcal{C}^-[f] = -(\mathcal{C}^+[f^c])^c$ and $\mathcal{C}^+[f] = -(\mathcal{C}^-[f^c])^c$*

(iv) *$\mathcal{C}^+[f]$ and $\mathcal{C}^-[f]$ are orthogonal, i.e. $(\mathcal{C}^+[f], \mathcal{C}^-[f])_{L_2} = 0$*

(v) *in engineering sciences it is said that "the Fourier spectrum of $\mathcal{C}^+[f]$ only contains positive frequencies", i.e.*

$$\mathcal{F}[\mathcal{C}^+[f]](\omega) \;=\; \frac{1}{2}\,(1 + \operatorname{sgn}(\omega))\mathcal{F}[f](\omega) \;=\; Y(\omega)\,\mathcal{F}[f](\omega)$$

whereas "the Fourier spectrum of $\mathcal{C}^-[f]$ only contains negative frequencies", i.e.

$$\mathcal{F}[\mathcal{C}^-[f]](\omega) \;=\; \frac{1}{2}\,(-1 + \operatorname{sgn}(\omega))\mathcal{F}[f](\omega) \;=\; -Y(-\omega)\,\mathcal{F}[f](\omega)$$

Now expressing the Cauchy integral in terms of the Poisson transform and its conjugate, it becomes apparent that the functions f and $i\mathcal{H}[f]$ play a symmetrical role – this idea will also be strengthened in Figure 2.2 (see Subsection 2.1.4). To that end, the convolution kernel of the Cauchy integral is rewritten as

$$i\,E(z) \;=\; -\frac{1}{2\pi i}\,\frac{1}{z} \;=\; \frac{1}{2}\,\frac{1}{\pi}\,\frac{y}{x^2+y^2} + \frac{i}{2}\,\frac{1}{\pi}\,\frac{x}{x^2+y^2}$$

in which we recognize the *Poisson kernel* $P(x,y) = \frac{1}{\pi}\,\frac{y}{x^2+y^2}$ and the *conjugate Poisson kernel* $Q(x,y) = \frac{1}{\pi}\,\frac{x}{x^2+y^2}$. The Cauchy integral of a function $f \in L_2(\mathbb{R})$ thus takes the form

$$\mathcal{C}[f] \;=\; \frac{1}{2}\,\mathcal{P}[f] + \frac{i}{2}\,\mathcal{Q}[f]$$

where

$$\mathcal{P}[f](x,y) \;=\; (P(\cdot,y) * f(\cdot))\,(x) \;=\; \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y}{(x-t)^2+y^2}\, f(t)\, dt$$

and

$$\mathcal{Q}[f](x,y) \;=\; (Q(\cdot,y) * f(\cdot))\,(x) \;=\; \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x-t}{(x-t)^2+y^2}\, f(t)\, dt$$

respectively are the *Poisson transform* and the *conjugate Poisson transform* of f. They enjoy the following properties:

(i) $\mathcal{P}[f]$ and $\mathcal{Q}[f]$ are harmonic in both $\mathbb{C}^+$ and $\mathbb{C}^-$

(ii) taking the supremum either in $\mathbb{C}^+$ or in $\mathbb{C}^-$, one has

$$\sup_{y \lesseqgtr 0} \int_{-\infty}^{+\infty} |\mathcal{P}[f](x,y)|^2\, dx < +\infty \quad \text{and} \quad \sup_{y \lesseqgtr 0} \int_{-\infty}^{+\infty} |\mathcal{Q}[f](x,y)|^2\, dx < +\infty$$

(iii) the Poisson kernel and the conjugate Poisson kernel (up to a minus sign in the lower half plane) form a Hilbert pair with respect to the variable x:

$$\mathcal{H}_{t\to x}[P(t,y)](x,y) \;=\; \begin{cases} Q(x,y)\,, & y>0 \\ -Q(x,y)\,, & y<0 \end{cases}$$

2.1.4 Hardy spaces of holomorphic functions

The Cauchy integral $\mathcal{C}[f]$ of a signal $f \in L_2(\mathbb{R})$ provides an example of a holomorphic function in the upper half plane $\mathbb{C}^+$ which is in the so–called *Hardy space* $H^2(\mathbb{C}^+)$. It is defined as follows.

Definition 2.2. *The Hardy space* $H^2(\mathbb{C}^+)$ *consists of those holomorphic functions* $F(z)$ *in* $\mathbb{C}^+$ *for which*

$$\sup_{y>0} \int_{-\infty}^{+\infty} |F(x+iy)|^2\, dx < +\infty$$

We know from Property 2.10 that the non–tangential boundary limit for $y \to 0+$ of $\mathcal{C}[f](z)$ is again in $L_2(\mathbb{R})$. The question arises if this is also the case for any function of $H^2(\mathbb{C}^+)$; the answer is positive.

Proposition 2.1. *The non–tangential boundary limit for $y \to 0+$ of a function $F(z) \in H^2(\mathbb{C}^+)$ exists a.e. and belongs to $L_2(\mathbb{R})$.*

A second question to investigate is whether the Cauchy integral of a non–tangential boundary limit of a function $F(z) \in H^2(\mathbb{C}^+)$, is precisely $F(z)$; again the answer is positive.

Proposition 2.2. *Let $F \in H^2(\mathbb{C}^+)$ and let $\lim\limits_{y \to 0+} F(z) = f(x)$, then $\mathcal{C}[f] = F$.*

So the functions in $L_2(\mathbb{R})$ which may be obtained as non–tangential boundary limit of functions in $H^2(\mathbb{C}^+)$, are special. This leads to the definition of another Hardy space, this time on the real line.

Definition 2.3. *The Hardy space $H^2(\mathbb{R})$ is the closure in $L_2(\mathbb{R})$ of the subspace of the non–tangential boundary limits for $y \to 0+$ of all functions in $H^2(\mathbb{C}^+)$.*

We immediately obtain

Corollary 2.5. *$H^2(\mathbb{R})$ is a closed subspace of $L_2(\mathbb{R})$ and hence itself a Hilbert space.*

It is also clear that both Hardy spaces $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{R})$ are intimately related; in fact they are identified with each other.

Proposition 2.3. *$H^2(\mathbb{R})$ and $H^2(\mathbb{C}^+)$ are isomorphic.*

From the considerations made above it follows that the Cauchy transform

$$\mathcal{C}^+ : L_2(\mathbb{R}) \longrightarrow H^2(\mathbb{R}); f \longmapsto \mathcal{C}^+[f]$$

is a projection; it is called the *Hardy projection*. It will be shown that this Hardy projection is an orthogonal projection. Note however that in a more general setting, where Hardy spaces are defined on the smooth boundary of bounded domains in Euclidean space, the Hardy projection is a skew projection, except for the Hardy space on the unit sphere where the Hardy projection also is orthogonal.

As $H^2(\mathbb{R})$ is a closed subspace of $L_2(\mathbb{R})$, the latter space may be decomposed into an orthogonal direct sum

$$L_2(\mathbb{R}) \;=\; H^2(\mathbb{R}) \oplus_\perp H^2(\mathbb{R})^\perp$$

whence there exist two orthogonal projection operators $\mathbb{P}^+ : L_2(\mathbb{R}) \to H^2(\mathbb{R})$ and $\mathbb{P}^- : L_2(\mathbb{R}) \to H^2(\mathbb{R})^\perp$; these are called the *Szegö projections*. It is then clear from Property 2.10 and Corollary 2.4 that for $f \in L_2(\mathbb{R})$

$$\mathbb{P}^+[f] \;=\; \mathcal{C}^+[f] \quad \text{and} \quad \mathbb{P}^-[f] \;=\; -\mathcal{C}^-[f]$$

as well as

$$\mathbb{P}^+[i\,\mathcal{H}[f]] \;=\; \mathcal{C}^+[f] \quad \text{and} \quad \mathbb{P}^-[i\,\mathcal{H}[f]] \;=\; \mathcal{C}^-[f]$$

So in this case, the Hardy en Szegö projections coincide. The orthogonal decomposition of $L_2(\mathbb{R})$ with respect to the Hardy spaces $H^2(\mathbb{R})$ and $H^2(\mathbb{R})^\perp$ is then visualized by the diagram in Figure 2.2. Note that when restricting to a real valued signal u and its associated real valued Hilbert transform $\mathcal{H}[u]$, their corresponding vector representations in the diagram must lay on the bisectors, since in that case $(u, \mathcal{H}[u])_{L_2} = 0$ (see Figure 2.3).

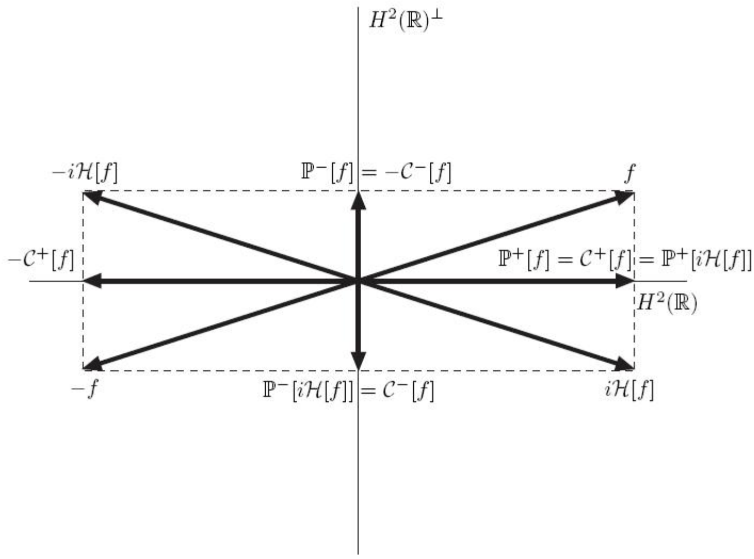

Figure 2.2: The orthogonal decomposition of $L_2(\mathbb{R})$ with respect to $H^2(\mathbb{R})$ and $H^2(\mathbb{R})^\perp$

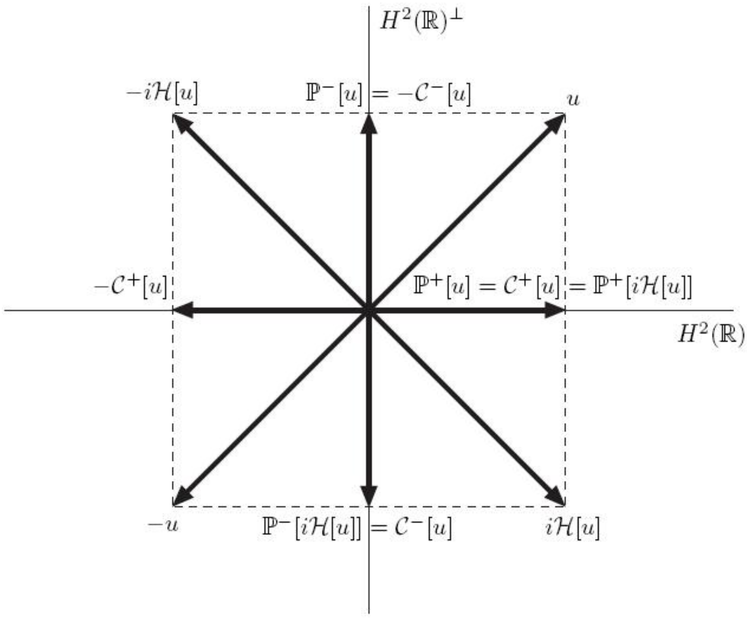

Figure 2.3: The orthogonal decomposition of $L_2(\mathbb{R})$ with respect to $H^2(\mathbb{R})$ and $H^2(\mathbb{R})^\perp$, for real valued functions

The Hardy space $H^2(\mathbb{R})$ is then characterized as follows.

Theorem 2.1. *A function $f \in L_2(\mathbb{R})$ belongs to $H^2(\mathbb{R})$ if and only if one of the following conditions is fulfilled:*

(i) $i\,\mathcal{H}[f] = f$

(ii) the Fourier spectrum of f contains only positive frequencies, *i.e.* $\operatorname{supp}\mathcal{F}[f] \subset [0, +\infty\,[$.

(iii) $\mathbb{P}^+[f] = \mathcal{C}^+[f] = f$

(iv) $\mathcal{C}[f] = \mathcal{P}[f] = i\,\mathcal{Q}[f]$ *in* $\mathbb{C}^+$

(v) $\mathcal{P}[f]$ *is holomorphic in* $\mathbb{C}^+$

(vi) $\mathcal{Q}[f]$ *is holomorphic in* $\mathbb{C}^+$

As already mentioned, the Hardy space $H^2(\mathbb{R})$, as a closed subspace of $L_2(\mathbb{R})$, is itself a Hilbert space. Moreover it possesses a so–called *reproducing kernel*, i.e. a function $K(x,t)$ such that for any $t \in \mathbb{R}$ fixed, $K(x,t) \in H^2(\mathbb{R})$ and

$$(K(\cdot,t), f(\cdot))_{L_2} \;=\; f(t)\;, \quad \forall f \in H^2(\mathbb{R})$$

This reproducing kernel is called the *Szegö kernel*; it is given by

$$S_y(x,t) \;=\; \frac{1}{2\pi i}\,\frac{1}{t-x-iy}\;, \quad y>0$$

in which we recognize the Cauchy kernel:

$$S_y(x,t) \;=\; \frac{i}{2\pi}\,\frac{1}{x-t+iy} \;=\; i\,E(x-t,y)\;, \quad y>0$$

It is proven that for each $t \in \mathbb{R}$ and $y > 0$ fixed, the Szegö kernel indeed belongs to $H^2(\mathbb{R})$ since $\mathcal{C}^+[S_y(x,t)](u) = S_y(u,t)$. The reproducing property of the Szegö kernel then follows from

$$\left(S_y(\cdot,t), f(\cdot)\right)_{L_2} \;=\; -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{1}{t-x+iy}\, f(x)\,dx \;=\; \mathcal{C}[f](t,y)\;, \quad y>0$$

since this Cauchy integral is in $H^2(\mathbb{C}^+)$ and its isomorphic image in $H^2(\mathbb{R})$ is

$$\lim_{y\to 0+} \left(S_y(\cdot,t), f(\cdot)\right)_{L_2} \;=\; \lim_{y\to 0+} \mathcal{C}[f](t,y) \;=\; f(t)\;, \quad t \in \mathbb{R}$$

The Szegö kernel $S_y(x,t)$ shows the symmetry property $S_y^c(x,t) = S_y(t,x)$ if $y > 0$ and moreover it holds in $\mathbb{C}^+$ that

$$\begin{aligned} S_y(x,t) + S_y^c(x,t) &= P(x-t,y) \\ S_y(x,t) - S_y^c(x,t) &= i\,Q(x-t,y) \end{aligned}$$

which links the Szegö kernel to the Poisson kernels. It is important to note that the Szegö kernel is the integral kernel for the Szegö projection $\mathbb{P}^+$ (which here coincides with the Hardy projection $\mathcal{C}^+$), since in $\mathbb{C}^+$

$$\mathcal{C}[f](z) \;=\; \left(S_y(\cdot,x), f(\cdot)\right)_{L_2} \;=\; \int_{-\infty}^{+\infty} S_y(x,t)\,f(t)\,dt$$

and hence

$$\mathbb{P}^+[f](x) \;=\; \mathcal{C}^+[f](x) \;=\; \lim_{y\to 0+}\int_{-\infty}^{+\infty} S_y(x,t)\,f(t)\,dt$$

In a similar way we can look for the reproducing kernel of the Hilbert space $H^2(\mathbb{R})^\perp$; it is called the *Garabedian kernel.* One has for $f \in L_2(\mathbb{R})$ and $y > 0$:

$$(\mathcal{C}[f^c](z))^c \;=\; \int_{-\infty}^{+\infty} S_y^c(x,t)\, f(t)\, dt$$

and hence, due to Corollary 2.4 (iii)

$$\mathbb{P}^-[f](x) \;=\; (\mathcal{C}^+[f^c](x))^c \;=\; \lim_{y\to 0+} \int_{-\infty}^{+\infty} S_y^c(x,t)\, f(t)\, dt$$

This means that the kernel function

$$L_y(x,t) \;=\; S_y^c(x,t) \;=\; \frac{1}{2\pi i}\,\frac{1}{x-t-iy}\;, \quad y>0$$

is the integral kernel for the Szegö projection $\mathbb{P}^-$. The Garabedian kernel $L_y(x,t)$ is clearly anti–holomorphic in $\mathbb{C}^+$ with respect to the variable $z = x+iy$, and it satisfies the symmetry property $L_y^c(x,t) = L_y(t,x)$. Moreover, it is indeed the reproducing kernel for $H^2(\mathbb{R})^\perp$, since for $y > 0$ and $t \in \mathbb{R}$ fixed, $L_y(x,t)$ belongs to $H^2(\mathbb{R})^\perp$ and

$$(L_y(\cdot,t), f(\cdot))_{L_2} \;=\; \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{1}{t-x-iy}\, f(x)\, dx \;=\; -\mathcal{C}[f](t,-y)\;, \quad y>0$$

This Cauchy integral is in the Hardy space $H^2(\mathbb{C}^-)$ which is isomorphic with $H^2(\mathbb{R})^\perp$. If $f \in H^2(\mathbb{R})^\perp$, then the isomorphic image of $-\mathcal{C}[f](t,-y)$ is clearly $-\mathcal{C}^-[f](t) = f(t)$. Note that we now have in $\mathbb{C}^+$

$$\begin{aligned} S_y(x,t) + L_y(x,t) &= P(x-t,y) \\ S_y(x,t) - L_y(x,t) &= i\,Q(x-t,y) \end{aligned}$$

2.1.5 Application: analytic signal

In one–dimensional signal processing, the Hilbert transform has become an indispensable tool for both global and local descriptions of a signal, yielding information on various independent signal properties. The instantaneous amplitude, phase and frequency are estimated by means of so–called quadrature filters, which allow to distinguish different frequency components and therefore locally refine the structure analysis. They are essentially based on the notion of *analytic* *signal*, a concept introduced by Gabor in 1946 (see [57]). An analytic signal is a complex signal, which consists of the linear combination of a bandpass filter, selecting a small part of the spectral information, and its Hilbert transform, the latter basically being the result of a phase shift by $-\pi/2$ on the original filter (see further on). Mathematically, if $u(t)$ is a real valued signal, then the corresponding analytic signal is the function

$$f(t) \;=\; u(t) + i\,\mathcal{H}[u](t)$$

The simplest example of an analytic signal was already used before the actual introduction of that nomenclature. The harmonic complex signal given by Euler's formula

$$\exp(i\omega t) \;=\; \cos(\omega t) + i\,\sin(\omega t)\ , \quad \omega \in \mathbb{R}$$

is commonly applied in electrical engineering since the 1890's and nowadays also in the theoretical description of various, not only electrical, systems. It can be shown that, mathematically in the distributional sense, $\sin(\omega t) = \cos(\omega t - \pi/2)$ is the Hilbert transform of $\cos(\omega t)$ and, since $\mathcal{H}^2 = -\mathbf{1}$, the Hilbert transform also changes $\sin(\omega t)$ into $-\cos(\omega t) = \sin(\omega t - \pi/2)$. That explains why the Hilbert transform is used as a tool for phase shifting a signal by $-\pi/2$, since important classes of real signals $u(t)$ are given by linear combinations in terms of $\cos(\omega t + \alpha)$.

Now note that if u is taken in $L_2(\mathbb{R})$, then it may be clear that its associated analytic signal f belongs to the Hardy space $H^2(\mathbb{R})$, while the complex conjugate $f^c = u - i\,\mathcal{H}[u]$, the so–called associated *anti–analytic signal*, belongs to $H^2(\mathbb{R})^\perp$. Conversely, if $f \in H^2(\mathbb{R})$, then $i\mathcal{H}[f] = f$, which leads to

$$i\mathcal{H}[\mathrm{Re}(f) + i\,\mathrm{Im}(f)] \;=\; \mathrm{Re}(f) + i\,\mathrm{Im}(f)$$

and thus

$$\mathcal{H}[\mathrm{Re}(f)] \;=\; \mathrm{Im}(f) \quad \text{and} \quad -\,\mathcal{H}[\mathrm{Im}(f)] \;=\; \mathrm{Re}(f)$$

and finally

$$f \;=\; (1 + i\,\mathcal{H})[\mathrm{Re}(f)]$$

which clearly has the structure of an analytic signal. With respect to the splitting of the function f into its even and odd parts, we get:

$$i\,\mathcal{H}[f_E + f_O] \;=\; f_E + f_O$$

or

$$i\,\mathcal{H}[f_E] \;=\; f_O \quad \text{and} \quad i\,\mathcal{H}[f_O] \;=\; f_E$$

which leads to the following structure of a function $f \in H^2(\mathbb{R})$

$$f \;=\; (1 + i\,\mathcal{H})[f_E] \;=\; (1 + i\,\mathcal{H})[f_O]$$

Further, as already mentioned, the analytic signal provides direct access to a real one–dimensional signal's *instantaneous amplitude*, *instantaneous phase* and *instantaneous frequency* (see also e.g. [81, 80]). They may be uniquely and conveniently defined introducing the notion of a phasor rotating in the Cartesian $(u, \mathcal{H}[u])$ plane, as shown in Figure 2.4. A change of co-ordinates from rectangular $(u, \mathcal{H}[u])$ to polar (ρ, θ) results into

$$u(t) \;=\; \rho(t)\,\cos(\theta(t))\,, \quad \mathcal{H}[u](t) \;=\; \rho(t)\,\sin(\theta(t)) \quad \text{and} \quad f(t) = \rho(t)\,\exp(i\theta(t))$$

yielding the definition of the instantaneous amplitude $\rho(t)$, the instantaneous phase $\theta(t)$ and the instantaneous frequency $\frac{d\theta}{dt}(t)$.

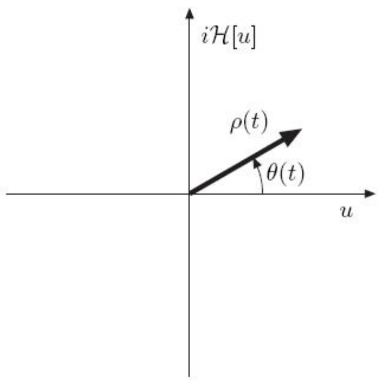

Figure 2.4: A phasor in the Cartesian $(u, \mathcal{H}[u])$ plane representing the analytic signal

To end this section, we introduce the notion of (anti–)causal signal and its relation with (the Fourier transform of) an analytic signal. Simply stated, a *causal signal* $f \in L_2(\mathbb{R})$ is a function with support contained in $[0, +\infty[$. So, the Fourier spectrum of an analytic signal $f \in H^2(\mathbb{R})$ is causal, since

$$\mathcal{F}[f](\omega) \;=\; \mathcal{F}[u + i\mathcal{H}[u]](\omega) \;=\; (1 + \operatorname{sgn}(\omega))\,\mathcal{F}[u](\omega) \;=\; 2\,Y(\omega)\,\mathcal{F}[u](\omega)$$

In other words: the negative frequencies of an analytic signal are vanishing, while the positive frequencies are doubled. An *anti–causal signal* has its support contained in $]-\infty, 0]$. Hence, the Fourier spectrum of an anti–analytic signal $f \in H^2(\mathbb{R})^\perp$ is anti–causal, since

$$\mathcal{F}[f](\omega) \;=\; \mathcal{F}[u - i\mathcal{H}[u]](\omega) \;=\; (1 - \mathrm{sgn}(\omega))\,\mathcal{F}[u](\omega) \;=\; 2\,Y(-\omega)\,\mathcal{F}[u](\omega)$$

Now, as $\mathcal{F}^2$ is a parity operator, i.e. $\mathcal{F}^2[f](t) = f(-t)$, the Fourier spectrum of a causal signal will be anti–analytic, while the Fourier spectrum of an anti–causal signal will be analytic. So, if $f \in L_2(\mathbb{R})$ is a causal signal, then its Fourier spectrum $F(\omega)$ shows the following properties:

(i) $\mathcal{H}[F](\omega) = i\,F(\omega)$

(ii) $\mathcal{H}[F_E](\omega) = i\,F_O(\omega)$

(iii) $\mathcal{H}[F_O](\omega) = i\,F_E(\omega)$

(iv) $F(\omega) = (1 - i\,\mathcal{H})[F_E] = (1 - i\,\mathcal{H})[F_O]$

which are sometimes quoted, especially in engineering textbooks, as "the absorption and dispersion spectra of a causal signal form a Hilbert pair". Similarly, for an anti–causal signal f one has the following properties of its Fourier transform F:

(i) $\mathcal{H}[F](\omega) = -i\,F(\omega)$

(ii) $\mathcal{H}[F_E](\omega) = -i\,F_O(\omega)$

(iii) $\mathcal{H}[F_O](\omega) = -i\,F_E(\omega)$

(iv) $F(\omega) = (1 + i\,\mathcal{H})[F_E] = (1 + i\,\mathcal{H})[F_O]$

2.2 Higher dimensional scalar Hilbert transforms

In the previous section it was explained how the analytic signal turns out to be a useful construct for extracting interesting features of the real one–dimensional signal from which it originates. Of course, a tool for obtaining this kind of information would be of much use for multidimensional signals as well. So there was – and still is – the need for suitable generalizations of the concept of analytic signal to higher dimensions and hence likewise of the concept of Hilbert transform. In this section we briefly discuss some of the existing generalizations in classical contexts and their properties, in order to have a clear view of the advantages and limitations of these attempts. For more detailed information about higher dimensional analytic signals we recommend a.o. [65, 64, 66, 38, 56].

2.2.1 Higher dimensional scalar Hilbert transforms

The total Hilbert transform

A first way to generalize the one–dimensional Hilbert kernel $\mathrm{Pv}\,\frac{1}{\pi t}$ to higher dimensions is by considering the tensor product $\bigotimes_{j=1}^{m} \mathrm{Pv}\,\frac{1}{\pi x_j}$ of the principal value kernels $\mathrm{Pv}\,\frac{1}{\pi x_j}$ in the real x_j–variables, $j = 1, \ldots, m$. For a suitable real signal f defined on $\mathbb{R}^m$, it induces the real m–dimensional *total Hilbert transform*

$$
\begin{aligned}
\mathcal{H}[f](\underline{y}) &= \frac{1}{\pi^m}\,\mathrm{Pv}\int_{\mathbb{R}^m} \frac{f(\underline{u})}{(y_1-u_1)(y_2-u_2)\ldots(y_m-u_m)}\,dV(\underline{u}) \\
&= \frac{1}{\pi^m} \lim_{\substack{\varepsilon_j\to 0+\\ j=1,\ldots,m}} \int_{|x_1|>\varepsilon_1}\cdots\int_{|x_m|>\varepsilon_m} \frac{f(\underline{y}-\underline{x})}{x_1x_2\ldots x_m}\,dV(\underline{x}) \\
&= \left(\left(\mathrm{Pv}\,\frac{1}{\pi x_1}\otimes\cdots\otimes \mathrm{Pv}\,\frac{1}{\pi x_m}\right) * f(\underline{x})\right)(\underline{y})
\end{aligned}
$$

with $dV(\underline{x}) = dx_1dx_2\ldots dx_m$ the Lebesgue measure on $\mathbb{R}^m$ and $\underline{x}$ the short-hand notation of $(x_1, \ldots, x_m)$.

Now we adopt for the Fourier transform in $\mathbb{R}^m$ the following definition

$$
\mathcal{F}[f](\underline{y}) = \int_{\mathbb{R}^m} f(\underline{x})\,\exp\left(-2\pi i\left\langle \underline{x},\underline{y}\right\rangle\right)\,dV(\underline{x}) \tag{2.5}
$$

with $\left\langle \underline{x},\underline{y}\right\rangle = \sum_{j=1}^m x_jy_j$ the classical scalar product in $\mathbb{R}^m$. Further, for a real signal f defined on $\mathbb{R}^m$, its Cauchy integral is defined in $(\mathbb{C}\setminus\mathbb{R})^m$ by

$$
\mathcal{C}[f](z_1,\ldots,z_m) = \left(\frac{-1}{2\pi i}\right)^m \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \frac{f(\xi_1,\ldots,\xi_m)}{\prod_{j=1}^m[(x_j-\xi_j)+i\,y_j]}\,dV(\underline{\xi})
$$

where we assumed each real x_j–axis to be embedded in a complex z_j–plane, $z_j = x_j + iy_j$, $j = 1, \ldots, m$. Note that this Cauchy integral is holomorphic in $(z_1, \ldots, z_m) \in (\mathbb{C}\setminus\mathbb{R})^m$.

The total Hilbert transform then satisfies the following properties.

Property 2.11.

P(1) *The total Hilbert transform is a convolution operator, which is equivalent with saying that the Hilbert transform commutes with translations, i.e.*

$$\tau_{\underline{a}}[\mathcal{H}[f]] \;=\; \mathcal{H}[\tau_{\underline{a}}[f]]$$

with $\tau_{\underline{a}}[f](\underline{x}) = f(\underline{x} - \underline{a})$, $\underline{a} \in \mathbb{R}^m$.

P(2) *The kernel of the total Hilbert transform is a homogeneous distribution of degree* $(-m)$*, which, for a convolution operator, is equivalent with saying that the total Hilbert transform commutes with dilations, i.e.*

$$d_a[\mathcal{H}[f]] \;=\; \mathcal{H}[d_a[f]]$$

with $d_a[f](\underline{x}) = \frac{1}{a^{m/2}} f(\frac{\underline{x}}{a})$, $a > 0$.

P(3) *The total Hilbert and Fourier transforms are interrelated in the following way:*

$$\begin{aligned}
\mathcal{F}[\mathcal{H}[f]](\underline{y}) &= (-i)^m \prod_{j=1}^{m} \operatorname{sgn}(y_j)\, \mathcal{F}[f](\underline{y}) \\
\mathcal{H}[\mathcal{F}[f]](\underline{y}) &= \mathcal{F}[i^m \prod_{j=1}^{m} \operatorname{sgn}(x_j)\, f(\underline{x})](\underline{y})
\end{aligned}$$

P(4) *The total Hilbert transform is a bounded linear operator on* $L_2(\mathbb{R}^m)$*, and is a fortiori norm preserving, i.e.*

$$\|\mathcal{H}[f]\|_{L_2}^2 \;\equiv\; \int_{\mathbb{R}^m} |\mathcal{H}[f](\underline{x})|^2\, dV(\underline{x}) \;=\; \int_{\mathbb{R}^m} |f(\underline{x})|^2\, dV(\underline{x}) \;=\; \|f\|_{L_2}^2$$

More generally:

$$\begin{aligned}
(\mathcal{H}[f], \mathcal{H}[g])_{L_2} &\equiv \int_{\mathbb{R}^m} \mathcal{H}[f]^c(\underline{x})\, \mathcal{H}[g](\underline{x})\, dV(\underline{x}) \\
&= \int_{\mathbb{R}^m} f^c(\underline{x})\, g(\underline{x})\, dV(\underline{x}) \;=\; (f, g)_{L_2}
\end{aligned}$$

P(5) *The total Hilbert transform* $\mathcal{H} : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$

(a) *is invertible with* $\mathcal{H}^{-1} = (-1)^m\, \mathcal{H}$.

(b) *is unitary, its adjoint being given by* $(-1)^m \mathcal{H}$.

(c) *satisfies the orthogonality relation* $(f^c, \mathcal{H}[f])_{L_2} = 0$ *if* m *is odd.*

P(6) *If for some* $j \in \{1, \ldots, m\}$ *it holds that for all* $(x_1, \ldots, x_m) \in \mathbb{R}^m$

(a) $f(x_1, \ldots, x_j, \ldots, x_m) = f(x_1, \ldots, -x_j, \ldots, x_m)$, *then*

$$\mathcal{H}[f](x_1, \ldots, x_j, \ldots, x_m) \; = \; -\mathcal{H}[f](x_1, \ldots, -x_j, \ldots, x_m)$$

(b) $f(x_1, \ldots, x_j, \ldots, x_m) = -f(x_1, \ldots, -x_j, \ldots, x_m)$, *then*

$$\mathcal{H}[f](x_1, \ldots, x_j, \ldots, x_m) \; = \; \mathcal{H}[f](x_1, \ldots, -x_j, \ldots, x_m)$$

P(7) *The total Hilbert transform commutes with differentiation, i.e. if* f *and* $\partial_{x_j} f$ *are in* $L_2(\mathbb{R}^m)$, *for some* $j \in \{1, \ldots, m\}$, *then*

$$\mathcal{H}[\partial_{x_j} f(\underline{x})](\underline{y}) \; = \; \partial_{y_j} \mathcal{H}[f](\underline{y})$$

P(8) *If* f *and* $x_j f$ *are in* $L_2(\mathbb{R}^m)$, *for some* $j \in \{1, \ldots, m\}$, *then*

$$\mathcal{H}[x_j f(\underline{x})](\underline{y}) \; = \; y_j \mathcal{H}[f](\underline{y}) - \frac{1}{\pi^m} \, \mathrm{Pv} \int_{\mathbb{R}^m} \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{1}{y_k - x_k} \, f(\underline{x}) \, dV(\underline{x})$$

P(9) *Stark's extension of Bedrosian's theorem for* $m = 2$ *(see [94]): let* f *and* g *be two–dimensional signals such that for some* $a_1, a_2 \in \mathbb{R}^+$

$$|F(y_1, y_2)| \; = \; 0 \; , \quad \textit{if } |y_1| > a_1 \textit{ and } |y_2| > a_2$$

and

$$|G(y_1, y_2)| \; = \; 0 \; , \quad \textit{if } |y_1| < a_1 \textit{ and } |y_2| < a_2$$

then

$$\mathcal{H}[fg](y_1, y_2) \; = \; f(y_1, y_2) \, \mathcal{H}[g](y_1, y_2)$$

P(10) *Let* $f \in L_2(\mathbb{R}^m)$, *then the non–tangential boundary limits of the Cauchy integral read*

$$\lim_{\substack{y_j \to 0+ \\ j=1,\ldots,m}} \mathcal{C}[f](z_1, \ldots, z_m) \; = \; \frac{1}{2} \, f(x_1, \ldots, x_m) + \frac{1}{2} \, i \, \mathcal{H}[f](x_1, \ldots, x_m)$$

$$\lim_{\substack{y_j \to 0- \\ j=1,\ldots,m}} \mathcal{C}[f](z_1, \ldots, z_m) \; = \; -\frac{1}{2} \, f(x_1, \ldots, x_m) + \frac{1}{2} \, i \, \mathcal{H}[f](x_1, \ldots, x_m)$$

P(11) *If the real signal $f(\underline{x})$ is* separable *in the co-ordinates $\underline{x}$, i.e. if it is given by the product of one–dimensional functions $f(\underline{x}) = f_1(x_1)f_2(x_2)\cdots f_m(x_m)$, then its total Hilbert transform has the form*

$$\mathcal{H}[f](\underline{y}) \;=\; \mathcal{H}[f_1](y_1)\,\mathcal{H}[f_2](y_2)\,\cdots\,\mathcal{H}[f_m](y_m)$$

Remark 2.3. *The optimal convolvability of the total Hilbert kernel $\bigotimes_{j=1}^{m} \mathrm{Pv}\,\frac{1}{\pi x_j}$ with tempered distributions has been studied in [2].*

The partial Hilbert transform

A second generalization of the one–dimensional Hilbert transform is the so–called *partial Hilbert transform $\mathcal{H}_{\underline{n}}$ with respect to a certain direction $\underline{n} \in \mathbb{R}^m$*, defined in frequency space by

$$\mathcal{F}[\mathcal{H}_{\underline{n}}[f]](\underline{y}) \;=\; -i\,\mathrm{sgn}\left(\langle \underline{y}, \underline{n}\rangle\right)\,\mathcal{F}[f](\underline{y})$$

For the special cases where $\underline{n}$ is taken to be one of the standard basis elements $\underline{e}_1 = (1,0,\ldots,0), \underline{e}_2 = (0,1,0,\ldots,0), \ldots, \underline{e}_m = (0,\ldots,0,1)$ in $\mathbb{R}^m$, the partial Hilbert transform $\mathcal{H}_{\underline{e}_j}$ with respect to $\underline{e}_j$ (with $j \in \{1,\ldots,m\}$) is given in the spatial domain by

$$\begin{aligned}
&\mathcal{H}_{\underline{e}_j}[f](\underline{y}) \\
&\quad= \left(\left(\delta(x_1)\otimes\cdots\otimes\delta(x_{j-1})\otimes \mathrm{Pv}\,\frac{1}{\pi x_j}\otimes\delta(x_{j+1})\otimes\cdots\otimes\delta(x_m)\right) * f(\underline{x})\right)(\underline{y}) \\
&\quad= \frac{1}{\pi}\,\mathrm{Pv}\int_{-\infty}^{+\infty}\frac{f(y_1,\ldots,y_{j-1},x_j,y_{j+1},\ldots,y_m)}{y_j - x_j}\,dx_j
\end{aligned}$$

The total Hilbert transform may thus be considered as the successive application of the partial Hilbert transforms with respect to all basis vectors $\underline{e}_j$, $j = 1,\ldots,m$, i.e.

$$\mathcal{H}[f] \;=\; \mathcal{H}_{\underline{e}_1}\left[\mathcal{H}_{\underline{e}_2}\left[\cdots\left[\mathcal{H}_{\underline{e}_m}[f]\right]\right]\right]$$

The partial Hilbert transform $\mathcal{H}_{\underline{e}_j}$ with respect to $\underline{e}_j$, $j = 1,\ldots,m$ then satisfies the following properties.

Property 2.12.

P(1) *The partial Hilbert transform $\mathcal{H}_{\underline{e}_j}$ is a convolution operator, which is equivalent with saying that the partial Hilbert transform $\mathcal{H}_{\underline{e}_j}$ commutes with translations, i.e. $\tau_{\underline{a}}[\mathcal{H}_{\underline{e}_j}[f]] = \mathcal{H}_{\underline{e}_j}[\tau_{\underline{a}}[f]]$, $\underline{a} \in \mathbb{R}^m$.*

P(2) *The kernel of the partial Hilbert transform* $\mathcal{H}_{\underline{e}_j}$ *is a homogeneous distribution of degree* $(-m)$*, which, for a convolution operator, is equivalent with saying that the Hilbert transform commutes with dilations, i.e.* $d_a[\mathcal{H}_{\underline{e}_j}[f]] = \mathcal{H}_{\underline{e}_j}[d_a[f]]$, $a > 0$.

P(3) *The partial Hilbert transform* $\mathcal{H}_{\underline{e}_j}$ *and the Fourier transform are interrelated in the following way:*

$$\begin{aligned} \mathcal{F}[\mathcal{H}_{\underline{e}_j}[f]](\underline{y}) &= -i\,\mathrm{sgn}(y_j)\,\mathcal{F}[f](\underline{y}) \\ \mathcal{H}_{\underline{e}_j}[\mathcal{F}[f]](\underline{y}) &= \mathcal{F}[i\,\mathrm{sgn}(x_j)\,f(\underline{x})](\underline{y}) \end{aligned}$$

P(4) *The partial Hilbert transform* $\mathcal{H}_{\underline{e}_j}$ *is a bounded linear operator on* $L_2(\mathbb{R}^m)$*, and is a fortiori norm preserving, i.e.* $\left\|\mathcal{H}_{\underline{e}_j}[f]\right\|_{L_2}^2 = \|f\|_{L_2}^2$.
More generally: $\left(\mathcal{H}_{\underline{e}_j}[f], \mathcal{H}_{\underline{e}_j}[g]\right)_{L_2} = (f,g)_{L_2}$.

P(5) *The partial Hilbert transform* $\mathcal{H}_{\underline{e}_j} : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$

(a) *is invertible with* $\mathcal{H}_{\underline{e}_j}^{-1} = -\mathcal{H}_{\underline{e}_j}$.

(b) *is unitary, its adjoint being given by* $-\mathcal{H}_{\underline{e}_j}$.

(c) *satisfies the orthogonality relation* $(f^c, \mathcal{H}_{\underline{e}_j}[f])_{L_2} = 0$.

P(6) *If for some* $k \in \{1, \ldots, m\}$ *it holds that for all* $(x_1, \ldots, x_m) \in \mathbb{R}^m$

(a) $f(x_1, \ldots, x_k, \ldots, x_m) = f(x_1, \ldots, -x_k, \ldots, x_m)$, *then*

$$\begin{aligned} &\mathcal{H}_{\underline{e}_j}[f](x_1, \ldots, x_k, \ldots, x_m) \\ &= \begin{cases} \mathcal{H}_{\underline{e}_j}[f](x_1, \ldots, -x_k, \ldots, x_m)\ , & \text{if } k \neq j \\ -\mathcal{H}_{\underline{e}_j}[f](x_1, \ldots, -x_j, \ldots, x_m)\ , & \text{if } k = j \end{cases} \end{aligned}$$

(b) $f(x_1, \ldots, x_k, \ldots, x_m) = -f(x_1, \ldots, -x_k, \ldots, x_m)$, *then*

$$\begin{aligned} &\mathcal{H}_{\underline{e}_j}[f](x_1, \ldots, x_k, \ldots, x_m) \\ &= \begin{cases} -\mathcal{H}_{\underline{e}_j}[f](x_1, \ldots, -x_k, \ldots, x_m)\ , & \text{if } k \neq j \\ \mathcal{H}_{\underline{e}_j}[f](x_1, \ldots, -x_j, \ldots, x_m)\ , & \text{if } k = j \end{cases} \end{aligned}$$

P(7) The partial Hilbert transform $\mathcal{H}_{\underline{e}_j}$ *commutes with differentiation, i.e. if* f *and* $\partial_{x_k} f$ *are in* $L_2(\mathbb{R}^m)$*, for some* $k \in \{1, \ldots, m\}$*, then*

$$\mathcal{H}_{\underline{e}_j}[\partial_{x_k} f(\underline{x})](\underline{y}) \;=\; \partial_{y_k} \mathcal{H}_{\underline{e}_j}[f](\underline{y})$$

P(8) If f *and* $x_k f$ *are in* $L_2(\mathbb{R}^m)$*, for some* $k \in \{1, \ldots, m\}$*, then*

$$\begin{aligned} &\mathcal{H}_{\underline{e}_j}[x_k f(\underline{x})](\underline{y}) \\ &= \begin{cases} y_k \mathcal{H}_{\underline{e}_j}[f](\underline{y}) \;, & \textit{if } k \neq j \\ y_j \mathcal{H}_{\underline{e}_j}[f](\underline{y}) - \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{+\infty} f(y_1, \ldots, x_j, \ldots, y_m)\, dx_j \;, & \textit{if } k = j \end{cases} \end{aligned}$$

P(9) If the real signal $f(\underline{x})$ *is* separable *in the co-ordinates* $\underline{x}$*, i.e. if it is given by* $f(\underline{x}) = f_1(x_1) f_2(x_2) \ldots f_m(x_m)$*, then its partial Hilbert transform* $\mathcal{H}_{\underline{e}_j}$ *has the form*

$$\mathcal{H}_{\underline{e}_j}[f](\underline{y}) \;=\; f_1(y_1)\, f_2(y_2)\, \ldots\, \mathcal{H}_{x_j \to y_j}[f_j(x_j)](y_j)\, \ldots\, f_m(y_m)$$

The Riesz transforms

A third way to generalize the one–dimensional Hilbert transform to $\mathbb{R}^m$ is presented in e.g. [70, 97]. Introducing the Euclidean norm

$$|\underline{x}| \;=\; \sqrt{\langle \underline{x}, \underline{x} \rangle} \;=\; (x_1^2 + x_2^2 + \cdots + x_m^2)^{\frac{1}{2}}$$

one considers the so–called *principal value distributions*

$$K(\underline{x}) \;=\; \mathrm{Pv}\, \frac{\Omega(\underline{x})}{|\underline{x}|^m} \tag{2.6}$$

where Ω is a real valued function defined on $\mathbb{R}^m$; notice that in Chapter 4, we allow Ω to take values in a Clifford algebra as well. Further, the following conditions are imposed upon Ω:

C(1) Ω is homogeneous of degree zero, i.e.

$$\Omega(a\underline{x}) \;=\; \Omega(\underline{x}) \;, \qquad \forall a > 0 \;, \forall \underline{x} \in \mathbb{R}^m$$

C(2) Ω is continuous on the unit sphere S^{m-1} in $\mathbb{R}^m$

C(3) the integral of Ω over the unit sphere vanishes

Note in particular that when $m = 1$ and $\Omega(t) = \frac{1}{\pi}\mathrm{sgn}(t)$, the convolution of $K(t) = \mathrm{Pv}\,\frac{1}{\pi t}$ with a real valued function f defined on $\mathbb{R}$ yields the one–dimensional Hilbert transform of f. In view of that observation, one defines, in general dimension and for general K of the form (2.6), the so–called *singular integral operator*

$$\mathcal{K}[f](\underline{y}) \;=\; \int_{\mathbb{R}^m} K(\underline{y}-\underline{x})\, f(\underline{x})\, dV(\underline{x})$$

It is then shown that $\mathcal{K}$ is a bounded linear operator on $L_2(\mathbb{R}^m)$ (see e.g. [97]).

An important class of singular integral operators in classical harmonic analysis is obtained by considering convolution kernels of the form

$$K(\underline{x}) \;=\; \mathrm{Pv}\,\frac{S_p(\underline{x})}{|\underline{x}|^{m+p}} \;=\; \mathrm{Pv}\,\frac{S_p\left(\frac{\underline{x}}{|\underline{x}|}\right)}{|\underline{x}|^m} \;\equiv\; \mathrm{Pv}\,\frac{\widetilde{S}_p\,(\underline{x})}{|\underline{x}|^m}$$

where S_p is a real valued (or Clifford algebra valued, in general), harmonic, homogeneous polynomial of degree $p \in \mathbb{N}_0 = \{0, 1, 2\ldots\}$; these kernels were introduced by Horváth in [70]. It may be clear that $\widetilde{S}_p$ satisfies the conditions C(1) and C(2). The cancellation condition C(3) is fulfilled as well, due to the orthogonality of the spherical harmonics with the constant function (see e.g. [97, Corollary IV.2.4]). The singular integral operators with such a convolution kernel K were then qualified by Horváth as *generalized Hilbert transforms.*

Important examples of such real valued generalized Hilbert transforms, are the *Riesz transforms* (introduced in [84]), defined by

$$R_j[f](\underline{y}) \;=\; \lim_{\varepsilon\to 0+} \frac{2}{a_{m+1}} \int_{|\underline{y}-\underline{x}|>\varepsilon} \frac{y_j - x_j}{|\underline{y}-\underline{x}|^{m+1}}\; f(\underline{x})\; dV(\underline{x})\;, \qquad j = 1, \ldots, m$$

where $a_{m+1} = \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)}$ denotes the area of the unit sphere S^m in $\mathbb{R}^{m+1}$. Note that in one dimension $m = 1$, the only Riesz transform obtained coincides with the one–dimensional Hilbert transform.

The Riesz transforms R_j, $j = 1, \ldots, m$, satisfy the following properties.

Property 2.13.

P(1) *The Riesz transform* R_j *is a convolution operator, which is equivalent with saying that the Riesz transform* R_j *commutes with translations, i.e.* $\tau_{\underline{a}}[R_j[f]] = R_j[\tau_{\underline{a}}[f]]$, $\underline{a} \in \mathbb{R}^m$.

P(2) *The kernel of the Riesz transform* R_j *is a homogeneous distribution of degree* $(-m)$*, which, for a convolution operator, is equivalent with saying that the Riesz transform* R_j *commutes with dilations, i.e.* $d_a[R_j[f]] = R_j[d_a[f]]$, $a > 0$.

P(3) *The Riesz transform* R_j *and the Fourier transform are interrelated in the following way:*

$$\begin{aligned} \mathcal{F}[R_j[f]](\underline{y}) &= -i\,\frac{y_j}{|\underline{y}|}\,\mathcal{F}[f](\underline{y}) \\ R_j[\mathcal{F}[f]](\underline{y}) &= \mathcal{F}[i\,\frac{x_j}{|\underline{x}|}\,f(\underline{x})](\underline{y}) \end{aligned}$$

P(4) *The Riesz transform* R_j *is a bounded linear operator on* $L_2(\mathbb{R}^m)$*, for which it holds that*

$$\sum_{j=1}^{m} \|R_j[f]\|_{L_2}^2 = \|f\|_{L_2}^2$$

More generally:

$$(f,g)_{L_2} = \sum_{j=1}^{m} (R_j[f], R_j[g])_{L_2}$$

P(5) *For the Riesz transform* $R_j : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$ *it holds that*

(a) $\displaystyle\sum_{j=1}^{m} R_j^2 = -\mathbf{1}$, $\mathbf{1}$ *being the identity operator.*

(b) *its adjoint is given by* $-R_j$.

P(6) *If for some* $k \in \{1,\ldots,m\}$ *it holds that for all* $(x_1,\ldots,x_m) \in \mathbb{R}^m$

(a) $f(x_1,\ldots,x_k,\ldots,x_m) = f(x_1,\ldots,-x_k,\ldots,x_m)$*, then*

$$\begin{aligned} & R_j[f](x_1,\ldots,x_k,\ldots,x_m) \\ &= \begin{cases} R_j[f](x_1,\ldots,-x_k,\ldots,x_m)\ , & \text{if } k \neq j \\ -R_j[f](x_1,\ldots,-x_j,\ldots,x_m)\ , & \text{if } k = j \end{cases} \end{aligned}$$

(b) $f(x_1, \ldots, x_k, \ldots, x_m) = -f(x_1, \ldots, -x_k, \ldots, x_m)$, *then*

$$R_j[f](x_1, \ldots, x_k, \ldots, x_m) = \left\{ \begin{array}{ll} -R_j[f](x_1, \ldots, -x_k, \ldots, x_m) \ , & \text{if } k \neq j \\ R_j[f](x_1, \ldots, -x_j, \ldots, x_m) \ , & \text{if } k = j \end{array} \right.$$

P(7) The Riesz transform R_j *commutes with differentiation, i.e. if* f *and* $\partial_{x_k} f$ *are in* $L_2(\mathbb{R}^m)$*, for some* $k \in \{1, \ldots, m\}$*, then*

$$R_j[\partial_{x_k} f(\underline{x})](\underline{y}) \ = \ \partial_{y_k} R_j[f](\underline{y})$$

P(8) If f *and* $x_k f$ *are in* $L_2(\mathbb{R}^m)$*, for some* $k \in \{1, \ldots, m\}$*, then*

$$R_j[x_k f(\underline{x})](\underline{y}) \ = \ y_k R_j[f](\underline{y}) - \frac{2}{a_{m+1}} \, \mathrm{Pv} \int_{\mathbb{R}^m} \frac{(y_j - x_j)(y_k - x_k)}{|\underline{y} - \underline{x}|^{m+1}} \, f(\underline{x}) \, dV(\underline{x})$$

Remark 2.4. *The optimal convolvability of the Riesz kernel* $\mathrm{Pv}\, \frac{x_j}{|\underline{x}|}$ *with tempered distributions has been studied in [2].*

2.2.2 Application: higher dimensional analytic signals

The total analytic signal

In a first generalization of the concept of analytic signal to higher dimensions, the total Hilbert transform underlies the construction of the so–called *total analytic signal* f_{tot} of a signal $f : \mathbb{R}^m \rightarrow \mathbb{R}$, given by

$$f_{tot} \ = \ f + i \, \mathcal{H}[f]$$

One of the major drawbacks of this construction is that the Fourier transform of f_{tot} reads

$$\mathcal{F}[f_{tot}](\underline{y}) \ = \ \left[1 - (-i)^{m+1} \prod_{j=1}^{m} \mathrm{sgn}(y_j) \right] \mathcal{F}[f](\underline{y})$$

So for m even it is not possible to suppress certain frequency components, as opposed to the classical one–dimensional case where the Fourier spectrum of the analytic signal is causal. Further, for the specific case of $m = 2$, it is also shown in [37] that the total Hilbert transform can no longer be interpreted as performing a phase shift of $-\pi/2$, since the cosine and sine function do not form a Hilbert pair. Finally, in the one–dimensional case, the original signal f is uniquely determined by its amplitude and phase. In [56] the total analytic signal is then even not seen as a valid two–dimensional generalization of the analytic signal, since it does not have a phase which can be related to the one–dimensional phase θ.

The partial analytic signal

A second possible way to introduce a higher dimensional analytic signal is the combination of the original signal with a partial Hilbert transform. The *partial analytic signal* f^{n}_{part} of a signal $f : \mathbb{R}^m \to \mathbb{R}$ with respect to the direction $\underline{n} \in \mathbb{R}^m$ is defined by

$$f^{n}_{part} \;=\; f + i\,\mathcal{H}_{\underline{n}}[f]$$

which in frequency domain reads

$$\mathcal{F}[f^{n}_{part}](\underline{y}) \;=\; \left[1 + \operatorname{sgn}\left(\langle \underline{y}, \underline{n}\rangle\right)\right] \mathcal{F}[f](\underline{y}) \;=\; \begin{cases} 2\,\mathcal{F}(\underline{y})\ , & \text{for } \langle \underline{y}, \underline{n}\rangle > 0 \\ \mathcal{F}(\underline{y})\ , & \text{for } \langle \underline{y}, \underline{n}\rangle = 0 \\ 0\ , & \text{for } \langle \underline{y}, \underline{n}\rangle < 0 \end{cases}$$

This result corresponds to the one–dimensional case where negative frequency components are suppressed while positive frequency components are doubled. Here a frequency $\underline{y}$ is called positive or negative (with respect to $\underline{n}$) if $\langle \underline{y}, \underline{n}\rangle > 0$ or $\langle \underline{y}, \underline{n}\rangle < 0$, respectively. So for the specific case of $\underline{n} = \underline{e}_j$, with $j \in \{1, \ldots, m\}$, the m–dimensional partial analytic signal $f^{\underline{e}_j}_{part}$ is obtained by omitting the half space $y_j < 0$ in the Fourier domain.

However, in [56] it is noticed that the partial Hilbert transform $\mathcal{H}_{\underline{n}}$ leads to amplitude and phase distortions unless the original signal considered only varies with respect to one direction and $\underline{n}$ is chosen properly with respect to that direction.

The single-orthant analytic signal

A third approach to the multidimensional analytic signal has been proposed by Hahn in [64]. Motivated by the one–sidedness of the one–dimensional analytic signal, he defines a higher dimensional complex signal the spectrum of which is zero everywhere except from one orthant of the frequency domain. In one dimension, an orthant is a half axis, in two dimensions a quadrant, in three dimensions an octant, and so on. So, let f be a real m–dimensional signal, then its corresponding *single-orthant analytic signal* f_{so_1} *with respect to the first orthant* $\{\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_j > 0, j = 1, \ldots, m\}$ is given in frequency domain by

$$\mathcal{F}[f_{so_1}](\underline{y}) \quad = \quad \prod_{j=1}^{m} [1 + \mathrm{sgn}(y_j)] \; \mathcal{F}[f](\underline{y}) \tag{2.7}$$

In the classical one–dimensional case the original signal could be recovered from the analytic signal by taking its real part. The same holds for the total and the partial analytic signals f_{tot} and f^{n}_{part}. However, this is no longer the case for the single-orthant analytic signal. To retrieve the original signal, one needs to consider 2^{m-1} single-orthant analytic signals constructed in different orthants (see [65]). We will demonstrate the reconstructability of f from two complex signals with single-orthant spectrum for two–dimensional signals. First of all, for $m = 2$, the expression (2.7) is written as

$$\begin{aligned}\mathcal{F}[f_{so_1}](\underline{y}) \quad &= \quad \mathcal{F}[f](\underline{y}) + \mathrm{sgn}(y_1)\mathrm{sgn}(y_2)\,\mathcal{F}[f](\underline{y}) \\ &\quad\quad + i\,\big(-i\,\mathrm{sgn}(y_1)\,\mathcal{F}[f](\underline{y}) - i\,\mathrm{sgn}(y_2)\,\mathcal{F}[f](\underline{y})\big)\end{aligned}$$

leading to the following definition for f_{so_1} in the spatial domain

$$f_{so_1} \quad = \quad f - \mathcal{H}[f] + i\,\big(\mathcal{H}_{\underline{e}_1}[f] + \mathcal{H}_{\underline{e}_2}[f]\big)$$

Another complex signal, now for the second quadrant, is given by

$$f_{so_2} \quad = \quad f + \mathcal{H}[f] - i\,\big(\mathcal{H}_{\underline{e}_1}[f] - \mathcal{H}_{\underline{e}_2}[f]\big)$$

From the two complex signals f_{so_1} and f_{so_2}, the original signal is then recovered by

$$f \quad = \quad \frac{1}{2}\,\mathrm{Re}(f_{so_1} + f_{so_2})$$

Considering both analytic signals f_{so_1} and f_{so_2} at the same time, a complete signal representation is obtained.

The hypercomplex signal

The most sophisticated approach to a higher dimensional analytic signal combines (the best of) the ideas of total and partial Hilbert transforms and the single-orthant spectrum. To this end, definition (2.7) will be modified in such a way that it keeps enough spectral information for reconstructing the input signal, while at the same time it still allows the suppression of all but one orthant of the spectrum. Therefore we will replace the Fourier transform by the so–called *hypercomplex Fourier transform*, given by

$$\mathcal{F}^h[f](\underline{y}) \;=\; \int_{\mathbb{R}^m} f(\underline{x}) \prod_{k=1}^{m} \exp(-2\pi\, i_k\, x_k y_k)\, dV(\underline{x})$$

the inverse hypercomplex Fourier transform then being defined as

$$f(\underline{x}) \;=\; \int_{\mathbb{R}^m} \mathcal{F}^h[f](\underline{y}) \prod_{k=1}^{m} \exp(2\pi\, i_{m+1-k}\, x_{m+1-k} y_{m+1-k})\, dV(\underline{x})$$

The symbols i_k may be seen as m copies of the imaginary unit i and thus obey the rules $i_k^2 = -1$, $k = 1, \ldots, m$. They generate a real, unital algebra of dimension 2^m where the basis consists of all products $i_{k_1} i_{k_2} \ldots i_{k_h}$, with $1 \leq k_1 < k_2 < \cdots < k_h \leq m$. The algebra can be commutative, i.e. $i_k i_l = i_l i_k$, or anticommutative, i.e. $i_k i_l = -i_l i_k$. In the latter case one has the Clifford algebra of the Euclidean space $\mathbb{R}^m$, which will be discussed in more detail in the following section.

Combining then the definition of the single-orthant analytic signal with the hypercomplex Fourier transform yields the so–called *hypercomplex signal* f_{so}^h of f, given in the frequency domain by

$$\mathcal{F}^h[f_{so}^h](\underline{y}) \;=\; \prod_{k=1}^{m} [1 + \mathrm{sgn}(y_k)]\; \mathcal{F}^h[f](\underline{y})$$

Note that f_{so}^h will take its values in the specific algebra chosen, i.e. either the commutative hypercomplex algebra or the Clifford algebra. It may then be shown that the original signal is contained in its hypercomplex signal as the real part, i.e. $\mathrm{Re}(f_{so}^h) = f$. The main drawback of the hypercomplex analytic signal is that it does not enjoy the flexibility of the partial analytic signal, depending on a direction $\underline{n}$. For a signal f only varying with respect to one direction, the latter can adjust its direction $\underline{n}$ to that specific direction. For the hypercomplex analytic signal this is however not possible. For more detailed information see e.g. [38, 56].

Finally, we quote from [56] that all previously introduced generalizations of the analytic signal in two dimensions fail to estimate in general the correct amplitude and phase of the original signal.

Part I

Hilbert transforms in orthogonal Clifford analysis

Chapter 3

The orthogonal Clifford toolbox

William Kingdon Clifford (1845–1879) introduced the algebras named after him in 1878 (see [41]). They may be seen as higher dimensional generalizations of the algebras of complex numbers and Hamilton's quaternions. Also the Pauli and Dirac matrices, used in quantum mechanics, have a natural representation in Clifford algebra. The importance of these algebras, which Clifford called geometric algebras, essentially lies in the fact that a new vector product, called geometric product, is introduced, incorporating the inner product as well as the wedge product of vectors.

Orthogonal (or standard) Clifford analysis is a higher dimensional function theory in a Clifford algebra setting, offering at the same time a refinement of classical harmonic analysis and a generalization of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, flat m–dimensional Euclidean space, it focusses on the null solutions of various special partial differential operators arising naturally within the Clifford algebra language, the most important of them being the so–called Dirac operator which is the first order vector valued differential operator given by $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \, \partial_{x_j}$. Here $(e_1, \ldots, e_m)$ forms an orthonormal basis for the quadratic space underlying the construction of the Clifford algebra. As the Dirac operator is invariant under the action of the special orthogonal group, we have chosen the name *orthogonal* Clifford analysis to refer to this setting.

In this introductory chapter we present the basic concepts and some results of orthogonal Clifford analysis which are necessary for the sequel. For an in–depth study of this higher dimensional function theory and its applications we refer to e.g. [23, 62, 60, 52, 77, 63, 88, 87, 8, 48, 82, 61].

3.1 Clifford algebra

Let, for $m \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$ such that $p + q = m$, $\mathbb{R}^{p,q}$ be the real vector space $\mathbb{R}^m$, endowed with a non–degenerate quadratic form of signature (p, q), and let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{p,q}$. Then the linear, real and associative *universal Clifford algebra* $\mathbb{R}_{p,q}$ constructed over $\mathbb{R}^{p,q}$ has a non–commutative multiplication governed by the rules:

$$\begin{aligned} e_j^2 &= 1 \ , \quad j = 1, \ldots, p \\ e_{p+j}^2 &= -1 \ , \quad j = 1, \ldots, q \\ e_j e_k + e_k e_j &= 0 \ , \quad j \neq k \ , \quad j, k = 1, \ldots, m \end{aligned}$$

For a set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\} = M$ with $1 \leq j_1 < j_2 < \cdots < j_h \leq m$, let $e_A = e_{j_1} e_{j_2} \ldots e_{j_h}$. Moreover, we put $e_\emptyset = 1$, the latter being the identity element of the algebra. Then $(e_A : A \subset M)$ is a canonical basis for the 2^m–dimensional Clifford algebra $\mathbb{R}_{p,q}$. Any Clifford number λ in $\mathbb{R}_{p,q}$ may thus be written as $\lambda = \sum_{A \subset M} e_A \lambda_A$, with $\lambda_A \in \mathbb{R}$, or still as $\lambda = \sum_{k=0}^{m} [\lambda]_k$ where $[\lambda]_k = \sum_{|A|=k} e_A \lambda_A$ is the so–called *k–vector part* of λ $(k = 0, 1, \ldots, m)$.

If we denote the subspace of k–vectors in $\mathbb{R}_{p,q}$ by $\mathbb{R}_{p,q}^k$, i.e. the image of $\mathbb{R}_{p,q}$ under the projection operator $[\cdot]_k$, one has the so–called *multivector structure decomposition*

$$\mathbb{R}_{p,q} = \mathbb{R}_{p,q}^0 \oplus \mathbb{R}_{p,q}^1 \oplus \cdots \oplus \mathbb{R}_{p,q}^m$$

leading to the identification of $\mathbb{R}$ with the subspace of real scalars $\mathbb{R}_{p,q}^0$ and of $\mathbb{R}^m$ with the subspace of real Clifford–vectors $\mathbb{R}_{p,q}^1$.

The Clifford number $e_M = e_1 e_2 \ldots e_m$ is mostly called the *pseudoscalar*; depending on the dimension m, this pseudoscalar (anti–)commutes with the k–vectors and squares up to ± 1.

Remark 3.1. *We note that there also exist* non–universal Clifford algebras. *This is the case if m is odd and if at the same time the pseudoscalar e_M is a real number. A basis for such a Clifford algebra is given by $(e_A : A \subset M, |A| \in 2\mathbb{N}_0)$ and the dimension consequently equals 2^{m-1}.*

In the sequel we will consider the real Clifford algebra $\mathbb{R}_{0,m}$ and the complex Clifford algebra $\mathbb{C}_m$ which may be seen as its complexification:

$$\mathbb{C}_m \;=\; \mathbb{C} \otimes \mathbb{R}_{0,m} \;=\; \mathbb{R}_{0,m} \oplus i\,\mathbb{R}_{0,m}$$

Let λ and μ be complex Clifford numbers; the (anti–)automorphisms on $\mathbb{C}_m$ leaving the multivector structure invariant are

(i) the *main involution* $\lambda \to \hat{\lambda}$, defined by

$$\begin{aligned} \widehat{(\lambda\mu)} &= \widehat{\lambda}\,\widehat{\mu} \\ \widehat{(e_A\,\lambda_A)} &= \widehat{e_A}\,\lambda_A\ , \qquad A \subset M \\ \widehat{e_j} &= -e_j\ , \qquad j = 1, \ldots, m \end{aligned}$$

(ii) the *reversion* $\lambda \to \lambda^*$, defined by

$$\begin{aligned} (\lambda\mu)^* &= \mu^*\,\lambda^* \\ (e_A\,\lambda_A)^* &= e_A^*\,\lambda_A\ , \qquad A \subset M \\ e_j^* &= e_j\ , \qquad j = 1, \ldots, m \end{aligned}$$

(iii) the *Hermitean conjugation* $\lambda \to \lambda^\dagger$, defined by

$$\begin{aligned} (\lambda\mu)^\dagger &= \mu^\dagger\,\lambda^\dagger \\ (e_A\,\lambda_A)^\dagger &= e_A^\dagger\,\lambda_A^c\ , \qquad A \subset M \\ e_j^\dagger &= -e_j\ , \qquad j = 1, \ldots, m \end{aligned}$$

In view of the decomposition $\mathbb{C}_m = \mathbb{R}_{0,m} \oplus i\,\mathbb{R}_{0,m}$, any complex Clifford number $\lambda \in \mathbb{C}_m$ may also be written as $\lambda = a + i\,b$ with $a, b \in \mathbb{R}_{0,m}$. Moreover, the restriction of the Hermitean conjugation to $\mathbb{R}_{0,m}$ coincides with the usual conjugation in $\mathbb{R}_{0,m}$, i.e. the main anti–involution for which

$$\overline{e_j} \;=\; -e_j\ , \qquad j = 1, \ldots, m$$

Hence, one may also write

$$\lambda^\dagger \;=\; (a + i\,b)^\dagger \;=\; \overline{a} - i\,\overline{b}$$

The Hermitean conjugation leads to a *Hermitean inner product* and its associated *norm* on $\mathbb{C}_m$, respectively given by

$$(\lambda, \mu) \;=\; [\lambda^\dagger \mu]_0 \quad \text{and} \quad |\lambda|^2 \;=\; [\lambda^\dagger \lambda]_0 \;=\; \sum_{A \subset M} |\lambda_A|^2$$

The following properties then hold:

$$|\lambda\mu| \leq 2^m \, |\lambda||\mu| \quad \text{and} \quad |\lambda + \mu| \leq |\lambda| + |\mu|$$

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$ by identifying the point $(x_1, \ldots, x_m) \in \mathbb{R}^m$ with the (Clifford–)vector variable $\underline{x}$ given by

$$\underline{x} \; = \; \sum_{j=1}^{m} e_j \, x_j$$

The (Clifford–)product of any two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{x}\,\underline{y} \quad = \quad \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}$$

with

$$\begin{aligned}
\underline{x} \bullet \underline{y} \quad &= \quad \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\,\underline{x}) \; = \; -\sum_{j=1}^{m} x_j y_j \; = \; -\langle \, \underline{x}, \underline{y} \, \rangle \\
\underline{x} \wedge \underline{y} \quad &= \quad \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x}) \; = \; \sum_{j=1}^{m} \sum_{k=j+1}^{m} e_j e_k \, (x_j y_k - x_k y_j)
\end{aligned}$$

being a scalar and a 2–vector (also called *bivector*), respectively. In particular we note that the square of a vector variable $\underline{x}$ is scalar valued and equals the norm squared up to a minus sign:

$$\underline{x}^2 \; = \; -\langle \, \underline{x}, \underline{x} \, \rangle \; = \; -|\underline{x}|^2 \; = \; -\sum_{j=1}^{m} x_j^2$$

Finally we introduce the Spin group in the following way. The *even subalgebra* $\mathbb{R}_{0,m}^{+}$ of the Clifford algebra $\mathbb{R}_{0,m}$ is defined by

$$\mathbb{R}_{0,m}^{+} \; = \; \bigoplus_{k=0}^{\left[\frac{m}{2}\right]} \mathbb{R}_{0,m}^{2k}$$

The *Clifford group* $\Gamma(m)$ of the Clifford algebra $\mathbb{R}_{0,m}$ consists of those invertible elements $a \in \mathbb{R}_{0,m}$ for which the action $a\underline{x}\overline{a}$ on a vector $\underline{x} \in \mathbb{R}_{0,m}^{1}$ again is a vector. Its subgroup $\Gamma^{+}(m)$ is the intersection of $\Gamma(m)$ with the even subalgebra

$\mathbb{R}^+_{0,m}$. The *Spin group* Spin(m) is the subgroup of $\Gamma^+(m)$ of those elements $s \in \Gamma^+(m)$ for which $ss^* = 1$, or equivalently,

$$\mathrm{Spin}(m) \;=\; \{s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} : \underline{\omega}_j \in S^{m-1}, j = 1, \ldots, 2\ell, \ell \in \mathbb{N}\}$$

The Spin group is a twofold covering group of the rotation group SO(m). Indeed, for any $T \in \mathrm{SO}(m)$, there exists $s \in \mathrm{Spin}(m)$ such that

$$T(\underline{x}) \;=\; s\underline{x}\overline{s} \;=\; (-s)\underline{x}(-\overline{s})\;, \qquad \forall \underline{x} \in \mathbb{R}^1_{0,m}$$

3.2 Clifford analysis

During the last fifty years, Clifford analysis has gradually developed to a comprehensive function theory offering a higher dimensional generalization of the theory of holomorphic functions of one complex variable. At the heart of Clifford analysis is the *Dirac operator*, which is a direct and elegant generalization to higher dimension of the Cauchy–Riemann operator in the complex plane. This Dirac operator in $\mathbb{R}^m$ is the elliptic, rotation invariant, (Clifford–)vector valued differential operator of first order defined by

$$\partial_{\underline{x}} \;=\; \sum_{j=1}^{m} e_j \, \partial_{x_j}$$

its fundamental solution being given by

$$E(\underline{x}) \;=\; \frac{1}{a_m} \frac{\overline{\underline{x}}}{|\underline{x}|^m}\;, \quad \underline{x} \neq \underline{0}$$

with

$$a_m \;=\; \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$$

the area of the unit sphere S^{m-1} in $\mathbb{R}^m$. This means that

(i) E is vector valued and belongs to $L_1^{\mathrm{loc}}(\mathbb{R}^m)$

(ii) $\lim\limits_{|\underline{x}| \to \infty} E(\underline{x}) = 0$

(iii) $\partial_{\underline{x}}\, E(\underline{x}) \;=\; E(\underline{x})\, \partial_{\underline{x}} \;=\; \delta(\underline{x})$ in distributional sense, δ being the classical δ–distribution in $\mathbb{R}^m$, i.e. for each test function ϕ defined on $\mathbb{R}^m$ and with values in $\mathbb{R}_{0,m}$, one has $\langle \delta, \phi \rangle = \phi(\underline{0})$.

It follows that the Dirac operator is invariant under the orthogonal group action, or, equally, under the action of $\mathrm{Spin}(m)$, which in Clifford language is explicited as follows. If $s \in \mathrm{Spin}(m)$ and $H(s)$ is its so–called *H–representation*, given for a Clifford algebra valued function f by

$$H(s)[f(\underline{x})] \;=\; s\, f(s^{-1}\underline{x}s)\, s^{-1}$$

then one has the commutation relation

$$\left[\partial_{\underline{x}}, H(s)\right] \;=\; 0$$

This invariance property actually shows that the Dirac operator is the appropriate operator to work with in the orthogonal setting.

Along with the Dirac operator comes the notion of monogenicity, a notion which is the multidimensional counterpart to the one of holomorphy in the complex plane. Considering functions f defined in $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ or in its complexification $\mathbb{C}_m$, we say that f is *(left) monogenic* in the open region Ω of $\mathbb{R}^m$ if and only if f is continuously differentiable in Ω and satisfies in Ω the equation $\partial_{\underline{x}} f = 0$. The notion of *right monogenicity* is defined in a similar way by letting act the Dirac operator from the right on the function considered. As

$$\overline{\partial_{\underline{x}} f} \;=\; \overline{f}\; \overline{\partial_{\underline{x}}} \;=\; -\overline{f}\partial_{\underline{x}}$$

a function f is (left) monogenic in Ω if and only if $\overline{f}$ is right monogenic in Ω. For example the fundamental solution E is left and right monogenic in $\mathbb{R}^m \setminus \{0\}$. Functions which are null solutions of the operator $\overline{\partial_{\underline{x}}}$ are called *anti–monogenic* with respect to $\partial_{\underline{x}}$. Further, since the Dirac operator factorizes the Laplace operator, i.e.

$$\Delta \;=\; \sum_{j=1}^{m} \partial^2_{x_j} \;=\; -\partial^2_{\underline{x}} \;=\; \partial_{\underline{x}}\,\overline{\partial_{\underline{x}}} \;=\; \overline{\partial_{\underline{x}}}\,\partial_{\underline{x}}$$

a(n) (anti–)monogenic function in Ω is harmonic as well, and hence C^∞ in Ω. So, monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. Note for instance that each harmonic function h can be split as

$$h(\underline{x}) \;=\; f(\underline{x}) + \underline{x}\, g(\underline{x})$$

with f and g monogenic.

Introducing spherical co-ordinates

$$
\begin{aligned}
x_1 &= r \cos\theta_1 \\
x_2 &= r \sin\theta_1 \cos\theta_2 \\
&\vdots \\
x_{m-1} &= r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{m-2} \cos\theta_{m-1} \\
x_m &= r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{m-2} \sin\theta_{m-1}
\end{aligned}
$$

where $0 < r < +\infty$, $0 < \theta_1, \ldots, \theta_{m-2} \leq \pi$, $0 < \theta_{m-1} < 2\pi$, for a point $(x_1, \ldots, x_m) \in \mathbb{R}^m$, we can write

$$
\underline{x} = r\underline{\omega}\,, \quad \text{with} \quad r = |\underline{x}| \quad \text{and} \quad \underline{\omega} \in S^{m-1}
$$

For the Dirac operator $\partial_{\underline{x}}$ the following spherical form is then obtained

$$
\partial_{\underline{x}} = \underline{\omega}\,\partial_r + \frac{1}{r}\partial_{\underline{\omega}} = \underline{\omega}\left(\partial_r - \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\right) \tag{3.1}
$$

where

$$
\partial_{\underline{\omega}} = \sum_{j=1}^{m} \frac{1}{\left|\frac{\partial\underline{\omega}}{\partial\theta_j}\right|^2} \frac{\partial\underline{\omega}}{\partial\theta_j}\,\partial_{\theta_j}
$$

while the Laplace operator takes the form

$$
\Delta = \partial_r^2 + \frac{m-1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta^*
$$

Δ^* being the Laplace–Beltrami operator. The form (3.1) of the Dirac operator can easily be rewritten as

$$
\partial_{\underline{x}} = \frac{\underline{\omega}}{r}\left(r\,\partial_r - \underline{\omega}\,\partial_{\underline{\omega}}\right) = \frac{\underline{\omega}}{r}\left(\mathbf{E}_{\underline{x}} + \mathbf{\Gamma}_{\underline{x}}\right)
$$

where

$$
\begin{aligned}
\mathbf{E}_{\underline{x}} &= r\,\partial_r = \sum_{j=1}^{m} x_j\,\partial_{x_j} \\
\mathbf{\Gamma}_{\underline{x}} &= -\underline{\omega}\,\partial_{\underline{\omega}} = -\underline{\omega} \wedge \partial_{\underline{\omega}} = -\sum_{i<j} e_i\,e_j\,(x_i\,\partial_{x_j} - x_j\,\partial_{x_i})
\end{aligned}
$$

are the (scalar) *Euler operator* and the (bivector valued) *spherical Dirac operator*, respectively.

Further, in Clifford analysis, extensive use is made of the standard tensorial Fourier transform (2.5) defined in Subsection 2.2.1. It satisfies the following calculation rules:

$$\begin{array}{rclrcl} \mathcal{F}[\partial_{\underline{x}}\, f(\underline{x})](\underline{y}) & = & 2\pi i\, \underline{y}\, \mathcal{F}[f](\underline{y}) & \mathcal{F}[f(\underline{x})\, \partial_{\underline{x}}](\underline{y}) & = & 2\pi i\, \mathcal{F}[f](\underline{y})\, \underline{y} \\ 2\pi i\, \mathcal{F}[\underline{x}\, f(\underline{x})](\underline{y}) & = & -\partial_{\underline{y}}\, \mathcal{F}[f](\underline{y}) & 2\pi i\, \mathcal{F}[f(\underline{x})\, \underline{x}](\underline{y}) & = & -\mathcal{F}[f](\underline{y})\, \partial_{\underline{y}} \\ \mathcal{F}[\delta(\underline{x})](\underline{y}) & = & 1 & \mathcal{F}[1](\underline{y}) & = & \delta(\underline{y}) \end{array} \tag{3.2}$$

We also consider the Clifford algebra valued inner product of functions f and g defined in $\mathbb{R}^m$ and taking values in the Clifford algebra $\mathbb{C}_m$:

$$\langle f, g \rangle \;=\; \int_{\mathbb{R}^m} f(\underline{x})^\dagger\, g(\underline{x})\, dV(\underline{x})$$

and moreover the associated norm

$$\|f\|_{L_2}^2 \;=\; [\langle f, f \rangle]_0 \;=\; \int_{\mathbb{R}^m} |f(\underline{x})|^2\, dV(\underline{x})$$

We also introduce the right Hilbert–module $L_2(\mathbb{R}^m)$ of *square integrable functions* f for which it holds that

$$\|f\|_{L_2} \;=\; \left(\int_{\mathbb{R}^m} |f(\underline{x})|^2\, dV(\underline{x}) \right)^{1/2} < \infty$$

Finally a fundamental role is played by the so–called inner and outer spherical monogenics. Let P_p be a real Clifford algebra valued, homogeneous polynomial of degree $p \in \mathbb{N}_0$, i.e.

$$P_p(\underline{x}) \;=\; \sum_{A \subset M} e_A\, P_{p,A}(\underline{x})$$

its components $P_{p,A}$ being real valued homogeneous polynomials of degree p in the variables $x_1, \ldots, x_m$. Furthermore, let P_p be left, respectively right, monogenic in $\mathbb{R}^m$, i.e.

$$\partial_{\underline{x}}\, P_p(\underline{x}) \;=\; 0\ , \quad \text{respectively} \quad P_p(\underline{x})\, \partial_{\underline{x}} \;=\; 0\ , \quad \forall \underline{x} \in \mathbb{R}^m$$

For $p = 0$, we put $P_0 = 1$. By taking restrictions to the unit sphere S^{m-1} of the polynomials $P_p(\underline{x})$, we obtain the left, respectively right, *inner spherical monogenics* $P_p(\underline{\omega})$. Conversely, given such an inner spherical monogenic $P_p(\underline{\omega})$ then obviously $r^p\, P_p(\underline{\omega}) = P_p(\underline{x})$ is a left, respectively right, monogenic homogeneous polynomial the restriction to the unit sphere of which is precisely $P_p(\underline{\omega})$. By spherical inversion, executed on a left, respectively right, monogenic P_p, the functions

$$Q_p^{(l)}(\underline{x}) \;=\; \frac{1}{r^{m+2p}}\,\underline{x}\,P_p(\underline{x}) \;=\; \frac{1}{r^{m+p-1}}\,\underline{\omega}\,P_p(\underline{\omega})$$

and

$$Q_p^{(r)}(\underline{x}) \;=\; \frac{1}{r^{m+2p}}\,P_p(\underline{x})\,\underline{x} \;=\; \frac{1}{r^{m+p-1}}\,P_p(\underline{\omega})\,\underline{\omega}$$

are homogeneous functions of order $-(m+p-1)$ in the complement of the origin which are left, respectively right, monogenic. Their restrictions to the unit sphere S^{m-1}, $\underline{\omega}P_p(\underline{\omega})$ and $P_p(\underline{\omega})\underline{\omega}$, are called *outer spherical monogenics*. Both the inner and the outer spherical monogenics are special cases of spherical harmonics. This implies that (see [52]) for each $p, q \in \mathbb{N}_0$:

$$\int_{S^{m-1}} \overline{P_p(\underline{\omega})}\,Q_q(\underline{\omega})\,dS(\underline{\omega}) \;=\; 0 \tag{3.3}$$

and for each $p \neq q \in \mathbb{N}_0$:

$$\int_{S^{m-1}} \overline{P_p(\underline{\omega})}\,P_q(\underline{\omega})\,dS(\underline{\omega}) \;=\; \int_{S^{m-1}} \overline{Q_p(\underline{\omega})}\,Q_q(\underline{\omega})\,dS(\underline{\omega}) \;=\; 0 \tag{3.4}$$

where $dS(\underline{\omega})$ denotes the Lebesgue measure on S^{m-1}. Moreover, both notions constitute a refinement of the concept of a spherical harmonic. Indeed, taking an arbitrary spherical harmonic $S_p(\underline{\omega})$, one may consider its unique orthogonal decomposition into an inner and an outer spherical monogenic, viz

$$S_p(\underline{\omega}) \;=\; P_p(\underline{\omega}) + Q_{p-1}(\underline{\omega})$$

where obviously $Q_{-1}(\underline{\omega}) \equiv 0$.

In the sequel we take the inner spherical monogenics to be vector valued, i.e. $P_p(\underline{x}) = \sum_{j=1}^m e_j\, P_{p,j}(\underline{x})$. Hence, if P_p is left monogenic it is right monogenic as well, since

$$P_p(\underline{x})\,\partial_{\underline{x}} \;=\; \overline{\overline{\partial_{\underline{x}}}\;\overline{P_p(\underline{x})}} \;=\; \overline{\partial_{\underline{x}}\,P_p(\underline{x})} \;=\; 0$$

Note that such kind of polynomials are easily obtained by considering

$$P_p(\underline{x}) \;=\; \partial_{\underline{x}}\, S_{p+1}(\underline{x})$$

where S_{p+1} is a scalar valued harmonic polynomial of degree $(p+1)$. We also remark that the formulae (3.3) and (3.4) in particular lead to the vanishing integrals

$$\int_{S^{m-1}} \underline{\omega}\, P_p(\underline{\omega})\, dS(\underline{\omega}) \;=\; \int_{S^{m-1}} P_p(\underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}) \;=\; 0\;, \qquad p \in \mathbb{N}_0$$

and

$$\int_{S^{m-1}} P_p(\underline{\omega})\, dS(\underline{\omega}) \;=\; 0\;, \qquad p \in \mathbb{N}$$

while of course for $p = 0$

$$\int_{S^{m-1}} dS(\underline{\omega}) \;=\; a_m$$

Further, these vector valued polynomials P_p enjoy the following calculus rules, being special cases of more general formulae, the proofs of which can be found in [11, 27], for inner spherical monogenics $P_p^{(k)}$ which are k–vector valued ($k \in M$).

Lemma 3.1. *Let P_p be a vector valued, monogenic, homogeneous polynomial of degree $p \in \mathbb{N}_0$, then the following formulae hold in $\mathbb{R}^m$:*

$$\begin{array}{rclcl}
\partial_{\underline{x}}\, P_p(\underline{x}) & = & P_p(\underline{x})\, \partial_{\underline{x}} & = & 0 \\
\partial_{\underline{x}}(\underline{x}\, P_p(\underline{x})) & = & (P_p(\underline{x})\, \underline{x})\partial_{\underline{x}} & = & -(m+2p)\, P_p(\underline{x}) \\
\partial_{\underline{x}}(P_p(\underline{x})\, \underline{x}) & = & (\underline{x}\, P_p(\underline{x}))\partial_{\underline{x}} & = & (m-2)\, P_p(\underline{x})\;, \qquad p \neq 0
\end{array}$$

$$\begin{array}{rcl}
\partial_{\underline{x}}(\underline{x}\, P_p(\underline{x})\, \underline{x}) & = & -(m+2p+2)\, P_p(\underline{x})\, \underline{x} - (m-2)\, \underline{x}\, P_p(\underline{x})\;, \quad p \neq 0 \\
(\underline{x}\, P_p(\underline{x})\, \underline{x})\partial_{\underline{x}} & = & -(m+2p+2)\, \underline{x}\, P_p(\underline{x}) - (m-2)\, P_p(\underline{x})\, \underline{x}\;, \quad p \neq 0
\end{array}$$

and also

$$\Delta\, P_p(\underline{x}) \;=\; \Delta(\underline{x}\; P_p(\underline{x})) \;=\; \Delta(P_p(\underline{x})\; \underline{x}) \;=\; 0$$

$$\Delta(\underline{x}\, P_p(\underline{x})\, \underline{x}) \;=\; 2(m-2)\, P_p(\underline{x})\;, \quad p \neq 0$$

Finally, also the following interesting and useful results have been obtained.

Proposition 3.1. *Let P_p be a vector valued, monogenic, homogeneous polynomial of degree $p \in \mathbb{N}_0$ and let $r = |\underline{x}|$, then for each $l \in \mathbb{N}_0$*

$$\textit{(i)} \quad \textit{(a)} \; P_p(\partial_{\underline{x}})\, r^{2l} \;=\; \begin{cases} 0\ , & \textit{if } l < p \\ 2^p \dfrac{l!}{(l-p)!}\, P_p(\underline{x})\, r^{2l-2p}\ , & \textit{if } l \geq p \end{cases}$$

$$\textit{(b)} \; P_p(\underline{x})\, \partial_{\underline{x}}^{2l}\delta(\underline{x}) \;=\; \begin{cases} 0\ , & \textit{if } l < p \\ 2^p \dfrac{l!}{(l-p)!}\, P_p(\partial_{\underline{x}})\, \partial_{\underline{x}}^{2l-2p}\delta(\underline{x})\ , & \textit{if } l \geq p \end{cases}$$

$$\textit{(ii)} \quad \textit{(a)} \; (-1)^p\, P_p(\partial_{\underline{x}})\, \underline{x}^{2l+1} \;=\; \begin{cases} 0\ , & \textit{if } l < p \\ 2^p \dfrac{l!}{(l-p)!}\, P_p(\underline{x})\, \underline{x}^{2l-2p+1}\ , & \textit{if } l \geq p \end{cases}$$

$$\textit{(b)} \; P_p(\underline{x})\, \partial_{\underline{x}}^{2l+1}\delta(\underline{x}) \;=\; \begin{cases} 0\ , & \textit{if } l < p \\ 2^p \dfrac{l!}{(l-p)!}\, P_p(\partial_{\underline{x}})\, \partial_{\underline{x}}^{2l-2p+1}\delta(\underline{x})\ , & \textit{if } l \geq p \end{cases}$$

$$\textit{(iii)} \quad \textit{(a)} \; (-1)^p\, \underline{x}^{2l+1}\, P_p(\partial_{\underline{x}}) \;=\; \begin{cases} 0\ , & \textit{if } l < p \\ 2^p \dfrac{l!}{(l-p)!}\, \underline{x}^{2l-2p+1}\, P_p(\underline{x})\ , & \textit{if } l \geq p \end{cases}$$

$$\textit{(b)} \; \partial_{\underline{x}}^{2l+1}\delta(\underline{x})\, P_p(\underline{x}) \;=\; \begin{cases} 0\ , & \textit{if } l < p \\ 2^p \dfrac{l!}{(l-p)!}\, \partial_{\underline{x}}^{2l-2p+1}\delta(\underline{x})\, P_p(\partial_{\underline{x}})\ , & \textit{if } l \geq p \end{cases}$$

Proof.

We first prove (i). The calculations being long and technical, we only sketch the main lines of the proof which proceeds in several steps. In order to make the formulae more compact, we will use multi-indices, being m–tuples of nonnegative integers $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$, and we put $|\underline{\alpha}| = \sum_{i=1}^m \alpha_i$.

STEP 1.
We write the vector valued monogenic homogeneous polynomial P_p of degree p as

$$P_p(\underline{x}) \;=\; \sum_{i=1}^m e_i \left(\sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}}\, F(\underline{\alpha})\, x_1^{\alpha_1} \ldots x_m^{\alpha_m} \right) \tag{3.5}$$

with

$$F(\underline{\alpha}) = \frac{|\underline{\alpha}|!}{\alpha_1! \dots \alpha_m!}$$

Then its assumed monogenicity leads to the following conditions on its coefficients: for any $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ with $|\underline{\alpha}| = p$ and for any $i \in \{1, \dots, m\}$ with $\alpha_i \geq 1$:

$$\left\{ \begin{array}{l} \displaystyle\sum_{k=1}^{m} b_{k,\widehat{\underline{\alpha}}_{i(1)}^{k(1)}} = 0 \\ b_{l,\underline{\alpha}} = b_{i,\widehat{\underline{\alpha}}_{i(1)}^{l(1)}} , \quad l = 1, \dots, m \end{array} \right. \tag{3.6}$$

where

$$\widehat{\underline{\alpha}}_{1(s_1)\,2(s_2)\,\dots\,m(s_m)}^{1(q_1)\,2(q_2)\,\dots\,m(q_m)} = (\alpha_1 + q_1 - s_1, \alpha_2 + q_2 - s_2, \dots, \alpha_m + q_m - s_m) = \underline{\alpha} + \underline{q} - \underline{s}$$

Eventually the conditions (3.6) imply that for any $i \in \{1, \dots, m\}$ with $\alpha_i \geq 2$:

$$\sum_{k=1}^{m} b_{i,\widehat{\underline{\alpha}}_{i(2)}^{k(2)}} = 0 \tag{3.7}$$

STEP 2.
As each term in the operator $P_p(\partial_{\underline{x}})$ is of the form $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_m}^{\alpha_m}$, with $\alpha_i \in \mathbb{N}_0$ $(i = 1, \dots, m)$ and $|\underline{\alpha}| = p$, we have explicitly calculated the action of such a term on r^{2l}, $l \in \mathbb{N}_0$, by a double induction argument both on the orders of derivation α_i ánd on the number of α_i's occurring (i.e., not being zero). The obtained result reads

$$\begin{aligned} & \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_m}^{\alpha_m} r^{2l} \\ &= \sum_{j=0}^{S_{\underline{\alpha}}} \left(\sum_{|\underline{\beta}|=j} a_{\alpha_1,\beta_1} \dots a_{\alpha_m,\beta_m} \, x_1^{\alpha_1 - 2\beta_1} \dots x_m^{\alpha_m - 2\beta_m} \right) [2l]_{2p-2j-2} \, r^{2l-2p+2j} \end{aligned} \tag{3.8}$$

where

$$S_{\underline{\alpha}} = \sum_{i=1}^{m} \frac{(\alpha_i)_e}{2} \quad \text{with} \quad (\alpha_i)_e = \left\{ \begin{array}{ll} \alpha_i , & \text{if } \alpha_i \in 2\mathbb{N}_0 \\ \alpha_i - 1 , & \text{if } \alpha_i \in 2\mathbb{N}_0 + 1 \end{array} \right.$$

$$a_{\alpha_i,\beta_i} = \left\{ \begin{array}{ll} \dfrac{1}{2^{\beta_i}} \dfrac{\alpha_i!}{\beta_i!(\alpha_i - 2\beta_i)!} , & \text{if } 0 \leq \beta_i \leq \frac{(\alpha_i)_e}{2} \\ 0 , & \text{otherwise} \end{array} \right.$$

and

$$[2l]_{2p-2j-2} \ = \ (2l)(2l-2)\ldots(2l-2p+2j+2)$$

In particular note that

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_m}^{\alpha_m}\, r^{2l} \ = \ 0 \ , \qquad \text{if } l < p - S_{\underline{\alpha}} \tag{3.9}$$

STEP 3.
We now let $P_p(\partial_{\underline{x}})$ act on r^{2l} for $l \geq p$. On account of (3.8) and of the proposed form (3.5) of $P_p(\underline{x})$ this yields

$$\sum_{i=1}^{m} e_i \sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}}\, F(\underline{\alpha}) \sum_{j=0}^{S_{\underline{\alpha}}} \left(\sum_{|\underline{\beta}|=j} a_{\underline{\alpha},\underline{\beta}}\, \underline{x}^{\underline{\alpha}-2\underline{\beta}} \right) [2l]_{2p-2j-2}\, r^{2(l-p+j)} \tag{3.10}$$

where we introduced the shorthand notations

$$a_{\underline{\alpha},\underline{\beta}} \ = \ a_{\alpha_1,\beta_1}\ldots a_{\alpha_m,\beta_m} \quad \text{and} \quad \underline{x}^{\underline{\alpha}-2\underline{\beta}} \ = \ x_1^{\alpha_1-2\beta_1}\ldots x_m^{\alpha_m-2\beta_m}$$

By isolating the term for $j=0$, expression (3.10) can be rewritten as

$$\begin{aligned} & 2^p\, \frac{l!}{(l-p)!}\, r^{2(l-p)}\, P_p(\underline{x}) \\ & + \sum_{i=1}^{m} e_i \sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}}\, F(\underline{\alpha}) \sum_{j=1}^{S_{\underline{\alpha}}} \left(\sum_{|\underline{\beta}|=j} a_{\underline{\alpha},\underline{\beta}}\, \underline{x}^{\underline{\alpha}-2\underline{\beta}} \right) [2l]_{2p-2j-2}\, r^{2(l-p+j)} \\ \equiv \ & 2^p\, \frac{l!}{(l-p)!}\, r^{2(l-p)}\, P_p(\underline{x}) + R_{p,l}(\underline{x}) \end{aligned} \tag{3.11}$$

We are then lead to the second part of formula (i)(a) if we can prove that each of the terms in $R_{p,l}$ is zero. So, we need to show for each $i \in \{1,\ldots,m\}$ that

$$\sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}}\, F(\underline{\alpha}) \sum_{j=1}^{S} \left(\sum_{|\underline{\beta}|=j} a_{\underline{\alpha},\underline{\beta}}\, \underline{x}^{\underline{\alpha}-2\underline{\beta}} \right) [2l]_{2p-2j-2}\, r^{2j} \ = \ 0 \tag{3.12}$$

where, for convenience, we have increased the upper boundary of the sum in j to $S = \max_{|\underline{\alpha}|=p} S_{\underline{\alpha}} = \frac{p_e}{2}$; this does not affect the sum. Now, taking into account that

$$r^{2j} \ = \ \sum_{|\underline{\theta}|=j} F(\underline{\theta})\, x_1^{2\theta_1}\ldots x_m^{2\theta_m}$$

the left-hand side of (3.12) may be written as a homogeneous polynomial of degree p in $(x_1, \ldots, x_m)$. So, expression (3.12) will hold if for each multi-index $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ for which $|\underline{\gamma}| = p$, the coefficient of $\underline{x}^{\underline{\gamma}}$ equals zero, i.e. if

$$\sum_{j=1}^{S} [2l]_{2p-2j-2} \sum_{|\underline{\theta}|=j} F(\underline{\theta}) \sum_{\substack{|\underline{\alpha}|=p, |\underline{\beta}|=j \\ \underline{\alpha}-2\underline{\beta}=\underline{\gamma}-2\underline{\theta}}} b_{i,\underline{\alpha}}\, F(\underline{\alpha})\, a_{\underline{\alpha},\underline{\beta}} \;=\; 0 \tag{3.13}$$

for each $i \in \{1, \ldots, m\}$ and for each multi-index $\underline{\gamma}$ for which $|\underline{\gamma}| = p$. Note that we only need to consider those multi-indices $\underline{\theta}$ for which

$$\gamma_j - 2\theta_j \;=\; \alpha_j - 2\beta_j \geq 0 \;, \qquad \forall j = 1, \ldots, m \tag{3.14}$$

since otherwise $a_{\underline{\alpha},\underline{\beta}} = 0$. Next, leaning upon

$$F(\underline{\alpha})\, a_{\underline{\alpha},\underline{\beta}} \;=\; F(\underline{\beta})\, \frac{p!}{2^j j!}\, \frac{1}{\prod_{k=1}^{m} (\gamma_k - 2\theta_k)!}$$

for the multi-indices $\underline{\alpha}$ and $\underline{\beta}$ satisfying the conditions under consideration in (3.13), expression (3.13) may be rewritten as

$$\sum_{j=1}^{S} \frac{[2l]_{2p-2j-2}}{2^j j!} \sum_{|\underline{\theta}|=j} \frac{F(\underline{\theta})}{\prod_{k=1}^{m} (\gamma_k - 2\theta_k)!} \sum_{|\underline{\beta}|=j} F(\underline{\beta})\, b_{i,\underline{\gamma}-2\underline{\theta}+2\underline{\beta}} \;=\; 0 \tag{3.15}$$

Now observe that every multi-index $\underline{\beta}$, for which $|\underline{\beta}| = j$, can be represented by a j–tuple $(s_1, \ldots, s_j) \in M^j$, in which each $l \in M$ will appear β_l times, in the following way:

$$\underline{\beta} \;=\; (\beta_1, \ldots, \beta_m) \;\longmapsto\; (s_1, \ldots, s_j) \;, \quad \text{if } \beta_1 e_1 + \cdots + \beta_m e_m = e_{s_1} + \cdots + e_{s_j}$$

It may be readily seen that this representation is not unique; with each multi-index $\underline{\beta}$ corresponds a number of

$$\binom{j}{\beta_1}\binom{j-\beta_1}{\beta_2}\binom{j-\beta_1-\beta_2}{\beta_3}\cdots\binom{j-\sum_{k=1}^{m-1}\beta_k}{\beta_m} \;=\; F(\underline{\beta})$$

such j–tuples $(s_1, \ldots, s_j)$. So, for each fixed i, $\underline{\gamma}$, j and $\underline{\theta}$ under the known conditions, we find that

$$\sum_{|\underline{\beta}|=j} F(\underline{\beta})\, b_{i,\underline{\gamma}-2\underline{\theta}+2\underline{\beta}} \;=\; \sum_{s_1,\ldots,s_j=1}^{m} b_{i,\underline{\gamma}-2\underline{\theta}+2e_{s_1}+2e_{s_2}+\cdots+2e_{s_j}} \tag{3.16}$$

Finally, on account of (3.14) and the conditions (3.7) of the coefficients of P_p, we find for each fixed $e_{s_2}, \ldots, e_{s_j}$ that

$$\sum_{s_1=1}^{m} b_{i,\underline{\gamma}-2\underline{\theta}+2e_{s_1}+2e_{s_2}+\cdots+2e_{s_j}} = 0$$

Hence (3.15) equals zero and $R_{p,l}$ indeed vanishes.

STEP 4.
Next, we consider the case where $l < p$. First, let $l < p - \frac{p_e}{2}$; then clearly, for each $\underline{\alpha}$ with $|\underline{\alpha}| = p$ one has $l < p - S_{\underline{\alpha}}$. Invoking (3.9) we thus have that

$$\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_m}^{\alpha_m} r^{2l} = 0 , \qquad \forall\, \underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \ , \ |\underline{\alpha}| = p$$

yielding $P_p(\partial_{\underline{x}}) r^{2l} = 0$. Next, take $p - \frac{p_e}{2} \leq l < p$. In this case, the arguments of step 3 may be rephrased quite literally, leading to an analogous result as in (3.11), however without the term for $j = 0$, since j will start from $p - l > 0$. So, also here $P_p(\partial_{\underline{x}}) r^{2l} = 0$, implying the first part of formula (i)(a) to hold.

STEP 5.
Finally, formula (i)(b) may be shown by conversion of formula (i)(a) to frequency space and invoking properties (3.2) of the Fourier transform. Indeed, in frequency domain the left-hand side of formula (i)(a) reads

$$\begin{aligned} \mathcal{F}[P_p(\partial_{\underline{x}})\, r^{2l}](\underline{y}) &= (2\pi i)^p\, P_p(\underline{y})\, \mathcal{F}[(-1)^l \underline{x}^{2l}](\underline{y}) \\ &= (-1)^l\, (2\pi i)^{p-2l}\, P_p(\underline{y})\, \partial_{\underline{y}}^{2l} \delta(\underline{y}) \end{aligned}$$

while the Fourier transform of the right-hand side of formula (i)(a) equals

$$\begin{aligned} \mathcal{F}[2^p \frac{l!}{(l-p)!}\, P_p(\underline{x})\, r^{2(l-p)}](\underline{y}) &= \left(\frac{i}{\pi}\right)^p \frac{l!}{(l-p)!}\, P_p(\partial_{\underline{y}})\, \mathcal{F}[(-1)^{l-p} \underline{x}^{2l-2p}](\underline{y}) \\ &= (-1)^l\, (2\pi i)^{p-2l}\, 2^p \frac{l!}{(l-p)!}\, P_p(\partial_{\underline{y}})\, \partial_{\underline{y}}^{2l-2p} \delta(\underline{y}) \end{aligned}$$

for $l \geq p$ and is zero for $l < p$.

The proofs of (ii) and (iii) running along similar lines, we only prove (ii). One has that in distributional sense

$$(-1)^p\, P_p(\partial_{\underline{x}})\, \underline{x}^{2l+1} = (-1)^p \left(P_p(\partial_{\underline{x}})\, \underline{x}^{2l}\right) \underline{x} = (-1)^{p+l} \left(P_p(\partial_{\underline{x}})\, r^{2l}\right) \underline{x}$$

Taking into account (i), this equals zero for $l < p$, while for $l \geq p$ the right-hand side can be rewritten as

$$(-1)^{p+l} \left(2^p \frac{l!}{(l-p)!} \, P_p(\underline{x}) \, r^{2(l-p)} \right) \underline{x} \;=\; 2^p \, \frac{l!}{(l-p)!} \, P_p(\underline{x}) \, \underline{x}^{2l-2p+1}$$

proving formula (ii)(a). Another conversion to frequency space will now lead to formula (ii)(b). ∎

Chapter 4

Four families of Clifford distributions in Euclidean space

In Subsection 2.1.2 we introduced the convolution kernel $\mathrm{Pv}\frac{1}{\pi t}$ for the Hilbert transform on the real line. Higher dimensional scalar valued generalizations were then presented in Section 2.2, by means of a traditional tensorial approach with a number of copies of the one–dimensional kernel or by means of the principal value kernels $K(\underline{x}) = \mathrm{Pv}\,\frac{\Omega(\underline{x})}{|\underline{x}|^m}$. In the framework of Clifford analysis, a multidimensional Hilbert kernel which is vector valued, was already considered by Horváth in his 1953 paper [69]. By adding all Riesz transforms R_j, each one multiplied with its corresponding basis vector e_j $(j \in M)$, he obtained the following Clifford–vector valued Hilbert transform:

$$\widetilde{\mathcal{H}} \;=\; \sum_{j=1}^{m} e_j\, R_j$$

which we will discuss in more detail in the following chapter. Its convolution kernel was recently studied by Delanghe in [50]. Together with Brackx and Sommen he realized however that this higher dimensional principal value distribution is but an example of vector valued Clifford distributions, out of an infinite collection of such kind of distributions, which moreover can be obtained by letting act the Dirac operator on a corresponding infinite set of classical real valued radial distributions. They were also aware of the fact that by making use of the well–known spherical means, which arise naturally by introducing spherical coordinates, a simple, powerful and highly efficient technique could be designed allowing to carry out the explicit calculations on the real line and exporting them to the original setting of Euclidean space. Finally they saw that introducing generalized spherical means involving spherical monogenics would give rise to much more general Clifford distributions, encompassing those already constructed in the special case where the degree of the spherical monogenic considered is zero. The original construction of all these families of distributions is treated in the papers [26, 25].

In the first section of this chapter we present the distribution "finite parts" on the real line, which is the first building block for the construction of the families of Clifford distributions mentioned above. Then, in the following section, the second building block is defined: the "classical" spherical means, which were already introduced in e.g. [92], and their generalizations, involving a spherical monogenic. Both in the classical and in the generalized case, special attention is paid to their interrelation by means of the multiplication with the vector $\underline{x}$ and the action of the Dirac operator. Also their derivatives in the origin are closely studied. In the third section we come to the construction of the families of distributions, depending on the "classical" spherical means. Several properties will be discussed, exposing strong connections between the two families under consideration. We also point out in a series of examples and some historical background that these higher dimensional distributions were already introduced, albeit dispersed, in the literature on harmonic analysis and on Clifford analysis. Classifying those distributions in families offers of course structural clarity and moreover admits to gather results and formulae which are spread over the literature. At the same time this unifying approach proves once more the power and elegance of Clifford analysis. In the last section, the families of distributions and their properties are generalized, making use of the generalized spherical means. They give rise to generalized Hilbert transforms which will be studied in the sequel of this thesis.

4.1 The distribution "finite parts" on the real line

One of the fundamental tools used in our theory of Clifford distributions is the distribution *"finite parts"* $\mathrm{Fp}\, t_+^\mu$ on the real line. Hence, for a better understanding, it is necessary to give its definition and to treat its main properties in some detail (see e.g. [90, 59, 22]).

Let μ be a complex parameter, t a real variable and consider the function

$$t_+^\mu \;=\; \begin{cases} t^\mu\,, & t > 0 \\ 0\,, & t < 0 \end{cases}$$

For $\mathrm{Re}(\mu) > -1$ the function t_+^μ is locally integrable and hence constitutes a regular distribution, its action on any test function ϕ given by

$$\langle\, t_+^\mu \,,\, \phi \,\rangle \;=\; \int_0^{+\infty} t^\mu\, \phi(t)\, dt$$

For $\mathrm{Re}(\mu) \leq -1$, the integral $\int_0^A t^\mu dt$ diverges for each real, positive number A. So in that case, one cannot define a regular distribution. Here we resort to the notion of finite part of a divergent integral: for $\mathrm{Re}(\mu) < -1$ or $\mu = -1 + i\, b$, with $b \in \mathbb{R} \setminus \{0\}$, one defines

$$\mathrm{Fp} \int_0^A t^\mu dt \;=\; \frac{A^{\mu+1}}{\mu+1}$$

while for $\mu = -1$ one puts

$$\mathrm{Fp} \int_0^A t^{-1} dt \;=\; \ln A$$

meaning that the unbounded terms $-\frac{0^{\mu+1}}{\mu+1}$, respectively $-\ln 0$, are dropped. In the same order of ideas one defines the distribution $\mathrm{Fp}\, t_+^\mu$ by

$$\begin{aligned} &\langle\, \mathrm{Fp}\, t_+^\mu \,,\, \phi \,\rangle \\ &= \int_0^{+\infty} t^\mu \left(\phi(t) - \phi(0) - \frac{\phi'(0)}{1!}\, t - \cdots - \frac{\phi^{(n-1)}(0)}{(n-1)!}\, t^{n-1} \right) dt \\ &= \lim_{\varepsilon \to 0+} \left(\int_\varepsilon^{+\infty} t^\mu\, \phi(t)\, dt + \phi(0)\, \frac{\varepsilon^{\mu+1}}{\mu+1} + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!}\, \frac{\varepsilon^{\mu+n}}{\mu+n} \right) \end{aligned} \tag{4.1}$$

when the parameter μ belongs to the strip $-n-1 < \mathrm{Re}(\mu) < -n$ for some $n \in \mathbb{N}$ or when $\mu = -n + i\,b$, with $b \in \mathbb{R} \setminus \{0\}$.

As a function of μ, t_+^μ is readily seen to be holomorphic in $\mathrm{Re}(\mu) > -1$, so by analytic continuation $\mathrm{Fp}\, t_+^\mu$ is holomorphic in $\mathbb{C} \setminus \{-1,-2,\ldots\}$; the singular points $\mu = -n$, $n \in \mathbb{N}$, are simple poles with residue

$$\operatorname*{Res}_{\mu=-n} \mathrm{Fp}\, t_+^\mu \;=\; \frac{(-1)^{n-1}}{(n-1)!}\, \delta^{(n-1)}(t)$$

There are two ways to deal with those singularities.

On the one hand, by slightly changing definition (4.1) for negative entire exponents, viz

$$\begin{aligned} \left\langle\, \mathrm{Fp}\, t_+^{-n} \;,\; \phi \,\right\rangle \;&=\; \lim_{\varepsilon \to 0+} \Bigg(\int_\varepsilon^{+\infty} t^{-n}\, \phi(t)\, dt + \phi(0)\, \frac{\varepsilon^{-n+1}}{-n+1} \\ &\qquad + \cdots + \frac{\phi^{(n-2)}(0)}{(n-2)!}\, \frac{\varepsilon^{-1}}{(-1)} + \frac{\phi^{(n-1)}(0)}{(n-1)!}\, \ln\varepsilon \Bigg) \end{aligned}$$

the so–called *monomial pseudofunctions* $\mathrm{Fp}\, t_+^{-n}$, $n \in \mathbb{N}$ are obtained. In this way the distribution $\mathrm{Fp}\, t_+^\mu$ is defined in the whole of the complex μ–plane, but it is only holomorphic in $\mathbb{C} \setminus \{-1,-2,\ldots\}$. It further enjoys the following properties which will be frequently used in Sections 4.3 and 4.4 while examining the properties of our families of distributions.

Proposition 4.1.

(i) Multiplication rule:

$$t\, \mathrm{Fp}\, t_+^\mu \;=\; \mathrm{Fp}\, t_+^{\mu+1} \;, \qquad \forall \mu \in \mathbb{C}$$

(ii) Derivative rule:

$$\begin{aligned} \frac{d}{dt}\, \mathrm{Fp}\, t_+^\mu \;&=\; \mu\, \mathrm{Fp}\, t_+^{\mu-1} \;, \qquad \mu \neq 0,-1,-2,\ldots \\ \frac{d}{dt}\, \mathrm{Fp}\, t_+^{-n} \;&=\; (-n)\, \mathrm{Fp}\, t_+^{-n-1} + (-1)^n \frac{1}{n!}\, \delta^{(n)}(t) \;, \qquad n = 0,1,2,\ldots \end{aligned}$$

On the other hand one may cope with the singularities of $\mathrm{Fp}\, t_+^\mu$ through the well–known technique of division by an appropriate Gamma–function which shows the same singularities in the complex μ–plane. Indeed, if the distribution $\frac{\mathrm{Fp}\, t_+^\mu}{\Gamma(\mu+1)}$ is defined at $\mu = -n$ ($n \in \mathbb{N}$) as the quotient of the respective residues

$$\left[\frac{\mathrm{Fp}\, t_+^\mu}{\Gamma(\mu+1)}\right]_{\mu=-n} = \delta^{(n-1)}(t)$$

it becomes an entire function of $\mu \in \mathbb{C}$.

In Sections 4.3 and 4.4 we will adopt the latter procedure of so–called *normalization*, more precisely in order to define normalized versions of all distributions under consideration.

4.2 The generalized spherical means

We now pass to the second building block in the definition of our families of distributions: the so–called generalized spherical means. The classical notion of spherical mean, presented in the next subsection, was already introduced in e.g. [72, 92]. In the papers [26, 25], these spherical means were generalized, making use of the vector valued, monogenic, homogeneous polynomials P_p of degree $p \in \mathbb{N}_0$ as introduced in Section 3.2. We define those generalizations and discuss their properties in the second subsection.

4.2.1 The generalized spherical mean operators $\Sigma^{(0)}$ and $\Sigma^{(1)}$

Let ϕ be a scalar valued test function defined on $\mathbb{R}^m$; putting $\underline{x} = r\,\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, we define the *generalized spherical means* (see e.g. [92])

$$\Sigma^{(0)}[\phi] = \frac{1}{a_m} \int_{S^{m-1}} \phi(r\underline{\omega})\, dS(\underline{\omega})$$

and

$$\Sigma^{(1)}[\phi] = \Sigma^{(0)}[\underline{\omega}\phi] = \frac{1}{a_m} \int_{S^{m-1}} \underline{\omega}\, \phi(r\underline{\omega})\, dS(\underline{\omega})$$

Note that $\Sigma^{(0)}[\phi]$ is nothing but the classical spherical mean introduced by John in [72].

The generalized spherical means enjoy the following properties. Proposition 4.2 may be proven straightforward making use of the spherical decomposition $\underline{x} = r\,\underline{\omega}$. The proofs of the other propositions may be found in e.g. [26, 25]. First we observe that both spherical means are interrelated by the multiplication with the vector $\underline{x}$ and by the action of the Dirac operator.

Proposition 4.2. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then*

(i) $\Sigma^{(0)}[\underline{x}\,\phi] \;=\; r\,\Sigma^{(1)}[\phi]$

(ii) $\Sigma^{(1)}[\underline{x}\,\phi] \;=\; -r\,\Sigma^{(0)}[\phi]$

Proposition 4.3. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then*

(i) $\Sigma^{(0)}[\partial_{\underline{x}}\,\phi] \;=\; \left(\partial_r + \dfrac{m-1}{r}\right)\Sigma^{(1)}[\phi]$

(ii) $\Sigma^{(1)}[\partial_{\underline{x}}\,\phi] \;=\; -\partial_r\,\Sigma^{(0)}[\phi]$

Next, for the behaviour of the derivatives of the spherical means at the origin $r = 0$, we introduce the constants

$$C(l) \;=\; \frac{2^{2l}l!}{(2l)!}\left(\frac{m}{2}+l-1\right)\ldots\left(\frac{m}{2}\right) \;=\; \frac{2^{2l}l!}{(2l)!}\,\frac{\Gamma(\frac{m}{2}+l)}{\Gamma(\frac{m}{2})}\;, \qquad l \in \mathbb{N}_0$$

in order to make the formulae more compact.

Proposition 4.4. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then the spherical mean $\Sigma^{(0)}[\phi]$ is an even, scalar valued test function on the real r–axis; its derivatives of odd order vanish at the origin $r = 0$, while for its derivatives of even order one has*

$$\left\{\partial_{\underline{x}}^{2l}\phi(\underline{x})\right\}_{\underline{x}=\underline{0}} \;=\; (-1)^l\,C(l)\,\left\{\partial_r^{2l}\Sigma^{(0)}[\phi]\right\}_{r=0}$$

or, equivalently, in terms of distributions:

$$\left\langle\; \partial_{\underline{x}}^{2l}\delta(\underline{x})\;,\;\phi(\underline{x})\;\right\rangle \;=\; (-1)^l\,C(l)\,\left\langle\; \partial_r^{2l}\delta(r)\;,\;\Sigma^{(0)}[\phi]\;\right\rangle$$

Proposition 4.5. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then the spherical mean $\Sigma^{(1)}[\phi]$ is an odd, vector valued test function on the real r–axis;*

its derivatives of even order vanish at the origin $r = 0$, while for the derivatives of odd order one has

$$\left\{ \partial_{\underline{x}}^{2l+1} \phi(\underline{x}) \right\}_{\underline{x}=\underline{0}} \ = \ (-1)^l \, C(l+1) \, \left\{ \partial_r^{2l+1} \Sigma^{(1)}[\phi] \right\}_{r=0}$$

or, equivalently, in terms of distributions:

$$\left\langle \ \partial_{\underline{x}}^{2l+1} \delta(\underline{x}) \ , \ \phi(\underline{x}) \ \right\rangle \ = \ (-1)^l \, C(l+1) \, \left\langle \ \partial_r^{2l+1} \delta(r) \ , \ \Sigma^{(1)}[\phi] \ \right\rangle$$

Note that in particular

$$\left\{ \Sigma^{(0)}[\phi] \right\}_{r=0} \ = \ \phi(\underline{0})$$

while

$$\left\{ \Sigma^{(1)}[\phi] \right\}_{r=0} \ = \ 0$$

or

$$\int_{S^{m-1}} \underline{\omega} \, dS(\underline{\omega}) \ = \ 0$$

The above expressions for the action of natural powers of the Dirac operator on the delta distribution then lead to the following interesting lemma for their products with natural powers of the vector variable $\underline{x}$. For the proof of the lemma we refer to [28]. To make the formulae more compact we introduce the *Pochhammer symbol*

$$(a)_n \ = \ \frac{\Gamma(a+1)}{\Gamma(a-n+1)}$$

with $a \in \mathbb{C}$ and $n \in \mathbb{N}$ such that neither a nor $a - n$ is a negative integer.

Lemma 4.1. *For $l \in \mathbb{N}$ and $k = 1, 2, \ldots, l$ the following formulae hold:*

(i) $\underline{x}^{2k} \, \partial_{\underline{x}}^{2l} \delta(\underline{x}) \ = \ 2^{2k} \, (l)_k \, (\frac{m}{2} + l - 1)_k \, \partial_{\underline{x}}^{2l-2k} \delta(\underline{x})$

(ii) $\underline{x}^{2k-1} \, \partial_{\underline{x}}^{2l} \delta(\underline{x}) \ = \ 2^{2k-1} \, (l)_k \, (\frac{m}{2} + l - 1)_{k-1} \, \partial_{\underline{x}}^{2l-2k+1} \delta(\underline{x})$

(iii) $\underline{x}^{2k} \, \partial_{\underline{x}}^{2l+1} \delta(\underline{x}) \ = \ \partial_{\underline{x}}^{2l+1} \delta(\underline{x}) \, \underline{x}^{2k} \ = \ 2^{2k} \, (l)_k \, (\frac{m}{2} + l)_k \, \partial_{\underline{x}}^{2l-2k+1} \delta(\underline{x})$

(iv) $\underline{x}^{2k+1} \, \partial_{\underline{x}}^{2l+1} \delta(\underline{x}) \ = \ \partial_{\underline{x}}^{2l+1} \delta(\underline{x}) \, \underline{x}^{2k+1} \ = \ 2^{2k+1} \, (l)_k \, (\frac{m}{2} + l)_{k+1} \, \partial_{\underline{x}}^{2l-2k} \delta(\underline{x})$

4.2.2 The generalized spherical mean operators $\Sigma_p^{(0)}$, $\Sigma_p^{(1)}$, $\Sigma_p^{(2)}$ and $\Sigma_p^{(3)}$

Now, the vector valued, monogenic, homogeneous polynomials P_p of degree $p \in \mathbb{N}_0$ come into play in order to generalize the spherical mean operators $\Sigma^{(0)}$ and $\Sigma^{(1)}$. Let again ϕ be a scalar valued test function defined on $\mathbb{R}^m$, then the spherical mean $\Sigma^{(0)}[\phi]$ is generalized, depending on the parity of p, as follows:

$$\Sigma_p^{(0)}[\phi] \; = \; r^{p-p_e}\, \Sigma^{(0)}[P_p(\underline{\omega})\, \phi(\underline{x})] \; = \; \frac{r^{p-p_e}}{a_m} \int_{S^{m-1}} P_p(\underline{\omega})\, \phi(\underline{x})\, dS(\underline{\omega})$$

where we have introduced the notation p_e ("even part of p") standing for $p_e = p$ if p is even, and for $p_e = p - 1$ if p is odd.

In the same way, the spherical mean $\Sigma^{(1)}[\phi]$ is generalized; depending on the multiplication of $P_p(\underline{\omega})$ from the left or the right we define the following two generalizations:

$$\begin{aligned}
\Sigma_p^{(1)}[\phi] \; &= \; r^{p-p_e}\, \Sigma^{(0)}[\underline{\omega}\, P_p(\underline{\omega})\, \phi(\underline{x})] \; = \; \frac{r^{p-p_e}}{a_m} \int_{S^{m-1}} \underline{\omega}\, P_p(\underline{\omega})\, \phi(\underline{x})\, dS(\underline{\omega}) \\
\Sigma_p^{(3)}[\phi] \; &= \; r^{p-p_e}\, \Sigma^{(0)}[P_p(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})] \; = \; \frac{r^{p-p_e}}{a_m} \int_{S^{m-1}} P_p(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})\, dS(\underline{\omega})
\end{aligned}$$

Finally, by considering the ultimate combination $\underline{\omega} P_p(\underline{\omega}) \underline{\omega}$, the picture is completed symmetrically by introducing the generalized spherical mean

$$\Sigma_p^{(2)}[\phi] \; = \; r^{p-p_e}\, \Sigma^{(0)}[\underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})] \; = \; \frac{r^{p-p_e}}{a_m} \int_{S^{m-1}} \underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})\, dS(\underline{\omega})$$

Note that if $p = 0$, then

$$\Sigma_0^{(0)}[\phi] \; = \; -\Sigma_0^{(2)}[\phi] \; = \; \Sigma^{(0)}[\phi] \quad \text{and} \quad \Sigma_0^{(1)}[\phi] \; = \; \Sigma_0^{(3)}[\phi] \; = \; \Sigma^{(1)}[\phi]$$

Remark 4.1. *It is clear that all generalized spherical means introduced above depend upon the monogenic polynomial P_p chosen. But in order to simplify the notation, we have abbreviated $\Sigma_{P_p}^{(k)}[\phi]$ to $\Sigma_p^{(k)}[\phi]$, $k = 0, 1, 2, 3$. This simplification however is justified in the perspective that monogenic polynomials of the same degree will give rise to seemingly identical formulae, in the sense that the arising constants will only depend on the degree and not on the specific polynomial involved.*

Remark 4.2. *These generalized spherical means may be generalized even further using a k–vector valued spherical monogenic $P_p^{(k)}(\underline{\omega})$, with $k \in M$ (see [11]). They then lead to families of distributions, embedding the ones presented in Section 4.4.*

In the propositions below it is shown how the spherical means are interrelated by the multiplication with the vector $\underline{x}$ and by the action of the Dirac operator. The proof of the first proposition again relies on the spherical decomposition $\underline{x} = r\,\underline{\omega}$. For the proof of the second proposition we once more refer to [26, 25].

Proposition 4.6. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then*

$$\text{(i)}\quad \Sigma_p^{(0)}[\phi\,\underline{x}] \;=\; -\Sigma_p^{(2)}[\underline{x}\,\phi] \;=\; r\,\Sigma_p^{(1)}[\phi]$$

$$\text{(ii)}\quad \Sigma_p^{(0)}[\underline{x}\,\phi] \;=\; -\Sigma_p^{(2)}[\phi\,\underline{x}] \;=\; r\,\Sigma_p^{(3)}[\phi]$$

$$\text{(iii)}\quad \Sigma_p^{(1)}[\phi\,\underline{x}] \;=\; \Sigma_p^{(3)}[\underline{x}\,\phi] \;=\; -r\,\Sigma_p^{(0)}[\phi]$$

$$\text{(iv)}\quad \Sigma_p^{(1)}[\underline{x}\,\phi] \;=\; \Sigma_p^{(3)}[\phi\,\underline{x}] \;=\; r\,\Sigma_p^{(2)}[\phi]$$

where $\Sigma_p^{(0)}[\phi\underline{x}]$ stands for $\Sigma^{(0)}[(\underline{x}\phi)P_p(\underline{\omega})]$ and $\Sigma_p^{(0)}[\underline{x}\phi]$ for $\Sigma^{(0)}[P_p(\underline{\omega})(\underline{x}\phi)]$.

Proposition 4.7. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then*

$$\text{(i)}\quad \Sigma_p^{(0)}[\phi\,\partial_{\underline{x}}] \;=\; \left(\partial_r + \frac{m+p_e-1}{r}\right)\Sigma_p^{(1)}[\phi]$$

$$\text{(ii)}\quad \Sigma_p^{(0)}[\partial_{\underline{x}}\,\phi] \;=\; \left(\partial_r + \frac{m+p_e-1}{r}\right)\Sigma_p^{(3)}[\phi]$$

$$\text{(iii)}\quad r\,\Sigma_p^{(1)}[\phi\,\partial_{\underline{x}}] \;=\; r\,\Sigma_p^{(3)}[\partial_{\underline{x}}\,\phi] \;=\; (-r\,\partial_r + 2p - p_e)\;\Sigma_p^{(0)}[\phi]$$

and for $p \neq 0$

$$\text{(iv)}\quad r\,\Sigma_p^{(1)}[\partial_{\underline{x}}\,\phi] \;=\; r\,\Sigma_p^{(3)}[\phi\,\partial_{\underline{x}}] \;=\; -(m-2)\,\Sigma_p^{(0)}[\phi] + (r\,\partial_r + m + p_e)\,\Sigma_p^{(2)}[\phi]$$

$$\text{(v)}\quad r\,\Sigma_p^{(2)}[\partial_{\underline{x}}\,\phi] \;=\; (-r\,\partial_r + 2p - p_e + 1)\,\Sigma_p^{(1)}[\phi] + (m-2)\,\Sigma_p^{(3)}[\phi]$$

$$\text{(vi)}\quad r\,\Sigma_p^{(2)}[\phi\,\partial_{\underline{x}}] \;=\; (m-2)\,\Sigma_p^{(1)}[\phi] + (-r\,\partial_r + 2p - p_e + 1)\,\Sigma_p^{(3)}[\phi]$$

where $\Sigma_p^{(0)}[\phi\partial_{\underline{x}}]$ stands for $\Sigma^{(0)}[(\partial_{\underline{x}}\phi)P_p(\underline{\omega})]$ and $\Sigma_p^{(0)}[\partial_{\underline{x}}\phi]$ for $\Sigma^{(0)}[P_p(\underline{\omega})(\partial_{\underline{x}}\phi)]$.

Next, the behaviour of the derivatives of the spherical means at the origin $r = 0$ is studied. Observe that the presence of the monogenic polynomials P_p will cause an extra number of derivatives to vanish at the origin.

Proposition 4.8. *If ϕ is a scalar valued test function defined on $\mathbb{R}^m$, then*

(i) the spherical means $\Sigma_p^{(0)}[\phi]$ and $\Sigma_p^{(2)}[\phi]$ are even test functions on the real r–axis; their derivatives of odd order vanish at the origin $r = 0$, while for their derivatives of even order one has for $l < p - \frac{p_e}{2}$:

$$\left\{\partial_r^{2l}\, \Sigma_p^{(0)}[\phi]\right\}_{r=0} = 0 \quad \textit{and} \quad \left\{\partial_r^{2l}\, \Sigma_p^{(2)}[\phi]\right\}_{r=0} = 0$$

while for $l \geq p - \frac{p_e}{2}$:

$$\begin{aligned} &\left\{\partial_r^{2l}\, \Sigma_p^{(0)}[\phi]\right\}_{r=0} \\ &= \frac{(2l)!}{(p_e+2l)!}\,\frac{(-1)^{\frac{p_e}{2}+l}}{C(\frac{p_e}{2}+l)} \left\langle\, P_p(\underline{x}) \left(\partial_{\underline{x}}^{p_e+2l}\delta(\underline{x})\right) \,,\, \phi(\underline{x}) \,\right\rangle \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\left\{\partial_r^{2l}\, \Sigma_p^{(2)}[\phi]\right\}_{r=0} \\ &= \frac{(2l)!}{(p_e+2l+2)!}\,\frac{(-1)^{\frac{p_e}{2}+l+1}}{C(\frac{p_e}{2}+l+1)} \left\langle\, \underline{x}\, P_p(\underline{x})\, \underline{x} \left(\partial_{\underline{x}}^{p_e+2l+2}\delta(\underline{x})\right) \,,\, \phi(\underline{x}) \,\right\rangle \end{aligned} \tag{4.3}$$

(ii) the spherical means $\Sigma_p^{(1)}[\phi]$ and $\Sigma_p^{(3)}[\phi]$ are odd test functions on the real r–axis; their derivatives of even order vanish at the origin $r = 0$, while for the derivatives of odd order one has for $l < p - \frac{p_e}{2}$:

$$\left\{\partial_r^{2l+1}\, \Sigma_p^{(1)}[\phi]\right\}_{r=0} = 0 \quad \textit{and} \quad \left\{\partial_r^{2l+1}\, \Sigma_p^{(3)}[\phi]\right\}_{r=0} = 0$$

while for $l \geq p - \frac{p_e}{2}$:

$$\begin{aligned} &\left\{\partial_r^{2l+1}\, \Sigma_p^{(1)}[\phi]\right\}_{r=0} \\ &= \frac{(2l+1)!}{(p_e+2l+1)!}\,\frac{(-1)^{\frac{p_e}{2}+l+1}}{C(\frac{p_e}{2}+l+1)} \left\langle\, \left(\partial_{\underline{x}}^{p_e+2l+1}\delta(\underline{x})\right) P_p(\underline{x}) \,,\, \phi(\underline{x}) \,\right\rangle \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\left\{ \partial_r^{2l+1} \, \Sigma_p^{(3)}[\phi] \right\}_{r=0} \\ &= \frac{(2l+1)!}{(p_e+2l+1)!} \, \frac{(-1)^{\frac{p_e}{2}+l+1}}{C(\frac{p_e}{2}+l+1)} \left\langle \, P_p(\underline{x}) \left(\partial_{\underline{x}}^{p_e+2l+1} \delta(\underline{x}) \right) \, , \, \phi(\underline{x}) \, \right\rangle \end{aligned} \tag{4.5}$$

Proof.
These results were already partly obtained in [26, 25], where the formulae (4.2) – (4.5) were shown to hold for all $l > 0$. Taking now into account Proposition 3.1, it is then easily seen that for $l < p - \frac{p_e}{2}$ the respective right-hand sides of (4.2), (4.4) and (4.5) vanish, so that indeed

$$\left\{ \partial_r^{2l} \, \Sigma_p^{(0)}[\phi] \right\}_{r=0} \; = \; \left\{ \partial_r^{2l+1} \, \Sigma_p^{(1)}[\phi] \right\}_{r=0} \; = \; \left\{ \partial_r^{2l+1} \, \Sigma_p^{(3)}[\phi] \right\}_{r=0} \; = \; 0$$

Further, leaning on Lemma 4.1 (ii) and then again taking into account Proposition 3.1, we find

$$\underline{x} \, P_p(\underline{x}) \, \underline{x} \left(\partial_{\underline{x}}^{p_e+2l+2} \delta(\underline{x}) \right) \; = \; (p_e+2l+2) \, \underline{x} \, P_p(\underline{x}) \left(\partial_{\underline{x}}^{p_e+2l+1} \delta(\underline{x}) \right) \; = \; 0$$

if $l < p - \frac{p_e}{2}$, whence in that case also

$$\left\{ \partial_r^{2l} \, \Sigma_p^{(2)}[\phi] \right\}_{r=0} \; = \; 0$$

■

4.3 The distributions T_λ^* and U_λ^*

The distributions $T_\lambda = \mathrm{Fp}\, r^\lambda$ and $U_\lambda = \mathrm{Fp}\, r^\lambda \, \underline{\omega}$ (with $\underline{x} = r\,\underline{\omega}$, $r = |\underline{x}|$ and $\lambda \in \mathbb{C}$) have been introduced in [26]. They have then been normalized and extensively studied in a series of papers [28, 29, 30, 19]. The idea behind their construction is disclosed in the following subsection and relies on the spherical co-ordinates, the fundamental distribution $\mathrm{Fp}\, r_+^\mu$ on the real r–axis and the generalized spherical means of Subsection 4.2.1. Subsequently, we present their definitions and some elegant properties. Then, particular examples of classical distributions in Euclidean space are presented which are embedded in our two families of distributions. We end this section with some short historical background since the distributions T_λ are of course very classical, in the sense that they were already introduced in the literature on harmonic analysis. Also the distributions U_λ, which are clearly Clifford–vector valued, will be discussed there; they have vectorial analogues which were introduced in the 1950's.

4.3.1 The idea

The underlying idea for the construction of those distributions mentioned above may be explained by considering the special case of a locally integrable radial function $T(r)$ on $\mathbb{R}^m$, $r = |\underline{x}|$. Its action as a regular distribution on a scalar valued test function ϕ defined on $\mathbb{R}^m$ is given by

$$\langle\, T(r)\ ,\ \phi(\underline{x})\,\rangle \;=\; \int_{\mathbb{R}^m} T(r)\ \phi(\underline{x})\, dV(\underline{x})$$

Introducing spherical co-ordinates, this integral takes the form

$$\int_0^{+\infty} T(r)\, r^{m-1}\, dr \int_{S^{m-1}} \phi(r\underline{\omega})\, dS(\underline{\omega}) \;=\; a_m \int_0^{+\infty} T(r)\, r^{m-1}\, \Sigma^{(0)}[\phi]\, dr$$

In particular for $T(r) = r^\lambda$, with $\mathrm{Re}(\lambda) > -m$, one gets

$$\left\langle\, r^\lambda\ ,\ \phi(\underline{x})\,\right\rangle \;=\; a_m \int_0^{+\infty} r^\mu\, \Sigma^{(0)}[\phi]\, dr \;=\; a_m \left\langle\, r_+^\mu\ ,\ \Sigma^{(0)}[\phi]\,\right\rangle$$

where we have set $\mu = \lambda + m - 1$. Summarizing, the action of the distribution $T(r) = r^\lambda$ in $\mathbb{R}^m$ is converted into an action of the distribution r_+^μ on the real line. Precisely this type of conversion procedure will be used for the definition of our Clifford distributions in $\mathbb{R}^m$.

4.3.2 Definition

Let λ be a complex parameter and let ϕ be a scalar valued test function defined on $\mathbb{R}^m$. We then define the scalar valued distributions T_λ and the vector valued distributions U_λ by:

$$\langle\, T_\lambda\ ,\ \phi\,\rangle \;=\; a_m \left\langle\, \mathrm{Fp}\, r_+^\mu\ ,\ \Sigma^{(0)}[\phi]\,\right\rangle \tag{4.6}$$

and

$$\langle\, U_\lambda\ ,\ \phi\,\rangle \;=\; a_m \left\langle\, \mathrm{Fp}\, r_+^\mu\ ,\ \Sigma^{(1)}[\phi]\,\right\rangle \tag{4.7}$$

where we have put $\mu = \lambda + m - 1$. It is then easily seen that both families of distributions inherit an infinite sequence of singular points from $\mathrm{Fp}\, r_+^\mu$, namely $\mu = -n$, $n \in \mathbb{N}$. However, on account of the vanishing at the origin of the even, respectively odd, derivatives of the spherical means $\Sigma^{(0)}[\phi]$, respectively

$\Sigma^{(1)}[\phi]$, in both cases half of those singularities disappears. Indeed, for $l \in \mathbb{N}_0$ the residue for $\lambda = -m-2l-1$ of (4.6) equals

$$a_m \left\langle \operatorname*{Res}_{\mu=-2l-2} \mathrm{Fp}\, r_+^\mu \ , \ \Sigma^{(0)}[\phi] \right\rangle \ = \ a_m \left\langle \frac{(-1)^{2l+1}}{(2l+1)!}\, \delta^{(2l+1)}(r) \ , \ \Sigma^{(0)}[\phi] \right\rangle \ = \ 0$$

while the residue for $\lambda = -m-2l$ of (4.7) yields

$$a_m \left\langle \operatorname*{Res}_{\mu=-2l-1} \mathrm{Fp}\, r_+^\mu \ , \ \Sigma^{(1)}[\phi] \right\rangle \ = \ a_m \left\langle \frac{(-1)^{2l}}{(2l)!}\, \delta^{(2l)}(r) \ , \ \Sigma^{(1)}[\phi] \right\rangle \ = \ 0$$

Removing the remaining singularities through the well–known technique of division by an appropriate Gamma–function showing the same singularities, results into the following normalizations:

$$\begin{cases} T^*_\lambda \ = \ \pi^{\frac{\lambda+m}{2}} \dfrac{T_\lambda}{\Gamma\left(\frac{\lambda+m}{2}\right)} \ , & \lambda \neq -m-2l \\[2ex] T^*_{-m-2l} \ = \ \dfrac{\pi^{\frac{m}{2}-l}}{2^{2l}\Gamma\left(\frac{m}{2}+l\right)} (-\Delta)^l \delta(\underline{x}) \ , & l \in \mathbb{N}_0 \end{cases}$$

and

$$\begin{cases} U^*_\lambda \ = \ \pi^{\frac{\lambda+m+1}{2}} \dfrac{U_\lambda}{\Gamma\left(\frac{\lambda+m+1}{2}\right)} \ , & \lambda \neq -m-2l-1 \\[2ex] U^*_{-m-2l-1} \ = \ -\dfrac{\pi^{\frac{m}{2}-l}}{2^{2l+1}\,\Gamma\left(\frac{m}{2}+l+1\right)}\, \partial_{\underline{x}}^{2l+1}\delta(\underline{x}) \ , & l \in \mathbb{N}_0 \end{cases}$$

So, up to a power of π, the distributions T^*_{-m-2l} are defined as the quotient of the residues for $\lambda = -m-2l$ of T_λ and $\Gamma(\frac{\lambda+m}{2})$; and the same procedure is applied for the distributions $U^*_{-m-2l-1}$, mutatis mutandis. For the removable poles, the corresponding distributions $T^*_{-m-2l-1}$ and U^*_{-m-2l} are defined by means of a limiting process

$$\begin{aligned} \langle\, T_{-m-2l-1} \ , \ \phi \,\rangle \ &= \ a_m \lim_{\mu \to -2l-2} \left\langle \mathrm{Fp}\, r_+^\mu \ , \ \Sigma^{(0)}[\phi] \right\rangle \\ \langle\, U_{-m-2l} \ , \ \phi \,\rangle \ &= \ a_m \lim_{\mu \to -2l-1} \left\langle \mathrm{Fp}\, r_+^\mu \ , \ \Sigma^{(1)}[\phi] \right\rangle \end{aligned}$$

where the limits at the right-hand sides exactly yield the monomial pseudofunctions $\mathrm{Fp}\, r_+^{-2l-2}$ and $\mathrm{Fp}\, r_+^{-2l-1}$. Summarizing, this normalization procedure results in two entire mappings $\lambda \mapsto T^*_\lambda$ and $\lambda \mapsto U^*_\lambda$ from $\mathbb{C}$ to the space $\mathcal{S}'(\mathbb{R}^m)$ of tempered distributions. The fundamental properties of the normalized distributions T^*_λ and U^*_λ are listed in the following subsection.

4.3.3 Properties

Multiplication with the vector $\underline{x}$ and action of the Dirac operator $\partial_{\underline{x}}$

We first consider on our two families of distributions the action of multiplication with the vector $\underline{x}$ and the action of the Dirac operator $\partial_{\underline{x}}$. Observe that both actions constitute a bijection *between* the two families. For the proofs, which rely on the properties of the finite parts distribution and the generalized spherical means, we refer to [26, 28].

Proposition 4.9. *For $\lambda \in \mathbb{C}$ one has*

(i) $\underline{x}\, T^*_\lambda \;=\; \dfrac{\lambda + m}{2\pi}\, U^*_{\lambda+1}$

(ii) $\underline{x}\, U^*_\lambda \;=\; U^*_\lambda\, \underline{x} \;=\; -T^*_{\lambda+1}$

Proposition 4.10. *For $\lambda \in \mathbb{C}$ one has*

(i) $\partial_{\underline{x}}\, T^*_\lambda \;=\; \lambda\, U^*_{\lambda-1}$

(ii) $\partial_{\underline{x}}\, U^*_\lambda \;=\; U^*_\lambda\, \partial_{\underline{x}} \;=\; -2\pi\, T^*_{\lambda-1}$

Fourier transform

Next, we have also examined the Fourier transforms of the distributions T^*_λ and U^*_λ, constituting a bijection *inside* each of the two families. They are essential for the study of convolution and product of the distributions under consideration (see further). The proof of the following proposition may be found in [28]: whenever λ equals one of the genuine singularities of the distributions T_λ and U_λ, one makes use of the properties (3.2) of the Fourier transform, while in the general case, classical results from [90, p. 257] are applied.

Proposition 4.11. *For $\lambda \in \mathbb{C}$ one has*

(i) $\mathcal{F}\,[T^*_\lambda] \;=\; T^*_{-\lambda-m}$

(ii) $\mathcal{F}\,[U^*_\lambda] \;=\; -i\, U^*_{-\lambda-m}$

Convolution

In [19], our aim was to construct the fundamental solution of an arbitrary complex power of the Dirac operator, these powers being defined as convolution operators with a kernel expressed in terms of T^*_λ– and/ or U^*_λ–distributions.

The desired fundamental solution was found, at least formally, in terms of the same families of distributions and is discussed in Subsection 4.3.4. In order to prove these results in a rigorous way, we first investigate the definition and properties of both the convolution and the product (see further) of arbitrary elements of the families of distributions under consideration, leading to a very attractive pattern of mutual relations between them.

In the sequel we will use the following subsets of $\mathbb{C} \times \mathbb{C}$:

$$
\begin{aligned}
\Phi_m &= \{(\alpha,\beta) \in \mathbb{C} \times \mathbb{C} : \mathrm{Re}(\alpha) > -m \ , \ \mathrm{Re}(\beta) > -m \ , \ \mathrm{Re}\,(\alpha+\beta) < -m\} \\
S_{m,1} &= \{(\alpha,\beta) \in \mathbb{C} \times \mathbb{C} : \alpha = -m-2k \ , \ k \in \mathbb{N}_0 \text{ and } \beta \neq 2l \ , \ l \in \mathbb{N}_0\} \\
S_{m,2} &= \{(\alpha,\beta) \in \mathbb{C} \times \mathbb{C} : \alpha \neq 2k \ , \ k \in \mathbb{N}_0 \text{ and } \beta = -m-2l \ , \ l \in \mathbb{N}_0\} \\
\Psi_3 &= \{(\alpha,\beta) \in (\mathbb{C} \times \mathbb{C}) \setminus (S_{m,1} \cup S_{m,2}) : m \geq 3 \text{ and} \\
&\quad (\mathrm{Re}(\alpha) \leq -m \ , \ -2 > \mathrm{Re}(\beta) > -m \\
&\quad \text{ or } \mathrm{Re}(\beta) \leq -m \ , \ -2 > \mathrm{Re}(\alpha) > -m)\} \\
\Psi_4 &= \{(\alpha,\beta) \in (\mathbb{C} \times \mathbb{C}) \setminus (S_{m,1} \cup S_{m,2}) : m \geq 4 \text{ and} \\
&\quad \mathrm{Re}(\alpha) \leq -m \ , \ \mathrm{Re}(\beta) \leq -m\}
\end{aligned}
$$

and

$$\Omega = \Phi_m \cup S_{m,1} \cup S_{m,2} \cup \Psi_3 \cup \Psi_4$$

Starting point are the Riesz potentials as introduced in [84]. For a complex parameter α and a scalar valued rapidly decreasing function f defined on $\mathbb{R}^m$ they are defined by

$$
\begin{aligned}
I^\alpha[f](\underline{y}) &= \frac{1}{2^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \,\mathrm{Fp} \int_{\mathbb{R}^m} \left|\underline{y}-\underline{x}\right|^{\alpha-m} \, f(\underline{x})\, dV(\underline{x}) \\
&= \frac{1}{2^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \left(\mathrm{Fp}\,|\underline{x}|^{\alpha-m} * f(\underline{x})\right)(\underline{y})
\end{aligned}
$$

Observe that the poles of $\mathrm{Fp}\,|\underline{x}|^{\alpha-m}$ are cancelled by the poles of $\Gamma\left(\frac{\alpha}{2}\right)$, so that $I^\alpha[f]$ is a holomorphic function for $\alpha \neq m+2l$ ($l \in \mathbb{N}_0$). It is clear that for $\alpha \neq -2l$ ($l \in \mathbb{N}_0$), these Riesz potentials may then be rewritten in terms of the distributions $T^*_{\alpha-m}$ in the following way:

$$I^\alpha[f] = \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{2^\alpha \pi^{\frac{\alpha+m}{2}}} \, T^*_{\alpha-m} * f \tag{4.8}$$

We also note that in [84, p. 20] and [68, Proposition 2.36], the following important property of the Riesz potentials has been proven.

Proposition 4.12. *For a scalar valued rapidly decreasing function f defined on $\mathbb{R}^m$ and for each couple $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$, satisfying* $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ *and* $\mathrm{Re}(\alpha + \beta) < m$*, the following identity holds:*

$$I^{\alpha}\left[I^{\beta}[f]\right] \quad = \quad I^{\alpha+\beta}[f]$$

The following result is then obtained and will be proven in a series of propositions.

Theorem 4.1. *For all $(\alpha, \beta) \in \Omega$, the convolution $T^*_\alpha * T^*_\beta$ is the tempered distribution given by*

$$T^*_\alpha * T^*_\beta \quad = \quad c_m(\alpha, \beta)\, T^*_{\alpha+\beta+m} \tag{4.9}$$

where

$$c_m(\alpha, \beta) \quad = \quad \pi^{\frac{m}{2}} \frac{\Gamma\left(-\frac{\alpha+\beta+m}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)} \tag{4.10}$$

Proposition 4.13. *Theorem 4.1 holds whenever the complex parameters α and β fulfil the conditions of the set Φ_m, viz*

(I) $\mathrm{Re}(\alpha) > -m$, $\mathrm{Re}(\beta) > -m$

(II) $\mathrm{Re}(\alpha + \beta) < -m$.

Proof.
The conditions (I) and (II) may be reformulated as

$$\mathrm{Re}(\alpha + m) > 0\ , \quad \mathrm{Re}(\beta + m) > 0 \quad \text{and} \quad \mathrm{Re}(\alpha + \beta + 2m) < m$$

such that, taking into account Proposition 4.12, the following identity holds for a scalar valued rapidly decreasing function f defined on $\mathbb{R}^m$:

$$I^{\alpha+m}\left[I^{\beta+m}[f]\right] \quad = \quad I^{\alpha+\beta+2m}[f] \tag{4.11}$$

Moreover, the conditions (I) and (II) also imply that

$$\alpha + m \neq -2j\ , \quad \beta + m \neq -2k \quad \text{and} \quad \alpha + \beta + 2m \neq -2l\ , \quad j, k, l \in \mathbb{N}_0$$

such that, taking into account (4.8) and putting in particular $f = \delta$, the left-hand side of (4.11) may be written as

$$\frac{\Gamma\left(-\frac{\alpha}{2}\right)}{2^{\alpha+m}\pi^{\frac{\alpha+2m}{2}}}\, T^*_\alpha * \left(\frac{\Gamma\left(-\frac{\beta}{2}\right)}{2^{\beta+m}\pi^{\frac{\beta+2m}{2}}}\, T^*_\beta * \delta\right) = \frac{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)}{2^{\alpha+\beta+2m}\pi^{\frac{\alpha+\beta+4m}{2}}}\, T^*_\alpha * T^*_\beta$$

and for the right-hand side of (4.11) we obtain

$$\frac{\Gamma\left(-\frac{\alpha+\beta+m}{2}\right)}{2^{\alpha+\beta+2m}\pi^{\frac{\alpha+\beta+3m}{2}}}\, T^*_{\alpha+\beta+m}$$

leading to the desired result. ■

Proposition 4.14. *Theorem 4.1 holds whenever the complex parameters α and β fulfil the conditions of the set $S_{m,1} \cup S_{m,2}$.*

Proof.
Since it is clear that the sets $S_{m,1}$ and $S_{m,2}$ only differ in an inversion of the roles of α and β, we restrict ourselves to the proof of the case $(\alpha, \beta) \in S_{m,1}$. So, let $\alpha = -m - 2k$ ($k \in \mathbb{N}_0$) and $\beta \in \mathbb{C}$ such that $\beta \neq 2l$ ($l \in \mathbb{N}_0$). First we infer from Proposition 4.10 that

$$\Delta^k T^*_\beta = (4\pi)^k \frac{\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma\left(\frac{\beta}{2}-k+1\right)}\, T^*_{\beta-2k}, \quad \forall k \in \mathbb{N}_0 \tag{4.12}$$

We then subsequently obtain

$$\begin{aligned} T^*_{-m-2k} * T^*_\beta &= \left(\frac{\pi^{\frac{m}{2}-k}}{2^{2k}\Gamma\left(\frac{m}{2}+k\right)}\,(-\Delta)^k\,\delta\right) * T^*_\beta \\ &= \frac{\pi^{\frac{m}{2}-k}}{2^{2k}\Gamma\left(\frac{m}{2}+k\right)}\,(-1)^k \Delta^k\, T^*_\beta = c_m(-m-2k, \beta)\, T^*_{\beta-2k} \end{aligned}$$

the last step on account of (4.12) and Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad \forall z \in \mathbb{C} \tag{4.13}$$

■

In a third proposition it will be shown that condition (II) of Proposition 4.13 can be omitted, provided that we modify condition (I) and put a restriction on the dimension m.

Proposition 4.15. *Theorem 4.1 holds whenever the complex parameters α and β fulfil the conditions of the set $\Psi_3 \cup \Psi_4$.*

Proof.

(i) First, we prove the proposition for arbitrary $(\alpha, \beta) \in \Psi_3$. To this end, let $\mathrm{Re}(\alpha) \leq -m$ and $-2 > \mathrm{Re}(\beta) > -m$, with $m \geq 3$; interchanging the roles of α and β does not affect the proof. Clearly, the chosen couple (α, β) satisfies condition (II) of Proposition 4.13 but does not fulfil condition (I). In the present situation, it is possible to find $l \in \mathbb{N}$, $\varepsilon \in]0, 2[$ and $r \in]0, m-2[$ such that

$$\begin{aligned} \mathrm{Re}(\alpha) &= -m - 2(l-1) - \varepsilon \\ \mathrm{Re}(\beta) &= -m + r \end{aligned}$$

Now the couple $(\alpha + 2l, \beta)$ fulfils conditions (I) and (II), resulting in

$$T^*_{\alpha+2l} * T^*_\beta = c_m(\alpha + 2l, \beta)\, T^*_{\alpha+\beta+m+2l}$$

As the convolution at the left-hand side is well–defined, application of the operator Δ^l to both sides leads to

$$\left(\Delta^l\, T^*_{\alpha+2l}\right) * T^*_\beta = c_m(\alpha + 2l, \beta)\, \left(\Delta^l\, T^*_{\alpha+\beta+m+2l}\right)$$

Formula (4.12) then yields

$$\begin{aligned} &(4\pi)^l\, \frac{\Gamma\left(\frac{\alpha}{2} + l + 1\right)}{\Gamma\left(\frac{\alpha}{2} + 1\right)}\, T^*_\alpha * T^*_\beta \\ &= c_m(\alpha + 2l, \beta)\, (4\pi)^l\, \frac{\Gamma\left(\frac{\alpha+\beta+m}{2} + l + 1\right)}{\Gamma\left(\frac{\alpha+\beta+m}{2} + 1\right)}\, T^*_{\alpha+\beta+m} \end{aligned}$$

from which (4.9) readily follows when taking into account Euler's reflection formula (4.13).

(ii) Similarly, for $(\alpha, \beta) \in \Psi_4$, it is possible to find $l_1, l_2 \in \mathbb{N}$, and $\varepsilon_1, \varepsilon_2 \in]0, 2[$ such that

$$\begin{aligned} \mathrm{Re}(\alpha) &= -m - 2(l_1 - 1) - \varepsilon_1 \\ \mathrm{Re}(\beta) &= -m - 2(l_2 - 1) - \varepsilon_2 \end{aligned}$$

Then the couple $(\alpha + 2l_1, \beta + 2l_2)$ fulfils conditions (I) and (II), allowing the proof to be completed as in (i). ■

Remark 4.3. *Notice that in part (i) of the previous proof, the condition (II) can not be fulfilled for the couple $(\alpha+2l,\beta)$ if $m=1,2$ or $0>\mathrm{Re}(\beta)>-2$. In the same way, in part (ii), the condition (II) can not be fulfilled for the couple $(\alpha+2l_1,\beta+2l_2)$ if $m=1,2,3$.*

Now, if $(\alpha,\beta)\notin\Omega$ then $T^*_\alpha * T^*_\beta$ does not exist as a genuine convolution anymore. However for each $\beta\in\mathbb{C}\setminus\{2j : j\in\mathbb{N}_0\}$, the expression $c_m(\alpha,\beta)\,T^*_{\alpha+\beta+m}$ is a holomorphic mapping of α, for all $\alpha\in\mathbb{C}\setminus\{2k : k\in\mathbb{N}_0\}$ such that $\alpha\neq-\beta-m+2l$, $l\in\mathbb{N}_0$. Hence, we may define the expression $T^*_\alpha * T^*_\beta$ at the left-hand side of (4.9) in this α–region by analytic continuation. Moreover, this reasoning clearly allows for the roles of α and β to be interchanged. This leads to a sound interpretation of the above result (4.9) in this case, involving a "$*$"–operation which, although not being a genuine convolution, still satisfies the basic properties of a convolution, as will be shown below in Corollary 4.1.

Similarly as above, we now give sense to the distributions $T^*_\alpha * U^*_\beta$, $U^*_\alpha * T^*_\beta$ and $U^*_\alpha * U^*_\beta$. In Theorem 4.2, we trace out the largest subset of $\mathbb{C}\times\mathbb{C}$ for which each of these distributions is a convolution in the classical sense. Next, for all admissible couples (α,β) not belonging to that subset, we provide a "natural" definition.

Theorem 4.2.

(i) For $(\alpha+1,\beta)\in\Omega$, the convolutions $U^*_\alpha * T^*_\beta$ and $T^*_\beta * U^*_\alpha$ are tempered distributions given by

$$U^*_\alpha * T^*_\beta \;=\; T^*_\beta * U^*_\alpha \;=\; c_m(\alpha-1,\beta)\,U^*_{\alpha+\beta+m} \tag{4.14}$$

(ii) For $(\alpha+1,\beta+1)\in\Omega$, the convolution $U^*_\alpha * U^*_\beta$ is a tempered distribution given by

$$U^*_\alpha * U^*_\beta \;=\; \frac{-2\pi}{\alpha+\beta+m}\,c_m(\alpha-1,\beta-1)\,T^*_{\alpha+\beta+m} \tag{4.15}$$

Proof.
We only treat the case of $U^*_\alpha * T^*_\beta$, the other cases being similar.
Take $(\alpha+1,\beta)\in\Omega$, then $T^*_{\alpha+1} * T^*_\beta$ is a genuine convolution and hence:

$$\partial_{\underline{x}}\left(T^*_{\alpha+1} * T^*_\beta\right) \;=\; \left(\partial_{\underline{x}} T^*_{\alpha+1}\right) * T^*_\beta$$

On account of (4.9) and Proposition 4.10, this leads to

$$(\alpha+\beta+m+1)\,c_m(\alpha+1,\beta)\,U^*_{\alpha+\beta+m} \;=\; (\alpha+1)\,U^*_\alpha * T^*_\beta$$

or

$$U^*_\alpha * T^*_\beta \;=\; c_m(\alpha-1,\beta)\, U^*_{\alpha+\beta+m}$$

■

Now note that for the admissible values of $(\alpha+1,\beta) \in \mathbb{C}\times\mathbb{C}$, viz

$$\alpha+1 \neq 2j \;, \quad \beta \neq 2k \quad \text{and} \quad \alpha+\beta+m+1 \neq 2l \;, \quad j,k,l \in \mathbb{N}_0$$

we already had that

$$T^*_{\alpha+1} * T^*_\beta \;=\; c_m(\alpha+1,\beta)\, T^*_{\alpha+\beta+m+1} \tag{4.16}$$

where the "$*$"–operator denotes a genuine convolution if $(\alpha+1,\beta) \in \Omega$. Then the action of the Dirac operator $\partial_{\underline{x}}$ on both sides of (4.16) produces

$$\partial_{\underline{x}} \left(T^*_{\alpha+1} * T^*_\beta\right) \;=\; (\alpha+1)\, c_m(\alpha-1,\beta)\, U^*_{\alpha+\beta+m}$$

which inspires the following definition.

Definition 4.1.

(i) For $(\alpha,\beta) \in \mathbb{C}\times\mathbb{C}$ such that

$$\alpha \neq 2j+1 \;, \quad \beta \neq 2k \quad \textit{and} \quad \alpha+\beta \neq -m+2l+1 \;, \quad j,k,l \in \mathbb{N}_0$$

one puts

$$U^*_\alpha * T^*_\beta \;=\; T^*_\beta * U^*_\alpha \;=\; c_m(\alpha-1,\beta)\, U^*_{\alpha+\beta+m}$$

(ii) For $(\alpha,\beta) \in \mathbb{C}\times\mathbb{C}$ such that

$$\alpha \neq 2j+1 \;, \quad \beta \neq 2k+1 \quad \textit{and} \quad \alpha+\beta \neq -m+2l, \quad j,k,l \in \mathbb{N}_0$$

one puts

$$U^*_\alpha * U^*_\beta \;=\; \frac{-2\pi}{\alpha+\beta+m}\, c_m(\alpha-1,\beta-1)\, T^*_{\alpha+\beta+m}$$

Now, the basic properties of convolutions also hold for the newly defined "$*$"–operation between members of the families of distributions $\mathbb{T} = \{T^*_\lambda : \lambda \in \mathbb{C}\}$ and $\mathbb{U} = \{U^*_\lambda : \lambda \in \mathbb{C}\}$.

Corollary 4.1. *Let X_1 and X_2 be two distributions of $\mathbb{T} \cup \mathbb{U}$. Then, as long as the distributions involved are defined, the following properties hold:*

(i) [Commutativity] $\quad X_1 * X_2 \;=\; X_2 * X_1$

(ii) [Derivation]
$$\begin{aligned} \partial_{\underline{x}} (X_1 * X_2) &= \left(\partial_{\underline{x}} X_1\right) * X_2 = X_1 * \left(\partial_{\underline{x}} X_2\right) \\ (X_1 * X_2)\, \partial_{\underline{x}} &= \left(X_1 \partial_{\underline{x}}\right) * X_2 = X_1 * \left(X_2 \partial_{\underline{x}}\right) \end{aligned}$$

Proof.

(i) This commutative law can readily be checked.

(ii) In all cases where the "$*$"–operator denotes a genuine convolution, the proof is trivial. In the other cases, restricting α and β to admissible values, one has e.g.

$$\partial_{\underline{x}} \left(T^*_\alpha * T^*_\beta\right) \;=\; (\alpha+\beta+m)\; c_m(\alpha,\beta)\; U^*_{\alpha+\beta+m-1}$$

The right-hand side equals

$$\frac{\alpha+\beta+m}{\alpha}\, \frac{c_m(\alpha,\beta)}{c_m(\alpha-2,\beta)} \left(\partial_{\underline{x}} T^*_\alpha\right) * T^*_\beta \;=\; \left(\partial_{\underline{x}} T^*_\alpha\right) * T^*_\beta$$

which also can be written as

$$\frac{\alpha+\beta+m}{\beta}\, \frac{c_m(\alpha,\beta)}{c_m(\alpha,\beta-2)}\, T^*_\alpha * \left(\partial_{\underline{x}} T^*_\beta\right) \;=\; T^*_\alpha * \left(\partial_{\underline{x}} T^*_\beta\right)$$

from which we may conclude that

$$\partial_{\underline{x}} \left(T^*_\alpha * T^*_\beta\right) \;=\; \left(\partial_{\underline{x}} T^*_\alpha\right) * T^*_\beta \;=\; T^*_\alpha * \left(\partial_{\underline{x}} T^*_\beta\right)$$

The other cases are treated similarly. ■

The above properties justify that the "$*$"–operation is called convolution as well, with a slight abuse of language. However, a careful interpretation is required in all cases.

Product

In general, the product of arbitrary distributions is not defined. However, if the convolution of two distributions f and g exists, one can always give meaning to the product of their Fourier transforms, since for $f, g \in \mathcal{S}'$ and $\phi \in \mathcal{S}$:

$$\langle\, \mathcal{F}[f] \cdot \mathcal{F}[g] \;,\; \phi \,\rangle \;=\; \langle\, \mathcal{F}[f * g] \;,\; \phi \,\rangle \;=\; \langle\, f * g \;,\; \mathcal{F}[\phi] \,\rangle$$

Hence, in view of the fact that the Fourier transform maps the set $\mathbb{T}$ (respectively $\mathbb{U}$) onto the set $\mathbb{T}$ (respectively $-i\,\mathbb{U}$), it makes sense to consider products of distributions T^*_λ and U^*_λ. We will give a rigorous definition for the distributions $T^*_\alpha \cdot T^*_\beta$, $T^*_\alpha \cdot U^*_\beta$, $U^*_\alpha \cdot T^*_\beta$ and $U^*_\alpha \cdot U^*_\beta$, for all allowed values of $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$. Note that we use the dot notation in order to emphasize that we are dealing with a product.

Theorem 4.3.

(i) *For $(-\alpha - m, -\beta - m) \in \Omega$, the product $T^*_\alpha \cdot T^*_\beta$ is a tempered distribution given by*

$$T^*_\alpha \cdot T^*_\beta \quad = \quad c_m(-\alpha - m, -\beta - m)\, T^*_{\alpha+\beta}$$

(ii) *For $(-\alpha - m + 1, -\beta - m) \in \Omega$, the products $U^*_\alpha \cdot T^*_\beta$ and $T^*_\beta \cdot U^*_\alpha$ are tempered distributions given by*

$$U^*_\alpha \cdot T^*_\beta \quad = \quad T^*_\beta \cdot U^*_\alpha \; = \; c_m(-\alpha - m - 1, -\beta - m)\, U^*_{\alpha+\beta}$$

(iii) *For $(-\alpha - m + 1, -\beta - m + 1) \in \Omega$, the product $U^*_\alpha \cdot U^*_\beta$ is a tempered distribution given by*

$$U^*_\alpha \cdot U^*_\beta \quad = \quad \frac{2\pi}{\alpha + \beta + m}\, c_m(-\alpha - m - 1, -\beta - m - 1)\, T^*_{\alpha+\beta}$$

Proof.
We only treat the product $T^*_\alpha \cdot T^*_\beta$, the other cases being similar.
Take $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$ such that $(-\alpha - m, -\beta - m) \in \Omega$, i.e. let (α, β) be a couple of complex parameters for which the distribution $T^*_{-\alpha-m} * T^*_{-\beta-m}$ is a genuine convolution (see Theorem 4.1). We then have

$$\begin{aligned} \mathcal{F}[T^*_{-\alpha-m}] \cdot \mathcal{F}[T^*_{-\beta-m}] \quad &= \quad \mathcal{F}\big[T^*_{-\alpha-m} * T^*_{-\beta-m}\big] \\ &= \quad c_m(-\alpha - m, -\beta - m)\, \mathcal{F}[T^*_{-\alpha-\beta-m}] \end{aligned}$$

which, by means of Proposition 4.11, reduces to

$$T^*_\alpha \cdot T^*_\beta \quad = \quad c_m(-\alpha - m, -\beta - m)\, T^*_{\alpha+\beta}$$

■

Now recall that for all $(-\alpha - m, -\beta - m) \notin \Omega$, such that

$$\alpha \neq -m - 2j\ , \quad \beta \neq -m - 2k \quad \text{and} \quad \alpha + \beta \neq -m - 2l\ , \quad j, k, l \in \mathbb{N}_0$$

the distribution $T^*_{-\alpha-m} * T^*_{-\beta-m}$ is still defined, and one has

$$\mathcal{F}\big[T^*_{-\alpha-m} * T^*_{-\beta-m}\big] \;=\; c_m(-\alpha-m,-\beta-m)\,T^*_{\alpha+\beta}$$

But, as the distribution $T^*_{-\alpha-m} * T^*_{-\beta-m}$ is no longer defined as a genuine convolution, we cannot rewrite the left-hand side as the product of two Fourier transforms. However, the above formula inspires a definition for the products of distributions in $\mathbb{T} \cup \mathbb{U}$ in these cases as well.

Definition 4.2.

(i) For $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C}$ *such that*

$$\alpha \neq -m-2j\ ,\quad \beta \neq -m-2k \quad \textit{and} \quad \alpha+\beta \neq -m-2l\ ,\quad j,k,l \in \mathbb{N}_0$$

one puts

$$T^*_\alpha \cdot T^*_\beta \;=\; c_m(-\alpha-m,-\beta-m)\,T^*_{\alpha+\beta}$$

(ii) For $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C}$ *such that*

$$\alpha \neq -m-2j-1\ ,\quad \beta \neq -m-2k \quad \textit{and} \quad \alpha+\beta \neq -m-2l-1\ ,\quad j,k,l \in \mathbb{N}_0$$

one puts

$$U^*_\alpha \cdot T^*_\beta \;=\; T^*_\beta \cdot U^*_\alpha \;=\; c_m(-\alpha-m-1,-\beta-m)\,U^*_{\alpha+\beta}$$

(iii) For $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C}$ *such that*

$$\alpha \neq -m-2j-1\ ,\quad \beta \neq -m-2k-1 \quad \textit{and} \quad \alpha+\beta \neq -m-2l\ ,\quad j,k,l \in \mathbb{N}_0$$

one puts

$$U^*_\alpha \cdot U^*_\beta \;=\; \frac{2\pi}{\alpha+\beta+m}\,c_m(-\alpha-m-1,-\beta-m-1)\,T^*_{\alpha+\beta}$$

By the above definition we have now given meaning to the distribution $T^*_\alpha \cdot T^*_\beta$ for all admissible $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C}$. However, one should always keep in mind that only for the couples (α,β) mentioned in Theorem 4.3, $T^*_\alpha \cdot T^*_\beta$ can be interpreted as a genuine product of distributions.

Remark 4.4. *On account of Corollary 4.1 (i), the "$\cdot$"-operator acting on admissible members of* $(\mathbb{T} \cup \mathbb{U}) \times (\mathbb{T} \cup \mathbb{U})$ *is commutative. In particular, one has, for allowed values of* α *and* β*, that* $U^*_\beta \cdot T^*_\alpha = T^*_\alpha \cdot U^*_\beta$.

4.3.4 Examples: some specific distributions in Euclidean space

To emphasize the unifying character of our theory we give some explicit examples showing that for specific choices of λ well–known classical and Clifford distributions are obtained.

Example 1. Higher dimensional signum distribution

For $\lambda = 0$ we find that

$$\begin{aligned}
&\langle\, U_0 \,,\, \phi \,\rangle \\
&= \; a_m \left\langle \, \mathrm{Fp}\, r_+^{m-1} \,,\, \Sigma^{(1)}[\phi] \, \right\rangle \; = \; \int_0^{+\infty} r^{m-1}\, dr \int_{S^{m-1}} \underline{\omega}\, \phi(r\underline{\omega})\, dS(\underline{\omega}) \\
&= \; \int_{\mathbb{R}^m} \underline{\omega}\, \phi(\underline{x})\, dV(\underline{x}) \; = \; \langle\, \underline{\omega} \,,\, \phi \,\rangle
\end{aligned}$$

showing that U_0 coincides with the locally integrable function $\underline{\omega}$. As a distribution, $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$ is the higher dimensional analogue of the one–dimensional signum distribution $\mathrm{sgn}(t)$ on the real line, introduced in Subsection 2.1.1.

Example 2. Higher dimensional principal value distribution

For $\lambda = -m$ we find that

$$\begin{aligned}
\langle\, U_{-m} \,,\, \phi \,\rangle \; &= \; a_m \left\langle \, \mathrm{Fp}\, r_+^{-1} \,,\, \Sigma^{(1)}[\phi] \, \right\rangle \\
&= \; \mathrm{Fp} \int_0^{+\infty} \frac{1}{r}\, dr \int_{S^{m-1}} \underline{\omega}\, \phi(r\underline{\omega})\, dS(\underline{\omega}) \; = \; \int_{\mathbb{R}^m} \frac{\underline{\omega}}{r^m} (\phi(\underline{x}) - \phi(\underline{0}))\, dV(\underline{x}) \\
&= \; \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^m \setminus B(\underline{0};\varepsilon)} \frac{\underline{\omega}}{r^m}\, \phi(\underline{x})\, dV(\underline{x}) \; \equiv \; \left\langle \, \mathrm{Pv}\, \frac{\underline{\omega}}{r^m} \,,\, \phi \, \right\rangle
\end{aligned}$$

so that

$$U_{-m}^* \; = \; U_{-m} \; = \; \mathrm{Pv}\, \frac{\underline{\omega}}{r^m}$$

which is the principal value kernel in $\mathbb{R}^m$, studied in [50]. It is seen as the higher dimensional analogue of the one–dimensional $\mathrm{Pv}\, \frac{1}{t}$ distribution on the real line, introduced in Subsection 2.1.1. Since the numerator $\underline{\omega}$ of that higher dimensional principal value kernel clearly is homogeneous of degree 0 and satisfies

$$\int_{S^{m-1}} \underline{\omega}\, dS(\underline{\omega}) \; = \; 0$$

(see Section 3.2), the distribution $\mathrm{Pv}\,\frac{\omega}{r^m}$ is a typical example of a convolution operator, giving rise to a singular integral operator in the sense of Horváth (see Subsection 2.2.1), viz

$$\begin{aligned}\widetilde{\mathcal{H}}[f](\underline{y}) &= \left(\frac{2}{a_{m+1}}\,\mathrm{Pv}\,\frac{\omega}{r^m} * f(\underline{x})\right)(\underline{y}) \\ &= \frac{2}{a_{m+1}}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^m\setminus B(\underline{0};\varepsilon)} \frac{\underline{y}-\underline{x}}{|\underline{y}-\underline{x}|}\,\frac{f(\underline{x})}{|\underline{y}-\underline{x}|^m}\,dV(\underline{x})\end{aligned}$$

This is nothing but the Clifford–vector valued Hilbert transform, introduced in e.g. [69, 60]. In the following chapter we study that operator in more detail.

Example 3. Fundamental solution of $\partial_{\underline{x}}^{\lambda}$, $\lambda \in \mathbb{C}$

In [19] the fundamental solution of an arbitrary complex power of the Dirac operator has been calculated in terms of the distributions T^*_λ and U^*_λ. For a complex number $\lambda \neq -m-n$ ($n \in \mathbb{N}_0$), the operator $\partial_{\underline{x}}^{\lambda}$ is a convolution operator, acting on tempered distributions f as follows (see [52]):

$$\begin{aligned}\partial_{\underline{x}}^{\lambda} f &= \left[\frac{1+e^{i\pi\lambda}}{2}\,\frac{2^\lambda\Gamma\left(\frac{\lambda+m}{2}\right)}{\pi^{\frac{-\lambda+m}{2}}}\,T^*_{-\lambda-m} - \frac{1-e^{i\pi\lambda}}{2}\,\frac{2^\lambda\Gamma\left(\frac{\lambda+m+1}{2}\right)}{\pi^{\frac{-\lambda+m+1}{2}}}\,U^*_{-\lambda-m}\right] * f \\ &= \frac{2^\lambda}{\pi^{\frac{m}{2}}}\,\mathrm{Fp}\,\frac{1}{|\underline{x}|^{\lambda+m}}\left[\frac{1+e^{i\pi\lambda}}{2}\,\frac{\Gamma\left(\frac{\lambda+m}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} - \frac{1-e^{i\pi\lambda}}{2}\,\frac{\Gamma\left(\frac{\lambda+m+1}{2}\right)}{\Gamma\left(-\frac{\lambda-1}{2}\right)}\,\underline{\omega}\right] * f\end{aligned}$$

We notice that complex powers of the Laplace operator were already defined in [68]; for $\mu \in \mathbb{C}\setminus\{-\frac{m}{2}-l,\, l\in\mathbb{N}_0\}$ and a tempered distribution f, one has

$$(-\Delta)^\mu f = \frac{2^{2\mu}\,\Gamma\left(\mu+\frac{m}{2}\right)}{\pi^{-\mu+\frac{m}{2}}}\,T^*_{-2\mu-m} * f$$

Note that in particular the "square root of the negative Laplacian" is given by

$$(-\Delta)^{\frac{1}{2}}[f](\underline{x}) = \left(\frac{4\pi}{a_{m+1}}T^*_{-m-1} * f\right)(\underline{x}) = \frac{-2}{a_{m+1}}\,\mathrm{Fp}\int_{\mathbb{R}^m}\frac{f(\underline{u})}{|\underline{x}-\underline{u}|^{m+1}}\,dV(\underline{u})$$

which is a scalar valued convolution operator, as opposed to the Clifford–vector valued Dirac operator $\partial_{\underline{x}}$ for which also holds $\partial_{\underline{x}}^2 = -\Delta$. Moreover, it is well–known (see e.g. [30]) that this scalar square root is the so–called *Hilbert–Dirac operator* $-\widetilde{\mathcal{H}}\partial_{\underline{x}}$, since

$$-\widetilde{\mathcal{H}}\partial_{\underline{x}}[f] = \frac{-2}{a_{m+1}}\,U^*_{-m} * \left(\partial_{\underline{x}} f\right) = \frac{-2}{a_{m+1}}\left(U^*_{-m}\partial_{\underline{x}}\right) * f = \frac{4\pi}{a_{m+1}}\,T^*_{-m-1} * f$$

We will now focus on the construction of the fundamental solutions E_λ of the complex powers of the Dirac operator $\partial_{\underline{x}}^\lambda$. For the moment we need to exclude the values $\lambda = -m-n$ ($n \in \mathbb{N}_0$) for which the operator $\partial_{\underline{x}}^\lambda$ is not yet defined. We then distinguish between several cases for the parameter $\lambda \in \mathbb{C}$, depending on the dimension m.

CASE A. $\lambda = 1, 2, \ldots, m-1$

For $\lambda = -m+1$ we find, according to Proposition 4.10, that

$$\partial_{\underline{x}} U^*_{-m+1} \;=\; -2\pi\, T^*_{-m} \;=\; -\pi a_m\, \delta(\underline{x})$$

which confirms that

$$-\frac{1}{\pi a_m} U^*_{-m+1} \;=\; \frac{1}{a_m} \frac{\overline{\omega}}{r^{m-1}} \;=\; \frac{1}{a_m} \frac{\overline{x}}{|\underline{x}|^m}$$

is the fundamental solution of the Dirac operator in $\mathbb{R}^m$ (see Section 3.2). Next, observe that we also have

$$\Delta\, T^*_{-m+2} \;=\; -\partial_{\underline{x}}((-m+2)\, U^*_{-m+1}) \;=\; -\pi a_m\, (m-2)\, \delta(\underline{x})$$

which confirms

$$-\frac{1}{\pi a_m} \frac{1}{m-2} T^*_{-m+2} \;=\; -\frac{1}{a_m} \frac{1}{m-2} \frac{1}{r^{m-2}}$$

being the fundamental solution of the Laplace operator in $\mathbb{R}^m$.
Proceeding in this way, we recursively find the fundamental solutions of natural powers of the Dirac operator (at least up to order $(m-1)$), and hence also of the Laplace operator:

$$\partial_{\underline{x}}^{2l} \left(\frac{1}{a_m} \frac{2}{(2\pi)^l} \frac{1}{m-2} \frac{1}{m-4} \cdots \frac{1}{m-2l}\, T^*_{-m+2l} \right) \;=\; \delta(\underline{x})$$

and

$$\partial_{\underline{x}}^{2l+1} \left(-\frac{1}{a_m} \frac{2}{(2\pi)^{l+1}} \frac{1}{m-2} \frac{1}{m-4} \cdots \frac{1}{m-2l}\, U^*_{-m+2l+1} \right) \;=\; \delta(\underline{x})$$

So, this leads to the fundamental solutions

$$E_{2l} \;=\; \frac{\Gamma\left(\frac{m}{2} - l\right)}{2^{2l} \pi^{\frac{m}{2}+l}}\, T^*_{-m+2l} \tag{4.17}$$

$$E_{2l+1} \;=\; -\frac{\Gamma\left(\frac{m}{2} - l\right)}{2^{2l+1} \pi^{\frac{m}{2}+l+1}}\, U^*_{-m+2l+1} \tag{4.18}$$

CASE B. $\lambda = m + n$, $n \in \mathbb{N}_0$

For *odd* dimension m, it is sufficient to notice that

$$T^*_{-2m-2n-1} * T^*_{2n+1} = \frac{\pi^m}{\Gamma\left(m+n+\frac{1}{2}\right)\Gamma\left(-n-\frac{1}{2}\right)}\,\delta(\underline{x})$$

and

$$U^*_{-2m-2n} * U^*_{2n} = \frac{\pi^{m+1}}{\Gamma\left(m+n+\frac{1}{2}\right)\Gamma\left(-n+\frac{1}{2}\right)}\,\delta(\underline{x})$$

Hence, in this case, in view of the definitions of $\partial_{\underline{x}}^{m+2n}$ and $\partial_{\underline{x}}^{m+2n+1}$ in terms of U^*_{-2m-2n} and $T^*_{-2m-2n-1}$, we arrive at the fundamental solutions:

$$E_{m+2n} = -\frac{\Gamma\left(\frac{1}{2}-n\right)}{2^{m+2n}\pi^{\frac{1}{2}+m+n}}\,U^*_{2n} \tag{4.19}$$

$$E_{m+2n+1} = \frac{\Gamma\left(-\frac{1}{2}-n\right)}{2^{m+2n+1}\pi^{\frac{1}{2}+m+n}}\,T^*_{2n+1} \tag{4.20}$$

In case of an *even* dimension m, note that (4.17) and (4.18) are ill–defined whenever $\lambda = m+n$ ($n \in \mathbb{N}_0$). Thus, one has to use different techniques in order to find a fundamental solution. As

$$\partial_{\underline{x}} \ln r = \frac{2}{a_m}\,U^*_{-1}$$

it is directly seen that, for even m, the fundamental solution of $\partial_{\underline{x}}^m$ is given by

$$E_m(\underline{x}) = -\frac{a_m}{(2\pi)^m}\,\ln r \tag{4.21}$$

This fact inspired us to propose, for the remaining natural powers of $\partial_{\underline{x}}$, a fundamental solution containing a logarithmic term, which has eventually lead to the following theorem.

Theorem 4.4. *Let the dimension m be even. Then, for all $n \in \mathbb{N}_0$, the fundamental solution E_{m+n} of $\partial_{\underline{x}}^{m+n}$ is given by*

$$E_{m+2n-1} = (p_{2n-1}\ln r + q_{2n-1})\,U^*_{2n-1} \tag{4.22}$$

$$E_{m+2n} = (p_{2n}\ln r + q_{2n})\,T^*_{2n} \tag{4.23}$$

with

$$
\begin{aligned}
p_{2n} &= \left(\frac{-1}{4\pi}\right)^n \frac{p_0}{n!}\\
q_{2n} &= -\left(\frac{-1}{4\pi}\right)^n \frac{p_0}{n!} \sum_{j=1}^{n} \left[\frac{1}{m+2n-2j} + \frac{1}{2n-2j+2}\right]\\
p_{2n+1} &= 2\left(\frac{-1}{4\pi}\right)^{n+1} \frac{p_0}{n!}\\
q_{2n+1} &= -2\left(\frac{-1}{4\pi}\right)^{n+1} \frac{p_0}{n!} \left\{\sum_{j=1}^{n} \left[\frac{1}{m+2n-2j} + \frac{1}{2n-2j+2}\right] + \frac{1}{m+2n}\right\}
\end{aligned}
$$

and

$$p_0 = -\frac{a_m}{(2\pi)^m}$$

Proof.
First, assuming that the desired fundamental solutions take the proposed forms (4.22) and (4.23), we establish recurrence relations between the coefficients (p_{2n}, q_{2n}) and (p_{2n-1}, q_{2n-1}). To this end, note that

$$\partial_{\underline{x}}^{m+2n} E_{m+2n}(\underline{x}) = \delta(\underline{x}) \iff \partial_{\underline{x}}(p_{2n} \ln r + q_{2n})\, T^*_{2n} = E_{m+2n-1}(\underline{x})$$

which, on account of (4.21) and the product rules from Theorem 4.3, leads to

$$\left\{\begin{aligned} p_{2n} &= \frac{1}{2n}\, p_{2n-1}\\ q_{2n} &= \frac{1}{2n}\left(q_{2n-1} - \frac{1}{2n}\, p_{2n-1}\right)\end{aligned}\right. \tag{4.24}$$

Similarly, one has

$$\partial_{\underline{x}}^{m+2n+1} E_{m+2n+1}(\underline{x}) = \delta(\underline{x}) \iff \partial_{\underline{x}}(p_{2n+1} \ln r + q_{2n+1}) U^*_{2n+1} = E_{m+2n}(\underline{x})$$

yielding

$$\left\{\begin{aligned} p_{2n+1} &= -\frac{1}{2\pi}\, p_{2n}\\ q_{2n+1} &= -\frac{1}{2\pi}\left(q_{2n} - \frac{1}{m+2n}\, p_{2n}\right)\end{aligned}\right. \tag{4.25}$$

Now, taking into account (4.21), one already has that

$$p_0 \;=\; -\frac{a_m}{(2\pi)^m} \qquad \text{and} \qquad q_0 \;=\; 0$$

Then, from (4.24) and (4.25), the desired closed expressions for all coefficients may be obtained by an induction argument. ■

CASE C. $\lambda \in \mathbb{C} \setminus \mathbb{N}$ and $\lambda \neq -m-n$, $n \in \mathbb{N}_0$

It remains to construct fundamental solutions E_λ for all non–natural powers of $\partial_{\underline{x}}$. Seen the definition of $\partial_{\underline{x}}^\lambda$ as a convolution operator in terms of $T^*_{-\lambda-m}$ and $U^*_{-\lambda-m}$ and the fact that

$$T^*_{-\lambda-m} * T^*_{\lambda-m} \;=\; \frac{\pi^m}{\Gamma\left(\frac{\lambda+m}{2}\right)\Gamma\left(\frac{-\lambda+m}{2}\right)}\,\delta(\underline{x})$$

and

$$U^*_{-\lambda-m} * U^*_{\lambda-m} \;=\; \frac{\pi^{m+1}}{\Gamma\left(\frac{\lambda+m+1}{2}\right)\Gamma\left(\frac{-\lambda+m+1}{2}\right)}\,\delta(\underline{x})$$

it seems rather natural to define E_λ as a linear combination of $T^*_{\lambda-m}$ and $U^*_{\lambda-m}$. Hence we put

$$E_\lambda \;=\; C_T(m,\lambda)\,T^*_{\lambda-m} + C_U(m,\lambda)\,U^*_{\lambda-m}$$

where the complex coefficients $C_T(m,\lambda)$ and $C_U(m,\lambda)$ still need to be determined in order to fulfil $\partial_{\underline{x}}^\lambda E_\lambda(\underline{x}) = \delta(\underline{x})$. The action of $\partial_{\underline{x}}^\lambda$ on the distribution $E_\lambda(\underline{x})$ produces four terms, two of which are "mixed", in the sense that they contain a T^*– as well as a U^*–distribution; they are given by

$$\begin{aligned}
&\frac{1+e^{i\pi\lambda}}{2}\,\frac{2^\lambda\Gamma\left(\frac{\lambda+m}{2}\right)}{\pi^{\frac{-\lambda+m}{2}}}\,C_U(m,\lambda)\,T^*_{-\lambda-m} * U^*_{\lambda-m} \\
&\quad = \frac{1+e^{i\pi\lambda}}{2}\,\frac{2^\lambda\Gamma\left(\frac{m+1}{2}\right)}{\pi^{-\frac{\lambda}{2}}\Gamma\left(\frac{-\lambda+m+1}{2}\right)}\,C_U(m,\lambda)\,U^*_{-m}
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1-e^{i\pi\lambda}}{2}\,\frac{2^\lambda\Gamma\left(\frac{\lambda+m+1}{2}\right)}{\pi^{\frac{-\lambda+m+1}{2}}}\,C_T(m,\lambda)\,U^*_{-\lambda-m} * T^*_{\lambda-m} \\
&\quad = -\frac{1-e^{i\pi\lambda}}{2}\,\frac{2^\lambda\Gamma\left(\frac{m+1}{2}\right)}{\pi^{-\frac{\lambda-1}{2}}\Gamma\left(\frac{-\lambda+m}{2}\right)}\,C_T(m,\lambda)\,U^*_{-m}
\end{aligned}$$

Clearly, we need these terms to cancel each other, leading to the condition

$$\frac{C_T(m,\lambda)}{C_U(m,\lambda)} = \pi^{\frac{1}{2}} \frac{1+e^{i\pi\lambda}}{1-e^{i\pi\lambda}} \frac{\Gamma\left(\frac{-\lambda+m}{2}\right)}{\Gamma\left(\frac{-\lambda+m+1}{2}\right)} \tag{4.26}$$

One can easily verify that putting

$$C_T(m,\lambda) = \frac{e^{-i\pi\lambda}+1}{2} \frac{\Gamma\left(\frac{-\lambda+m}{2}\right)}{2^{\lambda}\pi^{\frac{\lambda+m}{2}}}$$

and

$$C_U(m,\lambda) = \frac{e^{-i\pi\lambda}-1}{2} \frac{\Gamma\left(\frac{-\lambda+m+1}{2}\right)}{2^{\lambda}\pi^{\frac{\lambda+m+1}{2}}}$$

not only condition (4.26) is satisfied, but moreover

$$\partial_{\underline{x}}^{\lambda}\left[C_T(\lambda,m)T^*_{\lambda-m} + C_U(\lambda,m)U^*_{\lambda-m}\right] = \delta(\underline{x})$$

This means that for arbitrary $\lambda \in \mathbb{C}\setminus\mathbb{N}$ such that $\lambda \neq -m-n$ ($n \in \mathbb{N}_0$), the fundamental solution of $\partial_{\underline{x}}^{\lambda}$ is given by

$$E_\lambda = \frac{1+e^{-i\pi\lambda}}{2} \frac{\Gamma\left(\frac{-\lambda+m}{2}\right)}{2^{\lambda}\pi^{\frac{\lambda+m}{2}}} T^*_{\lambda-m} - \frac{1-e^{-i\pi\lambda}}{2} \frac{\Gamma\left(\frac{-\lambda+m+1}{2}\right)}{2^{\lambda}\pi^{\frac{\lambda+m+1}{2}}} U^*_{\lambda-m}$$

Note that the formulae (4.17) and (4.18), obtained in Case A, and the formulae (4.19) and (4.20), obtained in Case B for m odd, are included in the above formula as well. Only the fundamental solutions E_{m+n} for m even and with $n \in \mathbb{N}_0$, escape from this unifying structure.

CASE D. $\lambda = -m-n$, $n \in \mathbb{N}_0$

We are now able to define $\partial_{\underline{x}}^{-m-n}$ as the convolution operator

$$\partial_{\underline{x}}^{-m-n} f = E_{m+n} * f$$

We also put

$$E_{-m-n} = \partial_{\underline{x}}^{m+n}\delta$$

and observe that indeed

$$\partial_{\underline{x}}^{-m-n} E_{-m-n} = E_{m+n} * \partial_{\underline{x}}^{m+n}\delta = \delta$$

Note that, depending on the parity of the dimension m and the natural number n, E_{-m-n} is indeed a distribution in the $\mathbb{T}$– or the $\mathbb{U}$–family.

4.3.5 Historical comment

The radial distributions $T_\lambda = \mathrm{Fp}\, r^\lambda$, $r = |\underline{x}|$, $\lambda \in \mathbb{C}$, are of course well–known. In [84], Riesz introduced their normalizations R_α, given by

$$R_\alpha = \frac{1}{2^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \mathrm{Fp}\, r^{\alpha-m} = \frac{\Gamma\left(\frac{m-\alpha}{2}\right)}{2^\alpha \pi^{\frac{\alpha+m}{2}}} T^*_{\alpha-m}$$

for $\alpha \neq -2l$ and $\alpha \neq m+2l$, $l \in \mathbb{N}_0$, and

$$R_{-2l} = (-\Delta)^l \delta = \frac{2^{2l}\Gamma\left(\frac{m}{2}+l\right)}{\pi^{\frac{m}{2}-l}} T^*_{-m-2l}\ , \quad l \in \mathbb{N}_0$$

$$R_{m+2l} = \frac{2(-1)^l}{\pi^{\frac{m}{2}} 2^{m+2l}\Gamma\left(\frac{m}{2}+l\right) l!} r^{2l} \left(\log \frac{1}{\pi r} + A_{m,l}\right)\ , \qquad l \in \mathbb{N}_0$$

where

$$A_{m,l} = \frac{1}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{l} - C\right) + \frac{1}{2}\frac{\Gamma'\left(\frac{m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l\right)}\ , \qquad l \in \mathbb{N}_0$$

and C is Euler's constant. The corresponding Riesz potentials are then given by (see Subsection 4.3.3):

$$I^\alpha[f] = R_\alpha * f\ , \qquad \alpha \neq m+2l$$

So we note that, when ignoring the additional definition of R_{m+2l}, R_α shows simple poles at $\alpha = m+2l$, $l \in \mathbb{N}_0$. Moreover $\mathcal{F}[R_{-2l}]$ and $\mathcal{F}[R_{m+2l}]$, $l \in \mathbb{N}_0$, are no Riesz kernels anymore, whereas in our approach T_λ^* is an entire function of $\lambda \in \mathbb{C}$ and the Fourier transform constitutes a bijection in the $\mathbb{T}$–family.

In [71], Horváth introduced the vectorial kernels

$$\vec{N}_\alpha = -\vec{\nabla} R_{\alpha+1}$$

which, for $\alpha \neq -2l-1$ and $\alpha \neq m+2l-1$, $l \in \mathbb{N}_0$, are given by

$$\vec{N}_\alpha = \frac{1}{2^\alpha \pi^{\frac{m}{2}}} \frac{\Gamma\left(\frac{m-\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)} \frac{\vec{x}}{r^{m-\alpha+1}}$$

For admissible values of α and β, these kernels satisfy the convolution formulae

$$\vec{N}_\alpha * \vec{N}_\beta = -R_{\alpha+\beta}$$

where the convolution of the two vector valued distributions has to be taken in the sense of a scalar product. If the Euclidean vector $\vec{x}$ is identified with the Clifford–vector $\underline{x}$, then the Horváth kernels $\vec{N}_\alpha$ correspond to the Clifford distributions

$$\vec{N}_\alpha \;\approx\; \frac{\Gamma\left(\frac{m-\alpha+1}{2}\right)}{2^\alpha \pi^{\frac{\alpha+m+1}{2}}}\, U^*_{\alpha-m}$$

Again note that $\vec{N}_\alpha$ shows simple poles at $\alpha = m+2l-1$, $l \in \mathbb{N}_0$, whereas U^*_λ is an entire function of $\lambda \in \mathbb{C}$. Moreover, after identification, up to a minus sign, of the scalar product of two vectors with the inner product of two Clifford–vectors, the above convolution formula for the Horváth kernels $\vec{N}_\alpha$, finds its equivalent in our formula (4.15) for the convolution of U^*_λ–distributions where the Clifford geometric multiplication is involved.

In the special case where $\alpha = 1$ the Horváth kernel turns into the vectorial kernel of Newtonian force

$$\vec{N}_1 \;=\; \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}}}\, \frac{\vec{x}}{r^m}$$

in which one recognizes, after identification and up to a minus sign, the fundamental solution of the Dirac operator $\partial_{\underline{x}}$:

$$\vec{N}_1 \;\approx\; \frac{1}{a_m}\, \frac{\underline{\omega}}{r^{m-1}} \;=\; \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}+1}}\, U^*_{-m+1}$$

The relation $\vec{\nabla} \circ \vec{N}_1 \;=\; \delta$ in [71], where the nabla operator $\vec{\nabla}$ is acting as a divergence, is then the vectorial equivalent of the formula

$$\partial_{\underline{x}} \left(-\frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}+1}} U^*_{-m+1} \right) \;=\; \delta$$

where the Dirac operator $\partial_{\underline{x}}$ acts by geometric multiplication.

In the special case where $\alpha = 0$, the Horváth kernel

$$\vec{N}_0 \;=\; \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}\, \frac{\vec{x}}{r^{m+1}} \;=\; \frac{2}{a_{m+1}}\, \frac{\vec{x}}{r^{m+1}}$$

is a vectorial generalization to $\mathbb{R}^m$ of the $\mathrm{Pv}\frac{1}{\pi t}$ kernel on the real line. It satisfies the reciprocity formula

$$\vec{N}_0 * \vec{N}_0 \;=\; -\delta$$

which was proved first in [69]. This kernel leads to the vectorial Hilbert transform in $\mathbb{R}^m$, the components of which are the Riesz transforms R_j, $j = 1, \ldots, m$, given by

$$R_j[f](\vec{x}) \;=\; \lim_{\varepsilon\to 0+} \frac{2}{a_{m+1}} \int_{|\vec{x}-\vec{y}|>\varepsilon} \frac{x_j - y_j}{|\vec{x}-\vec{y}|^{m+1}}\, f(\vec{y})\, dV(\vec{y})$$

Note that $\vec{N}_0$ corresponds, up to a minus sign, to the Hilbert convolution kernel in the Clifford setting

$$-\vec{N}_0 \;\approx\; \frac{2}{a_{m+1}}\, \mathrm{Pv}\, \frac{\underline{\omega}}{r^m} \;=\; -\frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}}\, U^*_{-m}$$

(see Example 2 of previous subsection). To our knowledge, the first one to have considered the Hilbert convolution kernel as a function taking values in the subspace $\mathbb{R}^1_{0,m}$ of Clifford–vectors, is Horváth in [69].

Finally, the scalar convolution operator

$$\Xi \;=\; \vec{\nabla} \circ \vec{N}_0 \;=\; -\vec{\nabla} \circ (\vec{\nabla} R_1) \;=\; (-\Delta) R_1 \;=\; R_{-1}$$

satisfies

$$\Xi * \Xi \;=\; -\Delta$$

a result which already figures in [40]; one recognizes in Ξ the Hilbert–Dirac kernel $\frac{4\pi}{a_{m+1}}\, T^*_{-m-1}$ discussed above.

4.4 The distributions $T^*_{\lambda,p}$, $U^*_{\lambda,p}$, $V^*_{\lambda,p}$ and $W^*_{\lambda,p}$

The distributions $T_{\lambda,p} = \mathrm{Fp}\, r^\lambda\, P_p(\underline{x})$, $U_{\lambda,p} = \mathrm{Fp}\, r^\lambda\, \underline{\omega}\, P_p(\underline{x})$, $V_{\lambda,p} = \mathrm{Fp}\, r^\lambda\, P_p(\underline{x})\, \underline{\omega}$ and $W_{\lambda,p} = \mathrm{Fp}\, r^\lambda\, \underline{\omega}\, P_p(\underline{x})\, \underline{\omega}$ (with $\underline{x} = r\,\underline{\omega}$, $r = |\underline{x}|$ and $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$) were introduced in [26, 25]. They have then been normalized, extensively studied and applied in a series of papers [9, 10, 11, 13, 14, 21, 15]. These four families of Clifford distributions encompass the distributions of Section 4.3 in the sense that in the special case where the degree p of the involving vector valued, monogenic, homogeneous polynomial P_p is zero, the families of distributions $\mathbb{T}$ and $\mathbb{U}$ will be recovered.

In the first subsection these four families of Clifford distributions are introduced, making use of the same technique as in previous section. The basic ingredients for the construction again consist of the spherical co-ordinates and the fundamental distribution $\mathrm{Fp}\, r_+^{\mu}$ on the real r–axis. A next step, crucial for the generalization, is then the employment of the generalized spherical means of Subsection 4.2.2, involving vector valued, monogenic, homogeneous polynomials P_p of degree $p \in \mathbb{N}_0$. In the second subsection, we generalize the properties of Subsection 4.3.3. Finally, we look at some specific and useful members of the families under consideration.

4.4.1 Definition

Let ϕ be a scalar valued test function in $\mathbb{R}^m$ and, as before, let $\mu = \lambda + m - 1$. Then for all $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$, we define the distributions $T_{\lambda,p}$, $U_{\lambda,p}$, $V_{\lambda,p}$ and $W_{\lambda,p}$ as follows:

$$\begin{aligned}
\langle\, T_{\lambda,p} \ , \ \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \Sigma_p^{(0)}[\phi] \,\right\rangle \\
\langle\, U_{\lambda,p} \ , \ \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \Sigma_p^{(1)}[\phi] \,\right\rangle \\
\langle\, W_{\lambda,p} \ , \ \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \Sigma_p^{(2)}[\phi] \,\right\rangle \\
\langle\, V_{\lambda,p} \ , \ \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \Sigma_p^{(3)}[\phi] \,\right\rangle
\end{aligned}$$

Remark 4.5. *Following Remark 4.1 we also have abbreviated the notation T_{λ,P_p} to $T_{\lambda,p}$ and the like.*

It is clear that for each choice of $p \in \mathbb{N}_0$, the above introduced distributions all will inherit an infinite sequence of singular points in the complex λ–plane from the distribution finite parts on the real r–axis. Half of these singularities are directly seen to disappear, their residue being zero on account of the vanishing at the origin $r = 0$ of either the odd or the even order derivatives of the generalized spherical means (see Proposition 4.8). In addition, the appearance of the spherical monogenic causes a finite number of the remaining residues to be zero as well (see again Proposition 4.8). The same normalization procedure, already explained in previous section, will be applied: the eventual genuine simple poles will be removed through division by an appropriate Gamma–function. We only give the details of the method for the $T_{\lambda,p}$–distributions, the normalization of the other distributions being quite similar.

First of all, let for a moment $\lambda \neq -m - n$ and $\lambda \neq -m - n - p_e$, $n \in \mathbb{N}_0$. Then the connection between $T_{\lambda,p}$ and $T_\lambda = T_{\lambda,0}$ is obtained in a natural way from

$$\begin{aligned}
&\langle\, T_\lambda P_p(\underline{x}) \;,\; \phi(\underline{x}) \,\rangle \\
&= \langle\, T_\lambda \;,\; P_p(\underline{x})\phi(\underline{x}) \,\rangle \;=\; a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu} \;,\; \Sigma^{(0)}[P_p(\underline{x})\phi(\underline{x})] \,\right\rangle \\
&= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \;,\; \Sigma_p^{(0)}[\phi] \,\right\rangle \;=\; \langle\, T_{\lambda,p} \;,\; \phi \,\rangle
\end{aligned}$$

leading, at least for the values of λ mentioned above, to

$$T_{\lambda,p} \;=\; T_\lambda\, P_p \tag{4.27}$$

The other values of λ are not yet taken into account, as they seem to be simple poles of either the left- or the right-hand side of the relation (4.27). Further investigation of these assumed singularities is carried out in what follows.

It is obvious from the definition itself that $T_{\lambda,p}$, considered as a function of $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$, inherits the following infinite sequence of singularities (simple poles) from the finite parts distribution: $\mu + p_e = -n$, $n \in \mathbb{N}$, or equivalently, $\lambda = -m - p_e - n + 1$, $n \in \mathbb{N}$; the corresponding residues are given by

$$\underset{\lambda=-m-p_e-n+1}{\mathrm{Res}} \langle\, T_{\lambda,p} \;,\; \phi \,\rangle \;=\; a_m \left\langle\, \frac{(-1)^{(n-1)}}{(n-1)!}\, \delta_r^{(n-1)} \;,\; \Sigma_p^{(0)}[\phi] \,\right\rangle \tag{4.28}$$

According to the parity of n, we then distinguish between two cases.

CASE A. $n = 2l + 2$, $l \in \mathbb{N}_0$

In this case we rewrite (4.28) as

$$\underset{\lambda=-m-p_e-2l-1}{\mathrm{Res}} \langle\, T_{\lambda,p} \;,\; \phi \,\rangle \;=\; \frac{a_m}{(2l+1)!} \left\{ \partial_r^{2l+1}\, \Sigma_p^{(0)}[\phi] \right\}_{r=0}$$

the latter being zero on account of Proposition 4.8. Hence $T_{\lambda,p}$ shows no genuine poles whenever $n = 2l + 2$, or equivalently, whenever $\lambda = -m - p_e - 2l - 1$, $l \in \mathbb{N}_0$. Thus, the distributions $T_{-m-p_e-2l-1,p}$, $l \in \mathbb{N}_0$, can be defined by means of a limiting process:

$$\langle\, T_{-m-p_e-2l-1,p} \;,\; \phi \,\rangle \;=\; a_m \lim_{\mu \to -2l-2} \left\langle\, \mathrm{Fp}\, r_+^{\mu} \;,\; \Sigma_p^{(0)}[\phi] \,\right\rangle$$

where the limit at the right-hand side exactly yields the monomial pseudofunction $\mathrm{Fp}\, r_+^{-2l-2}$.

CASE B. $n = 2l + 1$, $l \in \mathbb{N}_0$

Substitution of these values of n in (4.28) yields

$$\underset{\lambda=-m-p_e-2l}{\mathrm{Res}} \langle\, T_{\lambda,p} \,,\, \phi \,\rangle \;=\; \frac{a_m}{(2l)!} \left\{ \partial_r^{2l}\, \Sigma_p^{(0)}[\phi] \right\}_{r=0}$$

Since, according to Proposition 4.8, the expression at the right-hand side equals zero for $l < p - \frac{p_e}{2}$, we conclude that $T_{\lambda,p}$ also has no genuine singularities in the case $\lambda = -m - p_e - 2l$ for $l = 0, 1, 2, \ldots, p - \frac{p_e}{2} - 1$; for this finite set of values, the distribution can be defined similarly as above by a limiting process, now involving the monomial pseudofunction $\mathrm{Fp}\, r_+^{-2l-1}$.

The results obtained are summarized in the following proposition.

Proposition 4.16. *Considered as a function of* $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$*, the distribution* $T_{\lambda,p}$ *shows simple poles at* $\lambda = -m - 2p - 2l$*,* $l \in \mathbb{N}_0$*, with residue*

$$\underset{\lambda=-m-2p-2l}{\mathrm{Res}} T_{\lambda,p} \;=\; a_m\, \frac{1}{(2p+2l)!}\, \frac{1}{C(p+l)}\, P_p(\underline{x}) \Delta^{p+l} \delta(\underline{x})$$

Remark 4.6. *The above considerations lead to the conclusion that multiplication of* T_λ *with* P_p *in (4.27) causes the removal of its singularities* $\lambda = -m - 2l$ *for* $l < p$*. Hence, the equality (4.27) may be holomorphically extended to all couples* (λ, p) *which do not fulfil the relation* $\lambda + 2p = -m - 2l$*,* $l \in \mathbb{N}_0$*. This means that, whenever* $T_{\lambda,p}$ *is well–defined, we may rewrite it as* $T_\lambda\, P_p$*.*

Now, the genuine singularities will be removed through the well–known technique of division by a deliberately chosen Gamma–function. Noting that the function $\Gamma(\frac{\lambda+m+2p}{2})$ shows exactly the same simple poles as $T_{\lambda,p}$, with residues

$$\underset{\lambda=-m-2p-2l}{\mathrm{Res}} \Gamma\left(\frac{\lambda+m+2p}{2}\right) \;=\; 2\frac{(-1)^l}{l!}$$

we are lead to the following definition of the normalized distributions $T^*_{\lambda,p}$:

$$\left\{ \begin{array}{lll} T^*_{\lambda,p} & = & \pi^{\frac{\lambda+m}{2}+p} \dfrac{T_{\lambda,p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)} \;, \quad \lambda \neq -m-2p-2l \\[2ex] T^*_{-m-2p-2l,p} & = & \dfrac{(-1)^l\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l\right)}\, P_p(\underline{x}) \Delta^{p+l} \delta(\underline{x}) \;, \quad l \in \mathbb{N}_0 \end{array} \right. \tag{4.29}$$

where, at the singularities of $T_{\lambda,p}$, the normalized distribution $T^*_{\lambda,p}$ is defined, up to constants, as the quotient of the residues involved.

As already discussed above, in this definition, $T_{\lambda,p}$ should be interpreted in terms of the monomial pseudofunction $\mathrm{Fp}\, r_+^{-n}$ whenever $\lambda = -m - p_e - n + 1$, $n \in \mathbb{N}$, but $\lambda \neq -m - 2p - 2l$, $l \in \mathbb{N}_0$. Moreover, one can verify that for $p = 0$ this definition is in accordance with the definition of $T^*_\lambda = T^*_{\lambda,0}$.

Adopting the same modus operandi leads to the following definitions of the normalized distributions $U^*_{\lambda,p}$, $V^*_{\lambda,p}$ and $W^*_{\lambda,p}$ with $l \in \mathbb{N}_0$:

$$\begin{cases} U^*_{\lambda,p} &= \pi^{\frac{\lambda+m+1}{2}+p} \dfrac{U_{\lambda,p}}{\Gamma\left(\frac{\lambda+m+1}{2}+p\right)}\,, \qquad \lambda \neq -m-2p-2l-1 \\[2ex] U^*_{-m-2p-2l-1,p} &= \dfrac{(-1)^{p+1} l! \pi^{\frac{m}{2}-l}}{2^{2p+2l+1}(p+l)!\Gamma\left(\frac{m}{2}+p+l+1\right)} \left(\partial_{\underline{x}}^{2p+2l+1}\delta(\underline{x})\right) P_p(\underline{x}) \end{cases}$$

$$\begin{cases} V^*_{\lambda,p} &= \pi^{\frac{\lambda+m+1}{2}+p} \dfrac{V_{\lambda,p}}{\Gamma\left(\frac{\lambda+m+1}{2}+p\right)}\,, \qquad \lambda \neq -m-2p-2l-1 \\[2ex] V^*_{-m-2p-2l-1,p} &= \dfrac{(-1)^{p+1} l! \pi^{\frac{m}{2}-l}}{2^{2p+2l+1}(p+l)!\Gamma\left(\frac{m}{2}+p+l+1\right)} P_p(\underline{x}) \left(\partial_{\underline{x}}^{2p+2l+1}\delta(\underline{x})\right) \end{cases}$$

$$\begin{cases} W^*_{\lambda,p} &= \pi^{\frac{\lambda+m}{2}+p} \dfrac{W_{\lambda,p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)}\,, \qquad \lambda \neq -m-2p-2l \\[2ex] W^*_{-m-2p-2l,p} &= \dfrac{(-1)^{l} l! \pi^{\frac{m}{2}-l}}{2^{2p+2l+2}(p+l+1)!\Gamma\left(\frac{m}{2}+p+l+1\right)} \underline{x} P_p(\underline{x}) \underline{x}\, \Delta^{p+l+1}\delta(\underline{x}) \end{cases}$$

For each $p \in \mathbb{N}_0$, we introduce the following families of distributions:

$$\mathbb{T}_p = \{T^*_{\lambda,p} : \lambda \in \mathbb{C}\}\,, \qquad \mathbb{U}_p = \{U^*_{\lambda,p} : \lambda \in \mathbb{C}\}$$
$$\mathbb{V}_p = \{V^*_{\lambda,p} : \lambda \in \mathbb{C}\}\,, \qquad \mathbb{W}_p = \{W^*_{\lambda,p} : \lambda \in \mathbb{C}\}$$

4.4.2 Properties

Multiplication with the vector $\underline{x}$ and action of the Dirac operator $\partial_{\underline{x}}$

First we consider on our four families of distributions the action of multiplication with the vector $\underline{x}$ and the action of the Dirac operator $\partial_{\underline{x}}$. Since the distributions involved are not necessarily scalar valued, it makes sense to distinguish between an action from the left or from the right.

Theorem 4.5. *For all* $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$ *one has*

$$\underline{x}\, T^*_{\lambda,p} \;=\; -W^*_{\lambda,p}\, \underline{x} \;=\; \frac{\lambda+m+2p}{2\pi}\, U^*_{\lambda+1,p}\,, \qquad \underline{x}\, U^*_{\lambda,p} \;=\; V^*_{\lambda,p}\, \underline{x} \;=\; -T^*_{\lambda+1,p}$$

$$T^*_{\lambda,p}\, \underline{x} \;=\; -\underline{x}\, W^*_{\lambda,p} \;=\; \frac{\lambda+m+2p}{2\pi}\, V^*_{\lambda+1,p}\,, \qquad U^*_{\lambda,p}\, \underline{x} \;=\; \underline{x}\, V^*_{\lambda,p} \;=\; W^*_{\lambda+1,p}$$

Proof.
We only calculate $\underline{x}\, T^*_{\lambda,p}$, the proof of the other formulae running along similar lines. Taking into account Proposition 4.6 (i), we get

$$\langle\, \phi \,,\, \underline{x}\, T_{\lambda,p} \,\rangle \;=\; \langle\, \phi\, \underline{x} \,,\, T_{\lambda,p} \,\rangle \;=\; a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \,,\, \Sigma_p^{(0)}[\phi\, \underline{x}] \,\right\rangle$$

$$=\; a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \,,\, r\, \Sigma_p^{(1)}[\phi] \,\right\rangle \;=\; \langle\, \phi \,,\, U_{\lambda+1,p} \,\rangle$$

where the singularities are located at $\lambda = -m-2p-2l$ for the distribution at the left-hand side and at $\lambda = -m-2p-2l-2$, $l \in \mathbb{N}_0$, for the distribution at the right-hand side. We then distinguish between three cases.
If $\lambda \neq -m-2p-2l$, $l \in \mathbb{N}_0$, then

$$\underline{x}\, T^*_{\lambda,p} \;=\; \frac{\pi^{\frac{\lambda+m}{2}+p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)}\, \underline{x}\, T_{\lambda,p} \;=\; \frac{\pi^{\frac{\lambda+m}{2}+p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)}\, U_{\lambda+1,p} \;=\; \frac{\lambda+m+2p}{2\pi}\, U^*_{\lambda+1,p}$$

Next, if $\lambda = -m-2p-2l$, $l \in \mathbb{N}$, then, by the definition of $T^*_{-m-2p-2l,p}$ and making use of Lemma 4.1 (ii), one has

$$\begin{aligned}
\underline{x}\, T^*_{-m-2p-2l,p} \;&=\; \frac{(-1)^p l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l\right)}\, \underline{x}\, \partial_{\underline{x}}^{2p+2l} \delta(\underline{x})\, P_p(\underline{x}) \\
&=\; \frac{(-1)^p l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l-1}(p+l-1)!\,\Gamma\left(\frac{m}{2}+p+l\right)}\, \partial_{\underline{x}}^{2p+2l-1} \delta(\underline{x})\, P_p(\underline{x}) \\
&=\; -\frac{l}{\pi}\, U^*_{-m-2p-2l+1,p}
\end{aligned}$$

the last equality following from the definition of $U^*_{-m-2p-2l+1,p}$ for $l \in \mathbb{N}$.
Finally, if $\lambda = -m-2p$, then again on account of Lemma 4.1 (ii) one has

$$\begin{aligned}
\underline{x}\, T^*_{-m-2p,p} \;&=\; \frac{(-1)^p \pi^{\frac{m}{2}}}{2^{2p} p!\,\Gamma\left(\frac{m}{2}+p\right)}\, \underline{x}\, \partial_{\underline{x}}^{2p} \delta(\underline{x})\, P_p(\underline{x}) \\
&=\; \frac{(-1)^p \pi^{\frac{m}{2}}}{2^{2p-1}(p-1)!\,\Gamma\left(\frac{m}{2}+p\right)}\, \partial_{\underline{x}}^{2p-1} \delta(\underline{x})\, P_p(\underline{x}) \;=\; 0
\end{aligned}$$

the last equality following from Proposition 3.1 (iii). ■

By a similar reasoning one can prove the following theorem.

Theorem 4.6.

(i) For all $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$ *one has*

$$\partial_{\underline{x}} T^*_{\lambda,p} = \lambda U^*_{\lambda-1,p} \tag{4.30}$$
$$T^*_{\lambda,p} \partial_{\underline{x}} = \lambda V^*_{\lambda-1,p} \tag{4.31}$$

(ii) For all $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$ *one has*

$$\partial_{\underline{x}} U^*_{\lambda,p} = V^*_{\lambda,p} \partial_{\underline{x}} = -2\pi T^*_{\lambda-1,p} \tag{4.32}$$

(iii) For all $(\lambda, p) \in \mathbb{C} \times \mathbb{N}$ *with* $\lambda \neq -m - 2p + 1$ *one has*

$$\begin{aligned} U^*_{\lambda,p} \partial_{\underline{x}} &= \partial_{\underline{x}} V^*_{\lambda,p} \\ &= \frac{2\pi}{\lambda + m + 2p - 1} \left[(m-2) T^*_{\lambda-1,p} + (\lambda - 1) W^*_{\lambda-1,p}\right] \end{aligned} \tag{4.33}$$

whereas

$$\begin{aligned} U^*_{-m-2p+1,p} \partial_{\underline{x}} &= \partial_{\underline{x}} V^*_{-m-2p+1,p} \\ &= \pi \left[(m-2) T_{-m-2p,p} - (m+2p) W_{-m-2p,p}\right] + 2\pi W^*_{-m-2p,p} \end{aligned} \tag{4.34}$$

(iv) For all $(\lambda, p) \in \mathbb{C} \times \mathbb{N}$ *one has*

$$\partial_{\underline{x}} W^*_{\lambda,p} = -(m-2)\, U^*_{\lambda-1,p} - (\lambda + m + 2p)\, V^*_{\lambda-1,p} \tag{4.35}$$
$$W^*_{\lambda,p} \partial_{\underline{x}} = -(\lambda + m + 2p)\, U^*_{\lambda-1,p} - (m-2)\, V^*_{\lambda-1,p} \tag{4.36}$$

Proof.
The proofs of the formulae (4.30)–(4.33) and (4.35)–(4.36) running along similar lines, we only illustrate the general method by calculating $\partial_{\underline{x}} T^*_{\lambda,p}$. We distinguish between two cases.
If $\lambda \neq -m - 2p - 2l$, $l \in \mathbb{N}_0$, then $T_{\lambda,p}$ is a vector valued distribution. So one has to consider $\partial_{\underline{x}} T_{\lambda,p}$ as a left distribution in order to be able to switch the action of the Dirac operator from the distribution to the test function. So, for any scalar valued test function ϕ defined on $\mathbb{R}^m$, one has

$$\langle \phi \,,\, \partial_{\underline{x}} T_{\lambda,p} \rangle = -\langle \phi \partial_{\underline{x}} \,,\, T_{\lambda,p} \rangle$$

which, by the definition of the distributions $T_{\lambda,p}$, equals

$$-a_m \left\langle \ \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \Sigma_p^{(0)}[\phi\, \partial_{\underline{x}}] \ \right\rangle$$

On account of Proposition 4.7 (i), we may rewrite this as

$$\begin{aligned} &- a_m \left\langle \ \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \left(\partial_r + \frac{m+p_e-1}{r}\right) \Sigma_p^{(1)}[\phi] \ \right\rangle \\ &\quad = \ a_m \left\langle \ \partial_r\, \mathrm{Fp}\, r_+^{\mu+p_e} \ , \ \Sigma_p^{(1)}[\phi] \ \right\rangle - a_m\,(m+p_e-1) \left\langle \ \mathrm{Fp}\, r_+^{\mu+p_e-1} \ , \ \Sigma_p^{(1)}[\phi] \ \right\rangle \end{aligned}$$

For the calculation of the first term at the right-hand side, we have to apply Proposition 4.1 (ii) and hence we need to distinguish between the subcases where $\mu + p_e \neq -n$ and where $\mu + p_e = -n$, $n \in \mathbb{N}_0$.
In the first subcase, by the definition of the distributions $U_{\lambda,p}$, we readily find

$$\langle \ \phi \ , \ \partial_{\underline{x}}\, T_{\lambda,p} \ \rangle \ = \ \langle \ \phi \ , \ \lambda\, U_{\lambda-1,p} \ \rangle \tag{4.37}$$

leading to (4.30), by the definition of $T^*_{\lambda,p}$ and $U^*_{\lambda-1,p}$ in terms of $T_{\lambda,p}$ and $U_{\lambda-1,p}$ respectively.
In the second subcase, where $\mu + p_e = -n$, $n \in \mathbb{N}_0$, we obtain

$$\begin{aligned} & a_m \left\langle \ \partial_r\, \mathrm{Fp}\, r_+^{-n} \ , \ \Sigma_p^{(1)}[\phi] \ \right\rangle \\ &\quad = \ a_m\,(-n) \left\langle \ \mathrm{Fp}\, r_+^{-n-1} \ , \ \Sigma_p^{(1)}[\phi] \ \right\rangle + a_m\, \frac{(-1)^n}{n!} \left\langle \ \delta_r^{(n)} \ , \ \Sigma_p^{(1)}[\phi] \ \right\rangle \end{aligned}$$

On account of Proposition 4.8 (ii), the second term at the right-hand side is then seen to vanish for all $n \in \mathbb{N}_0$. Indeed, this is immediately clear when n is even. When, on the other hand, n is odd, then, in view of the assumption made on λ, n can only take the values $n = 1, 3, 5, \ldots, 2p - p_e - 1$, turning that term also into zero. Thus it follows that also in this subcase (4.37) holds and hence does (4.30).
Next, if $\lambda = -m - 2p - 2l$, $l \in \mathbb{N}_0$, then, by the definition of $T^*_{-m-2p-2l,p}$, one has

$$\begin{aligned} \partial_{\underline{x}}\, T^*_{-m-2p-2l,p} \ &= \ \partial_{\underline{x}} \left(\frac{(-1)^p l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l\right)}\, \partial_{\underline{x}}^{2p+2l} \delta(\underline{x})\, P_p(\underline{x}) \right) \\ &= \ \frac{(-1)^p l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l\right)}\, \partial_{\underline{x}}^{2p+2l+1} \delta(\underline{x})\, P_p(\underline{x}) \\ &= \ (-m-2p-2l)\, U^*_{-m-2p-2l-1,p} \end{aligned}$$

leading to the desired formula (4.30).
Finally, for the proof of (4.34), we first apply the same working method as above. So, for $\lambda = -m-2p+1$ and $p \neq 0$, we find consecutively:

$$\begin{aligned}
\left\langle\, U_{-m-2p+1,p}\,\partial_{\underline{x}} \;,\; \phi \,\right\rangle \;&=\; -\left\langle\, U_{-m-2p+1,p} \;,\; \partial_{\underline{x}}\,\phi \,\right\rangle \\
&= -a_m \left\langle \mathrm{Fp}\, r_+^{-2p+p_e-1} \;,\; r\,\Sigma_p^{(1)}\,[\partial_{\underline{x}}\,\phi] \right\rangle \\
&= -a_m \left\langle \mathrm{Fp}\, r_+^{-2p+p_e-1} \;,\; -(m-2)\,\Sigma_p^{(0)}[\phi] + (r\,\partial_r + m + p_e)\;\Sigma_p^{(2)}[\phi] \right\rangle \\
&= (m-2)\,a_m \left\langle \mathrm{Fp}\, r_+^{-2p+p_e-1} \;,\; \Sigma_p^{(0)}[\phi] \right\rangle \\
&\quad + a_m \left\langle -(m+2p)\,\mathrm{Fp}\, r_+^{-2p+p_e-1} + \frac{(-1)^{2p-p_e}}{(2p-p_e)!}\,\delta^{(2p-p_e)}(r) \;,\; \Sigma_p^{(2)}[\phi] \right\rangle \\
&= \left\langle\, (m-2)\,T_{-m-2p,p} - (m+2p)\,W_{-m-2p,p} + 2\,W^*_{-m-2p,p} \;,\; \phi \,\right\rangle
\end{aligned}$$

the last step on account of (4.3) and the definition of $W^*_{-m-2p,p}$. It is clear that the value $\lambda = -m-2p+1$ yields a singularity for the distributions $T_{\lambda-1,p}$ and $W_{\lambda-1,p}$ at the right-hand side, but not for the distribution $U_{\lambda,p}$ at the left-hand side. Hence, we can only arrive at the following result:

$$\begin{aligned}
U^*_{-m-2p+1,p}\,\partial_{\underline{x}} \;&=\; \pi\,U_{-m-2p+1,p}\,\partial_{\underline{x}} \\
&= \pi\,[(m-2)\,T_{-m-2p,p} - (m+2p)\,W_{-m-2p,p}] + 2\pi\,W^*_{-m-2p,p}
\end{aligned}$$

■

Higher order Dirac derivatives of the distributions $T^*_{\lambda,p}$ and $U^*_{\lambda,p}$ may then readily be calculated. For our purpose, we only consider actions of the Dirac derivative from the left. Depending on the parity of the order of the derivative, the outcome belongs either to the $\mathbb{T}_p$– or to the $\mathbb{U}_p$–family of distributions. The proof follows by an induction argument on the natural parameter k.

Corollary 4.2.

(i) For all $(\lambda, p, k) \in \mathbb{C} \times \mathbb{N}_0 \times \mathbb{N}_0$ one has

$$\begin{aligned}
\partial_{\underline{x}}^{2k}\,T^*_{\lambda,p} \;&=\; (-2\pi)^k\,\lambda(\lambda-2)\ldots(\lambda-2k+2)\,T^*_{\lambda-2k,p} \\
\partial_{\underline{x}}^{2k+1}\,T^*_{\lambda,p} \;&=\; (-2\pi)^k\,\lambda(\lambda-2)\ldots(\lambda-2k+2)(\lambda-2k)\,U^*_{\lambda-2k-1,p} \qquad (4.38)
\end{aligned}$$

(ii) For all $(\lambda, p, k) \in \mathbb{C} \times \mathbb{N}_0 \times \mathbb{N}_0$ *one has*

$$\partial_{\underline{x}}^{2k}\, U^*_{\lambda,p} \;=\; (-2\pi)^k\, (\lambda-1)(\lambda-3)\ldots(\lambda-2k+1)\, U^*_{\lambda-2k,p} \qquad (4.39)$$

$$\partial_{\underline{x}}^{2k+1}\, U^*_{\lambda,p} \;=\; (-2\pi)^{k+1}\, (\lambda-1)(\lambda-3)\ldots(\lambda-2k+1)\, T^*_{\lambda-2k-1,p}$$

Fourier transform

The formulae already obtained in this subsection are particularly suited for the calculation of the Fourier transforms of all normalized distributions, up to one exception (see further); this was done in [9, 14]. It turns out that for the distributions in the $\mathbb{T}_p$–, $\mathbb{U}_p$– and $\mathbb{V}_p$–families, the Fourier spectrum remains in the same family, whereas for a distribution in the $\mathbb{W}_p$–family, a linear combination of a $T^*_{\lambda,p}$– and a $W^*_{\lambda,p}$–distribution is obtained.

For the calculation of the Fourier spectra of the distributions $T^*_{\lambda,p}$, we start from a classical result of [97, Theorem IV.4.1]: for those couples $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$ for which $\mathrm{Re}(\lambda)$ is restricted to the strip $-m-p < \mathrm{Re}(\lambda) < -p$, the following formula holds

$$\mathcal{F}[T_\lambda\, P_p(\underline{x})](\underline{y}) \;=\; i^{-p}\pi^{-\frac{m}{2}-\lambda-p}\, \frac{\Gamma\left(\frac{\lambda+m}{2}+p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\, T_{-\lambda-m-2p}\, P_p(\underline{y})$$

or, when taking into account (4.27) and Remark 4.6, we already have that

$$\mathcal{F}[T_{\lambda,p}] \;=\; i^{-p}\pi^{-\frac{m}{2}-\lambda-p}\, \frac{\Gamma\left(\frac{\lambda+m}{2}+p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\, T_{-\lambda-m-2p,p} \qquad (4.40)$$

Both sides of the above formula are meromorphic functions in the complex variable λ. Hence, by analytic continuation, this expression is valid in each open connected area containing the above mentioned strip, and where both sides exist. Singularities occur in (4.40) when $\lambda = -m-2p-2l$ for the distribution at the left-hand side and when $\lambda = 2l$ for the one at the right-hand side (twice with $l \in \mathbb{N}_0$). Naturally, the same singularities are also contained in the involved Gamma–functions. Consequently, (4.40) is seen to hold for λ belonging to the set Λ, which is defined as

$$\Lambda \;=\; \mathbb{C} \setminus (\{-m-2p-2l : l \in \mathbb{N}_0\} \cup \{2l : l \in \mathbb{N}_0\})$$

This smoothens the path for the following fundamental result.

Theorem 4.7. *For all $(\lambda, p) \in \mathbb{C}\times\mathbb{N}_0$, the Fourier transform of the distributions $T^*_{\lambda,p}$ is given by*

$$\mathcal{F}[T^*_{\lambda,p}] \;=\; i^{-p}\, T^*_{-\lambda-m-2p,p}$$

Proof.

Three cases have to be distinguished.

CASE A. $\lambda \in \Lambda$
On account of (4.40) we indeed have

$$\mathcal{F}[T^*_{\lambda,p}] \;=\; \frac{\pi^{\frac{\lambda+m}{2}+p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)}\,\mathcal{F}[T_{\lambda,p}] \;=\; i^{-p}\frac{\pi^{-\frac{\lambda}{2}}}{\Gamma\left(-\frac{\lambda}{2}\right)}\,T_{-\lambda-m-2p,p} \;=\; i^{-p}\,T^*_{-\lambda-m-2p,p}$$

CASE B. $\lambda = -m-2p-2l,\ l \in \mathbb{N}_0$
Exploiting the definition of $T^*_{-\lambda-m-2p,p}$ and the properties (3.2) of the Fourier transform, we arrive at

$$\begin{aligned}\mathcal{F}[T^*_{-m-2p-2l,p}](\underline{y}) \;&=\; \frac{(-1)^l l!\pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\Gamma\left(\frac{m}{2}+p+l\right)}\,\mathcal{F}[P_p(\underline{x})\,\Delta^{p+l}\delta(\underline{x})](\underline{y})\\ &=\; i^{-p}\,\frac{l!\pi^{\frac{m}{2}+l+p}}{2^p(p+l)!\Gamma\left(\frac{m}{2}+p+l\right)}\,P_p(\partial_{\underline{y}})\,\rho^{2p+2l}\end{aligned}$$

As $p+l \geq p$, Proposition 3.1 (i) leads to the desired result, i.e.

$$\mathcal{F}[T^*_{-m-2p-2l,p}] \;=\; i^{-p}\,\frac{\pi^{\frac{m}{2}+p+l}}{\Gamma\left(\frac{m}{2}+p+l\right)}\,\rho^{2l}\,P_p(\underline{y}) \;=\; i^{-p}\,T^*_{2l,p} \qquad (4.41)$$

In the above, we have used the notation $\rho = |\underline{y}|$.

CASE C. $\lambda = 2l,\ l \in \mathbb{N}_0$
This case follows by considering the action of the Fourier operator on (4.41):

$$\mathcal{F}[T^*_{2l,p}](\underline{y}) \;=\; i^p\,T^*_{-m-2p-2l,p}(-\underline{y}) \;=\; i^{-p}\,T^*_{-m-2p-2l,p}(\underline{y})$$

■

As the four families of normalized distributions are interrelated by the multiplication with the vector $\underline{x}$ and by the action of the Dirac operator $\partial_{\underline{x}}$, the Fourier transform of the distributions $U^*_{\lambda,p}$, $V^*_{\lambda,p}$ and $W^*_{\lambda,p}$ can be calculated leaning upon Theorem 4.7. However, for the specific value of $\lambda = -m-2p+1$, formula (4.33) does not hold, causing also a hiatus in the following theorem.

Theorem 4.8. *For all $(\lambda, p) \in \mathbb{C}\times\mathbb{N}_0$, the Fourier transform of the distributions $U^*_{\lambda,p}$, respectively $V^*_{\lambda,p}$, is given by*

$$\mathcal{F}[U^*_{\lambda,p}] = i^{-p-1}\, U^*_{-\lambda-m-2p,p} \tag{4.42}$$

$$\mathcal{F}[V^*_{\lambda,p}] = i^{-p-1}\, V^*_{-\lambda-m-2p,p} \tag{4.43}$$

while for $\lambda \neq 0$ and $p \in \mathbb{N}$, the Fourier transform of the distribution $W^*_{\lambda,p}$ is given by

$$\mathcal{F}[W^*_{\lambda,p}] = i^{-p}\frac{1}{\lambda}\left[(\lambda+m+2p)\, W^*_{-\lambda-m-2p,p} - (m-2)\, T^*_{-\lambda-m-2p,p}\right]$$

Proof.
Let $(\lambda, p) \in \mathbb{C}\times\mathbb{N}_0$. Substituting λ by $\lambda+1$ in (4.30) and then taking the Fourier transform of both sides yields

$$\begin{aligned}\mathcal{F}[(\lambda+1)\, U^*_{\lambda,p}](\underline{y}) &= \mathcal{F}[\partial_{\underline{x}} T^*_{\lambda+1,p}](\underline{y}) = 2\pi i\, \underline{y}\, \mathcal{F}[T^*_{\lambda+1,p}](\underline{y}) \\ &= 2\pi i^{-p+1}\, \underline{y}\, T^*_{-\lambda-m-2p-1,p} = (\lambda+1)\, i^{-p-1}\, U^*_{-\lambda-m-2p,p}\end{aligned}$$

the last step on account of Theorem 4.5. Formula (4.42) then immediately follows. The calculation of $\mathcal{F}[V^*_{\lambda,p}]$ runs along similar lines.
Next, on account of Theorem 4.5, we find

$$\mathcal{F}[W^*_{\lambda,p}](\underline{y}) = \mathcal{F}[U^*_{\lambda-1,p}\, \underline{x}](\underline{y}) = \frac{i}{2\pi}\, \mathcal{F}[U^*_{\lambda-1,p}](\underline{y})\, \partial_{\underline{y}} = \frac{i^{-p}}{2\pi}\, U^*_{-\lambda-m-2p+1,p}\, \partial_{\underline{y}}$$

Excluding the case $\lambda = 0$ and taking into account (4.33), we finally obtain

$$\mathcal{F}[W^*_{\lambda,p}] = i^{-p}\frac{1}{\lambda}\left[(\lambda+m+2p)\, W^*_{-\lambda-m-2p,p} - (m-2)\, T^*_{-\lambda-m-2p,p}\right]$$

■

Convolution

In Subsection 4.3.3 the convolvability of the distributions $T^*_{\lambda,0}$ and $U^*_{\lambda,0}$ has been studied. Here we proceed with this study by considering the convolution of arbitrary members of $\mathbb{T}_p \cup \mathbb{U}_p$.

The convolvability problem is tackled stepwise. First, in Lemma 4.2, a specific relation between $T^*_{\lambda,p}$ and $T^*_{\lambda+2p,0}$, respectively between $U^*_{\lambda,p}$ and $U^*_{\lambda+2p,0}$, is established, by means of which we will be able to convert new convolutions into already known ones. This lemma is then used to deal with convolutions within or in-between the $\mathbb{T}_p$– and $\mathbb{U}_p$–families where, for one of the involved distributions, we still have that $p = 0$. Finally, the main results are given in Theorem 4.9, completing the picture in the most general case where $p \neq 0$ for both distributions involved.

Lemma 4.2. *For each couple* $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$*, one has*

$$(-2)^p \frac{\Gamma\left(-\frac{\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2} - p\right)} T^*_{\lambda,p} = T^*_{\lambda+2p,0} P_p(\partial_{\underline{x}}) \tag{4.44}$$

$$(-2)^p \frac{\Gamma\left(-\frac{\lambda-1}{2}\right)}{\Gamma\left(-\frac{\lambda-1}{2} - p\right)} U^*_{\lambda,p} = U^*_{\lambda+2p,0} P_p(\partial_{\underline{x}}) \tag{4.45}$$

Proof.
We only prove the first equality, the proof of the second one running along similar lines. From (4.29) one can derive that

$$T^*_{\lambda,p} = \pi^p \frac{\Gamma\left(\frac{\lambda+m}{2}\right)}{\Gamma\left(\frac{\lambda+m}{2} + p\right)} T^*_{\lambda,0} P_p(\underline{x}) \tag{4.46}$$

for $\lambda \neq -m - 2l$, $l = 0, 1, \ldots, p-1$. Invoking Theorem 4.7 and some of the basic properties (3.2), we convert (4.46) to frequency space, which leads to

$$i^{-p} T^*_{-\lambda-m-2p,p} = \pi^p \frac{\Gamma\left(\frac{\lambda+m}{2}\right)}{\Gamma\left(\frac{\lambda+m}{2} + p\right)} \frac{i^p}{(2\pi)^p} T^*_{-\lambda-m,0} P_p(\partial_{\underline{x}})$$

Replacing λ by $-\lambda - m - 2p$, we obtain

$$(-2)^p T^*_{\lambda,p} = \frac{\Gamma\left(-\frac{\lambda}{2} - p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} T^*_{\lambda+2p,0} P_p(\partial_{\underline{x}})$$

for $\lambda \neq -2p + 2l$, $l = 0, \ldots, p-1$. Finally, rewriting this equality in the form of (4.44) reveals its validity for all couples (λ, p), since both sides reduce to zero whenever λ takes one of the values excluded above. ■

As announced, the previous lemma gives rise to a first generalization of the formulae (4.9), (4.14) and (4.15).

Lemma 4.3. *For each triplet* $(\lambda, \mu, p) \in \mathbb{C} \times \mathbb{C} \times \mathbb{N}$ *such that* *(i)* $\lambda \neq 2j$, $\mu \neq 2k$ *and* $\lambda + \mu \neq -m + 2l$, $j, k, l \in \mathbb{N}_0$, *one has*

$$T^*_{\lambda,p} * T^*_{\mu,0} = T^*_{\mu,0} * T^*_{\lambda,p} = c_m(\lambda,\mu)\, T^*_{\lambda+\mu+m,p}$$

(ii) $\lambda \neq 2j$, $\mu \neq 2k+1$ *and* $\lambda + \mu \neq -m + 2l + 1$, $j, k, l \in \mathbb{N}_0$, *one has*

$$\begin{aligned} T^*_{\lambda,p} * U^*_{\mu,0} &= c_m(\lambda,\mu-1)\, V^*_{\lambda+\mu+m,p} \\ U^*_{\mu,0} * T^*_{\lambda,p} &= c_m(\lambda,\mu-1)\, U^*_{\lambda+\mu+m,p} \end{aligned}$$

(iii) $\lambda \neq 2j+1$, $\mu \neq 2k$ *and* $\lambda + \mu \neq -m + 2l + 1$, $j, k, l \in \mathbb{N}_0$, *one has*

$$U^*_{\lambda,p} * T^*_{\mu,0} = T^*_{\mu,0} * U^*_{\lambda,p} = c_m(\lambda-1,\mu)\, U^*_{\lambda+\mu+m,p}$$

(iv) $\lambda \neq 2j+1$, $\mu \neq 2k+1$ *and* $\lambda + \mu \neq -m + 2l$, $j, k, l \in \mathbb{N}_0$, *one has*

$$U^*_{\mu,0} * U^*_{\lambda,p} = \frac{-2\pi}{\lambda+\mu+m}\, c_m(\lambda-1,\mu-1)\, T^*_{\lambda+\mu+m,p}$$

and if moreover $\lambda + \mu \neq -2m - 2p$, *one has*

$$\begin{aligned} U^*_{\lambda,p} * U^*_{\mu,0} &= \frac{2\pi}{(\lambda+\mu+2m+2p)(\lambda+\mu+m)}\, c_m(\lambda-1,\mu-1) \\ &\quad \times \left[(m-2)\, T^*_{\lambda+\mu+m,p} + (\lambda+\mu+m)\, W^*_{\lambda+\mu+m,p}\right] \end{aligned}$$

Proof.
We only treat the case of $T^*_{\lambda,p} * T^*_{\mu,0}$, the other cases being similar. First, take $\lambda \neq -2p + 2j$, $j = 0, 1, \ldots, p-1$. In that case, (4.44) can be rewritten as

$$T^*_{\lambda,p} = \frac{(-1)^p}{2^p} \frac{\Gamma\left(-\frac{\lambda}{2} - p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\, T^*_{\lambda+2p,0}\, P_p(\partial_{\underline{x}}) \tag{4.47}$$

Then, from (4.47), it follows that

$$\begin{aligned} T^*_{\lambda,p} * T^*_{\mu,0} &= \frac{(-1)^p}{2^p} \frac{\Gamma\left(-\frac{\lambda}{2} - p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} \left(T^*_{\lambda+2p,0}\, P_p(\partial_{\underline{x}}) * T^*_{\mu,0}\right) \\ &= \frac{(-1)^p}{2^p} \frac{\Gamma\left(-\frac{\lambda}{2} - p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\, P_p(\partial_{\underline{x}}) \left(T^*_{\lambda+2p,0} * T^*_{\mu,0}\right) \end{aligned}$$

In order for formula (4.9) to be applicable to the last expression, we need to assume, in addition to the premised conditions of (i), that $\lambda+\mu \neq -m-2p+2l$, $l = 0, 1, \ldots, p-1$. We are then lead to

$$T^*_{\lambda,p} * T^*_{\mu,0} \quad = \quad \frac{(-1)^p}{2^p} \frac{\Gamma\left(-\frac{\lambda}{2}-p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} c_m(\lambda+2p,\mu)\, P_p(\partial_{\underline{x}})\, T^*_{\lambda+\mu+m+2p,0}$$

from which the desired formula is easily obtained again exploiting (4.47):

$$\begin{aligned} T^*_{\lambda,p} * T^*_{\mu,0} &= \frac{\Gamma\left(-\frac{\lambda}{2}-p\right)\Gamma\left(-\frac{\lambda+\mu+m}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(-\frac{\lambda+\mu+m}{2}-p\right)} c_m(\lambda+2p,\mu)\, T^*_{\lambda+\mu+m,p} \\ &= c_m(\lambda,\mu)\, T^*_{\lambda+\mu+m,p} \end{aligned}$$

We now further examine the values $\lambda = -2p+2j$ and $\lambda+\mu = -m-2p+2l$, $j, l = 0, 1, \ldots, p-1$, which had to be excluded temporarily in the course of the proof. For these values, we may write $T^*_{-2p+2j,p} = \lim\limits_{\lambda\to -2p+2j} T^*_{\lambda,p}$, respectively $T^*_{-\mu-m-2p+2l,p} = \lim\limits_{\lambda\to-\mu-m-2p+2l} T^*_{\lambda,p}$, allowing us to repeat the procedure above, where we only effectuate the limit at the end of the calculations. ■

The previous lemma leads, in a second step, to more general results for the convolution of arbitrary $T^*_{\lambda,p}$– and/ or $U^*_{\lambda,p}$–distributions, apart from some exceptional values for the involved parameters which remain excluded. In order to make the formulae more compact, we introduce the constants

$$c_{m,p}(\lambda,\mu) \quad = \quad \frac{(-1)^p}{2^p} \pi^{\frac{m}{2}} \frac{\Gamma\left(-\frac{\lambda+\mu+m}{2}-p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(-\frac{\mu}{2}\right)}$$

with $c_{m,0}(\lambda,\mu) \equiv c_m(\lambda,\mu)$ (see 4.10).

Theorem 4.9. *For each 4–tuple $(\lambda,\mu,p,q) \in \mathbb{C}\times\mathbb{C}\times\mathbb{N}\times\mathbb{N}$ such that*

(i) $\lambda \neq 2j$ and $\mu \neq 2k$, $j, k \in \mathbb{N}_0$, one has

$$T^*_{\lambda,p} * T^*_{\mu,q} \quad = \quad c_{m,q}(\lambda,\mu)\, T^*_{\lambda+\mu+m+2q,p}\, P_q(\partial_{\underline{x}})$$

if $\lambda+\mu \neq -m-2q+2l$, $l \in \mathbb{N}_0$; and

$$T^*_{\lambda,p} * T^*_{\mu,q} \quad = \quad c_{m,p}(\lambda,\mu)\, P_p(\partial_{\underline{x}})\, T^*_{\lambda+\mu+m+2p,q}$$

if $\lambda + \mu \neq -m - 2p + 2l$, $l \in \mathbb{N}_0$.

(ii) $\lambda \neq 2j+1$, $\mu \neq 2k$ *and* $\lambda + \mu \neq -m - 2q + 2l + 1$, $j,k,l \in \mathbb{N}_0$, *one has*

$$U^*_{\lambda,p} * T^*_{\mu,q} \;=\; c_{m,q}(\lambda-1,\mu)\, U^*_{\lambda+\mu+m+2q,p}\, P_q(\partial_{\underline{x}})$$

(iii) $\lambda \neq 2j$, $\mu \neq 2k+1$ *and* $\lambda + \mu \neq -m - 2q + 2l + 1$, $j,k,l \in \mathbb{N}_0$, *one has*

$$T^*_{\lambda,p} * U^*_{\mu,q} \;=\; c_{m,q}(\lambda,\mu-1)\, V^*_{\lambda+\mu+m+2q,p}\, P_q(\partial_{\underline{x}}) \tag{4.48}$$

(iv) $\lambda \neq 2j+1$, $\mu \neq 2k+1$ *and* $\lambda + \mu \neq -m - 2q + 2l$, $j,k,l \in \mathbb{N}_0$, *one has*

$$\begin{aligned} U^*_{\lambda,p} * U^*_{\mu,q} \;=\; & \frac{2\pi}{(\lambda+\mu+2m+2p+2q)(\lambda+\mu+m+2q)}\, c_{m,q}(\lambda-1,\mu-1) \\ & \times \left[(m-2)\, T^*_{\lambda+\mu+m+2q,p} + (\lambda+\mu+m+2q)\, W^*_{\lambda+\mu+m+2q,p}\right] P_q(\partial_{\underline{x}}) \end{aligned}$$

if moreover $\lambda + \mu \neq -2m - 2p - 2q$.

Proof.
The proof directly follows from Lemma 4.2 and Lemma 4.3. ■

4.4.3 Examples: generalized Horváth kernels

In Subsection 2.2.1, we already introduced the important class of singular integral operators with convolution kernels (Horváth kernels) of the form

$$K(\underline{x}) \;=\; \mathrm{Pv}\,\frac{S_p(\underline{x})}{r^{m+p}} \;=\; \mathrm{Pv}\,\frac{S_p(\underline{\omega})}{r^m}$$

where S_p is a spherical harmonic of degree p. At that point we only considered scalar valued spherical harmonics. Now, a Clifford algebra valued refinement of those Horváth kernels is established by means of the families of Clifford distributions introduced above, giving rise to new generalized Hilbert transforms which will be discussed in Chapter 6.

When looking for principal value distributions amongst the above four families of distributions, it is clear that $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}_0$ have to be chosen in such a way that the resulting distributions are homogeneous of degree $(-m)$, this means: $\lambda + p + m = 0$. This results into the following four kinds of principal value distributions

$$\begin{aligned}
T_{-m-p,p} &= \mathrm{Fp}\, \frac{1}{r^m}\, P_p(\underline{\omega}) \;=\; \mathrm{Pv}\, \frac{P_p(\underline{\omega})}{r^m} \\
U_{-m-p,p} &= \mathrm{Fp}\, \frac{1}{r^m}\, \underline{\omega}\, P_p(\underline{\omega}) \;=\; \mathrm{Pv}\, \frac{\underline{\omega}\, P_p(\underline{\omega})}{r^m} \\
V_{-m-p,p} &= \mathrm{Fp}\, \frac{1}{r^m}\, P_p(\underline{\omega})\, \underline{\omega} \;=\; \mathrm{Pv}\, \frac{P_p(\underline{\omega})\, \underline{\omega}}{r^m} \\
W_{-m-p,p} &= \mathrm{Fp}\, \frac{1}{r^m}\, \underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega} \;=\; \mathrm{Pv}\, \frac{\underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega}}{r^m}
\end{aligned}$$

These distributions are homogeneous of degree $(-m)$ and the functions occurring in the numerator satisfy the cancellation condition

$$\int_{S^{m-1}} \Omega(\underline{\omega})\, dS(\underline{\omega}) \;=\; 0$$

$\Omega(\underline{\omega})$ being either of $P_p(\underline{\omega})$, $\underline{\omega}\, P_p(\underline{\omega})$, $P_p(\underline{\omega})\, \underline{\omega}$ or $\underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega}$. Note that, for $p = 0$, the distribution $U_{-m,0} = V_{-m,0} = \mathrm{Pv}\, \frac{\underline{\omega}}{r^m}$ is – up to a minus sign – exactly the kernel of the multidimensional Hilbert transform.

If the spherical monogenic P_p is realized under the action of the Dirac operator on a real valued spherical harmonic S_{p+1} of degree $(p+1)$, i.e. $P_p = \partial_{\underline{x}} S_{p+1}$, then the following decomposition formulae lead to the desired refinement of the Horváth kernels $\mathrm{Pv}\, \frac{S_{p+1}(\underline{\omega})}{r^m}$ and of the Clifford–vector valued singular integral kernel $\mathrm{Pv}\, \frac{\underline{\omega}\, S_{p+1}(\underline{\omega})}{r^m}$:

$$\begin{aligned}
\mathrm{Pv}\, \frac{S_{p+1}(\underline{\omega})}{r^m} &= -\frac{1}{2(p+1)}\, \left(U_{-m-p,p} + V_{-m-p,p}\right) \\
\mathrm{Pv}\, \frac{\underline{\omega}\, S_{p+1}(\underline{\omega})}{r^m} &= -\frac{1}{2(p+1)}\, \left(W_{-m-p,p} - T_{-m-p,p}\right)
\end{aligned}$$

For an overview of convolution kernels in Clifford analysis we refer to [29].

Chapter 5

The classical Clifford–Hilbert transform

In Chapter 2 we already discussed the classical one–dimensional Hilbert transform on the real line and some of its scalar valued higher dimensional extensions. In the introduction of the previous chapter we also mentioned that, to our knowledge, Horváth was the first to define a vector valued Hilbert transform on Euclidean space $\mathbb{R}^m$ using Clifford algebra (see [69]). This multidimensional Hilbert transform in the Clifford analysis setting was taken up again in the 1980's and further studied in e.g. [91, 60, 76, 50, 51]. It also plays a fundamental role in the study of Hardy spaces of monogenic functions, see e.g. [36, 77, 39, 4, 42, 49].

In the first section of this chapter, we present an alternative definition for the Clifford–vector valued Hilbert transform of Horváth, involving the multiplication with an extra basis vector. Its main properties are then examined; in particular its relationship with the Cauchy integral in $\mathbb{R}^{m+1}$ is disclosed, at the same time giving rise to a study of Hardy spaces of monogenic functions. Further, we also propose a multidimensional generalization of the concept of analytic signal in the Clifford analysis context. Finally, to conclude this section, we deal with the interaction between the Clifford–Hilbert transform and the Clifford–Radon transform, both transforms being protagonists in multidimensional signal analysis theory.

In the second section, the Clifford–Hilbert transform on closed surfaces in

Euclidean space $\mathbb{R}^m$ is introduced. It is shown that, in general, its properties are weaker then the ones of the Clifford–Hilbert transform on $\mathbb{R}^m$, except for the case of the unit sphere to which we will pay special attention.

5.1 The Clifford–Hilbert transform on $\mathbb{R}^m$

5.1.1 Definition and first properties

First we pass to $(m+1)$–dimensional space by introducing an additional basis vector e_0 which follows the usual multiplication rules, i.e. $e_0^2 = -1$ and it anti–commutes with the other basis vectors, viz $e_0 e_j + e_j e_0 = 0$, $j = 1, \ldots, m$. The variable $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ is then identified with the Clifford–vector

$$x \;=\; \sum_{j=0}^{m} e_j\, x_j \;=\; e_0\, x_0 + \underline{x}$$

in the Clifford algebra $\mathbb{R}_{0,m+1}$. Two open subspaces of $\mathbb{R}^{m+1}$ will be frequently used: the upper and lower half spaces $\mathbb{R}^{m+1}_{\pm}$, respectively given by

$$\begin{aligned} \mathbb{R}^{m+1}_{+} &= \{x = (x_0, \underline{x}) \in \mathbb{R}^{m+1} : x_0 > 0\} \\ \mathbb{R}^{m+1}_{-} &= \{x = (x_0, \underline{x}) \in \mathbb{R}^{m+1} : x_0 < 0\} \end{aligned}$$

Identifying Euclidean space $\mathbb{R}^m$ with the hyperplane $x_0 = 0$ in $\mathbb{R}^{m+1}$, we obtain that the boundaries of the spaces $\mathbb{R}^{m+1}_{\pm}$ are given by $\partial\mathbb{R}^{m+1}_{+} = \partial\mathbb{R}^{m+1}_{-} = \mathbb{R}^m$. Furthermore, the Dirac operator in $\mathbb{R}^{m+1}$ reads

$$\partial_x \;=\; \sum_{j=0}^{m} e_j\, \partial_{x_j} \;=\; e_0\, \partial_{x_0} + \partial_{\underline{x}}$$

Finally, in this chapter we consider functions defined in $\mathbb{R}^m$ and taking values in the Clifford algebra $\mathbb{R}_{0,m+1}$ or its complexification $\mathbb{C}_{m+1}$.

Now let $f \in L_2(\mathbb{R}^m)$; the *(Clifford–)Hilbert transform* $\mathcal{H}[f]$ of f on $\mathbb{R}^m$ is then defined by

$$\begin{aligned} \mathcal{H}[f](\underline{x}) &= \frac{2}{a_{m+1}}\, \overline{e_0}\, \mathrm{Pv} \int_{\mathbb{R}^m} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{\left|\underline{x} - \underline{y}\right|^{m+1}}\, f(\underline{y})\, dV(\underline{y}) \\ &= \frac{2}{a_{m+1}}\, \overline{e_0} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^m \setminus B(\underline{x};\varepsilon)} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{\left|\underline{x} - \underline{y}\right|^{m+1}}\, f(\underline{y})\, dV(\underline{y}) \end{aligned}$$

or, for an appropriate (see Remark 2.4) distribution f, by means of the convolution

$$\mathcal{H}[f](\underline{x}) \;=\; \overline{e_0}\,(H * f)\,(\underline{x}) \tag{5.1}$$

with H the convolution kernel given by the distribution

$$H(\underline{x}) \;=\; \frac{2}{a_{m+1}}\,\mathrm{Pv}\,\frac{\underline{\omega}}{r^m} \;=\; -\frac{2}{a_{m+1}}\,U^*_{-m,0} \tag{5.2}$$

In the previous chapter, we had already introduced the Clifford–vector valued Hilbert operator $\widetilde{\mathcal{H}}$. It may be clear that its relationship with the Clifford–Hilbert operator $\mathcal{H}$ is simply given by

$$\mathcal{H} \;=\; e_0\,\widetilde{\mathcal{H}} \tag{5.3}$$

Remark 5.1. *The motivation of introducing $\overline{e_0}$ in the formula (5.1) of the Clifford–Hilbert transform has its origin in the Clifford–Stokes theorem, which will be discussed in the following section. In the upper half space $\mathbb{R}^{m+1}_+$, the constant vector $\overline{e_0}$ plays the role of outward pointing unit normal vector in each point of the boundary $\mathbb{R}^m$. In the same order of ideas, when considering the Clifford–Hilbert transform on closed surfaces in $\mathbb{R}^m$, in the following section we will see that the outward pointing unit normal vector on the boundary of the closed surface under consideration will come into play.*

The Hilbert transform (5.1) then satisfies the following properties.

Property 5.1. *The Hilbert transform is a convolution operator, which is equivalent with saying that the Hilbert transform commutes with translations, i.e.*

$$\tau_{\underline{a}}[\mathcal{H}[f]] \;=\; \mathcal{H}[\tau_{\underline{a}}[f]]$$

with $\tau_{\underline{a}}[f](\underline{x}) = f(\underline{x} - \underline{a})$, $\underline{a} \in \mathbb{R}^m$.

Property 5.2. *The Hilbert kernel H is a homogeneous distribution of degree $(-m)$, which, for a convolution operator, is equivalent with saying that the Hilbert transform commutes with dilations, i.e.*

$$d_a[\mathcal{H}[f]] \;=\; \mathcal{H}[d_a[f]]$$

with $d_a[f](\underline{x}) = \frac{1}{a^{m/2}} f(\frac{\underline{x}}{a})$, $a > 0$.

Property 5.3. *The Hilbert and Fourier transforms are interrelated in the following way:*

$$\begin{aligned} \mathcal{F}[\mathcal{H}[f]](\underline{y}) &= \overline{e_0}\, i\, \underline{\xi}\, \mathcal{F}[f](\underline{y}) \\ \mathcal{H}[\mathcal{F}[f]](\underline{y}) &= e_0\, i\, \mathcal{F}[\underline{\omega}\, f(\underline{x})](\underline{y}) \end{aligned} \tag{5.4}$$

with $\underline{x} = r\underline{\omega}$, $\underline{y} = \rho\underline{\xi}$ *and* $\underline{\omega}, \underline{\xi} \in S^{m-1}$.

Proof.
Since the Hilbert transform is a convolution operator with kernel (5.2), we find, relying on Proposition 4.11 (ii), that on the one hand

$$\begin{aligned} \mathcal{F}[\mathcal{H}[f]](\underline{y}) &= \overline{e_0}\, \mathcal{F}[\frac{-2}{a_{m+1}}\, U^*_{-m,0} * f](\underline{y}) \ = \ \overline{e_0}\, \frac{-2}{a_{m+1}}\, \mathcal{F}[U^*_{-m,0}](\underline{y}) \cdot \mathcal{F}[f](\underline{y}) \\ &= \overline{e_0}\, \frac{2}{a_{m+1}}\, i\, U^*_{0,0} \cdot \mathcal{F}[f](\underline{y}) \ = \ \overline{e_0}\, i\, \underline{\xi}\, \mathcal{F}[f](\underline{y}) \end{aligned}$$

and on the other hand

$$\begin{aligned} \mathcal{H}[\mathcal{F}[f]](\underline{y}) &= \overline{e_0}\, \frac{-2}{a_{m+1}}\, U^*_{-m,0} * \mathcal{F}[f](\underline{y}) \ = \ \overline{e_0}\, \frac{-2}{a_{m+1}}\, i\, \mathcal{F}[U^*_{0,0}](\underline{y}) * \mathcal{F}[f](\underline{y}) \\ &= \overline{e_0}\, \frac{-2}{a_{m+1}}\, i\, \mathcal{F}[U^*_{0,0} \cdot f](\underline{y}) \ = \ e_0\, i\, \mathcal{F}[\underline{\omega}\, f(\underline{x})](\underline{y}) \end{aligned}$$

■

The Fourier symbol $\mathcal{F}[H](\underline{x}) = i\,\underline{\omega}$ of the Hilbert transform being a bounded function is equivalent with saying that the Hilbert transform is a bounded linear operator on $L_2(\mathbb{R}^m)$. More precisely one has:

Property 5.4. *The Hilbert transform is a bounded linear operator on* $L_2(\mathbb{R}^m)$*, and is a fortiori norm preserving, i.e.*

$$\|\mathcal{H}[f]\|_{L_2} = \|f\|_{L_2} \tag{5.5}$$

More generally, it also preserves the inner product

$$\langle \mathcal{H}[f], \mathcal{H}[g] \rangle = \langle f, g \rangle \tag{5.6}$$

Proof.
On account of Parseval's identity and (5.4) we find

$$\begin{aligned} \langle \mathcal{H}[f], \mathcal{H}[g] \rangle &= \langle \mathcal{F}[\mathcal{H}[f]], \mathcal{F}[\mathcal{H}[g]] \rangle \ = \ \langle \overline{e_0}\, i\, \underline{\omega}\, \mathcal{F}[f](\underline{x}), \overline{e_0}\, i\, \underline{\omega}\, \mathcal{F}[g](\underline{x}) \rangle \\ &= \int_{\mathbb{R}^m} \mathcal{F}[f](\underline{x})^\dagger\, \overline{\underline{\omega}}\, (-i)\, e_0\, \overline{e_0}\, i\, \underline{\omega}\, \mathcal{F}[g](\underline{x})\, dV(\underline{x}) \\ &= \langle \mathcal{F}[f], \mathcal{F}[g] \rangle \ = \ \langle f, g \rangle \end{aligned}$$

which proves (5.6) and at the same time also (5.5) since

$$\|\mathcal{H}[f]\|_{L_2} \;=\; [\langle \mathcal{H}[f], \mathcal{H}[f]\rangle]_0 \;=\; [\langle f, f\rangle]_0 \;=\; \|f\|_{L_2}$$

∎

Next, the inverse and the adjoint of the Hilbert transform may be calculated.

Property 5.5. *The Hilbert transform $\mathcal{H} : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$ is an involution, i.e. it is invertible with $\mathcal{H}^{-1} = \mathcal{H}$.*

Proof.
Applying formula (5.4) twice, we find in frequency space that

$$\mathcal{F}[\mathcal{H}^2[f]](\underline{x}) \;=\; \overline{e_0}\, i\, \underline{\omega}\, \mathcal{F}[\mathcal{H}[f]](\underline{x}) \;=\; \mathcal{F}[f](\underline{x})$$

So, the Hilbert transform $\mathcal{H}$ satisfies $\mathcal{H}^2 = \mathbf{1}$ on $L_2(\mathbb{R}^m)$ and may hence be addressed as an involution. ∎

Corollary 5.1. *The Hilbert transform $\mathcal{H} : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$ is unitary, its adjoint being given by $\mathcal{H}^* = \mathcal{H}$, i.e.*

$$\langle \mathcal{H}[f], g\rangle \;=\; \langle f, \mathcal{H}[g]\rangle \;, \qquad f, g \in L_2(\mathbb{R}^m)$$

Proof.
Let $\mathcal{H}^*$ be the unique adjoint of $\mathcal{H} : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$. Taking then into account (5.6), one has for all $f, g \in L_2(\mathbb{R}^m)$ that

$$\langle f, g\rangle \;=\; \langle \mathcal{H}[f], \mathcal{H}[g]\rangle \;=\; \langle f, \mathcal{H}^*[\mathcal{H}[g]]\rangle$$

This implies that $\mathcal{H}^*\mathcal{H} = \mathbf{1}$, or that $\mathcal{H}$ is unitary. Moreover, since $\mathcal{H}$ is an involution, this means that $\mathcal{H} = \mathcal{H}^*$. ∎

Taking into account the connection (5.3) between the Hilbert operators $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, the following relationship between the Hilbert operator $\mathcal{H}$ and the Dirac operator $\partial_{\underline{x}}$ is obtained:

Property 5.6. *The Hilbert transform anti–commutes with the Dirac operator, i.e. if f and $\partial_{\underline{x}} f$ are in $L_2(\mathbb{R}^m)$ or if f is an appropriate distribution, then*

$$\mathcal{H}[\partial_{\underline{x}} f(\underline{x})](\underline{y}) \;=\; -\partial_{\underline{y}} \left[\mathcal{H}[f](\underline{y})\right]$$

Proof.
For the Hilbert–Dirac operator $\widetilde{\mathcal{H}}\partial_{\underline{x}}$ one has that $\widetilde{\mathcal{H}}\partial_{\underline{x}} = \partial_{\underline{x}}\widetilde{\mathcal{H}}$ on account of Proposition 4.10 (ii). Hence:

$$\mathcal{H}\, \partial_{\underline{x}} \;=\; e_0\, \widetilde{\mathcal{H}}\, \partial_{\underline{x}} \;=\; e_0\, \partial_{\underline{x}}\, \widetilde{\mathcal{H}} \;=\; -\partial_{\underline{x}}\, e_0\, \widetilde{\mathcal{H}} \;=\; -\partial_{\underline{x}}\, \mathcal{H}$$

∎

5.1.2 Relationship with the Cauchy integral

Let $f \in L_2(\mathbb{R}^m)$, then its Cauchy integral $\mathcal{C}[f]$ is defined in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$ by

$$\mathcal{C}[f](x_0, \underline{x}) = (C(x_0, \cdot) * f(\cdot))(\underline{x}) = \int_{\mathbb{R}^m} C(x_0, \underline{x} - \underline{y})\, f(\underline{y})\, dV(\underline{y})$$

where the Cauchy kernel

$$C(x) = C(x_0, \underline{x}) = \frac{1}{a_{m+1}} \frac{\overline{x} e_0}{|x|^{m+1}} = \frac{1}{a_{m+1}} \frac{x_0 + e_0 \underline{x}}{|x_0 + \underline{x}|^{m+1}}, \qquad x \neq 0$$

is the fundamental solution of the Cauchy–Riemann operator D_x in $\mathbb{R}^{m+1}$, given by

$$D_x = \overline{e_0}\, \partial_x = \partial_{x_0} + \overline{e_0}\, \partial_{\underline{x}} \tag{5.7}$$

This means that

(i) $C(x)$ is ∂_x–monogenic in $\mathbb{R}^{m+1} \setminus \{0\}$

(ii) $\lim\limits_{|x| \to +\infty} C(x) = 0$

(iii) $D_x C(x) = \delta(x)$ in distributional sense

On account of those properties, it may be clear that the Cauchy integral is ∂_x–monogenic in both the upper half space $\mathbb{R}^{m+1}_+$ and the lower half space $\mathbb{R}^{m+1}_-$. Further, for a function $f \in L_2(\mathbb{R}^m)$, taking the supremum either in $\mathbb{R}^{m+1}_+$ or $\mathbb{R}^{m+1}_-$, we also have that

$$\sup_{x_0 \lessgtr 0} \int_{\mathbb{R}^m} |\mathcal{C}[f](x_0, \underline{x})|^2 \, dV(\underline{x}) < +\infty$$

Next, the Cauchy kernel is easily seen to decompose as

$$C(x) = C(x_0, \underline{x}) = \frac{1}{2} P(x_0, \underline{x}) + \frac{1}{2} \overline{e_0}\, Q(x_0, \underline{x}), \qquad x \neq 0$$

in which we have introduced the scalar valued Poisson kernel P given by

$$P(x) = P(x_0, \underline{x}) = \frac{2}{a_{m+1}} \frac{x_0}{|x_0 + \underline{x}|^{m+1}}, \qquad x \neq 0$$

and the vector valued conjugate Poisson kernel Q, given by

$$Q(x) \;=\; Q(x_0,\underline{x}) \;=\; \frac{2}{a_{m+1}}\,\frac{\overline{\underline{x}}}{|x_0+\underline{x}|^{m+1}}\;, \qquad x \neq 0$$

The Cauchy integral may then be rewritten as

$$\mathcal{C}[f](x_0,\underline{x}) \;=\; \frac{1}{2}\,\mathcal{P}[f](x_0,\underline{x}) + \overline{e_0}\,\frac{1}{2}\,\mathcal{Q}[f](x_0,\underline{x})\;, \qquad x_0 \neq 0$$

where

$$\mathcal{P}[f](x_0,\underline{x}) \;=\; \left(P(x_0,\cdot) * f(\cdot)\right)(\underline{x})\;, \qquad x_0 \neq 0$$

and

$$\mathcal{Q}[f](x_0,\underline{x}) \;=\; \left(Q(x_0,\cdot) * f(\cdot)\right)(\underline{x})\;, \qquad x_0 \neq 0$$

respectively are the Poisson transform and the conjugate Poisson transform of f. It then readily follows from the ∂_x–monogenicity of the Cauchy transform $\mathcal{C}[f]$ in $\mathbb{R}^{m+1}_+$ that $\mathcal{P}[f]$ and $\mathcal{Q}[f]$ are harmonic in $\mathbb{R}^{m+1}_+$ (and similarly in $\mathbb{R}^{m+1}_-$). Therefore we call $\mathcal{P}[f]$ and $\mathcal{Q}[f]$ *conjugate harmonics* in $\mathbb{R}^{m+1}_+$ (and similarly in $\mathbb{R}^{m+1}_-$), in the sense of [24].

The Hilbert transform now comes into play, when considering the non–tangential boundary limits of the Cauchy integral for $x_0 \to 0\pm$. Indeed, the following well–known distributional limits:

$$\begin{aligned}
\lim_{x_0\to 0\pm} P(x_0,\underline{x}) &= \lim_{x_0\to 0\pm} \frac{2}{a_{m+1}}\,\frac{x_0}{|x_0+\underline{x}|^{m+1}} = \pm\delta(\underline{x})\\
\lim_{x_0\to 0\pm} Q(x_0,\underline{x}) &= \lim_{x_0\to 0\pm} \frac{2}{a_{m+1}}\,\frac{\overline{\underline{x}}}{|x_0+\underline{x}|^{m+1}} = \frac{2}{a_{m+1}}\,\mathrm{Pv}\,\frac{\overline{\underline{\omega}}}{r^m} = H(\underline{x})
\end{aligned}$$

immediately lead to the following Plemelj–Sokhotzki formulae.

Property 5.7. *Let $f \in L_2(\mathbb{R}^m)$, then the non–tangential boundary limits of the Cauchy integral $\mathcal{C}[f]$ are given by*

$$\begin{aligned}
\mathcal{C}^+[f](\underline{x}) &\equiv \lim_{x_0\to 0+} \mathcal{C}[f](x_0,\underline{x}) = \frac{1}{2}\,f(\underline{x}) + \frac{1}{2}\,\mathcal{H}[f](\underline{x})\\
\mathcal{C}^-[f](\underline{x}) &\equiv \lim_{x_0\to 0-} \mathcal{C}[f](x_0,\underline{x}) = -\frac{1}{2}\,f(\underline{x}) + \frac{1}{2}\,\mathcal{H}[f](\underline{x})
\end{aligned}$$

For a function $f \in L_2(\mathbb{R}^m)$, we then call $\mathcal{C}^+[f]$ and $\mathcal{C}^-[f]$ its Cauchy transforms. They satisfy the following properties.

Corollary 5.2. *Let $f \in L_2(\mathbb{R}^m)$, then*

(i) $\mathcal{C}^+$ *and* $\mathcal{C}^-$ *are bounded linear operators on* $L_2(\mathbb{R}^m)$

(ii) $f = \mathcal{C}^+[f] - \mathcal{C}^-[f]$ *and* $\mathcal{H}[f] = \mathcal{C}^+[f] + \mathcal{C}^-[f]$

(iii) in $\mathbb{R}^{m+1}_+$ *one has* $\mathcal{C}[f] = \mathcal{C}[\mathcal{H}[f]]$*;*
in $\mathbb{R}^{m+1}_-$ *one has* $\mathcal{C}[f] = -\mathcal{C}[\mathcal{H}[f]]$

(iv) $\mathcal{C}^+[f] = e_0\,\mathcal{C}^-[e_0\, f]$ *and* $\mathcal{C}^-[f] = e_0\,\mathcal{C}^+[e_0\, f]$

(v) $\mathcal{C}^+[f]$ *and* $\mathcal{C}^-[f]$ *are orthogonal, i.e.* $\langle \mathcal{C}^+[f], \mathcal{C}^-[f] \rangle = 0$

(vi) the Fourier spectra of $\mathcal{C}^+[f]$ *and* $\mathcal{C}^-[f]$ *are respectively given by*

$$\begin{aligned}
\mathcal{F}[\mathcal{C}^+[f]](\underline{y}) &= \frac{1}{2}\,(1 + \overline{e_0}\, i\, \underline{\xi})\,\mathcal{F}[f](\underline{y}) \\
\mathcal{F}[\mathcal{C}^-[f]](\underline{y}) &= -\frac{1}{2}\,(1 - \overline{e_0}\, i\, \underline{\xi})\,\mathcal{F}[f](\underline{y})
\end{aligned}$$

5.1.3 Hardy spaces of monogenic functions

The Cauchy integral $\mathcal{C}[f]$ of a function $f \in L_2(\mathbb{R}^m)$ provides an example of a monogenic function in the upper half space $\mathbb{R}^{m+1}_+$ which belongs to the so–called Hardy space $H^2(\mathbb{R}^{m+1}_+)$, which is defined as follows.

Definition 5.1. *The Hardy space* $H^2(\mathbb{R}^{m+1}_+)$ *consists of functions* $F(x)$*, monogenic in* $\mathbb{R}^{m+1}_+$ *with respect to the Cauchy–Riemann operator* D_x*, for which*

$$\sup_{x_0>0} \int_{\mathbb{R}^m} |F(x_0, \underline{x})|^2 \, dV(\underline{x}) < +\infty$$

Taking into account Property 5.7, the non–tangential boundary limit for $x_0 \to 0+$ of $\mathcal{C}[f](x_0, \underline{x})$ again is in $L_2(\mathbb{R}^m)$. The question arises if this is also the case for any function of $H^2(\mathbb{R}^{m+1}_+)$; the answer is positive.

Proposition 5.1. *The non–tangential boundary limit for* $x_0 \to 0+$ *of a function* $F(x_0, \underline{x}) \in H^2(\mathbb{R}^{m+1}_+)$ *exists a.e. and belongs to* $L_2(\mathbb{R}^m)$*.*

A second question to investigate is whether the Cauchy integral of a non–tangential boundary limit of a function $F(x) \in H^2(\mathbb{R}^{m+1}_+)$, is precisely $F(x)$; again the answer is positive.

Proposition 5.2. *Let $F \in H^2(\mathbb{R}^{m+1}_+)$ and let $\lim_{x_0 \to 0+} F(x_0, \underline{x}) = f(\underline{x})$, then $\mathcal{C}[f] = F$.*

So the functions in $L_2(\mathbb{R}^m)$ which may be obtained as non–tangential boundary limit of functions in $H^2(\mathbb{R}^{m+1}_+)$, are special. This leads to the definition of another Hardy space, this time in $\mathbb{R}^m$ itself.

Definition 5.2. *The Hardy space $H^2(\mathbb{R}^m)$ is the closure in $L_2(\mathbb{R}^m)$ of the subspace of the non–tangential boundary limits for $x_0 \to 0+$ of all functions in $H^2(\mathbb{R}^{m+1}_+)$.*

We immediately obtain.

Corollary 5.3. *$H^2(\mathbb{R}^m)$ is a closed subspace of $L_2(\mathbb{R}^m)$ and hence itself a Hilbert space.*

It is also clear that both Hardy spaces $H^2(\mathbb{R}^{m+1}_+)$ and $H^2(\mathbb{R}^m)$ are intimately related; in fact they can be identified with each other.

Proposition 5.3. *$H^2(\mathbb{R}^m)$ and $H^2(\mathbb{R}^{m+1}_+)$ are isomorphic.*

From the considerations made above it follows that the Cauchy transform

$$\mathcal{C}^+ : L_2(\mathbb{R}^m) \longrightarrow H^2(\mathbb{R}^m) \; ; \; f \longmapsto \mathcal{C}^+[f]$$

is a projection; it is called the Hardy projection. It will be shown that, in the present context, this Hardy projection is an orthogonal projection. Note however that in a more general setting, where Hardy spaces are defined on the smooth boundary of bounded domains in Euclidean space, the Hardy projection will be a skew projection, except for the Hardy space on the unit sphere where the Hardy projection also is orthogonal (see the following section).

As $H^2(\mathbb{R}^m)$ is a closed subspace of $L_2(\mathbb{R}^m)$, the latter space may be decomposed into an orthogonal direct sum

$$L_2(\mathbb{R}^m) \;=\; H^2(\mathbb{R}^m) \oplus_\perp H^2(\mathbb{R}^m)^\perp \tag{5.8}$$

whence there exist two orthogonal projection operators $\mathbb{P}^+ : L_2(\mathbb{R}^m) \to H^2(\mathbb{R}^m)$ and $\mathbb{P}^- : L_2(\mathbb{R}^m) \to H^2(\mathbb{R}^m)^\perp$; these are called the Szegö projections. It is then clear from Property 5.7 and Corollary 5.2 that for $f \in L_2(\mathbb{R}^m)$

$$\mathbb{P}^+[f] \;=\; \mathcal{C}^+[f] \quad \text{and} \quad \mathbb{P}^-[f] \;=\; -\mathcal{C}^-[f]$$

as well as

$$\mathbb{P}^+[\mathcal{H}[f]] \;=\; \mathcal{C}^+[f] \quad \text{and} \quad \mathbb{P}^-[\mathcal{H}[f]] \;=\; \mathcal{C}^-[f]$$

So in this case, the Hardy en Szegö projections coincide. Moreover, it may be clear that in geometrical terms f and $\mathcal{H}[f]$ lie symmetrically with respect to $H^2(\mathbb{R}^m)$ (see also Figure 5.1).

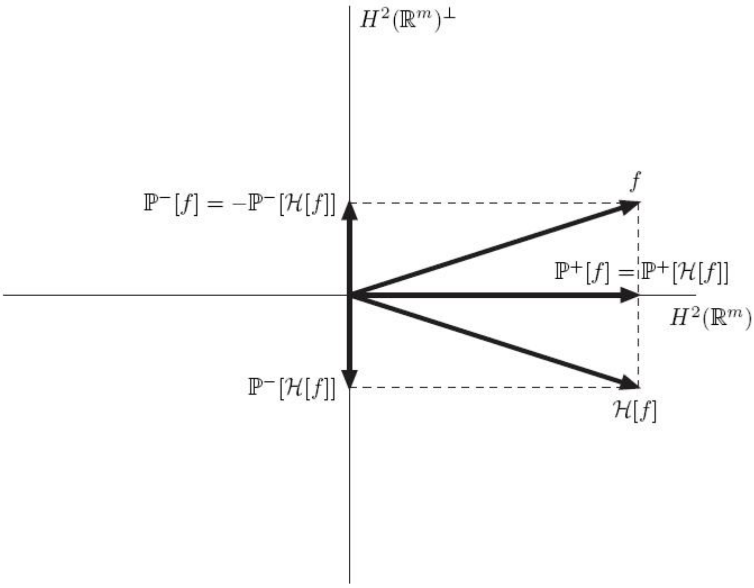

Figure 5.1: f and $\mathcal{H}[f]$ lie symmetrically w.r.t. $H^2(\mathbb{R}^m)$

Taking now into account Corollary 5.2 (iv), the orthogonal decomposition of $f \in L_2(\mathbb{R}^m)$ can be rewritten as

$$f \;=\; \mathbb{P}^+[f] + \mathbb{P}^-[f] \;=\; \mathcal{C}^+[f] - \mathcal{C}^-[f] \;=\; \mathcal{C}^+[f] + \overline{e_0}\,\mathcal{C}^+[e_0\, f]$$

such that the orthogonal decomposition (5.8) can be expressed in terms of the Hardy space $H^2(\mathbb{R}^m)$ only:

$$L_2(\mathbb{R}^m) \;=\; H^2(\mathbb{R}^m) \oplus_\perp \overline{e_0}\, H^2(\mathbb{R}^m)$$

The Hardy space $H^2(\mathbb{R}^m)$ is then characterized as follows.

Theorem 5.1. *A function $f \in L_2(\mathbb{R}^m)$ belongs to $H^2(\mathbb{R}^m)$ if and only if one of the following conditions is fulfilled:*

(i) $\mathcal{H}[f] = f$

(ii) $\mathcal{F}[f](\underline{y}) = \frac{1}{2}\,\mathcal{F}[(1 + \overline{e_0}\, i\, \underline{\omega})\, f(\underline{x})](\underline{y})$

(iii) $\mathbb{P}^+[f] = \mathcal{C}^+[f] = f$

(iv) $\mathcal{C}[f] = \mathcal{P}[f] = \overline{e_0}\,\mathcal{Q}[f]$ *in* $\mathbb{R}^{m+1}_+$

(v) $\mathcal{P}[f]$ *is monogenic in* $\mathbb{R}^{m+1}_+$

(vi) $\mathcal{Q}[f]$ *is monogenic in* $\mathbb{R}^{m+1}_+$

As already mentioned, the Hardy space $H^2(\mathbb{R}^m)$, as a closed subspace of $L_2(\mathbb{R}^m)$, is itself a Hilbert space. Moreover it possesses a reproducing kernel, i.e. a function $K(\underline{x}, \underline{y})$ such that for any $\underline{y} \in \mathbb{R}^m$ fixed, $K(\underline{x}, \underline{y}) \in H^2(\mathbb{R}^m)$ and

$$\big\langle K(\cdot, \underline{y}), f(\cdot) \big\rangle \;=\; f(\underline{y}) \ , \quad \forall f \in H^2(\mathbb{R}^m)$$

This reproducing kernel is called the Szegö kernel; it is given by

$$S_{x_0}(\underline{x}, \underline{y}) \;=\; \frac{1}{a_{m+1}} \, \frac{x_0 + \overline{e_0}\,(\underline{y} - \underline{x})}{\big|e_0\, x_0 + \underline{y} - \underline{x}\big|^{m+1}} \;=\; \overline{C(x_0, \underline{y} - \underline{x})} \ , \quad x_0 > 0$$

Clearly $S_{x_0}(\underline{x}, \underline{y})$ is monogenic in $\mathbb{R}^{m+1}_+$, for each $\underline{y} \in \mathbb{R}^m$ fixed, and moreover

$$\sup_{x_0 > 0} \int_{\mathbb{R}^m} \big|S_{x_0}(\underline{x}, \underline{y})\big|^2 \, dV(\underline{x}) < +\infty$$

so that the Szegö kernel belongs to $H^2(\mathbb{R}^{m+1}_+)$ for each $\underline{y} \in \mathbb{R}^m$ fixed. The reproducing property of the Szegö kernel then follows from

$$\big\langle S_{x_0}(\cdot, \underline{y}), f(\cdot) \big\rangle \;=\; \int_{\mathbb{R}^m} \overline{S_{x_0}(\underline{x}, \underline{y})}\, f(\underline{x})\, dV(\underline{x}) \;=\; \mathcal{C}[f](x_0, \underline{y}) \ , \quad x_0 > 0$$

since this Cauchy integral is in $H^2(\mathbb{R}^{m+1}_+)$ and its isomorphic image in $H^2(\mathbb{R}^m)$ is

$$\lim_{x_0 \to 0+} \big\langle S_{x_0}(\cdot, \underline{y}), f(\cdot) \big\rangle \;=\; \lim_{x_0 \to 0+} \mathcal{C}[f](x_0, \underline{y}) \;=\; f(\underline{y}) \ , \quad \underline{y} \in \mathbb{R}^m$$

The Szegö kernel $S_{x_0}(\underline{x},\underline{y})$ shows the symmetry property $\overline{S_{x_0}(\underline{x},\underline{y})} = S_{x_0}(\underline{y},\underline{x})$ if $x_0 > 0$ and moreover it holds in $\mathbb{R}^{m+1}_+$ that

$$\begin{aligned} S_{x_0}(\underline{x},\underline{y}) + \overline{S_{x_0}(\underline{x},\underline{y})} &= P(x_0,\underline{y}-\underline{x}) \\ S_{x_0}(\underline{x},\underline{y}) - \overline{S_{x_0}(\underline{x},\underline{y})} &= e_0\, Q(x_0,\underline{y}-\underline{x}) \end{aligned}$$

which links the Szegö kernel to the Poisson kernels. Moreover, it is important to note that the Szegö kernel is the integral kernel for the Szegö projection $\mathbb{P}^+$ (which here coincides with the Hardy projection $\mathcal{C}^+$), since in $\mathbb{R}^{m+1}_+$

$$\mathcal{C}[f](x_0,\underline{x}) \;=\; \langle S_{x_0}(\cdot,\underline{x}), f(\cdot)\rangle \;=\; \int_{\mathbb{R}^m} S_{x_0}(\underline{x},\underline{y})\, f(\underline{y})\, dV(\underline{y})$$

and hence

$$\mathbb{P}^+[f](\underline{x}) \;=\; \mathcal{C}^+[f](\underline{x}) \;=\; \lim_{x_0\to 0+} \int_{\mathbb{R}^m} S_{x_0}(\underline{x},\underline{y})\, f(\underline{y})\, dV(\underline{y})$$

In a similar way we can look for the reproducing kernel of the Hilbert space $H^2(\mathbb{R}^m)^\perp$, i.e. the orthogonal complement of $H^2(\mathbb{R}^m)$ in $L_2(\mathbb{R}^m)$; it is called the Garabedian kernel. One has for $f \in L_2(\mathbb{R}^m)$ and $x_0 > 0$:

$$\mathbb{P}^-[f](\underline{x}) \;=\; \overline{e_0}\,\mathcal{C}^+[e_0\, f](\underline{x}) \;=\; \lim_{x_0\to 0+} \int_{\mathbb{R}^m} \overline{S_{x_0}(\underline{x},\underline{y})}\, f(\underline{y})\, dV(\underline{y})$$

when taking into account Corollary 5.2 (iv). This means that the kernel function

$$L_{x_0}(\underline{x},\underline{y}) \;=\; \overline{S_{x_0}(\underline{x},\underline{y})} \;=\; \frac{1}{a_{m+1}}\, \frac{x_0 - \overline{e_0}\,(\underline{y}-\underline{x})}{\left|e_0\, x_0 + \underline{y} - \underline{x}\right|^{m+1}}\,, \qquad x_0 > 0$$

is the integral kernel for the Szegö projection $\mathbb{P}^-$. The Garabedian kernel $L_{x_0}(\underline{x},\underline{y})$ is clearly anti–monogenic in $\mathbb{R}^{m+1}_+$ with respect to the variable $(x_0,\underline{x})$, i.e.

$$\overline{D_x} L_{x_0}(\underline{x},\underline{y}) \;=\; 0$$

and it satisfies the symmetry property $\overline{L_{x_0}(\underline{x},\underline{y})} = L_{x_0}(\underline{y},\underline{x})$. Moreover, it is indeed the reproducing kernel for $H^2(\mathbb{R}^m)^\perp$, since for $x_0 > 0$ and $\underline{y} \in \mathbb{R}^m$ fixed, $L_{x_0}(\underline{x},\underline{y})$ belongs to $H^2(\mathbb{R}^m)^\perp$ and

$$\left\langle L_{x_0}(\cdot,\underline{y}), f(\cdot)\right\rangle \;=\; \int_{\mathbb{R}^m} \overline{L_{x_0}(\underline{x},\underline{y})}\, f(\underline{x})\, dV(\underline{x}) \;=\; -\mathcal{C}[f](-x_0,\underline{y})$$

This Cauchy integral is in the Hardy space $H^2(\mathbb{R}^{m+1}_-)$ which is isomorphic with $H^2(\mathbb{R}^m)^\perp$. If $f \in H^2(\mathbb{R}^m)^\perp$, then the isomorphic image of $-\mathcal{C}[f](-x_0, \underline{y})$ clearly is $-\mathcal{C}^-[f](\underline{y}) = f(\underline{y})$. Note that we now have in $\mathbb{R}^{m+1}_+$

$$\begin{aligned} S_{x_0}(\underline{x}, \underline{y}) + L_{x_0}(\underline{x}, \underline{y}) &= P(x_0, \underline{y} - \underline{x}) \\ S_{x_0}(\underline{x}, \underline{y}) - L_{x_0}(\underline{x}, \underline{y}) &= e_0\, Q(x_0, \underline{y} - \underline{x}) \end{aligned}$$

linking the Szegö and the Garabedian kernel with the (conjugate) Poisson kernel.

5.1.4 Application: analytic signal

Due to the splitting of the Clifford algebra

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e_0}\, \mathbb{R}_{0,m}$$

any function f taking values in $\mathbb{R}_{0,m+1}$, may be written as

$$f = u + \overline{e_0}\, v$$

where u and v take their values in $\mathbb{R}_{0,m}$. We introduce the involution defined by

$$\breve{f} = (u + \overline{e_0}\, v)\breve{} = u - \overline{e_0}\, v$$

So, if u and v are in $L_2(\mathbb{R}^m)$, then their Hilbert transforms $\mathcal{H}[u]$ and $\mathcal{H}[v]$ take their values in $\overline{e_0}\, \mathbb{R}_{0,m}$ and one has $(\mathcal{H}[u])\breve{} = -\mathcal{H}[u]$ and $(\mathcal{H}[v])\breve{} = -\mathcal{H}[v]$. We immediately observe that

$$\mathcal{H}[f] = \mathcal{H}[u] - \overline{e_0}\, \mathcal{H}[v] \qquad \text{and} \qquad \mathcal{H}[\breve{f}] = \mathcal{H}[u] + \overline{e_0}\, \mathcal{H}[v]$$

whence

$$\mathcal{H}[\breve{f}] = -(\mathcal{H}[f])\breve{}$$

It also follows that

$$\mathcal{C}^+[\breve{f}] = -(\mathcal{C}^-[f])\breve{} \qquad \text{and} \qquad \mathcal{C}^-[\breve{f}] = -(\mathcal{C}^+[f])\breve{}$$

Note that, in particular, if f is D_x–monogenic in a certain region, then $\breve{f}$ is anti–monogenic, and vice versa:

$$D_x f = 0 \iff \overline{D_x}\breve{f} = 0$$

with $\overline{D_x} = \partial_{x_0} - \overline{e_0}\, \partial_{\underline{x}} = \breve{D}_x$.

In a similar way as in the one–dimensional case, one could now introduce the notion of a multidimensional analytic signal in the Clifford setting. This is a signal of the form

$$f(\underline{x}) \;=\; u(\underline{x}) + \mathcal{H}[u](\underline{x})$$

where u is $\mathbb{R}_{0,m}$ valued and $\mathcal{H}[u]$ is $\overline{e_0}\,\mathbb{R}_{0,m}$ valued. We then call

$$\breve{f}(\underline{x}) \;=\; u(\underline{x}) - \mathcal{H}[u](\underline{x})$$

an anti–analytic signal. (Anti–)analytic signals show the following properties.

Property 5.8. *Let $u \in L_2(\mathbb{R}^m)$ and $f = u + \mathcal{H}[u]$, then*

(i) $\mathcal{H}[f] = f$ *and* $\mathcal{H}[\breve{f}] = -\breve{f}$

(ii) $\mathcal{C}^+[f] = f$ *and* $\mathcal{C}^-[f] = 0$*;*
$\mathcal{C}^+[\breve{f}] = 0$ *and* $\mathcal{C}^-[\breve{f}] = -\breve{f}$

(iii) $f = 2\,\mathcal{C}^+[u] = 2\,\mathcal{C}^+[\mathcal{H}[u]]$*;*
$-\breve{f} = 2\,\mathcal{C}^-[u] = -2\,\mathcal{C}^-[\mathcal{H}[u]]$

(iv) f *and* $\breve{f}$ *are orthogonal, i.e.* $\left\langle f, \breve{f} \right\rangle = 0$

(v) the Fourier spectra of f *and* $\breve{f}$ *are respectively given by*

$$\begin{aligned} \mathcal{F}[f](\underline{y}) &= (1 + \overline{e_0}\, i\, \underline{\xi})\, \mathcal{F}[u](\underline{y}) \\ \mathcal{F}[\breve{f}](\underline{y}) &= (1 - \overline{e_0}\, i\, \underline{\xi})\, \mathcal{F}[u](\underline{y}) \end{aligned}$$

Finally note that if u is a $\mathbb{R}_{0,m}$ valued function in $L_2(\mathbb{R}^m)$, then it is clear that its associated analytic signal $f = u + \mathcal{H}[u]$ belongs to the Hardy space $H^2(\mathbb{R}^m)$, while its associated anti–analytic signal $\breve{f} = u - \mathcal{H}[u]$ belongs to $H^2(\mathbb{R}^m)^{\perp}$. Conversely, if $f \in H^2(\mathbb{R}^m)$, then $\mathcal{H}[f] = f$, which leads to

$$\mathcal{H}[u + \overline{e_0}\, v] \;=\; u + \overline{e_0}\, v$$

and hence

$$\mathcal{H}[u] \;=\; \overline{e_0}\, v \quad \text{and} \quad \mathcal{H}[\overline{e_0}\, v] \;=\; u$$

and finally

$$f \;=\; (1 + \mathcal{H})[u]$$

which clearly has the structure of an analytic signal.

5.1.5 Relationship with the Radon transform

Introduction

In his 1917 paper [83], Johann Radon posed and solved the problem of reconstructing a function of two variables $f(x, y)$ if its integrals over arbitrary lines are given. In that original manuscript, the integral over the line with equation $x\cos(\phi) + y\sin(\phi) = p$, is written as

$$F(p, \phi) \quad = \quad \int_{s=-\infty}^{\infty} f(p\cos(\phi) - s\sin(\phi), p\sin(\phi) + s\cos(\phi))\, ds$$

which formally can be rewritten in the following form

$$F(p, \phi) \quad = \quad \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \delta(x\cos(\phi) + y\sin(\phi) - p)\, f(x, y)\, dx\, dy$$

Later on, Radon himself also considered analogues of his transform in higher dimensions. The idea of integrating over arbitrary lines was then translated into the more general concept of integrating over arbitrary hyperplanes, leading to the transform which assigns to a given function f defined in $\mathbb{R}^m$ the totality of its integrals over all hyperplanes in $\mathbb{R}^m$. One of the main problems of integral geometry is then to reconstruct the function f from the information contained in these "sliced profiles". Nowadays, the integral transform $f \mapsto F$ is called the *Radon transform* and the corresponding operator is usually denoted by $\mathcal{R}$.

For a detailed treatment of the theory of Radon transforms we refer to the classical works [59, 58, 67], while applications are extensively treated in [45] and the references therein. More in general, people have also studied integrals of functions over surfaces belonging to a special class, such as spheres (see [72]), quadrics (see [44]), or even over zeros of higher order homogeneous polynomials (see [78, 47]).

The Radon transform in Clifford analysis

For a sufficiently smooth function f defined in $\mathbb{R}^m$ and taking values in the Clifford algebra $\mathbb{R}_{0,m+1}$, its *(Clifford–)Radon transform* $\mathcal{R}[f]$ is defined as

$$\mathcal{R}[f](\alpha) \quad = \quad \int_{\alpha} f(\underline{x})\, d\ell(\underline{x}) \tag{5.9}$$

with α a hyperplane in $\mathbb{R}^m$ and $d\ell(\underline{x})$ the Lebesgue measure on that hyperplane. However, this definition is not practical to work with in calculations. Therefore, we have chosen for the following more convenient and popular definition, which is shown to be equivalent with (5.9) (see [45]). Let $\underline{n} \in S^{m-1}$ be a unit normal vector of the hyperplane α and let $s \in \mathbb{R}$ be the oriented distance of the hyperplane α to the origin, i.e. the hyperplane α is given by the equation $\langle \underline{x}, \underline{n} \rangle - s = 0$, then the Radon transform of f may also be rewritten as

$$\mathcal{R}[f](\underline{n}, s) \;=\; \int_{\mathbb{R}^m} \delta(\langle \underline{x}, \underline{n} \rangle - s)\, f(\underline{x})\, dV(\underline{x}) \tag{5.10}$$

Stricto sensu we have allowed for some abuse of mathematical language at the right-hand side of the above formula; however all results obtained are rigorous.

The Radon transform (5.10) is linear in two different, yet related senses. Indeed, for suitable $\mathbb{R}_{0,m+1}$ valued functions $f = \sum_A e_A\, f_A$ and g defined in $\mathbb{R}^m$ and real scalars a and b, it is clear that on the one hand

$$\mathcal{R}[a\, f + b\, g](\underline{n}, s) \;=\; a\, \mathcal{R}[f](\underline{n}, s) + b\, \mathcal{R}[g](\underline{n}, s)$$

while on the other hand also

$$\mathcal{R}[f](\underline{n}, s) \;=\; \mathcal{R}[\sum_A e_A\, f_A](\underline{n}, s) \;=\; \sum_A e_A\, \mathcal{R}[f_A](\underline{n}, s) \tag{5.11}$$

A well–known and important formula, relating the Radon transform to the Fourier transform, is given in the following lemma.

Lemma 5.1. *For a real variable u and a unit vector $\underline{n}$ one has*

$$\mathcal{F}[f(\underline{x})](u\, \underline{n}) \;=\; \mathcal{F}_{s \to u}[\mathcal{R}[f](\underline{n}, s)](u) \tag{5.12}$$

where f is a suitable, sufficiently smooth $\mathbb{R}_{0,m+1}$ valued function in $\mathbb{R}^m$.

Proof.
One subsequently has

$$\begin{aligned}
\mathcal{F}_{s \to u}\left[\mathcal{R}[f](\underline{n}, s)\right](u) \;&=\; \int_{-\infty}^{\infty} \mathcal{R}[f](\underline{n}, s)\, \exp(-2\pi i\, u\, s)\, ds \\
&= \int_{-\infty}^{\infty} \left[\int_{\mathbb{R}^m} \delta(\langle \underline{x}, \underline{n} \rangle - s)\, f(\underline{x})\, dV(\underline{x})\right] \exp(-2\pi i\, u\, s)\, ds \\
&= \int_{\mathbb{R}^m} f(\underline{x})\, dV(\underline{x}) \int_{-\infty}^{\infty} \delta(\langle \underline{x}, \underline{n} \rangle - s)\, \exp(-2\pi i\, u\, s)\, ds \\
&= \int_{\mathbb{R}^m} f(\underline{x})\, \exp(-2\pi i\, u\, \langle \underline{x}, \underline{n} \rangle)\, dV(\underline{x}) \;=\; \mathcal{F}[f(\underline{x})](u\, \underline{n})
\end{aligned}$$

yielding the desired result. ∎

This lemma shows that the m–dimensional Fourier transform of a suitable function may be obtained through subsequent application of the Radon transform and the one–dimensional Fourier transform. The result is known as the *central–slice theorem.*

Some other elementary properties of the Radon transform are listed in Lemma 5.2. The proofs for scalar valued functions can be found e.g. in [46]; the proofs for $\mathbb{R}_{0,m+1}$ valued functions run along similar lines, keeping in mind the linearity (5.11) of the Radon transform.

Lemma 5.2. *Let $(\underline{n}, s) \in S^{m-1} \times \mathbb{R}$ and let f, f_1, f_2 be sufficiently smooth $\mathbb{R}_{0,m+1}$ valued functions in $\mathbb{R}^m$, then*

(i) $\mathcal{R}[f(\underline{x}+\underline{t})](\underline{n}, s) = \mathcal{R}[f(\underline{x})](\underline{n}, s + \langle \underline{t}, \underline{n}\rangle)\ , \quad \underline{t} \in \mathbb{R}^m$

(ii) $\mathcal{R}[\partial_{x_j} f(\underline{x})](\underline{n}, s) = n_j\, \partial_s \mathcal{R}[f(\underline{x})](\underline{n}, s)\ , \quad j = 1, \ldots, m$

(iii) $\mathcal{R}[(f_1 * f_2)(\underline{x})](\underline{n}, s) = (\mathcal{R}[f_1](\underline{n}, \cdot) * \mathcal{R}[f_2](\underline{n}, \cdot))(s)$

We want to draw the reader's attention to the fact that in property (iii) the m–dimensional convolution at the left-hand side is converted into a one–dimensional convolution, applying only to the oriented distance, while the role of the unit vector $\underline{n}$ is reduced to the one of a parameter.

Interaction between the Hilbert and the Radon transform

As a consequence of formula (5.12), in [56] a link was established between the two–dimensional Hilbert transform (therein referred to as Riesz transform) of a two–dimensional signal and the one–dimensional Hilbert transforms of the so–called *intrinsically one–dimensional profiles* obtained from the Radon transform (see equation (4.34) in [56]). We have generalized this result to higher dimension in our Clifford context.

Proposition 5.4. *For a suitable function f defined on $\mathbb{R}^m$ $(m > 1)$ and taking values in the Clifford algebra $\mathbb{R}_{0,m+1}$, one has*

$$\mathcal{R}[\mathcal{H}[f]](\underline{n}, s) = e_0\, \underline{n}\, \mathcal{H}_{u \to s}[\mathcal{R}[f](\underline{n}, u)](s)\ , \quad (\underline{n}, s) \in S^{m-1} \times \mathbb{R} \tag{5.13}$$

The above formula has to be read in the following way: the Radon transform of the m–dimensional Clifford–Hilbert transform of a given function f, evaluated in some hyperplane, can be obtained, up to some factor involving the normal vector to that hyperplane, by taking the one–dimensional Hilbert transform of the Radon transform of f, evaluated in the oriented distance of the hyperplane to the origin. We have proven this proposition by a suitable adaptation of the techniques used in [56].

Proof.
We will show (5.13) by passing to one–dimensional frequency space where the unit vector $\underline{n}$ plays the role of a parameter. In a first step we make use of relation (5.12) between the Fourier and the Radon transform and of relation (5.4) between the Fourier and the Hilbert transform, yielding

$$\begin{aligned} \mathcal{F}_{s\to q}\left\{\mathcal{R}[\mathcal{H}[f]](\underline{n},s)\right\}(q) &= \mathcal{F}[\mathcal{H}[f](\underline{x})](q\,\underline{n}) \\ = \overline{e_0}\,i\,\frac{q\,\underline{n}}{|q\,\underline{n}|}\,\mathcal{F}[f](q\,\underline{n}) &= \overline{e_0}\,i\,\underline{n}\,\mathrm{sgn}(q)\,\mathcal{F}[f](q\,\underline{n}) \end{aligned}$$

We proceed by noticing that $(-i)\,\mathrm{sgn}(q)$ is the Fourier symbol of the one–dimensional Hilbert kernel on the real line, and by applying once more (5.12), now on the Fourier transform of f, which leads us to

$$\begin{aligned} &\mathcal{F}_{s\to q}\left\{\mathcal{R}[\mathcal{H}[f]](\underline{n},s)\right\}(q) \\ &\quad= e_0\,\underline{n}\,\mathcal{F}_{s\to q}[H(s)](q)\cdot\mathcal{F}_{s\to q}[\mathcal{R}[f](\underline{n},s)](q) \\ &\quad= e_0\,\underline{n}\,\mathcal{F}_{s\to q}[(H(u)*\mathcal{R}[f](\underline{n},u))(s)](q) \end{aligned}$$

So in frequency space one has

$$\mathcal{F}_{s\to q}\left\{\mathcal{R}[\mathcal{H}[f]](\underline{n},s)\right\}(q) \quad = \quad e_0\,\underline{n}\,\mathcal{F}_{s\to q}\left\{\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)\right\}(q)$$

from which formula (5.13) immediately can be deduced. ■

5.2 The Clifford–Hilbert transform on closed surfaces in $\mathbb{R}^m$

In this section we consider the definition and the fundamental properties of the (Clifford–)Hilbert transform on closed surfaces in Euclidean space. Extra attention is paid to the special case of the Hilbert transform on the unit sphere, since only on that particular closed surface the Hilbert transform will be unitary.

First we set some notations. We denote by Ω some open subset of $\mathbb{R}^m$. Then, an m–dimensional compact differentiable and oriented manifold $\Gamma \subset \Omega$ with C^∞ smooth boundary $\partial\Gamma$ is considered. Further, Γ^+ will stand for the interior of Γ, and Γ^- for the exterior of Γ with respect to Ω, i.e. $\Gamma^- = \Omega \setminus \Gamma$.

Subsequently, we denote by $\widetilde{d\sigma_{\underline{x}}}$ the vector valued *oriented surface element* on $\partial\Gamma$ given by the following differential form of order $(m-1)$:

$$\widetilde{d\sigma_{\underline{x}}} \;=\; \sum_{j=1}^{m} e_j\,(-1)^{j-1}\,\widetilde{\widehat{dx_j}}$$

with

$$\widetilde{\widehat{dx_j}} \;=\; dx_1 \wedge \cdots \wedge [dx_j] \wedge \cdots \wedge dx_m$$

where $[\cdot]$ denotes omitting that particular differential, i.e.

$$dx_1 \wedge \cdots \wedge [dx_j] \wedge \cdots \wedge dx_m \;=\; dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_m$$

Alternatively, if $dS(\underline{x})$ stands for the classical surface element on $\partial\Gamma$ and $\nu(\underline{x})$ for the outward pointing (with respect to Γ^+) unit normal vector in $\underline{x}$ on $\partial\Gamma$, then the surface element $\widetilde{d\sigma_{\underline{x}}}$ may also be expressed as

$$\widetilde{d\sigma_{\underline{x}}} \;=\; \nu(\underline{x})\,dS(\underline{x})$$

The corresponding *oriented volume element* on Γ reads

$$\widetilde{dV}(\underline{x}) \;=\; dx_1 \wedge \cdots \wedge dx_n$$

For functions f and g defined on $\partial\Gamma$ and with values in the Clifford algebra $\mathbb{C}_m$, we then introduce the $\mathbb{C}_m$ valued inner product

$$\langle f, g\rangle_{\partial\Gamma} \;=\; \int_{\partial\Gamma} f^\dagger(\underline{x})\,g(\underline{x})\,dS(\underline{x})$$

and moreover the associated norm

$$\|f\|^2_{L_2(\partial\Gamma)} \;=\; [\langle f, f\rangle]_0 \;=\; \int_{\partial\Gamma} |f(\underline{x})|^2\; dS(\underline{x})$$

We also consider the right Hilbert–module $L_2(\partial\Gamma)$ of square integrable functions f defined on $\partial\Gamma$ for which it holds that $\|f\|_{L_2(\partial\Gamma)} < +\infty$. To make the formulae more compact, in what follows, we drop "$\partial\Gamma$" in the notation for the inner product and the norm.

Now, in order to arrive at the definition of the Hilbert transform on $\partial\Gamma$, we list some elementary function theoretical results in Clifford analysis (see e.g. [23]). First of all, the theorem of Stokes may be formulated as follows.

Theorem 5.2 (Clifford–Stokes theorem). *Let f and g be functions in $C^1(\Omega)$ and let $\Gamma \subset \Omega$ be an m–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} f(\underline{x})\, \widetilde{d\sigma_{\underline{x}}}\, g(\underline{x}) \;=\; \int_{\Gamma} \left[(f(\underline{x})\partial_{\underline{x}})\, g(\underline{x}) + f(\underline{x})\, (\partial_{\underline{x}} g(\underline{x})) \right] \widetilde{dV}(\underline{x})$$

As an immediate consequence one obtains the basic theorem of Cauchy.

Theorem 5.3 (Clifford–Cauchy theorem). *Let the function f be $\partial_{\underline{x}}$–monogenic in Ω and let $\Gamma \subset \Omega$ be an m–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} \widetilde{d\sigma_{\underline{x}}}\, f(\underline{x}) \;=\; 0$$

Next, a Cauchy–Pompeiu formula is obtained.

Theorem 5.4 (Cauchy–Pompeiu formula). *Let f be a function in $C^1(\Omega)$ and let $\Gamma \subset \Omega$ be an m–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} E(\underline{x}-\underline{y})\, \widetilde{d\sigma_{\underline{x}}}\, f(\underline{x}) - \int_{\Gamma} E(\underline{x}-\underline{y}) \left[\partial_{\underline{x}} f(\underline{x})\right] \widetilde{dV}(\underline{x}) \;=\; \begin{cases} 0\,, & \underline{y} \in \Gamma^- \\ f(\underline{y})\,, & \underline{y} \in \Gamma^+ \end{cases}$$

In particular, for $\partial_{\underline{x}}$–monogenic functions this leads to:

Corollary 5.4 (Cauchy integral formula). *Let the function f be $\partial_{\underline{x}}$–monogenic in Ω and let $\Gamma \subset \Omega$ be an m–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} E(\underline{x}-\underline{y})\, \widetilde{d\sigma_{\underline{x}}}\, f(\underline{x}) \;=\; \begin{cases} 0\,, & \underline{y} \in \Gamma^- \\ f(\underline{y})\,, & \underline{y} \in \Gamma^+ \end{cases}$$

Now let $f \in L_2(\partial\Gamma)$; its Cauchy integral $\mathcal{C}[f]$ in $\Gamma^\pm$ is defined by

$$\mathcal{C}[f](\underline{y}) \quad = \quad \int_{\partial\Gamma} E(\underline{x} - \underline{y})\, \widetilde{d\sigma_{\underline{x}}}\, f(\underline{x}) \ , \qquad \underline{y} \in \Gamma^\pm$$

Clearly, this Cauchy integral is $\partial_{\underline{x}}$–monogenic in $\Gamma^\pm$. The Hilbert transform $\mathcal{H}[f]$ then pops up in a natural way when considering the non–tangential boundary limits of the Cauchy integral $\mathcal{C}[f]$, leading to the Plemelj–Sokhotzki formulae:

$$\mathcal{C}^+[f](\underline{u}) \ \equiv \ \lim_{\substack{\underline{y}\to\underline{u} \\ \underline{y}\in\Gamma^+}} \mathcal{C}[f](\underline{y}) \ = \ \frac{1}{2}\, f(\underline{u}) + \frac{1}{2}\, \mathcal{H}[f](\underline{u}) \ , \quad \underline{u} \in \partial\Gamma \tag{5.14}$$

$$\mathcal{C}^-[f](\underline{u}) \ \equiv \ \lim_{\substack{\underline{y}\to\underline{u} \\ \underline{y}\in\Gamma^-}} \mathcal{C}[f](\underline{y}) \ = \ -\frac{1}{2}\, f(\underline{u}) + \frac{1}{2}\, \mathcal{H}[f](\underline{u}) \ , \quad \underline{u} \in \partial\Gamma$$

where the limits are taken in L_2 sense and where the Hilbert transform $\mathcal{H}[f]$ for functions $f \in L_2(\partial\Gamma)$ is given by the principal value integral

$$\mathcal{H}[f](\underline{u}) \quad = \quad 2\ \mathrm{Pv} \int_{\partial\Gamma} E(\underline{x} - \underline{u})\, \widetilde{d\sigma_{\underline{x}}}\, f(\underline{x}) \ , \qquad \underline{u} \in \partial\Gamma \tag{5.15}$$

We list its main properties, apart from the above defining one. For their proofs we refer to e.g. [39, 42, 4].

Property 5.9.

P(1) $\mathcal{H}$ *is a bounded linear operator on* $L_2(\partial\Gamma)$

P(2) $\mathcal{H}^2 = \mathbf{1}$ *on* $L_2(\partial\Gamma)$

P(3) $\mathcal{H}^* = \nu\, \mathcal{H}\, \nu$ *on* $L_2(\partial\Gamma)$

P(4) for $f \in L_2(\partial\Gamma)$*, one has that* $\mathcal{H}[f] = f$ *if and only if* $f \in H^2(\partial\Gamma)$

The last property P(4) deserves some more explanation. For the open set Γ^+ one can consider the Hardy space $H^2(\Gamma^+)$ of $\partial_{\underline{x}}$–monogenic Clifford algebra valued functions, viz

$$H^2(\Gamma^+) \quad = \quad \{f : \Gamma^+ \to \mathbb{C}_m \ : \ \partial_{\underline{x}} f = 0 \text{ in } \Gamma^+ \text{ and } f_{\partial\Gamma} \in L_2(\partial\Gamma)\}$$

where $f_{\partial\Gamma}$ denotes the non–tangential boundary limit of f. It is well–known that $H^2(\Gamma^+)$ entails the Hardy space $H^2(\partial\Gamma)$ as the closure in $L_2(\partial\Gamma)$ of the space of all non–tangential boundary limits of all functions in $H^2(\Gamma^+)$. Moreover, both spaces $H^2(\Gamma^+)$ and $H^2(\partial\Gamma)$ are isomorphic, the isomorphism being obtained explicitly by means of the Cauchy integral in the following way. For a given $f \in H^2(\partial\Gamma)$ its Cauchy integral $\mathcal{C}[f]$ belongs to $H^2(\Gamma^+)$ and

$$\lim_{\substack{\underline{y}\to\underline{u} \\ \underline{y}\in\Gamma^+}} \mathcal{C}[f](\underline{y}) \;=\; f(\underline{u}) \;, \qquad \underline{u} \in \partial\Gamma$$

in the L_2 sense, so that $\mathcal{C}[f]$ may be seen as the $\partial_{\underline{x}}$–monogenic extension of f to Γ^+. Finally, as $H^2(\partial\Gamma)$ is a closed subspace of $L_2(\partial\Gamma)$, the latter space may be decomposed into an orthogonal direct sum

$$L_2(\partial\Gamma) \;=\; H^2(\partial\Gamma) \oplus_\perp H^2(\partial\Gamma)^\perp \tag{5.16}$$

We also point out that in general, property P(3) has some severe consequences. Firstly, in general, the Hilbert transform on closed surfaces in $\mathbb{R}^m$ is not unitary, as opposed to the Hilbert transform on $\mathbb{R}^m$. Secondly, the Hardy projection $\mathcal{C}^+$ is a linear bounded operator, mapping $L_2(\partial\Gamma)$ onto $H^2(\partial\Gamma)$, which is, in general, only a skew projection. Indeed, every function $f \in L_2(\partial\Gamma)$ may be decomposed as $f = \mathcal{C}^+[f] + (-\mathcal{C}^-[f])$, but clearly $\mathcal{C}^-[f] \notin H^2(\partial\Gamma)^\perp$. Invoking now property P(3), the orthogonal decomposition (5.16) can be expressed in terms of the Hardy space $H^2(\partial\Gamma)$ only:

Proposition 5.5.

$$L_2(\partial\Gamma) \;=\; H^2(\partial\Gamma) \oplus_\perp \nu\, H^2(\partial\Gamma)$$

Proof.
We need to characterize the orthogonal complement $H^2(\partial\Gamma)^\perp$ of the space $H^2(\partial\Gamma)$. To this end, take an arbitrary function $f \in L^2(\partial\Gamma)$ and note that $f + \mathcal{H}[f] \in H^2(\partial\Gamma)$, since $\mathcal{H}[f + \mathcal{H}[f]] = f + \mathcal{H}[f]$. Thus, for any $g \in H^2(\partial\Gamma)^\perp$, one has

$$\langle f + \mathcal{H}[f], g\rangle \;=\; 0 \tag{5.17}$$

which may be rewritten as

$$\langle f, g + \mathcal{H}^*[g]\rangle \;=\; 0 \tag{5.18}$$

whence $\mathcal{H}^*[g] = -g$. Conversely, take $g \in L_2(\partial\Gamma)$ such that $\mathcal{H}^*[g] = -g$, then g fulfils (5.18) and hence also (5.17). From this, we may conclude that

$g \in H^2(\partial\Gamma)^\perp$, since any function in $H^2(\partial\Gamma)$ can always be written in the form $f + \mathcal{H}[f]$, $f \in L_2(\partial\Gamma)$. So one has that

$$g \in H^2(\partial\Gamma)^\perp \quad \Longleftrightarrow \quad \mathcal{H}^*[g] \;=\; -g$$

On account of property P(3) and of the fact that $\nu^2 = -1$, this is seen to be equivalent to $\mathcal{H}[\nu g] = \nu g$, or still to $\nu g \in H^2(\partial\Gamma)$, in view of property P(4). Once more invoking $\nu^2 = -1$ we thus have shown that

$$g \in H^2(\partial\Gamma)^\perp \quad \Longleftrightarrow \quad g \in \nu H^2(\partial\Gamma)$$

■

Now considering the two orthogonal projection operators, the so–called Szegö projections, $\mathbb{P}^+ : L_2(\partial\Gamma) \to H^2(\partial\Gamma)$ and $\mathbb{P}^- : L_2(\partial\Gamma) \to H^2(\partial\Gamma)^\perp$, every function $f \in L_2(\partial\Gamma)$ may be orthogonally decomposed as

$$f \;=\; \mathbb{P}^+[f] + \mathbb{P}^-[f] \tag{5.19}$$

As a direct consequence of the previous proposition, the orthogonal decomposition (5.19) of f may also be written in terms of the Szegö projection $\mathbb{P}^+$ only.

Corollary 5.5. *Every function $f \in L_2(\partial\Gamma)$ may be orthogonally decomposed as*

$$f \;=\; \mathbb{P}^+[f] - \nu\,\mathbb{P}^+[\nu f] \tag{5.20}$$

Proof.
If the function f belongs to $L_2(\partial\Gamma)$, then the same holds for the function νf. Clearly, the orthogonal decomposition of the latter function is given by

$$\nu\, f \;=\; \mathbb{P}^+[\nu f] + \mathbb{P}^-[\nu f]$$

Multiplying both sides with $(-\nu)$ then yields

$$f \;=\; -\nu\,\mathbb{P}^+[\nu f] - \nu\,\mathbb{P}^-[\nu f] \tag{5.21}$$

Taking into account Proposition 5.5, we find that $-\nu\,\mathbb{P}^+[\nu f] \in H^2(\partial\Gamma)^\perp$ and $-\nu\,\mathbb{P}^-[\nu f] \in H^2(\partial\Gamma)$. Comparing then the expressions (5.19) and (5.21) for f, leads to

$$\begin{aligned} \mathbb{P}^+[f] &= -\nu\,\mathbb{P}^-[\nu f] \\ \mathbb{P}^-[f] &= -\nu\,\mathbb{P}^+[\nu f] \end{aligned}$$

from which the desired decomposition (5.20) follows. ■

We end this section by considering the special case of $\Gamma^+ = \overset{\circ}{B}(\underline{0};1)$, i.e. the open unit ball in $\mathbb{R}^m$. In this case $\partial\Gamma = S^{m-1}$, i.e. the unit sphere in $\mathbb{R}^m$, and at each point $\underline{\omega} \in S^{m-1}$, one has $\nu(\underline{\omega}) = \underline{\omega}$.

The Hilbert transform $\mathcal{H}[f]$ of a function $f \in L_2(S^{m-1})$ thus reads

$$\begin{aligned} \mathcal{H}[f](\underline{\xi}) &= \frac{2}{a_m}\,\mathrm{Pv}\int_{S^{m-1}} \frac{\underline{\xi}-\underline{\omega}}{|\underline{\xi}-\underline{\omega}|^m}\,\underline{\omega}\, f(\underline{\omega})\, dS(\underline{\omega}) \\ &= \frac{2}{a_m}\,\mathrm{Pv}\int_{S^{m-1}} \frac{1+\underline{\xi}\underline{\omega}}{|1+\underline{\omega}\underline{\xi}|^m}\, f(\underline{\omega})\, dS(\underline{\omega}) \end{aligned}$$

where $\underline{\xi} \in S^{m-1}$. It was then shown in e.g. [51, Theorem 2.1] that only in this case the Hilbert transform is unitary. Indeed, one readily finds:

$$\mathcal{H}^*[f](\underline{\xi}) \;=\; \underline{\xi}\,\mathcal{H}[\underline{\omega} f(\underline{\omega})](\underline{\xi}) \;=\; \mathcal{H}[f](\underline{\xi})$$

Chapter 6

Generalized Clifford–Hilbert transforms on $\mathbb{R}^m$ involving spherical monogenics

In this chapter we treat two possible generalizations of the Clifford–Hilbert transform in $\mathbb{R}^m$, making use of the families of Clifford distributions already introduced in Section 4.4, and aiming at preserving in these approaches as many of the traditional properties of the Clifford–Hilbert transform as possible. It is shown that in each of both cases some of the properties – different ones – are inevitably lost. Nevertheless we twice obtain a new bounded singular integral operator on L_2 or on appropriate Sobolev spaces.

The two generalizations are presented in the first section of this chapter (see [10, 13]). In the first approach the Clifford–Hilbert transform on $\mathbb{R}^m$ is generalized by using convolution kernels which were already briefly discussed in Subsection 4.4.3. There it was shown that they constitute a refinement of the generalized Hilbert kernels introduced by Horváth in [70]. We will now investigate our generalized Hilbert convolution kernels more thoroughly. The resulting generalized Hilbert transforms are shown to be no longer unitary operators, yet they remain bounded singular operators on $L_2(\mathbb{R}^m)$.

The second approach is based on the intimate relationship between the Hilbert transform and the Cauchy integral and starts with the construction of a generalized Cauchy integral in $\mathbb{R}^{m+1}$ involving a distribution from one of the above mentioned families as a generalized Cauchy kernel. A new generalized Hilbert transform in $\mathbb{R}^m$ is then defined as part of the L_2 or distributional boundary limits of the generalized Cauchy integral considered, and it is shown to be a bounded operator on Sobolev spaces $W_2^n(\mathbb{R}^m)$.

Finally a connection is established between both generalizations through the action of a higher order Dirac derivative.

The relationship between the Clifford–Radon transform and the classical Clifford–Hilbert transform on $\mathbb{R}^m$ was treated in Subsection 5.1.5. The second section in this chapter is devoted to the action of the Radon transform on the two types of generalized Hilbert operators mentioned above (see [15]).

6.1 Two types of generalized Hilbert transforms on $\mathbb{R}^m$

6.1.1 First generalization

For $p \in \mathbb{N}$, we recall the specific distributions introduced in Subsection 4.4.3:

$$\begin{array}{rclcl} T_{-m-p,p} & = & \mathrm{Fp}\,\dfrac{1}{r^m}\,P_p(\underline{\omega}) & = & \mathrm{Pv}\,\dfrac{P_p(\underline{\omega})}{r^m} \\ U_{-m-p,p} & = & \mathrm{Fp}\,\dfrac{1}{r^m}\,\underline{\omega}\,P_p(\underline{\omega}) & = & \mathrm{Pv}\,\dfrac{\underline{\omega}\,P_p(\underline{\omega})}{r^m} \\ V_{-m-p,p} & = & \mathrm{Fp}\,\dfrac{1}{r^m}\,P_p(\underline{\omega})\,\underline{\omega} & = & \mathrm{Pv}\,\dfrac{P_p(\underline{\omega})\,\underline{\omega}}{r^m} \\ W_{-m-p,p} & = & \mathrm{Fp}\,\dfrac{1}{r^m}\,\underline{\omega}\,P_p(\underline{\omega})\,\underline{\omega} & = & \mathrm{Pv}\,\dfrac{\underline{\omega}\,P_p(\underline{\omega})\,\underline{\omega}}{r^m} \end{array} \tag{6.1}$$

for which it holds that

$$\begin{array}{rcl} \mathrm{Pv}\,\dfrac{S_{p+1}(\underline{\omega})}{r^m} & = & -\dfrac{1}{2(p+1)}\,(U_{-m-p,p} + V_{-m-p,p}) \\ \mathrm{Pv}\,\dfrac{\underline{\omega}\,S_{p+1}(\underline{\omega})}{r^m} & = & -\dfrac{1}{2(p+1)}\,(W_{-m-p,p} - T_{-m-p,p}) \end{array}$$

where $P_p(\underline{x}) = \partial_{\underline{x}} S_{p+1}(\underline{x})$ is a vector valued spherical monogenic, S_{p+1} being a scalar valued spherical harmonic. Note that these distributions are homogeneous of degree $(-m)$ and that the functions in the respective numerators of (6.1) all satisfy the cancellation condition

$$\int_{S^{m-1}} \Omega(\underline{\omega})\, dS(\underline{\omega}) \;=\; 0$$

$\Omega(\underline{\omega})$ being either of $P_p(\underline{\omega})$, $\underline{\omega}\, P_p(\underline{\omega})$, $P_p(\underline{\omega})\, \underline{\omega}$ or $\underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega}$.

Next, we introduce the convolution operators

$$\begin{aligned} \mathcal{T}_p &= T_{-m-p,p} * f \qquad & \mathcal{U}_p &= U_{-m-p,p} * f \\ \mathcal{V}_p &= V_{-m-p,p} * f \qquad & \mathcal{W}_p &= W_{-m-p,p} * f \end{aligned} \tag{6.2}$$

which are direct generalizations of the Clifford–Hilbert transform $\mathcal{H}$ on $\mathbb{R}^m$. The following properties are then immediately obtained.

Property 6.1. *The generalized Hilbert transforms (6.2) commute with translations, which is an equivalent statement to their definition as convolution operators.*

Property 6.2. *The generalized Hilbert transforms (6.2) commute with dilations, which, for convolution operators, is an equivalent statement to the above consideration that their kernels (6.1) are homogeneous distributions of degree $(-m)$.*

Now taking into account Theorems 4.7 and 4.8, the Fourier symbols of the generalized Hilbert kernels (6.1), given by

$$\begin{aligned} \mathcal{F}[T_{-m-p,p}] &= i^{-p}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\, P_p(\underline{\omega}) \\ \mathcal{F}[U_{-m-p,p}] &= i^{-p-1}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, \underline{\omega}\, P_p(\underline{\omega}) \\ \mathcal{F}[V_{-m-p,p}] &= i^{-p-1}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, P_p(\underline{\omega})\, \underline{\omega} \\ \mathcal{F}[W_{-m-p,p}] &= i^{-p-2}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)} \left(\underline{\omega}\, P_p(\underline{\omega})\, \underline{\omega} - \frac{m-2}{p}\, P_p(\underline{\omega}) \right) \end{aligned} \tag{6.3}$$

are homogeneous of degree 0 and moreover are bounded functions. We then have

Property 6.3. *The generalized Hilbert transforms (6.2) and the Fourier transform are interrelated in the following way:*

$$\begin{aligned}
\mathcal{F}[\mathcal{T}_p[f]] &= i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{\omega})\,\mathcal{F}[f] \\
\mathcal{T}_p[\mathcal{F}[f]] &= i^{p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,\mathcal{F}[P_p(\underline{\omega})\,f] \\
\mathcal{F}[\mathcal{U}_p[f]] &= i^{-p-1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,\underline{\omega}\,P_p(\underline{\omega})\,\mathcal{F}[f] \\
\mathcal{U}_p[\mathcal{F}[f]] &= i^{p+1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,\mathcal{F}[\underline{\omega}\,P_p(\underline{\omega})\,f] \\
\mathcal{F}[\mathcal{V}_p[f]] &= i^{-p-1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,P_p(\underline{\omega})\,\underline{\omega}\,\mathcal{F}[f] \\
\mathcal{V}_p[\mathcal{F}[f]] &= i^{p+1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,\mathcal{F}[P_p(\underline{\omega})\,\underline{\omega}\,f] \\
\mathcal{F}[\mathcal{W}_p[f]] &= i^{-p-2}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)}\left(\underline{\omega}\,P_p(\underline{\omega})\,\underline{\omega} - \frac{m-2}{p}\,P_p(\underline{\omega})\right)\mathcal{F}[f] \\
\mathcal{W}_p[\mathcal{F}[f]] &= i^{p+2}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)}\,\mathcal{F}\left[\left(\underline{\omega}\,P_p(\underline{\omega})\,\underline{\omega} - \frac{m-2}{p}\,P_p(\underline{\omega})\right)\,f\right]
\end{aligned}$$

Property 6.4. *The generalized Hilbert transforms (6.2) are bounded linear operators on $L_2(\mathbb{R}^m)$.*

Now we investigate whether these new operators will show some appropriate analogues of the remaining properties of the classical Clifford–Hilbert transform on $\mathbb{R}^m$. To this end we closely examine the kernel $T_{-m-p,p}$ and the corresponding operator $\mathcal{T}_p$.

A first traditional property of the Hilbert transform is that it squares to unity, or more precisely: to the identity operator. Here we have

$$\mathcal{T}_p^2[f] \;=\; (T_{-m-p} * T_{-m-p}) * f$$

whence we should investigate whether the convolution of the kernel T_{-m-p} with itself equals the delta distribution. Now, since

$$\mathcal{F}[T_{-m-p,p} * T_{-m-p,p}] \;=\; (\mathcal{F}[T_{-m-p,p}])^2 \;\neq\; 1 \tag{6.4}$$

this is seen not to be the case, from which we may conclude that $\mathcal{T}_p^2 \neq \mathbf{1}$. Next, the traditional Hilbert transform is a unitary operator. In order to investigate whether the generalized Hilbert transform $\mathcal{T}_p$ fulfils this property, we first compute its adjoint $\mathcal{T}_p^*$. For functions $f, g \in L_2(\mathbb{R}^m)$ one finds

$$\begin{aligned}
\big\langle f, \mathcal{T}_p^*[g]\big\rangle \;&=\; \langle \mathcal{T}_p[f], g\rangle \;=\; \int_{\mathbb{R}^m} \mathcal{T}_p[f](\underline{x})^\dagger\, g(\underline{x})\, dV(\underline{x}) \\
&=\; \int_{\mathbb{R}^m} \left[\mathrm{Pv} \int_{\mathbb{R}^m} f(\underline{y})^\dagger \, \frac{P_p(\underline{x}-\underline{y})^\dagger}{|\underline{x}-\underline{y}|^{m+p}}\, dV(\underline{y})\right] g(\underline{x})\, dV(\underline{x}) \\
&=\; \int_{\mathbb{R}^m} f(\underline{y})^\dagger \left[(-1)^{p+1}\, \mathrm{Pv} \int_{\mathbb{R}^m} \frac{P_p(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^{m+p}}\, g(\underline{x})\, dV(\underline{x})\right] dV(\underline{y}) \\
&=\; \big\langle f, (-1)^{p+1}\mathcal{T}_p[g]\big\rangle
\end{aligned}$$

leading to

$$\mathcal{T}_p^* \;=\; (-1)^{p+1}\mathcal{T}_p$$

From this result we may also conclude, in view of (6.4), that $\mathcal{T}_p\mathcal{T}_p^* = \mathcal{T}_p^*\mathcal{T}_p \neq \mathbf{1}$, whence $\mathcal{T}_p$ is not a unitary operator. Finally, we fail to establish an analogue of Property 5.7 as well, since it has turned out being impossible to find a generalized Cauchy kernel in $\mathbb{R}^{m+1} \setminus \{0\}$, for which a part of the boundary limits precisely equals the generalized Hilbert kernel $T_{-m-p,p}$.

Similar conclusions hold for the other generalized kernels (6.1).

6.1.2 Second generalization

Subsequent to the observations above and in particular to the failing of the very crucial Property 5.7, we now want to find a type of generalized Hilbert kernel which actually is part of the non–tangential boundary limits of some generalized Cauchy kernel. To that end, we define the function

$$\begin{aligned}
C_p(x) \;=\; C_p(x_0, \underline{x}) \;&=\; \frac{1}{a_{m+1,p}}\, \frac{\overline{x} e_0}{|x|^{m+1+2p}}\, P_p(\underline{x}) \\
&=\; \frac{1}{a_{m+1,p}}\, \frac{x_0 + e_0\underline{x}}{|x_0+\underline{x}|^{m+1+2p}}\, P_p(\underline{x})\,, \qquad x \neq 0
\end{aligned}$$

involving a homogeneous polynomial P_p of degree $p \in \mathbb{N}_0$ on $\mathbb{R}^m$ which we take to be vector valued and monogenic, and where

$$a_{m+1,p} \;=\; \frac{(-1)^p}{2^p}\,\frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}+p\right)} \tag{6.5}$$

In the next proposition we show that these functions C_p constitute good candidates for generalized Cauchy kernels.

Proposition 6.1. *The function C_p satisfies the following properties:*

(i) $C_p \in L_1^{\text{loc}}(\mathbb{R}^{m+1})$ *and* $\lim_{|x|\to\infty} C_p(x) = 0$, $\forall p \in \mathbb{N}$

(ii) $D_x\, C_p(x) = P_p(\partial_{\underline{x}})\,\delta(x)$ *in distributional sense,* $\forall p \in \mathbb{N}$

(iii) for $p = 0$, C_0 *coincides with the traditional Cauchy kernel* C

Proof.
The proof of (i) being straightforward, we focus on the proofs of (ii) and (iii). First recall that in $\mathbb{R}^m$ the following formula holds for each couple $(\lambda, p) \in \mathbb{C}\times\mathbb{N}$ (see (4.32)):

$$\partial_{\underline{x}} U^*_{\lambda,p} \;=\; -2\pi\, T^*_{\lambda-1,p}$$

Hence, passing to $\mathbb{R}^{m+1}$ and using the tilde-notation for the corresponding families of distributions there, we still have that

$$\partial_x \widetilde{U}^*_{\lambda,p} \;=\; -2\pi\, \widetilde{T}^*_{\lambda-1,p}$$

Applying this formula in the specific case where $\lambda = -m-2p$, $p \in \mathbb{N}$, one gets

$$\partial_x \widetilde{U}^*_{-m-2p,p} \;=\; -2\pi\, \widetilde{T}^*_{-m-2p-1,p} \tag{6.6}$$

which, invoking the definitions for the normalized distributions in Subsection 4.4.1 – however with m being replaced by $(m+1)$ –, can be rewritten as

$$\partial_x \left(\frac{\pi}{\Gamma(1)}\,\widetilde{U}_{-m-2p,p}\right) \;=\; -2\pi\left(\frac{\pi^{\frac{m+1}{2}}}{2^{2p}p!\Gamma\left(\frac{m+1}{2}+p\right)}\,\widetilde{P}_p(x)\,\Delta^p\delta(x)\right)$$

or as

$$\partial_x\left(\frac{\omega}{|x|^{m+2p}}\,\widetilde{P}_p(x)\right) \;=\; -\frac{1}{2^p p!}\,a_{m+1,p}\,\widetilde{P}_p(x)\,\partial_x^{2p}\delta(x) \tag{6.7}$$

with $x = |x|\,\omega$, $\omega \in S^m$. Now take in particular

$$\widetilde{P}_p(x) = e_0\,P_p(\underline{x}) \tag{6.8}$$

Although that specific polynomial $\widetilde{P}_p$ is not vector valued, it may be clear that formula (6.6) still holds. Then taking into account the relationship (5.7) between ∂_x and D_x, and substituting (6.8) in (6.7) yields

$$D_x\left(\frac{\overline{x}e_0}{|x|^{m+1+2p}}\,P_p(\underline{x})\right) = \frac{1}{2^p p!}a_{m+1,p}\,P_p(\underline{x})\,\partial_x^{2p}\delta(x) \tag{6.9}$$

On the other hand we know from Proposition 3.1 that in $\mathbb{R}^m$

$$P_p(\underline{x})\,\partial_{\underline{x}}^{2p}\delta(\underline{x}) = 2^p p!\,P_p(\partial_{\underline{x}})\,\delta(\underline{x}) \tag{6.10}$$

which can be rewritten in $\mathbb{R}^{m+1}$ as $\widetilde{P}_p(x)\,\partial_x^{2p}\delta(x) = 2^p p!\,\widetilde{P}_p(\partial_x)\,\delta(x)$. Again taking $\widetilde{P}_p(x) = e_0 P_p(\underline{x})$ gives

$$P_p(\underline{x})\,\partial_x^{2p}\delta(x) = 2^p p!\,P_p(\partial_{\underline{x}})\,\delta(x) \tag{6.11}$$

Finally, substitution of (6.11) in the right-hand side of (6.9) yields the desired result of (ii):

$$D_x\,C_p(x) = P_p(\partial_{\underline{x}})\,\delta(x)$$

Now, as for $p = 0$ one has that $P_0(\underline{x}) = 1$ and $a_{m+1,0} = a_{m+1}$, this implies

$$C_0(x) = \frac{1}{a_{m+1}}\,\frac{\overline{x}e_0}{|x|^{m+1}}$$

which is precisely the standard Cauchy kernel in Clifford analysis. ■

As a nice additional result, using a similar method as in the previous proof, one also can construct a generalized fundamental solution for the Dirac operator $\partial_{\underline{x}}$ in $\mathbb{R}^m$, viz

$$E_p(\underline{x}) = \frac{1}{a_{m,p}}\,\frac{\underline{x}\,P_p(\underline{x})}{|\underline{x}|^{m+2p}} = -\frac{1}{\pi\,a_{m,p}}\,U^*_{-m-2p+1,p}$$

for which

$$\partial_{\underline{x}}E_p(\underline{x}) = P_p(\partial_{\underline{x}})\delta(\underline{x})$$

and $E_0 = E$, the standard fundamental solution of the Dirac operator (see Section 3.2).

In the next proposition we calculate the non–tangential distributional boundary limits for $x_0 \to 0\pm$ of the generalized Cauchy kernels $C_p(x_0, \underline{x})$, $p \in \mathbb{N}_0$. To this end we first formulate an auxiliary result in the following lemma.

Lemma 6.1. *For $p \in \mathbb{N}_0$ one has*

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}} = \frac{1}{2^{p+1}p!}\, a_{m+1,p}\, \partial_{\underline{x}}^{2p} \delta(\underline{x}) \tag{6.12}$$

$$\lim_{x_0 \to 0-} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}} = -\frac{1}{2^{p+1}p!}\, a_{m+1,p}\, \partial_{\underline{x}}^{2p} \delta(\underline{x}) \tag{6.13}$$

Proof.
First notice that $|-\widetilde{x_0} + \underline{x}| = |\widetilde{x_0} + \underline{x}|$, since the scalar $\widetilde{x_0}$ and the vector $\underline{x}$, both considered as elements of $\mathbb{R}^{m+1}$, are orthogonal. Hence, when substituting x_0 by $-\widetilde{x_0}$ and then taking into account (6.12), the equation (6.13) may be proven immediately:

$$\lim_{x_0 \to 0-} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}} = -\lim_{\widetilde{x_0} \to 0+} \frac{\widetilde{x_0}}{|\widetilde{x_0} + \underline{x}|^{m+1+2p}} = -\frac{1}{2^{p+1}p!}\, a_{m+1,p}\, \partial_{\underline{x}}^{2p} \delta(\underline{x})$$

So we are reduced to the proof of (6.12). Therefore we apply induction on p. Clearly, for $p = 0$, (6.12) stands for the well–known distributional limit

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1}} = \frac{1}{2} a_{m+1}\, \delta(\underline{x})$$

Next, assume (6.12) to be valid for $(p-1)$, i.e. one has

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m-1+2p}} = \frac{1}{2^{p}(p-1)!}\, a_{m+1,p-1}\, \partial_{\underline{x}}^{2p-2} \delta(\underline{x})$$

By the action of the Dirac operator on both sides of this equality one obtains

$$\lim_{x_0 \to 0+} \partial_{\underline{x}} \left(\frac{x_0}{|x_0 + \underline{x}|^{m-1+2p}} \right) = \frac{1}{2^{p}(p-1)!}\, a_{m+1,p-1}\, \partial_{\underline{x}}^{2p-1} \delta(\underline{x}) \tag{6.14}$$

On the other hand, one can directly calculate that

$$\lim_{x_0 \to 0+} \partial_{\underline{x}} \left(\frac{x_0}{|x_0 + \underline{x}|^{m-1+2p}} \right) = -(m-1+2p) \lim_{x_0 \to 0+} \frac{x_0 \underline{x}}{|x_0 + \underline{x}|^{m+1+2p}} \tag{6.15}$$

Comparing (6.14) and (6.15) leads to

$$\begin{aligned} &\frac{1}{2^p(p-1)!}\, a_{m+1,p-1}\, \partial_{\underline{x}}^{2p-1} \delta(\underline{x}) \\ &\quad = \; -(m-1+2p)\, \underline{x} \lim_{x_0 \to 0+} \frac{x_0}{|x_0+\underline{x}|^{m+1+2p}} \end{aligned} \tag{6.16}$$

Using Lemma 4.1 (ii) one has that $\underline{x}\, \partial_{\underline{x}}^{2p} \delta(\underline{x}) = 2p\, \partial_{\underline{x}}^{2p-1}\delta(\underline{x})$. Thus (6.16) may be rewritten as

$$\frac{1}{2^{p+1}p!}\, a_{m+1,p-1}\, \underline{x}\, \partial_{\underline{x}}^{2p} \delta(\underline{x}) \quad = \quad -(m-1+2p)\, \underline{x} \lim_{x_0\to 0+} \frac{x_0}{|x_0+\underline{x}|^{m+1+2p}}$$

leading to the desired result

$$\lim_{x_0\to 0+} \frac{x_0}{|x_0+\underline{x}|^{m+1+2p}} \quad = \quad \frac{1}{2^{p+1}p!}\, a_{m+1,p}\, \partial_{\underline{x}}^{2p}\delta(\underline{x})$$

when we invoke the definition (6.5) of $a_{m+1,p}$. ■

Proposition 6.2. *For each $p \in \mathbb{N}_0$ one has*

$$\begin{aligned} C_p(0+,\underline{x}) \;&\equiv\; \lim_{x_0\to 0+} C_p(x_0,\underline{x}) \;=\; \frac{1}{2}\, P_p(\partial_{\underline{x}})\delta(\underline{x}) + \overline{e_0}\, \frac{1}{2}\, H_p(\underline{x}) \\ C_p(0-,\underline{x}) \;&\equiv\; \lim_{x_0\to 0-} C_p(x_0,\underline{x}) \;=\; -\frac{1}{2}\, P_p(\partial_{\underline{x}})\delta(\underline{x}) + \overline{e_0}\, \frac{1}{2}\, H_p(\underline{x}) \end{aligned}$$

where

$$H_p(\underline{x}) \quad = \quad \frac{2}{a_{m+1,p}}\, \mathrm{Fp}\, \frac{\underline{\omega}\, P_p(\underline{\omega})}{r^{m+p}} \quad = \quad -\frac{2}{a_{m+1,p}}\, U^*_{-m-2p,p}$$

Proof.
We only calculate $C_p(0+,\underline{x})$, the computation for $C_p(0-,\underline{x})$ running along similar lines. Multiplying both sides of (6.12) with $P_p(\underline{x})$ and applying (6.10) yields

$$\lim_{x_0\to 0+} \frac{x_0\, P_p(\underline{x})}{|x_0+\underline{x}|^{m+1+2p}} \quad = \quad \frac{1}{2}\, a_{m+1,p}\, P_p(\partial_{\underline{x}})\, \delta(\underline{x}) \tag{6.17}$$

Next, on account of Lebesgue's dominated convergence theorem, one may show that in distributional sense

$$\lim_{x_0\to 0+} e_0\, \frac{\underline{x}\, P_p(\underline{x})}{|x_0+\underline{x}|^{m+1+2p}} \quad = \quad e_0\, \mathrm{Fp}\, \frac{\underline{\omega}\, P_p(\underline{\omega})}{r^{m+p}} \tag{6.18}$$

Expressions (6.17) and (6.18) then result into the following distributional limit

$$\begin{aligned} C_p(0+,\underline{x}) &= \lim_{x_0\to 0+} \frac{1}{a_{m+1,p}} \frac{x_0\, P_p(\underline{x})}{|x_0+\underline{x}|^{m+1+2p}} + \lim_{x_0\to 0+} \frac{1}{a_{m+1,p}} \frac{e_0\, \underline{x}\, P_p(\underline{x})}{|x_0+\underline{x}|^{m+1+2p}} \\ &= \frac{1}{2} P_p(\partial_{\underline{x}})\delta(\underline{x}) + \frac{1}{a_{m+1,p}}\, e_0 \,\mathrm{Fp}\, \frac{\underline{\omega}\, P_p(\underline{\omega})}{r^{m+p}}, \end{aligned}$$

which had to be proved. ∎

The distribution H_p arising in the previous proposition allows for the definition of a generalized Hilbert transform $\mathcal{H}_p$, given by

$$\mathcal{H}_p[f] \quad = \quad \overline{e_0}\, H_p * f$$

Now we compare its properties with those of the standard Clifford–Hilbert transform $\mathcal{H}$. First of all it readily follows that:

Property 6.5. *The generalized Hilbert transform $\mathcal{H}_p$ commutes with translations, which is an equivalent statement to its definition as convolution operator.*

Next, the kernel H_p being a homogeneous distribution of degree $(-m-p)$ means that the operator $\mathcal{H}_p$ is not dilation invariant. Further, taking into account (4.42), its Fourier symbol

$$\mathcal{F}[H_p] \quad = \quad -\frac{2}{a_{m+1,p}}\, i^{-p-1}\, U^*_{0,p} \tag{6.19}$$

not being a bounded function, the operator $\mathcal{H}_p$ will not be bounded on $L_2(\mathbb{R}^m)$. However, the Fourier symbol is polynomial of degree p, implying that $\mathcal{H}_p$ is a bounded operator between the Sobolev spaces $W_2^n(\mathbb{R}^m) \to W_2^{n-p}(\mathbb{R}^m)$, for $n \geq p$ (see [96, Proposition VI.5]). This is also confirmed in Property 6.6 resulting from the following proposition.

Proposition 6.3. *The generalized Cauchy integral $\mathcal{C}_p$ maps the Sobolev space $W_2^n(\mathbb{R}^m)$ into the Hardy space $H^2(\mathbb{R}^{m+1}_+)$, for each natural number $n \geq p$.*

Proof.
First of all we recall that the Hardy spaces $H^2(\mathbb{R}^{m+1}_+)$ and $H^2(\mathbb{R}^m)$ are isomorphic (see Proposition 5.3); each element of the latter space can be identified with the non–tangential limit $\lim_{x_0\to 0+} F(x_0,\underline{x})$ of a function F in $H^2(\mathbb{R}^{m+1}_+)$. Moreover, see Theorem 5.1, $H^2(\mathbb{R}^m)$ can be characterized as follows

$$g \in H^2(\mathbb{R}^m) \quad \Longleftrightarrow \quad \left\{ \begin{array}{ll} \text{(C1)} & g \in L_2(\mathbb{R}^m) \\ & \\ \text{(C2)} & \mathcal{H}[g] = g \end{array} \right.$$

So, it is necessary and sufficient to prove that, for each $f \in W_2^n(\mathbb{R}^m)$, $n \geq p$,

$$\lim_{x_0 \to 0+} \mathcal{C}_p[f](x_0, \underline{x}) \;=\; \frac{1}{2}\, P_p(\partial_{\underline{x}}) f(\underline{x}) + \frac{1}{2}\, \mathcal{H}_p[f](\underline{x}) \tag{6.20}$$

satisfies conditions (C1) and (C2).
For such a function f one immediately has that $P_p(\partial_{\underline{x}})f$ belongs to the Sobolev space $W_2^{n-p}(\mathbb{R}^m) \subset L_2(\mathbb{R}^m)$. Then, invoking (4.45) one finds that

$$\mathcal{H}[P_p(\partial_{\underline{x}})f] \;=\; \overline{e_0}\, \frac{-2}{a_{m+1}}\, U^*_{-m,0} * P_p(\partial_{\underline{x}})f \;=\; \overline{e_0}\, \frac{-2}{a_{m+1,p}}\, U^*_{-m-2p,p} * f \;=\; \mathcal{H}_p[f]$$

This implies that

$$\mathcal{H}_p[f] \;=\; \mathcal{H}[P_p(\partial_{\underline{x}})f] \;\in L_2(\mathbb{R}^m) \tag{6.21}$$

whence condition (C1) is fulfilled. Now we examine whether condition (C2) is satisfied as well, i.e. we check whether

$$\mathcal{H}\left[\lim_{x_0 \to 0+} \mathcal{C}_p[f](x_0, \underline{x})\right] \;=\; \lim_{x_0 \to 0+} \mathcal{C}_p[f](x_0, \underline{x})$$

As $\mathcal{H}$ is an involution and relying on the equality contained in (6.21), one indeed finds that

$$\mathcal{H}\left[\frac{1}{2}\, P_p(\partial_{\underline{x}})f + \frac{1}{2}\, \mathcal{H}_p[f]\right] \;=\; \frac{1}{2}\, \mathcal{H}_p[f] + \frac{1}{2}\, P_p(\partial_{\underline{x}})f$$

which completes the proof. ■

Property 6.6. *The generalized Hilbert transform $\mathcal{H}_p$ is a bounded linear operator between the Sobolev spaces $W_2^n(\mathbb{R}^m)$ and $W_2^{n-p}(\mathbb{R}^m)$, for each natural number $n \geq p$.*

Proof.
Take a function f in $W_2^n(\mathbb{R}^m)$, with $n \geq p$. The previous proposition then already shows that $\mathcal{H}_p[f]$ belongs to $L_2(\mathbb{R}^m)$ (see (6.21)). Moreover, relying on

$$\mathcal{H}[\partial_{x_j} f](\underline{y}) \;=\; \partial_{y_j} \mathcal{H}[f](\underline{y}) \;, \qquad j = 1, \ldots, m$$

one finds that, for each multi-index $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$ with $|\underline{\alpha}| = n - p$,

$$\partial^{\underline{\alpha}}_{\underline{y}} \mathcal{H}_p[f](\underline{y}) \;=\; \partial^{\underline{\alpha}}_{\underline{y}} \mathcal{H}[P_p(\partial_{\underline{x}})f](\underline{y}) \;=\; \mathcal{H}[\partial^{\underline{\alpha}}_{\underline{x}} P_p(\partial_{\underline{x}})f](\underline{y}) \in L_2(\mathbb{R}^m)$$

Thus, a fortiori, $\mathcal{H}_p[f]$ belongs to $W_2^{n-p}(\mathbb{R}^m)$. ∎

As for further traditional properties, we see that the generalized Hilbert transform $\mathcal{H}_p$ does not square to the identity operator. Indeed, one has that

$$\mathcal{H}_p^2[f] \;=\; \overline{e_0} H_p * (\overline{e_0} H_p * f) \;=\; (-H_p * H_p) * f$$

whence we should investigate whether the convolution of the kernel H_p with itself equals minus the delta distribution. Taking into account (4.30) and (4.48), one finds that

$$\begin{aligned} U^*_{-m-2p,p} * U^*_{-m-2p,p} &= \frac{-1}{m+2p-1}\, \partial_{\underline{x}} \left(T^*_{-m-2p+1,p} * U^*_{-m-2p,p}\right) \\ &= \frac{(-1)^{p+1}}{2^{p+1}} \pi^{\frac{m}{2}} \frac{\Gamma\left(\frac{m}{2}+p\right)}{\left[\Gamma\left(\frac{m}{2}+p+\frac{1}{2}\right)\right]^2}\, \partial_{\underline{x}} V^*_{-m-2p+1,p}\, P_p(\partial_{\underline{x}}) \end{aligned}$$

such that, making use of the properties (3.2) of the Fourier transform and of (4.43) and Theorem 4.5, one has

$$\begin{aligned} \mathcal{F}\left[H_p * H_p\right](\underline{y}) &= \frac{4}{(a_{m+1,p})^2}\, \mathcal{F}\left[U^*_{-m-2p,p} * U^*_{-m-2p,p}\right](\underline{y}) \\ &= (-1)^{p+1}\, 2^{2p}\, \pi^{-\frac{m}{2}+p}\, \Gamma\left(\frac{m}{2}+p\right) W^*_{0,p}\, P_p(\underline{y}) \end{aligned}$$

from which we may conclude that $\mathcal{H}_p^2 \neq \mathbf{1}$.

For functions $f, g \in L_2(\mathbb{R}^m)$, we then calculate the adjoint $\mathcal{H}_p^*$ of the generalized Hilbert transform $\mathcal{H}_p$. The direct calculation

$$\begin{aligned} \langle \mathcal{H}_p[f], g\rangle &= \int_{\mathbb{R}^m} \mathcal{H}_p[f](\underline{x})^\dagger\, g(\underline{x})\, dV(\underline{x}) \\ &= \int_{\mathbb{R}^m} \left[\frac{2}{a_{m+1,p}} \operatorname{Fp} \int_{\mathbb{R}^m} f(\underline{y})^\dagger\, \frac{\overline{P_p(\underline{x}-\underline{y})}\; \overline{(\overline{\underline{x}}-\overline{\underline{y}})}}{\left|\underline{x}-\underline{y}\right|^{m+2p+1}}\, e_0\, dV(\underline{y})\right] g(\underline{x})\, dV(\underline{x}) \\ &= \int_{\mathbb{R}^m} f(\underline{y})^\dagger \left[(-1)^p\, \frac{2}{a_{m+1,p}}\, \overline{e_0} \operatorname{Fp} \int_{\mathbb{R}^m} \frac{P_p(\underline{y}-\underline{x})(\overline{\underline{y}}-\overline{\underline{x}})}{\left|\underline{y}-\underline{x}\right|^{m+2p+1}}\, g(\underline{x})\, dV(\underline{x})\right] dV(\underline{y}) \\ &= \left\langle f, \overline{e_0}\, (-1)^{p+1} \frac{2}{a_{m+1,p}}\, V^*_{-m-2p,p} * g \right\rangle \end{aligned}$$

leads to the adjoint $\mathcal{H}_p^*$ being given by

$$\mathcal{H}_p^* \;=\; \overline{e_0}\, H_p^* * f$$

with the convolution kernel

$$H_p^* \;=\; (-1)^{p+1}\,\frac{2}{a_{m+1,p}}\,V^*_{-m-2p,p} \;=\; (-1)^p\,\frac{2}{a_{m+1,p}}\,\mathrm{Fp}\frac{P_p(\underline{\omega})\overline{\underline{\omega}}}{r^{m+p}}$$

Applying then the same calculation method as on previous page one finds

$$\mathcal{F}[H_p * H_p^*](\underline{y}) \;=\; -2^{2p}\,\pi^{-\frac{m}{2}+p}\,\Gamma\left(\frac{m}{2}+p\right)\,U^*_{-1,p}\,P_p(\underline{y})\,\underline{y} \;\neq\; -1$$

from which we conclude that the generalized Hilbert transform $\mathcal{H}_p$ is not unitary.

However, the main objective for this second generalization is fulfilled on account of Proposition 6.2: H_p pops up as a part of the boundary limits of a generalized Cauchy kernel C_p, an analogue of the "classical" Property 5.7. Finally, a link with the first type of generalized Hilbert transforms is established.

Proposition 6.4. *The generalized Hilbert kernel H_p can be written as a higher order Dirac derivative, say $\partial_{\underline{x}}^p$, of the generalized Hilbert kernels of the first kind $T_{-m-p,p}$ and $U_{-m-p,p}$, depending on the parity of p. More specifically one has that, for a suitable function f and a natural number p*

(i) if p is odd, then

$$\mathcal{H}_p[f] \;=\; \overline{e_0}\,\frac{-2^{\frac{p-1}{2}}\,\Gamma\left(\frac{m+p}{2}\right)}{\pi^{\frac{m+1}{2}}\,(p-2)!!}\,\partial_{\underline{x}}^p T_{-m-p,p} * f$$

(ii) if p is even, then

$$\mathcal{H}_p[f] \;=\; \overline{e_0}\,\frac{-2^{\frac{p}{2}}\,\Gamma\left(\frac{m+p+1}{2}\right)}{\pi^{\frac{m+1}{2}}\,(p-1)!!}\,\partial_{\underline{x}}^p U_{-m-p,p} * f$$

Proof.
The proofs of (i) and (ii) running along similar lines, we only prove (i).
So, let p be an odd number. Formula (4.38) for the specific choice of $\lambda = -m-p$ and $2k+1 = p$ then reads

$$\begin{aligned}
\partial_{\underline{x}}^p\,T^*_{-m-p,p} &= (-2\pi)^{\frac{p-1}{2}}\,(-m-p)(-m-p-2)\ldots(-m-2p+1)\,U^*_{-m-2p,p} \\
&= -2^p\pi^{\frac{p-1}{2}}\,\frac{\Gamma\left(\frac{m+2p+1}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,U^*_{-m-2p,p}
\end{aligned}$$

The normalized distribution $T^*_{-m-p,p}$ may be written in terms of the generalized Hilbert kernel $T_{-m-p,p}$ as follows (see (4.29)):

$$T^*_{-m-p,p} \;=\; \frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)}\, T_{-m-p,p} \;=\; \frac{(2\pi)^{\frac{p-1}{2}}}{(p-2)!!}\, T_{-m-p,p}$$

Taking into account those two previous results, one finally arrives at following expression for the generalized Hilbert kernel H_p:

$$H_p \;=\; -\frac{2}{a_{m+1,p}}\, U^*_{-m-2p,p} \;=\; -\frac{2^{\frac{p-1}{2}}\,\Gamma\left(\frac{m+p}{2}\right)}{\pi^{\frac{m+1}{2}}\,(p-2)!!}\, \partial^p_{\underline{x}}\, T_{-m-p,p}$$

■

6.2 Relationship with the Radon transform

We will now calculate the action of the Radon transform on the generalized Hilbert transforms introduced in the previous section. In the following proposition, the Radon transform of the generalized Hilbert transforms of the first kind is considered. Again, for the proof, we rely on a suitable adaptation of the techniques used in [56].

Proposition 6.5. *For a suitable function f defined on $\mathbb{R}^m$ and with values in the Clifford algebra $\mathbb{R}_{0,m+1}$, one has for p even:*

$$\begin{aligned}
\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s) &= i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\, P_p(\underline{n})\,\mathcal{R}[f](\underline{n},s)\\
\mathcal{R}[\mathcal{U}_p[f]](\underline{n},s) &= i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, \underline{n}\,P_p(\underline{n})\,\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)\\
\mathcal{R}[\mathcal{V}_p[f]](\underline{n},s) &= i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, P_p(\underline{n})\,\underline{n}\,\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)\\
\mathcal{R}[\mathcal{W}_p[f]](\underline{n},s) &= i^{-p-2}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)}\left(\underline{n}\,P_p(\underline{n})\,\underline{n} - \frac{m-2}{p}\,P_p(\underline{n})\right)\\
&\quad \times \mathcal{R}[f](\underline{n},s)
\end{aligned}$$

while for p odd:

$$\begin{aligned}
\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s) &= i^{-p+1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)\\
\mathcal{R}[\mathcal{U}_p[f]](\underline{n},s) &= i^{-p-1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,\underline{n}\,P_p(\underline{n})\,\mathcal{R}[f](\underline{n},s)\\
\mathcal{R}[\mathcal{V}_p[f]](\underline{n},s) &= i^{-p-1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,P_p(\underline{n})\,\underline{n}\,\mathcal{R}[f](\underline{n},s)\\
\mathcal{R}[\mathcal{W}_p[f]](\underline{n},s) &= i^{-p-1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)}\left(\underline{n}\,P_p(\underline{n})\,\underline{n}-\frac{m-2}{p}\,P_p(\underline{n})\right)\\
&\quad\times\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)
\end{aligned}$$

Before proving these formulae, we first make some preliminary remarks on their structure. First of all, it is worth noticing that the last factor in the above results is affected by the parity of p, which is the degree of homogeneity of the spherical monogenic involved; more specifically, this parity dependence is reflected in the fact that the outcome involves either the Radon transform of the considered function, or the Hilbert transform of that Radon transform. Secondly, if we isolate the parity dependent part in all formulae, then we easily recognize the remaining factor at the right-hand side to be the Fourier symbol of one of the considered operators $\mathcal{T}_p$, $\mathcal{U}_p$, $\mathcal{V}_p$ or $\mathcal{W}_p$ (see (6.3)).

Proof.
We only calculate $\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s)$ as the other computations run along similar lines. Taking into account Lemma 5.1 and Theorem 4.7, in frequency space one has

$$\begin{aligned}
\mathcal{F}_{s\to q}\left[\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s)\right](q) &= \mathcal{F}\left[T_{-m-p,p}*f\right](q\,\underline{n})\\
= \mathcal{F}\left[T_{-m-p,p}\right](q\,\underline{n})\cdot\mathcal{F}\left[f\right](q\,\underline{n}) &= i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,T_{-p,p}(q\underline{n})\cdot\mathcal{F}\left[f\right](q\,\underline{n})
\end{aligned}$$

As the spherical decomposition of $q\underline{n}$ reads $|q|\cdot(\mathrm{sgn}(q)\underline{n})$, we then easily get

$$T_{-p,p}(q\underline{n}) \;=\; \mathrm{Fp}\,|q|^{-p}\,P_p(q\,\underline{n}) \;=\; (\mathrm{sgn}(q))^p\,P_p(\underline{n})$$

leading to

$$\begin{aligned} &\mathcal{F}_{s\to q}\left[\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s)\right](q) \\ &= \quad i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,(\mathrm{sgn}(q))^p\,\mathcal{F}_{s\to q}\left[\mathcal{R}[f](\underline{n},s)\right](q) \end{aligned} \tag{6.22}$$

where we applied Lemma 5.1 on the Fourier transform of f. Further, as

$$(\mathrm{sgn}(q))^p \quad = \quad \begin{cases} 1\,, & \text{if } p \text{ even} \\ \mathrm{sgn}(q)\,, & \text{if } p \text{ odd} \end{cases}$$

one has to distinguish between two cases.

CASE A. p even
In this case, formula (6.22) turns into

$$\mathcal{F}_{s\to q}\left[\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s)\right](q) \quad = \quad i^{-p}\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathcal{F}_{s\to q}\left[\mathcal{R}[f](\underline{n},s)\right](q)$$

from which the desired formula follows, taking the one–dimensional inverse Fourier transform, viz

$$\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s) \quad = \quad i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathcal{R}[f](\underline{n},s)$$

CASE B. p odd
Noticing that $(-i)\,\mathrm{sgn}(q)$ is the Fourier symbol of the one–dimensional Hilbert kernel on the real line, formula (6.22) may be rewritten as

$$\begin{aligned} &\mathcal{F}_{s\to q}\left[\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s)\right](q) \\ &= \quad i^{-p}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathrm{sgn}(q)\,\mathcal{F}_{s\to q}\left[\mathcal{R}[f](\underline{n},s)\right](q) \\ &= \quad i^{-p+1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathcal{F}_{s\to q}\left[H(s)\right](q)\,\mathcal{F}_{s\to q}\left[\mathcal{R}[f](\underline{n},s)\right](q) \\ &= \quad i^{-p+1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathcal{F}_{s\to q}\left\{\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)\right\}(q) \end{aligned}$$

Then taking the one–dimensional inverse Fourier transform, viz

$$\mathcal{R}[\mathcal{T}_p[f]](\underline{n},s) \quad = \quad i^{-p+1}\,\pi^{\frac{m}{2}}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{n})\,\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)$$

we obtain the desired result. ∎

In the following proposition the Radon transform of the generalized Hilbert transform $\mathcal{H}_p$ is calculated. We will prove the formula in two different ways.

Proposition 6.6. *For a suitable function f defined on $\mathbb{R}^m$ and with values in the Clifford algebra $\mathbb{R}_{0,m+1}$, one has*

$$\mathcal{R}[\mathcal{H}_p[f]](\underline{n},s) \quad = \quad e_0\,\underline{n}\,P_p(\underline{n})\,\partial_s^p\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s) \tag{6.23}$$

Proof 1.
The first way to prove formula (6.23), is by means of similar techniques as in the previous proposition. So again, the calculations are done in one–dimensional frequency space:

$$\begin{aligned}
\mathcal{F}_{s\to q}\left\{\mathcal{R}[\mathcal{H}_p[f]](\underline{n},s)\right\}(q) \quad &= \quad \overline{e_0}\,\mathcal{F}\left[H_p * f\right](q\,\underline{n}) \\
&= \quad i^{p+1}\,(2\pi)^p\,\overline{e_0}\,U_{0,p}(q\,\underline{n})\,\mathcal{F}\left[f\right](q\,\underline{n}) \qquad (6.24)
\end{aligned}$$

where in the last step we have used (6.19). Note that one may write

$$\begin{aligned}
U_{0,p}(q\,\underline{n}) \;&=\; \mathrm{Fp}\,|q|^0\,(\mathrm{sgn}(q)\,\underline{n})\,P_p(q\,\underline{n}) \;=\; \underline{n}\,P_p(\underline{n})\,q^p\,\mathrm{sgn}(q) \\
&=\; i\,\underline{n}\,P_p(\underline{n})\,q^p\,\mathcal{F}_{s\to q}\left[H(s)\right](q) \\
&=\; i^{-p+1}\,(2\pi)^{-p}\,\underline{n}\,P_p(\underline{n})\,\mathcal{F}_{s\to q}\left[\frac{d^p}{ds^p}H(s)\right](q) \qquad (6.25)
\end{aligned}$$

Substitution of (6.25) into (6.24) then yields

$$\begin{aligned}
&\mathcal{F}_{s\to q}\left\{\mathcal{R}[\mathcal{H}_p[f]](\underline{n},s)\right\}(q) \\
&= \quad i^2\,\overline{e_0}\,\underline{n}\,P_p(\underline{n})\left(\mathcal{F}_{s\to q}\left[\frac{d^p}{ds^p}H(s)\right]\,\mathcal{F}_{s\to q}\left[\mathcal{R}[f](\underline{n},s)\right]\right)(q) \\
&= \quad e_0\,\underline{n}\,P_p(\underline{n})\,\mathcal{F}_{s\to q}\left\{\partial_s^p\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)\right\}(q)
\end{aligned}$$

from which formula (6.23) can be deduced taking the one–dimensional inverse Fourier transform. ■

Proof 2.
The second way to prove the proposition is through a direct calculation, on the one hand making use of the relation (6.21) between the generalized Hilbert transform $\mathcal{H}_p$ and the classical Clifford–Hilbert transform $\mathcal{H}$, and on the other hand taking into account Lemma 5.2 (ii). For a function $f \in W_2^n(\mathbb{R}^m)$ one has

$$\begin{aligned}\mathcal{R}[\mathcal{H}_p[f]](\underline{n}, s) &= \mathcal{R}[\mathcal{H}[P_p(\partial_{\underline{x}})f]](\underline{n}, s) \\ &= e_0\, \underline{n}\, \mathcal{H}_{u\to s}[\mathcal{R}[P_p(\partial_{\underline{x}})f](\underline{n}, u)](s) \qquad (6.26)\end{aligned}$$

where in the last step we made use of (5.13). Now invoking the linearity (5.11) of the Radon transform, the fact that P_p is a vector valued homogeneous polynomial of degree p and Lemma 5.2 (ii), we are lead to

$$\mathcal{R}[P_p(\partial_{\underline{x}})f](\underline{n}, u) = P_p(\underline{n})\, \partial_u^p\, \mathcal{R}[f](\underline{n}, u) \qquad (6.27)$$

Substitution of (6.27) in (6.26) gives

$$\begin{aligned}\mathcal{R}[\mathcal{H}_p[f]](\underline{n}, s) &= e_0\, \underline{n}\, \mathcal{H}_{u\to s}[P_p(\underline{n})\, \partial_u^p\, \mathcal{R}[f](\underline{n}, u)](s) \\ &= e_0\, \underline{n}\, P_p(\underline{n})\, \partial_s^p \mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n}, u)](s)\end{aligned}$$

■

Part II

Hilbert transforms in anisotropic Clifford analysis

Chapter 7

The anisotropic Clifford toolbox

In Part I, (generalized) multidimensional Hilbert transforms have been constructed in $\mathbb{R}^m$ in the framework of orthogonal Clifford analysis. The considered Hilbert transforms, usually obtained as a part of the boundary limit of an associated Cauchy integral in $\mathbb{R}^{m+1}$, might be characterized as isotropic, since the metric in the underlying space is the standard Euclidean one.

This introductory chapter adopts the idea of an anisotropic (also called metric dependent or metrodynamical) Clifford setting, which offers the possibility of adjusting the co-ordinate system to preferential and not necessarily mutually orthogonal directions. This is achieved by means of a so–called metric tensor. The idea is in fact not completely new since Clifford analysis on manifolds with local metric tensors was already considered in e.g. [60, 75, 42], while in [56] a specific three–dimensional metric tensor, leaving the third dimension unaltered, was introduced for analyzing two–dimensional signals and textures. Of a more recent character, however, is the detailed development of Clifford analysis in a global metric dependent setting (see [35, 53]). It should be clear that this has opened a new domain in Clifford analysis, offering a framework for a new kind of applications such as texture analysis. In this context, we mention e.g. the anisotropic Clifford–Hermite wavelets introduced in [35, 53] and moreover the anisotropic multidimensional Hilbert transform, which will be thoroughly discussed in the next chapter.

The outline of this chapter is as follows. We first introduce the notion of metric tensor which will give rise to two bases, a covariant one and a contravariant one. Then, a Clifford algebra will be constructed, depending on the metric tensor involved, and all necessary definitions and results of orthogonal Clifford analysis will be extended to this metric dependent setting. We introduce e.g. the concepts of Dirac operator, monogenicity and Laplace operator. We end this chapter with the definition and study of the so–called anisotropic Fourier transform, the metric dependent analogue of the classical Fourier transform (2.5). For a more detailed account about anisotropic Clifford analysis we once more refer the reader to [35, 53].

7.1 The metric tensor

Let $\widetilde{G} = (g_{kl})_{k,l=0,\ldots,m} \in \mathbb{R}^{(m+1)\times(m+1)}$ be a real, symmetric and positive definite tensor, which will be referred to as *metric tensor of order* $(m+1)$, and consider its corresponding subtensor $G = (g_{kl})_{k,l=1,\ldots,m} \in \mathbb{R}^{m\times m}$, i.e.

$$\widetilde{G} \;=\; \begin{pmatrix} g_{00} & \cdots & & g_{0m} \\ \vdots & & & \\ & & G & \\ g_{0m} & & & \end{pmatrix} \tag{7.1}$$

Notice that G is a metric tensor as well, however of order m, which is obtained by simply taking the restriction of $\widetilde{G}$ to $\mathbb{R}^m$, the latter being identified with the hyperplane $x^0 = 0$ of $\mathbb{R}^{m+1}$. Furthermore, let $\widetilde{G}^{-1} = (g^{kl})_{k,l=0,\ldots,m}$ denote the reciprocal, or inverse, tensor of $\widetilde{G}$, i.e.

$$\sum_{s=0}^{m} g_{ks}\, g^{sl} \;=\; \delta_{kl} \;, \qquad k,l = 0,\ldots,m$$

In the following lemma, a criterion is given for the specific (and interesting) case where the inverse G^{-1} of G is included as a part of $\widetilde{G}^{-1}$, i.e. where one has $G^{-1} = (g^{kl})_{k,l=1,\ldots,m}$. The geometric consequences are discussed in Remark 7.1.

Lemma 7.1. *Let $\widetilde{G}$ be a metric tensor of order $(m+1)$, given by (7.1). The reciprocal of its corresponding subtensor G is given by*

$$G^{-1} \;=\; (g^{kl})_{k,l=1,\ldots,m}$$

if and only if the following conditions are simultaneously fulfilled:

(C1) $g_{00}\, g^{00} \;=\; 1$

(C2) $g_{01} \;=\; \ldots \;=\; g_{0m} \;=\; 0$

Proof.
It may be clear that conditions (C1)–(C2) directly lead to $G^{-1} = (g^{kl})_{k,l=1,\ldots,m}$. Additionally, (C2) also implies that $g^{01} \;=\; \ldots \;=\; g^{0m} \;=\; 0$. Now the inverse implication is addressed. First, the assumption that $G^{-1} = (g^{kl})_{k,l=1,\ldots,m}$ may be rewritten as

$$\sum_{s=1}^{m} g_{ks}\, g^{sl} \;=\; \delta_{kl} \;, \qquad k,l = 1,\ldots,m \tag{7.2}$$

This implies that the reciprocity of $\widetilde{G}$ and $\widetilde{G}^{-1}$ may be expressed in a block matrix structure, viz

$$\begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix} \begin{pmatrix} g^{00} & \underline{u}'^T \\ \underline{u}' & G^{-1} \end{pmatrix} \;=\; \begin{pmatrix} 1 & \underline{0}^T \\ \underline{0} & \mathbb{E}_m \end{pmatrix} \tag{7.3}$$

where $\underline{u}^T$, $\underline{u}'^T$ and $\underline{0}^T$ respectively denote the $(1\times m)$ row matrices $(g_{01}\;\ldots\;g_{0m})$, $(g^{01}\;\ldots\;g^{0m})$ and $(0\;\ldots\;0)$ and $\mathbb{E}_m$ is the unity tensor of order m. Explicit calculation of the left-hand side of (7.3) then yields the following equations on the level of the tensor entries:

$$\sum_{s=0}^{m} g_{0s}\, g^{s0} \;=\; 1 \tag{7.4}$$

$$\sum_{s=0}^{m} g_{0s}\, g^{sl} \;=\; 0 \;, \qquad l = 1,\ldots,m$$

$$\sum_{s=0}^{m} g_{ks}\, g^{s0} \;=\; 0 \;, \qquad k = 1,\ldots,m \tag{7.5}$$

$$\sum_{s=0}^{m} g_{ks}\, g^{sl} \;=\; \delta_{kl} \;, \qquad k,l = 1,\ldots,m \tag{7.6}$$

In view of (7.2), the left-hand side of (7.6) may be turned into

$$g_{k0}\, g^{0l} + \sum_{s=1}^{m} g_{ks}\, g^{sl} \;=\; g_{k0}\, g^{0l} + \delta_{kl} \;, \qquad k,l = 1,\ldots,m$$

which leads to the condition

$$g_{k0}\, g^{0l} \;=\; 0 \;=\; g_{0k}\, g^{l0} \;, \qquad k,l = 1,\ldots,m \tag{7.7}$$

seen also the symmetry of $\widetilde{G}$. Combination of (7.7) for $k = l$ with (7.4) then immediately results in condition (C1). Next, condition (C2) can be proven by *reductio ad absurdum*. Assume that there exists an index $\kappa \in \{1,\ldots,m\}$ for which $g_{0\kappa} \neq 0$, then (7.7) implies that $g^{0l} = g^{l0} = 0$ for all $l = 1,\ldots,m$. Hence, for each $k = 1,\ldots,m$, (7.5) reduces to $g_{0k}\, g^{00} = 0$. This leads to a contradiction for $k = \kappa$ since $g_{0\kappa} \neq 0$ and $g^{00} \neq 0$ on account of condition (C1). ■

7.2 Anisotropic Clifford analysis

In the vector space $\mathbb{R}^{m+1}$ a covariant basis $(e_k) = (e_0,\ldots,e_m)$ and a contravariant basis $(e^l) = (e^0,\ldots,e^m)$ are considered, corresponding to each other through the metric tensor $\widetilde{G}$, i.e.

$$e_k \;=\; \sum_{l=0}^{m} g_{kl}\, e^l \qquad \text{and} \qquad e^l \;=\; \sum_{k=0}^{m} g^{lk}\, e_k$$

The *universal anisotropic Clifford algebra* $(\mathbb{R}_{0,m+1},\widetilde{G})$ is then constructed over $(\mathbb{R}^{m+1},\widetilde{G})$, with a non–commutative multiplication governed by

$$\begin{aligned} e_k\, e_l + e_l\, e_k &= -2\, g_{kl} \;, & k,l &= 0,\ldots,m \\ e^k\, e^l + e^l\, e^k &= -2\, g^{kl} \;, & k,l &= 0,\ldots,m \\ e_k\, e^l + e^l\, e_k &= -2\, \delta_{kl} \;, & k,l &= 0,\ldots,m \end{aligned}$$

Sometimes the notion *anisotropic* is also referred to as *metric dependent* or *metrodynamical*.

Remark 7.1. *The above multiplication rules, together with Lemma 7.1, learn that the specific case where $G^{-1} = (g^{kl})_{k,l=1,\ldots,m}$ corresponds to the geometric situation where the e_0–direction in $\mathbb{R}^{m+1}$ will be perpendicular to the $\mathbb{R}^m$–plane spanned by $(e_1,\ldots,e_m)$. Of course, the same then holds for the position of e^0–direction with respect to the $\mathbb{R}^m$–plane spanned by $(e^1,\ldots,e^m)$. For $m = 2$ this corresponds to the application considered in [56].*

For a set $A = \{i_1,\ldots,i_h\} \subset \{0,\ldots,m\}$ with $0 \leq i_1 < i_2 < \cdots < i_h \leq m$, one puts $e_A = e_{i_1} e_{i_2} \ldots e_{i_h}$. Moreover, $e_\emptyset = 1$ is the identity element. In this way a covariant basis for the anisotropic Clifford algebra $(\mathbb{R}_{0,m+1}, \widetilde{G})$ is constructed by means of which any $a \in \mathbb{R}_{0,m+1}$ may be written as

$$a = \sum_A a_A \, e_A \ , \qquad a_A \in \mathbb{R}$$

or still as

$$a = \sum_{k=0}^{m+1} [a]_k \ , \qquad [a]_k = \sum_{|A|=k} a_A \, e_A$$

where the terms $[a]_k$ correspond to the so–called covariant k–vectors, with $k = 0, 1, \ldots, m+1$. Alternatively, also a contravariant basis may be considered for the Clifford algebra.

A point $(x^0, \ldots, x^m) \in \mathbb{R}^{m+1}$ will be identified with the covariant Clifford (1–)vector $\sum_{k=0}^m e_k \, x^k$. The above multiplication rules then lead to the decomposition of the Clifford product of two covariant Clifford–vectors $x = \sum_{k=0}^m e_k \, x^k$ and $y = \sum_{k=0}^m e_k \, y^k$ as

$$x \, y = -\langle \, x, y \, \rangle_{\widetilde{G}} + x \wedge y$$

with

$$\langle \, x, y \, \rangle_{\widetilde{G}} = \sum_{k=0}^m \sum_{l=0}^m g_{kl} \, x^k \, y^l \tag{7.8}$$

defining a scalar, symmetric bilinear form associated to the metric tensor $\widetilde{G}$ which replaces the classical scalar product

$$\langle \, x, y \, \rangle = \sum_{k=0}^m x^k \, y^k \tag{7.9}$$

and with

$$x \wedge y = \frac{1}{2} \, x^k y^l \, (e_k e_l - e_l e_k)$$

a bivector. The norm of a vector x then is given by

$$|x|_{\widetilde{G}} = \sqrt{\langle \, x, x \, \rangle_{\widetilde{G}}}$$

Obviously, when $\widetilde{G} = \mathbb{E}_{m+1}$, one recovers the traditional Clifford algebra stemming from the standard Euclidean metric, and (7.8) reduces to (7.9).

In this metric dependent context, the *anisotropic Dirac operator* is introduced as the contravariant Clifford–vector valued differential operator of first order, given by

$$\partial_{x,\widetilde{G}} \;=\; \sum_{k=0}^{m} e^k \, \partial_{x^k}$$

with fundamental solution

$$E_{\widetilde{G}}(x) \;=\; \frac{1}{a_{m+1}} \, \frac{\overline{x}}{|x|_{\widetilde{G}}^{m+1}}$$

as well as the *anisotropic Laplace operator*

$$\Delta_{\widetilde{G}} \;=\; -\partial_{x,\widetilde{G}} \, \partial_{x,\widetilde{G}} \;=\; \sum_{k=0}^{m} \sum_{l=0}^{m} g^{kl} \, \partial_{x^k} \, \partial_{x^l}$$

with fundamental solution

$$F_{\widetilde{G}}(x) \;=\; -\frac{1}{(m-1)a_{m+1}} \, \frac{1}{|x|_{\widetilde{G}}^{m-1}}$$

In the above, $\overline{\cdot}$ denotes the usual conjugation in $\mathbb{R}_{0,m+1}$, defined as the main anti–involution for which $\overline{e_k} = -e_k$ (and thus also $\overline{e^k} = -e^k$), $k = 0, \ldots, m$. In particular for a vector x one has $\overline{x} = -x$.

A function defined on $\mathbb{R}^{m+1}$ and taking values in $\mathbb{R}_{0,m+1}$, is then called *$\widetilde{G}$–monogenic* in the open region Ω of $\mathbb{R}^{m+1}$ if and only if f is continuously differentiable in Ω and satisfies in Ω the equation $\partial_{x,\widetilde{G}} f \;=\; 0$. As the Dirac operator factorizes the Laplace operator $\Delta_{\widetilde{G}}$, a $\widetilde{G}$–monogenic function in Ω is *$\widetilde{G}$–harmonic*, and so are its components.

In what follows, we also refer to the *anisotropic Cauchy–Riemann operator*

$$D_{x,\widetilde{G}} \;=\; \overline{e_0} \, \partial_{x,\widetilde{G}}$$

and the corresponding function

$$C_{\widetilde{G}}(x) \;=\; C_{\widetilde{G}}(x^0, \underline{x}) \;=\; \frac{1}{a_{m+1}} \, \frac{\overline{x} \, e^0}{|x|_{\widetilde{G}}^{m+1}} \;=\; \frac{1}{a_{m+1}} \, \frac{\overline{e_0} e^0 \, x^0 + \overline{e^0} \, \overline{\underline{x}}}{|e_0 x^0 + \underline{x}|_{\widetilde{G}}^{m+1}}$$

almost, but not entirely, covering the notion of fundamental solution of $D_{x,\widetilde{G}}$, since

$$D_{x,\widetilde{G}}\, C_{\widetilde{G}}(x) \;=\; \overline{e_0}\, e^0\, \delta(x)$$

Here, in an obvious notation,

$$\underline{x} \;=\; \sum_{k=1}^{m} e_k\, x^k$$

is a covariant Clifford–vector in $\mathbb{R}^m$, the latter still being identified with the hyperplane $x^0 = 0$ of $(\mathbb{R}^{m+1}, \widetilde{G})$. Note that, as

$$D_{x,\widetilde{G}} \;=\; \overline{e_0}\, \partial_{x,\widetilde{G}} \qquad \text{and} \qquad \partial_{x,\widetilde{G}} \;=\; \frac{1}{g_{00}}\, e_0\, D_{x,\widetilde{G}}$$

$\widetilde{G}$–monogenicity may equally be expressed with respect to the Cauchy–Riemann operator.

The function $C_{\widetilde{G}}$ above is then easily seen to split into

$$C_{\widetilde{G}}(x) \;=\; \frac{1}{2}\left(\overline{e_0}\, e^0\, P_{\widetilde{G}}(x) + \overline{e^0}\, Q_{\widetilde{G}}(x)\right)\;, \qquad x^0 \neq 0$$

where

$$P_{\widetilde{G}}(x) \;=\; P_{\widetilde{G}}(x^0, \underline{x}) \;=\; \frac{2}{a_{m+1}}\, \frac{x^0}{|x|_{\widetilde{G}}^{m+1}}\;, \qquad x^0 \neq 0$$

is the scalar valued *anisotropic Poisson kernel* and

$$Q_{\widetilde{G}}(x) \;=\; Q_{\widetilde{G}}(x^0, \underline{x}) \;=\; \frac{2}{a_{m+1}}\, \frac{\overline{\underline{x}}}{|x|_{\widetilde{G}}^{m+1}}\;, \qquad x^0 \neq 0$$

is the vector valued *anisotropic conjugate Poisson kernel.* It then readily follows from the $\widetilde{G}$–monogenicity of $C_{\widetilde{G}}$ in $\mathbb{R}^{m+1}_+$ that $P_{\widetilde{G}}$ and $Q_{\widetilde{G}}$ are $\widetilde{G}$–harmonic in $\mathbb{R}^{m+1}_+$ (and similarly in $\mathbb{R}^{m+1}_-$). In accordance with the definition of conjugate harmonicity in the sense of [24], we call them *$\widetilde{G}$–conjugate harmonic functions.*

The above functions may be used as the kernels for metric dependent counterparts of well–known integral transforms. Indeed, for an appropriate function $f \in L_2(\mathbb{R}^m)$, we may define its *anisotropic Cauchy integral* by

$$\mathcal{C}_{\widetilde{G}}[f] \;=\; C_{\widetilde{G}} * f$$

which is $\widetilde{G}$–monogenic in $\mathbb{R}^{m+1}_+$ (and in $\mathbb{R}^{m+1}_-$). Analogously we introduce its *anisotropic Poisson* and *($\widetilde{G}$–)conjugate Poisson transforms* as the $\widetilde{G}$–harmonic functions

$$\mathcal{P}_{\widetilde{G}}[f] \;=\; P_{\widetilde{G}} * f \qquad \text{and} \qquad \mathcal{Q}_{\widetilde{G}}[f] \;=\; Q_{\widetilde{G}} * f$$

such that

$$\mathcal{C}_{\widetilde{G}}[f] \;=\; \frac{\overline{e_0}\, e^0}{2}\, \mathcal{P}_{\widetilde{G}}[f] + \frac{\overline{e^0}}{2}\, \mathcal{Q}_{\widetilde{G}}[f]$$

either in $\mathbb{R}^{m+1}_+$ or in $\mathbb{R}^{m+1}_-$.

7.3 The anisotropic Fourier transform

In the isotropic case the Fourier transform on $\mathbb{R}^m$ was defined by

$$\begin{aligned} \mathcal{F}[f](\underline{x}) &= \int_{\mathbb{R}^m} \exp\left(-2\pi i \left\langle\, \underline{x}, \underline{y} \,\right\rangle\right)\, f(\underline{y})\, dV(\underline{y}) \\ &= \int_{\mathbb{R}^m} \exp\left(-2\pi i\, \underline{x}^T \underline{y}\right)\, f(\underline{y})\, dV(\underline{y}) \end{aligned} \tag{7.10}$$

where $\left\langle\, \underline{x}, \underline{y} \,\right\rangle$ denotes the restriction of the classical scalar product (7.9) to $\mathbb{R}^m$ (identified with $x^0 = 0$) and, in the last equality, the vectors $\underline{x}$ and $\underline{y}$ are interpreted as column matrices. In a natural way, this leads to the following definition of the *anisotropic Fourier transform* on $(\mathbb{R}^m, G)$:

$$\begin{aligned} \mathcal{F}_G[f](\underline{x}) &= \int_{\mathbb{R}^m} \exp\left(-2\pi i \left\langle\, \underline{x}, \underline{y} \,\right\rangle_G\right)\, f(\underline{y})\, dV(\underline{y}) \\ &= \int_{\mathbb{R}^m} \exp\left(-2\pi i\, \underline{x}^T G \underline{y}\right)\, f(\underline{y})\, dV(\underline{y}) \end{aligned} \tag{7.11}$$

where the restriction of the scalar product (7.8) to $\mathbb{R}^m$ comes into play. Due to the symmetric character of G, one then immediately finds the following link between the two Fourier transforms:

$$\mathcal{F}_G[f](\underline{x}) \;=\; \mathcal{F}[f](G\underline{x}) \tag{7.12}$$

To show that the definition (7.11) is meaningful in the metric dependent context, it is checked how this anisotropic Fourier transform behaves with respect to multiplication with the variable $\underline{x}$ in $(\mathbb{R}^m, G)$ and, by duality, with respect to the action of the anisotropic Dirac operator $\partial_{\underline{x},G}$ in $(\mathbb{R}^m, G)$.

Proposition 7.1. *The anisotropic Fourier transform $\mathcal{F}_G$ satisfies the following calculation rules:*

(i) the multiplication rule:

$$2\pi i \, \mathcal{F}_G[\underline{y}\, f(\underline{y})](\underline{x}) \;=\; -\partial_{\underline{x},G}\, \mathcal{F}_G[f](\underline{x})$$

(ii) the differentiation rule:

$$\mathcal{F}_G[\partial_{\underline{y},G}\, f(\underline{y})](\underline{x}) \;=\; 2\pi i\, \underline{x}\, \mathcal{F}_G[f](\underline{x})$$

Proof.
First of all, if we put

$$\underline{u} \;\equiv\; G\,\underline{x} \;=\; \sum_{j=1}^{m} e_j \sum_{k=1}^{m} g_{jk}\, x^k$$

then

$$\partial_{\underline{u}} \;=\; \sum_{j=1}^{m} e_j\, \partial_{u^j} \;=\; \sum_{j=1}^{m} e_j \sum_{k=1}^{m} g^{kj}\, \partial_{x^k} \;=\; \sum_{k=1}^{m} e^k\, \partial_{x^k} \;=\; \partial_{\underline{x},G}$$

Hence, taking now into account (7.12) and the properties (3.2) of the isotropic Fourier transform, we consecutively find

$$2\pi i\, \mathcal{F}_G[\underline{y}\, f(\underline{y})](\underline{x}) \;=\; 2\pi i\, \mathcal{F}[\underline{y}\, f(\underline{y})](\underline{u}) \;=\; -\partial_{\underline{u}}\, \mathcal{F}[f](\underline{u}) \;=\; -\partial_{\underline{x},G}\, \mathcal{F}_G[f](\underline{x})$$

and

$$\mathcal{F}_G[\partial_{\underline{y},G}\, f(\underline{y})](\underline{x}) \;=\; \mathcal{F}[\partial_{\underline{y},G}\, f(\underline{y})](\underline{u}) \;=\; 2\pi i \sum_{j=1}^{m} e^j\, u^j\, \mathcal{F}[f](\underline{u}) \;=\; 2\pi i\, \underline{x}\, \mathcal{F}_G[f](\underline{x})$$

■

Chapter 8

The anisotropic Hilbert transform

In the previous chapter we have introduced the basic language of anisotropic Clifford analysis. In this setting, a new anisotropic multidimensional Hilbert transform in $\mathbb{R}^m \subset \mathbb{R}^{m+1}$ can be defined (see our papers [12, 16]), which is shown to possess formally the same properties as the isotropic Hilbert operator introduced in Chapter 5. We note however that a special case of such an anisotropic Hilbert transform, fitting into this general framework, was already introduced and used for two–dimensional image processing in [56].

In this chapter we first present our definition of the anisotropic Hilbert transform in $\mathbb{R}^m$, arising naturally as a part of the non–tangential boundary limit of the anisotropic Cauchy integral in $\mathbb{R}^{m+1}$. Next, we study the main properties of the former operator. In the second section, for a specific tempered distribution in $\mathbb{R}^m$, both its isotropic and anisotropic Hilbert transform are calculated in order to examine the influence of the metric tensor considered. Finally, a striking result to be mentioned is that the associated anisotropic Cauchy integral in $\mathbb{R}^{m+1}$ is no longer uniquely determined, but may stem from a diversity of metric tensors of order $(m + 1)$.

8.1 Definition and properties

For the introduction of the anisotropic Hilbert kernel, one needs to calculate the distributional limits of $P_{\widetilde{G}}(x^0, \underline{x})$ and $Q_{\widetilde{G}}(x^0, \underline{x})$ for $x^0 \to 0+$. The outcome of those limits is presented in Proposition 8.1, which is preceded by the following two auxiliary results.

Lemma 8.1. *The determinant of a metric tensor $\widetilde{G}$ of order $(m+1)$, given by (7.1), is related to the determinant of its corresponding subtensor G in the following way:*

$$\det(\widetilde{G}) \;=\; \det(G)\,\left(g_{00} - \underline{u}^T\, G^{-1}\,\underline{u}\right) \tag{8.1}$$

where $\underline{u}^T$ denotes the row matrix $(g_{01}\;\ldots\;g_{0m})$.

Proof.
As the restriction G of the metric tensor $\widetilde{G}$ to $\mathbb{R}^m$ is a metric tensor as well, one may write $G = B^T B$, with $B \in \mathrm{GL}(m;\mathbb{R})$. Defining $\underline{v} = (B^T)^{-1}\underline{u}$, the tensor $\widetilde{G}$ may then be factorized as

$$\widetilde{G} \;=\; \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & B^T B \end{pmatrix} \;=\; \begin{pmatrix} 1 & \underline{0}^T \\ \underline{0} & B^T \end{pmatrix} \begin{pmatrix} g_{00} & \underline{v}^T \\ \underline{v} & \mathbb{E}_m \end{pmatrix} \begin{pmatrix} 1 & \underline{0}^T \\ \underline{0} & B \end{pmatrix}$$

from which it follows that

$$\begin{aligned} \det(\widetilde{G}) \;&=\; \det(B^T)\,\det\begin{pmatrix} g_{00} & \underline{v}^T \\ \underline{v} & \mathbb{E}_m \end{pmatrix}\det(B) \\ &=\; \det(B^T B)\,\left(g_{00} - \underline{v}^T\underline{v}\right) \;=\; \det(G)\,\left(g_{00} - \underline{u}^T\,G^{-1}\,\underline{u}\right) \end{aligned}$$

■

Lemma 8.2. *Let $\widetilde{G}$ be a metric tensor of order $(m+1)$, given by (7.1), and let $\hat{x} = e_0 + \underline{x}$, then*

$$\int_{\mathbb{R}^m} \frac{dV(\underline{x})}{|\hat{x}|_{\widetilde{G}}^{m+1}} \;=\; \frac{a_{m+1}}{2\sqrt{\det(\widetilde{G})}}$$

Proof.
As the restriction G of the metric tensor $\widetilde{G}$ to $\mathbb{R}^m$ is a metric tensor as well, there exists an orthogonal $(m \times m)$ matrix A such that

$$A^T G A \;=\; \mathrm{diag}(\mu_1^2, \ldots, \mu_m^2)$$

with $\mu_1^2, \ldots, \mu_m^2$ the strictly positive (not necessarily different) eigenvalues of G. Then, introducing a new integration variable $\underline{x}'$ by means of the transformation

$$\underline{x} \;=\; A\underline{x}' - G^{-1}\underline{u}$$

and interpreting the vectors $\underline{x}$ and $\underline{x}'$ as column matrices, one has

$$\begin{aligned}
|\hat{x}|^2_{\widetilde{G}} &= \langle\, \hat{x}, \hat{x} \,\rangle_{\widetilde{G}} \;=\; \begin{pmatrix} 1 & \underline{x}^T \end{pmatrix} \widetilde{G} \begin{pmatrix} 1 \\ \underline{x} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \underline{x}'^T A^T - \underline{u}^T G^{-1} \end{pmatrix} \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix} \begin{pmatrix} 1 \\ A\underline{x}' - G^{-1}\underline{u} \end{pmatrix} \\
&= \sum_{j=1}^{m} \left(\mu_j x'^j\right)^2 + \left(g_{00} - \underline{u}^T G^{-1}\underline{u}\right)
\end{aligned}$$

Once again introducing a new variable, now through the transformation

$$\underline{x}' \;=\; \sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\, \mathrm{diag}(\mu_1^{-1}, \ldots, \mu_m^{-1})\, \underline{x}''$$

and moreover invoking (8.1), one arrives at

$$|\hat{x}|^2_{\widetilde{G}} \;=\; \langle\, \hat{x}, \hat{x} \,\rangle_{\widetilde{G}} \;=\; \frac{\det(\widetilde{G})}{\det(G)} \left(\sum_{j=1}^{m} \left(x''^j\right)^2 + 1 \right)$$

Furthermore, the volume elements $dV(\underline{x})$ and $dV(\underline{x}'')$ are then seen to correspond as follows:

$$\begin{aligned}
dV(\underline{x}) &= |\det A|\; dV(\underline{x}') \;=\; \left(\sqrt{\frac{\det(\widetilde{G})}{\det(G)}} \right)^m \left(\prod_{j=1}^{m} \mu_j^{-1} \right) dV(\underline{x}'') \\
&= \frac{\left(\sqrt{\det(\widetilde{G})}\right)^m}{\left(\sqrt{\det(G)}\right)^{m+1}}\, dV(\underline{x}'')
\end{aligned}$$

So eventually one has

$$\int_{\mathbb{R}^m} \frac{dV(\underline{x})}{|\hat{x}|^{m+1}_{\widetilde{G}}} \;=\; \frac{1}{\sqrt{\det(\widetilde{G})}} \int_{\mathbb{R}^m} \frac{dV(\underline{x}'')}{\left[1 + \sum_{j=1}^{m} (x''^j)^2\right]^{\frac{m+1}{2}}} \;=\; \frac{a_{m+1}}{2\sqrt{\det(\widetilde{G})}}$$

the last equality leaning on the classical result (see e.g. [97, Lemma I.1.17])

$$\int_{\mathbb{R}^m} \frac{dV(\underline{x}'')}{\left[1+\sum_{j=1}^m (x''^j)^2\right]^{\frac{m+1}{2}}} \quad = \quad \frac{a_{m+1}}{2}$$

■

Proposition 8.1. *In distributional sense one has*

$$\lim_{x^0 \to 0+} P_{\widetilde{G}}(x^0, \underline{x}) \quad = \quad \frac{1}{\sqrt{\det(\widetilde{G})}}\, \delta(\underline{x})$$

$$\lim_{x^0 \to 0+} Q_{\widetilde{G}}(x^0, \underline{x}) \quad = \quad \frac{1}{\sqrt{\det(\widetilde{G})}}\, H_{G,c}(\underline{x})$$

with

$$H_{G,c}(\underline{x}) \quad = \quad \frac{2c}{a_{m+1}}\, \mathrm{Pv}\, \frac{\overline{x}}{|\underline{x}|_G^{m+1}} \ , \qquad c = \sqrt{\det(\widetilde{G})}$$

Proof.
First consider the distributional limit of $P_{\widetilde{G}}(x^0, \underline{x})$. It is well–known that, if a real valued integrable function h defined on $\mathbb{R}^m$ satisfies the property

$$\int_{\mathbb{R}^m} h(\underline{x})\, dV(\underline{x}) \quad = \quad 1 \tag{8.2}$$

then one has in distributional sense

$$\lim_{x^0 \to 0+} \widetilde{h}(x^0, \underline{x}) \quad = \quad \delta(\underline{x})$$

where

$$\widetilde{h}(x^0, \underline{x}) \quad = \quad \frac{1}{(x^0)^m}\, h\left(\frac{\underline{x}}{x^0}\right) \ , \quad x^0 > 0$$

As Lemma 8.2 implies that the specific integrable function

$$h(\underline{x}) \quad = \quad \frac{2\sqrt{\det(\widetilde{G})}}{a_{m+1}}\, \frac{1}{|\hat{x}|_{\widetilde{G}}^{m+1}}$$

satisfies property (8.2), it follows that

$$\lim_{x^0 \to 0+} P_{\widetilde{G}}(x^0, \underline{x}) \;=\; \frac{1}{\sqrt{\det(\widetilde{G})}} \lim_{x^0 \to 0+} \widetilde{h}(x^0, \underline{x}) \;=\; \frac{1}{\sqrt{\det(\widetilde{G})}}\, \delta(\underline{x})$$

Next, since $Q_{\widetilde{G}}(x^0, \underline{x}) \in L_1^{\rm loc}(\mathbb{R}^m)$ for each $x^0 > 0$, it defines a regular distribution whose action on a test function ϕ in $\mathbb{R}^m$ is given by

$$\left\langle\, Q_{\widetilde{G}}(x^0, \underline{x}) \;,\; \phi(\underline{x}) \,\right\rangle \;=\; \int_{\mathbb{R}^m} Q_{\widetilde{G}}(x^0, \underline{x})\, \phi(\underline{x})\, dV(\underline{x})$$

Taking the limit for $x^0 \to 0+$ then results in

$$\begin{aligned} \left\langle\, \lim_{x^0 \to 0+} Q_{\widetilde{G}}(x^0, \underline{x}) \;,\; \phi(\underline{x}) \,\right\rangle &= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^m \setminus B(\underline{0};\varepsilon)} \lim_{x^0 \to 0+} Q_{\widetilde{G}}(x^0, \underline{x})\, \phi(\underline{x})\, dV(\underline{x}) \\ &= \mathrm{Pv} \int_{\mathbb{R}^m} \frac{2}{a_{m+1}}\, \frac{\overline{\underline{x}}}{|\underline{x}|_G^{m+1}}\, \phi(\underline{x})\, dV(\underline{x}) \;=\; \left\langle\, \frac{1}{\sqrt{\det(\widetilde{G})}}\, H_{G,c}(\underline{x}) \;,\; \phi(\underline{x}) \,\right\rangle \end{aligned}$$

which completes the proof. ∎

The previous proposition directly leads us to the distributional limits of the anisotropic Poisson transform and its ($\widetilde{G}$–)conjugate, viz

$$\begin{aligned} \lim_{x^0 \to 0+} \mathcal{P}_{\widetilde{G}}[f] &= \frac{1}{\sqrt{\det(\widetilde{G})}}\, f \\ \lim_{x^0 \to 0+} \mathcal{Q}_{\widetilde{G}}[f] &= \frac{1}{\sqrt{\det(\widetilde{G})}}\, H_{G,c} * f \end{aligned}$$

whence

$$\lim_{x^0 \to 0+} \mathcal{C}_{\widetilde{G}}[f] \;=\; \frac{1}{\sqrt{\det(\widetilde{G})}} \left(\frac{1}{2}\, \overline{e_0} e^0\, f + \frac{1}{2}\, \overline{e^0}\, H_{G,c} * f \right)$$

Similarly, for $x^0 \to 0-$, one obtains

$$\lim_{x^0 \to 0-} \mathcal{C}_{\widetilde{G}}[f] \;=\; \frac{1}{\sqrt{\det(\widetilde{G})}} \left(-\frac{1}{2}\, \overline{e_0} e^0\, f + \frac{1}{2}\, \overline{e^0}\, H_{G,c} * f \right)$$

The above results may be seen as the *anisotropic Plemelj–Sokhotzki formulae*. For a function $f \in L_2(\mathbb{R}^m)$ (or a tempered distribution), they give rise to the definition of the *anisotropic Hilbert transform*, defined as

$$\mathcal{H}_{G,c}[f] \;=\; \overline{e^0}\, H_{G,c} * f$$

by means of which the Plemelj–Sokhotzki formulae can be rewritten as

$$\lim_{x^0 \to 0\pm} \mathcal{C}_{\widetilde{G}}[f] \;=\; \frac{1}{\sqrt{\det(\widetilde{G})}} \left(\pm \frac{1}{2}\, \overline{e_0} e^0\, f + \frac{1}{2}\, \mathcal{H}_{G,c}[f] \right) \tag{8.3}$$

For $m = 2$, such an anisotropic Hilbert transform was considered in [56], however for the special case where the e_0–direction in $\mathbb{R}^3$ is chosen perpendicular to the $\mathbb{R}^2$–plane spanned by (e_1, e_2). This corresponds to a $\widetilde{G}$–matrix of order 3 in which $g_{01} = g_{02} = 0$ (see also Remark 7.1).

The following properties of the Hilbert transform $\mathcal{H}_{G,c}$ may then be proven:

Proposition 8.2.

P(1) *The anisotropic Hilbert transform $\mathcal{H}_{G,c}$ commutes with translations, which is an equivalent statement to its definition as a convolution operator.*

P(2) *The anisotropic Hilbert transform $\mathcal{H}_{G,c}$ commutes with dilations, which, for a convolution operator, is equivalent to its kernel $H_{G,c}$ being a homogeneous distribution of degree $(-m)$.*

P(3) *The anisotropic Hilbert transform $\mathcal{H}_{G,c}$ is a bounded linear operator on $L_2(\mathbb{R}^m)$, which is equivalent to its anisotropic Fourier symbol*

$$\mathcal{F}_G[H_{G,c}](\underline{x}) \;=\; \sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\; i\, \frac{\underline{x}}{|\underline{x}|_G} \tag{8.4}$$

being a bounded function.

P(4) *Up to a metric related constant, the anisotropic Hilbert transform $\mathcal{H}_{G,c}$ squares to unity, i.e.*

$$(\mathcal{H}_{G,c})^2 \;=\; g^{00}\, \frac{\det(\widetilde{G})}{\det(G)}\, \mathbf{1}$$

P(5) The anisotropic Hilbert transform $\mathcal{H}_{G,c}$ is self–adjoint, i.e.

$$\langle \mathcal{H}_{G,c}[f], g\rangle \;=\; \langle f, \mathcal{H}_{G,c}[g]\rangle \;, \qquad f, g \in L_2(\mathbb{R}^m)$$

Proof.
The proof of properties P(1), P(2) and P(5) is rather straightforward, starting from the definition of $\mathcal{H}_{G,c}$ and taking into account the anisotropic setting. Next, the calculation of the Fourier symbol in P(3) is established by invoking the factorization of the positive definite tensor G as

$$G \;=\; B^T B \;, \qquad B \in \mathrm{GL}(m, \mathbb{R})$$

We then have

$$\begin{aligned}
&\mathcal{F}_G[H_{G,c}](\underline{x}) \\
&= \sqrt{\det(\widetilde{G})}\,\frac{2}{a_{m+1}}\int_{\mathbb{R}^m} \exp(-2\pi i\,\underline{x}^T G\underline{y})\,\mathrm{Pv}\,\frac{\overline{\underline{y}}}{\left[\underline{y}^T G\underline{y}\right]^{\frac{m+1}{2}}}\,dV(\underline{y}) \\
&= \sqrt{\det(\widetilde{G})}\,\frac{2}{a_{m+1}}\int_{\mathbb{R}^m} \exp(-2\pi i\,(B\underline{x})^T B\underline{y})\,\mathrm{Pv}\,\frac{\overline{\underline{y}}}{\left[(B\underline{y})^T B\underline{y}\right]^{\frac{m+1}{2}}}\,dV(\underline{y})
\end{aligned}$$

Putting $\underline{y}' = B\underline{y}$ one arrives at

$$\begin{aligned}
&\mathcal{F}_G[H_{G,c}](\underline{x}) \\
&= \sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\,B^{-1}\,\frac{2}{a_{m+1}}\int_{\mathbb{R}^m} \exp(-2\pi i(B\underline{x})^T\underline{y}')\,\mathrm{Pv}\,\frac{\overline{\underline{y}'}}{\left[\underline{y}'^T\underline{y}'\right]^{\frac{m+1}{2}}}\,dV(\underline{y}')
\end{aligned}$$

such that the anisotropic Fourier transform of the anisotropic Hilbert kernel can be rewritten in terms of the isotropic Fourier transform of the isotropic Hilbert kernel, i.e.

$$\mathcal{F}_G[H_{G,c}](\underline{x}) \;=\; \sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\,B^{-1}\,\mathcal{F}[H](B\underline{x})$$

with the isotropic Hilbert kernel being given by

$$H(\underline{x}) \;=\; \frac{2}{a_{m+1}}\,\mathrm{Pv}\,\frac{\overline{\underline{x}}}{(\langle\,\underline{x}, \underline{x}\,\rangle)^{\frac{m+1}{2}}}$$

Its Fourier symbol $\mathcal{F}[H]$ reads (see also (5.4))

$$\mathcal{F}[H](\underline{x}) \;=\; i\,\frac{\underline{x}}{\sqrt{\langle\, \underline{x}, \underline{x}\,\rangle}}$$

yielding

$$\mathcal{F}_G[H_{G,c}](\underline{x}) \;=\; \sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\, B^{-1}\, i\, \frac{B\underline{x}}{\sqrt{\langle\, B\underline{x}, B\underline{x}\,\rangle}} \;=\; \sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\, i\, \frac{\underline{x}}{|\underline{x}|_G}$$

Finally, property P(4) then results from a conversion to the Fourier domain. Indeed,

$$\begin{aligned}
\mathcal{F}_G\left[\mathcal{H}^2_{G,c}[f]\right] &= \mathcal{F}_G\left[\overline{e^0}\, H_{G,c} * \mathcal{H}_{G,c}[f]\right] \;=\; \mathcal{F}_G\left[\overline{e^0}\, H_{G,c}\right] \mathcal{F}_G\left[\mathcal{H}_{G,c}[f]\right] \\
&= \mathcal{F}_G\left[\overline{e^0}\, H_{G,c}\right] \mathcal{F}_G\left[\overline{e^0}\, H_{G,c} * f\right] \;=\; \mathcal{F}_G\left[\overline{e^0} H_{G,c}\right]^2 \mathcal{F}_G[f]
\end{aligned}$$

whence

$$\mathcal{F}_G\left[\mathcal{H}^2_{G,c}[f]\right] \;=\; -\frac{\det(\widetilde{G})}{\det(G)}\, g^{00}\, \frac{\underline{x}^2}{|\underline{x}|^2_G}\, \mathcal{F}_G[f] \;=\; g^{00}\, \frac{\det(\widetilde{G})}{\det(G)}\, \mathcal{F}_G[f]$$

■

Notice that, due to properties P(4)–P(5), the operator

$$\sqrt{\frac{\det(G)}{g^{00}\, \det(\widetilde{G})}}\, \mathcal{H}_{G,c}$$

is unitary.

8.2 Example

Consider in $\mathbb{R}^m$ the tempered distribution

$$f(\underline{x}) \;=\; \exp\left(2\pi i \,\langle\, \underline{a}, \underline{x}\,\rangle\right)$$

where $\underline{a}$ is a given, nonzero vector. Both the isotropic Hilbert transform of f and its anisotropic counterpart, defined respectively by

$$\mathcal{H}[f](\underline{y}) \;=\; \overline{e_0}\, \frac{2}{a_{m+1}} \left(\mathrm{Pv}\, \frac{\underline{x}}{(\underline{x}^T \underline{x})^{\frac{m+1}{2}}} * f(\underline{x}) \right)(\underline{y})$$

and

$$\mathcal{H}_{G,c}[f](\underline{y}) \;=\; \overline{e^0}\,\sqrt{\det(\widetilde{G})}\,\frac{2}{a_{m+1}}\left(\mathrm{Pv}\,\frac{\overline{\underline{x}}}{(\underline{x}^T\,G\,\underline{x})^{\frac{m+1}{2}}} * f(\underline{x})\right)(\underline{y})$$

will be calculated, in order to illustrate the differences between both cases on a concrete example. Note that the above formulae show once more how $\mathcal{H}_{G,c}$ reduces to $\mathcal{H}$ when $\widetilde{G} = \mathbb{E}_{m+1}$, seen also the fact that $e_0 = e^0$ in that case.

We first consider the isotropic case. Using definition (7.10), the isotropic Fourier transform of f reads

$$\mathcal{F}[f](\underline{y}) \;=\; \delta(\underline{y} - \underline{a})$$

leading to

$$\mathcal{F}\left[\mathcal{H}[f]\right](\underline{y}) \;=\; i\,\overline{e_0}\,\frac{\underline{y}}{|\underline{y}|}\,\delta(\underline{y}-\underline{a}) \;=\; i\,\overline{e_0}\,\frac{\underline{a}}{|\underline{a}|}\,\delta(\underline{y}-\underline{a})$$

and eventually to

$$\mathcal{H}[f](\underline{x}) \;=\; i\,\overline{e_0}\,\frac{\underline{a}}{|\underline{a}|}\,\exp\left(2\pi i\,\langle\,\underline{a},\underline{x}\,\rangle\right)$$

In the anisotropic case, the Fourier transform is defined by (7.11) or the equivalent form (7.12), so that

$$\mathcal{F}_G[f](\underline{y}) \;=\; \mathcal{F}[f](G\underline{y}) \;=\; \delta(G\underline{y} - \underline{a})$$

and thus

$$\mathcal{F}_G\left[\mathcal{H}_{G,c}[f]\right](\underline{y}) \;=\; \overline{e^0}\,i\,\sqrt{\frac{\det(\widetilde{G})}{\det(G)}}\,\frac{G^{-1}\underline{a}}{|G^{-1}\underline{a}|_G}\,\delta(G\underline{y}-\underline{a})$$

with

$$|G^{-1}\underline{a}|_G \;=\; \left[\left(G^{-1}\underline{a}\right)^T\,G\,\left(G^{-1}\underline{a}\right)\right]^{\frac{1}{2}} \;=\; \left[\underline{a}^T\,G^{-1}\,\underline{a}\right]^{\frac{1}{2}}$$

Subsequent calculations learn that

$$\begin{aligned}\mathcal{F}_G^{-1}\left[\delta(G\underline{y}-\underline{a})\right](\underline{x}) \;&=\; \int_{\mathbb{R}^m}\exp\left(2\pi i\,\underline{x}^T G\underline{y}\right)\delta(G\underline{y}-\underline{a})\,dV(\underline{y})\\ &=\; \frac{1}{\det(G)}\int_{\mathbb{R}^m}\exp\left(2\pi i\,\underline{x}^T\underline{y}'\right)\delta(\underline{y}'-\underline{a})\,dV(\underline{y}') \;=\; \frac{1}{\det(G)}\,\exp\left(2\pi i\,\langle\,\underline{a},\underline{x}\,\rangle\right)\end{aligned}$$

Hence

$$\mathcal{H}_{G,c}[f](\underline{x}) \;=\; i\,\overline{e^0}\,\sqrt{\frac{\det(\widetilde{G})}{(\det(G))^3}}\,\frac{G^{-1}\underline{a}}{|G^{-1}\underline{a}|}\,\exp\left(2\pi i\,\langle\,\underline{a},\underline{x}\,\rangle\right)$$

Anticipating the next section, we already observe that in the above formula the anisotropic Hilbert transform of our specific chosen distribution not only depends on the chosen metric tensor G in $\mathbb{R}^m$ but also on the determinant of the "mother" metric $\widetilde{G}$ in $\mathbb{R}^{m+1}$.

8.3 Conclusion

As is the case for the definition in the isotropic setting, the present anisotropic Hilbert kernel $H_{G,c}$ has been obtained in a constructive way, by taking distributional limits of a $\widetilde{G}$–harmonic function in $\mathbb{R}^{m+1}_{+}$, which is one of the two conjugate harmonic parts in which the $\widetilde{G}$–monogenic anisotropic Cauchy kernel $C_{\widetilde{G}}$ splits. The resulting anisotropic Hilbert transform $\mathcal{H}_{G,c}[f] = \overline{e^0}H_{G,c} * f$ depends on the underlying metric in two different ways:

(1) the determinant of the "mother" metric $\widetilde{G}$ on $\mathbb{R}^{m+1}$ arises as an explicit factor in the expression for the kernel

and

(2) the induced metric G on $\mathbb{R}^m$ implicitly comes into play through the denominator of the kernel, since $|\underline{x}|_G^{m+1}$ can be rewritten as $\left[\underline{x}^T\,G\,\underline{x}\right]^{\frac{m+1}{2}}$.

The particularity of this metric dependence may also be seen in the Fourier domain, where the metric G not only arises in the Fourier symbol (8.4) of $\mathcal{H}_{G,c}$, but is also hidden in the definition of the Fourier transform itself, while the "mother" metric $\widetilde{G}$ again only pops up through its determinant.

The above observations raise the question whether there exists a one–to–one correspondence between a given anisotropic Hilbert transform $\mathcal{H}_{G,c}$ on $\mathbb{R}^m$ and the associated anisotropic Cauchy integral $\mathcal{C}_{\widetilde{G}}$ on $\mathbb{R}^{m+1}$ from which it originates, or in other words: does the anisotropic Hilbert transform contain enough geometrical information to completely determine the "mother" metric $\widetilde{G}$? One may already intuitively feel that the answer is negative, since only the induced metric G and the determinant $\det(\widetilde{G})$ seem to be involved.

To answer this question properly, we consider, for a given G and $\det(\widetilde{G})$, the equation

$$g_{00} - \underline{u}^T\, G^{-1}\, \underline{u} \;=\; \frac{\det(\widetilde{G})}{\det(G)}$$

derived from (8.1). If we want $\widetilde{G}$ to be uniquely determined, then this equation should have a unique solution $(g_{00}, \underline{u}^T)$, which clearly is not the case, since we directly see that

$$g_{00} \;=\; \frac{\det(\widetilde{G})}{\det(G)}\;, \qquad \underline{u}^T \;=\; \underline{0}^T$$

and

$$g_{00} \;=\; \frac{\det(\widetilde{G})}{\det(G)} + (G^{-1})_{11}\;, \qquad \underline{u}^T \;=\; (1\,0\,\ldots\,0)$$

already constitute two different solutions, and others may be found straightaway.

We conclude that, given a Hilbert kernel

$$H_{G,c}(\underline{x}) \;=\; \frac{2c}{a_{m+1}}\, \mathrm{Pv}\, \frac{\overline{\underline{x}}}{|\underline{x}|_G^{m+1}}$$

which depends on the metric tensor G of order m and on the constant $c > 0$, it is part of the boundary limit of a Cauchy kernel $C_{\widetilde{G}}$ in $\mathbb{R}^{m+1}$, with

$$\widetilde{G} \;=\; \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix}$$

where $(g_{00}, \underline{u}^T)$ are characterized, but not uniquely determined, by the equation

$$g_{00} - \underline{u}^T\, G^{-1}\, \underline{u} \;=\; \frac{c^2}{\det(G)}$$

Part III

Hilbert transforms in Hermitean Clifford analysis

Chapter 9

The Hermitean Clifford toolbox

In a series of recent papers, so–called Hermitean Clifford analysis has emerged as a new and successful branch of Clifford analysis, offering a refinement of the traditional, also called orthogonal, case; it focusses on the simultaneous null solutions, called Hermitean monogenic functions, of two Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ which do not longer factorize but still decompose the Laplace operator in the sense that $4(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}}) = \Delta$ and which are invariant under the action of a realization of the unitary group. The study of Hermitean Dirac operators was initiated in [86, 85, 89, 43]; a systematic development of the associated function theory, including the invariance properties with respect to the underlying Lie groups and Lie algebras, is still in full progress, see e.g. [34, 6, 7, 54].

In this introductory chapter we first present the elementary objects of Hermitean Clifford analysis which originate in a natural way by introducing a so–called complex structure, which has proven to be the appropriate instrument for converting notions from the orthogonal setting into their Hermitean counterparts. In the second section a splitting of the Hermitean monogenic system is considered, which has already been studied in [7], leading to the so–called homogeneous parts of complex spinor space.

9.1 Hermitean Clifford analysis: the basic ingredients

We reconsider the complex Clifford algebra $\mathbb{C}_m$, already introduced in Chapter 3, which may be seen as the complexification of the real Clifford algebra $\mathbb{R}_{0,m}$, i.e.

$$\mathbb{C}_m \;=\; \mathbb{C} \otimes \mathbb{R}_{0,m} \;=\; \mathbb{R}_{0,m} \oplus i\,\mathbb{R}_{0,m}$$

An elegant way for introducing the setting of Hermitean Clifford analysis consists in considering $\mathbb{C}_m$ as a Hermitean space, i.e. endowing it with a so–called *complex structure*, i.e. a specific $\mathrm{SO}(m)$ element J for which it is required that $J^2 = -\mathbb{E}_m$ (see [6, 7]). It is then immediately seen that the requirement $\det(J)^2 = (-1)^m$ forces the dimension m to be even, i.e. from now on we take $m = 2n$. Moreover, without loss of generality, the generators $e_1, \ldots, e_{2n}$ of the Clifford algebra may always be chosen in such a way that J is given by the matrix

$$J \;=\; \begin{pmatrix} 0 & \mathbb{E}_n \\ -\mathbb{E}_n & 0 \end{pmatrix}$$

or, equivalently, when J is identified with the operator associated to the matrix, its action upon those generators is given by

$$J[e_j] \;=\; -e_{n+j} \qquad \text{and} \qquad J[e_{n+j}] \;=\; e_j \;, \qquad j = 1, \ldots, n$$

With J one may then associate two projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ which produce the main protagonists of the Hermitean setting by acting upon the corresponding objects in the orthogonal framework. It is precisely here that originates the statement describing Hermitean Clifford analysis as a refinement of orthogonal Clifford analysis. Indeed, the considered projection operators cause a direct sum decomposition of the vector space $\mathbb{R}^{2n}$ (or its complexification $\mathbb{C}^{2n}$), viz

$$\mathbb{R}^{2n} \;=\; \frac{1}{2}(\mathbf{1} + iJ)[\mathbb{R}^{2n}] \oplus \frac{1}{2}(\mathbf{1} - iJ)[\mathbb{R}^{2n}]$$

into two isotropic subspaces, whence all concepts from the orthogonal setting will be split accordingly.

First of all, the so–called *Witt basis* elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for the complex Clifford algebra $\mathbb{C}_{2n}$ are obtained through the action of $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal basis elements $(e_j)_{j=1}^{2n}$, i.e.

$$\begin{aligned}\mathfrak{f}_j &= \frac{1}{2}(\mathbf{1}+iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j})\ , \quad j=1,\ldots,n\\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1}-iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j})\ , \quad j=1,\ldots,n\end{aligned}$$

Notice that the dagger notation is well–chosen since the Hermitean conjugate (see Chapter 3 for the definition) $(\mathfrak{f}_j)^\dagger$ of $\mathfrak{f}_j$ indeed yields $\mathfrak{f}_j^\dagger$:

$$(\mathfrak{f}_j)^\dagger = \left[\frac{1}{2}(e_j - i\,e_{n+j})\right]^\dagger = \frac{1}{2}(-e_j - i\,e_{n+j}) = \mathfrak{f}_j^\dagger\ , \qquad j=1,\ldots,n$$

Taking into account the multiplication rules for the orthogonal basis elements

$$e_je_k + e_ke_j = -2\,\delta_{jk}\ , \qquad j,k=1,\ldots,2n$$

the Witt basis elements are seen to satisfy the Grassmann identities

$$\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0\ , \qquad j,k=1,\ldots,n$$

including their isotropy when $j=k$, as well as the duality identities

$$\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}\ , \qquad j,k=1,\ldots,n$$

Next, we rewrite the Euclidean vector $\underline{X} = (X_1,\ldots,X_{2n})$ in $\mathbb{R}^{0,2n}$ as

$$\underline{X} = (x_1,\ldots,x_n,y_1,\ldots,y_n)$$

and we identify it, as usual, with the Clifford–vector $\underline{X} = \sum_{j=1}^n (e_j\,x_j + e_{n+j}\,y_j)$. By means of the action of the complex structure J, we now associate to $\underline{X}$ the so–called *twisted vector* $\underline{X}|$, i.e.

$$\underline{X}| = J[\underline{X}] = \sum_{j=1}^n (e_j\,y_j - e_{n+j}\,x_j)$$

Observe that the Clifford–vectors $\underline{X}$ and $\underline{X}|$ anti–commute, since the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal with respect to the standard Euclidean scalar product. The actions of the projection operators on the Clifford–vector $\underline{X}$ then produce the *Hermitean Clifford–variable* $\underline{Z}$ and its Hermitean conjugate $\underline{Z}^\dagger$:

$$\begin{aligned}\underline{Z} &= \frac{1}{2}(\mathbf{1}+iJ)[\underline{X}] = \frac{1}{2}(\underline{X} + i\,\underline{X}|)\\ \underline{Z}^\dagger &= -\frac{1}{2}(\mathbf{1}-iJ)[\underline{X}] = -\frac{1}{2}(\underline{X} - i\,\underline{X}|)\end{aligned}$$

which may also be rewritten in terms of the Witt basis elements as

$$\underline{Z} \;=\; \sum_{j=1}^{n} \mathfrak{f}_j\, z_j \quad \text{and} \quad \underline{Z}^\dagger \;=\; (\underline{Z})^\dagger \;=\; \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\, z_j^c$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, \ldots, n$. In terms of the Hermitean vector variables $\underline{Z}$ and $\underline{Z}^\dagger$, the orthogonal vector variables $\underline{X}$ and $\underline{X}|$ are decomposed as follows:

$$\underline{X} \;=\; \underline{Z} - \underline{Z}^\dagger \qquad \text{and} \qquad \underline{X}| \;=\; \frac{1}{i}\,(\underline{Z} + \underline{Z}^\dagger)$$

Finally, the *Hermitean Dirac operators* $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are derived from the orthogonal Dirac operator $\partial_{\underline{X}}$ in the following way:

$$\begin{aligned} \partial_{\underline{Z}^\dagger} &= \frac{1}{4}\,(\mathbf{1} + iJ)[\partial_{\underline{X}}] &&= \frac{1}{4}\,(\partial_{\underline{X}} + i\,\partial_{\underline{X}|}) \\ \partial_{\underline{Z}} &= -\frac{1}{4}\,(\mathbf{1} - iJ)[\partial_{\underline{X}}] &&= -\frac{1}{4}\,(\partial_{\underline{X}} - i\,\partial_{\underline{X}|}) \end{aligned}$$

where we have introduced the so–called *twisted Dirac operator*

$$\partial_{\underline{X}|} \;=\; J[\partial_{\underline{X}}] \;=\; \sum_{j=1}^{n} (e_j\,\partial_{y_j} - e_{n+j}\,\partial_{x_j})$$

As was the case with $\partial_{\underline{X}}$, a notion of monogenicity may be associated in a natural way to $\partial_{\underline{X}|}$ as well. Again passing to the Witt basis, the Hermitean Dirac operators may then be expressed as

$$\partial_{\underline{Z}} \;=\; \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\,\partial_{z_j} \quad \text{and} \quad \partial_{\underline{Z}^\dagger} \;=\; (\partial_{\underline{Z}})^\dagger \;=\; \sum_{j=1}^{n} \mathfrak{f}_j\,\partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and their complex conjugates $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ in the complex z_j–planes, $j = 1, \ldots, n$. In terms of the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, the orthogonal Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ are decomposed as follows:

$$\partial_{\underline{X}} \;=\; 2\,(\partial_{\underline{Z}^\dagger} - \partial_{\underline{Z}}) \qquad \text{and} \qquad \partial_{\underline{X}|} \;=\; \frac{2}{i}\,(\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}})$$

For further use, observe that the Hermitean vector variables and Dirac operators are isotropic, on account of the Witt basis properties, i.e.

$$(\underline{Z})^2 \;=\; (\underline{Z}^\dagger)^2 \;=\; 0 \quad \text{and} \quad (\partial_{\underline{Z}})^2 \;=\; (\partial_{\underline{Z}^\dagger})^2 \;=\; 0$$

whence the Laplacian $\Delta \;=\; -\partial_{\underline{X}}^2 \;=\; -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta \;=\; 4\,(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}})$$

while also

$$\underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger\underline{Z} \;=\; |\underline{Z}|^2 \;=\; |\underline{Z}^\dagger|^2 \;=\; |\underline{X}|^2 \;=\; |\underline{X}|\,|^2$$

In this setting, a continuously differentiable function g on an open region Ω of $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called a *(left) Hermitean monogenic* (or *h–monogenic*) function in Ω if and only if it simultaneously is $\partial_{\underline{X}}$– and $\partial_{\underline{X}|}$–monogenic in Ω, i.e. it satisfies in Ω the system

$$\partial_{\underline{X}}\, g \;=\; 0 \;=\; \partial_{\underline{X}|}\, g \tag{9.1}$$

or equivalently, the system

$$\partial_{\underline{Z}}\, g \;=\; 0 \;=\; \partial_{\underline{Z}^\dagger}\, g \tag{9.2}$$

As has been mentioned in Chapter 3, the reason to speak of orthogonal Clifford analysis, when considering the function theory centred around the null solutions of the orthogonal Dirac operator $\partial_{\underline{X}}$, is that the underlying group invariance is given by the SO$(2n)$ group, which is doubly covered, see [23], by the Spin group of the Clifford algebra. For the group invariance underlying Hermitean Clifford analysis, one may, seen the splitting of $\partial_{\underline{X}}$ into $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, expect a subgroup of SO$(2n)$ to come into play. Indeed, it has been proven, see [6], that one has to consider the group $\mathrm{SO}_J(2n)$ of all SO$(2n)$ elements commuting with the complex structure J, a group which is isomorphic to the unitary group U(n). This group $\mathrm{SO}_J(2n)$ is doubly covered by the group $\mathrm{Spin}_J(2n)$, also denoted as $\widetilde{\mathrm{U}}(n)$ of all Spin elements commuting with

$$s_J \;=\; s_1 \ldots s_n \;, \qquad \text{where } s_k = \frac{1}{\sqrt{2}}(1 - e_k e_{n+k})\;, \quad k = 1, \ldots, n$$

which is the Spin element corresponding to the complex structure J. A nice and useful characterization of $\widetilde{\mathrm{U}}(n)$ is given by

$$\widetilde{\mathrm{U}}(n) \;=\; \{s \in \mathrm{Spin}(2n) \mid \exists\, \theta \geq 0 : \overline{s}I = \exp(-i\theta)I\}$$

involving the self–adjoint primitive idempotent

$$I \;=\; \mathfrak{f}_1\mathfrak{f}_1^\dagger\,\mathfrak{f}_2\mathfrak{f}_2^\dagger\ldots\mathfrak{f}_n\mathfrak{f}_n^\dagger \tag{9.3}$$

The fact that its associated action leaves the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ invariant will be commented on in more detail in the following section.

Finally, when considering all possible combinations of actions of the orthogonal, respectively Hermitean, Dirac operators on the orthogonal, respectively Hermitean, vector variables, the following Clifford number, the so–called *spin Euler operator*, frequently appears

$$\beta \;=\; \sum_{j=1}^{n}\mathfrak{f}_j^\dagger\mathfrak{f}_j \;=\; \frac{1}{2}\left(n + i\sum_{j=1}^{n} e_j e_{n+j}\right)$$

Lemma 9.1.

(i) For the actions of the orthogonal Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ from the left and the right on the orthogonal vector variables $\underline{X}$ and $\underline{X}|$ we find

$$\partial_{\underline{X}}\,\underline{X} \;=\; \underline{X}\,\partial_{\underline{X}} \;=\; -2n \;=\; \partial_{\underline{X}|}\,\underline{X}| \;=\; \underline{X}|\,\partial_{\underline{X}|}$$

and

$$\partial_{\underline{X}}\,\underline{X}| \;=\; -\underline{X}|\,\partial_{\underline{X}} \;=\; 2i\,(2\beta - n) \;=\; -\partial_{\underline{X}|}\,\underline{X} \;=\; \underline{X}\,\partial_{\underline{X}|} \tag{9.4}$$

(ii) For the actions of the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ from the left and the right on the Hermitean vector variables $\underline{Z}$ and $\underline{Z}^\dagger$ we find

$$\partial_{\underline{Z}}\,\underline{Z} \;=\; n - \underline{Z}\,\partial_{\underline{Z}} \;=\; \beta \;=\; n - \partial_{\underline{Z}^\dagger}\underline{Z}^\dagger \;=\; \underline{Z}^\dagger\,\partial_{\underline{Z}^\dagger}$$

and

$$\partial_{\underline{Z}}\,\underline{Z}^\dagger \;=\; \underline{Z}^\dagger\,\partial_{\underline{Z}} \;=\; 0 \;=\; \partial_{\underline{Z}^\dagger}\underline{Z} \;=\; \underline{Z}\,\partial_{\underline{Z}^\dagger}$$

9.2 Splitting of the h–monogenic system

In Subsection 11.1.4 we will consider functions taking values in the so–called "n–homogeneous part of complex spinor space" of the complex Clifford algebra $\mathbb{C}_{2n}$. In order to arrive in a natural way to that specific subset of $\mathbb{C}_{2n}$, in this subsection the splitting of the h–monogenic system (9.1) is briefly examined. For a more profound study, we refer to [7].

For $j = 1, \ldots, n$ we define the following mutually commuting, self–adjoint idempotents

$$\begin{aligned} I_j &= \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}(1 - i\, e_j e_{n+j}) \\ K_j &= \mathfrak{f}_j^\dagger \mathfrak{f}_j = \frac{1}{2}(1 + i\, e_j e_{n+j}) \end{aligned}$$

for which it moreover holds that

$$I_j + K_j = 1 , \qquad j = 1, \ldots, n$$

and thus

$$\prod_{j=1}^{n}(I_j + K_j) = 1$$

the left-hand side consisting of 2^n terms, each one being a self–adjoint idempotent annihilating all other terms since

$$I_j K_j = K_j I_j = 0 , \qquad j = 1, \ldots, n$$

In this way, a decomposition of the complex Clifford algebra $\mathbb{C}_{2n}$ may be obtained as a direct sum of 2^n components, all of them being mutually isomorphic minimal left ideals, called *complex spinor spaces*:

$$\mathbb{C}_{2n} \equiv \mathbb{C}_{2n}\left(\prod_{j=1}^{n}(I_j + K_j)\right) = \mathbb{C}_{2n}(I_1 \ldots I_n) + \cdots + \mathbb{C}_{2n}(K_1 \ldots K_n) \quad (9.5)$$

If one realizes the Clifford algebra in the usual way through representation by $(2^n \times 2^n)$ matrices with complex entries, each of the terms at the right-hand side of (9.5) will correspond to the subspace of matrices with only one nontrivial column. In [7] it is then shown that the Dirac equation $\partial_{\underline{X}} g = 0$ for $\mathbb{C}_{2n}$ valued functions may be split into 2^n independent subsystems for functions with values in the corresponding 2^n minimal left ideals. All these subsystems are equivalent to each other, so that their solutions will have the same properties. So, this implies that the study of properties of solutions of the Dirac equation for Clifford algebra valued functions reduces to a characterization of the solutions of the same equation, considered for functions with values in a standard model of complex spinor space. In what follows, the standard model which will be considered, is $\mathbb{CS}_n \equiv \mathbb{C}_{2n} I$, where the specific primitive idempotent I (see (9.3)) may now be rewritten as

$$I \; = I_1 \, I_2 \ldots I_n$$

Clearly, all other components in (9.5) may be used as alternative realizations for complex spinor space. As further

$$e_j \, I \;=\; i \, e_{n+j} \, I \;=\; -\mathfrak{f}_j^\dagger I \quad \text{and} \quad \mathfrak{f}_j I = 0 \;, \qquad j = 1, \ldots, n$$

we also have that

$$\mathbb{CS}_n \;\cong\; \mathbb{C}_n I \quad \text{and} \quad \mathbb{CS}_n \;\cong\; (\mathbb{C}\Lambda_n^\dagger) I$$

where $\mathbb{C}_n$ is the complex Clifford algebra generated by $\{e_1, \ldots, e_n\}$ and where $\mathbb{C}\Lambda_n^\dagger$ denotes the Grassmann algebra generated by $\{\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger\}$.

When considering the h–monogenic system (9.2) as a refinement of the Dirac equation, it thus suffices to study only spinor valued solutions as well. Observe that this allows us to formulate the invariance under the action of the unitary group $\mathrm{U}(n)$, or more precisely: of its Clifford realization $\widetilde{\mathrm{U}}(n)$, explicitly in terms of the so–called L–representation $L(s)$ of arbitrary Spin elements s (see e.g. [23]):

$$L(s) \, [g(\underline{X})] \;=\; s \, g(\overline{s} \underline{X} s)$$

We then have, see [34], that

$$\left[\partial_{\underline{Z}}, L(s)\right] \;=\; 0 \quad \text{and} \quad \left[\partial_{\underline{Z}^\dagger}, L(s)\right] \;=\; 0 \;, \qquad s \in \widetilde{\mathrm{U}}(n)$$

The important question to be addressed here is whether it would be possible to split the system (9.2) for functions with values in $\mathbb{CS}_n$ into still smaller subsystems, while preserving the invariance. First, to answer that question, it has been shown in [7] that $\mathbb{CS}_n$, considered as a $\widetilde{\mathrm{U}}(n)$–module, decomposes as

$$\mathbb{CS}_n \;=\; \bigoplus_{j=1}^{n} \mathbb{CS}_n^{(j)} \;=\; \bigoplus_{j=1}^{n} (\mathbb{C}\Lambda_n^\dagger)^{(j)} I \tag{9.6}$$

into the $\widetilde{\mathrm{U}}(n)$–invariant and irreducible subspaces

$$\mathbb{CS}_n^{(j)} \;=\; (\mathbb{C}\Lambda_n^\dagger)^{(j)} I \;, \qquad j = 0, \ldots, n$$

consisting of j–vectors from $\mathbb{C}\Lambda_n^\dagger$ multiplied by the idempotent I. Therefore, the spaces $\mathbb{CS}_n^{(j)}$ are also called the *j–homogeneous parts* of the spinor space $\mathbb{CS}_n$. Then, it was observed that the respective actions, i.e. multiplications from the left, of the Witt basis elements $\mathfrak{f}_k^\dagger$ and $\mathfrak{f}_k$ on the spaces $\mathbb{CS}_n^{(j)}$ behave like creation and annihilation operators respectively, i.e.

$$\mathfrak{f}_k^\dagger : \mathbb{CS}_n^{(j)} \longrightarrow \mathbb{CS}_n^{(j+1)} \ , \qquad k = 1, \ldots, n \ , \ j = 0, \ldots, n$$

and

$$\mathfrak{f}_k : \mathbb{CS}_n^{(j)} \longrightarrow \mathbb{CS}_n^{(j-1)} \ , \qquad k = 1, \ldots, n \ , \ j = 0, \ldots, n$$

where we have put by definition $\mathbb{CS}_n^{(-1)} = \mathbb{CS}_n^{(n+1)} = \{0\}$. Clearly, this eventually leads to the system for h–monogenic spinor valued functions to be split into n independent subsystems for functions with values in the homogeneous parts $\mathbb{CS}_n^{(j)}$, $j = 0, \ldots, n$.

Proposition 9.1. *Let for a function $g : \Omega \subset \mathbb{R}^{2n} \to \mathbb{CS}_n$ denote $g = \sum_{j=0}^n g_j$ its decomposition into its homogeneous spinor parts, i.e. according to the direct sum decomposition (9.6) of $\mathbb{CS}_n$. Then one has*

(i) the system $\partial_{\underline{Z}} g = 0$ is equivalent with the set of subsystems $\partial_{\underline{Z}} g_j = 0$, $j = 0, \ldots, n$

(ii) the system $\partial_{\underline{Z}^\dagger} g = 0$ is equivalent with the set of subsystems $\partial_{\underline{Z}^\dagger} g_j = 0$, $j = 0, \ldots, n$

Note however that the obtained subsystems are not mutually equivalent, so that one has to study all individual values of j when trying to characterize the solutions of the original h–monogenic system.

Finally, we want to draw attention to the important cases corresponding with the values $j = 0$ and $j = n$. In the first case, the function g takes the form

$$g(\underline{Z}, \underline{Z}^\dagger) \ = \ g_0(\underline{Z}, \underline{Z}^\dagger) I$$

where g_0 is a smooth complex valued function on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. For such a function the first equation $\partial_{\underline{Z}^\dagger}[g] = 0$ in the h–monogenic system (9.2) is directly seen to be trivially fulfilled, while the solutions of the remaining equation $\partial_{\underline{Z}}[g] = 0$, or equivalently

$$\partial_{z_k}[g_0] \ = \ 0 \ , \qquad k = 1, \ldots, n$$

are exactly all anti–holomorphic functions g_0 of n complex variables $(z_1, \ldots, z_n)$. Similarly, in the symmetric case where $j = n$, we have

$$g(\underline{Z}, \underline{Z}^\dagger) \;=\; g_n(\underline{Z}, \underline{Z}^\dagger)\mathfrak{f}_1^\dagger \cdots \mathfrak{f}_n^\dagger I$$

where g_n is a smooth complex valued function on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, showing that $\partial_{\underline{Z}}[g] = 0$ will be trivially fulfilled, while the solutions of $\partial_{\underline{Z}^\dagger}[g] = 0$ will be all holomorphic functions g_n of n complex variables $(z_1, \ldots, z_n)$. Those observations are important, since it means that the theory of Hermitean monogenic functions not only refines orthogonal Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, even encompassing some of its results.

Chapter 10

Hermitean Hilbert transforms

While studying Clifford–Hermite wavelets in the context of Hermitean Clifford analysis, see [31, 32], the authors came across a new kind of operator, obtained as the composition of two orthogonal Clifford–Hilbert transforms. The resulting operator, denoted with $\mathcal{K}$, was shown to possess some typical properties of a classical Hilbert transform as well. In our paper [18], we have further extensively investigated this $\mathcal{K}$–transform, reobtaining it as the commutator of two new Hermitean Clifford–Hilbert transforms.

In the first section of this chapter, we introduce, next to the Clifford–Hilbert transform already presented in Chapter 5, a second orthogonal Clifford–Hilbert transform, its kernel being obtained by the action of the complex structure J on the orthogonal Clifford–Hilbert kernel. Through the action of the operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal Clifford–Hilbert kernel we get two new isotropic Hermitean Hilbert transforms. The commutator of the latter transforms then gives rise to a new Hilbert–like operator, the $\mathcal{K}$–transform, which is studied in the second section. Its connections and similarities with the standard Clifford–Hilbert transforms as well as with the newly introduced Hermitean Hilbert transforms are explicitly investigated, and in particular new Hardy spaces associated to this operator are defined and characterized. Some results also allow for a nice geometric interpretation. In the last section the concept of multidimensional analytic signal is revised. The left multiplication with the basis vector e_0

is seen to act as a mapping between the different Hardy spaces considered in this chapter.

10.1 Two isotropic Hermitean Hilbert transforms

In this section, we first construct and study a second Hilbert transform, next to the classical Clifford–Hilbert transform, by – roughly spoken – letting act the complex structure J on the latter transform. Taking then two deliberate linear combinations of both transforms we arrive at two new isotropic Hilbert transforms in Hermitean Clifford analysis.

First we pass to $(2n+1)$–dimensional space by introducing the supplementary unit vector e_0, squaring up to (-1) and orthogonal to all of $(e_j)_{j=1}^{2n}$. The real variable $X = (t, \underline{X})$ in $\mathbb{R}^{2n+1}$ and an *associated variable* $X| = (t, \underline{X}|)$ are then identified with the vectors

$$X \ = \ e_0\, t + \underline{X} \qquad \text{and} \qquad X| \ = \ e_0\, t + \underline{X}|$$

in the real Clifford algebra $\mathbb{R}_{0,2n+1}$. In the same order of ideas, we define a Dirac operator and an *associated Dirac operator* in $\mathbb{R}^{2n+1}$, viz

$$\partial_X \ = \ e_0\, \partial_t + \partial_{\underline{X}} \qquad \text{and} \qquad \partial_{X|} \ = \ e_0\, \partial_t + \partial_{\underline{X}|}$$

Finally, we identify $\mathbb{R}^{2n}$ with the hyperplane $\{(t, \underline{X}) : t = 0\}$ in $\mathbb{R}^{2n+1}$.

In Section 5.1 we already introduced and studied the Clifford–Hilbert transform $\mathcal{H}[f]$ for functions $f \in L_2(\mathbb{R}^{2n})$ in the framework of orthogonal Clifford analysis. In the present setting of Hermitean Clifford analysis, a second Hilbert transform, the so–called *twisted Hilbert transform* $\mathcal{H}|[f]$ of f, was then considered in [32, 1]. Its kernel $H|$ arises naturally by means of the action of the complex structure J on the classical Clifford–Hilbert kernel H, viz

$$H|(\underline{X}) \ = \ J[H(\underline{X})] \ = \ \frac{2}{a_{2n+1}}\, \mathrm{Pv}\, \frac{\overline{\underline{X}|}}{|\underline{X}|^{2n+1}} \ = \ \frac{2}{a_{2n+1}}\, \mathrm{Pv}\, \frac{\overline{\underline{X}|}}{|\underline{X}||^{2n+1}} \ = \ H(J[\underline{X}])$$

and the twisted Hilbert transform itself is then given by

$$\mathcal{H}|[f](\underline{X}) \ = \ \overline{e_0}\, (H| * f)\, (\underline{X})$$

At the same time, one may also define for the ∂_X–monogenic Cauchy integral $\mathcal{C}[f]$ in $\mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$, introduced in Section 5.1, the *associated Cauchy integral*

$$\mathcal{C}|[f](X) \;=\; \mathcal{C}|[f](t,\underline{X}) \;=\; \frac{1}{a_{2n+1}} \int_{\mathbb{R}^{2n}} \frac{t + e_0\,(\underline{X}| - \underline{U}|)}{|t + \underline{X} - \underline{U}|^{2n+1}}\, f(\underline{U})\, dV(\underline{U}) \;, \quad t \neq 0$$

which is monogenic in $\mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$ with respect to the associated Dirac operator $\partial_{X|}$. In upper halfspace $\mathbb{R}^{2n+1}_+ = \{(t,\underline{X}) : t > 0\}$ we may then consider the Hardy space $H|^2(\mathbb{R}^{2n+1}_+)$ of $\partial_{X|}$–monogenic Clifford algebra valued functions F for which

$$\sup_{t>0} \int_{\mathbb{R}^{2n}} |F(t,\underline{X})|^2\, dV(\underline{X}) < +\infty$$

The Hardy space $H|^2(\mathbb{R}^{2n+1}_+)$ entails the Hardy space $H|^2(\mathbb{R}^{2n})$ as the closure in $L_2(\mathbb{R}^{2n})$ of the space of all non–tangential boundary limits for $t \to 0+$ of all functions in $H|^2(\mathbb{R}^{2n+1}_+)$, and moreover, both spaces $H|^2(\mathbb{R}^{2n+1}_+)$ and $H|^2(\mathbb{R}^{2n})$ are isomorphic. The isomorphism is established by means of the Cauchy integral $\mathcal{C}|[f] \in H|^2(\mathbb{R}^{2n+1}_+)$ in the following way: for a given $h \in H|^2(\mathbb{R}^{2n})$ one has

$$\lim_{t \to 0+} \mathcal{C}|[h](t,\underline{X}) \;=\; h(\underline{X})$$

in the L_2 sense of non–tangential boundary limits, such that $\mathcal{C}|[h]$ may be seen as a $\partial_{X|}$–monogenic extension of h in $\mathbb{R}^{2n+1}_+$. More generally, for $f \in L_2(\mathbb{R}^{2n})$, its Cauchy integral $\mathcal{C}|[f]$ still exists and belongs to $H|^2(\mathbb{R}^{2n+1}_+)$ (and in fact also to $H^2(\mathbb{R}^{2n+1}_-)$, defined similarly), but one obtains the Plemelj–Sokhotzki formulae

$$\mathcal{C}|^+[f](\underline{X}) \;\equiv\; \lim_{t \to 0+} \mathcal{C}|[f](t,\underline{X}) \;=\; \frac{1}{2}\, f(\underline{X}) + \frac{1}{2}\, \mathcal{H}|[f](\underline{X}) \qquad (10.1)$$

$$\mathcal{C}|^-[f](\underline{X}) \;\equiv\; \lim_{t \to 0-} \mathcal{C}|[f](t,\underline{X}) \;=\; -\frac{1}{2}\, f(\underline{X}) + \frac{1}{2}\, \mathcal{H}|[f](\underline{X}) \qquad (10.2)$$

for its non–tangential boundary limits, also called *associated Hardy projections*. As $H|^2(\mathbb{R}^{2n})$ is a closed subspace of $L_2(\mathbb{R}^{2n})$, an orthogonal decomposition of the latter space with respect to the inner product

$$\langle f, g \rangle \;=\; \int_{\mathbb{R}^{2n}} f(\underline{X})^\dagger\, g(\underline{X})\, dV(\underline{X})$$

is obtained, viz

$$\begin{array}{ccccc} L_2(\mathbb{R}^{2n}) & = & H|^2(\mathbb{R}^{2n}) & \oplus_\perp & H|^2(\mathbb{R}^{2n})^\perp \\ f & = & \mathbb{P}|^+[f] & + & \mathbb{P}|^-[f] \end{array}$$

where $\mathbb{P}|^+$ and $\mathbb{P}|^-$ are the so–called *associated Szegö projections* on $H|^2(\mathbb{R}^{2n})$ and its orthogonal complement $H|^2(\mathbb{R}^{2n})^\perp$ respectively. These projections may be further explicited in terms of the twisted Hilbert transform, viz

$$\begin{aligned} \mathbb{P}|^+[f] &= \frac{1}{2}(\mathbf{1}+\mathcal{H}|)[f] \\ \mathbb{P}|^-[f] &= \frac{1}{2}(\mathbf{1}-\mathcal{H}|)[f] \end{aligned}$$

Observe that, in the present case of halfspace, the associated Hardy projections coincide with the associated Szegö projections. Explicitly, one has

$$\mathcal{C}|^+[f] \;=\; \mathbb{P}|^+[f] \qquad \text{and} \qquad \mathcal{C}|^-[f] \;=\; -\mathbb{P}|^-[f]$$

The twisted Hilbert transform then shows the usual Clifford–Hilbert transform properties, the proofs of which run along similar lines as the analogous properties in Section 5.1, mutatis mutandis.

Property 10.1.

P(1) The twisted Hilbert transform $\mathcal{H}|$ commutes with translations, which is an equivalent statement to its definition as a convolution operator.

P(2) The twisted Hilbert transform $\mathcal{H}|$ commutes with dilations, which, for a convolution operator, is equivalent to its kernel $H|$ being a homogeneous distribution of degree $(-2n)$.

P(3) The twisted Hilbert transform $\mathcal{H}|$ is a bounded linear operator on $L_2(\mathbb{R}^{2n})$, which is equivalent to its Fourier symbol

$$\mathcal{F}[H|](\underline{X}) \;=\; i\,\frac{\underline{X}|}{|\underline{X}|}$$

being a bounded function.

P(4) The twisted Hilbert transform $\mathcal{H}|$ is involutory on $L_2(\mathbb{R}^{2n})$, i.e. $\mathcal{H}|^2 = \mathbf{1}$.

P(5) The twisted Hilbert transform $\mathcal{H}|$ is self–adjoint on $L_2(\mathbb{R}^{2n})$, i.e.

$$\langle \mathcal{H}|[f], g\rangle \;=\; \langle f, \mathcal{H}|[g]\rangle \;, \qquad f,g \in L_2(\mathbb{R}^{2n})$$

P(6) The twisted Hilbert transform $\mathcal{H}|[f]$ for a function $f \in L_2(\mathbb{R}^{2n})$ arises in a natural way when considering the non–tangential boundary limits (10.1) and (10.2) of the associated Cauchy integral $\mathcal{C}|[f]$ in $\mathbb{R}^{2n+1}$.

P(7) The twisted Hilbert transform $\mathcal{H}|$ anti–commutes with the twisted Dirac operator $\partial_{\underline{X}|}$, i.e. if f and $\partial_{\underline{X}|} f$ are in $L_2(\mathbb{R}^{2n})$, then

$$\mathcal{H}| \left[\partial_{\underline{X}|} f(\underline{X})\right](\underline{Y}) \;=\; -\partial_{\underline{Y}|} \left[\mathcal{H}|[f](\underline{Y})\right]$$

P(8) For $f \in L_2(\mathbb{R}^{2n})$, one has that $\mathcal{H}|[f] = f$ if and only if $f \in H|^2(\mathbb{R}^{2n})$.

A completion of the interrelating picture between the (twisted) Hilbert transform and the (twisted) Dirac operator is given in Property 10.2, which is preceded by following auxiliary result where the spherical mean operator $\Sigma^{(0)}$, introduced in Section 4.2, again pops up.

Lemma 10.1. *If ϕ is a scalar valued test function defined on $\mathbb{R}^{2n}$, then*

$$\begin{aligned}
\Sigma^{(0)} \left[\underline{\Omega}| \left(\partial_{\underline{\Omega}} \phi\right)\right] \;&=\; \Sigma^{(0)} \left[\left(\partial_{\underline{\Omega}|} \phi\right) \underline{\Omega}\right] \\
&=\; 2i\,(2\beta - n)\, \Sigma^{(0)} \left[\phi\right] + 2n\, \Sigma^{(0)} \left[\underline{\Omega}|\, \underline{\Omega}\, \phi\right] \\
&=\; -\Sigma^{(0)} \left[\underline{\Omega} \left(\partial_{\underline{\Omega}|}\, \phi\right)\right] \;=\; -\Sigma^{(0)} \left[\left(\partial_{\underline{\Omega}} \phi\right) \underline{\Omega}|\right]
\end{aligned} \tag{10.3}$$

and

$$\begin{aligned}
\Sigma^{(0)} \left[\underline{\Omega}| \left(\partial_{\underline{X}} \phi\right)\right] \;&=\; \Sigma^{(0)} \left[\left(\partial_{\underline{X}|} \phi\right) \underline{\Omega}\right] \\
&=\; 2i\,(2\beta - n)\, \frac{1}{r}\, \Sigma^{(0)} \left[\phi\right] + \left(\partial_r + \frac{2n}{r}\right) \Sigma^{(0)} \left[\underline{\Omega}|\, \underline{\Omega}\, \phi\right] \\
&=\; -\Sigma^{(0)} \left[\underline{\Omega} \left(\partial_{\underline{X}|} \phi\right)\right] \;=\; -\Sigma^{(0)} \left[\left(\partial_{\underline{X}} \phi\right) \underline{\Omega}|\right]
\end{aligned} \tag{10.4}$$

Proof.
Let ϕ be a scalar valued test function defined on $\mathbb{R}^{2n}$. Consider the ball $B(\underline{0};\rho)$ with arbitrary radius $\rho > 0$ and apply the Clifford–Stokes theorem 5.2 to obtain

$$\begin{aligned}
&\int_{B(\underline{0};\rho)} \left[\underline{X}|\; \phi(\underline{X})\right] \partial_{\underline{X}}\, \widetilde{dV}(\underline{X}) \\
&= \int_{\partial B(\underline{0};\rho)} \underline{X}|\, \phi(\underline{X})\, \widetilde{d\sigma}_{\underline{X}} \;=\; \rho^{2n} \int_{S^{2n-1}} \underline{\Omega}|\, \underline{\Omega}\, \phi(\rho\underline{\Omega})\, dS(\underline{\Omega})
\end{aligned} \tag{10.5}$$

where we have used the spherical decomposition $\underline{X} = r\underline{\Omega}$, and the fact that $\underline{\Omega}$ is the outward pointing unit normal vector to S^{2n-1}. On the other hand, for the left-hand side of (10.5) we also get

$$\int_{B(\underline{0};\rho)} \left[\left(\underline{X}|\, \partial_{\underline{X}}\right) \phi(\underline{X}) + \underline{X}|\, \left(\partial_{\underline{X}}\, \phi(\underline{X})\right)\right] \widetilde{dV}(\underline{X})$$

Then, taking into account (9.4), the first term in the above expression equals

$$-2i\,(2\beta-n)\int_0^\rho r^{2n-1}\,dr\int_{S^{2n-1}}\phi(r\underline{\Omega})\,dS(\underline{\Omega})$$

while introducing the spherical decomposition (3.1) of the orthogonal Dirac operator and applying integration by parts, the second term equals

$$\begin{aligned}
&\int_0^\rho r^{2n}\,dr\int_{S^{2n-1}}\underline{\Omega}|\left(\underline{\Omega}\,\partial_r\phi(r\underline{\Omega})+\frac{1}{r}\,\partial_{\underline{\Omega}}\,\phi(r\underline{\Omega})\right)dS(\underline{\Omega})\\
&= \rho^{2n}\int_{S^{2n-1}}\underline{\Omega}|\,\underline{\Omega}\,\phi(\rho\underline{\Omega})\,dS(\underline{\Omega})-2n\int_0^\rho r^{2n-1}\,dr\int_{S^{2n-1}}\underline{\Omega}|\,\underline{\Omega}\,\phi(r\underline{\Omega})\,dS(\underline{\Omega})\\
&\quad+\int_0^\rho r^{2n-1}\,dr\int_{S^{2n-1}}\underline{\Omega}|\,\left(\partial_{\underline{\Omega}}\,\phi(r\underline{\Omega})\right)\,dS(\underline{\Omega})
\end{aligned}$$

such that

$$\begin{aligned}
&\int_{B(\underline{0};\rho)}[\underline{X}|\;\phi(\underline{X})]\,\partial_{\underline{X}}\,\widetilde{dV}(\underline{X})\\
&= \rho^{2n}\int_{S^{2n-1}}\underline{\Omega}|\,\underline{\Omega}\,\phi(\rho\underline{\Omega})\,dS(\underline{\Omega})+\int_0^\rho r^{2n-1}\,dr\left[(-2n)\int_{S^{2n-1}}\underline{\Omega}|\,\underline{\Omega}\,\phi(r\underline{\Omega})\,dS(\underline{\Omega})\right.\\
&\quad\left.+\int_{S^{2n-1}}\underline{\Omega}|\,\left(\partial_{\underline{\Omega}}\,\phi(r\underline{\Omega})\right)\,dS(\underline{\Omega})-2i\,(2\beta-n)\int_{S^{2n-1}}\phi(r\underline{\Omega})\,dS(\underline{\Omega})\right]
\end{aligned}$$

As the first term in the last expression equals the right-hand side of the above Stokes formula (10.5), we arrive at

$$\begin{aligned}
a_{2n}\,\Sigma^{(0)}\left[\underline{\Omega}|\,\left(\partial_{\underline{\Omega}}\phi\right)\right] &= \int_{S^{2n-1}}\underline{\Omega}|\,\left(\partial_{\underline{\Omega}}\,\phi(r\underline{\Omega})\right)\,dS(\underline{\Omega})\\
&= 2i\,(2\beta-n)\int_{S^{2n-1}}\phi(r\underline{\Omega})\,dS(\underline{\Omega})+2n\int_{S^{2n-1}}\underline{\Omega}|\,\underline{\Omega}\,\phi(r\underline{\Omega})\,dS(\underline{\Omega})\\
&= a_{2n}\,2i\,(2\beta-n)\,\Sigma^{(0)}\,[\phi]+a_{2n}\,2n\,\Sigma^{(0)}\,[\underline{\Omega}|\,\underline{\Omega}\,\phi]
\end{aligned}$$

This already proofs one of the equalities in (10.3). The proofs for the other equalities run along similar lines. Next, one has

$$\begin{aligned}
a_{2n}\,\Sigma^{(0)}\left[\underline{\Omega}|\,\left(\partial_{\underline{X}}\phi\right)\right] &= \int_{S^{2n-1}}\underline{\Omega}|\left(\underline{\Omega}\,\partial_r\phi(r\underline{\Omega})+\frac{1}{r}\,\partial_{\underline{\Omega}}\phi(r\underline{\Omega})\right)dS(\underline{\Omega})\\
&= a_{2n}\,\partial_r\Sigma^{(0)}\,[\underline{\Omega}|\,\underline{\Omega}\,\phi]+a_{2n}\,\frac{1}{r}\,\Sigma^{(0)}\left[\underline{\Omega}|\,\left(\partial_{\underline{\Omega}}\phi\right)\right]\\
&= a_{2n}\,2i\,(2\beta-n)\,\frac{1}{r}\,\Sigma^{(0)}\,[\phi]+a_{2n}\left(\partial_r+\frac{2n}{r}\right)\Sigma^{(0)}\,[\underline{\Omega}|\,\underline{\Omega}\,\phi]
\end{aligned}$$

the last step on account of (10.3). This already proves one of the equalities in (10.4). The proofs for the other equalities run along similar lines. ■

Property 10.2.

(i) The Hilbert transform $\mathcal{H}$ commutes with the twisted Dirac operator $\partial_{\underline{X}|}$, i.e. if f and $\partial_{\underline{X}|} f$ are in $L_2(\mathbb{R}^{2n})$, then

$$\mathcal{H}\left[\partial_{\underline{X}|} f(\underline{X})\right](\underline{Y}) \;=\; \partial_{\underline{Y}|}\left[\mathcal{H}[f](\underline{Y})\right]$$

(ii) The twisted Hilbert transform $\mathcal{H}|$ commutes with the Dirac operator $\partial_{\underline{X}}$, i.e. if f and $\partial_{\underline{X}} f$ are in $L_2(\mathbb{R}^{2n})$, then

$$\mathcal{H}|\left[\partial_{\underline{X}} f(\underline{X})\right](\underline{Y}) \;=\; \partial_{\underline{Y}}\left[\mathcal{H}|[f](\underline{Y})\right]$$

Proof.
We only prove (i), the proof of (ii) running along similar lines. Let ϕ be a scalar valued test function defined on $\mathbb{R}^{2n}$. Taking into account (10.4), we find in distributional sense that the convolution kernel H and the twisted Dirac operator $\partial_{\underline{X}|}$ anti–commute:

$$\begin{aligned}
\left\langle\, H(\underline{X})\, \partial_{\underline{X}|}\;,\; \phi(\underline{X}) \,\right\rangle \;&=\; \frac{2}{a_{2n+1}} \left\langle\, U^{*}_{-2n}\;,\; \partial_{\underline{X}|}\, \phi(\underline{X}) \,\right\rangle \\
&=\; \frac{2a_{2n}}{a_{2n+1}} \left\langle\, \mathrm{Fp}\, r_{+}^{-1}\;,\; \Sigma^{(0)}\left[\underline{\Omega}\left(\partial_{\underline{X}|}\phi(\underline{X})\right)\right] \right\rangle \\
&=\; -\frac{2a_{2n}}{a_{2n+1}} \left\langle\, \mathrm{Fp}\, r_{+}^{-1}\;,\; \Sigma^{(0)}\left[\left(\partial_{\underline{X}|}\phi(\underline{X})\right)\underline{\Omega}\right] \right\rangle \\
&=\; -\frac{2}{a_{2n+1}} \left\langle\, \partial_{\underline{X}|}\, \phi(\underline{X})\;,\; U^{*}_{-2n} \,\right\rangle \;=\; -\left\langle\, \phi(\underline{X})\;,\; \partial_{\underline{X}|}\, H(\underline{X}) \,\right\rangle
\end{aligned}$$

The desired result then immediately follows:

$$\begin{aligned}
\mathcal{H}\left[\partial_{\underline{X}|} f(\underline{X})\right](\underline{Y}) \;&=\; \overline{e_0}\,\left(H(\underline{X})\, \partial_{\underline{X}|} * f(\underline{X})\right)(\underline{Y}) \\
&=\; -\overline{e_0}\,\left(\partial_{\underline{X}|}\, H(\underline{X}) * f(\underline{X})\right)(\underline{Y}) \;=\; \partial_{\underline{Y}|}\,\left[\mathcal{H}[f](\underline{Y})\right]
\end{aligned}$$

■

As a last step towards the definition of new Hermitean Hilbert transforms, the following lemma is crucial (see also [32]).

Lemma 10.2. *The Hilbert transforms $\mathcal{H}$ and $\mathcal{H}|$ anti–commute.*

Proof.
From the observation in frequency space that

$$\begin{aligned}\mathcal{F}\left\{\mathcal{H}\left[\mathcal{H}|[f]\right]\right\}(\underline{X}) &= \overline{e_0}\, i\, \frac{\underline{X}}{|\underline{X}|}\, \mathcal{F}\left[\mathcal{H}|[f]\right](\underline{X}) \;=\; -\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^2}\,\mathcal{F}[f](\underline{X}) \\ &= \frac{\underline{X}|\underline{X}}{|\underline{X}|^2}\,\mathcal{F}[f](\underline{X}) \;=\; -\mathcal{F}\left\{\mathcal{H}|\left[\mathcal{H}[f]\right]\right\}(\underline{X})\end{aligned}$$

we derive that $\mathcal{H}\,\mathcal{H}| \;=\; -\mathcal{H}|\,\mathcal{H}$. ∎

In previous chapter, we introduced the Hermitean Clifford–variables $\underline{Z}$ and $\underline{Z}^{\dagger}$ by means of the action of the projection operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal Clifford–vector $\underline{X}$. In the same order of ideas we now introduce two Hermitean Hilbert transforms on $L_2(\mathbb{R}^{2n})$ by letting act the same projection operators on the Clifford–Hilbert transform $\mathcal{H}$ where the action is in fact defined on the convolution kernel H, resulting into

$$\mathfrak{H} \;=\; \frac{1}{2}\,(\mathbf{1} + i\,J)[\mathcal{H}] \;=\; \frac{1}{2}\,(\mathcal{H} + i\,\mathcal{H}|) \tag{10.6}$$

$$\mathfrak{H}^{\dagger} \;=\; -\frac{1}{2}\,(\mathbf{1} - i\,J)[\mathcal{H}] \;=\; -\frac{1}{2}\,(\mathcal{H} - i\,\mathcal{H}|) \tag{10.7}$$

or more explicitly, their action on a function $f \in L_2(\mathbb{R}^{2n})$ is given by

$$\begin{aligned}\mathfrak{H}[f] &= e_0\,\frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\underline{Z}}{r^{2n+1}} * f \\ \mathfrak{H}^{\dagger}[f] &= e_0\,\frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\underline{Z}^{\dagger}}{r^{2n+1}} * f\end{aligned}$$

with $r \,=\, |\underline{Z}| \,=\, |\underline{Z}^{\dagger}| \,=\, |\underline{X}| \,=\, |\underline{X}||$. We list a number of properties of these Hermitean Hilbert transforms, the proofs of which follow directly when taking into account their definitions (10.6) and (10.7), and the properties of the Hilbert transforms $\mathcal{H}$ and $\mathcal{H}|$.

Property 10.3.

P(1) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^{\dagger}$ commute with translations, which is an equivalent statement to their definition as convolution operators.*

P(2) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ commute with dilations, which, for convolution operators, is equivalent to their kernels being homogeneous distributions of degree $(-2n)$.*

P(3) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ are bounded linear operators on $L_2(\mathbb{R}^{2n})$, which is equivalent to their Fourier symbols*

$$\begin{aligned}\mathcal{F}\left[\frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\underline{Z}}{r^{2n+1}}\right](\underline{Z},\underline{Z}^\dagger) &= -i\,\frac{\underline{Z}}{|\underline{Z}|}\\ \mathcal{F}\left[\frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\underline{Z}^\dagger}{r^{2n+1}}\right](\underline{Z},\underline{Z}^\dagger) &= -i\,\frac{\underline{Z}^\dagger}{|\underline{Z}|}\end{aligned}$$

being bounded functions.

P(4) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ are isotropic on $L_2(\mathbb{R}^{2n})$, i.e. $\mathfrak{H}^2 = \mathbf{0}$ and $(\mathfrak{H}^\dagger)^2 = \mathbf{0}$.*

P(5) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ are self–adjoint on $L_2(\mathbb{R}^{2n})$, i.e. for $f, g \in L_2(\mathbb{R}^{2n})$ one has*

$$\langle \mathfrak{H}[f], g\rangle = \langle f, \mathfrak{H}[g]\rangle \qquad \text{and} \qquad \left\langle \mathfrak{H}^\dagger[f], g\right\rangle = \left\langle f, \mathfrak{H}^\dagger[g]\right\rangle$$

P(6) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ are interrelated in the following way:*

$$\mathfrak{H}\,\mathfrak{H}^\dagger + \mathfrak{H}^\dagger\,\mathfrak{H} = -\mathbf{1} \tag{10.8}$$

$$\mathfrak{H}\,\mathfrak{H}^\dagger - \mathfrak{H}^\dagger\,\mathfrak{H} = i\,\mathcal{H}\,\mathcal{H}| \tag{10.9}$$

P(7) *The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ on the one hand and the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ on the other hand are interrelated in the following way:*

$$\begin{aligned}\mathfrak{H}\,\partial_{\underline{Z}} + \partial_{\underline{Z}^\dagger}\,\mathfrak{H}^\dagger &= \mathbf{0} = \mathfrak{H}\,\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}}\,\mathfrak{H}^\dagger\\ \partial_{\underline{Z}}\,\mathfrak{H} + \mathfrak{H}^\dagger\,\partial_{\underline{Z}^\dagger} &= \mathbf{0} = \partial_{\underline{Z}^\dagger}\,\mathfrak{H} + \mathfrak{H}^\dagger\,\partial_{\underline{Z}}\end{aligned}$$

Observe in particular property P(4) expressing the isotropy of the Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$, and expression (10.9). Precisely these results will play an important role in the introduction of a new Hilbert type transform in the Hermitean Clifford analysis setting.

10.2 The $\mathcal{K}$–transform

In [32] the following new Hilbert type transform has been introduced:

$$\mathcal{K} \;=\; i\,\mathcal{H}\,\mathcal{H}| \;=\; -i\,\mathcal{H}|\,\mathcal{H} \tag{10.10}$$

As the operator $\mathcal{K}$ results from the composition of two convolution operators, it is itself a convolution operator, i.e.

$$\mathcal{K}[f] \;=\; K * f \tag{10.11}$$

A precise calculation of its kernel $K(\underline{X})$ is obtained in the following proposition.

Proposition 10.1.

$$K(\underline{X}) \;=\; \frac{(n-1)!}{\pi^n}\left[\frac{a_{2n}}{2n}\,(2\beta-n)\,\delta(\underline{X}) + i\,n\,\mathrm{Fp}\,\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^{2n+2}} + (2\beta-n)\,\mathrm{Fp}\,\frac{1}{|\underline{X}|^{2n}}\right]$$

Proof.
As the $\mathcal{K}$–transform (10.10) of a function $f \in L_2(\mathbb{R}^m)$ may be written as

$$\mathcal{K}[f] \;=\; i\,\overline{e_0}\,H * (\overline{e_0}\,H| * f) \;=\; (i\,H * H|) * f$$

its convolution kernel K is given by

$$K \;=\; i\,H * H|$$

In order now to calculate the above distributional convolution, we first rewrite the twisted Hilbert kernel $H|$, making use of Proposition 4.10 (i), but for the twisted Dirac operator $\partial_{\underline{X}|}$, viz

$$H|(\underline{X}) \;=\; \frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\overline{\underline{X}|}}{|\underline{X}|^{2n+1}} \;=\; \frac{2}{a_{2n+1}}\,\frac{1}{2n-1}\,T^*_{-2n+1}\,\partial_{\underline{X}|}$$

So, for the kernel K we already find

$$K(\underline{X}) \;=\; -\frac{1}{(a_{2n+1})^2}\,\frac{4i}{2n-1}\,\left(U^*_{-2n} * T^*_{-2n+1}\right)\partial_{\underline{X}|} \;=\; -\frac{i\,(n-1)!}{2\pi^{n+1}}\,U^*_{-2n+1}\,\partial_{\underline{X}|}$$

the last step on account of (4.14). Next, let ϕ be a scalar valued test function defined on $\mathbb{R}^{2n}$. Then, taking into account (10.4), one has that

$$\begin{aligned}
\left\langle\, U^*_{-2n+1}\,\partial_{\underline{X}|}\ ,\ \phi(\underline{X})\,\right\rangle \ &=\ -\pi\,\left\langle\, U_{-2n+1}\ ,\ \partial_{\underline{X}|}\,\phi(\underline{X})\,\right\rangle\\
&=\ -\pi\,a_{2n}\left\langle\ \mathrm{Fp}\,r_+^0\ ,\ \Sigma^{(0)}\left[\underline{\Omega}\left(\partial_{\underline{X}|}\phi\right)\right]\ \right\rangle\\
&=\ -\pi\,a_{2n}\left\langle\ \mathrm{Fp}\,r_+^0\ ,\ -2i\,(2\beta-n)\,\frac{1}{r}\,\Sigma^{(0)}\left[\phi\right]+\left(\partial_r+\frac{2n}{r}\right)\Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]\ \right\rangle\\
&=\ -\pi\,a_{2n}\left\langle\ \mathrm{Fp}\,r_+^0\ ,\ \partial_r\Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]\ \right\rangle \qquad (10.12)\\
&\quad -\pi\,a_{2n}\left\langle\ \mathrm{Fp}\,r_+^0\ ,\ \frac{1}{r}\,\left(2n\,\Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]-2i\,(2\beta-n)\,\Sigma^{(0)}\left[\phi\right]\right)\ \right\rangle \qquad (10.13)
\end{aligned}$$

In order to compute the first term (10.12) of the right-hand side, let $\underline{\Xi}$ and $\underline{\Xi}^\dagger$ be the Hermitean counterparts of $\underline{\Omega}$ and $\underline{\Omega}|$. Then it was shown in [33] that

$$\int_{S^{2n-1}}\underline{\Omega}\,\underline{\Omega}|\,dS(\underline{\Omega})\ =\ i\int_{S^{2n-1}}\left(2\,\underline{\Xi}^\dagger\,\underline{\Xi}-1\right)\,dS(\underline{\Omega})\ =\ \frac{i}{n}\,a_{2n}\,(2\beta-n)$$

leading to

$$\begin{aligned}
&\left\langle\ \mathrm{Fp}\,r_+^0\ ,\ \partial_r\Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]\ \right\rangle\ =\ -\left\langle\ \partial_r\mathrm{Fp}\,r_+^0\ ,\ \Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]\ \right\rangle\\
&\quad=\ -\left\langle\ \delta(r)\ ,\ \Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]\ \right\rangle\ =\ -\frac{1}{a_{2n}}\left(\int_{S^{2n-1}}\underline{\Omega}\,\underline{\Omega}|\,dS(\underline{\Omega})\right)\phi(\underline{0})\\
&\quad=\ -\frac{i}{n}\,(2\beta-n)\,\langle\ \delta(\underline{X})\ ,\ \phi(\underline{X})\ \rangle
\end{aligned}$$

For the second term (10.13) of the right-hand side we notice that

$$\begin{aligned}
&2n\,\Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right](0)-2i\,(2\beta-n)\,\Sigma^{(0)}\left[\phi\right](0)\\
&\quad=\ \frac{2n}{a_{2n}}\left(\int_{S^{2n-1}}\underline{\Omega}\,\underline{\Omega}|\,dS(\underline{\Omega})\right)\phi(\underline{0})-2i\,(2\beta-n)\,\phi(\underline{0})\ =\ 0
\end{aligned}$$

such that for the specific function

$$\psi(r)\ =\ 2n\,\Sigma^{(0)}\left[\underline{\Omega}\,\underline{\Omega}|\,\phi\right]-2i\,(2\beta-n)\,\Sigma^{(0)}\left[\phi\right]$$

we may write

$$\left\langle\ \mathrm{Fp}\,r_+^0\ ,\ \frac{1}{r}\,\psi(r)\ \right\rangle\ =\ \left\langle\ \mathrm{Fp}\,r_+^{-1}\ ,\ \psi(r)\ \right\rangle$$

on account of [26, Lemma 1.1]. This leads to

$$\left\langle \operatorname{Fp} r_+^0 \ , \ \frac{1}{r} \left(2n\, \Sigma^{(0)} \left[\underline{\Omega}\, \underline{\Omega}|\, \phi \right] - 2i\, (2\beta - n)\, \Sigma^{(0)} \left[\phi \right] \right) \right\rangle$$

$$= 2n \left\langle \operatorname{Fp} r_+^{-1} \ , \ \Sigma^{(0)} \left[\underline{\Omega}\, \underline{\Omega}|\, \phi \right] \right\rangle - 2i\, (2\beta - n) \left\langle \operatorname{Fp} r_+^{-1} \ , \ \Sigma^{(0)} \left[\phi \right] \right\rangle$$

$$= \frac{2n}{a_{2n}} \left\langle \operatorname{Fp} \frac{\underline{X}\, \underline{X}|}{|\underline{X}|^{2n+2}} \ , \ \phi(\underline{X}) \right\rangle - \frac{2i}{a_{2n}} (2\beta - n) \left\langle \operatorname{Fp} \frac{1}{|\underline{X}|^{2n}} \ , \ \phi(\underline{X}) \right\rangle$$

Hence we get

$$U^*_{-2n+1}\, \partial_{\underline{X}|} \ = \ 2\pi i \left[\frac{a_{2n}}{2n} (2\beta - n)\, \delta(\underline{X}) + i\, n \operatorname{Fp} \frac{\underline{X}\, \underline{X}|}{|\underline{X}|^{2n+2}} + (2\beta - n) \operatorname{Fp} \frac{1}{|\underline{X}|^{2n}} \right]$$

such that finally

$$K(\underline{X}) \ = \ \frac{(n-1)!}{\pi^n} \left[\frac{a_{2n}}{2n} (2\beta - n)\, \delta(\underline{X}) + i\, n \operatorname{Fp} \frac{\underline{X}\underline{X}|}{|\underline{X}|^{2n+2}} + (2\beta - n) \operatorname{Fp} \frac{1}{|\underline{X}|^{2n}} \right]$$

■

Moreover, when taking now into account (10.9), the $\mathcal{K}$–transform may be immediately seen as the commutator of the isotropic Hilbert operators $\mathfrak{H}$ and $\mathfrak{H}^\dagger$, viz

$$\mathcal{K} \ = \ \left[\mathfrak{H}, \mathfrak{H}^\dagger \right] \ = \ \mathfrak{H}\, \mathfrak{H}^\dagger - \mathfrak{H}^\dagger\, \mathfrak{H} \tag{10.14}$$

The following characteristic properties of the $\mathcal{K}$–transform may then justify that this transform is indeed a new Hilbert–like operator. Unfortunately, we have not succeeded in constructing an h–monogenic Cauchy–like integral, the non–tangential boundary limits of which give rise to the $\mathcal{K}$–transform. Indeed, such a Hermitean Cauchy integral would have to be defined on $\mathbb{R}^{2n+2}$ (causing a jump of two dimensions at once), since the Hermitean framework requires all involved vector spaces to be even dimensional. Moreover, it is by no means clear how to construct a mutual fundamental solution of both Hermitean Dirac operators, which would then act as a Cauchy kernel.

Property 10.4.

P(1) *The $\mathcal{K}$–transform commutes with translations, which is an equivalent statement to its definition as a convolution operator.*

P(2) *The $\mathcal{K}$–transform commutes with dilations, which, for a convolution operator, is equivalent to its kernel K being a homogeneous distribution of degree $(-2n)$.*

P(3) *The $\mathcal{K}$–transform is a bounded linear operator on $L_2(\mathbb{R}^{2n})$, which is equivalent to its Fourier symbol*

$$\mathcal{F}\left[K\right](\underline{X}) \;=\; i\,\frac{\underline{X}|\,\underline{X}}{|\underline{X}|^2}$$

being a bounded function.

P(4) *The $\mathcal{K}$–transform is involutory on $L_2(\mathbb{R}^{2n})$, i.e. $\mathcal{K}^2 = \mathbf{1}$.*

P(5) *The $\mathcal{K}$–transform is self–adjoint on $L_2(\mathbb{R}^{2n})$, i.e.*

$$\langle \mathcal{K}[f], g\rangle \;=\; \langle f, \mathcal{K}[g]\rangle \;, \qquad f, g \in L_2(\mathbb{R}^{2n})$$

P(6) *The $\mathcal{K}$–transform anti–commutes with both $\mathcal{H}$ and $\mathcal{H}|$, i.e.*

$$\mathcal{H}\,\mathcal{K} + \mathcal{K}\,\mathcal{H} \;=\; \mathbf{0} \qquad \textit{and} \qquad \mathcal{H}|\,\mathcal{K} + \mathcal{K}\,\mathcal{H}| \;=\; \mathbf{0}$$

P(7) *The $\mathcal{K}$–transform anti–commutes with both the orthogonal Dirac operator $\partial_{\underline{X}}$ and the twisted Dirac operator $\partial_{\underline{X}|}$, i.e. if f, $\partial_{\underline{X}} f$ and $\partial_{\underline{X}|} f$ are in $L_2(\mathbb{R}^{2n})$, then*

$$\begin{aligned}\mathcal{K}\left[\partial_{\underline{X}} f(\underline{X})\right](\underline{Y}) &= -\partial_{\underline{Y}}\left[\mathcal{K}[f](\underline{Y})\right]\\ \mathcal{K}\left[\partial_{\underline{X}|} f(\underline{X})\right](\underline{Y}) &= -\partial_{\underline{Y}|}\left[\mathcal{K}[f](\underline{Y})\right]\end{aligned}$$

Proof.
Properties P(1) and P(2) immediately follow from the definition (10.11) of the $\mathcal{K}$–transform as convolution operator and the specific expression for its kernel K, calculated in Proposition 10.1. Properties P(3), P(4) and P(5) were already proven in [32, Lemma 2.1]. For the proof of properties P(6) and P(7) we make use of the definition (10.10) of the $\mathcal{K}$–transform. Property P(6) then directly follows when taking into account that the Hilbert transforms $\mathcal{H}$ and $\mathcal{H}|$ are involutions. Recalling Property 5.6, Property 10.1 P(7) and Property 10.2, viz

$$\begin{aligned}\mathcal{H}\,\partial_{\underline{X}} + \partial_{\underline{X}}\,\mathcal{H} &= \mathbf{0} \qquad & \mathcal{H}\,\partial_{\underline{X}|} - \partial_{\underline{X}|}\mathcal{H} &= \mathbf{0}\\ \mathcal{H}|\partial_{\underline{X}|} + \partial_{\underline{X}|}\mathcal{H}| &= \mathbf{0} \qquad & \mathcal{H}|\partial_{\underline{X}} - \partial_{\underline{X}}\,\mathcal{H}| &= \mathbf{0}\end{aligned}$$

property P(7) may be shown in a straightforward way. ■

Corollary 10.1. *For the Hardy spaces $H^2(\mathbb{R}^{2n})$ and $H|^2(\mathbb{R}^{2n})$, defined in terms of the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ respectively, one has*

$$H^2(\mathbb{R}^{2n}) \cap H|^2(\mathbb{R}^{2n}) \ = \ \{0\}$$

Proof.
Let g belong to both Hardy spaces. It then should hold simultaneously that $\mathcal{H}[g] = g$ and $\mathcal{H}|[g] = g$, from which we infer that

$$\mathcal{K}[g] \ = \ i\,\mathcal{H}\,\mathcal{H}|[g] \ = \ i\,\mathcal{H}[g] \ = \ i\,g$$

Hence

$$\mathcal{K}^2[g] \ = \ \mathcal{K}[ig] \ = \ i\,\mathcal{K}[g] \ = \ -g$$

However, as $\mathcal{K}^2 = \mathbf{1}$, we also have that $\mathcal{K}^2[g] = g$, whence $g = 0$. ■

Corollary 10.2. *The operators $\frac{1}{2}(\mathbf{1} \pm \mathcal{K})$ are projection operators on $L_2(\mathbb{R}^{2n})$.*

Proof.
One can immediately check that

$$\left(\frac{1}{2}\,(\mathbf{1} \pm \mathcal{K})\right)^2 \ = \ \frac{1}{2}\,(\mathbf{1} \pm \mathcal{K})$$

and that

$$\frac{1}{2}\,(\mathbf{1} + \mathcal{K})\,\frac{1}{2}\,(\mathbf{1} - \mathcal{K}) \ = \ \frac{1}{2}\,(\mathbf{1} - \mathcal{K})\,\frac{1}{2}\,(\mathbf{1} + \mathcal{K}) \ = \ \mathbf{0}$$

■

In this way a new orthogonal decomposition of $L_2(\mathbb{R}^{2n})$ is obtained. Indeed, putting

$$\begin{aligned} K^2(\mathbb{R}^{2n}) \ &= \ \frac{1}{2}\,(\mathbf{1} + \mathcal{K})[L_2(\mathbb{R}^{2n})] \\ K^2(\mathbb{R}^{2n})^{\perp} \ &= \ \frac{1}{2}\,(\mathbf{1} - \mathcal{K})[L_2(\mathbb{R}^{2n})] \end{aligned}$$

we obtain

$$\begin{array}{ccccc} L_2(\mathbb{R}^{2n}) & = & K^2(\mathbb{R}^{2n}) & \oplus_{\perp} & K^2(\mathbb{R}^{2n})^{\perp} \\ f & = & \frac{1}{2}\,(\mathbf{1} + \mathcal{K})[f] & + & \frac{1}{2}\,(\mathbf{1} - \mathcal{K})[f] \end{array}$$

with

$$\left\langle \frac{1}{2}\,(\mathbf{1} + \mathcal{K})[f], \frac{1}{2}\,(\mathbf{1} - \mathcal{K})[f] \right\rangle \ = \ 0$$

Moreover, the closed subspaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ may be characterized in a similar way as the traditional Hardy spaces.

Proposition 10.2. *For $f \in L_2(\mathbb{R}^{2n})$, one has that*

(i) $\mathcal{K}[f] = f$ *if and only if* $f \in K^2(\mathbb{R}^{2n})$

(ii) $\mathcal{K}[f] = -f$ *if and only if* $f \in K^2(\mathbb{R}^{2n})^\perp$

(iii) $\mathcal{K}[f] = -i\,\mathcal{H}|[f]$ *if and only if* $f \in H^2(\mathbb{R}^{2n})$

(iv) $\mathcal{K}[f] = i\,\mathcal{H}|[f]$ *if and only if* $f \in H^2(\mathbb{R}^{2n})^\perp$

(v) $\mathcal{K}[f] = i\,\mathcal{H}[f]$ *if and only if* $f \in H|^2(\mathbb{R}^{2n})$

(vi) $\mathcal{K}[f] = -i\,\mathcal{H}[f]$ *if and only if* $f \in H|^2(\mathbb{R}^{2n})^\perp$

Then, it directly follows from (10.8) and the definition (10.14) of $\mathcal{K}$ that the projection operators defining the subspaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ may be expressed in terms of the Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$.

Proposition 10.3. *One has*

$$\frac{1}{2}\,(\mathbf{1} + \mathcal{K}) \;=\; -\mathfrak{H}^\dagger\,\mathfrak{H} \qquad \textit{and} \qquad \frac{1}{2}\,(\mathbf{1} - \mathcal{K}) \;=\; -\mathfrak{H}\,\mathfrak{H}^\dagger$$

Quite remarkable, however, is the following observation. The Hilbert transforms $\mathcal{H}$ and $\mathcal{H}|$, as well as the operator $\mathcal{K}$ are bijective on $L_2(\mathbb{R}^{2n})$, since they are bounded linear operators on $L_2(\mathbb{R}^{2n})$ with $\mathcal{H}^{-1} = \mathcal{H}$, $\mathcal{H}|^{-1} = \mathcal{H}|$ and $\mathcal{K}^{-1} = \mathcal{K}$. The Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ on the contrary can not be injective since they are isotropic, so their kernels should be nontrivial; they are determined in the following proposition.

Proposition 10.4. *One has*

$$\operatorname{Ker}\mathfrak{H}^\dagger \;=\; K^2(\mathbb{R}^{2n}) \qquad \textit{and} \qquad \operatorname{Ker}\mathfrak{H} \;=\; K^2(\mathbb{R}^{2n})^\perp$$

Proof.
If $f \in \operatorname{Ker}\mathfrak{H}^\dagger$, then it holds that

$$\frac{1}{2}\,(\mathbf{1} - \mathcal{K})[f] \;=\; -\mathfrak{H}\,\mathfrak{H}^\dagger[f] \;=\; 0$$

or $\mathcal{K}[f] = f$, whence $f \in K^2(\mathbb{R}^{2n})$. Conversely, if $f \in K^2(\mathbb{R}^{2n})$, then

$$f \;=\; \mathcal{K}[f] \;=\; i\,\mathcal{H}\,\mathcal{H}|[f]$$

from which it follows that

$$\mathcal{H}[f] \;=\; i\,\mathcal{H}^2\,\mathcal{H}|[f] \;=\; i\,\mathcal{H}|[f]$$

whence $\mathfrak{H}^\dagger[f] = -\frac{1}{2}(\mathcal{H}[f] - i\,\mathcal{H}|[f]) = 0$. A similar argument may be applied to $\operatorname{Ker}\mathfrak{H}$. ■

Summarizing, we obtain the following characterizations of the Hardy–like spaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$.

Theorem 10.1.

(a) A function $f \in L_2(\mathbb{R}^{2n})$ belongs to $K^2(\mathbb{R}^{2n})$ if and only if one of the following conditions is satisfied:

(i) $\mathcal{K}[f] = f$

(ii) $\mathfrak{H}^\dagger[f] = 0$

(iii) $\mathfrak{H}[f] = \mathcal{H}[f] = i\,\mathcal{H}|[f]$

(b) A function $f \in L_2(\mathbb{R}^{2n})$ belongs to $K^2(\mathbb{R}^{2n})^\perp$ if and only if one of the following conditions is satisfied:

(i) $\mathcal{K}[f] = -f$

(ii) $\mathfrak{H}[f] = 0$

(iii) $\mathfrak{H}^\dagger[f] = -\mathcal{H}[f] = i\,\mathcal{H}|[f]$

Corollary 10.3. *For $k, \ell \in L_2(\mathbb{R}^{2n})$, one has that*

(i) $k \in K^2(\mathbb{R}^{2n})$ if and only if $\mathcal{H}[k] \in K^2(\mathbb{R}^{2n})^\perp$

(ii) $k \in K^2(\mathbb{R}^{2n})$ if and only if $\mathcal{H}|[k] \in K^2(\mathbb{R}^{2n})^\perp$

(iii) $\ell \in K^2(\mathbb{R}^{2n})^\perp$ if and only if $\mathcal{H}[\ell] \in K^2(\mathbb{R}^{2n})$

(iv) $\ell \in K^2(\mathbb{R}^{2n})^\perp$ if and only if $\mathcal{H}|[\ell] \in K^2(\mathbb{R}^{2n})$

Proof.
We only prove property (i), the proofs of (ii), (iii) and (iv) proceeding along similar lines. If $k \in K^2(\mathbb{R}^{2n})$ then $i\,\mathcal{H}|[k] = \mathcal{H}[k]$ and hence

$$\mathcal{K}\,[\mathcal{H}[k]] \;=\; -i\,\mathcal{H}|\,\mathcal{H}\,\mathcal{H}[k] \;=\; -i\,\mathcal{H}|[k] \;=\; -\mathcal{H}[k]$$

from which it follows that $\mathcal{H}[k] \in K^2(\mathbb{R}^{2n})^\perp$. Conversely, if $\mathcal{H}[k] \in K^2(\mathbb{R}^{2n})^\perp$ then

$$i\,\mathcal{H}|\,[\mathcal{H}[k]] \;=\; -\mathcal{H}\,[\mathcal{H}[k]] \;=\; -k$$

or $-\mathcal{K}[k] = -k$ meaning that $k \in K^2(\mathbb{R}^{2n})$. ■

There is a nice geometric interpretation of the above corollary. Indeed, let us recall that a function $f \in L_2(\mathbb{R}^{2n})$ and its Hilbert transform $\mathcal{H}[f]$ lie symmetrically with respect to $H^2(\mathbb{R}^{2n})$ (see Figure 5.1). If in particular f is chosen to belong to $K^2(\mathbb{R}^{2n})$ then $\mathcal{H}[f] \in K^2(\mathbb{R}^{2n})^\perp$, i.e. a function $f \in K^2(\mathbb{R}^{2n})$ and its Hilbert transform $\mathcal{H}[f]$ are orthogonal. The spaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ may thus be considered as being the bisector spaces of $H^2(\mathbb{R}^{2n})$ and $H^2(\mathbb{R}^{2n})^\perp$ (see Figure 10.1). At the same time, $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ are also the bisector spaces of the associated Hardy spaces $H|^2(\mathbb{R}^{2n})$ and $H|^2(\mathbb{R}^{2n})^\perp$.

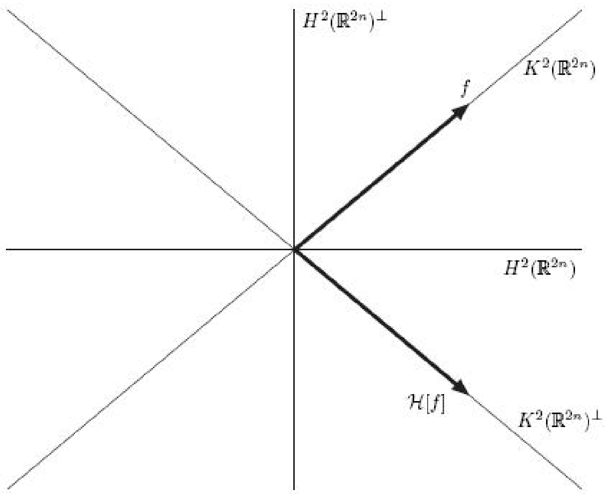

Figure 10.1: $f \in K^2(\mathbb{R}^{2n})$

Finally, the $L_2(\mathbb{R}^{2n})$ decompositions with respect to the Hilbert transform $\mathcal{H}$ and with respect to the new integral transform $\mathcal{K}$ can be matched together, resulting into the following schemes. Take $f \in L_2(\mathbb{R}^{2n})$, then on the one hand

$$f \;=\; h + g\;, \qquad h \in H^2(\mathbb{R}^{2n})\;, \qquad g \in H^2(\mathbb{R}^{2n})^\perp$$

and on the other hand

$$f \;=\; k + \ell \;, \qquad k \in K^2(\mathbb{R}^{2n}) \;, \qquad \ell \in K^2(\mathbb{R}^{2n})^\perp$$

where moreover the components h, g, k and ℓ may be decomposed themselves as well, viz

$$\begin{array}{rcllll} h & = & h_k + h_\ell \;, & h_k \in K^2(\mathbb{R}^{2n}) \;, & h_\ell \in K^2(\mathbb{R}^{2n})^\perp \\ g & = & g_k + g_\ell \;, & g_k \in K^2(\mathbb{R}^{2n}) \;, & g_\ell \in K^2(\mathbb{R}^{2n})^\perp \end{array}$$

and

$$\begin{array}{rcllll} k & = & k_h + k_g \;, & k_h \in H^2(\mathbb{R}^{2n}) \;, & k_g \in H^2(\mathbb{R}^{2n})^\perp \\ \ell & = & \ell_h + \ell_g \;, & \ell_h \in H^2(\mathbb{R}^{2n}) \;, & \ell_g \in H^2(\mathbb{R}^{2n})^\perp \end{array}$$

where obviously the following relations hold:

$$k \;=\; h_k + g_k \;, \quad \ell \;=\; h_\ell + g_\ell \qquad \text{and} \qquad h \;=\; k_h + \ell_h \;, \quad g \;=\; k_g + \ell_g$$

We thus obtain

$$f \;=\; h_k + g_k + h_\ell + g_\ell \tag{10.15}$$

$$\mathcal{H}[f] \;=\; h_k - g_k + h_\ell - g_\ell \tag{10.16}$$

$$\mathcal{K}[f] \;=\; h_k + g_k - h_\ell - g_\ell \tag{10.17}$$

and

$$i\,\mathcal{H}|[f] \;=\; \mathcal{H}\mathcal{K}[f] \;=\; -h_k + g_k + h_\ell - g_\ell \tag{10.18}$$

since

$$\mathcal{H}[h_k] \;=\; h_\ell \;, \quad \mathcal{H}[g_k] \;=\; -g_\ell \;, \quad \mathcal{H}[h_\ell] \;=\; h_k \;, \quad \mathcal{H}[g_\ell] \;=\; -g_k$$

and

$$\mathcal{K}[h_k] \;=\; h_k \;, \quad \mathcal{K}[g_k] \;=\; g_k \;, \quad \mathcal{K}[h_\ell] \;=\; -h_\ell \;, \quad \mathcal{K}[g_\ell] \;=\; -g_\ell$$

Furthermore, the above results (10.15)–(10.18) show that

$$\begin{array}{rcl} \mathfrak{H}[f] & = & \dfrac{1}{2}\,(\mathcal{H} + i\,\mathcal{H}|)[f] \;=\; h_\ell - g_\ell \\ \mathfrak{H}^\dagger[f] & = & -\dfrac{1}{2}\,(\mathcal{H} - i\,\mathcal{H}|)[f] \;=\; g_k - h_k \end{array}$$

as it should, since

$$\mathfrak{H}[f] \;=\; \mathfrak{H}[h_k+g_k] \;=\; \mathcal{H}[h_k+g_k] \;=\; h_\ell - g_\ell$$

and

$$\mathfrak{H}^\dagger[f] \;=\; \mathfrak{H}^\dagger[h_\ell+g_\ell] \;=\; -\mathcal{H}[h_\ell+g_\ell] \;=\; -h_k + g_k$$

on account of Theorem 10.1. Invoking (10.15)–(10.18), we may also express the components (10.15) of f in terms of the respective projection operators $\frac{1}{2}\,(\mathbf{1}\pm\mathcal{K})$ and $\frac{1}{2}\,(\mathbf{1}\pm\mathcal{H})$, leading to

$$\begin{aligned}
h_k &= \frac{1}{4}\,(\mathbf{1}+\mathcal{H}+\mathcal{K}-\mathcal{H}\,\mathcal{K})\,[f] = \frac{(\mathbf{1}+\mathcal{K})}{2}\,\frac{(\mathbf{1}+\mathcal{H})}{2}[f]\\
g_k &= \frac{1}{4}\,(\mathbf{1}-\mathcal{H}+\mathcal{K}+\mathcal{H}\,\mathcal{K})\,[f] = \frac{(\mathbf{1}+\mathcal{K})}{2}\,\frac{(\mathbf{1}-\mathcal{H})}{2}[f]\\
h_\ell &= \frac{1}{4}\,(\mathbf{1}+\mathcal{H}-\mathcal{K}+\mathcal{H}\,\mathcal{K})\,[f] = \frac{(\mathbf{1}-\mathcal{K})}{2}\,\frac{(\mathbf{1}+\mathcal{H})}{2}[f]\\
g_\ell &= \frac{1}{4}\,(\mathbf{1}-\mathcal{H}-\mathcal{K}-\mathcal{H}\,\mathcal{K})\,[f] = \frac{(\mathbf{1}-\mathcal{K})}{2}\,\frac{(\mathbf{1}-\mathcal{H})}{2}[f]
\end{aligned}$$

which is in accordance with the definitions of these projections. Similarly, we also have

$$\begin{aligned}
f &= k_h + k_g + \ell_h + \ell_g && (10.19)\\
\mathcal{H}[f] &= k_h - k_g + \ell_h - \ell_g && (10.20)\\
\mathcal{K}[f] &= k_h + k_g - \ell_h - \ell_g && (10.21)
\end{aligned}$$

and

$$i\,\mathcal{H}|[f] \;=\; \mathcal{H}\mathcal{K}[f] \;=\; k_h - k_g - \ell_h + \ell_g \qquad (10.22)$$

since

$$\mathcal{H}[k_h] \;=\; k_h\;, \quad \mathcal{H}[k_g] \;=\; -k_g\;, \quad \mathcal{H}[\ell_h] \;=\; \ell_h\;, \quad \mathcal{H}[\ell_g] \;=\; -\ell_g$$

and

$$\mathcal{K}[k_h] \;=\; k_g\;, \quad \mathcal{K}[k_g] \;=\; k_h\;, \quad \mathcal{K}[\ell_h] \;=\; -\ell_g \quad \mathcal{K}[\ell_g] \;=\; -\ell_h$$

This yields, as an alternative decomposition of f by subsequent projections,

$$\begin{aligned} k_h &= \frac{1}{4}\,(\mathbf{1}+\mathcal{H}+\mathcal{K}+\mathcal{H}\,\mathcal{K})\,[f] \;=\; \frac{(\mathbf{1}+\mathcal{H})}{2}\,\frac{(\mathbf{1}+\mathcal{K})}{2}[f] \\ k_g &= \frac{1}{4}\,(\mathbf{1}-\mathcal{H}+\mathcal{K}-\mathcal{H}\,\mathcal{K})\,[f] \;=\; \frac{(\mathbf{1}-\mathcal{H})}{2}\,\frac{(\mathbf{1}+\mathcal{K})}{2}[f] \\ \ell_h &= \frac{1}{4}\,(\mathbf{1}+\mathcal{H}-\mathcal{K}-\mathcal{H}\,\mathcal{K})\,[f] \;=\; \frac{(\mathbf{1}+\mathcal{H})}{2}\,\frac{(\mathbf{1}-\mathcal{K})}{2}[f] \\ \ell_g &= \frac{1}{4}\,(\mathbf{1}-\mathcal{H}-\mathcal{K}+\mathcal{H}\,\mathcal{K})\,[f] \;=\; \frac{(\mathbf{1}-\mathcal{H})}{2}\,\frac{(\mathbf{1}-\mathcal{K})}{2}[f] \end{aligned}$$

which also was to be expected. Furthermore we then have

$$\begin{aligned} \mathfrak{H}[f] &= k_h - k_g \\ \mathfrak{H}^\dagger[f] &= \ell_g - \ell_h \end{aligned}$$

again in agreement with Theorem 10.1.

A following nice result is that the $\mathcal{K}$–transform may be considered as a commutator of the Hardy projections $\mathcal{C}^\pm$ and $\mathcal{C}|^\pm$.

Proposition 10.5. *On $L_2(\mathbb{R}^{2n})$, one has that*

$$\frac{1}{2}\,\mathcal{K} \;=\; [\mathcal{C}^+, i\mathcal{C}|^+] \;=\; [\mathcal{C}^-, i\mathcal{C}|^-]$$

Proof.
For a function $f \in L_2(\mathbb{R}^{2n})$, the result is directly obtained from the following calculations:

$$\begin{aligned} \mathcal{C}^+\left(i\,\mathcal{C}|^+\right)[f] &= \frac{1}{4}\,(i\,\mathbf{1}+i\,\mathcal{H}|+i\,\mathcal{H}+i\,\mathcal{H}\,\mathcal{H}|)\,[f] \\ \left(i\,\mathcal{C}|^+\right)\mathcal{C}^+[f] &= \frac{1}{4}\,(i\,\mathbf{1}+i\,\mathcal{H}+i\,\mathcal{H}|+i\,\mathcal{H}|\,\mathcal{H})\,[f] \\ \mathcal{C}^-\left(i\,\mathcal{C}|^-\right)[f] &= \frac{1}{4}\,(i\,\mathbf{1}-i\,\mathcal{H}|-i\,\mathcal{H}+i\,\mathcal{H}\,\mathcal{H}|)\,[f] \\ \left(i\,\mathcal{C}|^-\right)\mathcal{C}^-[f] &= \frac{1}{4}\,(i\,\mathbf{1}-i\,\mathcal{H}-i\,\mathcal{H}|+i\,\mathcal{H}|\,\mathcal{H})\,[f] \end{aligned}$$

■

Finally, it is also possible to make the Hermitean Hilbert transforms $\mathfrak{H}$ and $\mathfrak{H}^\dagger$ apparent as part of a boundary limit of a suitable combination of a $\partial_{\underline{X}}$–monogenic and a $\partial_{\underline{X}|}$–monogenic function in halfspace.

Proposition 10.6. *On $L_2(\mathbb{R}^{2n})$, one has that*

(i) $(\mathcal{C}^+ + i\,\mathcal{C}|^+)\,[f] \;=\; \frac{1+i}{2}\,f + \mathfrak{H}[f]$

(ii) $(\mathcal{C}^+ - i\,\mathcal{C}|^+)\,[f] \;=\; \frac{1-i}{2}\,f - \mathfrak{H}^\dagger[f]$

(iii) $(\mathcal{C}^- + i\,\mathcal{C}|^-)\,[f] \;=\; -\frac{1+i}{2}\,f + \mathfrak{H}[f]$

(iv) $(\mathcal{C}^- - i\,\mathcal{C}|^-)\,[f] \;=\; -\frac{1-i}{2}\,f - \mathfrak{H}^\dagger[f]$

10.3 Analytic signals

In Chapter 2, we introduced the concept of analytic signal in one–dimensional signal analysis. We recall that given a real signal $u(t)$ depending on the time variable $t \in \mathbb{R}$ and its one–dimensional Hilbert transform $\mathcal{H}[u](t)$, then its corresponding analytic signal is given by

$$f(t) \;=\; u(t) + i\,\mathcal{H}[u](t)$$

As $\mathcal{H}^2 = -\mathbf{1}$, this analytic signal satisfies the condition

$$i\mathcal{H}[f] \;=\; f$$

which means that if the signal u has finite energy, i.e. $u \in L_2(\mathbb{R})$, then its associated analytic signal f belongs to the Hardy space $H^2(\mathbb{R})$. At the same time the complex conjugated signal

$$f^c(t) \;=\; u(t) - i\,\mathcal{H}[u](t)$$

will then belong to the orthogonal complement $H^2(\mathbb{R})^\perp$ since

$$i\,\mathcal{H}[f^c] \;=\; -(u - i\,\mathcal{H}[u]) \;=\; -f^c$$

So one could say that in the one–dimensional case complex conjugation maps the Hardy spaces $H^2(\mathbb{R})$ and $H^2(\mathbb{R})^\perp$ onto each other.

In the multidimensional case it is directly seen from the definitions of the Hilbert transforms $\mathcal{H}$ and $\mathcal{H}|$ themselves that for functions $f : \mathbb{R}^{2n} \to \mathbb{C}_{2n}$ it holds that

$$\mathcal{H}[e_0\,f] \;=\; -e_0\,\mathcal{H}[f] \qquad \text{and} \qquad \mathcal{H}|[e_0\,f] \;=\; -e_0\,\mathcal{H}|[f]$$

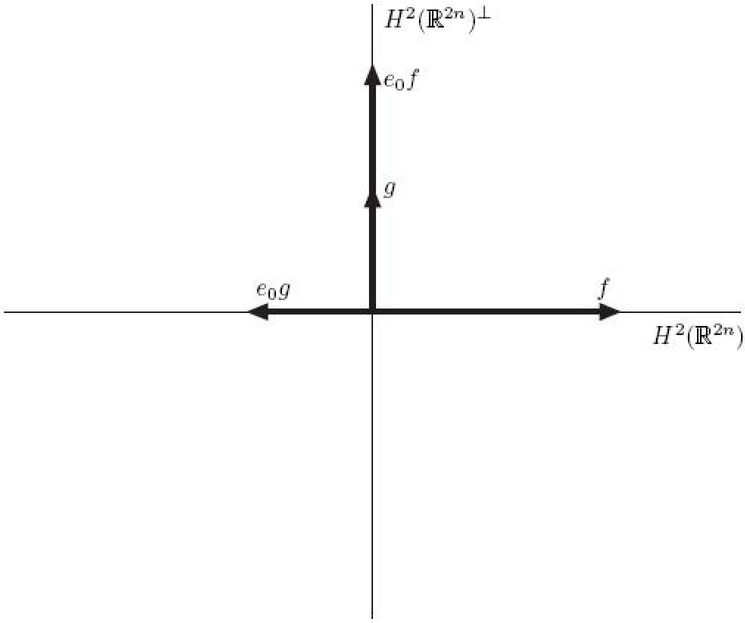

Figure 10.2: $f \in H^2(\mathbb{R}^{2n})$ and $g \in H^2(\mathbb{R}^{2n})^\perp$

It follows that left multiplication by e_0, i.e. $T_{e_0} : f \to e_0\, f$, will map the Hardy spaces $H^2(\mathbb{R}^{2n})$ and $H|^2(\mathbb{R}^{2n})$ onto their orthogonal complements and vice versa (see Figure 10.2). Indeed, for a function $u : \mathbb{R}^{2n} \to \mathbb{C}_{2n}$ with finite energy, its associated analytic signal is given by

$$f(\underline{X}) \;=\; u(\underline{X}) + \mathcal{H}[u](\underline{X})$$

which belongs to $H^2(\mathbb{R}^{2n})$. Note that f takes values in the Clifford algebra $\mathbb{C}_{2n} \oplus e_0\, \mathbb{C}_{2n} = \mathbb{C}_{2n+1}$. The action of the map T_{e_0} then results into

$$T_{e_0}[f] \;=\; e_0\, u + e_0\, \mathcal{H}[u] \;=\; -\mathcal{H}[e_0\, u] + e_0\, u$$

where now $e_0\, u$ takes values in $e_0\, \mathbb{C}_{2n}$, while $-\mathcal{H}[e_0\, u] = e_0\, \mathcal{H}[u]$ takes values in $\mathbb{C}_{2n}$. Moreover

$$\mathcal{H}\left[T_{e_0}[f]\right] \;=\; \mathcal{H}[e_0\, u] - e_0\, u \;=\; -T_{e_0}[f]$$

which means that $T_{e_0}[f]$ belongs to $H^2(\mathbb{R}^{2n})^\perp$. More generally, the geometric interpretation of the action of T_{e_0} on a function $u \in L_2(\mathbb{R}^{2n})$ is illustrated in Figure 10.3.

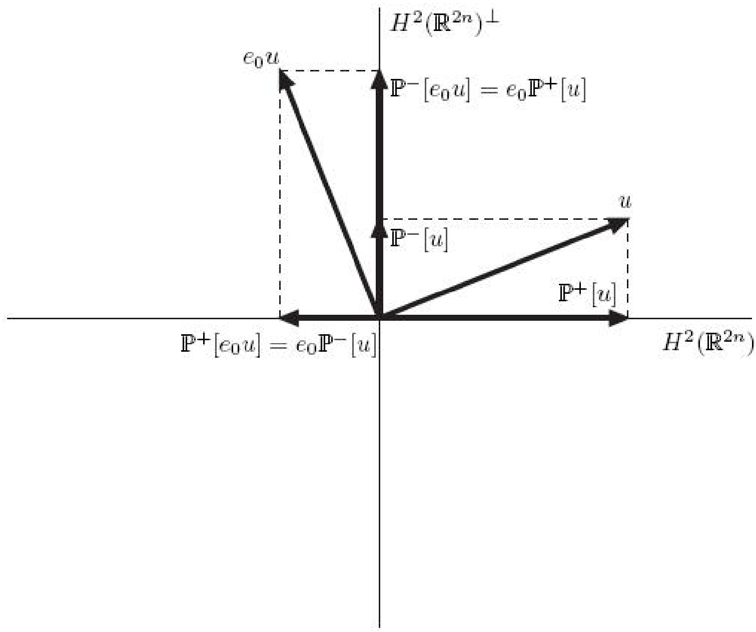

Figure 10.3: $u \in L_2(\mathbb{R}^{2n})$

However, for the new spaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ of Hardy type, it holds that

$$\mathcal{K}[e_0\, f] \;=\; i\,\mathcal{H}\,\mathcal{H}|[e_0\, f] \;=\; i\,\mathcal{H}\,[-e_0\,\mathcal{H}|[f]] \;=\; e_0\, i\,\mathcal{H}\,\mathcal{H}|[f] \;=\; e_0\,\mathcal{K}[f]$$

implying that T_{e_0} will map both $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ onto themselves.

Chapter 11

The matrical Hermitean Hilbert transform

In orthogonal Clifford analysis (see Chapter 5), the Clifford–Cauchy integral formula has proven to be a corner stone of the function theory, as is the case for the traditional Cauchy formula for holomorphic functions in the complex plane. Recently, in our paper [20], a Hermitean Clifford–Cauchy integral formula is established in the framework of circulant (2×2) matrix functions. Furthermore, in our follow–up paper [17], by means of the same matrix approach, a new Hermitean Hilbert transform is introduced arising naturally as part of the non–tangential boundary limit of the Hermitean Clifford–Cauchy integral. The resulting matrix Hilbert operator is shown to satisfy properly adapted analogues of the characteristic properties of the Hilbert transform in classical analysis and orthogonal Clifford analysis.

The outline of this chapter is as follows. Section 11.1 is devoted to the Hermitean Clifford–Cauchy integral formula. To arrive at that formula, in Subsection 11.1.2 and 11.1.3, the main results obtained in Section 5.2 are translated in the Hermitean Clifford analysis framework and some aspects are included of the indispensable framework of circulant (2×2) matrix functions in which the Cauchy integral formula is constructed. Finally, in Subsection 11.1.4, the obtained Hermitean Clifford–Cauchy integral formula turns out to reduce to the traditional Martinelli–Bochner formula for holomorphic functions of several complex variables in the special case of $\mathbb{CS}_n^{(n)}$ valued functions.

The aim of Section 11.2 is to show that the non–tangential boundary limits of the Hermitean Clifford–Cauchy integral reveal a new Hilbert–like matrix operator. It is proven that this operator has a close connection to the Hilbert transform in the orthogonal case, showing some quite similar properties as well.

11.1 Cauchy and Martinelli–Bochner integral formulae in Hermitean Clifford analysis

11.1.1 Introduction

The Cauchy integral formula, a key result for the theory of holomorphic functions in the complex plane, may be generalized to the case of several complex variables in two ways: either one takes a holomorphic kernel and an integral over the distinguished boundary $\partial_0 \widetilde{D} = \prod_{j=1}^n \partial \widetilde{D}_j$ of a polydisk $\widetilde{D} = \prod_{j=1}^n \widetilde{D}_j$ in $\mathbb{C}^n$, leading to the formula

$$
\begin{aligned}
&f(z_1, \ldots, z_n) \\
&\quad = \frac{1}{(2\pi i)^n} \int_{\partial_0 \widetilde{D}} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} \, d\xi_1 \wedge \cdots \wedge d\xi_n \,, \qquad z_j \in \overset{\circ}{\widetilde{D}}_j
\end{aligned}
$$

or one takes an integral over the (piecewise) smooth boundary ∂D of a bounded domain D in $\mathbb{C}^n$ in combination with the Martinelli–Bochner kernel, see e.g. [74], which is not holomorphic anymore but still harmonic, resulting into

$$
f(z) = \int_{\partial D} f(\xi) \, U(\xi, z) \,, \qquad z \in \overset{\circ}{D} \tag{11.1}
$$

with

$$
\begin{aligned}
&U(\xi, z) \\
&\quad = \tfrac{(n-1)!}{(2\pi i)^n} \textstyle\sum_{j=1}^n (-1)^{j-1} \tfrac{\xi_j^c - z_j^c}{|\xi - z|^{2n}} \, d\xi_1^c \wedge \cdots \wedge \left[d\xi_j^c \right] \wedge \cdots \wedge d\xi_n^c \wedge d\xi_1 \wedge \cdots \wedge d\xi_n
\end{aligned}
$$

The history of formula (11.1), obtained independently and through different methods by Martinelli and by Bochner, has been described in detail in [73]. It reduces to the traditional Cauchy integral formula when $n = 1$; for $n > 1$, it is related to the double layer potential, while at the same time, it establishes a connection between harmonic and holomorphic functions.

A third alternative for a generalization of the Cauchy integral formula was then offered in Section 5.2, where functions defined in Euclidean space $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and taking values in a Clifford algebra are considered. In this framework the Cauchy kernel appearing in the Clifford–Cauchy formula is monogenic, up to a pointwise singularity, while the integral remains being taken over the complete boundary:

$$f(\underline{Y}) \quad = \quad \int_{\partial D} E(\underline{X} - \underline{Y}) \, \widetilde{d\sigma}_{\underline{X}} \, f(\underline{X}) \ , \qquad \underline{Y} \in \overset{\circ}{D} \tag{11.2}$$

This Clifford–Cauchy integral formula is a corner stone in the function theoretic development of orthogonal Clifford analysis.

Naturally a Cauchy integral formula for Hermitean monogenic functions taking values in the complex Clifford algebra $\mathbb{C}_{2n}$ is essential in the further development of this function theory. A first result in this direction was obtained in [93], however for functions which are null solutions of only one of the Hermitean Dirac operators and moreover presenting a "fake" – as termed by the authors – Cauchy kernel, failing to be monogenic.

In this section a Cauchy integral formula for Hermitean monogenic functions is established. However, from the start of our quest, it was clear that the formula aimed at could not have a traditional form as in (11.2). Indeed, as was already mentioned in the previous chapter, it is not clear how to obtain a mutual fundamental solution of both Hermitean Dirac operators, which could act as a Cauchy kernel. Moreover, as observed in Section 9.2, when the functions considered do not take their values in the whole Clifford algebra $\mathbb{C}_{2n}$, but in the n–homogeneous part $\mathbb{CS}_n^{(n)}$ of complex spinor space $\mathbb{CS}_n$, h–monogenicity is equivalent with holomorphy in the complex variables $(z_1, \ldots, z_n)$. This means that, in particular, the Martinelli–Bochner formula (11.1) should be included as a special case in any Hermitean Cauchy integral formula to be established. It turned out that a matrix approach is the key to obtain the desired result.

11.1.2 Clifford–Stokes and Clifford–Cauchy theorems

The aim of this subsection is to translate the orthogonal Clifford–Stokes Theorem 5.2 and the orthogonal Clifford–Cauchy Theorem 5.3 into the Hermitean Clifford analysis framework.

First we set some notations. In the remainder of this chapter, we will, as in Section 5.2, denote by Ω some open region in $\mathbb{R}^{2n}$, and we consider a $2n$–dimensional compact differentiable and oriented manifold $\Gamma \subset \Omega$ with C^∞ smooth boundary $\partial\Gamma$. Further, Γ^+ will stand for the interior of Γ and Γ^- for the exterior of Γ with respect to Ω, i.e. $\Gamma^- = \Omega \setminus \Gamma$.

Subsequently, to the vector valued oriented surface element $\widetilde{d\sigma}_{\underline{X}}$ on $\partial\Gamma$, given by the following differential form of order $(2n-1)$:

$$\widetilde{d\sigma}_{\underline{X}} \;=\; \sum_{j=1}^{n} e_j\,(-1)^{j-1}\,\widetilde{\widehat{dx_j}} + \sum_{j=1}^{n} e_{n+j}\,(-1)^{n+j-1}\,\widetilde{\widehat{dy_j}}$$

we associate its twisted analogue $\widetilde{d\sigma}_{\underline{X}|}$, defined as

$$\widetilde{d\sigma}_{\underline{X}|} \;=\; J[\widetilde{d\sigma}_{\underline{X}}] \;=\; \sum_{j=1}^{n} e_j\,(-1)^{n+j-1}\,\widetilde{\widehat{dy_j}} - \sum_{j=1}^{n} e_{n+j}\,(-1)^{j-1}\,\widetilde{\widehat{dx_j}}$$

Here

$$\begin{aligned}
\widetilde{\widehat{dx_j}} &= dx_1 \wedge \cdots \wedge [dx_j] \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \\
\widetilde{\widehat{dy_j}} &= dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge [dy_j] \wedge \cdots \wedge dy_n
\end{aligned}$$

reflecting the original consecutive ordering of the variables $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Alternatively, if $dS(\underline{X})$ stands for the classical surface element on $\partial\Gamma$ and $\nu(\underline{X})$ for the outward pointing (with respect to Γ^+) unit normal vector in $\underline{X}$ on $\partial\Gamma$, then it is well–known that the surface element $\widetilde{d\sigma}_{\underline{X}}$ is also given by

$$\widetilde{d\sigma}_{\underline{X}} \;=\; \nu(\underline{X})\, dS(\underline{X})$$

so that we may write for $\widetilde{d\sigma}_{\underline{X}|}$

$$\widetilde{d\sigma}_{\underline{X}|} \;=\; \nu|(\underline{X})\, dS(\underline{X}) \;=\; J[\nu(\underline{X})]\, dS(\underline{X})$$

The corresponding oriented volume elements on Γ then read

$$\begin{aligned}
\widetilde{dV}(\underline{X}) &= dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \\
\widetilde{dV}(\underline{X}|) &= dy_1 \wedge \cdots \wedge dy_n \wedge (-dx_1) \wedge \cdots \wedge (-dx_n)
\end{aligned}$$

for which it is easily checked that

$$\widetilde{dV}(\underline{X}) \;=\; \widetilde{dV}(\underline{X}|) \tag{11.3}$$

In orthogonal Clifford analysis the theorems of Stokes and Cauchy were already formulated in Section 5.2. Clearly, both theorems may be restated for $\underline{X}|$, $\widetilde{d\sigma}_{\underline{X}|}$ and $\partial_{\underline{X}|}$, leading to their "twisted" formulations below, where (11.3) has been taken into account. We summarize as follows.

Theorem 11.1 (Clifford–Stokes theorem)**.** *Let f and g be functions in $C^1(\Omega)$ and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\begin{aligned}
\int_{\partial\Gamma} f(\underline{X})\,\widetilde{d\sigma}_{\underline{X}}\; g(\underline{X}) \;&=\; \int_\Gamma \left[(f(\underline{X})\partial_{\underline{X}})\,g(\underline{X}) + f(\underline{X})\,(\partial_{\underline{X}} g(\underline{X}))\right]\,\widetilde{dV}(\underline{X})\\
\int_{\partial\Gamma} f(\underline{X})\,\widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X}) \;&=\; \int_\Gamma \left[(f(\underline{X})\partial_{\underline{X}|})\,g(\underline{X}) + f(\underline{X})\,(\partial_{\underline{X}|} g(\underline{X}))\right]\,\widetilde{dV}(\underline{X})
\end{aligned}$$

Corollary 11.1 (Clifford–Cauchy theorem)**.** *Let the function g be in $C^1(\Omega)$ and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

(i) if g is $\partial_{\underline{X}}$–monogenic in Ω, one has

$$\int_{\partial\Gamma} \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}) \;=\; 0$$

(ii) if g is $\partial_{\underline{X}|}$–monogenic in Ω, one has

$$\int_{\partial\Gamma} \widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X}) \;=\; 0$$

We now introduce the Hermitean counterparts of the pair of oriented surface elements $(\widetilde{d\sigma}_{\underline{X}}, \widetilde{d\sigma}_{\underline{X}|})$. To this end, we should note that we may also write

$$\begin{aligned}
\widetilde{d\sigma}_{\underline{X}} \;&=\; (-1)^{\frac{n(n-1)}{2}}\, d\sigma_{\underline{X}}\\
\widetilde{d\sigma}_{\underline{X}|} \;&=\; (-1)^{\frac{n(n-1)}{2}}\, d\sigma_{\underline{X}|}
\end{aligned}$$

with the alternative pair of surface elements $(d\sigma_{\underline{X}}, d\sigma_{\underline{X}|})$ only involving a re-ordering of the variables according to n complex planes, i.e.

$$\begin{aligned} d\sigma_{\underline{X}} &= \sum_{j=1}^{n} [e_j \, (dx_1 \wedge dy_1) \wedge \cdots \wedge ([dx_j] \wedge dy_j) \wedge \cdots \wedge (dx_n \wedge dy_n)] \\ &\quad + \sum_{j=1}^{n} [-e_{n+j} \, (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_j \wedge [dy_j]) \wedge \cdots \wedge (dx_n \wedge dy_n)] \end{aligned}$$

and

$$\begin{aligned} d\sigma_{\underline{X}|} &= \sum_{j=1}^{n} [-e_{n+j} \, (dx_1 \wedge dy_1) \wedge \cdots \wedge ([dx_j] \wedge dy_j) \wedge \cdots \wedge (dx_n \wedge dy_n)] \\ &\quad + \sum_{j=1}^{n} [-e_j \, (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_j \wedge [dy_j]) \wedge \cdots \wedge (dx_n \wedge dy_n)] \end{aligned}$$

It is then easily seen that

$$\widetilde{d\sigma}_{\underline{X}} - i \, \widetilde{d\sigma}_{\underline{X}|} = (-1)^{\frac{n(n-1)}{2}} (d\sigma_{\underline{X}} - i \, d\sigma_{\underline{X}|}) = (-1)^{\frac{n(n-1)}{2}} (-4) \left(\frac{i}{2}\right)^n \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \widehat{dz_j}$$

while

$$\widetilde{d\sigma}_{\underline{X}} + i \, \widetilde{d\sigma}_{\underline{X}|} = (-1)^{\frac{n(n-1)}{2}} (d\sigma_{\underline{X}} + i \, d\sigma_{\underline{X}|}) = (-1)^{\frac{n(n-1)}{2}} (-4) \left(\frac{i}{2}\right)^n \sum_{j=1}^{n} \mathfrak{f}_j \, \widehat{dz_j^c}$$

with

$$\begin{aligned} \widehat{dz_j} &= (dz_1 \wedge dz_1^c) \wedge \cdots \wedge ([dz_j] \wedge dz_j^c) \wedge \cdots \wedge (dz_n \wedge dz_n^c) \\ \widehat{dz_j^c} &= (dz_1 \wedge dz_1^c) \wedge \cdots \wedge (dz_j \wedge [dz_j^c]) \wedge \cdots \wedge (dz_n \wedge dz_n^c) \end{aligned}$$

This observation leads to the definition of the *Hermitean oriented surface elements*

$$\begin{aligned} d\sigma_{\underline{Z}} &= \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \widehat{dz_j} \\ d\sigma_{\underline{Z}^\dagger} &= \sum_{j=1}^{n} \mathfrak{f}_j \, \widehat{dz_j^c} \end{aligned}$$

for which it holds that

$$d\sigma_{\underline{Z}} = -\frac{1}{4}(-2i)^n \left(d\sigma_{\underline{X}} - i\, d\sigma_{\underline{X}|}\right)$$
$$d\sigma_{\underline{Z}^\dagger} = -\frac{1}{4}(-2i)^n \left(d\sigma_{\underline{X}} + i\, d\sigma_{\underline{X}|}\right)$$

or equivalently

$$d\sigma_{\underline{Z}} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\widetilde{d\sigma}_{\underline{X}} - i\, \widetilde{d\sigma}_{\underline{X}|}\right) \tag{11.4}$$
$$d\sigma_{\underline{Z}^\dagger} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\widetilde{d\sigma}_{\underline{X}} + i\, \widetilde{d\sigma}_{\underline{X}|}\right) \tag{11.5}$$

Note that we in fact have applied the same technique as introduced in Chapter 9, by means of the projection operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ acting on $d\sigma_{\underline{X}}$, up to a deliberately chosen multiplicative constant.

We also consider the associated volume element $dW(\underline{Z}, \underline{Z}^\dagger)$ defined as

$$dW(\underline{Z}, \underline{Z}^\dagger) = (dz_1 \wedge dz_1^c) \wedge (dz_2 \wedge dz_2^c) \wedge \cdots \wedge (dz_n \wedge dz_n^c)$$

reflecting integration over the respective complex z_j–planes, $j = 1, \ldots, n$. One has that

$$\widetilde{dV}(\underline{X}) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n dW(\underline{Z}, \underline{Z}^\dagger) \tag{11.6}$$

A first result is then easily obtained.

Theorem 11.2 (Hermitean Clifford–Stokes theorems). *Let f and g be functions in $C^1(\Omega)$ and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} f\, d\sigma_{\underline{Z}}\, g = \int_\Gamma \left[(f\, \partial_{\underline{Z}}\ )\, g + f\, (\partial_{\underline{Z}}\ g)\right] dW(\underline{Z}, \underline{Z}^\dagger)$$
$$\int_{\partial\Gamma} f\, (-d\sigma_{\underline{Z}^\dagger})\, g = \int_\Gamma \left[(f\, \partial_{\underline{Z}^\dagger})\, g + f\, (\partial_{\underline{Z}^\dagger} g)\right] dW(\underline{Z}, \underline{Z}^\dagger)$$

Proof.
Start from the orthogonal Clifford–Stokes theorems and invoke the expressions (11.4) and (11.5), as well as the relation (11.6) between the orthogonal volume element $\widetilde{dV}(\underline{X})$ and the Hermitean volume element $dW(\underline{Z}, \underline{Z}^\dagger)$. ■

Theorem 11.3 (Hermitean Clifford–Cauchy theorems)**.** *Let the function g be* h*–monogenic in Ω and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} d\sigma_{\underline{Z}}\, g \;=\; 0 \qquad \textit{and} \qquad \int_{\partial\Gamma} d\sigma_{\underline{Z}^\dagger}\, g \;=\; 0$$

Proof.
Start from the orthogonal Clifford–Cauchy theorems and invoke the expressions (11.4) and (11.5), or alternatively, take $f = 1$ and g an h–monogenic function in the above Hermitean Clifford–Stokes theorems. ■

11.1.3 Cauchy integral formulae

The fundamental solutions of the orthogonal Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$, i.e. the orthogonal Cauchy kernels, are respectively given by

$$E\,(\underline{X}) \;=\; \frac{1}{a_{2n}}\,\frac{\overline{\underline{X}}}{|\underline{X}|^{2n}} \tag{11.7}$$

$$E|(\underline{X}) \;=\; \frac{1}{a_{2n}}\,\frac{\overline{\underline{X}|}}{|\underline{X}|^{2n}} \tag{11.8}$$

Explicitly, this means

$$\partial_{\underline{X}}\, E\,(\underline{X}) \;=\; \delta(\underline{X}) \tag{11.9}$$

$$\partial_{\underline{X}|} E|(\underline{X}) \;=\; \delta(\underline{X}|) \;=\; \delta(\underline{X}) \tag{11.10}$$

with

$$\lim_{|\underline{X}|\to\infty} E(\underline{X}) \;=\; 0 \quad \text{and} \quad \lim_{|\underline{X}|\to\infty} E|(\underline{X}) \;=\; 0 \tag{11.11}$$

For the remaining combinations between the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ on the one hand and the fundamental solutions E and $E|$ on the other hand, we find

Lemma 11.1.

$$\partial_{\underline{X}} E|(\underline{X}) = -\frac{i\,(2\beta-n)}{n}\,\delta(\underline{X}) + \frac{2n}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^{2n+2}} - \frac{2i\,(2\beta-n)}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{|\underline{X}|^{2n}} \tag{11.12}$$

$$\partial_{\underline{X}|} E(\underline{X}) = \frac{i\,(2\beta-n)}{n}\,\delta(\underline{X}) + \frac{2n}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{X}|\underline{X}}{|\underline{X}|^{2n+2}} + \frac{2i\,(2\beta-n)}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{|\underline{X}|^{2n}} \tag{11.13}$$

Proof.
We only give the proof for (11.12), the proof of (11.13) running along similar lines. In distributional sense we may write $E|(\underline{X})$ as

$$E|(\underline{X}) \;=\; \frac{1}{a_{2n}}\,\mathrm{Fp}\,\frac{\overline{\underline{X}|}}{|\underline{X}|^{2n}} \;=\; \frac{1}{a_{2n}}\,T_{-2n}\,\overline{\underline{X}|}$$

with T_{-2n} a member of the $\mathbb{T}$–family of distributions introduced in Chapter 4. Taking into consideration various results of that same chapter, the action of $\partial_{\underline{X}} T_{-2n}$ on a scalar valued test function ϕ in $\mathbb{R}^{2n}$ is given by

$$\begin{aligned}
\left\langle\, \partial_{\underline{X}}\, T_{-2n}\,,\, \phi \,\right\rangle \;&=\; -\left\langle\, T_{-2n}\,,\, \partial_{\underline{X}}\, \phi \,\right\rangle\\
&=\; -a_{2n}\left\langle\, \mathrm{Fp}\, r_+^{-1}\,,\, \Sigma^{(0)}[\partial_{\underline{X}}\, \phi] \,\right\rangle\\
&=\; -a_{2n}\left\langle\, \mathrm{Fp}\, r_+^{-1}\,,\, \left(\partial_r + \frac{2n-1}{r}\right)\Sigma^{(1)}[\phi] \,\right\rangle\\
&=\; -a_{2n}\left\langle\, \delta'(r) + (2n)\,\mathrm{Fp}\, r_+^{-2}\,,\, \Sigma^{(1)}[\phi] \,\right\rangle\\
&=\; \left\langle\, -\frac{a_{2n}}{2n}\,\partial_{\underline{X}}\,\delta(\underline{X}) - (2n)\,U_{-2n-1}\,,\, \phi \,\right\rangle
\end{aligned}$$

Then, when taking into account (9.4), we find

$$\left(\partial_{\underline{X}}\, T_{-2n}\right)\overline{\underline{X}|} \;=\; -\frac{i(2\beta-n)}{n}\,a_{2n}\,\delta(\underline{X}) + (2n)\,\mathrm{Fp}\,\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^{2n+2}}$$

and finally we arrive at

$$\begin{aligned}
\partial_{\underline{X}} E|(\underline{X}) \;&=\; \frac{1}{a_{2n}}\left[\left(\partial_{\underline{X}}\, T_{-2n}\right)\overline{\underline{X}|} + T_{-2n}\left(\partial_{\underline{X}}\,\overline{\underline{X}|}\right)\right]\\
&=\; -\frac{i\,(2\beta-n)}{n}\,\delta(\underline{X}) + \frac{2n}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^{2n+2}} - \frac{2i\,(2\beta-n)}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{|\underline{X}|^{2n}}
\end{aligned}$$

■

For a function $g \in C^0(\partial\Gamma)$, its Cauchy integral $\mathcal{C}[g]$ in $\Gamma^\pm$ was already defined in Section 5.2, viz

$$\mathcal{C}[g](\underline{Y}) \;=\; \int_{\partial\Gamma} E(\underline{X}-\underline{Y})\,\widetilde{d\sigma}_{\underline{X}}\,g(\underline{X})\,, \qquad \underline{Y} \in \Gamma^\pm$$

At the same time, one may also define on $\Gamma^\pm$ the $\partial_{\underline{X}|}$–monogenic associated Cauchy integral $\mathcal{C}|[g]$, given by

$$\mathcal{C}|[g](\underline{Y}) \;=\; \int_{\partial\Gamma} E|(\underline{X}-\underline{Y})\, \widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X})\ , \qquad \underline{Y} \in \Gamma^\pm$$

The boundedness of both integrals is guaranteed by (11.11).

Similarly as above, we now introduce the Hermitean counterparts to the pair of fundamental solutions $(E, E|)$, by putting

$$\begin{aligned} \mathcal{E} &= -\,(E + i\,E|) \\ \mathcal{E}^\dagger &= \,(E - i\,E|) \end{aligned}$$

Explicitly this yields

$$\begin{aligned} \mathcal{E}\ (\underline{Z}, \underline{Z}^\dagger) &= \frac{2}{a_{2n}}\, \frac{\underline{Z}}{|\underline{Z}|^{2n}} \\ \mathcal{E}^\dagger(\underline{Z}, \underline{Z}^\dagger) &= \frac{2}{a_{2n}}\, \frac{\underline{Z}^\dagger}{|\underline{Z}|^{2n}} \end{aligned}$$

with

$$\lim_{|\underline{Z}|\to\infty} \mathcal{E}(\underline{Z}, \underline{Z}^\dagger) \;=\; 0 \quad \text{and} \quad \lim_{|\underline{Z}|\to\infty} \mathcal{E}^\dagger(\underline{Z}, \underline{Z}^\dagger) \;=\; 0 \tag{11.14}$$

Note however that $\mathcal{E}$ and $\mathcal{E}^\dagger$ are not the fundamental solutions to the respective Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$! Indeed, invoking (11.9)–(11.10) and (11.12)–(11.13), one obtains

Lemma 11.2.

$$\begin{aligned} \partial_{\underline{Z}}\ \mathcal{E}(\underline{Z}, \underline{Z}^\dagger) &= \frac{\beta}{n}\,\delta(\underline{Z}) + \frac{2\beta}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{r^{2n}} - \frac{2n}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{Z}^\dagger\underline{Z}}{r^{2n+2}} \\ \partial_{\underline{Z}^\dagger}\mathcal{E}(\underline{Z}, \underline{Z}^\dagger) &= 0 \end{aligned}$$

and

$$\begin{aligned} \partial_{\underline{Z}}\ \mathcal{E}^\dagger(\underline{Z}, \underline{Z}^\dagger) &= 0 \\ \partial_{\underline{Z}^\dagger}\mathcal{E}^\dagger(\underline{Z}, \underline{Z}^\dagger) &= \frac{n-\beta}{n}\,\delta(\underline{Z}) + \frac{2(n-\beta)}{a_{2n}}\,\mathrm{Fp}\,\frac{1}{r^{2n}} - \frac{2n}{a_{2n}}\,\mathrm{Fp}\,\frac{\underline{Z}\,\underline{Z}^\dagger}{r^{2n+2}} \end{aligned}$$

A first attempt at constructing a Hermitean Cauchy integral formula has been undertaken in [93], however presenting the "fake" – as termed by the authors themselves – Cauchy kernel $\frac{1}{2}\mathcal{E}^\dagger = \frac{1}{2}(E - iE|)$, which obviously fails to be h–monogenic.

Nevertheless it is clear that, in order to establish the desired formula, the functions $\mathcal{E}$ and $\mathcal{E}^\dagger$ will need to be involved. Indeed, surprisingly, combining the above calculations, we are lead to the following result.

Theorem 11.4. *Introducing the particular circulant* (2×2) *matrices*

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix} \quad \textit{and} \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

one obtains that

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{\mathcal{E}}(\underline{Z}, \underline{Z}^\dagger) = \boldsymbol{\delta}(\underline{Z})$$

This means that $\boldsymbol{\mathcal{E}}$ may be considered as a fundamental solution of $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$, the latter concept being reinterpreted in a matrical context. It is precisely this simple observation which has lead us to the idea of a matrix approach to arrive at a Cauchy integral formula in the Hermitean setting. Also note, as another remarkable fact, that the Dirac matrix $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$ in some sense factorizes the Laplacian, since

$$4\,\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \left(\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right)^{\dagger} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$

where we have introduced the Hermitean conjugation on circulant elements of $(\mathbb{C}_{2n})^{2\times 2}$, defined as follows:

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}^{\dagger} = \begin{pmatrix} a_1^\dagger & a_2^\dagger \\ a_2^\dagger & a_1^\dagger \end{pmatrix}$$

with $a_1, a_2 \in \mathbb{C}_{2n}$.

Thus, in the same setting of circulant (2×2) matrices we associate, with continuously differentiable functions g_1 and g_2 defined in Ω and taking values in $\mathbb{C}_{2n}$, the matrix function

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \tag{11.15}$$

Definition 11.1. *We call $\boldsymbol{G}_2^1$ (left)* **H***–monogenic if and only if it satisfies the system*

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1 \;=\; \boldsymbol{O} \tag{11.16}$$

where $\boldsymbol{O}$ denotes the matrix with zero entries.

The above system (11.16) for **H**–monogenicity explicitly reads

$$\begin{cases} \partial_{\underline{Z}}\ [g_1] + \partial_{\underline{Z}^\dagger}[g_2] \;=\; 0 \\ \partial_{\underline{Z}^\dagger}[g_1] + \partial_{\underline{Z}}\ [g_2] \;=\; 0 \end{cases}$$

Choosing in particular $g_1 = g$ and $g_2 = g^\dagger$, it is clear that, in general, the **H**–monogenicity of the corresponding matrix function

$$\boldsymbol{G} \;=\; \begin{pmatrix} g & g^\dagger \\ g^\dagger & g \end{pmatrix}$$

will not imply the h–monogenicity of the function g and vice versa. As a simple example consider the matrix function $\boldsymbol{\mathcal{E}}$ for which we have found above that it is **H**–monogenic in $\mathbb{R}^{2n} \setminus \{\underline{0}\}$, while clearly the function $\mathcal{E}$ is not h–monogenic. An exception to this general remark occurs in the special case of scalar (i.e. complex) valued functions, where h–monogenicity (of g) and **H**–monogenicity (of $\boldsymbol{G}$) are easily found to be equivalent notions.

Another special yet very important case occurs when considering the diagonal matrix function

$$\boldsymbol{G}_0 \;=\; \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

Since its **H**–monogenicity is readily seen to be equivalent with the h–monogenicity of the function g, this specific matrix will form the key for the construction of a Hermitean Cauchy integral formula. A first step in this direction is the reformulation of the Hermitean Clifford–Stokes theorems, established in Theorem 11.2, in a matrical form. To this end, we still introduce the matrix

$$\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \;=\; \begin{pmatrix} d\sigma_{\underline{Z}} & -d\sigma_{\underline{Z}^\dagger} \\ -d\sigma_{\underline{Z}^\dagger} & d\sigma_{\underline{Z}} \end{pmatrix}$$

which will play the role of the differential form. We then have the following result.

Theorem 11.5 (Hermitean Clifford–Stokes theorem)**.** *Let f_1, f_2, g_1 and g_2 be arbitrary functions in $C^1(\Omega)$ and consider the corresponding matrix functions $\boldsymbol{F}_2^1$ and $\boldsymbol{G}_2^1$ of the form (11.15); let as above $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$. It then holds that*

$$\begin{aligned}
&\int_{\partial\Gamma} \boldsymbol{F}_2^1(\underline{X})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&\quad = \int_\Gamma \left[\left(\boldsymbol{F}_2^1(\underline{X})\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right) \boldsymbol{G}_2^1(\underline{X}) + \boldsymbol{F}_2^1(\underline{X}) \left(\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right)\right] dW(\underline{Z},\underline{Z}^\dagger)
\end{aligned}$$

Proof.
The proof follows by taking deliberate combinations of the Hermitean Clifford–Stokes formulae found in Theorem 11.2. ■

From now on we reserve the notations $\underline{Y}$ and $\underline{Y}|$ for Clifford–vectors associated to points in $\Gamma^\pm$. Their Hermitean counterparts are denoted by

$$\begin{aligned}
\underline{V} &= \frac{1}{2}(\mathbf{1} + i\,J)[\underline{Y}] &= \frac{1}{2}(\underline{Y} + i\,\underline{Y}|) \\
\underline{V}^\dagger &= -\frac{1}{2}(\mathbf{1} - i\,J)[\underline{Y}] &= -\frac{1}{2}(\underline{Y} - i\,\underline{Y}|)
\end{aligned}$$

The following Hermitean Cauchy–Pompeiu formula is then established.

Theorem 11.6 (Hermitean Cauchy–Pompeiu formula)**.** *Let g_1 and g_2 be functions in $C^1(\Omega)$ and let $\boldsymbol{G}_2^1$ be the corresponding matrix function of the form (11.15); let as above $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$. It then holds that*

$$\begin{aligned}
&\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&- \int_\Gamma \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z},\underline{Z}^\dagger) \\
&= \begin{cases} \boldsymbol{O}\,, & \textit{if } \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \boldsymbol{G}_2^1(\underline{Y})\,, & \textit{if } \underline{Y} \in \Gamma^+ \end{cases}
\end{aligned}$$

Proof.
First, let $\underline{Y} = \underline{V} - \underline{V}^\dagger \in \Gamma^-$. In this case one has that, considered as functions

of $\underline{X} = \underline{Z} - \underline{Z}^\dagger$,

$$\mathcal{E}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) = \frac{2}{a_{2n}} \frac{\underline{Z} - \underline{V}}{|\underline{Z} - \underline{V}|^{2n}} = \frac{2}{a_{2n}} \frac{\underline{Z} - \underline{V}}{|\underline{X} - \underline{Y}|^{2n}}$$

and

$$\mathcal{E}^\dagger(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) = \frac{2}{a_{2n}} \frac{\underline{Z}^\dagger - \underline{V}^\dagger}{|\underline{Z} - \underline{V}|^{2n}} = \frac{2}{a_{2n}} \frac{\underline{Z}^\dagger - \underline{V}^\dagger}{|\underline{X} - \underline{Y}|^{2n}}$$

are continuously differentiable in Γ, so that the Hermitean Clifford–Stokes Theorem 11.5 can be applied, yielding the desired statement, since one has in Γ that

$$\boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) = \boldsymbol{O}$$

Next, let $\underline{Y} = \underline{V} - \underline{V}^\dagger \in \Gamma^+$, and take $R > 0$ such that $B(\underline{Y}; R) \subset \Gamma$. Invoking the previous case, we may then write

$$\begin{aligned} &\int_{\partial(\Gamma \setminus B(\underline{Y};R))} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\ &- \int_{\Gamma \setminus B(\underline{Y};R)} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z}, \underline{Z}^\dagger) = \boldsymbol{O} \end{aligned} \tag{11.17}$$

Taking limits for $R \to 0$ the second term at the left-hand side yields

$$\begin{aligned} &\lim_{R \to 0} \int_{\Gamma \setminus B(\underline{Y};R)} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z}, \underline{Z}^\dagger) \\ &\quad = \int_{\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z}, \underline{Z}^\dagger) \end{aligned}$$

since the integrand only contains functions which are integrable on Γ. Furthermore we may write for the first term at the right-hand side of (11.17):

$$
\begin{aligned}
&\int_{\partial(\Gamma\setminus B(\underline{Y};R))} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&\quad= \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&\qquad - \int_{\partial B(\underline{Y};R)} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&\quad= \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&\qquad - \frac{2}{a_{2n}R^{2n}} \int_{\partial B(\underline{Y};R)} \boldsymbol{G}_{\underline{Z}-\underline{V}}\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})
\end{aligned}
$$

where

$$
\boldsymbol{G}_{\underline{Z}-\underline{V}} = \begin{pmatrix} \underline{Z}-\underline{V} & \underline{Z}^\dagger-\underline{V}^\dagger \\ \underline{Z}^\dagger-\underline{V}^\dagger & \underline{Z}-\underline{V} \end{pmatrix}
$$

In order to calculate the last integral in the above expression, we apply once more the Hermitean Clifford–Stokes Theorem 11.5:

$$
\begin{aligned}
&\frac{2}{a_{2n}R^{2n}} \int_{\partial B(\underline{Y};R)} \boldsymbol{G}_{\underline{Z}-\underline{V}}\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) \\
&= \frac{2}{a_{2n}R^{2n}} \int_{B(\underline{Y};R)} \left[\boldsymbol{G}_{\underline{Z}-\underline{V}}\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right] \boldsymbol{G}_2^1(\underline{X})\, dW(\underline{Z},\underline{Z}^\dagger) \\
&+ \frac{2}{a_{2n}R^{2n}} \int_{B(\underline{Y};R)} \boldsymbol{G}_{\underline{Z}-\underline{V}} \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z},\underline{Z}^\dagger)
\end{aligned}
\tag{11.18}
$$

For the integrand of the first integral at the right-hand side of (11.18), we observe that in the ball $B(\underline{Y};R)$

$$
\boldsymbol{G}_{\underline{Z}-\underline{V}}\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} (\underline{Z}\,\partial_{\underline{Z}}) + (\underline{Z}^\dagger\partial_{\underline{Z}^\dagger}) & (\underline{Z}^\dagger\partial_{\underline{Z}}) + (\underline{Z}\,\partial_{\underline{Z}^\dagger}) \\ (\underline{Z}^\dagger\partial_{\underline{Z}}) + (\underline{Z}\,\partial_{\underline{Z}^\dagger}) & (\underline{Z}\,\partial_{\underline{Z}}) + (\underline{Z}^\dagger\partial_{\underline{Z}^\dagger}) \end{pmatrix} = n\,\mathbb{E}_2
$$

on account of Lemma 9.1 (ii). Moreover, since the functions g_1 and g_2 are continuously differentiable on Ω, we may also write for $\underline{X} \in B(\underline{Y};R)$:

$$
\boldsymbol{G}_2^1(\underline{X}) = \boldsymbol{G}_2^1(\underline{Y}) + \mathcal{O}(\rho)
$$

with

$$\rho \;=\; |\underline{Y}-\underline{X}| \leq R \qquad \text{and} \qquad \lim_{\rho\to 0} \mathcal{O}(\rho) \;=\; \boldsymbol{O}$$

By direct calculation we then obtain

$$\begin{aligned}
&\frac{2}{a_{2n}R^{2n}} \int_{B(\underline{Y};R)} \left[\boldsymbol{G}_{\underline{Z}-\underline{V}}\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right] \boldsymbol{G}_2^1(\underline{X})\, dW(\underline{Z},\underline{Z}^\dagger) \\
&\quad= \frac{2n}{a_{2n}R^{2n}} \left(\boldsymbol{G}_2^1(\underline{Y}) \int_{B(\underline{Y};R)} dW(\underline{Z},\underline{Z}^\dagger) + \int_{B(\underline{Y};R)} \mathcal{O}(\rho)\, dW(\underline{Z},\underline{Z}^\dagger)\right) \\
&\quad= \frac{2n}{a_{2n}R^{2n}} (-2i)^n\, (-1)^{\frac{n(n-1)}{2}}\, \boldsymbol{G}_2^1(\underline{Y}) \int_{B(\underline{Y};R)} \widetilde{dV}(\underline{X}) \\
&\qquad+ \frac{2n}{a_{2n}R^{2n}} (-2i)^n\, (-1)^{\frac{n(n-1)}{2}} \int_0^R \mathcal{O}(\rho)\, \rho^{2n-1}\, d\rho \int_{S^{2n-1}} dS(\underline{\Omega}) \\
&\quad= (2i)^n\, (-1)^{\frac{n(n+1)}{2}} \left(\boldsymbol{G}_2^1(\underline{Y}) + 2n\, \mathcal{O}(R)\right)
\end{aligned}$$

such that eventually

$$\begin{aligned}
&\lim_{R\to 0} \frac{2}{a_{2n}R^{2n}} \int_{B(\underline{Y};R)} \left[\boldsymbol{G}_{\underline{Z}-\underline{V}}\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right] \boldsymbol{G}_2^1(\underline{X})\, dW(\underline{Z},\underline{Z}^\dagger) \\
&\quad= (2i)^n\, (-1)^{\frac{n(n+1)}{2}}\, \boldsymbol{G}_2^1(\underline{Y})
\end{aligned}$$

The proof may now be finished since the second term at the right-hand side of (11.18) is shown to converge to $\boldsymbol{O}$ for $R \to 0$, i.e. putting

$$\begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \;=\; \frac{2}{a_{2n}R^{2n}} \int_{B(\underline{Y};R)} \boldsymbol{G}_{\underline{Z}-\underline{V}} \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z},\underline{Z}^\dagger)$$

we may find that

$$\lim_{R\to 0} M_1 \;=\; 0 \qquad \text{and} \qquad \lim_{R\to 0} M_2 \;=\; 0 \tag{11.19}$$

Indeed, g_1 and g_2 being continuously differentiable on Ω implies that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\sup_{\underline{X}\in B(\underline{Y};R)} \left|\partial_{\underline{Z}}\, g_1(\underline{X})\right| \leq C \qquad \text{and} \qquad \sup_{\underline{X}\in B(\underline{Y};R)} \left|\partial_{\underline{Z}^\dagger}\, g_1(\underline{X})\right| \leq C \\
&\sup_{\underline{X}\in B(\underline{Y};R)} \left|\partial_{\underline{Z}}\, g_2(\underline{X})\right| \leq C \qquad \text{and} \qquad \sup_{\underline{X}\in B(\underline{Y};R)} \left|\partial_{\underline{Z}^\dagger}\, g_2(\underline{X})\right| \leq C
\end{aligned}$$

The following estimation for M_1 can then be derived:

$$\begin{aligned}
|M_1| &\leq \frac{2^{3n+1}}{a_{2n} R^{2n}} \int_{B(\underline{Y};R)} \Big[|\underline{Z} - \underline{V}| \, \left(\left|\partial_{\underline{Z}}\, g_1(\underline{X})\right| + \left|\partial_{\underline{Z}^\dagger}\, g_2(\underline{X})\right| \right) \\
&\qquad\qquad + |\underline{Z}^\dagger - \underline{V}^\dagger| \, \left(\left|\partial_{\underline{Z}}\, g_2(\underline{X})\right| + \left|\partial_{\underline{Z}^\dagger}\, g_1(\underline{X})\right| \right) \Big] \, \widetilde{dV}(\underline{X}) \\
&\leq \frac{2^{3n+3} C}{a_{2n} R^{2n-1}} \int_{B(\underline{Y};R)} \widetilde{dV}(\underline{X}) \\
&= \frac{2^{3n+2} C}{n} \, R
\end{aligned}$$

and the same estimation holds for M_2:

$$|M_2| \leq \frac{2^{3n+2} C}{n} \, R$$

The limits (11.19) then follow immediately. ■

This theorem then leads to the following Hermitean Cauchy integral formulae for $\mathbf{H}$–monogenic matrix functions $\boldsymbol{G}_2^1$ and h–monogenic functions g, respectively.

Theorem 11.7 (Hermitean Cauchy integral formula I)**.** *If the matrix function* $\boldsymbol{G}_2^1$ *is* $\mathbf{H}$*–monogenic in* Ω *then*

$$\begin{aligned}
&\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) \, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \, \boldsymbol{G}_2^1(\underline{X}) \\
&\quad = \begin{cases} \boldsymbol{O} \ , & \text{if } \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n \, \boldsymbol{G}_2^1(\underline{Y}) \ , & \text{if } \underline{Y} \in \Gamma^+ \end{cases}
\end{aligned}$$

Proof.
Apply Theorem 11.6 while taking into account the $\mathbf{H}$–monogenicity of the matrix function $\boldsymbol{G}_2^1$. ■

Theorem 11.8 (Hermitean Cauchy integral formula II)**.** *If the function* g *is* h*–monogenic in* Ω *then*

$$\begin{aligned}
&\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger) \, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} \, \boldsymbol{G}_0(\underline{X}) \\
&\quad = \begin{cases} \boldsymbol{O} \ , & \text{if } \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n \, \boldsymbol{G}_0(\underline{Y}) \ , & \text{if } \underline{Y} \in \Gamma^+ \end{cases}
\end{aligned} \tag{11.20}$$

The previous theorem may be considered as a Hermitean Cauchy integral formula for the h–monogenic function g; therefore the matrix function $\boldsymbol{\mathcal{E}}$ appearing in this formula is called the Hermitean Cauchy kernel. For functions $g_1, g_2, g \in C^0(\partial\Gamma)$ the following Hermitean Cauchy integrals $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0]$ and $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1]$ are then defined

$$\begin{aligned}
\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1](\underline{Y}) &= \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\ , \qquad \underline{Y} \in \Gamma^\pm \\
\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0](\underline{Y}) &= \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_0(\underline{X})\ , \qquad \underline{Y} \in \Gamma^\pm
\end{aligned}$$

which are both $\mathbf{H}$–monogenic in $\Gamma^\pm$, i.e.

$$\boldsymbol{\mathcal{D}}_{(\underline{V},\underline{V}^\dagger)}\, \boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1](\underline{Y}) = \boldsymbol{O} \qquad \text{in } \Gamma^\pm$$

and

$$\boldsymbol{\mathcal{D}}_{(\underline{V},\underline{V}^\dagger)}\, \boldsymbol{\mathcal{C}}[\boldsymbol{G}_0](\underline{Y}) = \boldsymbol{O} \qquad \text{in } \Gamma^\pm$$

Notice that both integrals converge, also in case of Ω being unbounded, on account of (11.14). Further, calculations reveal that both Hermitean Cauchy integrals can be expressed in terms of the orthogonal Cauchy integrals $\mathcal{C}$ and $\mathcal{C}|$, viz

$$\begin{aligned}
\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1] &= (-1)^{\frac{n(n+1)}{2}} (2i)^n \left[\frac{1}{2} \begin{pmatrix} \mathcal{C}[g_1-g_2] & -\mathcal{C}[g_1-g_2] \\ -\mathcal{C}[g_1-g_2] & \mathcal{C}[g_1-g_2] \end{pmatrix} \right. \\
&\qquad \left. + \frac{1}{2} \begin{pmatrix} \mathcal{C}|[g_1+g_2] & \mathcal{C}|[g_1+g_2] \\ \mathcal{C}|[g_1+g_2] & \mathcal{C}|[g_1+g_2] \end{pmatrix} \right] \qquad (11.21) \\
\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0] &= (-1)^{\frac{n(n+1)}{2}} (2i)^n \left[\frac{1}{2} \begin{pmatrix} \mathcal{C}[g] & -\mathcal{C}[g] \\ -\mathcal{C}[g] & \mathcal{C}[g] \end{pmatrix} \right. \\
&\qquad \left. + \frac{1}{2} \begin{pmatrix} \mathcal{C}|[g] & \mathcal{C}|[g] \\ \mathcal{C}|[g] & \mathcal{C}|[g] \end{pmatrix} \right] \qquad (11.22)
\end{aligned}$$

Now taking in particular an $\mathbf{H}$–monogenic matrix function $\boldsymbol{G}_2^1$ in Ω, i.e.

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1 = \boldsymbol{O}$$

or equivalently

$$\begin{cases} \partial_{\underline{X}}\, [g_1 - g_2] = 0 \\ \partial_{\underline{X}|} [g_1 + g_2] = 0 \end{cases}$$

then $\mathcal{C}[g_1 - g_2] = g_1 - g_2$ and $\mathcal{C}|[g_1 + g_2] = g_1 + g_2$ in Γ^+. Taking into account (11.21), one obtains that

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1] \;=\; (-1)^{\frac{n(n+1)}{2}}(2i)^n\,\boldsymbol{G}_2^1 \qquad \text{in } \Gamma^+$$

in accordance with Theorem 11.7. On the other hand, taking an h–monogenic function g in Ω, for which

$$\mathcal{C}[g] \;=\; g \;=\; \mathcal{C}|[g] \qquad \text{in } \Gamma^+$$

yields

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0] \;=\; (-1)^{\frac{n(n+1)}{2}}(2i)^n\,\boldsymbol{G}_0 \qquad \text{in } \Gamma^+$$

on account of (11.22), thus confirming Theorem 11.8.

Finally, we already announce that the subject of Section 11.2 will be the study of the non–tangential boundary limits in L_2 sense of the Hermitean Cauchy integrals $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1](\underline{Y})$ and $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0](\underline{Y})$, leading to Hermitean Clifford–Hardy spaces and to a matrical Hermitean Hilbert transform.

11.1.4 The Martinelli–Bochner formula revisited

In this section we will restrict ourselves to functions g on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, taking values in the homogeneous n–space $\mathbb{CS}_n^{(n)}$ of spinor space $\mathbb{CS}_n$, i.e. the functions g under consideration will take the following form:

$$g(\underline{Z}, \underline{Z}^\dagger) \;=\; g_n(\underline{Z}, \underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I \tag{11.23}$$

where g_n is a smooth complex valued function on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. It was argued in Section 9.2 that h–monogenicity for such a function g is equivalent to holomorphy of the corresponding scalar function g_n in the n complex variables $(z_1, \ldots, z_n)$. Thus, considering functions g of the form (11.23) establishes a connection between Hermitean Clifford analysis and the theory of holomorphic functions of several complex variables, whence it is interesting to investigate the true nature of the Hermitean Cauchy integral formula obtained in the previous section for this type of functions.

To this end, we will explicitly calculate the left-hand side of formula (11.20), taking g to be of the form (11.23). One obtains

$$\begin{aligned}\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\,\boldsymbol{G}_0(\underline{X}) &= \begin{pmatrix} \sum_{k=1}^n (\widehat{dz_k\,\mathfrak{f}_k^\dagger})\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\ldots\mathfrak{f}_n^\dagger\,I & * \\ -\sum_{k=1}^n (\widehat{dz_k^c\,\mathfrak{f}_k})\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\ldots\mathfrak{f}_n^\dagger\,I & * \end{pmatrix} \\ &= \begin{pmatrix} 0 & * \\ -d\sigma_{\underline{Z}^\dagger}\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\ldots\mathfrak{f}_n^\dagger\,I & * \end{pmatrix}\end{aligned}$$

where the second column has not been written, since it only duplicates the first one (in reversed order) seen the circulant structure of the involved matrices. The matrix entry $[\cdot_{11}]$ reduces to zero on account of the anti–commutation relations and the isotropy of the Witt basis elements $\mathfrak{f}_k^\dagger$, $k = 1, \ldots, n$. Further calculation yields

$$\begin{aligned}&\boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\,\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\,\boldsymbol{G}_0(\underline{X}) \\ &\quad= \begin{pmatrix} -\mathcal{E}^\dagger(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\,d\sigma_{\underline{Z}^\dagger}\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\ldots\mathfrak{f}_n^\dagger\,I & * \\ -\mathcal{E}\;(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\,d\sigma_{\underline{Z}^\dagger}\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\ldots\mathfrak{f}_n^\dagger\,I & * \end{pmatrix}\end{aligned}$$

where, putting $\rho = |\underline{Z}-\underline{V}|$,

$$\mathcal{E}^\dagger(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\,d\sigma_{\underline{Z}^\dagger} = \frac{2}{a_{2n}\rho^{2n}}\left(\sum_{j=1}^n \mathfrak{f}_j^\dagger\,(z_j^c - v_j^c)\right)\left(\sum_{k=1}^n \mathfrak{f}_k\,\widehat{dz_k^c}\right)$$

whence

$$\begin{aligned}&\mathcal{E}^\dagger(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\,d\sigma_{\underline{Z}^\dagger}\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger\ldots\mathfrak{f}_n^\dagger\,I \\ &\qquad= \frac{2}{a_{2n}\rho^{2n}}\sum_{j=1}^n (z_j^c - v_j^c)\,\widehat{dz_j^c}\,g_n(\underline{Z},\underline{Z}^\dagger)\,\mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger\ldots\mathfrak{f}_n^\dagger\,I\end{aligned}$$

where we have invoked the duality identities for the Witt basis elements and once more their isotropy. Similarly

$$\mathcal{E}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\,d\sigma_{\underline{Z}^\dagger} = \frac{2}{a_{2n}\rho^{2n}}\left(\sum_{j=1}^n \mathfrak{f}_j\,(z_j^c - v_j^c)\right)\left(\sum_{k=1}^n \mathfrak{f}_k\,\widehat{dz_k^c}\right)$$

whence

$$\begin{aligned}
&\mathcal{E}(\underline{Z}-\underline{V},\underline{Z}^\dagger-\underline{V}^\dagger)\, d\sigma_{\underline{Z}^\dagger}\, g_n(\underline{Z},\underline{Z}^\dagger)\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I \\
&\qquad = \frac{2}{a_{2n}\rho^{2n}} \sum_{j\neq k} (z_j^c - v_j^c)\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger)\, \mathfrak{f}_j \mathfrak{f}_k\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I
\end{aligned}$$

Thus, the Hermitean Cauchy integral formula (11.20) for $\underline{Y} \in \Gamma^+$ yields two statements. The first one reads

$$(-1)^{\frac{n(n+1)}{2}}\, g_n(\underline{V},\underline{V}^\dagger) \;=\; -\int_{\partial\Gamma} \frac{(n-1)!}{(2\pi i)^n}\, \frac{1}{\rho^{2n}} \sum_{j=1}^{n} (z_j^c - v_j^c)\, \widehat{dz_j^c}\, g_n(\underline{Z},\underline{Z}^\dagger)$$

which exactly coincides with the Martinelli–Bochner formula (11.1), when taking into account the appropriate reordering of the involved differential forms. Here we have used $a_{2n} = \frac{2\pi^n}{(n-1)!}$. The second statement is

$$0 \;=\; -\int_{\partial\Gamma} \frac{2}{a_{2n}}\, \frac{1}{\rho^{2n}} \sum_{j\neq k} (z_j - v_j)\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger)\, \mathfrak{f}_j\, \mathfrak{f}_k\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I$$

which, by means of some more Witt basis calculations, decomposes into

$$\int_{\partial\Gamma} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) \;=\; \int_{\partial\Gamma} \frac{z_k - v_k}{\rho^{2n}}\, \widehat{dz_j^c}\, g_n(\underline{Z},\underline{Z}^\dagger)\ ,\ \ j,k = 1,\ldots,n\ ,\ \ j \neq k$$

a result which is proved through direct computation in the following proposition.

Proposition 11.1. *If g_n is a complex valued funcion, holomorphic in $(z_1,\ldots,z_n)$, then for $\underline{Y} \in \Gamma^+$:*

$$\int_{\partial\Gamma} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) \;=\; \int_{\partial\Gamma} \frac{z_k - v_k}{\rho^{2n}}\, \widehat{dz_j^c}\, g_n(\underline{Z},\underline{Z}^\dagger)\ , \qquad j,k = 1,\ldots,n$$

where $\rho = |\underline{Z} - \underline{V}|$.

Proof.
Consider the second equation in Theorem 11.2

$$\int_{\partial\Gamma} f\,(-d\sigma_{\underline{Z}^\dagger})\, g \;=\; \int_{\Gamma} \left[(f\,\partial_{\underline{Z}^\dagger})\, g + f\,(\partial_{\underline{Z}^\dagger}\, g)\right]\, dW(\underline{Z},\underline{Z}^\dagger)$$

For complex valued functions f and g, this expression reduces to

$$\int_{\partial\Gamma} f\,(-\widehat{dz_j^c})\, g \;=\; \int_{\Gamma} \left[(\partial_{z_j^c} f)\, g + f\,(\partial_{z_j^c} g)\right]\, dW(\underline{Z},\underline{Z}^\dagger)\ ,\ \ j = 1,\ldots,n \quad (11.24)$$

First consider the case of the ball, i.e. $\Gamma = B(\underline{Y};\varepsilon)$ and $\partial\Gamma = \partial B(\underline{Y};\varepsilon)$, with $\varepsilon > 0$. Then for all $j,k = 1,\ldots,n$ we apply the "scalar" Stokes formula (11.24):

$$\begin{aligned}
&\int_{\partial B(\underline{Y};\varepsilon)} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) \\
&= \frac{1}{\varepsilon^{2n}} \int_{\partial B(\underline{Y};\varepsilon)} (z_j - v_j)\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) \\
&= -\frac{1}{\varepsilon^{2n}} \int_{B(\underline{Y};\varepsilon)} \Big[(\partial_{z_k^c}(z_j - v_j))\, g_n(\underline{Z},\underline{Z}^\dagger) + (z_j - v_j)\,(\partial_{z_k^c} g_n(\underline{Z},\underline{Z}^\dagger)) \Big]\, dW(\underline{Z},\underline{Z}^\dagger)
\end{aligned}$$

Since $\partial_{z_k^c}(z_j - v_j) = 0$, for all $j,k = 1,\ldots,n$ and since g_n is holomorphic in $(z_1,\ldots,z_n)$, we are then lead to

$$\int_{\partial B(\underline{Y};\varepsilon)} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) \;=\; 0 \tag{11.25}$$

Now consider $\widetilde{\Gamma} = \Gamma \setminus B(\underline{Y};\varepsilon)$ with boundary $\partial\widetilde{\Gamma} = \partial\Gamma \cup \partial B(\underline{Y};\varepsilon)$. Again we apply the "scalar" Stokes formula (11.24):

$$\int_{\partial\widetilde{\Gamma}} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n \;=\; -\int_{\widetilde{\Gamma}} \left[(\partial_{z_k^c} \frac{z_j - v_j}{\rho^{2n}})\, g_n + \frac{z_j - v_j}{\rho^{2n}}\, (\partial_{z_k^c} g_n) \right] dW(\underline{Z},\underline{Z}^\dagger)$$

On account of (11.25), the left-hand side of the above equation equals

$$\begin{aligned}
&\int_{\partial\Gamma} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) - \int_{\partial B(\underline{Y};\varepsilon)} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger) \\
&= \int_{\partial\Gamma} \frac{z_j - v_j}{\rho^{2n}}\, \widehat{dz_k^c}\, g_n(\underline{Z},\underline{Z}^\dagger)
\end{aligned}$$

while for the right-hand side one obtains

$$n \int_{\widetilde{\Gamma}} \frac{1}{\rho^{2n+2}} (z_j - v_j)(z_k - v_k)\, g_n(\underline{Z},\underline{Z}^\dagger)\, dW(\underline{Z},\underline{Z}^\dagger)$$

on account of the holomorphy of g_n in $(z_1,\ldots,z_n)$ and

$$\partial_{z_k^c}\left(\rho^2\right) \;=\; \partial_{z_k^c}\left[\sum_{j=1}^{n} (z_j - v_j)(z_j^c - v_j^c)\right] \;=\; z_k - v_k$$

So, for all $j,k=1,\ldots,n$

$$\begin{aligned}
&\int_{\partial\Gamma} \frac{z_j - v_j}{\rho^{2n}} \, \widehat{dz_k^c} \, g_n(\underline{Z}, \underline{Z}^\dagger) \\
&\quad = \; n \int_{\widetilde{\Gamma}} \frac{1}{\rho^{2n+2}} (z_j - v_j)(z_k - v_k) \, g_n(\underline{Z}, \underline{Z}^\dagger) \, dW(\underline{Z}, \underline{Z}^\dagger) \\
&\quad = \; \int_{\partial\Gamma} \frac{z_k - v_k}{\rho^{2n}} \, \widehat{dz_j^c} \, g_n(\underline{Z}, \underline{Z}^\dagger)
\end{aligned}$$

■

11.2 A matrix Hilbert transform in Hermitean Clifford analysis

11.2.1 The twisted Clifford–Hilbert transform on closed surfaces in $\mathbb{R}^{2n}$

For a function $g \in L_2(\partial\Gamma)$ its Clifford–Hilbert transform on $\partial\Gamma$ was already introduced in Section 5.2 as part of the L_2 non–tangential boundary limits of the Cauchy integral $\mathcal{C}[g]$, viz

$$\mathcal{C}^{\pm}[g](\underline{U}) \;=\; \lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^{\pm}}} \mathcal{C}[g](\underline{Y}) \;=\; \pm\frac{1}{2}\, g(\underline{U}) + \frac{1}{2}\,\mathcal{H}[g](\underline{U}) \;, \quad \underline{U}\in\partial\Gamma \tag{11.26}$$

with

$$\mathcal{H}[g](\underline{U}) \;=\; 2\ \mathrm{Pv} \int_{\partial\Gamma} E(\underline{X}-\underline{U})\, \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}) \;, \qquad \underline{U}\in\partial\Gamma$$

Adopting the same idea, the twisted Clifford–Hilbert transform $\mathcal{H}|[g]$ may be immediately defined as part of the L_2 non–tangential boundary limits of the associated Cauchy integral $\mathcal{C}|[g]$, viz

$$\mathcal{C}|^{\pm}[g](\underline{U}) \;=\; \lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^{\pm}}} \mathcal{C}|[g](\underline{Y}) \;=\; \pm\frac{1}{2}\, g(\underline{U}) + \frac{1}{2}\,\mathcal{H}|[g](\underline{U}) \;, \quad \underline{U}\in\partial\Gamma \tag{11.27}$$

Explicitly, the twisted Hilbert transform is given by the principal value integral

$$\mathcal{H}|[g](\underline{U}) \;=\; 2\ \mathrm{Pv} \int_{\partial\Gamma} E|(\underline{X}-\underline{U})\, \widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X}) \;, \qquad \underline{U}\in\partial\Gamma$$

Taking into account Property 5.9, the following main properties of the twisted Hilbert transform on $\partial\Gamma$, apart from the above defining one, may be proven instantly, mutatis mutandis.

Property 11.1. *One has*

P(1) $\mathcal{H}|$ *is a bounded linear operator on* $L_2(\partial\Gamma)$

P(2) $\mathcal{H}|^2 = \mathbf{1}$ *on* $L_2(\partial\Gamma)$

P(3) $\mathcal{H}|^* = \nu|\,\mathcal{H}|\,\nu|$ *on* $L_2(\partial\Gamma)$

P(4) for $g \in L_2(\partial\Gamma)$*, one has that* $\mathcal{H}|[g] = g$ *if and only if* $g \in H|^2(\partial\Gamma)$

The last property P(4) deserves some more explanation. For the open set Γ^+ one can consider the Hardy space $H|^2(\Gamma^+)$ of $\partial_{\underline{X}|}$–monogenic Clifford algebra valued functions, viz

$$H|^2(\Gamma^+) \;=\; \{g : \Gamma^+ \to \mathbb{C}_{2n} \;:\; \partial_{\underline{X}|} g = 0 \text{ in } \Gamma^+ \text{ and } g_{\partial\Gamma} \in L_2(\partial\Gamma)\}$$

The Hardy space $H|^2(\Gamma^+)$ then entails the Hardy space $H|^2(\partial\Gamma)$ as the closure in $L_2(\partial\Gamma)$ of the space of all non–tangential boundary limits of all functions in $H|^2(\Gamma^+)$. Moreover, both spaces $H|^2(\Gamma^+)$ and $H|^2(\partial\Gamma)$ are isomorphic, the isomorphism being obtained explicitly by means of the Cauchy integral in the following way. For a given $g \in H|^2(\partial\Gamma)$ its associated Cauchy integral $\mathcal{C}|[g]$ belongs to $H|^2(\Gamma^+)$ and

$$\lim_{\substack{\underline{Y}\to\underline{U}\\ \underline{Y}\in\Gamma^+}} \mathcal{C}|[g](\underline{Y}) \;=\; g(\underline{U})\;, \qquad \underline{U} \in \partial\Gamma$$

in the L_2 sense, so that $\mathcal{C}|[g]$ may be seen as the $\partial_{\underline{X}|}$–monogenic extension of g to Γ^+.

11.2.2 A matrix Hilbert transform in Hermitean Clifford analysis

A first attempt at constructing Hermitean Hilbert transforms for functions in $L_2(\mathbb{R}^{2n})$ has been undertaken in the previous chapter. However, although the obtained transforms showed some nice and satisfactory properties, one big issue remained unsolved at that moment: it seemed impossible to construct in the Hermitean context an h–monogenic Cauchy integral, such that those Hermitean

Hilbert transforms could be retrieved as part of its non–tangential boundary limits.

In the context of this chapter, if only that class of functions $g \in L_2(\partial\Gamma)$ would be considered for which $\mathcal{H}[g] = \mathcal{H}|[g]$, then of course the h–monogenic Cauchy integral is trivially given by $\mathcal{C}[g]$ which in this case coincides with $\mathcal{C}|[g]$. Indeed, for such functions g one has that $\mathcal{C}[g] - \mathcal{C}|[g]$ is a harmonic function in $\Gamma^{\pm}$ with boundary limit equal to zero, as $\mathcal{C}^{\pm}[g] = \mathcal{C}|^{\pm}[g]$. On account of the maximum and the minimum principle for harmonic functions this yields that $\mathcal{C}[g] = \mathcal{C}|[g]$ in $\Gamma^{\pm}$.

In the general case, it appeared that the matrix approach which has been introduced in the previous chapter, is the key to obtain the desired result.

Given functions $g_1, g_2, g \in L_2(\partial\Gamma)$, we will investigate in this section the non–tangential boundary behaviour of the Hermitean Cauchy integrals $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1]$ and $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0]$. To that end, we introduce the matrix operator

$$\boldsymbol{\mathcal{H}} \;=\; \frac{1}{2}\begin{pmatrix} \mathcal{H}+\mathcal{H}| & -\mathcal{H}+\mathcal{H}| \\ -\mathcal{H}+\mathcal{H}| & \mathcal{H}+\mathcal{H}| \end{pmatrix}$$

its action on the matrix functions $\boldsymbol{G}_2^1$ and $\boldsymbol{G}_0$ being given by matrix multiplication, followed by an operator action on the level of the entries, e.g.

$$\begin{aligned} \boldsymbol{\mathcal{H}}[\boldsymbol{G}_0] &= \frac{1}{2}\begin{pmatrix} \mathcal{H}+\mathcal{H}| & -\mathcal{H}+\mathcal{H}| \\ -\mathcal{H}+\mathcal{H}| & \mathcal{H}+\mathcal{H}| \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (\;\mathcal{H}+\mathcal{H}|)[g] & (-\mathcal{H}+\mathcal{H}|)[g] \\ (-\mathcal{H}+\mathcal{H}|)[g] & (\;\mathcal{H}+\mathcal{H}|)[g] \end{pmatrix} \end{aligned}$$

Now expressing $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1]$ in terms of $\mathcal{C}[g_1 - g_2]$ and $\mathcal{C}|[g_1 + g_2]$ and $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0]$ in terms of $\mathcal{C}[g]$ and $\mathcal{C}|[g]$, as in (11.21) and (11.22), respectively, and taking into account the classical Plemelj–Sokhotzki formulae (11.26) and (11.27), the following results are obtained.

Proposition 11.2. *For functions $g_1, g_2 \in L_2(\partial\Gamma)$, the non–tangential boundary limits of its Hermitean Cauchy integral $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1]$ are given by*

$$\begin{aligned} \boldsymbol{\mathcal{C}}^{\pm}[\boldsymbol{G}_2^1](\underline{U}) &= \lim_{\substack{\underline{Y}\to\underline{U} \\ \underline{Y}\in\Gamma^{\pm}}} \boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1](\underline{Y}) \\ &= (-1)^{\frac{n(n+1)}{2}}(2i)^n \left(\pm\frac{1}{2}\,\boldsymbol{G}_2^1(\underline{U}) + \frac{1}{2}\,\boldsymbol{\mathcal{H}}[\boldsymbol{G}_2^1](\underline{U})\right), \qquad \underline{U}\in\partial\Gamma \end{aligned}$$

Corollary 11.2. *For a function $g \in L_2(\partial\Gamma)$, the non–tangential boundary limits of its Hermitean Cauchy integral $\boldsymbol{\mathcal{C}}[\boldsymbol{G}_0]$ are given by*

$$\begin{aligned}\boldsymbol{\mathcal{C}}^{\pm}[\boldsymbol{G}_0](\underline{U}) &= \lim_{\substack{\underline{Y}\to\underline{U}\\ \underline{Y}\in\Gamma^{\pm}}} \boldsymbol{\mathcal{C}}[\boldsymbol{G}_0](\underline{Y}) \\ &= (-1)^{\frac{n(n+1)}{2}}(2i)^n\left(\pm\frac{1}{2}\,\boldsymbol{G}_0(\underline{U}) + \frac{1}{2}\,\boldsymbol{\mathcal{H}}[\boldsymbol{G}_0](\underline{U})\right), \qquad \underline{U}\in\partial\Gamma\end{aligned}$$

We call the matrix operator $\boldsymbol{\mathcal{H}}$ the *matrical Hermitean Hilbert transform*. The matrix function $\boldsymbol{G}_0$ being a special case of $\boldsymbol{G}_2^1$, we now only focus on the last one. Our aim is to establish for that matrical Hilbert transform the traditional properties, similar to those mentioned in Property 5.9. To this end, we first create the proper framework for dealing with circulant (2×2) matrix functions. First of all, we introduce the vector space

$$\boldsymbol{L_2}(\partial\Gamma) = \left\{\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \,:\, g_1, g_2 \in L_2(\partial\Gamma)\right\}$$

on which, inspired by the $\mathbb{C}_{2n}$ valued inner product $\langle\cdot,\cdot\rangle$ on $L_2(\partial\Gamma)$ given by

$$\langle f, g\rangle = \int_{\partial\Gamma} f(\underline{X})^{\dagger}\, g(\underline{X})\, dS(\underline{X})$$

we introduce the following bilinear form:

$$\begin{aligned}&\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}} \,:\, \boldsymbol{L_2}(\partial\Gamma)\times\boldsymbol{L_2}(\partial\Gamma)\longrightarrow(\mathbb{C}_{2n})^{2\times 2}\ ;\\ &\left(\begin{pmatrix} f_1 & f_2 \\ f_2 & f_1\end{pmatrix},\begin{pmatrix} g_1 & g_2 \\ g_2 & g_1\end{pmatrix}\right)\longmapsto\begin{pmatrix}\langle f_1,g_1\rangle+\langle f_2,g_2\rangle & \langle f_1,g_2\rangle+\langle f_2,g_1\rangle\\ \langle f_1,g_2\rangle+\langle f_2,g_1\rangle & \langle f_1,g_1\rangle+\langle f_2,g_2\rangle\end{pmatrix}\end{aligned}$$

In the lemma below it is stated that $\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}}$ is a $(\mathbb{C}_{2n})^{2\times 2}$ valued inner product.

Lemma 11.3. *For $\boldsymbol{F}_2^1, \boldsymbol{G}_2^1, \boldsymbol{K}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ and $\lambda\in\mathbb{C}$ one has*

(i) $\left\langle\boldsymbol{F}_2^1, \lambda\boldsymbol{G}_2^1+\boldsymbol{K}_2^1\right\rangle_{\boldsymbol{L_2}} = \lambda\left\langle\boldsymbol{F}_2^1,\boldsymbol{G}_2^1\right\rangle_{\boldsymbol{L_2}} + \left\langle\boldsymbol{F}_2^1,\boldsymbol{K}_2^1\right\rangle_{\boldsymbol{L_2}}$

(ii) if for all $\boldsymbol{F}_2^1\in\boldsymbol{L_2}(\partial\Gamma)$: $\left\langle\boldsymbol{F}_2^1,\boldsymbol{G}_2^1\right\rangle_{\boldsymbol{L_2}} = \boldsymbol{O}$, then $\boldsymbol{G}_2^1 = \boldsymbol{O}$

(iii) $\left(\left\langle\boldsymbol{F}_2^1,\boldsymbol{G}_2^1\right\rangle_{\boldsymbol{L_2}}\right)^{\dagger} = \left\langle\boldsymbol{G}_2^1,\boldsymbol{F}_2^1\right\rangle_{\boldsymbol{L_2}}$

Proof.

Let $\lambda \in \mathbb{C}$ and let $\boldsymbol{F}_2^1, \boldsymbol{G}_2^1, \boldsymbol{K}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ be given explicitly by

$$\boldsymbol{F}_2^1 = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}, \quad \boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{K}_2^1 = \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}$$

with $f_j, g_j, k_j \in L_2(\partial\Gamma)$, $j = 1, 2$.

(i) Taking into account that $\langle \cdot, \cdot \rangle$ is an inner product, one subsequently has that

$$\begin{aligned}
&\left\langle \boldsymbol{F}_2^1, \lambda \boldsymbol{G}_2^1 + \boldsymbol{K}_2^1 \right\rangle_{\boldsymbol{L_2}} \\
&= \left\langle \begin{pmatrix} f_1 & * \\ f_2 & * \end{pmatrix}, \begin{pmatrix} \lambda g_1 + k_1 & * \\ \lambda g_2 + k_2 & * \end{pmatrix} \right\rangle_{\boldsymbol{L_2}} \\
&= \begin{pmatrix} \langle f_1, \lambda g_1 + k_1 \rangle + \langle f_2, \lambda g_2 + k_2 \rangle & * \\ \langle f_1, \lambda g_2 + k_2 \rangle + \langle f_2, \lambda g_1 + k_1 \rangle & * \end{pmatrix} \\
&= \begin{pmatrix} \lambda \langle f_1, g_1 \rangle + \langle f_1, k_1 \rangle + \lambda \langle f_2, g_2 \rangle + \langle f_2, k_2 \rangle & * \\ \lambda \langle f_1, g_2 \rangle + \langle f_1, k_2 \rangle + \lambda \langle f_2, g_1 \rangle + \langle f_2, k_1 \rangle & * \end{pmatrix} \\
&= \lambda \begin{pmatrix} \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle & * \\ \langle f_1, g_2 \rangle + \langle f_2, g_1 \rangle & * \end{pmatrix} + \begin{pmatrix} \langle f_1, k_1 \rangle + \langle f_2, k_2 \rangle & * \\ \langle f_1, k_2 \rangle + \langle f_2, k_1 \rangle & * \end{pmatrix} \\
&= \lambda \left\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1 \right\rangle_{\boldsymbol{L_2}} + \left\langle \boldsymbol{F}_2^1, \boldsymbol{K}_2^1 \right\rangle_{\boldsymbol{L_2}}
\end{aligned}$$

where in the matrices the second column has not been written, since it only duplicates the first column (in reversed order) seen the circulant structure of the involved matrices.

(ii) Let $\boldsymbol{G}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ and suppose that for all $\boldsymbol{F}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$: $\left\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1 \right\rangle_{\boldsymbol{L_2}} = \boldsymbol{O}$. This implies that

$$\left\{ \begin{array}{rcl} \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle & = & 0 \\ \langle f_1, g_2 \rangle + \langle f_2, g_1 \rangle & = & 0 \end{array} \right. \tag{11.28}$$

for all $f_1, f_2 \in L_2(\partial\Gamma)$. If we take in particular $f_2 = 0$, then system (11.28) reduces for all $f_1 \in L_2(\partial\Gamma)$ to

$$\left\{ \begin{array}{rcl} \langle f_1, g_1 \rangle & = & 0 \\ \langle f_1, g_2 \rangle & = & 0 \end{array} \right.$$

This can only be the case if both $g_1 = 0$ and $g_2 = 0$, so if $\boldsymbol{G}_2^1 = \boldsymbol{O}$.

(iii) Taking into account that $\langle \cdot, \cdot \rangle$ is an inner product, one has that

$$
\begin{aligned}
&\left(\left\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1 \right\rangle_{\boldsymbol{L_2}}\right)^\dagger \\
&= \begin{pmatrix} \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle & * \\ \langle f_1, g_2 \rangle + \langle f_2, g_1 \rangle & * \end{pmatrix}^\dagger = \begin{pmatrix} \langle f_1, g_1 \rangle^\dagger + \langle f_2, g_2 \rangle^\dagger & * \\ \langle f_1, g_2 \rangle^\dagger + \langle f_2, g_1 \rangle^\dagger & * \end{pmatrix} \\
&= \begin{pmatrix} \langle g_1, f_1 \rangle + \langle g_2, f_2 \rangle & * \\ \langle g_1, f_2 \rangle + \langle g_2, f_1 \rangle & * \end{pmatrix} = \left\langle \boldsymbol{G}_2^1, \boldsymbol{F}_2^1 \right\rangle_{\boldsymbol{L_2}}
\end{aligned}
$$

where in the matrices the second column has again been omitted, since it only duplicates the first column (in reversed order) seen the circulant structure of the involved matrices. ■

Next, we also consider the Hardy spaces

$$
\begin{aligned}
\boldsymbol{H^2}(\Gamma^+) &= \Big\{ \boldsymbol{G}_2^1 : \Gamma^+ \to (\mathbb{C}_{2n})^{2\times 2} \ : \ \boldsymbol{\mathcal{D}}_{(\underline{Z}, \underline{Z}^\dagger)} \boldsymbol{G}_2^1 = \boldsymbol{O} \text{ in } \Gamma^+ \\
&\qquad \text{and} \ \ g_{1|\partial\Gamma}, g_{2|\partial\Gamma} \in L_2(\partial\Gamma) \Big\}
\end{aligned}
$$

and $\boldsymbol{H^2}(\partial\Gamma)$, being the closure in $\boldsymbol{L_2}(\partial\Gamma)$ of the set of boundary values of elements of $\boldsymbol{H^2}(\Gamma^+)$.

Finally, we need a matrix analogue of the outward pointing unit normal vector. An apt choice for our purpose is

$$
\boldsymbol{\mathcal{V}} = \frac{1}{2} \begin{pmatrix} \nu + \nu| & -\nu + \nu| \\ -\nu + \nu| & \nu + \nu| \end{pmatrix}
$$

observing that indeed $\boldsymbol{\mathcal{V}}^2 = -\mathbb{E}_2$. The Hermitean Hilbert transform $\boldsymbol{\mathcal{H}}$ then satisfies the following properties.

Theorem 11.9. *One has*

$\mathcal{P}$(1) $\boldsymbol{\mathcal{H}}$ *is a bounded linear operator on* $\boldsymbol{L_2}(\partial\Gamma)$

$\mathcal{P}$(2) $\boldsymbol{\mathcal{H}}^2 = \mathbb{E}_2$ *on* $\boldsymbol{L_2}(\partial\Gamma)$

$\mathcal{P}$(3) $\boldsymbol{\mathcal{H}}^* = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{V}}$ *on* $\boldsymbol{L_2}(\partial\Gamma)$

$\boldsymbol{\mathcal{P}}\mathbf{(4)}$ *for* $\boldsymbol{G}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$*, one has that* $\boldsymbol{\mathcal{H}}[\boldsymbol{G}_2^1] = \boldsymbol{G}_2^1$ *if and only if* $\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$

Proof.

$\boldsymbol{\mathcal{P}}\mathbf{(1)}$ Follows from the fact that both $\mathcal{H}$ and $\mathcal{H}|$ are bounded linear operators on $L_2(\partial\Gamma)$.

$\boldsymbol{\mathcal{P}}\mathbf{(2)}$ As both $\mathcal{H}$ and $\mathcal{H}|$ are involutory operators on $L_2(\partial\Gamma)$, one obtains:

$$\begin{aligned}
\boldsymbol{\mathcal{H}}^2 &= \frac{1}{2}\begin{pmatrix} \mathcal{H}+\mathcal{H}| & -\mathcal{H}+\mathcal{H}| \\ -\mathcal{H}+\mathcal{H}| & \mathcal{H}+\mathcal{H}| \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathcal{H}+\mathcal{H}| & -\mathcal{H}+\mathcal{H}| \\ -\mathcal{H}+\mathcal{H}| & \mathcal{H}+\mathcal{H}| \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} \mathcal{H}^2+\mathcal{H}|^2+\mathcal{H}^2+\mathcal{H}|^2 & -\mathcal{H}^2+\mathcal{H}|^2-\mathcal{H}^2+\mathcal{H}|^2 \\ -\mathcal{H}^2+\mathcal{H}|^2-\mathcal{H}^2+\mathcal{H}|^2 & \mathcal{H}^2+\mathcal{H}|^2+\mathcal{H}^2+\mathcal{H}|^2 \end{pmatrix} \\
&= \mathbb{E}_2
\end{aligned}$$

$\boldsymbol{\mathcal{P}}\mathbf{(3)}$ Let $\boldsymbol{F}_2^1$ and $\boldsymbol{G}_2^1$ belong to $\boldsymbol{L_2}(\partial\Gamma)$ and be given explicitly by

$$\boldsymbol{F}_2^1 = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

One then has

$$\begin{aligned}
&\langle \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1], \boldsymbol{G}_2^1 \rangle_{\boldsymbol{L_2}} \\
&= \left\langle \frac{1}{2}\begin{pmatrix} \mathcal{H}+\mathcal{H}| & * \\ -\mathcal{H}+\mathcal{H}| & * \end{pmatrix}\begin{pmatrix} f_1 & * \\ f_2 & * \end{pmatrix}, \begin{pmatrix} g_1 & * \\ g_2 & * \end{pmatrix} \right\rangle_{\boldsymbol{L_2}} \\
&= \left\langle \frac{1}{2}\begin{pmatrix} (\mathcal{H}+\mathcal{H}|)[f_1]+(-\mathcal{H}+\mathcal{H}|)[f_2] & * \\ (\mathcal{H}+\mathcal{H}|)[f_2]+(-\mathcal{H}+\mathcal{H}|)[f_1] & * \end{pmatrix}, \begin{pmatrix} g_1 & * \\ g_2 & * \end{pmatrix} \right\rangle_{\boldsymbol{L_2}} \\
&= \frac{1}{2}\begin{pmatrix} k_1 & * \\ k_2 & * \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
k_1 &= \langle (\mathcal{H}+\mathcal{H}|)[f_1]+(-\mathcal{H}+\mathcal{H}|)[f_2], g_1 \rangle \\
&\quad + \langle (\mathcal{H}+\mathcal{H}|)[f_2]+(-\mathcal{H}+\mathcal{H}|)[f_1], g_2 \rangle \\
&= \langle f_1, (\mathcal{H}^*+\mathcal{H}|^*)[g_1]+(-\mathcal{H}^*+\mathcal{H}|^*)[g_2] \rangle \\
&\quad + \langle f_2, (\mathcal{H}^*+\mathcal{H}|^*)[g_2]+(-\mathcal{H}^*+\mathcal{H}|^*)[g_1] \rangle
\end{aligned}$$

and

$$\begin{aligned} k_2 &= \langle (\mathcal{H}+\mathcal{H}|)[f_1] + (-\mathcal{H}+\mathcal{H}|)[f_2], g_2\rangle \\ &\quad + \langle (\mathcal{H}+\mathcal{H}|)[f_2] + (-\mathcal{H}+\mathcal{H}|)[f_1], g_1\rangle \\ &= \langle f_1, (\mathcal{H}^*+\mathcal{H}|^*)[g_2] + (-\mathcal{H}^*+\mathcal{H}|^*)[g_1]\rangle \\ &\quad + \langle f_2, (\mathcal{H}^*+\mathcal{H}|^*)[g_1] + (-\mathcal{H}^*+\mathcal{H}|^*)[g_2]\rangle \end{aligned}$$

So:

$$\begin{aligned} &\left\langle \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1], \boldsymbol{G}_2^1\right\rangle_{\boldsymbol{L_2}} \\ &= \left\langle \begin{pmatrix} f_1 & * \\ f_2 & * \end{pmatrix}, \frac{1}{2}\begin{pmatrix} (\mathcal{H}^*+\mathcal{H}|^*)[g_1] + (-\mathcal{H}^*+\mathcal{H}|^*)[g_2] & * \\ (\mathcal{H}^*+\mathcal{H}|^*)[g_2] + (-\mathcal{H}^*+\mathcal{H}|^*)[g_1] & * \end{pmatrix}\right\rangle_{\boldsymbol{L_2}} \\ &= \left\langle \boldsymbol{F}_2^1, \boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1]\right\rangle_{\boldsymbol{L_2}} \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{\mathcal{H}}^* &= \frac{1}{2}\begin{pmatrix} \mathcal{H}^*+\mathcal{H}|^* & -\mathcal{H}^*+\mathcal{H}|^* \\ -\mathcal{H}^*+\mathcal{H}|^* & \mathcal{H}^*+\mathcal{H}|^* \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \nu\,\mathcal{H}\,\nu + \nu|\,\mathcal{H}|\,\nu| & -\nu\,\mathcal{H}\,\nu + \nu|\,\mathcal{H}|\,\nu| \\ -\nu\,\mathcal{H}\,\nu + \nu|\,\mathcal{H}|\,\nu| & \nu\,\mathcal{H}\,\nu + \nu|\,\mathcal{H}|\,\nu| \end{pmatrix} \\ &= \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{V}} \end{aligned}$$

$\boldsymbol{\mathcal{P}(4)}$ Let $\boldsymbol{G}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$. Then $\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$ if and only if $\boldsymbol{G}_2^1$ is the non–tangential boundary limit of a certain $\boldsymbol{F}_2^1 \in \boldsymbol{H^2}(\Gamma^+)$, i.e. if and only if there exists a matrix function $\boldsymbol{F}_2^1 : \Gamma^+ \to (\mathbb{C}_{2n})^{2\times 2}$ such that

$$\lim_{\Gamma^+ \xrightarrow{NT} \partial\Gamma} \boldsymbol{F}_2^1 = \boldsymbol{G}_2^1 \quad \text{and} \quad \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\,\boldsymbol{F}_2^1 = \boldsymbol{O} \tag{11.29}$$

Now taking the explicit forms

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{F}_2^1 = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}$$

the characterization (11.29) of $\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$ may be given in terms of the matrix entries as follows: there exist functions $f_1, f_2 : \Gamma^+ \to \mathbb{C}_{2n}$ such that

$$\left\{\begin{aligned} \lim_{\Gamma^+ \xrightarrow{NT} \partial\Gamma} (f_1 - f_2) &= g_1 - g_2 \\ \lim_{\Gamma^+ \xrightarrow{NT} \partial\Gamma} (f_1 + f_2) &= g_1 + g_2 \end{aligned}\right. \qquad \text{and} \qquad \left\{\begin{aligned} \partial_{\underline{X}}\,[f_1 - f_2] &= 0 \\ \partial_{\underline{X}|}[f_1 + f_2] &= 0 \end{aligned}\right.$$

which is equivalent with

$$g_1 - g_2 \in H^2(\partial\Gamma) \quad \text{and} \quad g_1 + g_2 \in H|^2(\partial\Gamma)$$

or with

$$\mathcal{H}[g_1 - g_2] \;=\; g_1 - g_2 \quad \text{and} \quad \mathcal{H}|[g_1 + g_2] \;=\; g_1 + g_2 \tag{11.30}$$

when taking into account P(4) of Property 5.9 and 11.1. This ends the proof since (11.30) is equivalent to $\boldsymbol{\mathcal{H}}[\boldsymbol{G}_2^1] = \boldsymbol{G}_2^1$. ■

These properties then lead to the following orthogonal decomposition of $\boldsymbol{L_2}(\partial\Gamma)$ in terms of the Hardy space $\boldsymbol{H^2}(\partial\Gamma)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\boldsymbol{L_2}}$.

Proposition 11.3.

$$\boldsymbol{L_2}(\partial\Gamma) \;=\; \boldsymbol{H^2}(\partial\Gamma) \oplus_\perp \boldsymbol{\mathcal{V}} \boldsymbol{H^2}(\partial\Gamma)$$

Proof.
We need to characterize the orthogonal complement, denoted $\left(\boldsymbol{H^2}(\partial\Gamma)\right)^\perp$, of the space $\boldsymbol{H^2}(\partial\Gamma)$. To this end, take an arbitrary matrix function $\boldsymbol{F}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ and note that $\boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1] \in \boldsymbol{H^2}(\partial\Gamma)$, since $\boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1]] = \boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1]$. Thus, for any $\boldsymbol{G}_2^1 \in \left(\boldsymbol{H^2}(\partial\Gamma)\right)^\perp$, one has

$$\left\langle \boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1], \boldsymbol{G}_2^1 \right\rangle_{\boldsymbol{L_2}} \;=\; \boldsymbol{O} \tag{11.31}$$

which may be rewritten as

$$\left\langle \boldsymbol{F}_2^1, \boldsymbol{G}_2^1 + \boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1] \right\rangle_{\boldsymbol{L_2}} \;=\; \boldsymbol{O} \tag{11.32}$$

whence $\boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1] = -\boldsymbol{G}_2^1$. Conversely, take arbitrarily $\boldsymbol{G}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$ such that $\boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1] = -\boldsymbol{G}_2^1$, then $\boldsymbol{G}_2^1$ fulfils (11.32) and hence also (11.31). From this, we may conclude that $\boldsymbol{G}_2^1 \in \left(\boldsymbol{H^2}(\partial\Gamma)\right)^\perp$, since any function in $\boldsymbol{H^2}(\partial\Gamma)$ can always be written in the form $\boldsymbol{F}_2^1 + \boldsymbol{\mathcal{H}}[\boldsymbol{F}_2^1]$, $\boldsymbol{F}_2^1 \in \boldsymbol{L_2}(\partial\Gamma)$. So one has that

$$\boldsymbol{G}_2^1 \in \left(\boldsymbol{H^2}(\partial\Gamma)\right)^\perp \quad \Longleftrightarrow \quad \boldsymbol{\mathcal{H}}^*[\boldsymbol{G}_2^1] \;=\; -\boldsymbol{G}_2^1$$

On account of property $\boldsymbol{\mathcal{P}(3)}$ and of the fact that $\boldsymbol{\mathcal{V}}^2 \;=\; -\mathbb{E}_2$, this is seen to be equivalent to $\boldsymbol{\mathcal{H}}[\boldsymbol{\mathcal{V}}\boldsymbol{G}_2^1] = \boldsymbol{\mathcal{V}}\boldsymbol{G}_2^1$, or still to $\boldsymbol{\mathcal{V}}\boldsymbol{G}_2^1 \in \boldsymbol{H^2}(\partial\Gamma)$, in view of property $\boldsymbol{\mathcal{P}(4)}$. Once more invoking $\boldsymbol{\mathcal{V}}^2 = -\mathbb{E}_2$ we thus have shown that

$$\boldsymbol{G}_2^1 \in \left(\boldsymbol{H^2}(\partial\Gamma)\right)^\perp \quad \Longleftrightarrow \quad \boldsymbol{G}_2^1 \in \boldsymbol{\mathcal{V}}\boldsymbol{H^2}(\partial\Gamma)$$

■

Conclusion

In this doctoral thesis we accomplished the following results.

First of all we elaborated the four families of distributions in orthogonal Clifford analysis which were already considered in [26, 25] by Brackx, Delanghe and Sommen. After normalizing those distributions, we studied their interrelation by means of the multiplication with the vector $\underline{x}$, the action of the Dirac operator and the Fourier transform. We also obtained convolution and multiplication formulae for couples of those normalized distributions. The former formulae were then used to construct the fundamental solutions of complex powers of the Dirac operator.

Next, we constructed two possible generalizations of the Clifford–Hilbert transform on $\mathbb{R}^m$. Their kernels were deliberately chosen from the abovementioned distributions, in such a way that the corresponding convolution operators preserve as much traditional properties of the classical Clifford–Hilbert transform as possible. In the first approach for generalization it was shown that the kernels constitute a refinement of the generalized Hilbert kernels introduced by Horváth in [70]. Our resulting generalized Hilbert transforms were shown to be no longer unitary operators, yet they remain bounded singular operators on $L_2(\mathbb{R}^m)$. The second approach was based on the intimate relationship between the Hilbert transform and the Cauchy integral and started with the construction of a generalized Cauchy integral on $\mathbb{R}^{m+1}$ involving a distribution from one of the aforementioned families as a generalized Cauchy kernel. A new generalized Hilbert transform on $\mathbb{R}^m$ was then defined as part of the L_2 or distributional boundary limits of the generalized Cauchy integral considered, and it was shown to be a bounded operator on the Sobolev spaces $W_2^n(\mathbb{R}^m)$. Finally, we dealt with the action of the Radon transform on the two types of generalized Hilbert operators.

Further, inspired by the two–dimensional anisotropic Hilbert transform introduced in [56], we presented in the setting of anisotropic Clifford analysis, a Hilbert transform on $\mathbb{R}^m$, arising naturally as a part of the non–tangential boundary limits of the anisotropic Cauchy integral on $\mathbb{R}^{m+1}$. We could show that this new operator possesses formally the same properties as the classical Clifford–Hilbert operator. Moreover, we found that there is no one-to-one correspondence between a given anisotropic Hilbert transform and the associated anisotropic Cauchy integral from which it originates. The latter may stem from a diversity of metric tensors in $\mathbb{R}^{m+1}$.

To conclude, we examined Hilbert transforms in de framework of Hermitean Clifford analysis. First of all, on the half space, we extensively studied the Hilbert–like operator $\mathcal{K}$ which was introduced in [31, 32]. In particular, new Hardy spaces associated to this operator were defined and characterized. Secondly, on bounded domains, we could obtain a Cauchy integral formula for Hermitean monogenic functions in the framework of circulant (2×2) matrices. As an additional result, the obtained Hermitean Clifford–Cauchy integral formula turned out to reduce to the traditional Martinelli–Bochner formula for holomorphic functions of several complex variables when considering the special case of functions taking values in the n–homogeneous part of complex spinor space. This means that the theory of Hermitean monogenic functions not only refines orthogonal Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, even encompassing some of its results. A new Hermitean Clifford–Hilbert transform on closed surfaces in $\mathbb{R}^m$ then arised naturally as part of the non–tangential boundary limits of the Hermitean Clifford–Cauchy integral. The resulting matrix Hilbert operator was shown to satisfy properly adapted analogues of the characteristic properties of the orthogonal Clifford–Hilbert transform on closed surfaces in $\mathbb{R}^m$.

Summarizing, we have established and thoroughly investigated suitable generalizations of the multidimensional Hilbert transform in three main branches of Clifford analysis which are studied nowadays: Euclidean, anisotropic and Hermitean Clifford analysis.

Nederlandse samenvatting

In deze doctoraatsverhandeling bestuderen we enkele specifieke families van meerdimensionale distributies in het kader van orthogonale cliffordanalyse. De observatie dat de convolutiekern van de klassieke clifford–hilberttransformatie tot één van die families behoort, leidt vervolgens tot de constructie van verschillende veralgemeningen van die kern, zorgvuldig gerekruteerd uit de distributies van de verschillende families. Dit alles wordt behandeld in Deel I. Vervolgens, in Deel II, introduceren en bestuderen we een nieuwe hilberttransformatie in het kader van de metriekafhankelijke cliffordanalyse. Tot slot is Deel III gewijd aan een studie van verscheidene, nieuwe hilberttransformaties in het kader van hermitische cliffordanalyse.

We geven nu een meer gedetailleerd overzicht van de inhoud van deze scriptie.

In het inleidend Hoofdstuk 2 steken we van wal met de presentatie van de klassieke hilberttransformatie op de reële rechte. Deze transformatie vindt haar toepassingen in de theoretische beschrijving van vele elektronische componenten en systemen en wordt daar eerder bestempeld als (analoge of digitale) hilbertfilter. In het bijzonder vermelden we het gebruik van de hilberttransformatie in de constructie van het zogenaamd analytisch signaal waarvan uitvoerig gebruik wordt gemaakt in de theorie van signalen, schakelingen en systemen in elektronische apparatuur. Wij belichten echter meer de theoretische kant van de hilberttransformatie. In de eerste paragraaf brengen we de fundamentele en karakteriserende eigenschappen van de hilberttransformatie op de reële rechte in herinnering. Die eigenschappen fungeren in het verdere verloop van deze thesis als toetsstenen: er wordt onderzocht of de meerdimensionale veralgemeningen van die originele, eendimensionale hilberttransformatie voldoen aan op gepaste manier geherformuleerde analoge eigenschappen. Speciale aandacht wordt besteed aan de relatie tussen de hilberttransformatie op de reële rechte en de cauchyintegraal in het complexe vlak. Op een natuurlijke manier leidt dit tot twee welbekende, isomorfe hardyruimten. De eerste hardyruimte bevat holomorfe functies gedefinieerd in het bovenste halfvlak van het complexe vlak; de tweede hardyruimte, gedefinieerd op de reële rechte, bevat functies die invariant zijn onder de hilberttransformatie. We beëindigen dan de eerste paragraaf met de notie van analytisch signaal, waarbij de hilberttransformatie een onontbeerlijk werktuig is voor zowel globale als lokale beschrijving van een reëelwaardig signaal. In de tweede paragraaf komen een aantal bestaande, meerdimensionale, scalairwaardige veralgemeningen van de eendimensionale hilberttransformatie aan bod teneinde een bespreking te houden over de voordelen en beperkingen van hun geassocieerde analytische signalen. Tegelijkertijd worden hun eigenschappen getoetst aan die van de hilberttransformatie op de reële rechte. Voor nadere toelichting over laatst genoemde transformatie, refereren we de lezer aan o.a. [98, 79, 5, 60, 66]. Meerdimensionale, scalairwaardige hilberttransformaties zijn geïntroduceerd in bijvoorbeeld [99, 95, 55, 66].

Het opzet van Deel I is tweeledig. Enerzijds introduceren we in Hoofdstuk 4 enkele specifieke families van distributies in het kader van orthogonale cliffordanalyse. Alhoewel een aantal van die distributies al eerder opdook in de literatuur over harmonische analyse en cliffordanalyse, biedt het classificeren ervan in families de mogelijkheid om resultaten en formules, die erg verspreid liggen in de literatuur, te bundelen. Tevens bekomen we met behulp van deze aanpak ook originele resultaten. Anderzijds is Hoofdstuk 6 gewijd aan een extensieve studie van nieuwe, meerdimensionale hilberttransformaties waarvan de convolutiekernen nauwgezet geselecteerd worden uit de hierboven vermelde distributieverzamelingen.

We vatten echter Deel I aan met het inleidend Hoofdstuk 3 om de lezer te voorzien van de noodzakelijke taal van orthogonale cliffordalgebra en cliffordanalyse. De algebra's die we behandelen vinden hun oorsprong in het artikel [41] van W. K. Clifford en werden aanvankelijk geometrische algebra's genoemd, omdat ze binnen één enkele structuur zowel het inwendig als het uitwendig product van vectoren bevatten. Ze veralgemenen o.a. zowel de uitwendige algebra van Grassmann als Hamiltons algebra van quaternionen. Een van de meest eenvoudige, niet–triviale cliffordalgebra's is de algebra van complexe getallen, verkregen door constructie van de universele cliffordalgebra over het veld van de reële getallen. Een welbekend resultaat in complexe analyse luidt dat de tweedimensionale laplaciaan kan worden ontbonden als het product van de cauchy–riemannoperator met zijn complex toegevoegde. Derhalve zijn holomorfe functies, dit zijn nuloplossingen van de cauchy–riemannoperator, ook harmonisch. Het is in deze zin dat orthogonale cliffordanalyse te voorschijn komt als een natuurlijke veralgemening tot hogere dimensie van complexe analyse in het vlak. Is $(e_1, \ldots, e_m)$ een orthonormale basis voor de euclidische ruimte $\mathbb{R}^m$, dan wordt het punt $(x_1, \ldots, x_m)$ van $\mathbb{R}^m$ geïdentificeerd met de cliffordvectorvariabele

$$\underline{x} \;=\; \sum_{j=1}^{m} e_j \, x_j$$

Bovendien wordt een elegante veralgemening tot hogere dimensie van de cauchy–riemannoperator geïntroduceerd, namelijk de zogenaamde diracoperator

$$\partial_{\underline{x}} \;=\; \sum_{j=1}^{m} e_j \, \partial_{x_j}$$

Orthogonale cliffordanalyse is dan een meerdimensionale functietheorie in het kader van een zekere cliffordalgebra en geconcentreerd rond het begrip van de zogenaamd monogene functies, dit zijn nuloplossingen van bovenstaande cliffordvectorwaardige diracoperator. Aangezien de diracoperator de laplaciaan factoriseert, zijn monogene functies ook harmonisch en bovendien houden hun eigenschappen een verfijning in van deze van harmonische functies. We vermelden ook nog dat de gekende fundamentele oplossing van de diracoperator gegeven wordt door

$$E(\underline{x}) \;=\; \frac{1}{a_m} \, \frac{\overline{\underline{x}}}{|\underline{x}|^m} \;, \quad \underline{x} \neq \underline{0}$$

met a_m de oppervlakte van de eenheidssfeer S^{m-1} in $\mathbb{R}^m$ en $\bar{\cdot}$ de gebruikelijke toevoeging in de cliffordalgebra gedefinieerd als $\overline{e_j} = -e_j$, $j = 1, \ldots, m$. We sluiten dit hoofdstuk af met de zogenaamd sferische monogenen omdat zij een fundamentele rol spelen bij de constructie van onze veralgemeende hilberttransformaties. Deze sferische monogenen worden bekomen als restricties tot de eenheidssfeer S^{m-1} van homogene, monogene veeltermfuncties $P_p(\underline{x})$ van een zekere graad p die we cliffordvectorwaardig nemen. We merken hier tot slot op dat cliffordanalyse de laatste decennia meer en meer aan interesse gewonnen heeft en zelfs is uitgegroeid tot een op zichzelf staande onderzoekstak binnen de klassieke analyse. Een diepgaande studie van cliffordanalyse waarbij de parallellen getrokken worden tussen de klassieke complexe functietheorie enerzijds en die monogene functietheorie anderzijds, kan worden teruggevonden in het toonaangevende boek [23] van Brackx, Delanghe en Sommen. Voor een verdere studie van deze hoger dimensionale functietheorie en haar toepassingen citeren we volgende referenties: [62, 60, 52, 77, 63, 88, 87, 8, 48, 82, 61].

Vervolgens presenteren we in Hoofdstuk 4 een aantal verwante families van clifforddistributies die reeds beschouwd werden in [26, 25] door Brackx, Delanghe en Sommen. Een van de meest frappante, gemeenschappelijke eigenschappen van die distributies is de manier van inwerking op een scalairwaardige testfunctie ϕ in de euclidische ruimte. Introduceren we sferische coördinaten

$$\underline{x} \;=\; r\,\underline{\omega}\;, \quad r \;=\; |\underline{x}| > 0\;, \quad \underline{\omega} \in S^{m-1}$$

dan laten de gekende sferische gemiddelden

$$\begin{aligned}
\Sigma^{(0)}[\phi] &= \frac{1}{a_m}\int_{S^{m-1}} \phi(r\underline{\omega})\,dS(\underline{\omega}) \\
\Sigma^{(1)}[\phi] &= \Sigma^{(0)}[\underline{\omega}\phi] \;=\; \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\,\phi(r\underline{\omega})\,dS(\underline{\omega})
\end{aligned}$$

waar $dS(\underline{\omega})$ staat voor de lebesguemaat op S^{m-1}, de ontwikkeling toe van een eenvoudige, doch krachtige en uiterst efficiënte methode die het mogelijk maakt om de distributionele actie in het originele kader van de euclidische ruimte om te zetten in een actie die plaatsvindt op de reële rechte met behulp van de distributie "finite parts". Voor een complexe parameter λ worden de scalairwaardige distributies T_λ en de cliffordvectorwaardige distributies U_λ dan als volgt gedefinieerd:

$$\begin{aligned}
\langle\, T_\lambda\;,\;\phi\,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu}\;,\;\Sigma^{(0)}[\phi]\,\right\rangle \\
\langle\, U_\lambda\;,\;\phi\,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu}\;,\;\Sigma^{(1)}[\phi]\,\right\rangle
\end{aligned}$$

met $\mu = \lambda + m - 1$. Bovenstaande auteurs veralgemeenden ook de sferische gemiddelden door in hun definitie sferische monogenen te introduceren:

$$\begin{aligned}
\Sigma_p^{(0)}[\phi] &= r^{p-p_e}\,\Sigma^{(0)}[P_p(\underline{\omega})\,\phi(\underline{x})] = \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} P_p(\underline{\omega})\,\phi(\underline{x})\,dS(\underline{\omega}) \\
\Sigma_p^{(1)}[\phi] &= r^{p-p_e}\,\Sigma^{(0)}[\underline{\omega}\,P_p(\underline{\omega})\,\phi(\underline{x})] \;=\; \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} \underline{\omega}\,P_p(\underline{\omega})\,\phi(\underline{x})\,dS(\underline{\omega}) \\
\Sigma_p^{(2)}[\phi] &= r^{p-p_e}\,\Sigma^{(0)}[\underline{\omega}\,P_p(\underline{\omega})\,\underline{\omega}\,\phi(\underline{x})] \;=\; \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} \underline{\omega}\,P_p(\underline{\omega})\,\underline{\omega}\,\phi(\underline{x})\,dS(\underline{\omega}) \\
\Sigma_p^{(3)}[\phi] &= r^{p-p_e}\,\Sigma^{(0)}[P_p(\underline{\omega})\,\underline{\omega}\,\phi(\underline{x})] \;=\; \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} P_p(\underline{\omega})\,\underline{\omega}\,\phi(\underline{x})\,dS(\underline{\omega})
\end{aligned}$$

met $p_e = p$ als p even is en $p_e = p - 1$ als p oneven is. Dit leidt tot meer algemene families van clifforddistributies, gedefinieerd als

$$\begin{aligned}
\langle\, T_{\lambda,p} \,,\, \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \,,\, \Sigma_p^{(0)}[\phi] \,\right\rangle \\
\langle\, U_{\lambda,p} \,,\, \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \,,\, \Sigma_p^{(1)}[\phi] \,\right\rangle \\
\langle\, W_{\lambda,p} \,,\, \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \,,\, \Sigma_p^{(2)}[\phi] \,\right\rangle \\
\langle\, V_{\lambda,p} \,,\, \phi \,\rangle &= a_m \left\langle\, \mathrm{Fp}\, r_+^{\mu+p_e} \,,\, \Sigma_p^{(3)}[\phi] \,\right\rangle
\end{aligned}$$

waarin de distributies T_λ en U_λ vervat zijn in het speciale geval waar de graad p van de beschouwde sferische monogeen nul genomen wordt, namelijk

$$T_\lambda = T_{\lambda,0} = -W_{\lambda,0} \qquad \text{en} \qquad U_\lambda = U_{\lambda,0} = V_{\lambda,0}$$

De beschouwde families van distributies erven een aftelbaar aantal singuliere punten van de distributie "finite parts". Door middel van de gekende techniek van deling door een geschikte gammafunctie die dezelfde singulariteiten vertoont, hebben wij die singuliere punten kunnen verwijderen, wat resulteert in de volgende normaliseringen van bovenstaande veralgemeende distributies, met $l \in \mathbb{N}_0$:

$$\begin{cases}
T^*_{\lambda,p} = \pi^{\frac{\lambda+m}{2}+p} \dfrac{T_{\lambda,p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)}\,, & \lambda \neq -m-2p-2l \\[2ex]
T^*_{-m-2p-2l,p} = \dfrac{(-1)^l\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l\right)}\, P_p(\underline{x})\Delta^{p+l}\delta(\underline{x})
\end{cases}$$

$$\begin{cases}
U^*_{\lambda,p} = \pi^{\frac{\lambda+m+1}{2}+p} \dfrac{U_{\lambda,p}}{\Gamma\left(\frac{\lambda+m+1}{2}+p\right)}\,, & \lambda \neq -m-2p-2l-1 \\[2ex]
U^*_{-m-2p-2l-1,p} = \dfrac{(-1)^{p+1}\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l+1}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l+1\right)} \left(\partial_{\underline{x}}^{2p+2l+1}\delta(\underline{x})\right) P_p(\underline{x})
\end{cases}$$

$$\begin{cases}
V^*_{\lambda,p} = \pi^{\frac{\lambda+m+1}{2}+p} \dfrac{V_{\lambda,p}}{\Gamma\left(\frac{\lambda+m+1}{2}+p\right)}\,, & \lambda \neq -m-2p-2l-1 \\[2ex]
V^*_{-m-2p-2l-1,p} = \dfrac{(-1)^{p+1}\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l+1}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l+1\right)}\, P_p(\underline{x}) \left(\partial_{\underline{x}}^{2p+2l+1}\delta(\underline{x})\right)
\end{cases}$$

$$\left\{\begin{array}{lcl} W^*_{\lambda,p} & = & \pi^{\frac{\lambda+m}{2}+p} \dfrac{W_{\lambda,p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)} \ , \qquad \lambda \neq -m-2p-2l \\ W^*_{-m-2p-2l,p} & = & \dfrac{(-1)^l l! \pi^{\frac{m}{2}-l}}{2^{2p+2l+2}(p+l+1)!\Gamma\left(\frac{m}{2}+p+l+1\right)} \underline{x} P_p(\underline{x}) \underline{x}\, \Delta^{p+l+1} \delta(\underline{x}) \end{array}\right.$$

Tot slot komt een grondige studie van hun eigenschappen aan bod die sterke verbanden blootlegt tussen de verschillende families.

In Hoofdstuk 5 voeren we de klassieke clifford–hilberttransformatie in en haar karakteriserende eigenschappen. Voor zover ons bekend, was Horváth de eerste om een vectorwaardige hilberttransformatie te definiëren in de euclidische ruimte $\mathbb{R}^m$, gebruik makend van een cliffordalgebra (zie [69]). Deze meerdimensionale hilberttransformatie in het kader van cliffordanalyse werd opnieuw bestudeerd vanaf de jaren 1980, zie [91, 60, 76, 50, 51], maar ook bijvoorbeeld [36, 77, 39, 4, 42, 49] voor de fundamentele rol die deze transformatie speelt in de studie van hardyruimten van monogene functies. In de eerste paragraaf van dit hoofdstuk stellen we een alternatieve definitie voor van de cliffordvectorwaardige hilberttransformatie van Horváth. Voor een functie $f \in L_2(\mathbb{R}^m)$ wordt de clifford–hilberttransformatie $\mathcal{H}[f]$ op $\mathbb{R}^m$ gegeven door

$$\begin{aligned} \mathcal{H}[f](\underline{x}) &= \frac{2}{a_{m+1}}\, \overline{e_0}\, \mathrm{Pv} \int_{\mathbb{R}^m} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{\left|\underline{x}-\underline{y}\right|^{m+1}} f(\underline{y})\, dV(\underline{y}) \\ &= \frac{2}{a_{m+1}}\, \overline{e_0} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^m \setminus B(\underline{x};\varepsilon)} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{\left|\underline{x}-\underline{y}\right|^{m+1}} f(\underline{y})\, dV(\underline{y}) \end{aligned}$$

of, voor een geschikte distributie f, door middel van de convolutie

$$\mathcal{H}[f] \ = \ \overline{e_0}\, H * f$$

waarbij H de convolutiekern is, gedefinieerd door

$$H(\underline{x}) \ = \ \frac{2}{a_{m+1}}\, \mathrm{Pv}\, \frac{\underline{\overline{\omega}}}{r^m} \ = \ -\frac{2}{a_{m+1}}\, U^*_{-m,0}$$

We vermelden haar belangrijkste eigenschappen.

P(1) De clifford–hilberttransformatie commuteert met translaties, wat equivalent is met haar definitie als convolutieoperator.

P(2) De clifford–hilberttransformatie commuteert met dilataties, wat equivalent is met het feit dat haar convolutiekern homogeen is van de graad $(-2n)$.

P(3) De clifford–hilberttransformatie is een begrensde, lineaire operator op $L_2(\mathbb{R}^m)$, wat equivalent is met het feit dat haar fouriersymbool

$$\mathcal{F}[H](\underline{x}) \;=\; i\underline{\omega}$$

een begrensde functie is.

P(4) De clifford–hilberttransformatie is een involutie op $L_2(\mathbb{R}^m)$.

P(5) De clifford–hilberttransformatie is unitair op $L_2(\mathbb{R}^m)$.

P(6) De clifford–hilberttransformatie anticommuteert met de diracoperator.

P(7) De clifford–hilberttransformatie is een deel van de niet–tangentiële randwaarden van de cauchyintegraal in $\mathbb{R}^{m+1}$.

In het bijzonder geven we nadere toelichting bij eigenschap P(7). Voor een functie $f \in L_2(\mathbb{R}^m)$ wordt de cauchyintegraal $\mathcal{C}[f]$ in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$ gegeven door

$$\mathcal{C}[f](x_0, \underline{x}) \;=\; (C(x_0, \cdot) * f(\cdot))\,(\underline{x}) \;=\; \int_{\mathbb{R}^m} C(x_0, \underline{x} - \underline{y})\, f(\underline{y})\, dV(\underline{y})$$

waarbij de cauchykern C, gedefinieerd door

$$C(x) \;=\; C(x_0, \underline{x}) \;=\; \frac{1}{a_{m+1}} \frac{\overline{x} e_0}{|x|^{m+1}} \;=\; \frac{1}{a_{m+1}} \frac{x_0 + e_0 \underline{x}}{|x_0 + \underline{x}|^{m+1}}\;, \qquad x \neq 0$$

de fundamentele oplossing voorstelt van de cauchy–riemannoperator in $\mathbb{R}^{m+1}$:

$$D_x \;=\; \partial_{x_0} + \overline{e_0}\, \partial_{\underline{x}}$$

De niet–tangentiële randwaarden van de cauchyintegraal leiden dan tot de gekende plemelj–sokhotzkiformules:

$$\begin{aligned} \lim_{x_0 \to 0+} \mathcal{C}[f](x_0, \underline{x}) &= \frac{1}{2}\, f(\underline{x}) + \frac{1}{2}\, \mathcal{H}[f](\underline{x}) \\ \lim_{x_0 \to 0-} \mathcal{C}[f](x_0, \underline{x}) &= -\frac{1}{2}\, f(\underline{x}) + \frac{1}{2}\, \mathcal{H}[f](\underline{x}) \end{aligned}$$

Door middel van deze betrekkingen kan dan ook een mooi verband gelegd worden met de theorie van hardyruimten. Vervolgens komen we terug op het concept analytisch signaal waarvoor we een meerdimensionale veralgemening voorstellen in de context van cliffordanalyse. Ter afsluiting van deze paragraaf onderzoeken we de interactie tussen twee hoofdrolspelers in meerdimensionale signaalanalysetheorie, namelijk de clifford–hilberttransformatie en de clifford–radontransformatie. Voor een geschikte cliffordalgebrawaardige functie f, vinden we volgend verband tussen beide transformaties:

$$\mathcal{R}[\mathcal{H}[f]](\underline{n}, s) \ = \ e_0 \, \underline{n} \, \mathcal{H}_{u \to s}[\mathcal{R}[f](\underline{n}, u)](s) \ , \quad (\underline{n}, s) \in S^{m-1} \times \mathbb{R}$$

waarbij de clifford–radontransformatie $\mathcal{R}[f]$ van f gedefinieerd wordt door

$$\mathcal{R}[f](\underline{n}, s) \ = \ \int_{\mathbb{R}^m} \delta(\langle \underline{x}, \underline{n} \rangle - s) \, f(\underline{x}) \, dV(\underline{x})$$

De bovenstaande notatie $\langle \cdot, \cdot \rangle$ staat voor voor het klassiek scalair product in de euclidische ruimte $\mathbb{R}^m$. In de tweede paragraaf beschouwen we een open gebied Ω in $\mathbb{R}^m$ en daarin bevat een m–dimensionale, compacte, afleidbare en geöriënteerde variëteit Γ met C^∞–gladde rand $\partial\Gamma$. Voorts introduceren we het klassieke oppervlakelement $dS(\underline{x})$ op $\partial\Gamma$ en daaraan geassocieerd het vectorwaardige oppervlakelement $\widetilde{d\sigma_{\underline{x}}} = \nu(\underline{x}) dS(\underline{x})$, met $\nu(\underline{x})$ de uitwendig gerichte eenheidsnormaalvector op $\underline{x} \in \partial\Gamma$. Tot slot voeren we nog Γ^+ en Γ^- in als het inwendige van Γ, respectievelijk, het uitwendige van Γ ten opzichte van Ω. Voor een functie $f \in L_2(\partial\Gamma)$ wordt de clifford–cauchyintegraal $\mathcal{C}[f]$ in $\Gamma^\pm$ dan gedefinieerd door

$$\mathcal{C}[f](\underline{y}) \ = \ \int_{\partial\Gamma} E(\underline{x} - \underline{y}) \, \widetilde{d\sigma_{\underline{x}}} \, f(\underline{x}) \ , \qquad \underline{y} \in \Gamma^\pm$$

De clifford–hilberttransformatie $\mathcal{H}[f]$ op $\partial\Gamma$ komt vervolgens op een natuurlijke manier te voorschijn wanneer de niet–tangentiële randwaarden van de clifford–cauchyintegraal $\mathcal{C}[f]$, i.e. plemelj–sokhotzkiformules, beschouwd worden:

$$\lim_{\substack{\underline{y} \to \underline{u} \\ \underline{y} \in \Gamma^+}} \mathcal{C}[f](\underline{y}) \ = \ \frac{1}{2} f(\underline{u}) + \frac{1}{2} \mathcal{H}[f](\underline{u}) \ , \ \ \underline{u} \in \partial\Gamma$$

$$\lim_{\substack{\underline{y} \to \underline{u} \\ \underline{y} \in \Gamma^-}} \mathcal{C}[f](\underline{y}) \ = \ -\frac{1}{2} f(\underline{u}) + \frac{1}{2} \mathcal{H}[f](\underline{u}) \ , \ \ \underline{u} \in \partial\Gamma$$

De clifford–hilberttransformatie $\mathcal{H}[f]$ wordt dan gedefinieerd door de cauchyhoofdwaarde–integraal

$$\mathcal{H}[f](\underline{u}) \ = \ 2 \ \mathrm{Pv} \int_{\partial\Gamma} E(\underline{x} - \underline{u}) \, \widetilde{d\sigma_{\underline{x}}} \, f(\underline{x}) \ , \qquad \underline{u} \in \partial\Gamma$$

en voldoet verder aan de volgende eigenschappen.

P(1) De clifford–hilberttransformatie is een begrensde, lineaire operator op $L_2(\partial\Gamma)$.

P(2) De clifford–hilberttransformatie is een involutie op $L_2(\partial\Gamma)$.

P(3) De clifford–hilberttransformatie heeft als toegevoegde operator $\mathcal{H}^* = \nu\,\mathcal{H}\,\nu$ op $L_2(\partial\Gamma)$.

P(4) De clifford–hilberttransformatie karakteriseert de hardyruimte $H^2(\partial\Gamma)$ als de verzameling van functies $f \in L_2(\partial\Gamma)$ waarvoor $\mathcal{H}[f] = f$.

Het moge duidelijk zijn dat, in het algemeen, haar eigenschappen zwakker zijn dan die van de clifford–hilberttransformatie op $\mathbb{R}^m$, behalve voor het geval van de eenheidssfeer waaraan we speciale aandacht besteden.

Deel I wordt besloten met onze constructie van twee veralgemeningen van de clifford–hilberttransformatie op $\mathbb{R}^m$. Hun kernen worden met zorg gekozen uit de clifforddistributies geïntroduceerd in Hoofdstuk 4, zodat de ermee corresponderende convolutieoperatoren een maximaal aantal traditionele, maar aan de huidige meerdimensionale situatie aangepaste, eigenschappen van de clifford–hilberttransformatie behouden. In de eerste aanpak tot veralgemening wordt aangetoond dat de geselecteerde distributies als een verfijning mogen worden beschouwd van de veralgemeende hilbertkernen van Horváth in [70]. Onze corresponderende veralgemeende hilberttransformaties

$$\begin{aligned} \mathcal{T}_p &= T_{-m-p,p} * f \\ \mathcal{U}_p &= U_{-m-p,p} * f \\ \mathcal{V}_p &= V_{-m-p,p} * f \\ \mathcal{W}_p &= W_{-m-p,p} * f \end{aligned}$$

blijven begrensde singuliere operatoren op $L_2(\mathbb{R}^m)$; toch blijken ze echter niet langer unitair te zijn. De tweede aanpak is gebaseerd op de intieme relatie tussen de hilberttransformatie en de cauchyintegraal. Opnieuw wordt beroep gedaan op de distributies uit Hoofdstuk 4. Een van die distributies werpt zich namelijk op als goede kandidaat om als veralgemeende cauchykern te fungeren voor de constructie van een veralgemeende cauchyintegraal in $\mathbb{R}^{m+1}$. Voor een zeker natuurlijk getal p en een functie $f \in L_2(\mathbb{R}^m)$, introduceren we de veralgemeende cauchyintegraal $\mathcal{C}_p[f]$ in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$ als

$$\mathcal{C}_p[f](x_0, \underline{x}) = (C_p(x_0, \cdot) * f(\cdot))\,(\underline{x}) = \int_{\mathbb{R}^m} C_p(x_0, \underline{x} - \underline{y})\, f(\underline{y})\, dV(\underline{y})$$

waarbij de veralgemeende cauchykern C_p, gegeven door

$$C_p(x) = C_p(x_0, \underline{x}) = \frac{1}{a_{m+1,p}} \frac{\bar{x} e_0}{|x|^{m+1+2p}} P_p(\underline{x})$$
$$= \frac{1}{a_{m+1,p}} \frac{x_0 + e_0 \underline{x}}{|x_0 + \underline{x}|^{m+1+2p}} P_p(\underline{x}) \, , \qquad x \neq 0$$

met

$$a_{m+1,p} = \frac{(-1)^p}{2^p} \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2} + p\right)}$$

in distributionele zin voldoet aan

$$D_x \, C_p(x) = P_p(\partial_{\underline{x}}) \, \delta(x)$$

De niet–tangentiële randwaarden van de veralgemeende cauchyintegraal leiden dan tot een veralgemening van de plemelj–sokhotzkiformules:

$$\lim_{x_0 \to 0+} \mathcal{C}_p[f](x_0, \underline{x}) = \frac{1}{2} P_p(\partial_{\underline{x}}) f(\underline{x}) + \frac{1}{2} \mathcal{H}_p[f](\underline{x})$$
$$\lim_{x_0 \to 0-} \mathcal{C}_p[f](x_0, \underline{x}) = -\frac{1}{2} P_p(\partial_{\underline{x}}) f(\underline{x}) + \frac{1}{2} \mathcal{H}_p[f](\underline{x})$$

met $\mathcal{H}_p[f]$ een nieuwe, veralgemeende hilberttransformatie in $\mathbb{R}^m$, gegeven door

$$\mathcal{H}_p[f] = \overline{e_0} \, H_p * f$$

waarbij H_p de convolutiekern is, gedefinieerd als

$$H_p(\underline{x}) = \frac{2}{a_{m+1,p}} \, \mathrm{Fp} \, \frac{\underline{\bar{\omega}} \, P_p(\underline{\omega})}{r^{m+p}} = -\frac{2}{a_{m+1,p}} \, U^*_{-m-2p,p}$$

Er wordt aangetoond dat deze nieuwe hilberttransformatie een begrensde operator is op de sobolevruimten $W_2^n(\mathbb{R}^m)$. Met behulp van de actie van hogere orde diracafgeleiden kunnen we ook een verband leggen tussen beide veralgemeningen. De laatste paragraaf wordt gewijd aan de actie van de clifford–radontransformatie op de twee types van veralgemeende hilberttransformaties. Het resultaat van de actie van de clifford–radontransformatie op de veralgemeende hilberttransformaties $\mathcal{T}_p[f]$, $\mathcal{U}_p[f]$, $\mathcal{V}_p[f]$ en $\mathcal{W}_p[f]$ hangt af van de pariteit van p. Voor p even verkrijgen we:

$$\begin{aligned}
\mathcal{R}[\mathcal{T}_p[f]](\underline{n}, s) &= i^{-p}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\, P_p(\underline{n})\, \mathcal{R}[f](\underline{n}, s) \\
\mathcal{R}[\mathcal{U}_p[f]](\underline{n}, s) &= i^{-p}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, \underline{n}\, P_p(\underline{n})\, \mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n}, u)](s) \\
\mathcal{R}[\mathcal{V}_p[f]](\underline{n}, s) &= i^{-p}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, P_p(\underline{n})\, \underline{n}\, \mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n}, u)](s) \\
\mathcal{R}[\mathcal{W}_p[f]](\underline{n}, s) &= i^{-p-2}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)} \left(\underline{n}\, P_p(\underline{n})\, \underline{n} - \frac{m-2}{p}\, P_p(\underline{n})\right) \\
&\quad \times \mathcal{R}[f](\underline{n}, s)
\end{aligned}$$

terwijl voor p oneven:

$$\begin{aligned}
\mathcal{R}[\mathcal{T}_p[f]](\underline{n}, s) &= i^{-p+1}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\, P_p(\underline{n})\, \mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n}, u)](s) \\
\mathcal{R}[\mathcal{U}_p[f]](\underline{n}, s) &= i^{-p-1}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, \underline{n}\, P_p(\underline{n})\, \mathcal{R}[f](\underline{n}, s) \\
\mathcal{R}[\mathcal{V}_p[f]](\underline{n}, s) &= i^{-p-1}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\, P_p(\underline{n})\, \underline{n}\, \mathcal{R}[f](\underline{n}, s) \\
\mathcal{R}[\mathcal{W}_p[f]](\underline{n}, s) &= i^{-p-1}\, \pi^{\frac{m}{2}}\, \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{m+p}{2}+1\right)} \left(\underline{n}\, P_p(\underline{n})\, \underline{n} - \frac{m-2}{p}\, P_p(\underline{n})\right) \\
&\quad \times \mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n}, u)](s)
\end{aligned}$$

De inwerking van de clifford–radontransformatie op de veralgemeende hilberttransformatie $\mathcal{H}_p[f]$ leidt tot

$$\mathcal{R}[\mathcal{H}_p[f]](\underline{n}, s) \;=\; e_0\, \underline{n}\, P_p(\underline{n})\, \partial_s^p \mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n}, u)](s)$$

De tot hiertoe beschouwde (veralgemeende) hilberttransformaties op $\mathbb{R}^m$ mogen worden getypeerd als isotroop, vermits de aangewende metriek in de onderliggende ruimte de standaard euclidische metriek is. Deel II adopteert nu het idee van een anisotrope (ook genoemd metriekafhankelijke of metrodynamische) cliffordcontext. Dit biedt de mogelijkheid om het assenstelsel aan te passen aan bepaalde, niet noodzakelijk loodrechte, voorkeursrichtingen, bijvoorbeeld aanwezig in de te analyseren structuren of signalen. In dit nieuwe domein van cliffordanalyse (zie bijvoorbeeld [35, 53]) construeren wij een zogenaamde anisotrope clifford–hilberttransformatie.

In het inleidend Hoofdstuk 7 worden de basisbegrippen van anisotrope cliffordanalyse voorgesteld. We introduceren eerst de notie metrische tensor als een reële, symmetrische en positief definiete tensor $\widetilde{G} = (g_{kl})_{k,l=0,\ldots,m}$ van orde $(m+1)$ die twee bases in $\mathbb{R}^{m+1}$, een covariante basis $(e_0, \ldots, e_m)$ en een contravariante basis $(e^0, \ldots, e^m)$, met elkaar in verband brengt:

$$e_k = \sum_{l=0}^{m} g_{kl}\, e^l \quad \text{en} \quad e^l = \sum_{k=0}^{m} g^{lk}\, e_k \ , \quad \text{met} \quad \widetilde{G}^{-1} = (g^{kl})_{k,l=0,\ldots,m}$$

Vervolgens wordt een cliffordalgebra geconstrueerd, afhankelijk van die metrische tensor, en alle voor ons noodzakelijke definities en resultaten van orthogonale cliffordanalyse worden aangepast aan dit metriekafhankelijk kader. Onder andere worden de concepten diracoperator, monogeniciteit en laplaceoperator ingevoerd. We vermelden bijvoorbeeld dat het klassiek scalair product vervangen wordt door de symmetrische bilineaire vorm

$$\langle\, x, y \,\rangle_{\widetilde{G}} = \sum_{k=0}^{m} \sum_{l=0}^{m} g_{kl}\, x^k\, y^l$$

We beëindigen dit hoofdstuk met de definitie en studie van de zogenaamde anisotrope fouriertransformatie, de metriekafhankelijke versie van de klassieke fouriertransformatie in de euclidische ruimte.

In Hoofdstuk 8 beschouwen we dan de deeltensor $G = (g_{kl})_{k,l=1,\ldots,m}$ in $\mathbb{R}^{m\times m}$ van de metrische tensor $\widetilde{G}$ in $\mathbb{R}^{(m+1)\times(m+1)}$. Onze nieuwe anisotrope hilberttransformatie in $\mathbb{R}^m$ wordt gedefinieerd door

$$\mathcal{H}_{G,c}[f] = \overline{e^0}\, H_{G,c} * f$$

waarbij de convolutiekern $H_{G,c}$ gegeven wordt door

$$H_{G,c}(\underline{x}) = \frac{2c}{a_{m+1}}\, \mathrm{Pv}\, \frac{\overline{\underline{x}}}{(\langle\, \underline{x}, \underline{x} \,\rangle_G)^{\frac{m+1}{2}}} \ , \qquad c = \sqrt{\det(\widetilde{G})}$$

Er wordt aangetoond dat deze anisotrope hilberttransformatie formeel gezien dezelfde eigenschappen vertoont als haar isotrope tegenhanger uit Paragraaf 5.1.

In het bijzonder observeren we dat eenzelfde anisotrope hilberttransformatie in $\mathbb{R}^m$ kan worden bekomen als deel van de randwaarden van meerdere anisotrope cauchyintegralen in $\mathbb{R}^{m+1}$, afhangend van een diversiteit aan metrische tensors van orde $(m+1)$.

In Deel III staat de ontwikkeling centraal van nieuwe meerdimensionale hilberttransformaties in het kader van hermitische cliffordanalyse, een vrij recente tak binnen cliffordanalyse (zie bijvoorbeeld [86, 85, 89, 43, 34, 6, 7, 54]).

Het inleidend Hoofdstuk 9 behandelt de basisingrediënten van hermitische cliffordanalyse, een nieuwe en succesvolle functietheorie die een verfijning inhoudt van orthogonale cliffordanalyse. Zij concentreert zich op de gemeenschappelijke nuloplossingen, genoemd hermitisch monogene (of h–monogene) functies, van twee hermitische diracoperatoren die invariant zijn onder de actie van een realisatie van de unitaire groep. In de eerste paragraaf voeren we een zogenaamde complexe structuur J in, die een even dimensie $m = 2n$ aan de vectorruimte $\mathbb{R}^m$ oplegt en die aangewend wordt om de elementaire objecten van orthogonale cliffordanalyse om te zetten in het nieuwe hermitische kader. Zo wordt de actie van J op de voortbrengers $(e_1, \ldots, e_{2n})$ van $\mathbb{R}^{2n}$ gegeven door

$$J[e_j] \;=\; -e_{n+j} \qquad \text{en} \qquad J[e_{n+j}] \;=\; e_j \;, \qquad j = 1, \ldots, n$$

Herschrijven we dan de euclidische vector $\underline{X} = (X_1, \ldots, X_{2n})$ in $\mathbb{R}^{0,2n}$ als

$$\underline{X} \;=\; (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

en identificeren we deze, zoals gewoonlijk, met de cliffordvector

$$\underline{X} \;=\; \sum_{j=1}^{n} (e_j\, x_j + e_{n+j}\, y_j)$$

dan kunnen we aan $\underline{X}$ de geroteerde vector $\underline{X}|$ associëren door inwerking van de complexe structuur J op $\underline{X}$, i.e.

$$\underline{X}| \;=\; J[\underline{X}] \;=\; \sum_{j=1}^{n} (e_j\, y_j - e_{n+j}\, x_j)$$

Op dezelfde manier wordt ook een geroteerde diracoperator ingevoerd:

$$\partial_{\underline{X}|} \;=\; J[\partial_{\underline{X}}] \;=\; \sum_{j=1}^{n} (e_j\, \partial_{y_j} - e_{n+j}\, \partial_{x_j})$$

De inwerking van de projectieoperatoren $\pm\frac{1}{2}(\mathbf{1}\pm iJ)$ op $\underline{X}$ leidt dan tot de introductie van de hermitische cliffordvariabele $\underline{Z}$ en zijn hermitisch toegevoegde $\underline{Z}^\dagger$:

$$\begin{aligned} \underline{Z} &= \frac{1}{2}(\mathbf{1}+iJ)[\underline{X}] = \frac{1}{2}(\underline{X}+i\,\underline{X}|) \\ \underline{Z}^\dagger = (\underline{Z})^\dagger &= -\frac{1}{2}(\mathbf{1}-iJ)[\underline{X}] = -\frac{1}{2}(\underline{X}-i\,\underline{X}|) \end{aligned}$$

De inwerking van de operatoren $\pm\frac{1}{4}(\mathbf{1}\pm iJ)$ op $\partial_{\underline{X}}$ leidt tot de invoering van de hermitische diracoperatoren

$$\begin{aligned} \partial_{\underline{Z}^\dagger} = (\partial_{\underline{Z}})^\dagger &= \frac{1}{4}(\mathbf{1}+iJ)[\partial_{\underline{X}}] = \frac{1}{4}(\partial_{\underline{X}}+i\,\partial_{\underline{X}|}) \\ \partial_{\underline{Z}} &= -\frac{1}{4}(\mathbf{1}-iJ)[\partial_{\underline{X}}] = -\frac{1}{4}(\partial_{\underline{X}}-i\,\partial_{\underline{X}|}) \end{aligned}$$

We noemen een functie g dan h–monogeen als ze voldoet aan het stelsel

$$\partial_{\underline{Z}}\, g = 0 = \partial_{\underline{Z}^\dagger}\, g$$

In de tweede paragraaf wordt een opsplitsing van bovenstaand h–monogeen stelsel beschouwd die al bestudeerd was in [7] en die leidt tot de zogenaamde homogene delen van een complexe spinorruimte.

Tijdens de bestudering van clifford–hermitewavelets in de context van hermitische cliffordanalyse, zie [31, 32], ontdekten de auteurs onverwacht een nieuw type operator, verkregen als samenstelling van twee orthogonale hilberttransformaties op $\mathbb{R}^{2n}$. De resulterende operator, $\mathcal{K}$ genoteerd, bleek te voldoen aan enkele typische eigenschappen van een klassieke hilberttransformatie. In Hoofdstuk 10 voeren wij een dieper onderzoek uit naar deze $\mathcal{K}$–transformatie. In de eerste paragraaf introduceren we daarom, naast the klassieke clifford–hilberttransformatie, een tweede orthogonale clifford–hilberttransformatie, $\mathcal{H}|$ genoteerd, op $\mathbb{R}^{2n}$, gegeven door

$$\mathcal{H}|[f] = \overline{e_0}\, H| * f$$

waarvan de kern verkregen wordt door de actie van de complexe structuur J op de klassieke orthogonale clifford–hilbertkern:

$$H|(\underline{X}) = J[H(\underline{X})] = \frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\overline{\underline{X}|}}{|\underline{X}|^{2n+1}} = \frac{2}{a_{2n+1}}\,\mathrm{Pv}\,\frac{\overline{\underline{X}|}}{|\underline{X}||^{2n+1}} = H(J[\underline{X}])$$

De eerder vermelde $\mathcal{K}$–transformatie wordt dan gedefinieerd als

$$\mathcal{K} \;=\; i\,\mathcal{H}\,\mathcal{H}| \;=\; -i\,\mathcal{H}|\,\mathcal{H}$$

Vervolgens ontstaan twee nieuwe, isotrope, hermitische hilberttransformaties door inwerking van de projectieoperatoren $\pm\frac{1}{2}(\mathbf{1}\pm iJ)$ op de orthogonale clifford–hilbertkern. De $\mathcal{K}$–transformatie wordt dan opnieuw verkregen, nu als commutator van die twee hermitische clifford–hilberttransformaties. In de tweede paragraaf worden dan alle verbanden en overeenkomsten tussen alle hierboven vermelde transformaties grondig onderzocht. Bijzondere aandacht gaat daarbij uit naar de introductie en karakterisering van nieuwe hardyruimten geassocieerd aan de $\mathcal{K}$–transformatie. Een aantal resultaten leent zich ook tot een mooie meetkundige interpretatie. In de laatste paragraaf wordt het begrip analytisch signaal herbekeken.

Beschouwen we nu opnieuw Ω en Γ zoals geïntroduceerd in Paragraaf 5.2. In orthogonale cliffordanalyse fungeert de clifford–cauchyintegraalformule

$$g(\underline{Y}) \;=\; \int_{\partial\Gamma} E(\underline{X}-\underline{Y})\,\widetilde{d\sigma_{\underline{X}}}\,g(\underline{X})\;, \qquad \underline{Y}\in\Gamma^+$$

voor monogene functies g als hoeksteen van die functietheorie, net zoals dit het geval is voor de traditionele cauchyformule voor holomorfe functies in het complexe vlak. Het is evident dat een cauchyintegraalformule voor h–monogene functies essentieel is voor de verder ontwikkeling van hermitische cliffordanalyse, maar pogingen tot dusver voor het verkrijgen van deze formule waren niet bevredigend. In de eerste paragraaf van Hoofdstuk 11 komen wij tot het gewenste resultaat, echter in het kader van circulaire (2×2)–matrixfuncties. Zonder te veel in detail te treden voeren we de nodige notaties in om te komen tot onze formule. Om te beginnen worden voor de fundamentele oplossingen E en $E| = J[E]$ van de diracoperatoren $\partial_{\underline{X}}$, respectievelijk $\partial_{\underline{X}|}$, hermitische tegenhangers gepresenteerd, namelijk

$$\mathcal{E} \;=\; -\,(E+i\,E|) \quad \text{en} \quad \mathcal{E}^\dagger \;=\; (E-i\,E|)$$

We merken op dat $\mathcal{E}$ en $\mathcal{E}^\dagger$ niet de fundamentele oplossingen voorstellen van de respectieve hermitische diracoperatoren $\partial_{\underline{Z}}$ en $\partial_{\underline{Z}^\dagger}$. Vervolgens worden hermitische tegenhangers ingevoerd voor de vectorwaardige oppervlakelementen $\widetilde{d\sigma_{\underline{X}}}$

en $\widetilde{d\sigma}_{\underline{X}|} = J[\widetilde{d\sigma}_{\underline{X}}]$, namelijk

$$\begin{aligned} d\sigma_{\underline{Z}} &= -\frac{1}{4}\,(-1)^{\frac{n(n+1)}{2}}\,(2i)^n\,(\widetilde{d\sigma}_{\underline{X}} - i\,\widetilde{d\sigma}_{\underline{X}|}) \\ d\sigma_{\underline{Z}^\dagger} &= -\frac{1}{4}\,(-1)^{\frac{n(n+1)}{2}}\,(2i)^n\,(\widetilde{d\sigma}_{\underline{X}} + i\,\widetilde{d\sigma}_{\underline{X}|}) \end{aligned}$$

Introduceren we daarna de circulaire (2×2)–matrices

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix} \quad \text{en} \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

dan observeren we dat

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\,\boldsymbol{\mathcal{E}}(\underline{Z},\underline{Z}^\dagger) = \boldsymbol{\delta}(\underline{Z})$$

We interpreteren deze uitdrukking als volgt: in de huidige matrixcontext mag $\boldsymbol{\mathcal{E}}$ worden beschouwd als een fundamentele oplossing van $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$. We associëren dan aan de cliffordalgebrawaardige, continu afleidbare functies g_1 en g_2 de matrixfunctie

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

We noemen dan $\boldsymbol{G}_2^1$ (links) **H**–monogeen als en slechts als voldaan is aan $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_2^1 = \boldsymbol{O}$, met $\boldsymbol{O}$ de nulmatrix. Echter, in het speciale geval van een diagonale matrixfunctie

$$\boldsymbol{G}_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

zien we dat **H**–monogeniteit van de matrixfunctie $\boldsymbol{G}_0$ equivalent is met h–monogeniteit van de functie g. Introduceren we tot slot nog de matrix

$$\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} d\sigma_{\underline{Z}} & -d\sigma_{\underline{Z}^\dagger} \\ -d\sigma_{\underline{Z}^\dagger} & d\sigma_{\underline{Z}} \end{pmatrix}$$

die de rol zal spelen van matrixdifferentiaalvorm, dan zien de hermitische clifford–cauchyintegraalformules voor **H**–monogene matrixfuncties $\boldsymbol{G}_2^1$ en h–monogene functies g er als volgt uit:

$$\begin{aligned} &\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\,\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\,\boldsymbol{G}_2^1(\underline{X}) \\ &\quad = \begin{cases} \boldsymbol{O}\,, & \text{als } \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}}\,(2i)^n\,\boldsymbol{G}_2^1(\underline{Y})\,, & \text{als } \underline{Y} \in \Gamma^+ \end{cases} \end{aligned}$$

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_0(\underline{X})$$
$$= \begin{cases} \boldsymbol{O}\ , & \text{als } \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \boldsymbol{G}_0(\underline{Y})\ , & \text{als } \underline{Y} \in \Gamma^+ \end{cases}$$

Als een bijkomend resultaat blijkt onze hermitische clifford–cauchyintegraalformule zich te herleiden tot de traditionele martinelli–bochnerformule voor holomorfe functies van meerdere complexe variabelen door het speciale geval van functies die waarden aannemen in het n–homogene deel van een complexe spinorruimte te beschouwen. Dit betekent dat de theorie van h–monogene functies niet enkel orthogonale cliffordanalyse (en dus ook harmonische analyse) verfijnt, maar ook sterke verbanden heeft met en resultaten omvat van de theorie van functies van meerdere complexe variabelen. Vermits de matrixfunctie $\boldsymbol{\mathcal{E}}$ die opduikt in bovenstaande formules fundamentele oplossing is van de diracmatrix $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$, bestempelen we die als hermitische cauchykern. Voor continue functies g_1 en g_2 definiëren we dan de volgende hermitische clifford–cauchyintegraal:

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1](\underline{Y}) \ = \ \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}, \underline{Z}^\dagger - \underline{V}^\dagger)\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})\ , \qquad \underline{Y} \in \Gamma^\pm$$

In de tweede paragraaf komen we uiteindelijk terecht bij de definitie van een nieuwe hermitische clifford–hilberttransformatie op $\partial\Gamma$, gegeven door de matrixoperator

$$\boldsymbol{\mathcal{H}} \ = \ \frac{1}{2} \begin{pmatrix} \mathcal{H} + \mathcal{H}| & -\mathcal{H} + \mathcal{H}| \\ -\mathcal{H} + \mathcal{H}| & \mathcal{H} + \mathcal{H}| \end{pmatrix}$$

die op een natuurlijke manier opduikt als deel van de niet–tangentiële randwaarden van de hermitische clifford–cauchyintegraal. Voor functies $g_1, g_2 \in L_2(\partial\Gamma)$ verkrijgen we namelijk

$$\lim_{\substack{\underline{Y} \to \underline{U} \\ \underline{Y} \in \Gamma^\pm}} \boldsymbol{\mathcal{C}}[\boldsymbol{G}_2^1](\underline{Y}) \ = \ (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\pm \frac{1}{2}\, \boldsymbol{G}_2^1(\underline{U}) + \frac{1}{2}\, \boldsymbol{\mathcal{H}}[\boldsymbol{G}_2^1](\underline{U}) \right)\ , \qquad \underline{U} \in \partial\Gamma$$

De matrixhilbertoperator blijkt dan tevens de karakteriserende, geherformuleerde eigenschappen te bezitten van de clifford–hilberttransformatie zoals gepresenteerd in Paragraaf 5.2.

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