View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Ghent University Academic Bibliography

Inspect every piece of pseudoscience and you will find
a security blanket, a thumb to suck, a skirt to hold.
What does the scientist have to offer in exchange?
Uncertainty! Insecurity!

— Isaac Asimov

I think people get it upside down when they say the unambiguous is the reality and
the ambiguous is merely uncertainty about what is really unambiguous.
Let's turn it around the other way: the ambiguous is the reality and the unambiguous is
merely a special case of it, where we finally manage to pin down some very special aspect.

— David Bohm

But ignorance of the different causes involved in the production of events,
as well as their complexity, taken together with the imperfection of analysis,
prevents our reaching the same certainty about the vast majority of phenomena.

— Pierre-Simon Laplace

Contents

Preface

Dear reader,

Maybe you are interested in algebraic structures. Maybe you are interested in logic. Maybe you know me and you are curious about what I have been doing the last four and half years. Or maybe you are a student who has been ordered to summarize a PhD-dissertation, and you chose the one with the fewest number of pages.

In the last case, allow me to warn you that this PhD-dissertation might not only be the one with the fewest number of pages, but also the one with the greatest density of formulas and proofs. Indeed, according to my promotor Etienne Kerre, this dissertation is the most compact he has ever supervised. This might not be so special, were it not that more than 25 students already have obtained their doctoral degree under his guidance. Unfortunately for students to come, he is about to retire soon. A very well deserved pension, if I may say so, even though in my eyes he can go on for at least another 10 students. Maybe – however, never say never – I am his last doctoral student, and as people say “the last mile is the longest" (they also tend to say “the last straw breaks the camel’s back", but let’s hope Etienne’s back just bended a little and did not break). However, he has been a great help for me, not only in giving guidance to my scientific research, but also in assisting me with all kinds of administrative work. I would really like to thank him, for all these efforts and for the patience he has had with me. Furthermore, I wish him all the best with his retirement!

Not yet retired are my other two supervisors, Chris Cornelis and Glad Deschrijver. Nevertheless, I wish them all the best as well. I am very grateful I could always count on them for checking proofs, helping out with LaTeX, finding the right formulations and terminology, recommending interesting papers, suggesting approaches to tackle the problems I encountered, and so on. If they had not done their job so well, I might even have listened to them and leave Ghent for a stay abroad in another institution.

However, I did meet researchers from other institutions at conferences and workshops. Some of them were of great help to me: we had fruitful discussions, they sent me papers, suggested methods or computer software I could use and/or helped me out when there was something I did not understand. In particular, I would like to thank Félix Bou, Petr Cintula, Didier Dubois, Francesc Esteva, Nick Galatos, Lluis Godo, Petr Hájek, Ulrich Höhle, Rostislav Horčík, Afrodita Iorgulescu, Sándor Jenei, Peter Jipsen, George Metcalfe, Carles Noguera and Hiroakira Ono.

My colleagues in Ghent deserve an acknowledgement as well. Our conversations, the lunch

breaks, the poker games, the conferences we attended together, the other activities we did, it has always been pleasant spending time with them. Yes, even the expensive paintball afternoon. In particular, I want to thank Yun Shi, with whom I cooperated on some papers about fuzzy implicators. I wish her good luck finding a new job, so she can stay in Belgium.

My parents should not be forgotten either. Without them I would not even be here, let alone write a PhD-dissertation. I cannot thank them enough for their everlasting support.

Finally, I want to express my gratitude to the BOF (Bijzonder Onderzoeksfonds) and FWO (Research Foundation - Flanders) for supplying me with the financial support I needed for my research activities and an expensive paintball afternoon.

Last but not least, I would like to thank you, reader of this preface, for reading also the remainder of this dissertation. Enjoy!

Bart Van Gasse
Labour Day/May Day/International Workers Day, 2010

Chapter 1

Introduction

Classical logic is a two-valued logic: propositions in this logic are either true or false. In the first case, the truth value 1 is attributed to the proposition, while in the second case the truth value is 0. Given the truth values of two propositions p and q, it is possible to derive the truth values of the negation 'not p' (and 'not q'), the conjunction 'p and q', the disjunction 'p or q' and the implication 'p implies q'. These formulas are denoted as $\neg p$, $p\&q$, $p \vee q$ and $p \to q$. The truth values are calculated using the operations[1] $\neg$, $*$, $\sqcup$ and $\Rightarrow$:

Table 1.1: Truth tables of the operations in classical logic.

x	y	$\neg x$	$x * y$	$x \sqcup y$	$x \Rightarrow y$
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	1	1

For more complicated formulas the truth values can be computed in the same way. For example, if p is true and q is false, then the truth value of $(p \vee q) \to ((p \to q) \to q)$ is calculated as follows: the truth value of $p \to q$ is $1 \Rightarrow 0 = 0$, so the truth value of $(p \to q) \to q$ is $0 \Rightarrow 0 = 1$. The truth value of $p \vee q$ is $1 \sqcup 0 = 1$. So we conclude that the truth value of $(p \vee q) \to ((p \to q) \to q)$ is $1 \Rightarrow 1 = 1$. Interestingly, the truth value of this formula is always 1, even if other truth values are attributed to p and q. Such formulas are called tautologies. If a formula φ is a tautology, this is denoted as $\models \varphi$. More generally, for a set of formulas Γ, $\Gamma \models \varphi$ means "no matter what truth values are attributed to the propositions, if the truth values of the formulas in Γ are 1, then the truth value of φ is 1".

The two values 0 and 1, together with the defined operations, form a Boolean algebra [47]. Therefore we say that this Boolean algebra is the semantics of classical logic. Saying that $(p \vee q) \to ((p \to q) \to q)$ is a tautology in classical logic, is the same as saying that $(x \sqcup y) \Rightarrow ((x \Rightarrow y) \Rightarrow y) = 1$ is an identity in this Boolean algebra (meaning "whatever value of the Boolean algebra we give to x and y, the calculation of $(x \sqcup y) \Rightarrow ((x \Rightarrow y) \Rightarrow y)$ yields 1").

[1] Note that we use different symbols, to distinguish the logical connectives from the corresponding operations. Only for the negation we employ the same symbol.

Now, identities in this Boolean algebra are also identities in every other Boolean algebra[2] (we say that this Boolean algebra generates all Boolean algebras). Therefore classical logic does not only have the Boolean algebra with two elements as semantics, but also the whole variety of Boolean algebras: the general semantics of classical logic consists of all Boolean algebras.

Interestingly it is also possible to describe classical logic without using semantics. This is done with axioms and deduction rules, which allow to prove a formula from a set of formulas. When a formula φ is provable from a theory Γ, this is denoted as $\Gamma \vdash \varphi$. Two important results in classical logic are soundness (if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$) and completeness (if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$). We write this shortly as $\Gamma \vdash \varphi$ iff[3] $\Gamma \models \varphi$.

Now, for the truth values of several propositions one might prefer more than the two options 0 (false) and 1 (true). Indeed, for vague propositions like 'it is raining hard', it would be useful if one could attribute an intermediate truth value, somewhere between 'false' and 'true'. It was Zadeh [81] who came up with the idea of fuzzy sets, in which every element of the unit interval $[0, 1]$ could serve as a truth value, instead of only 0 and 1. The operations for the negation, conjunction, disjunction and implication were generalized to this setting. Later Goguen [39] replaced the structure of the unit interval by an arbitrary bounded lattice to allow for incomparabilities among elements, and triangular norms and conorms [52, 68] are quite common nowadays as generalized representations of logical conjunction and disjunction, respectively. An interesting class of these generalizations, especially from the logical point of view, are MTL-algebras [28]. In these structures, the operations modelling (strong) conjunction and implication are connected by the residuation principle. These MTL-algebras form the general semantics of monoidal t-norm based logic (MTL, [28]), in the same way Boolean algebras form the general semantics of classical logic. Similarly as for classical logic, these general semantics can be restricted. Indeed, MTL is also sound and complete w.r.t. standard MTL-algebras, i.e., MTL-algebras on the unit interval. Therefore MTL is called a (formal) fuzzy logic[4]. But it is definitely not the only fuzzy logic. Indeed, by adding more axioms and/or deduction rules to the axioms and deduction rule of MTL, we obtain other fuzzy logics. It is even possible to retrieve classical logic in this way. Semantically speaking, this means that Boolean algebras are special cases of MTL-algebras. Some well-known fuzzy logics, situated between MTL and classical logic, are Hájek's Basic Logic (BL) [42], Łukasiewicz Logic (Ł) [55] and Gödel Logic (GL) [25, 38]. Also Intuitionistic Logic (IL) [43] can be seen as a fuzzy logic. These logics are sound and complete w.r.t. BL-algebras, MV-algebras (or, equivalently, Wajsberg algebras [34]), G-algebras and Heyting algebras, respectively. We refer to [29] for a comprehensive overview of these and other logics. Other general references on fuzzy logics are [14, 41, 40, 42, 73].

In [82], Zadeh introduced type-2 fuzzy sets, a generalization of fuzzy sets. The idea behind these structures is that they provide a way to express incomplete as well as graded knowledge; as opposed to fuzzy sets, which only express gradedness, not incompleteness. Unfortunately, type-2 fuzzy sets are quite complicated to work with. Therefore often interval-valued fuzzy sets are used [16, 26, 37, 56, 57]. These special cases of type-2 fuzzy sets are easier to handle. Indeed, truth values in this setting are closed subintervals of the unit interval, and such an interval is deter-

[2] This follows, e.g., from Theorem 4.15.

[3] The notation 'iff' is short for 'if, and only if'. It means that two statements are equivalent and is often used in this work.

[4] A paper that discusses which logics should be called fuzzy logics and which not, is [4].

mined by just two values: its lower and upper bound. The aim of this work is to develop a logic that has intervals as truth values. The intended semantics are residuated lattices on the set of closed subintervals of the unit interval. We call this set the triangularization of the unit interval. A particular subset of this triangularization is its so-called diagonal, consisting of those intervals for which the lower and upper bound coincide. These intervals are called exact intervals and represent truth values of propositions about which the knowledge is complete. Intuitively, the truth values of formulas constructed with these propositions should be exact intervals as well (because in these cases, the situation is similar to working with formulas in fuzzy logics). The semantics of so-called interval-valued fuzzy logics have already been examined by different authors. Especially interval-valued triangular norms, triangular conorms and implicators have received ample attention. Most of these authors [1, 7, 26, 37, 44] only consider interval-valued operations that map the diagonal on the diagonal, although the most general definitions of triangular norms, triangular conorms and implicators allow other operations as well [15, 21, 19, 48]. Generally speaking, interval-valued operations satisfy not as many properties as operations on the unit interval. For example, standard interval-valued residuated lattices can never satisfy prelinearity [16]. A lot of other properties can hold though. There are even interval-valued implicators that satisfy all the Smets-Magrez axioms [17, 70].
At this point we should also mention that interval-valued fuzzy sets are equivalent with Atanassov's intuitionistic fuzzy sets [2], which was formally proven in [20].

The three main chapters of this work are conceived as follows:

- In Chapter 2 we elaborate the theory of interval-valued residuated lattices, which include the intended semantics of interval-valued fuzzy logics.
- In Chapter 3 we give the definition of triangle algebras, which are algebraic structures describing interval-valued residuated lattices. We study the properties of triangle algebras in great detail, because these triangle algebras will form the general semantics of our interval-valued fuzzy logics.
- In Chapter 4 we then introduce several interval-valued fuzzy logics and examine their properties, in particular the soundness and completeness w.r.t. the intended and the general semantics.

Most results in this work were presented at international conferences and workshops, and are published in scientific journals [74, 75, 76, 77, 78, 79, 80].

Before we continue, we recall some algebraic concepts that will be used in this work.

- An algebra of type $(n_1, n_2, \ldots, n_m)$, with $n_1, n_2, \ldots, n_m$ non-negative integers, is a structure $(A, f_1, \ldots, f_m)$ in which A is a set, f_1 an n_1-ary operation on A, ... and f_m an n_m-ary operation on A. If n_i is 0, then f_i is a constant.
- A reduct of an algebra is an algebra on the same set, but in which some of the operations are left out. An algebra $\mathscr{A}$ is an expansion of an algebra $\mathscr{B}$ if $\mathscr{B}$ is a reduct of $\mathscr{A}$.
- A subalgebra of an algebra $\mathscr{A} = (A, f_1, \ldots, f_m)$ is an algebra on a subset A' of A in which all operations of $\mathscr{A}$ are restricted to A'. Of course, this is only possible if A' is closed under all these operations, i.e., if for every operation f_i of $\mathscr{A}$, $f_i(a_1, \ldots, a_n) \in A'$ whenever the arguments $a_1, \ldots, a_n$ are in A' (with n the arity of f_i).

- The product of two algebras $\mathscr{A} = (A, f_1, \ldots, f_m)$ and $\mathscr{B} = (B, g_1, \ldots, g_m)$ of the same type, is the algebra (also of the same type) on $A \times B$ (the set of all couples (a, b), with $a \in A$ and $b \in B$) with operations $f_i \times g_i$ defined by $(f_i \times g_i)(a, b) = (f_i(a), g_i(b))$. Similarly for products of more than two algebras.

- A morphism from an algebra $\mathscr{A} = (A, f_1, \ldots, f_m)$ to an algebra $\mathscr{B} = (B, g_1, \ldots, g_m)$ of the same type, is a mapping h from A to B such that $h(f_i(a_1, \ldots, a_n)) = g_i(h(a_1), \ldots, h(a_n))$ for all operations f_i of $\mathscr{A}$ and all $a_1, \ldots, a_n$ in A (with n the arity of f_i).

- An embedding of an algebra $\mathscr{A} = (A, f_1, \ldots, f_m)$ in an algebra $\mathscr{B} = (B, g_1, \ldots, g_m)$ of the same type, is a morphism h from A to B such that $h(a_1) \neq h(a_2)$ whenever $a_1 \neq a_2$.

- An isomorphism from an algebra $\mathscr{A} = (A, f_1, \ldots, f_m)$ to an algebra $\mathscr{B} = (B, g_1, \ldots, g_m)$ of the same type, is an embedding of A in B such that for every element b of B, $b = h(a)$ for some a in A.

CHAPTER 2

Interval-valued structures

The most general semantics of fuzzy logics do not only contain algebraic structures on the unit interval, they consist of all residuated lattices. For interval-valued fuzzy logics, the situation is comparable: the most general semantics are interval-valued residuated lattices. In this chapter, we propose a definition of these structures and give some examples. We start with Section 2.1 about partially ordered sets and introduce triangularizations of these structures in Section 2.2. Then, in Section 2.3 we focus on particular partially ordered sets, named lattices. We also introduce triangular lattices, which describe the triangularizations of lattices with identities. In Section 2.4 we add two binary connectives to the lattice structure: a product and an implication. We give the basic definitions and properties of the obtained residuated lattices. In Section 2.5 we introduce several kinds of filters of residuated lattices and show the connections between them. Finally, in Section 2.6 we give the definition of interval-valued residuated lattices.

2.1 Partially ordered sets

Definition 2.1 *A partially ordered set (shortly: poset) is a couple $(P, \leqslant)$, in which P is a set and $\leqslant$ a reflexive, anti-symmetric and transitive binary relation on P. In other words, $\leqslant \subseteq P^2$ and – using the notation $x \leqslant y$ for $(x, y) \in \leqslant$ and saying 'x is smaller than or equal to y' – for all x, y and z in P*

- *$x \leqslant x$,*
- *if $x \leqslant y$ and $y \leqslant x$, then $x = y$,*
- *if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.*

The partial order $\leqslant$ is called a total order (or alternatively, a linear order) if for all x and y in P, $x \leqslant y$ or $y \leqslant x$. In this case the poset $(P, \leqslant)$ is called a chain.

We denote $x \geqslant y$ (and say 'x is greater than or equal to y') iff $y \leqslant x$, $x < y$ (and say 'x is smaller than y') iff $x \leqslant y$ and $x \neq y$, and $x > y$ (and say 'x is greater than y') iff $x < y$. Two elements x and y are called incomparable if $x \nleqslant y$ and $y \nleqslant x$. A poset is a chain iff it has no incomparable elements.
Note that if we restrict a partial order $\leqslant$ on a set P to a subset $P' \subseteq P$, then the resulting binary relation is a partial order on P'.

Definition 2.2 *Let $(P, \leqslant)$ be a poset, $\emptyset \subset A \subseteq P$ and $a \in P$. Then a is an upper (resp. lower) bound of A if, for all $x \in P$, $x \leqslant a$ (resp. $x \geqslant a$). The subset A is called bounded in $(P, \leqslant)$ if it has an upper as well as a lower bound. The poset $(P, \leqslant)$ is called bounded if P is bounded in $(P, \leqslant)$.*

A subset A can have at most one upper (resp. lower) bound that is smaller (resp. greater) than every other upper (resp. lower) bound. If such an element exists, it is called the supremum (resp. infimum) of A and denoted $\sup A$ (resp. $\inf A$).
A subset A can have at most one upper (resp. lower) bound that belongs to A. If such an element exists, it is called the maximum or greatest element (resp. minimum or smallest element) of A and denoted $\max A$ (resp. $\min A$). In this case, the supremum (resp. infimum) of A also exists and equals $\max A$ (resp. $\min A$).

Of particular interest are posets in which suprema and infima always exist.

- Posets in which the supremum and infimum of every two elements exist, are called lattices. In Section 2.3 we will propose an equational representation for these structures. In lattices, the supremum and infimum of all non-empty finite subsets exist and it holds that

$$\sup\{x_1, \ldots, x_n\} = \sup\{x_1, \sup\{x_2, \ldots \sup\{x_{n-1}, x_n\} \ldots\} \text{ and}$$
$$\inf\{x_1, \ldots, x_n\} = \inf\{x_1, \inf\{x_2, \ldots \inf\{x_{n-1}, x_n\} \ldots\}.$$

- Posets in which the supremum and infimum[1] of all subsets (including the empty set) exist, are called complete. Obviously, complete posets are lattices. The existence of $\sup\emptyset$ and $\inf\emptyset$ ensure that complete lattices have a minimum and a maximum, which implies that complete lattices are bounded.

Finite lattices are always complete. Indeed, in this case the suprema and infima of all non-empty subsets exist and the supremum and infimum of the empty set exist because they are equal to the infimum and the supremum of all elements.

A useful way to represent finite posets is by Hasse diagrams. In such a diagram the elements of the poset are represented by dots. The dots representing x and y are connected by a line iff x is smaller than y and there are no elements z such that $x < z$ and $z < y$ (or vice versa, y is smaller than x and there are no elements z such that $y < z$ and $z < x$). The dot corresponding to $\max\{x, y\}$ must be placed higher than the dot corresponding to $\min\{x, y\}$. So there are no horizontal lines in a Hasse diagram. The partial order $\leqslant$ can be recovered from the diagram in the following way: $x \leqslant y$ iff there is a path in the Hasse diagram from dot x to dot y that always goes upwards. Several examples can be found in this work; the first three are in Figure 2.2. As an example, in the third poset (L_3) $0 \leqslant a$ (there is an upwards going path from 0 to a, via b), but c is incomparable to a.

Now we define two notions which will be useful to model conjunction and implication in the logics that we define later.

Definition 2.3 *Let $(P, \leqslant)$ be a poset which has an upper bound* 1.

- *A triangular norm (t-norm, for short) [68] on $(P, \leqslant)$, is a binary, increasing, commutative and associative operator $T : P^2 \to P$ that satisfies $T(x, 1) = 1$, for all x in P.*

[1] Actually the existence of all suprema (resp. infima) implies the existence of all infima (resp. suprema) because one has that $\inf A = \sup\{c \in P \mid c \text{ is a lower bound of } A\}$ (resp. $\sup A = \inf\{c \in P \mid c \text{ is an upper bound of } A\}$). Therefore one only needs to verify one of the two conditions.

- *If for every pair (x,y) in P^2, $\sup\{z \in P \mid T(x,z) \leqslant y\}$ exists, then the map I_T defined by $I_T(x,y) = \sup\{z \in P \mid T(x,z) \leqslant y\}$ is called the residual implicator of T [52]. A t-norm T is called residuated [52] if it has a residual implicator satisfying $I_T(x,y) = \max\{z \in P \mid T(x,z) \leqslant y\}$, in other words if for any pair (x,y) in P^2 the set $\{z \in P \mid T(x,z) \leqslant y\}$ has a maximum.*

Obviously, in a complete poset every t-norm has a residual implicator. On a bounded poset $(P,\leqslant)$ (with smallest element 0 and greatest element 1), at least one t-norm always exists:

$$T(x,y) = \begin{cases} 0, & \text{if } x \neq 1 \neq y, \\ \min\{x,y\}, & \text{otherwise}^2. \end{cases}$$

It is called the drastic t-norm, and obviously it is the smallest possible t-norm.
Note that the residual implicator of a t-norm (if it exists) is always decreasing in the first argument and increasing in the second. Moreover, $I_T(1,y) = y$, and $I_T(x,y) = 1$ if $x \leqslant y$. So on a bounded poset, a residual implicator of a t-norm satisfies the following definition of implicator.

Definition 2.4 *Let $(P,\leqslant)$ be a bounded poset with smallest element 0 and greatest element 1. An implicator on $(P,\leqslant)$ is a binary operator I on P for which $I(0,0) = 1$, $I(1,0) = 0$, $I(1,1) = 1$ and $I(x_2,y_1) \leqslant I(x_1,y_2)$ if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.*

Example 2.5 *The unit interval $[0,1]$ equipped with the usual ordering $\leqslant$ is a poset. It is complete and linear. This particular poset is of great importance for many formal fuzzy logics, because the truth values are elements of $[0,1]$.*
The residuated t-norms on this poset are exactly the left-continuous (w.r.t. the usual Euclidean metric) t-norms [52].

2.2 Triangularizations

Definition 2.6 *Given any poset $\mathscr{P} = (P,\leqslant)$, we can define its triangularization $\mathbb{T}(\mathscr{P}) = (Int(\mathscr{P}), \preceq)$ in the following way:*

- *$Int(\mathscr{P}) = \{[p_1,p_2] \mid (p_1,p_2) \in P^2$ and $p_1 \leqslant p_2\}$,*
- *$[p_1,p_2] \preceq [q_1,q_2]$ iff $p_1 \leqslant q_1$ and $p_2 \leqslant q_2$, for all $[p_1,p_2]$ and $[q_1,q_2]$ in $Int(\mathscr{P})$.*

The elements of $Int(\mathscr{P})$ are called the intervals of $\mathscr{P}$.
The first and the second projection pr_1 and pr_2 are the mappings from $Int(\mathscr{P})$ to P, defined by $\mathrm{pr}_1([x_1,x_2]) = x_1$ and $\mathrm{pr}_2([x_1,x_2]) = x_2$, for all $[x_1,x_2]$ in $Int(\mathscr{P})$.
The vertical and the horizontal projection pr_v and pr_h are the mappings from $Int(\mathscr{P})$ to $Int(\mathscr{P})$, defined by $\mathrm{pr}_v([x_1,x_2]) = [x_1,x_1]$ and $\mathrm{pr}_h([x_1,x_2]) = [x_2,x_2]$, for all $[x_1,x_2]$ in $Int(\mathscr{P})$.

It is straightforward to verify that for any poset $\mathscr{P}$, $\mathbb{T}(\mathscr{P})$ is also a poset. Moreover, the original poset $(P,\leqslant)$ is contained in $\mathbb{T}(\mathscr{P})$ in some way: indeed, the mapping $i : P \longrightarrow Int(\mathscr{P})$ defined by $i(p) = [p,p]$ for all p in P, is injective and preserves the ordering (if $p \leqslant q$, then $i(p) = [p,p] \preceq [q,q] = i(q)$). The image $i(P)$ consists of the intervals $[p_1,p_2]$ in $Int(\mathscr{P})$ for

[2] By $x \neq 1 \neq y$, we mean $x \neq 1$ AND $1 \neq y$. So in this case, 'otherwise' means $x = 1$ OR $y = 1$.
Similarly for, e.g., $a < 0.5 \leqslant b \neq c$: this means $a < 0.5$ AND $0.5 \leqslant b$ AND $b \neq c$.

which $p_1 = p_2$. The elements of $i(P)$ are called exact intervals. The subset $i(P)$ of $Int(\mathscr{P})$ is often referred to as the diagonal of $\mathbb{T}(\mathscr{P})$. Note that $\mathrm{pr}_v = i \circ \mathrm{pr}_1$ and $\mathrm{pr}_h = i \circ \mathrm{pr}_2$, and that $i(P) = \mathrm{pr}_v(Int(\mathscr{P})) = \mathrm{pr}_h(Int(\mathscr{P}))$.

Example 2.7 *The poset that will be of central interest in this work, is*[3] $\mathbb{T}([0,1],\leqslant)$*: the subintervals of the unit interval. This poset is complete and its order is not linear. Its graphical representation as a triangle is shown in Figure 2.1. The diagonal is the hypothenuse of this triangle. Note that*

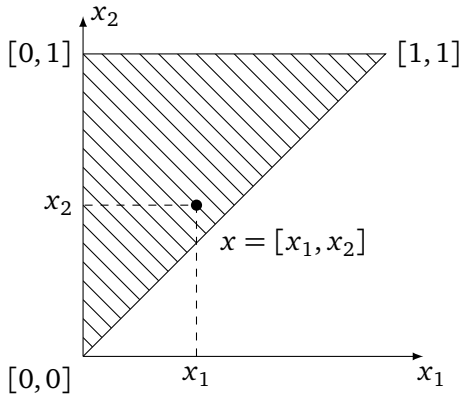

Figure 2.1: The lattice $\mathbb{T}([0,1],\leqslant)$

the shape of this representation is triangular. This holds for all triangularizations of bounded linear posets, hence the name 'triangularization'.

The following properties can be proven directly from the definitions:

- if $\mathscr{P}$ is bounded, then $\mathbb{T}(\mathscr{P})$ is bounded,
- if $\mathscr{P}$ is a lattice, then $\mathbb{T}(\mathscr{P})$ is a lattice,
- if $\mathscr{P}$ is complete, then $\mathbb{T}(\mathscr{P})$ is complete,
- if $\mathscr{P}$ is finite, then $\mathbb{T}(\mathscr{P})$ is finite.

Instead of $\preceq$, one can also define other orderings on $Int(\mathscr{P})$, e.g.

- $[p_1,p_2] \subseteq [q_1,q_2]$ iff $q_1 \leqslant p_1$ and $p_2 \leqslant q_2$. Note that if intervals $[p_1,p_2]$ are interpreted as sets $\{p \in P | p_1 \leqslant p \leqslant p_2\}$ (instead of couples), this corresponds to the usual inclusion of sets. This ordering is called the imprecision ordering [26]. Note that the exact intervals are the least imprecise elements. Note that in this case the ordering $\leqslant$ on P is not preserved under the injection i: if $p \neq q$, then $i(p)$ is incomparable (w.r.t. $\subseteq$) with $i(q)$, even if $p \leqslant q$ or $q \leqslant p$.
- $[p_1,p_2] \preceq_2 [q_1,q_2]$ iff $[p_1,p_2] = [q_1,q_2]$ or $p_2 \leqslant q_1$. This relation is a subset of $\preceq$: if $[p_1,p_2] \preceq_2 [q_1,q_2]$ then $[p_1,p_2] \preceq [q_1,q_2]$. It is called the strong truth ordering (by contrast with the weak truth ordering $\preceq$) in [26].

[3] Note that, to be absolutely correct, we should write $\mathbb{T}(([0,1],\leqslant))$ instead of $\mathbb{T}([0,1],\leqslant)$. However, we shall only use one pair of brackets in this work, as no confusion can occur.

The embedding i preserves the ordering not only in case of the weak truth ordering $\preceq$, but also for the strong truth ordering $\preceq_2$. However, because the reflexivity of $\preceq_2$ seems a bit artificial (note, e.g., that $[a,c] \preceq_2 [a,c]$ and $[a,c] \preceq_2 [c,c]$, but $[a,c] \not\preceq_2 [b,c]$ if $a < b < c$), from now on we will always use $\preceq$ and refer to it as the truth ordering. Moreover, as it is very unlikely to cause any confusion, we will use $\leqslant$ to denote the ordering on the original poset as well as on its triangularization.

If $\mathscr{P} = (P, \leqslant)$ is a poset which has a greatest element 1, then $\mathbb{T}(\mathscr{P})$ is also a poset, with greatest element $[1,1]$. So the definition of t-norm can be applied to this case. A simple class of examples is obtained by taking two t-norms T_1 and T_2 on $\mathscr{P}$ satisfying $T_1(x,y) \leqslant T_2(x,y)$ for all x and y in P, and defining $\mathscr{T}$ as follows: $\mathscr{T}([x_1,x_2],[y_1,y_2]) = [T_1(x_1,y_1), T_2(x_2,y_2)]$. It is straightforward to verify that this procedure indeed yields a t-norm on $\mathscr{P}$. Also note that $T_1 \leqslant T_2$ is a necessary condition, because otherwise we would obtain values $[T_1(x_1,y_1), T_2(x_2,y_2)]$ for which $T_1(x_1,y_1) \not\leqslant T_2(x_2,y_2)$. The t-norms defined in this way are called t-representable t-norms [19].

Proposition 2.8 *Let $\mathscr{P} = (P, \leqslant)$ be a poset which has a greatest element* 1. *The t-representable t-norms on $\mathbb{T}(\mathscr{P})$ are exactly the t-norms on $\mathbb{T}(\mathscr{P})$ that are increasing in both arguments w.r.t. the inclusion*[4] *$\subseteq$ (the imprecision ordering).*

Proof. Assume $\mathscr{T}$ is a t-norm that increases w.r.t. the inclusion. Now take x $(= [x_1,x_2])$ and y in $Int(\mathscr{P})$. Because $[x_1,x_1] \leqslant x$ and $[x_1,x_1] \subseteq x$, we have both $\mathscr{T}([x_1,x_1],y) \leqslant \mathscr{T}(x,y)$ and $\mathscr{T}([x_1,x_1],y) \subseteq \mathscr{T}(x,y)$. Thus we have $\mathrm{pr}_1\mathscr{T}([x_1,x_1],y) \leqslant \mathrm{pr}_1\mathscr{T}(x,y)$ and $\mathrm{pr}_1\mathscr{T}(x,y) \leqslant \mathrm{pr}_1\mathscr{T}([x_1,x_1],y)$. Therefore $\mathrm{pr}_1\mathscr{T}(x,y) = \mathrm{pr}_1\mathscr{T}([x_1,x_1],y)$. In other words: the first projection of $\mathscr{T}$ is independent of the second projection of the first argument. By the symmetry of $\mathscr{T}$, it is also independent of the second projection of the second argument.
In a completely similar way we can prove that the second projection of $\mathscr{T}$ is independent of the first projections of the arguments, so we can conclude that $\mathscr{T}([x_1,x_2],[y_1,y_2])$ has the form $[T_1(x_1,y_1), T_2(x_2,y_2)]$. One can easily check that T_1 and T_2 are necessarily t-norms (on $\mathscr{P}$), with $T_1 \leqslant T_2$.
Conversely, it is straightforward to show that t-representable t-norms on $\mathbb{T}(\mathscr{P})$ are increasing w.r.t. the inclusion. □

In a similar way as in Proposition 2.8 it can be proven that the implicators on $\mathbb{T}(\mathscr{P})$ (with $\mathscr{P}$ a bounded poset) that are increasing in both arguments w.r.t. the inclusion are exactly the implicators of the form $\mathscr{I}([x_1,x_2],[y_1,y_2]) = [I_1(x_2,y_1), I_2(x_1,y_2)]$, with I_1 and I_2 implicators on $\mathscr{P}$ and $I_1 \leqslant I_2$.

2.3 Lattices

As already mentioned in Section 2.1, lattices are posets in which the supremum and infimum of every two elements exist. However, often another definition is used.

Definition 2.9 *A lattice is an algebra $(L, \sqcap, \sqcup)$ of type $(2,2)$ (i.e., both $\sqcap$ and $\sqcup$ are binary operations on L) such that $\sqcap$ ('meet') and $\sqcup$ ('join') are idempotent, commutative and associative*

[4] In [3] 'interval triangular norms' are defined as what we call t-norms on $\mathbb{T}(\mathscr{P})$ that are increasing w.r.t. the inclusion. This proposition shows that these are always t-representable, and vice versa.

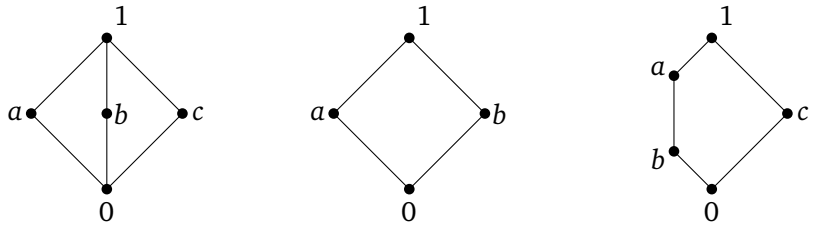

Figure 2.2: Hasse diagrams of the lattices L_1, L_2 and L_3.

operations satisfying the following absorption laws: for all x and y in L, $x \sqcup (x \sqcap y) = x$ and $x \sqcap (x \sqcup y) = x$.
The lattice order $\leqslant$ is defined by $x \leqslant y$ iff $x \sqcap y = x$ (or, equivalently, iff $x \sqcup y = y$), for all x and y in L.

One can easily check that if $(P, \leqslant)$ is a poset in which the supremum and infimum of every two elements exist, then $(P, \inf, \sup)$ is a lattice and the lattice order coincides with $\leqslant$. And conversely, if $(L, \sqcap, \sqcup)$ is a lattice, then the lattice order is a partial order on L, and the supremum and infimum w.r.t. this order are exactly $\sqcap$ and $\sqcup$. So the two given definitions are equivalent. Therefore we can (and will) use the properties and terminology of posets also for lattices without any problems. An example of a property that is easier to express in terms of $\sqcap$ and $\sqcup$ than in terms of $\leqslant$ is distributivity.

Definition 2.10 *Let $\mathscr{L} = (L, \sqcap, \sqcup)$ be a lattice.*

- *We say $\mathscr{L}$ is distributive if $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ and $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$, for all x, y and z in L.*

- *If $\mathscr{L}$ is a complete lattice, we say $\mathscr{L}$ is infinitely distributive if $x \sqcap \sup Y = \sup\{x \sqcap y | y \in Y\}$ and $x \sqcup \inf Y = \inf\{x \sqcup y | y \in Y\}$, for all x in L and $Y \subseteq L$.*

- *Let $a \in L$. We say that a is $\sqcup$-irreducible if for all b and c in L, $a = b \sqcup c$ implies $b = a$ or $c = a$; and similarly a is $\sqcap$-irreducible if for all b and c in L, $a = b \sqcap c$ implies $b = a$ or $c = a$.*

Obviously each infinitely distributive lattice is complete and distributive, but the converse is not true. (See e.g. Section 3.3 in [30])
Proposition 2.12 summarizes a few well-known properties about lattices. First we give some examples of small lattices.

Example 2.11 *The three lattices in Figure 2.2 are finite (and therefore also complete and bounded). None of them is a chain. In L_2, the elements 0, a and b are $\sqcup$-irreducible and a, b and 1 are $\sqcap$-irreducible. L_2 is distributive, but L_1 and L_3 are not (in both cases, $(a \sqcap b) \sqcup (a \sqcap c) < a \sqcap (b \sqcup c)$).*

Proposition 2.12 *Let $\mathscr{L} = (L, \sqcap, \sqcup)$ be a lattice.*

1. *If $x \sqcap (y \sqcup z) \leqslant (x \sqcap y) \sqcup (x \sqcap z)$ for all x and y in L, then $\mathscr{L}$ is distributive.*

2. *If $x \sqcup (y \sqcap z) \geqslant (x \sqcup y) \sqcap (x \sqcup z)$ for all x and y in L, then $\mathscr{L}$ is distributive.*

3. *$\mathscr{L}$ is distributive if and only if none of its sublattices*[5] *is isomorphic to L_1 or L_3.*

4. *If $\mathscr{L}$ is complete, $x \sqcap \sup Y \leqslant \sup\{x \sqcap y | y \in Y\}$ and $x \sqcup \inf Y \geqslant \inf\{x \sqcup y | y \in Y\}$, for all $\{x\} \cup Y \subseteq L$, then $\mathscr{L}$ is infinitely distributive.*

5. *If $\mathscr{L}$ is a chain, then $\mathscr{L}$ is distributive and each of its elements is both $\sqcap$-irreducible and $\sqcup$-irreducible.*

6. *If each element of L is $\sqcap$-irreducible, then $\mathscr{L}$ is a chain.*

7. *If each element of L is $\sqcup$-irreducible, then $\mathscr{L}$ is a chain.*

8. *If $\mathscr{L}$ is a complete chain, then $\mathscr{L}$ is infinitely distributive.*

9. *If $\mathscr{L}$ is finite and distributive, then $\mathscr{L}$ is infinitely distributive.*

Because lattices can be seen as posets, we can consider their triangularizations. One easily observes that the infimum (resp. supremum) on the triangularization of a lattice is obtained by taking the infimum (resp. supremum) of the first and second projections. More precisely: for any lattice $\mathscr{L} = (L, \sqcap, \sqcup)$, the infimum $\bigsqcap$ and supremum $\bigsqcup$ on its triangularization $\mathbb{T}(\mathscr{L})$ are given by

- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$,
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$,

for all $[x_1, x_2]$ and $[y_1, y_2]$ in $Int(\mathscr{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leqslant x_2\}$. Note that we use big $\bigsqcap$- and $\bigsqcup$-symbols for the intervals of the triangularization, and small $\sqcap$- and $\sqcup$-symbols for the elements of the original lattice.
More generally: $\inf X = [\inf \mathrm{pr}_1(X), \inf \mathrm{pr}_2(X)]$ and $\sup X = [\sup \mathrm{pr}_1(X), \sup \mathrm{pr}_2(X)]$, for all $X \subseteq Int(\mathscr{L})$ (whenever the left hand side or the right hand side of these identities exist).
Using these properties, it follows immediately that $\mathbb{T}(\mathscr{L})$ is a lattice iff $\mathscr{L}$ is a lattice, that i (as defined in Section 2.2) is a homomorphism from $(L, \sqcap, \sqcup)$ to $(Int(\mathscr{L}), \bigsqcap, \bigsqcup)$, and that the set $i(L)$ of exact intervals is therefore closed under $\bigsqcap$ and $\bigsqcup$ and forms a sublattice.
Moreover, also distributivity and infinite distributivity are properties inherited by triangularizations: if $\mathscr{L}$ is (infinitely) distributive, then $\mathbb{T}(\mathscr{L})$ is (infinitely) distributive too (and the converse holds as well).

Proposition 2.13 *Let $\mathscr{L} = (L, \sqcap, \sqcup)$ be a lattice and $[l_1, l_2]$ an element of $Int(\mathscr{L})$.*

- *$[l_1, l_2]$ is $\bigsqcap$-reducible iff l_1 or l_2 is $\sqcap$-reducible, or $(\exists l \in L)(l_1 < l_2 < l)$.*
- *$[l_1, l_2]$ is $\bigsqcup$-reducible iff l_1 or l_2 is $\sqcup$-reducible, or $(\exists l \in L)(l < l_1 < l_2)$.*

[5] A sublattice of $(L, \sqcap, \sqcup)$ is a subset of L that is closed under the meet and join operations. Note that this is not the same as being a subset that is a lattice under the original order (but possibly with different join and meet operations). Indeed, consider as an example the subset $\{0, a, b, 1\}$ in the lattice depicted in Figure 2.4 on page 18. Although the poset corresponding to the restricted order is a lattice (which looks like L_2 in Figure 2.2), this subset does not form a sublattice of the original lattice because the suprema of a and b do not coincide ($d \neq 1$).

Proof. We will prove only the first equivalence, as the second is analogous.
First suppose l_2 is $\sqcap$-reducible, i.e., there exist a and b in L such that $a \sqcap b = l_2$ with $a \neq l_2 \neq b$. In this case we find $[l_1, a] \sqcap [l_1, b] = [l_1, l_2]$ with $[l_1, a] \neq [l_1, l_2] \neq [l_1, b]$, which means $[l_1, l_2]$ is $\sqcap$-reducible.
Now suppose l_1 is $\sqcap$-reducible, i.e., there exist a and b in L such that $a \sqcap b = l_1$ with $a \neq l_1 \neq b$. We consider three cases:

- If $l_2 = l_1$, then $[l_1, l_2] = [a, a] \sqcap [b, b]$ with $[a, a] \neq [l_1, l_2] \neq [b, b]$, which means $[l_1, l_2]$ is $\sqcap$-reducible.
- If $l_2 > l_1$ and $l \leqslant l_2$ for all l in L, then $[l_1, l_2] = [a, l_2] \sqcap [b, l_2]$ with $[a, l_2] \neq [l_1, l_2] \neq [b, l_2]$, which means $[l_1, l_2]$ is $\sqcap$-reducible.
- If $l_2 > l_1$ and there is an l_0 in L such that $l_0 \nleqslant l_2$, then $[l_1, l_2] = [l_2, l_2] \sqcap [l_1, l_2 \sqcup l_0]$ with $[l_2, l_2] \neq [l_1, l_2] \neq [l_1, l_2 \sqcup l_0]$, which means $[l_1, l_2]$ is $\sqcap$-reducible. Note that for this case, we did not need the $\sqcap$-reducibility of l_1.

Finally suppose $(\exists l \in L)(l_1 < l_2 < l)$. We can use the same argument as in the itemization above.

Now we prove the other direction. Suppose $[l_1, l_2]$ is $\sqcap$-reducible. It suffices to show that if l_1 and l_2 are both $\sqcap$-irreducible, then $(\exists l \in L)(l_1 < l_2 < l)$.
Because there exist $[a_1, a_2]$ and $[b_1, b_2]$ (both different from $[l_1, l_2]$) such that $[l_1, l_2] = [a_1, a_2] \sqcap [b_1, b_2] = [a_1 \sqcap b_1, a_2 \sqcap b_2]$, we find (using the $\sqcap$-irreducibility of l_1 and l_2): ($l_1 = a_1$, $l_2 < a_2$, $l_1 < b_1$ and $l_2 = b_2$) or ($l_1 < a_1$, $l_2 = a_2$, $l_1 = b_1$ and $l_2 < b_2$). In the first case we find $l_1 < b_1 \leqslant b_2 = l_2 < a_2$, in the second case we find $l_1 < a_1 \leqslant a_2 = l_2 < b_2$. In both cases, $l_1 < l_2$, and $(\exists l \in L)(l_2 < l)$. □

We already know (see Section 2.1) that $\mathbb{T}(\mathscr{L})$ is bounded iff $\mathscr{L}$ is bounded. In this case, the smallest (resp. greatest) element of $\mathscr{L}$ is usually denoted by 0 (resp. 1), and the smallest (resp. greatest) element of $\mathbb{T}(\mathscr{L})$ by $[0, 0]$ (resp. $[1, 1]$). We will see that the element $[0, 1]$ will play an important role later on, especially in Chapter 3 where we will formalize particular triangularizations of bounded lattices using identities. At this point, we can already give some easily verified properties valid for triangularizations of bounded lattices.

Proposition 2.14 *Let $\mathscr{L} = (L, \sqcap, \sqcup)$ be a bounded lattice (with smallest element 0 and greatest element 1) and $\mathbb{T}(\mathscr{L})$ its triangularization. Then the following properties are satisfied, for all x and y in $Int(\mathscr{L})$:*

$$\begin{array}{ll}
\mathrm{pr}_v(x) \leqslant x, & x \leqslant \mathrm{pr}_h(x), \\
\mathrm{pr}_v(x) \leqslant \mathrm{pr}_v(\mathrm{pr}_v(x)), & \mathrm{pr}_h(\mathrm{pr}_h(x)) \leqslant \mathrm{pr}_h(x), \\
\mathrm{pr}_v(x \sqcap y) = \mathrm{pr}_v(x) \sqcap \mathrm{pr}_v(y), & \mathrm{pr}_h(x \sqcap y) = \mathrm{pr}_h(x) \sqcap \mathrm{pr}_h(y), \\
\mathrm{pr}_v(x \sqcup y) = \mathrm{pr}_v(x) \sqcup \mathrm{pr}_v(y), & \mathrm{pr}_h(x \sqcup y) = \mathrm{pr}_h(x) \sqcup \mathrm{pr}_h(y), \\
\mathrm{pr}_v([0, 1]) = [0, 0], & \mathrm{pr}_h([0, 1]) = [1, 1], \\
\mathrm{pr}_v(\mathrm{pr}_h(x)) = \mathrm{pr}_h(x), & \mathrm{pr}_h(\mathrm{pr}_v(x)) = \mathrm{pr}_v(x).
\end{array}$$

Moreover, we have $x \leqslant y$ iff $\mathrm{pr}_v(x) \leqslant \mathrm{pr}_v(y)$ and $\mathrm{pr}_h(x) \leqslant \mathrm{pr}_h(y)$.
Furthermore, $x = (\mathrm{pr}_h(x) \sqcap [0, 1]) \sqcup \mathrm{pr}_v(x) = (\mathrm{pr}_v(x) \sqcup [0, 1]) \sqcap \mathrm{pr}_h(x)$.

For any triangularization $(Int(\mathscr{L}), \sqcap, \sqcup)$ of a bounded lattice $\mathscr{L} = (L, \sqcap, \sqcup)$ (with smallest element 0 and greatest element 1), we call $(Int(\mathscr{L}), \sqcap, \sqcup, \mathrm{pr}_v, \mathrm{pr}_h, [0, 0], [0, 1], [1, 1])$ the extended triangularization of $\mathscr{L}$.

Definition 2.15 *A triangular lattice is an algebra* $(L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ *of type* $(2,2,1,1,0,0,0)$ *such that* $(L, \sqcap, \sqcup)$ *is a bounded lattice with smallest element* 0 *and greatest element* 1 *such that*

$$\begin{array}{llll}
(T.1) & \nu x \leqslant x, & (T.1') & x \leqslant \mu x, \\
(T.2) & \nu x \leqslant \nu\nu x, & (T.2') & \mu\mu x \leqslant \mu x, \\
(T.3) & \nu(x \sqcap y) = \nu x \sqcap \nu y, & (T.3') & \mu(x \sqcap y) = \mu x \sqcap \mu y, \\
(T.4) & \nu(x \sqcup y) = \nu x \sqcup \nu y, & (T.4') & \mu(x \sqcup y) = \mu x \sqcup \mu y, \\
(T.5) & \nu u = 0, & (T.5') & \mu u = 1, \\
(T.6) & \nu\mu x = \mu x, & (T.6') & \mu\nu x = \nu x, \\
(T.10) & x = \nu x \sqcup (\mu x \sqcap u), & (T.10') & x = \mu x \sqcap (\nu x \sqcup u).
\end{array}$$

The unary operators ν *and* μ *are called the necessity and possibility operator, respectively.*

Note that from (T.10) it follows that for all x and y in L,

$$x = y \text{ whenever } \nu x = \nu y \text{ and } \mu x = \mu y. \tag{2.1}$$

From (T.1) and (T.2), it is clear that in a triangular lattice, always $\nu\nu x = \nu x$. Similarly, $\mu\mu x = \mu x$. Each of (T.3) and (T.4) implies that ν is an increasing operator. In the same way, (T.3') or (T.4') force μ to be increasing too. Other properties that follow easily are $\nu 1 = 1$ and $\mu 0 = 0$: $\nu 1 = \nu\mu u = \mu u = 1$ and $\mu 0 = \mu\nu u = \nu u = 0$. Together with (T.1), (T.1'), (T.2), (T.2'), (T.3) and (T.4'), they imply μ is a closure operator, and ν is an interior operator. Both are also lattice morphisms.
Note that (T.1')–(T.4') are conditions for μ, which is similar to the modal possibility operator; they are dual to (T.1)–(T.4) for ν, which is similar to the modal necessity operator. Only (T.4) and (T.3') are different: in the modal setting, they are in general not true; and one doesn't want them to be true either (see e.g. [83]). In general, we do not require dependency of μ on ν.The conditions (T.5) and (T.5') express the complete lack of knowledge about u: its necessity is 0, but its possibility is 1. The conditions (T.6) and (T.6') are known in modal logics as the S5-principles [61].

Remark 2.16 *The reason we called the last two conditions (T.10) and (T.10') instead of (T.7) and (T.7') is that we would like to keep the same notations as in our latest papers [77, 78, 79, 80]. In these papers, (T.7) was already the name for another property (see Chapter 3).*
Also note that not all properties are independent: (T.1), (T.2') and (T.10') actually follow from the other properties[6]*. Indeed, (T.1) immediately follows from (T.10). From (T.10), (T.4'), (T.3'), (T.5'), (T.6'), (T.1) and (T.1') we also get* $\mu(x) = \mu(\nu x \sqcup (\mu x \sqcap u)) = \mu\nu x \sqcup (\mu\mu x \sqcap \mu u) = \mu\nu x \sqcup \mu\mu x = \mu\mu x$. *For (T.10') we apply (2.1): using (T.1)-(T.6) and (T.1')-(T.6'), it is easy to verify that* $\nu x = \nu(\mu x \sqcap (\nu x \sqcup u))$ *and* $\mu x = \mu(\mu x \sqcap (\nu x \sqcup u))$.
Note that for (T.6) and (T.6') we could also have sufficed with the weaker axioms (T.6w) $\mu x \leqslant \nu\mu x$ *and (T.6'w)* $\mu\nu x \leqslant \nu x$*, because of (T.1) and (T.1').*

So Proposition 2.14 tells us that if $\mathscr{L} = (L, \sqcap, \sqcup)$ is a bounded lattice (with smallest element 0 and greatest element 1), then $(Int(\mathscr{L}), \sqcap, \sqcup, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ is a triangular lattice. The converse holds as well, up to isomorphism. Before we prove this proposition, we first prove some properties of triangular lattices.

The direct images of a triangular lattice under the necessity and the possibility operators are identical, and every element of this image is invariant under both operators:

[6] Similarly also (T.1'), (T.2) and (T.10) follow from the other properties.

Lemma 2.17 *Let $\mathscr{L} = (L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice. The following statements are equivalent for all x in L:*

1. *$x = \nu y$ for some y in L,*
2. *$x = \mu y$ for some y in L,*
3. *$x = \nu x$,*
4. *$x = \mu x$,*
5. *$\nu x = \mu x$.*

Proof. Let x be in L.

(1) implies (2): If $x = \nu y$, then $x = \mu\nu y$ (using (T.6')).
(2) implies (3): If $x = \mu z$, then $\nu x = \nu\mu z = \mu z = x$ (using (T.6)).
(3) implies (4): If $x = \nu x$, then $\mu x = \mu\nu x = \nu x = x$ (using (T.6')).
(4) implies (5): If $x = \mu x$, then $\nu x = \nu\mu x = \mu x$ (using (T.6)).
(5) implies (1): Finally, if $\nu x = \mu x$, then $x = \nu x$ because $\nu x \leqslant x \leqslant \mu x$ (using (T.1) and (T.1')). □

Definition 2.18 *The set of exact elements $E(\mathscr{L})$ of a triangular lattice $\mathscr{L}$ is $\{x \in L \mid \nu x = x\}$.*

As a consequence of Lemma 2.17 and Definition 2.15, we find:

Proposition 2.19 *Let $\mathscr{L} = (L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice. Then $E(\mathscr{L}) = \nu(L) = \mu(L) = \nu(E(\mathscr{L})) = \mu(E(\mathscr{L}))$. This set contains 0 and 1, but not u (unless in the trivial case when $|L| = 1$).*

The set of exact elements of a triangular lattice is closed under all the defined (unary and binary) operators:

Corollary 2.20 *Let $\mathscr{L} = (L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice. Then $E(\mathscr{L})$ is closed under $\sqcap$ and $\sqcup$, and therefore $\mathscr{E}(\mathscr{L}) = (E(\mathscr{L}), \sqcap, \sqcup)$ (in which the binary operators are restricted to $E(\mathscr{L})$) is a bounded lattice.*

Proof. By (T.3), $\nu(x \sqcap y) = \nu x \sqcap \nu y = x \sqcap y$ for every x and y in $E(\mathscr{L})$. So $E(\mathscr{L})$ is closed under $\sqcap$. Using (T.4) we can prove in an analogous way that $E(\mathscr{L})$ is closed under $\sqcup$. The operator ν is a surjective homomorphism from $(L, \sqcap, \sqcup, 0, 1)$ to $(E(\mathscr{L}), \sqcap, \sqcup, 0, 1)$, so the bounded lattice structure is preserved.

□

Proposition 2.21 *Let $(L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice. Then $(L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ is isomorphic to the extended triangularization of a bounded lattice. Conversely, every extended triangularization of a bounded lattice is a triangular lattice.*

Proof. Indeed, with any triangular lattice $\mathscr{L} = (L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$, we can associate the extended triangularization $(Int(\mathscr{E}(\mathscr{L})), \bigsqcap, \bigsqcup, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ of its sublattice $\mathscr{E}(\mathscr{L}) = (E(\mathscr{L}), \sqcap, \sqcup)$ of exact elements. We now show that the mapping $\chi\colon L \to Int(\mathscr{E}(\mathscr{L}))$ defined as $\chi(x) = [\nu x, \mu x]$ is an isomorphism. Note that $\chi(0) = [0,0]$, $\chi(u) = [0,1]$ and $\chi(1) = [1,1]$ (using (T.1), (T.1'), (T.5), (T.5')). Because $[x_1, x_2] = \chi(x_1 \sqcup (u \sqcap x_2))$ for any $[x_1, x_2]$ in $Int(\mathscr{E}(\mathscr{L}))$, χ is a surjection. And it is an injection too, because of (2.1). Furthermore, because of conditions (T.3), (T.3'), (T.4) and (T.4'), $\chi(x \sqcap y) = [\nu(x \sqcap y), \mu(x \sqcap y)] = [\nu x \sqcap \nu y, \mu x \sqcap \mu y] = [\nu x, \mu x] \bigsqcap [\nu y, \mu y] = \chi(x) \bigsqcap \chi(y)$, and analogously $\chi(x \sqcup y) = \chi(x) \bigsqcup \chi(y)$ for all x and y in L. And finally, $\chi(\nu x) = [\nu\nu x, \mu\nu x] = [\nu x, \nu x] = \mathrm{pr}_v[\nu x, \mu x] = \mathrm{pr}_v(\chi(x))$ and analogously $\chi(\mu x) = \mathrm{pr}_h(\chi(x))$. So χ is indeed an isomorphism.
The converse statement immediately follows from Proposition 2.14 and Definition 2.15. □

This means we have found a way to represent triangularizations of bounded lattices by a class of algebraic structures defined only with identities: the variety[7] of triangular lattices. We will give an alternative definition for triangular lattices in Proposition 2.24. First we show some more properties of triangular lattices.

It is possible to express νx and μy in terms of the ordering of $\mathscr{L}$ and the set of exact elements $E(\mathscr{L})$:

Proposition 2.22 *Let $\mathscr{L} = (L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice. Then $\nu x = \sup\{y \in E(\mathscr{L}) \mid y \leqslant x\}$ and $\mu x = \inf\{y \in E(\mathscr{L}) \mid x \leqslant y\}$.*

Proof. On the one hand, $\nu x \leqslant \sup\{y \in E(\mathscr{L}) \mid y \leqslant x\}$ because $\nu x \in \{y \in E(\mathscr{L}) \mid y \leqslant x\}$. On the other hand, $\sup\{y \in E(\mathscr{L}) \mid y \leqslant x\} \leqslant \nu x$ because, for every y in $E(\mathscr{L})$, $y = \nu y \leqslant \nu x$ if $y \leqslant x$.
The proof of the second part is analogous. □

It is also possible to describe the ordering on a triangular lattice in terms of the restricted ordering on the set of exact elements:

Proposition 2.23 *In a triangular lattice $(L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$, $x \leqslant y$ is equivalent to ($\nu x \leqslant \nu y$ and $\mu x \leqslant \mu y$) for all x and y in L.*

Proof. If $x \leqslant y$, then $\nu x \leqslant \nu y$ and $\mu x \leqslant \mu y$ because ν and μ are increasing (this follows from (T.3) and (T.3')).
Now suppose $\nu x \leqslant \nu y$ and $\mu x \leqslant \mu y$. Then we find, using (T.3) and (T.3'), $\nu(x \sqcap y) = \nu x \sqcap \nu y = \nu x$ and $\mu(x \sqcap y) = \mu x \sqcap \mu y = \mu x$. So by (2.1), $x \sqcap y = x$. □

As two special cases of Proposition 2.23, $x \leqslant u$ iff $\nu x = 0$, and $u \leqslant x$ iff $\mu x = 1$.

Proposition 2.24 *In a triangular lattice $(L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$, we can find a subset D of L satisfying*

- *D is closed under $\sqcap$ and $\sqcup$,*
- *$\{0, 1\} \subseteq D$,*
- *for all l in L, $\max\{x \in D \mid x \leqslant l\}$ and $\min\{x \in D \mid x \geqslant l\}$ exist,*
- *if $(d_1, d_2) \in D^2$ and $d_1 \leqslant d_2$, then there exists a unique l in L such that $\max\{x \in D \mid x \leqslant l\} = d_1$ and $\min\{x \in D \mid x \geqslant l\} = d_2$.*

[7] The class is called a variety because it can be defined simply by identities.

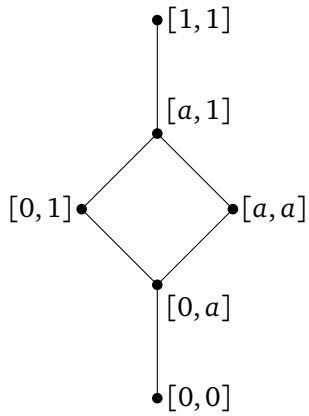

Figure 2.3: The triangularization of a lattice with three elements.

Conversely, any bounded lattice in which such a subset D can be found, has the structure of a triangular lattice.

Proof. Indeed, let $(L, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice. Take for D the set of exact elements, and note that $\max\{x \in D \mid x \leqslant l\} = \nu l$ and $\min\{x \in D \mid x \geqslant l\} = \mu l$.
Conversely, we show that any bounded lattice in which such a subset D can be found, is a triangular lattice. Indeed, define νl as $\max\{x \in D \mid x \leqslant l\}$ and μl as $\min\{x \in D \mid x \geqslant l\}$ for all l in L, and u as the unique element l in L for which $\nu l = 0$ and $\mu l = 1$. Then it is straightforward to verify that the defined structure satisfies (T.1)-(T.6) and (T.1')-(T.6'). Indeed, immediately from the definition of ν, we have (T.1), $\nu l \in D$, $\nu d = d$ for all $d \in D$ and $\nu x \leqslant \nu y$ if $x \leqslant y$. Similarly for μ. So we already have (T.1), (T.2), (T.6), (T.1'), (T.2'), (T.6'), $\nu(x \sqcap y) \leqslant \nu x \sqcap \nu y$, $\mu(x \sqcap y) \leqslant \mu x \sqcap \mu y$, $\nu x \sqcup \nu y \leqslant \nu(x \sqcup y)$ and $\mu x \sqcup \mu y \leqslant \mu(x \sqcup y)$. Because D is closed under $\sqcap$, we also find (T.3): $\nu x \sqcap \nu y = \nu(\nu x \sqcap \nu y) \leqslant \nu(x \sqcap y)$. Similarly for (T.3'), (T.4) and (T.4'). (T.5) and (T.5') follow from the definition of u. Using these properties, we now easily find $\nu x = \nu(\nu x \sqcup (\mu x \sqcap u))$ and $\mu x = \mu(\nu x \sqcup (\mu x \sqcap u))$. Therefore $\nu x \sqcup (\mu x \sqcap u)$ is the unique element l in L such that $\nu l = d_1$ and $\mu l = d_2$, with $d_1 = \nu x$ and $d_2 = \mu x$ ($d_1 \leqslant d_2$ and $(d_1, d_2) \in D^2$). This implies $\nu x \sqcup (\mu x \sqcap u) = x$. □

Remark 2.25 *As we have seen, by Proposition 2.14 triangularizations of bounded lattices can be equipped with operators ν and μ, and a constant u such that the resulting structure is a triangular lattice. Indeed, taking $\nu = \mathrm{pr}_v$, $\mu = \mathrm{pr}_h$ and $u = [0, 1]$ always does this job. In many cases, this is the only way to obtain such a triangular lattice. But not always, as the next example shows.*

Example 2.26 *On the triangularization of a bounded chain $\mathscr{L}$ with three elements (see Figure 2.3), say 0, a and 1, we define u as $[a, a]$ and ν and μ by $\nu[0, 0] = \nu[0, a] = \nu[a, a] = \mu[0, 0] = [0, 0]$, $\nu[0, 1] = \nu[a, 1] = \mu[0, 1] = \mu[0, a] = [0, 1]$ and $\nu[1, 1] = \mu[a, a] = \mu[a, 1] = \mu[1, 1] = [1, 1]$. Then $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \nu, \mu, [0, 0], [a, a], [1, 1])$ is a triangular lattice in which $u \neq [0, 1]$.*

We do have, however, that ν and μ are determined whenever u is fixed as $[0, 1]$.

Proposition 2.27 *If $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangular lattice on a triangularization of a bounded lattice $\mathscr{L}$, then $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$ for every $[x_1, x_2]$ in $Int(\mathscr{L})$.*

Proof. We first show that $\nu[x_1,x_2] = \nu[x_1,1]$ for every $[x_1,x_2]$ in $Int(\mathscr{L})$, using (T.5) and (T.4): $\nu[x_1,x_2] = \nu[x_1,x_2] \bigsqcup [0,0] = \nu[x_1,x_2] \bigsqcup \nu[0,1] = \nu([x_1,x_2] \bigsqcup [0,1]) = \nu[x_1,1]$. So $\nu[x_1,x_2]$ does not depend on x_2.
Suppose there exists an interval $[a_1,a_2]$ in $Int(\mathscr{L})$ for which $\nu[a_1,a_2] = [b_1,b_2]$ and $[b_1,b_2] \neq [a_1,a_1]$. Because $[b_1,b_2] = \nu[a_1,a_2] = \nu[a_1,a_1] \leqslant [a_1,a_1]$, we have $b_1 < a_1$ (if b_1 were equal to a_1, then $a_1 = b_1 \leqslant b_2 \leqslant a_1$, which would imply $[b_1,b_2] = [a_1,a_1]$).
We can now show that $\nu[a_1,1] = \nu[b_1,1]$ and $\mu[a_1,1] = \mu[b_1,1]$, which is in contradiction with (2.1) and $[a_1,1] \neq [b_1,1]$.

- $\nu[a_1,1] = \nu[a_1,a_2] = \nu\nu[a_1,a_2] = \nu[b_1,b_2] = \nu[b_1,1]$ (using (T.2))
- Because $[0,1] \leqslant [b_1,1] < [a_1,1]$, $[1,1] = \mu[0,1] \leqslant \mu[b_1,1] \leqslant \mu[a_1,1]$.

So we conclude that our assumption was false. There exists no interval $[a_1,a_2]$ in $Int(\mathscr{L})$ such that $\nu[a_1,a_2] \neq [a_1,a_1]$.
The proof for μ is completely analogous. □

Combining Proposition 2.27 with Proposition 2.29, we see that on triangularizations of bounded linear lattices with at least four elements, there is only one possibility for u, ν and μ in order to get a triangular lattice, namely $\nu = \mathrm{pr}_v$, $\mu = \mathrm{pr}_h$ and $u = [0,1]$. Before Proposition 2.29, we first prove Lemma 2.28.

Lemma 2.28 *If $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \nu, \mu, [0,0], u, [1,1])$ is a triangular lattice on a triangularization $(Int(\mathscr{L}), \bigsqcap, \bigsqcup)$ of a bounded lattice $\mathscr{L}$ in which 0 is $\sqcap$-irreducible and 1 $\sqcup$-irreducible, then $u = [0,1]$ or $u = [a,a]$ (with $0 < a < 1$ if $\mathscr{L}$ has at least two elements).*

Proof. First suppose $u = [a,a]$ for some a in L. If $a = 0$, then $[1,1] = \mu u = \mu[0,0] = \mu\nu u = \nu u = [0,0]$, a contradiction if $\mathscr{L}$ has at least two elements. Similarly a cannot be 1 if $\mathscr{L}$ has at least two elements.
Now suppose that $u = [u_1,u_2]$, with $u_1 < u_2$. Then $u \neq [u_2,u_2]$. So, according to (2.1), it cannot be both the case that $\nu[u_2,u_2] = [0,0]$ and $\mu[u_2,u_2] = [1,1]$ (because of (T.5) and (T.5')). As we know $[1,1] = \mu u \leqslant \mu[u_2,u_2]$ (because μ is increasing), necessarily $\nu[u_2,u_2] > [0,0]$. Similarly $\mu[u_1,u_1] < [1,1]$.
From $[0,0] = \nu[0,u_2] = \nu([0,1] \sqcap [u_2,u_2]) = \nu[0,1] \sqcap \nu[u_2,u_2]$ and the fact that $[0,0]$ is $\sqcap$-irreducible, we can then derive that $\nu[0,1] = [0,0]$; and from $[1,1] = \mu[u_1,1] = \mu([0,1] \sqcup [u_1,u_1]) = \mu[0,1] \sqcup \mu[u_1,u_1]$ and the fact that $[1,1]$ is $\sqcup$-irreducible, that $\mu[0,1] = [1,1]$.
This means that $\nu[0,1] = \nu u$ and $\mu[0,1] = \mu u$, so by (2.1) we obtain $[0,1] = u$. □

Unfortunately the conditions in Lemma 2.28 that 0 is $\sqcap$-irreducible and 1 is $\sqcup$-irreducible, cannot be left out. A counterexample is the triangularization of the lattice in Figure 2.4. Different triangular lattices can be defined on this triangularization (one with $u = [0,1]$ and one with $u = [a,d]$). This example was found using a computer program called Prover9/Mace4 [8], which also was very helpful to find other (counter)examples given in this work.

Proposition 2.29 *If $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \nu, \mu, [0,0], u, [1,1])$ is a triangular lattice on a triangularization $(Int(\mathscr{L}), \bigsqcap, \bigsqcup)$ of a bounded linear lattice $\mathscr{L}$ with at least four elements, then $u = [0,1]$.*

[8] `http://www.cs.unm.edu/~mccune/mace4/`

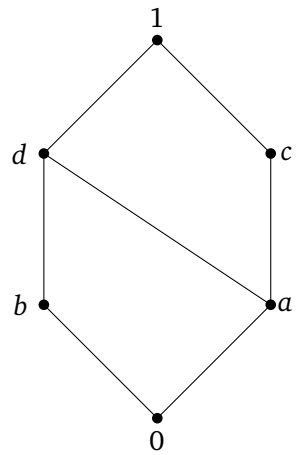

Figure 2.4: A triangular lattice with $u = [a, d]$ exist on the triangularization of this lattice.

Proof. Because in every bounded linear lattice, 0 is $\sqcap$-irreducible and 1 $\sqcup$-irreducible, we know from Lemma 2.28 that if u is not $[0,1]$, then it must be of the form $[a,a]$, with $0 < a < 1$. We now show that this is impossible.
So assume $u = [a,a]$, with $0 < a < 1$. There exists at least one element different from 0, a and 1, say b. We will suppose $0 < a < b < 1$ (the case $0 < b < a < 1$ is analogous). Then $[1,1] = \mu[a,a] \leqslant \mu[a,b]$. So $\mu[a,b] = \mu[a,a]$. Because of the linearity of $\mathscr{L}$, there are two possibilities for $\nu[a,b]$ (as for any element of $Int(\mathscr{L})$): $[0,a] \leqslant \nu[a,b]$ or $\nu[a,b] < [a,a]$. The first one can not be the case, because then from (T.1'), (T.5'), (T.3'), (T.6') and (T.1) it would follow that $[0,1] \leqslant \mu[0,1] = \mu[0,1] \sqcap \mu[a,a] = \mu[0,a] \leqslant \mu\nu[a,b] = \nu[a,b] \leqslant [a,b]$ (a contradiction, as $b < 1$). Therefore $\nu[a,b]$ has to be strictly smaller than $[a,a]$. In this case, $\nu[a,b] = \nu\nu[a,b] \leqslant \nu[a,a] = [0,0]$. But then $\nu[a,b] = \nu[a,a]$. As we already know $\mu[a,b] = \mu[a,a]$ too, (2.1) would imply that $[a,b] = [a,a]$, a contradiction. So our assumption that u was not $[0,1]$, must be false. □

Unfortunately the condition in Proposition 2.29 that the bounded lattice is linear, cannot be left out. Even the $\sqcap$-irreducibility of 0 and the $\sqcup$-irreducibility of 1 do not suffice. A counterexample is the triangularization of the lattice in Figure 2.5. Different triangular lattices can be defined on this triangularization (one with $u = [0,1]$, one with $u = [a,a]$ and one with $u = [b,b]$).

We note that in this example as well as in Example 2.26 and the example from Figure 2.4, the sublattices of exact elements (see Definition 2.18) are isomorphic. It seems likely that this always holds, i.e., if different triangular lattices exist on the same bounded lattice, then the sublattices of exact elements of these triangular lattices are isomorphic. However, currently this is still an open problem.

Example 2.30 *The unit interval* $[0,1]$ *with the usual ordering* $\leqslant$ *is linear and therefore the infimum and supremum of two elements are just the minimum and maximum of these elements. Its triangularization* $\mathbb{T}([0,1], \min, \max)$ *is complete (see Example 2.7). It is infinitely distributive (and thus also distributive). From Proposition 2.13 we derive that the exact intervals are* $\bigsqcap$*-irreducible and* $\bigsqcup$*-irreducible. The only other* $\bigsqcap$*-irreducible elements are the intervals of the form* $[x,1]$*, the only other* $\bigsqcup$*-irreducible elements are the intervals of the form* $[0,x]$*.*
Because of Propositions 2.27 and 2.29, there is only one way to define a triangular lattice on $\mathbb{T}([0,1], \min, \max)$*.*

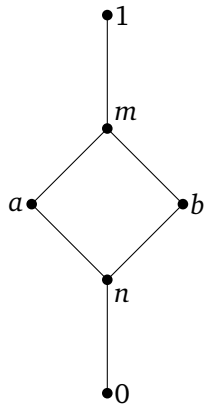

Figure 2.5: Different triangular lattices exist on the triangularization of this lattice.

Several t-norms can be defined on triangularizations of bounded lattices. A class that will be of great importance later on, consists of t-norms constructed from a t-norm on and an element of the original lattice: if T is a t-norm on a lattice $\mathscr{L} = (L, \sqcap, \sqcup)$ which has a greatest element 1, and t is an element of L, then $\mathscr{T}_{T,t}$, defined by

$$\mathscr{T}_{T,t}(x, y) = [T(x_1, y_1), T(T(x_2, y_2), t) \sqcup T(x_1, y_2) \sqcup T(x_2, y_1)], \tag{2.2}$$

for all x and y in $Int(\mathscr{L})$, is a t-norm on $\mathbb{T}(\mathscr{L})$. These t-norms were introduced by Deschrijver and Kerre in [21]. Note that for $t = 1$, we obtain a t-representable t-norm. For $t = 0$ (if $\mathscr{L}$ has a smallest element 0) we get the so-called pseudo-t-representable t-norms. Also note that $\mathscr{T}_{T,t}([x, x], [y, y]) = [T(x, y), T(x, y)] \in i(L)$ (with i the injection from Section 2.2) and that (if $\mathscr{L}$ has a smallest element 0) $t = \mathrm{pr}_2(\mathscr{T}_{T,t}([0, 1], [0, 1]))$. It can be verified that, whatever t is chosen, T is residuated iff $\mathscr{T}_{T,t}$ is residuated. In this case the residual implicator $\mathscr{I}_{T,t}$ of $\mathscr{T}_{T,t}$ is given by:

$$\mathscr{I}_{T,t}([x_1, x_2], [y_1, y_2]) = [I_T(x_1, y_1) \sqcap I_T(x_2, y_2), I_T(T(x_2, t), y_2) \sqcap I_T(x_1, y_2)],$$

with I_T the residual implicator of T. On the triangularization of the unit interval (on which every t-norm has a residual implicator), we prove that this expression is also valid for t-norms that are not residuated.

Proposition 2.31 *Let T be a t-norm on* $([0, 1], \min, \max)$ *and* $t \in [0, 1]$. *The residual implicator $\mathscr{I}_{T,t}$ of $\mathscr{T}_{T,t}$ is given by:*

$$\mathscr{I}_{T,t}([x_1, x_2], [y_1, y_2]) = [\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, t), y_2), I_T(x_1, y_2))],$$

with I_T the residual implicator of T.

Proof. We define first $C = \{[z_1, z_2] \in L^I \mid T(z_1, x_1) \leqslant y_1 \text{ and } T(T(z_2, x_2), t) \leqslant y_2 \text{ and } T(z_1, x_2) \leqslant y_2 \text{ and } T(z_2, x_1) \leqslant y_2\}$. Then $\mathscr{I}_{T,t}([x_1, x_2], [y_1, y_2]) = \bigvee C$. We can rewrite $\bigvee C$ as

$$\alpha := \Big[\bigvee\{z_1 \in [0, 1] \mid T(z_1, x_1) \leqslant y_1 \text{ and } T(z_1, x_2) \leqslant y_2\}\,, \\ \bigvee\{z_2 \in [0, 1] \mid T(T(z_2, x_2), t) \leqslant y_2 \text{ and } T(z_2, x_1) \leqslant y_2\}\Big]\,.$$

Indeed: every $[z_1, z_2]$ in C is clearly smaller than or equal to α. If $[a_1, a_2]$ is greater than or equal to every $[z_1, z_2]$ in C, it is certainly greater than or equal to every $[z_1, z_1]$ in C, and greater than or equal to every $[0, z_2]$ in C. This means $\bigvee\{z_1 \in [0,1] \mid T(z_1, x_1) \leqslant y_1 \text{ and } T(z_1, x_2) \leqslant y_2\} \leqslant a_1$ and $\bigvee\{z_2 \in [0,1] \mid T(T(z_2, x_2), t) \leqslant y_2 \text{ and } T(z_2, x_1) \leqslant y_2\} \leqslant a_2$, so $\alpha \leqslant [a_1, a_2]$.
It is easy to check that α is smaller than or equal to

$$\begin{aligned}\beta := \Big[&\min\Big(\bigvee\{z \in [0,1] \mid T(z, x_1) \leqslant y_1\}, \bigvee\{z \in [0,1] \mid T(z, x_2) \leqslant y_2\}\Big),\\ &\min\Big(\bigvee\{z \in [0,1] \mid T(T(z, x_2), t) \leqslant y_2\}, \bigvee\{z \in [0,1] \mid T(z, x_1) \leqslant y_2\}\Big)\Big]\\ = [&\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, t), y_2), I_T(x_1, y_2))].\end{aligned}$$

Suppose now that $\alpha < \beta$. Then $\alpha_1 < \beta_1$ or $\alpha_2 < \beta_2$, so

$$\begin{aligned}&\bigvee\{z_1 \in [0,1] \mid T(z_1, x_1) \leqslant y_1 \text{ and } T(z_1, x_2) \leqslant y_2\}\\ &< \bigvee\{z \in [0,1] \mid T(z, x_1) \leqslant y_1\} \text{ and}\end{aligned}$$

$$\begin{aligned}&\bigvee\{z_1 \in [0,1] \mid T(z_1, x_1) \leqslant y_1 \text{ and } T(z_1, x_2) \leqslant y_2\}\\ &< \bigvee\{z \in [0,1] \mid T(z, x_2) \leqslant y_2\},\end{aligned}$$

or

$$\begin{aligned}&\bigvee\{z_2 \in [0,1] \mid T(T(z_2, x_2), t) \leqslant y_2 \text{ and } T(z_2, x_1) \leqslant y_2\}\\ &< \bigvee\{z \in [0,1] \mid T(T(z, x_2), t) \leqslant y_2\} \text{ and}\end{aligned}$$

$$\begin{aligned}&\bigvee\{z_2 \in [0,1] \mid T(T(z_2, x_2), t) \leqslant y_2 \text{ and } T(z_2, x_1) \leqslant y_2\}\\ &< \bigvee\{z \in [0,1] \mid T(z, x_1) \leqslant y_2\}.\end{aligned}$$

The proof for both cases is analogous. We take the first one. Because of the characterization of the supremum for reals, we know that there exist z_1 and z_2 in $[0,1]$ such that $T(z_1, x_1) \leqslant y_1$, $T(z_2, x_2) \leqslant y_2$ and $z_1, z_2 > \bigvee\{z \in [0,1] \mid T(z, x_1) \leqslant y_1 \text{ and } T(z, x_2) \leqslant y_2\}$. If we take $z_0 := \min(z_1, z_2)$, we find $T(z_0, x_1) \leqslant y_1$, $T(z_0, x_2) \leqslant y_2$ and $z_0 > \bigvee\{z \in [0,1] \mid T(z, x_1) \leqslant y_1 \text{ and } T(z, x_2) \leqslant y_2\}$. This is a contradiction. □

2.4 Residuated lattices

For most formal fuzzy logics the semantics require not only a partial order on the set of truth values, but also some extra operations that model 'AND' (the strong conjunction - the infimum being the weak conjunction) and 'IMPLIES' (the implication). A very commonly used structure - and also the basic structure that we will use in this work - is that of residuated lattices.

Definition 2.32 *A residuated lattice*[9] *[22] is an algebra* $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *in which* $\sqcap, \sqcup, *$ *and* $\Rightarrow$ *are binary operators on the set* L *and*

[9] In the literature (e.g. in [44]), the name residuated lattice is sometimes used for structures more general than what we call residuated lattices. In the most general terminology, our structures would be called bounded integral commutative residuated lattices.

- *$(L, \sqcap, \sqcup)$ is a bounded lattice with 0 as smallest and 1 as greatest element,*
- *$*$ is commutative and associative, with 1 as neutral element, and*
- *$x * y \leqslant z$ iff $x \leqslant y \Rightarrow z$ for all x, y and z in L (residuation principle).*

The residuation principle is also known as the adjointness property [5] or the Galois correspondence [71]. The binary operations $*$ and $\Rightarrow$ are called product and implication, respectively. We will use the notations $\neg x$ for $x \Rightarrow 0$ (negation), $x \Leftrightarrow y$ for $(x \Rightarrow y) \sqcap (y \Rightarrow x)$ and x^n for $\underbrace{x * x * \ldots * x}_{n \text{ times}}$.
Moreover, we assume $x^0 = 1$.
An element x of L is idempotent iff $x * x = x$, it is nilpotent iff there exists an integer n such that $x^n = 0$.

In [5] Belohlávek proved that a residuated lattice can be defined alternatively as a bounded lattice (with 0 as smallest and 1 as greatest element) with two binary operations $*$ and $\Rightarrow$ satisfying the residuation principle, $1 \Rightarrow x = x$ and the exchange principle $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$. He also gave another alternative definition, using only identities[10]: a bounded lattice (with 0 as smallest and 1 as greatest element) with a commutative and associative binary operation $*$ and a binary operation $\Rightarrow$ such that

- $x * 1 = x$,
- $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$,
- $(x * (x \Rightarrow y)) \sqcup y = y$ and
- $x \Rightarrow (x \sqcup y) = 1$.

He also proved that it is impossible to give such an alternative definition (with only identities) using less than three variables.
The connection between (subvarieties of) residuated lattices and formal fuzzy logics will be discussed in Chapter 4. In this (and the next) section, we will give the basic definitions and properties of residuated lattices and investigate the residuated lattices $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $(L, \sqcap, \sqcup)$ is a triangularization of a bounded lattice.

Proposition 2.33 *In a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, $*$ is increasing in both arguments and $\Rightarrow$ is decreasing in the first and increasing in the second argument. Furthermore the following inequalities and identities hold, for every x, y and z in L:*

1. *$x * y \leqslant x \sqcap y$,*
2. *$\neg x \sqcup y \leqslant x \Rightarrow y$,*
3. *$x * \neg y \leqslant \neg(x \Rightarrow y)$,*
4. *$x \leqslant y \Rightarrow (x * y)$,*
5. *$x * (x \Rightarrow y) \leqslant x \sqcap y$ (in particular: $x * \neg x = 0$),*
6. *$x \sqcup y \leqslant (x \Rightarrow y) \Rightarrow y$ (in particular: $x \leqslant \neg\neg x$),*

[10] So the class of residuated lattices is a variety.

7. $\neg\neg\neg x = \neg x$,

8. $(x \Rightarrow y) * z \leqslant x \Rightarrow (y * z)$,

9. $x \Rightarrow (y \sqcap z) = (x \Rightarrow y) \sqcap (x \Rightarrow z)$,

10. $(x \sqcup y) \Rightarrow z = (x \Rightarrow z) \sqcap (y \Rightarrow z)$ *(in particular:* $\neg(x \sqcup y) = \neg x \sqcap \neg y$*)*,

11. $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$ *(in particular:* $\neg(x * y) = x \Rightarrow \neg y$*)*,

12. $x \Rightarrow y$ *is equal to the largest element* z *in* L *that satisfies* $x * z \leqslant y$, *so we have* $x \Rightarrow y = \sup\{z \in L \mid x * z \leqslant y\}$ *(in particular:* $1 \Rightarrow y = y$*)*

13. *if* $x \leqslant y$ *then* $\neg y \leqslant \neg x$,

14. $x \leqslant y$ *iff* $x \Rightarrow y = 1$, *(in particular:* $x = 0$ *iff* $\neg x = 1$*)*

15. $x = y$ *iff* $x \Leftrightarrow y = 1$,

16. $x * (y \sqcup z) = (x * y) \sqcup (x * z)$,

17. $(x \Rightarrow y) * (y \Rightarrow z) \leqslant x \Rightarrow z$,

18. $x \Rightarrow y \leqslant (x * z) \Rightarrow (y * z)$.

Moreover, in residuated lattices with involutive negation ($\neg\neg x = x$)*, for every* x *and* y *in* L,

19. $\neg(x \sqcap y) = \neg x \sqcup \neg y$,

20. $x \Rightarrow y = \neg y \Rightarrow \neg x$,

21. $\neg(x \Rightarrow y) = x * \neg y$.

The proofs can be found in, e.g., [71]. Property 19 and the particular case of 10 are known as the de Morgan laws. Property 14 is called confinement principle, property 20 contrapositivity. The following kinds of residuated lattices are useful for this work.

Definition 2.34

- *An MTL-algebra [28] is a prelinear residuated lattice, i.e., a residuated lattice in which* $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ *for all* x *and* y *in* L.

- *A BL-algebra [42] is a divisible MTL-algebra, i.e., an MTL-algebra in which* $x \sqcap y = x * (x \Rightarrow y)$ *for all* x *and* y *in* L. *The weaker property* $x \sqcap y = (x * (x \Rightarrow y)) \sqcup (y * (y \Rightarrow x))$ *is called weak divisibility*[11] *in this work.*

- *An MV-algebra [10, 11] is a BL-algebra in which the negation is an involution, i.e.,* $(x \Rightarrow 0) \Rightarrow 0 = x$ *for all* x *in* L.

- *A Heyting-algebra, or pseudo-Boolean algebra [67], is a residuated lattice in which* $x * x = x$ *(contraction) for all* x *in* L, *or, equivalently, in which* $x * y = x \sqcap y$ *for all* x *and* y *in* L.

[11] This property is also mentioned in [35]. In that paper, another (unrelated) weakened form of divisibility (in MTL-algebras), called $D_\vee$, is examined.

- *A G-algebra [42] is a prelinear Heyting-algebra.*
- *A Boolean algebra [47] is an MV-algebra that is also a Heyting-algebra.*

If a residuated lattice satisfies $x \sqcup y = ((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x)$, for all x and y in L, then it is called $\sqcup$-definable [28, 29]. The stronger property $x \sqcup y = (x \Rightarrow y) \Rightarrow y$ is called strong $\sqcup$-definability[12] *in this work. Other interesting properties are the law of excluded middle*[13] *($x \sqcup \neg x = 1$), pseudocomplementation ($x \sqcap \neg x = 0$), cancellation ($\neg x \sqcup ((x \Rightarrow (x * y)) \Rightarrow y) = 1$), weak cancellation ($\neg(x * y) \sqcup ((x \Rightarrow (x * y)) \Rightarrow y) = 1$) and weak nilpotent minimum ($\neg(x * y) \sqcup ((x \sqcap y) \Rightarrow (x * y)) = 1$).*

In residuated lattices, divisibility is equivalent to the following property: if $x \leqslant y$, then there exists a z such that $x = y * z$ (see, e.g., [44]). Using this equivalence, it is easy to see that a Heyting algebra is always divisible. Being divisible residuated lattices, Heyting algebras are always distributive [44]. Some other connections between the properties in Definition 2.34 are the following (see, e.g., [29, 42, 44, 71]).

- A residuated lattice is an MV-algebra iff it is divisible and the negation is an involution.
- A residuated lattice with involutive negation is $\sqcup$-definable iff it is weak divisible.
- If a residuated lattice is prelinear or divisible, then it is distributive.
- A divisible and $\sqcup$-definable residuated lattice, is prelinear.
- A prelinear residuated lattice is $\sqcup$-definable and weak divisible.
- A residuated lattice is a Boolean algebra iff it is a Heyting-algebra with involutive negation. It satisfies all of the properties mentioned here: distributivity, (weak) divisibility, (strong) $\sqcup$-definability, prelinearity, (weak) cancellation, pseudocomplementation and weak nilpotent minimum. Boolean algebras form the semantics of classical logic, which can be seen as the strongest possible formal fuzzy logic.

In Section 2.5 we will see that these connections can often be generalized.

In a residuated lattice, the operator $*$ is always a residuated t-norm, with $\Rightarrow$ as its residual implicator. Conversely, if T is a residuated t-norm on a bounded lattice $(L, \sqcap, \sqcup)$, then $(L, \sqcap, \sqcup, T, I_T, 0, 1)$ is a residuated lattice. Note however that not all t-norms are residuated. In complete lattices $(L, \sqcap, \sqcup)$, a t-norm T is residuated (and therefore induces a residuated lattice) iff it satisfies $T(x, \sup Y) = \sup\{T(x, y) | y \in Y\}$, for all x in L and $Y \subseteq L$ [71]. Note that this implies the following: on a finite lattice $\mathscr{L} = (L, \sqcap, \sqcup)$, a Heyting-algebra exists iff $\mathscr{L}$ is distributive. Also note that some of the given properties can be satisfied by a t-norm and its residual implicator (if it exists), even if it is not a residuated t-norm. For example, if in a bounded lattice $\mathscr{L}$ every two incomparable elements a and b satisfy $a \sqcup b = 1$, then any t-norm T having a residual implicator I_T satisfies $I_T(x, y) \sqcup I_T(y, x) = 1$. This follows from the remarks before Definition 2.4.

Example 2.35 *Let T be a t-norm on $([0, 1], \min, \max)$. It is well-known (see, e.g., [29, 42] that*

- *T is residuated iff T is left-continuous,*

[12] Strong $\sqcup$-definable residuated lattices are exactly MV-algebras [44].
[13] Residuated lattices satisfying the law of excluded middle are exactly Boolean algebras.

- $([0,1], \min, \max, T, I_T, 0, 1)$ *is an MTL-algebra*[14] *iff* T *is left-continuous,*
- $([0,1], \min, \max, T, I_T, 0, 1)$ *is a BL-algebra iff* T *is continuous,*
- $([0,1], \min, \max, T, I_T, 0, 1)$ *is an MV-algebra iff* T *is conjugated to the Łukasiewicz t-norm* T_W*, i.e., iff there exists a strictly increasing bijection* $\varphi : [0,1] \longrightarrow [0,1]$ *such that* $T(x,y) = \varphi^{-1} T_W(\varphi(x), \varphi(y))$*, where* $T_W(x,y) = \max(0, x+y-1)$.

These structures are called standard MTL-algebras, standard BL-algebras and standard MV-algebras, respectively. Similarly for other kinds of residuated lattice (except for Boolean algebras[15]*). For example, a standard Heyting-algebra is a Heyting-algebra of the form* $([0,1], \min, \max, T, I_T, 0, 1)$*. Note that there is only one standard Heyting-algebra, namely* $([0,1], \min, \max, \min, I_{\min}, 0, 1)$*. Also note that the only t-norm conjugated to* min *is* min *itself, and that the only t-norm conjugated to the drastic t-norm is the drastic t-norm itself.*

As triangularizations of bounded lattices are bounded lattices, we can consider residuated lattices on these structures (see also (2.2) at the end of Section 2.3). In most cases, these residuated lattices are not prelinear. For example, no MTL-algebra exists on $\mathbb{T}([0,1], \min, \max)$, because of the $\sqcup$-irreducibility of $[1,1]$ and Proposition 2.33(14). However, some (non-residuated) t-norms can satisfy prelinearity, as the following proposition shows.

Proposition 2.36 *Let* T *be a t-norm on* $([0,1], \min, \max)$*. Then* $(\mathbb{T}([0,1], \min, \max), \mathscr{T}_{T,t}, \mathscr{I}_{T,t}, [0,0], [1,1])$ *is prelinear iff* T *is the drastic t-norm on* $[0,1]$.

Proof. Suppose first that T is the drastic t-norm, i.e. the smallest possible t-norm on $[0,1]$. Then T is given by

$$T(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then its residual implicator is

$$I_T(x,y) = \bigvee \{z \in [0,1] \mid T(z,x) \leqslant y\} = \begin{cases} 1, & \text{if } x \neq 1, \\ y, & \text{if } x = 1. \end{cases}$$

We want to prove that

$$\begin{aligned} [1,1] = &[\min(I_T(x_1,y_1), I_T(x_2,y_2)), \min(I_T(T(x_2,t),y_2), I_T(x_1,y_2))] \vee \\ &[\min(I_T(y_1,x_1), I_T(y_2,x_2)), \min(I_T(T(y_2,t),x_2), I_T(y_1,x_2))]. \end{aligned}$$

It suffices to show that, for every x and y in $Int([0,1], \min, \max)$, $\min(I_T(x_1,y_1), I_T(x_2,y_2)) = 1$ or $\min(I_T(y_1,x_1), I_T(y_2,x_2)) = 1$. If this were not true, then (for example) $I_T(x_2,y_2) \neq 1$. In this case, necessarily $x_2 = 1$. Therefore $I_T(y_2,x_2) = 1$. This implies $I_T(y_1,x_1) \neq 1$ (as we supposed that $\min(I_T(y_1,x_1), I_T(y_2,x_2)) \neq 1$), hence $y_1 = 1$ and so $y_2 = 1$. This is a contradiction. The other cases are similar.

[14] Because $([0,1], \min, \max)$ is linear, every residuated lattice on this lattice is automatically an MTL-algebra.

[15] No Boolean algebra exists on $([0,1], \min, \max)$. The only linear Boolean algebras are the trivial one (with one element) and the standard Boolean algebra, with two elements.

Conversely, suppose that $(\mathbb{T}([0,1],\min,\max),\mathscr{T}_{T,t},\mathscr{I}_{T,t},[0,0],[1,1])$ is prelinear. We know that (taking $x_1=0$, $x_2=1$ and $y_1=y_2$)

$$\max(\min(I_T(0,y_1),I_T(1,y_1)),\min(I_T(y_1,0),I_T(y_1,1)))=1,$$

which reduces to $\max(y_1,I_T(y_1,0))=1$. In other words: for every $y\neq 1$, $I_T(y,0)=1$. Now take any x and y in $[0,1[$. Then $x<1=I_T(y,0)=\bigvee\{z\in[0,1]\mid T(z,y)\leqslant 0\}$. So there must exist a z_0 in $[0,1]$ such that $x\leqslant z_0$ and $T(z_0,y)\leqslant 0$, which implies that $T(x,y)=0$. We conclude that T must be the drastic t-norm. □

2.5 Filters of residuated lattices

Filters are subsets of partially ordered sets that satisfy some defining properties. The concept is also used in partially ordered algebraic structures with more connectives (lattices, residuated lattices, MTL-algebras, ...), in particular in the algebraic semantics of formal fuzzy logics. Filters are of crucial importance in the proof of the (chain) completeness of these logics (see Chapter 4). Filters are also particularly interesting because they are closely related to congruence relations; with every filter we can associate a quotient algebra. If we want such a quotient algebra to satisfy particular properties, we need to impose extra conditions on the filter. To prove similar results in the interval-valued setting later on, we will need specific kinds of filters (see Section 3.6). In the current section we already give the results that can be obtained in general, for all residuated lattices.

In bounded partially ordered sets at most one filter can be a singleton, namely $\{1\}$. Stating that this singleton is a specific kind of filter, is equivalent to requiring the algebraic structure to fulfill a particular property. For example, $\{1\}$ is a prime filter of a residuated lattice if, and only if, that residuated lattice is linear (i.e., totally ordered). In this way, we can translate connections between properties on algebraic structures to connections between properties of the filter $\{1\}$. Often these connections can be generalized to any filter (not necessarily $\{1\}$). We will give several examples later on in this subsection.

Prime filters are also interesting because they can be used to prove that MTL-algebras are subdirect products of linear residuated lattices [28].

Definition 2.37 *[42, 44, 51, 53, 54, 65, 72] A filter of a residuated lattice $\mathscr{L}=(L,\sqcap,\sqcup,*,\Rightarrow,0,1)$ is a non-empty subset F of L satisfying*

- *for all x and y in L: if $x\in F$ and $x\leqslant y$, then $y\in F$,*
- *for all x and y in F: $x*y\in F$ (i.e., F is closed under $*$).*

A Boolean filter (BF) of $\mathscr{L}$ is a filter F satisfying

- *for all x in L: $x\sqcup\neg x\in F$.*

A prime filter (PF) of $\mathscr{L}$ is a filter F satisfying

- *for all x and y in L: $x\Rightarrow y\in F$ or $y\Rightarrow x\in F$ (or both).*

A prime filter of the second kind[16] *(PF2) is a filter F satisfying*

[16] In [53], this kind of filter was called a prime filter. In general however, this definition and the usual definition of prime filter are not equivalent. Therefore we call it 'of the second kind'.

- *for all x and y in L: if $x \sqcup y \in F$, then $x \in F$ or $y \in F$ (or both).*

A positive implicative filter of $\mathscr{L}$ is a subset F of L such that

- $1 \in F$,
- *for all x, y and z in F: if $(x * y) \Rightarrow z \in F$ and $x \Rightarrow y \in F$, then $x \Rightarrow z \in F$.*

An alternative definition for a filter F (see e.g. [71]) of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is the following:

- $1 \in F$,
- for all x and y in L: if $x \in F$ and $x \Rightarrow y \in F$, then $y \in F$.

An alternative definition for a Boolean filter F of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is the following: F is a filter of $\mathscr{L}$ satisfying

- for all x, y and z in L: if $(x * \neg z) \Rightarrow y \in F$ and $y \Rightarrow z \in F$, then $x \Rightarrow z \in F$.

This alternative definition was proven in [54], where these filters were called implicative filters. It was also shown in [54] that a Boolean filter is always a positive implicative filter and that also the converse is true in residuated lattices with an involutive negation $\neg$. A positive implicative filter of a residuated lattice is always a filter of that residuated lattice [54].

Finite filters and finite prime filters of the second kind

If $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and $l \in L$ such that $l * l = l$, then it is not difficult to see that $F_l := \{x \in L | l \leqslant x\}$ is a filter of $\mathscr{L}$. In fact, finite filters of residuated lattices are always of this form.

Proposition 2.38 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice and F a finite filter of $\mathscr{L}$. Then $F = F_l$ for some idempotent l in L.*

Proof. If F is a finite filter of $\mathscr{L}$, then the product m of all its elements (a finite number) must also be an element of F. So $m * m \in F$ and therefore m (as the product of all elements of F) is smaller than or equal to $m * m$ (because of Proposition 2.33(1)), which implies $m * m = m$. □

Note that this property does not hold for infinite filters. For example, $]0, 1]$ is a filter of $([0, 1], \min, \max, \min, \Rightarrow_{\min}, 0, 1)$, but $]0, 1]$ is not of the form F_l for any $l \in [0, 1]$.
Finite prime filters of the second kind of residuated lattices are always of the form F_l, with l idempotent and $\sqcup$-irreducible. Conversely, filters (even infinite ones) of this form will always be prime filters of the second kind.

Proposition 2.39 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice. If l is an idempotent and $\sqcup$-irreducible element of L, then F_l is a prime filter of the second kind of $\mathscr{L}$.*

Proof. Suppose $a \sqcup b \in F_l$. Then $l \leqslant a \sqcup b$ and therefore, using Proposition 2.33(16), $l = l * l \leqslant l * (a \sqcup b) = (l * a) \sqcup (l * b) \leqslant l$. So $l = (l * a) \sqcup (l * b)$, which implies $l = l * a$ or $l = l * b$. So $l \leqslant a$ or $l \leqslant b$, which means exactly that $a \in F_l$ or $b \in F_l$. □

Boolean filters of the second kind and prime filters of the third kind

Some particular kinds of filters appear quite frequently in our investigation of the connections between different kinds of filters. Therefore we give them a name and introduce them in the next definition.

Definition 2.40 *A Boolean filter of the second kind (BF2) of a residuated lattice* $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *is a filter of* $\mathscr{L}$ *satisfying*

- *for all* x *in* L*:* $x \in F$ *or* $\neg x \in F$ *(or both).*

A prime filter of the third kind (PF3) of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *is a filter of* $\mathscr{L}$ *satisfying*

- *for all* x *and* y *in* L*:* $(x \Rightarrow y) \sqcup (y \Rightarrow x) \in F$.

It can easily be seen that there is always a trivial example of each kind of filter defined before, namely L; and that every filter always contains the greatest element 1.

Definition 2.41 *Let* S *be a set,* $\mathscr{C}$ *be a subset of its powerset* $\mathscr{P}(S)$ *and* $\mathscr{C} \subseteq \mathscr{D} \subseteq \mathscr{P}(S)$.

- *We say* $\mathscr{C}$ *satisfies the intersection property iff* $\mathscr{C}$ *is closed under arbitrary intersections.*
- *We say* $\mathscr{C}$ *satisfies the monotonicity property w.r.t.* $\mathscr{D}$ *iff* $S_2 \in \mathscr{C}$ *whenever* $S_1 \in \mathscr{C}$ *and* $S_1 \subseteq S_2 \in \mathscr{D}$.

In this section, $\mathscr{D}$ will always be the set of filters (of a particular lattice). Therefore we will just write 'monotonicity property' instead of 'monotonicity property w.r.t. the set of filters'.
Proposition 2.42 summarizes some properties of filters that follow straight from the definitions.

Proposition 2.42 *Let* $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *be a residuated lattice.*

- *The intersection property holds for the set of filters of* $\mathscr{L}$*, the set of prime filters of the third kind of* $\mathscr{L}$*, and the set of Boolean filters of* $\mathscr{L}$.
- *The monotonicity property holds for the set of filters of* $\mathscr{L}$*, the set of prime filters of* $\mathscr{L}$*, the set of prime filters of the third kind of* $\mathscr{L}$*, the set of Boolean filters of* $\mathscr{L}$*, and the set of Boolean filters of the second kind of* $\mathscr{L}$.

For PF, PF2 and BF2 the intersection property does not hold, as will be shown in some of the examples.
In Example 2.43 we will show that the monotonicity property does not hold for prime filters of the second kind.[17] Note that the monotonicity property suggests a special role for the singleton $\{1\}$, which is always a filter. Indeed, in order to show that all filters in a residuated lattice $\mathscr{L}$ are PF (PF3, BF or BF2, resp.), it suffices to show that $\{1\}$ is PF (PF3, BF or BF2, resp.). Therefore we first take a closer look at the filter $\{1\}$. The proofs of the following statements immediately follow from the definitions.

- It is a Boolean filter iff $\mathscr{L}$ is a Boolean algebra. In this case, all filters are Boolean filters.

[17] Another reason why PF2 can be regarded as the odd one out amongst PF, PF2, PF3, BF and BF2, is that the condition 'if $x \sqcup y \in F$, then $x \in F$ or $y \in F$' in its definition only depends on the lattice structure, not on the product or implication.

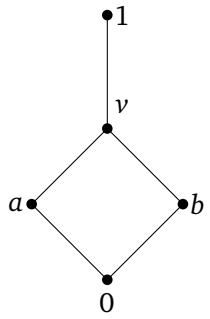

Figure 2.6: Doubleton $\{v, 1\}$ is a filter of the Heyting-algebra on this lattice, but not a prime filter of the second kind. The singleton $\{1\}$ is a prime filter of the second kind, but not a prime filter.

- It is a Boolean filter of the second kind iff $\mathscr{L}$ is the Boolean algebra with two elements (or the trivial Boolean algebra with one element). In this case, all filters are Boolean filters of the second kind.
- It is a prime filter iff $\mathscr{L}$ is linearly ordered. In this case, all filters are prime filters.
- It is a prime filter of the second kind iff 1 is $\sqcup$-irreducible in $\mathscr{L}$. Note that in general, this does not imply that all filters are prime filters of the second kind. A counterexample will be given in Example 2.43.
- It is a prime filter of the third kind iff $\mathscr{L}$ is an MTL-algebra. In this case, all filters are prime filters of the third kind.
- It is a positive implicative filter iff[18] $\mathscr{L}$ is a Heyting-algebra. In this case, all filters are positive implicative filters.

Now, before we further investigate the connections between the different kinds of filters, we first give some simple but useful examples[19].

Examples of filters of residuated lattices

Example 2.43 *Three residuated lattices exist on the lattice in Figure 2.6 (lattice 5.1.20 in [50]). If we consider the Heyting-algebra, the filters are* $\{1\}$, $\{v, 1\}$, $\{a, v, 1\}$, $\{b, v, 1\}$ *and* $\{0, a, b, v, 1\}$. *Note that* $\{1\}$ *is PF2, but* $\{v, 1\}$ *is not. So the monotonicity property does not hold for PF2. Also note that* $\{1\}$ *is not PF3, as prelinearity does not hold:* $(a \Rightarrow b) \sqcup (b \Rightarrow a) = b \sqcup a = v$. *The filters* $\{a, v, 1\}$, $\{b, v, 1\}$ *and* $\{0, a, b, v, 1\}$ *are PF, PF2, PF3, BF and BF2. Note however that their intersection* $\{v, 1\}$ *is not PF, PF2 nor BF2. So the intersection property does not hold for PF, PF2 and BF2.*

Example 2.44 *Only one residuated lattice exists on the lattice displayed in Figure 2.7 (lattice 4.1.5 in [50]). This is a Boolean algebra (isomorphic to the powerset* $\mathscr{P}(\{a, b\})$ *endowed with intersection and union). So* $\{1\}$ *is a Boolean filter. But it is not a prime filter of the second kind. Similarly as in Example 2.43, the filters* $\{a, 1\}$, $\{b, 1\}$ *and* $\{0, a, b, 1\}$ *are PF, PF2, PF3, BF and BF2, but their intersection* $\{1\}$ *is not PF, PF2 nor BF2.*

[18] This follows from the equivalence between (1) and (4) in Proposition 2.53.

[19] The small examples can be found in [50]. This website contains all residuated lattices (according to the most general definition) up to size 6.

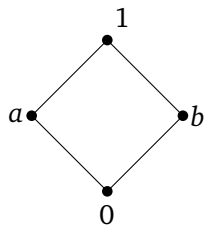

Figure 2.7: A Boolean algebra in which 1 is not ⊔-irreducible.

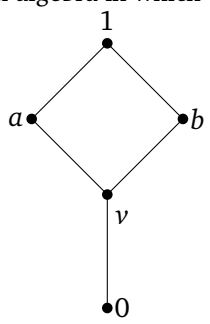

Figure 2.8: The residuated lattices on this lattice are prelinear. But they are not Boolean algebras, and 1 is not ⊔-irreducible.

Example 2.45 *On a three-element chain (lattice 3.1.3 in [50])), there are two residuated lattices. One in which the middle element is idempotent (a Heyting-algebra) and one in which it is nilpotent. In both cases, $\{1\}$ is a prime filter but not a Boolean filter. Note that there are three filters in the first case and two in the second.*

Example 2.46 *Two residuated lattices exist on the lattice in Figure 2.8 (lattice 5.1.17 in [50]): a Heyting-algebra and one in which $v * v = 0$. In both cases $\{1\}$ is PF3, but not PF, PF2, BF nor BF2.*

Example 2.47 *Seven different residuated lattices exist on the lattice in Figure 2.9. Exactly one of these is prelinear. In this MTL-algebra, the product $*$ is defined as follows:*

$$x * y = \begin{cases} x \sqcap y, & \text{if } x = 1,\ y = 1 \text{ or } n \leqslant x \sqcap y \\ a, & \text{if } \{x, y\} = \{a, c\} \text{ or } \{x, y\} = \{m, c\} \\ b, & \text{if } \{x, y\} = \{b, d\} \text{ or } \{x, y\} = \{m, d\} \\ 0, & \text{otherwise.} \end{cases}$$

One of the six non-prelinear residuated lattices is the Heyting-algebra. The filter $\{1\}$ is not a prime filter of the third kind of this residuated lattice. It is not a prime filter of the second kind either.

Remark 2.48 *Prelinearity can be seen as some kind of 'approximation' of linearity (remind that prelinear residuated lattices are subdirect products of linear residuated lattices [28]) and linearity is independent of the product and implication (in other words: if one residuated lattice on a lattice is linear, all residuated lattices on that lattice will be linear). So one might expect that this holds for prelinearity as well. Example 2.47 shows however that this is not the case, because a prelinear as well as a non-prelinear residuated lattice can exist on the same lattice. Note that for Boolean algebras, we do have such a property. If a Boolean algebra exists on a specific lattice, all residuated lattices on that lattice will be Boolean algebras. To be more precise, there will be only one residuated lattice: the*

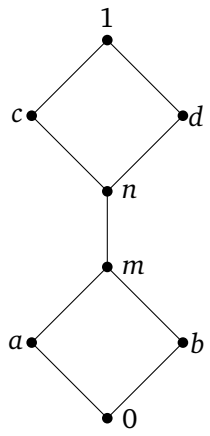

Figure 2.9: The Heyting-algebra is not prelinear, and 1 is not $\sqcup$-irreducible.

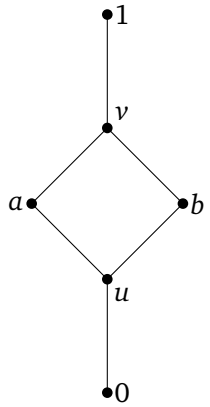

Figure 2.10: In some of the residuated lattices on this lattice, all filters are PF2.

Boolean algebra. The proof of this property is very short. Indeed, suppose $(L, \sqcap, \sqcup, \sqcap, \Rightarrow_\sqcap, 0, 1)$ *is a Boolean algebra and* $(L, \sqcap, \sqcup, *, \Rightarrow_*, 0, 1)$ *a residuated lattice. Then, because of Proposition 2.33(1 and 5), for every* x *in* L*,* $x * (x \Rightarrow_\sqcap 0) \leqslant x \sqcap (x \Rightarrow_\sqcap 0) = 0$ *and therefore* $x \Rightarrow_\sqcap 0 \leqslant x \Rightarrow_* 0$*. So* $1 = x \sqcup (x \Rightarrow_\sqcap 0) \leqslant x \sqcup (x \Rightarrow_* 0)$*.*

Example 2.49 *There exist thirteen different*[20] *residuated lattices on the lattice in Figure 2.10 (6.1.75 in [50]). None of them is prelinear, so* $\{1\}$ *is not PF3 in any of them. One of them is determined by* $a * a = b * b = u$ *and* $a * b = 0$*. The filters of this residuated lattice are* $\{1\}$ *and* $\{0, u, a, b, v, 1\}$*. Both are prime filters of the second kind.*

These examples give us some negative results. The most important[21] for us are:

- a filter can be PF2, but not PF3; (Example 2.43)

[20] They are listed in [50] under 6.1.75, together with seven other structures that are not commutative. Three of them are discussed in [29] as well.

[21] We call these negative results the most important because – together with the positive results in the next subsection – they imply all negative results amongst the connections between PF, PF2, PF3, BF and BF2. For example, PF3 cannot imply PF because BF does not imply PF2 and as we will see, BF implies PF3 and PF implies PF2.

- a filter can be BF, but not PF2; (Example 2.44)
- a filter can be PF, but not BF. (Example 2.45)

Moreover, in each of these three cases, the residuated lattice can be chosen in such a way that $\{1\}$ can serve as the sought-after filter. For example, in Example 2.45, $\{1\}$ is a prime but not a Boolean filter.
Other negative results are:

- $\{1\}$ being PF2 does not imply all filters being PF2; (Example 2.43)
- all filters being PF2 does not imply $\{1\}$ being PF3. (Example 2.49)

Now we continue with the positive results.

Connections between the different kinds of filters of residuated lattices

It is easy to see that a prime filter is always a prime filter of the third kind, and that a Boolean filter of the second kind is always a Boolean filter.

Proposition 2.50 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice. Every prime filter of $\mathscr{L}$ is also a prime filter of the second kind.*
If $\mathscr{L}$ is an MTL-algebra, then every prime filter of the second kind of $\mathscr{L}$ is also a prime filter.

Proof. Suppose F is a prime filter of the residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, and $a \sqcup b \in F$. We know that F contains $a \Rightarrow b$ or $b \Rightarrow a$ (or both). Without loss of generality, we assume $a \Rightarrow b \in F$. Then, using Proposition 2.33(16), $(a*(a \Rightarrow b)) \sqcup (b*(a \Rightarrow b)) = (a \sqcup b)*(a \Rightarrow b) \in F$. This implies $b \in F$, because $(a * (a \Rightarrow b)) \sqcup (b * (a \Rightarrow b)) \leqslant b$ (Proposition 2.33(5)).
Now suppose F is a prime filter of the second kind of the MTL-algebra $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, and $a, b \in L$. Because $(a \Rightarrow b) \sqcup (b \Rightarrow a) = 1 \in F$, either $a \Rightarrow b$ or $b \Rightarrow a$ (or both) must be in F. □

Corollary 2.51 *The prime filters of a residuated lattice $\mathscr{L}$ are exactly the filters of $\mathscr{L}$ that are at the same time prime filters of the second kind and prime filters of the third kind.*

In residuated lattices that are not MTL-algebras, prime filters of the second kind are in general not prime filters. As a counterexample we can take the filter $\{1\}$ in Example 2.43.
In fact, we have not been able to find any non-prelinear residuated lattice in which all prime filters of the second kind are prime filters. Such an example (if it should exist) cannot be finite, due to the next proposition.

Proposition 2.52 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice in which every prime filter of the second kind is also a prime filter. If L is finite or 1 is $\sqcup$-irreducible in $\mathscr{L}$, then $\mathscr{L}$ is prelinear.*

Proof. First note that $\mathscr{L}$ is linear if $|L| \leqslant 3$, so we only need to consider residuated lattices with 4 elements or more.
If 1 is $\sqcup$-irreducible, then $\{1\}$ is a prime filter of the second kind of $\mathscr{L}$ and by assumption also a prime filter of $\mathscr{L}$. This means that $\mathscr{L}$ is linear (and therefore also prelinear). So in the rest of the proof we can assume L is finite.

Denote $B := \{x \in L | x < 1 \text{ and } (\forall y \in L)(\text{if } x < y \text{ then } y = 1)\}$. In other words, B is the subset of elements of L that are covered by 1. In the Hasse-diagram of $\mathscr{L}$, these are the elements that are connected directly with 1. Because L is finite, B is not empty. We consider two cases:

- $|B| = 1$. Say, $B = \{b\}$. Then for all x in L: $x = 1$ or $x \leqslant b$. This implies that 1 is $\sqcup$-irreducible; and we already proved this case.

- $|B| > 1$. For any $b \in B$, we will construct a prime filter of the second kind of $\mathscr{L}$ that does not contain b. By assumption this filter will be a prime filter of $\mathscr{L}$ and thus also a prime filter of the third kind of $\mathscr{L}$. So for any $b \in B$ we obtain a prime filter of the third kind of $\mathscr{L}$ that does not contain b. The intersection of these filters is also a prime filter of the third kind of $\mathscr{L}$ and it does not contain any element of B. Thus this intersection must be $\{1\}$. Saying that $\{1\}$ is a prime filter of the third kind of $\mathscr{L}$ exactly means that $\mathscr{L}$ is prelinear.
  Take any b in B. Because B contains at least two elements, we can choose $a_1 \in B$ such that $b \neq a_1$. Note that $b \sqcup a_1 = 1$. If a_1 is idempotent and $\sqcup$-irreducible, we are done. Because in this case $\{a_1, 1\}$ is a prime filter of the second kind (Proposition 2.39). If this is not the case, we will show that we can find an element a_2 in L such that $a_2 < a_1$ and $b \sqcup a_2 = 1$. If a_2 is idempotent and $\sqcup$-irreducible, we are done. If not, we can find an element a_3 such that $a_3 < a_2$ and $b \sqcup a_3 = 1$. And so on. Because L is finite, this process must stop at some point. This is only possible if we arrive at an element a_k which is idempotent and $\sqcup$-irreducible, giving us the desired prime filter of the second kind, namely $\{x \in L | a_k \leqslant x\}$.
  What remains to prove is thus: if $a_n \in L$ is such that $b \sqcup a_n = 1$ and ($a_n * a_n < a_n$ or $a_n = c \sqcup d$ with $c \neq a_n \neq d$), then there exists an element a_{n+1} in L such that $a_{n+1} < a_n$ and $b \sqcup a_{n+1} = 1$. We have to consider two cases:

  1. $b \sqcup a_n = 1$ and $a_n * a_n < a_n$. Then we choose $a_{n+1} := a_n * a_n$. Indeed, $a_{n+1} < a_n$ and (using Proposition 2.33(1 and 16)) $1 = b \sqcup a_n = b \sqcup (a_n * (b \sqcup a_n)) = b \sqcup ((a_n * b) \sqcup (a_n * a_n)) = b \sqcup (a_n * a_n) = b \sqcup a_{n+1}$.
  2. $b \sqcup a_n = 1$ and $a_n = c \sqcup d$ with $c \neq a_n \neq d$. Suppose $b \sqcup c = b$ and $b \sqcup d = b$. Then $b \sqcup a_n = b \sqcup (c \sqcup d) = (b \sqcup c) \sqcup (b \sqcup d) = b < 1$, a contradiction. So at least one of $b \sqcup c$ and $b \sqcup d$ must be strictly greater than b, and therefore (by definition of B) equal to 1, say $b \sqcup c = 1$. Then we can choose $a_{n+1} := c$.

□

It might be true that MTL-algebras (including all infinite ones) are exactly those residuated lattices in which prime filters and prime filters of the second kind coincide, but this remains an open problem.
Because for any two elements x and y in a residuated lattice it holds that $x \leqslant y \Rightarrow x$ and $\neg x \leqslant x \Rightarrow y$ (Proposition 2.33(2)), it follows that every Boolean filter of the second kind of a residuated lattice is also a prime filter of that residuated lattice, and that every Boolean filter is also a prime filter of the third kind.
Clearly a Boolean filter of the second kind is also any of the other kinds of filters we defined. Note that any Boolean filter that is at the same time a prime filter of the second kind, is automatically a Boolean filter of the second kind.

Considering all examples and all the above implications, we see that only six different situations can occur for a filter F of a residuated lattice $\mathscr{L}$:

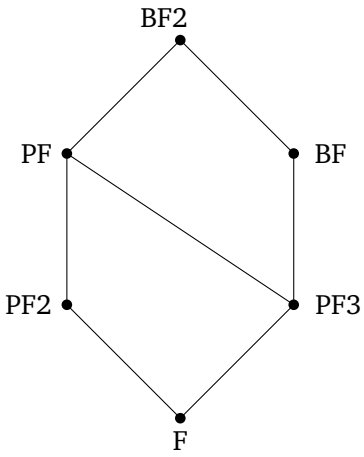

Figure 2.11: The six possibilities for a filter.

- F is a filter of $\mathscr{L}$, but F is not PF, PF2, PF3, BF nor BF2;
- F is PF2, but not PF, PF3, BF nor BF2;
- F is PF3, but not PF, PF2, BF nor BF2;
- F is PF (and therefore also PF2 and PF3), but not BF nor BF2;
- F is BF (and therefore also PF3), but not PF, PF2 nor BF2;
- F is BF2 (and therefore also PF, PF2, PF3 and BF).

The filter $\{1\}$ is an example of the first five situations in Examples 2.47 (the Heyting-algebra), 2.43, 2.46, 2.45 and 2.44, respectively. It is an example of the last situation in the standard Boolean algebra with two elements.
The lattice in Figure 2.11 gives a schematic summary of these situations. If $A \leqslant B$ in this lattice, this means "every filter which is B, is also A". And if $A \sqcup B = C$ in this lattice, this means "every filter which is A as well as B, is also C".

Positive implicative filters, pseudocomplementation filters and involution filters

In this paragraph, we give some alternative characterizations for positive implicative filters and show the connection with the other kinds of filters.

Proposition 2.53 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice and F a filter of $\mathscr{L}$. Then the following statements are equivalent:*

1. *F is a positive implicative filter,*
2. *$(x \sqcap (x \Rightarrow y)) \Rightarrow y \in F$ for all x and y in L,*
3. *$(x \sqcap y) \Rightarrow (x * y) \in F$ for all x and y in L,*
4. *$x \Rightarrow (x * x) \in F$ for all x in L.*

Proof.

- (1) $\Rightarrow$ (2): Suppose F is a positive implicative filter of $\mathscr{L}$. Take x and y arbitrarily in L. Because $((x \sqcap (x \Rightarrow y)) * x) \Rightarrow y = 1 \in F$ and $(x \sqcap (x \Rightarrow y)) \Rightarrow x = 1 \in F$ (using Proposition 2.33(5 and 14)), we conclude from the definition of a positive implicative filter that $(x \sqcap (x \Rightarrow y)) \Rightarrow y$ must be in F.

- (2) $\Rightarrow$ (3): For any x and y in L, $(x \sqcap (x \Rightarrow (x * y))) \Rightarrow (x * y)$ is an element of F. Because $y \leqslant x \Rightarrow (x * y)$ (Proposition 2.33(4)), $(x \sqcap y) \Rightarrow (x * y)$ is greater than or equal to $(x \sqcap (x \Rightarrow (x * y))) \Rightarrow (x * y)$ and therefore also an element of F.

- (3) $\Rightarrow$ (4): This follows immediately by taking $x = y$.

- (4) $\Rightarrow$ (1): Choose any x, y and z in L such that $(x * y) \Rightarrow z \in F$ and $x \Rightarrow y \in F$. Then $(x \Rightarrow (x * x)) * (x \Rightarrow y) * ((x * y) \Rightarrow z) \in F$. Because (using Proposition 2.33(18 and 17)) $(x \Rightarrow (x * x)) * (x \Rightarrow y) * ((x * y) \Rightarrow z) \leqslant (x \Rightarrow (x * x)) * ((x * x) \Rightarrow (x * y)) * ((x * y) \Rightarrow z) \leqslant x \Rightarrow z$, also $x \Rightarrow z \in F$.

□

Taking $F = \{1\}$ in Proposition 2.53 shows that Heyting-algebras are exactly residuated lattices in which $x \sqcap (x \Rightarrow y) \leqslant y$ for all x and y. Or, equivalently, in which $x \sqcap (x \Rightarrow y) = x \sqcap y$ for all x and y. We call this property generalized pseudocomplementation, as the special case $x \sqcap \neg x = 0$ is called pseudocomplementation.

Another immediate corollary of Proposition 2.53 is that the intersection and monotonicity property[22] hold for positive implicative filters.
On one hand, Boolean filters (and therefore also Boolean filters of the second kind) are always positive implicative filters. On the other hand, there are no connections with the other kinds of filters we considered:

- a prime filter is not necessarily a positive implicative filter (take for example $\{1\}$ in the residuated lattice with nilpotent middle element in Example 2.45),

- a prime filter which is also a positive implicative filter is not necessarily a Boolean filter (take for example $\{1\}$ in the residuated lattice with idempotent middle element in Example 2.45),

- a prime filter of the second kind which is also a positive implicative filter is not necessarily a prime filter of the third kind (take for example $\{1\}$ in Example 2.43).

If we denote pos. impl. filter, pos. impl. filter + PF2, pos. impl. filter + PF3, and pos. impl. filter + PF shortly by A, B, C and D, respectively, the situation can be graphically represented as the lattice in Figure 2.12. Just after Definition 2.37 we already mentioned that Boolean filters and positive implicative filters of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ coincide if $\mathscr{L}$ has an involutive negation. This can be generalized a bit more.

Definition 2.54 *A pseudocomplementation filter of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a filter containing $\neg(x \sqcap \neg x)$ for all x in L.*

[22]The monotonicity property for positive implicative filters was already proven (in another way) in [54].

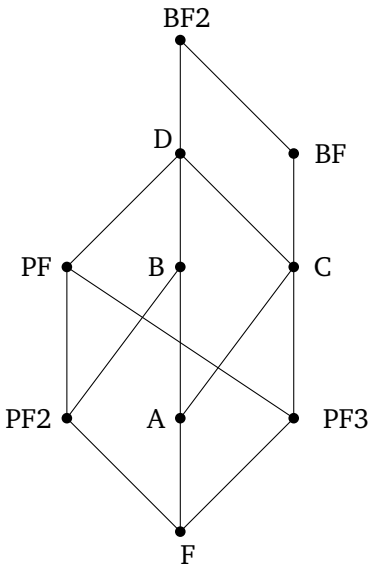

Figure 2.12: The ten possibilities for a filter if positive implicativeness is also considered.

Note that a filter F of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a pseudocomplementation filter of $\mathscr{L}$ iff F contains $\neg(\neg x \sqcap \neg\neg x)$ for all x in L. Indeed, a pseudocomplementation filter obviously contains $\neg(\neg x \sqcap \neg\neg x)$ for all x in L, and conversely, if a filter contains $\neg(\neg x \sqcap \neg\neg x)$ for all x in L, then it also contains $\neg(\neg x \sqcap x)$ for all x in L, because $\neg(\neg x \sqcap \neg\neg x) \leqslant \neg(x \sqcap \neg x)$ (using Proposition 2.33(6)).
Also note that pseudocomplementation filters enjoy the intersection and monotonicity property.
Positive implicative filters are special cases of pseudocomplementation filters (to see this, take $y = 0$ in the second equivalence in Proposition 2.53). If the negation $\neg$ is involutive, then $\neg(x \sqcap \neg x) = x \sqcup \neg x$ and we can easily conclude that in this case Boolean filters coincide with pseudocomplementation filters. Note however that this sufficient condition is not a necessary condition. Indeed, consider the linear residuated lattice with four elements 0, a, b, 1 determined by $x * x = 0$ for all x different from 1. This residuated lattice does not have an involutive negation. Yet there is only one pseudocomplementation filter (namely $\{0, a, b, 1\}$), which is obviously also a Boolean filter. Also note that not all pseudocomplementation filters are positive implicative filters. Indeed, consider the linear residuated lattice with four elements 0, a, b, 1 ($0 < a < b < 1$) determined by $a * a = b * b = a$. Then $\{1\}$ is not a positive implicative filter ($(b \sqcap (b \Rightarrow a)) \Rightarrow a = b \neq 1$), although $\neg(0 \sqcap \neg 0) = \neg(a \sqcap \neg a) = \neg(b \sqcap \neg b) = \neg(1 \sqcap \neg 1) = 1$.

Finally we also introduce involution filters.

Definition 2.55 *An involution filter of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a filter of $\mathscr{L}$ that contains $\neg\neg x \Rightarrow x$ for every x in L.*

Involution filters obviously satisfy the intersection and monotonicity property.

Proposition 2.56 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice and F a filter of $\mathscr{L}$.*

- *If F is a Boolean filter of $\mathscr{L}$, then F is an involution filter of $\mathscr{L}$.*
- *If F is a pseudocomplementation filter of $\mathscr{L}$ and at the same time an involution filter of $\mathscr{L}$, then F is a Boolean filter of $\mathscr{L}$.*

Proof.

- Immediately, because from Proposition 2.33(2 and 7) it follows that $x \sqcup \neg x \leqslant \neg\neg x \Rightarrow x$ for all x in L.
- Choose any x in L. We know that $\neg\neg(x \sqcup \neg x) \Rightarrow (x \sqcup \neg x) \in F$. Because (using Proposition 2.33(10)) $\neg\neg(x \sqcup \neg x) = \neg(\neg x \sqcap \neg\neg x) = ((\neg x) \sqcap (\neg x) \Rightarrow 0) \Rightarrow 0 \in F$, $x \sqcup \neg x \in F$.

□

Congruence relations and quotient algebras

Given a filter F of a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, we can define the relation $\sim_F$ as follows: for all x and y in L, $x \sim_F y$ iff $x \Rightarrow y \in F$ and $y \Rightarrow x \in F$. This relation is a congruence on $\mathscr{L}$, i.e., $\sim_F$ is an equivalence relation (reflexive, symmetric and transitive) and if $x_1 \sim_F x_2$ and $y_1 \sim_F y_2$, then $(x_1 \sqcap y_1) \sim_F (x_2 \sqcap y_2)$, $(x_1 \sqcup y_1) \sim_F (x_2 \sqcup y_2)$, $(x_1 * y_1) \sim_F (x_2 * y_2)$, $(x_1 \Rightarrow y_1) \sim_F (x_2 \Rightarrow y_2)$. This can be proven as in [42].
As a corollary, we can meaningfully consider the quotient algebra $\mathscr{L}_F = (L_F, \sqcap_F, \sqcup_F, *_F, \Rightarrow_F, [0]_F, [1]_F)$, in which L_F is the set of equivalence classes $[x]_F$ (induced by $\sim_F$) in L, and for any x and y in L, $[x]_F = \{y \in L \mid x \sim_F y\}$, $[x]_F \circ_F [y]_F = [x \circ y]_F$ (for $\circ \in \{\sqcap, \sqcup, *, \Rightarrow\}$). This structure is again a residuated lattice, as all defining properties are inherited from $\mathscr{L}$. The congruence relations $\sim_{\{1\}}$ and $\sim_L$ are trivial: for all x and y in L, $x \sim_{\{1\}} y$ iff $x = y$; and $x \sim_L y$ always holds.

Based on the simple fact that $x \in F$ iff $[x]_F = [1]_F$, we immediately obtain the following properties, for every residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ and filter F of $\mathscr{L}$.

- The quotient algebra $\mathscr{L}_{\sim_F}$ is linear iff F is a prime filter of $\mathscr{L}$.
- $[1]_F$ is $\sqcup_F$-irreducible in the quotient algebra $\mathscr{L}_{\sim_F}$ iff F is a prime filter of the second kind of $\mathscr{L}$.
- The quotient algebra $\mathscr{L}_{\sim_F}$ is prelinear iff F is a prime filter of the third kind of $\mathscr{L}$.
- The quotient algebra $\mathscr{L}_{\sim_F}$ is a Boolean algebra iff F is a Boolean filter of $\mathscr{L}$.
- The quotient algebra $\mathscr{L}_{\sim_F}$ is a Boolean algebra with one or two elements iff F is a Boolean filter of the second kind of $\mathscr{L}$.
- The quotient algebra $\mathscr{L}_{\sim_F}$ is a Heyting-algebra iff F is a positive implicative filter of $\mathscr{L}$.
- The quotient algebra $\mathscr{L}_{\sim_F}$ has an involutive negation iff F is an involution filter of $\mathscr{L}$.

2.6 Interval-valued residuated lattices

Recall that we are working towards a variety of algebraic structures suitable as semantics for a logic with intervals as truth values. At this point it might seem a good idea to choose residuated lattices on triangularizations (or, equivalently, residuated lattices on triangular lattices). However, note that in these structures the set of exact intervals is not necessarily closed under the product and implication. We give three examples.

Example 2.57 *The following structures are residuated lattices on triangularizations.*

- *Indeed, if we take on the triangularization from Example 2.26 the product $*$ determined by $[1/2,1/2]*[1/2,1/2]=[0,1/2]$, $[1/2,1/2]*[0,1]=[0,1/2]$ and $[0,1]*[0,1]=[0,0]$ (see 6.1.75(9) in [50]), and its residual implicator $\Rightarrow$, then $[1/2,1/2]\Rightarrow[0,0]=[0,1/2]$. So the set $\{[0,0],[1/2,1/2],[1,1]\}$ of exact intervals is not closed under $*$, nor under $\Rightarrow$.*

- *With the t-norm $\mathscr{T}_{T_P,T_M}$, which is defined by $\mathscr{T}_{T_P,T_M}([x_1,x_2],[y_1,y_2])=[x_1y_1,\min(x_2,y_2)]$, and its residual implicator $\mathscr{I}_{T_P,T_M}$, the structure $(Int([0,1],\min,\max),\bigsqcap,\bigsqcup,\mathscr{T}_{T_P,T_M},\mathscr{I}_{T_P,T_M},[0,0],[1,1])$ is a residuated lattice on $\mathscr{L}^I$. But $\mathscr{T}_{T_P,T_M}([0.5,0.5],[0.3,0.3])=[0.15,0.3]$, so the set of exact intervals is not closed under $\mathscr{T}_{T_P,T_M}$ (although it is closed under $\mathscr{I}_{T_P,T_M}$).*

- *For $\mathscr{T}_1$, defined[23] by $\mathscr{T}_1([x_1,x_2],[y_1,y_2])=[\max(0,x_1+y_1-1),\max(0,2x_1+y_2-2,2y_1+x_2-2,x_1+y_1-1)]$, it is the other way around: the set of exact intervals is closed under this t-norm, but not under its residual implicator $\mathscr{I}_1$, which is given by $\mathscr{I}_1([x_1,x_2],[y_1,y_2])=[\min(1,1+y_1-x_1,1+(y_2-x_2)/2),\min(1,2+y_2-2x_1)]$. Also in this case however, $(Int([0,1],\min,\max),\bigsqcap,\bigsqcup,\mathscr{T}_1,\mathscr{I}_1,[0,0],[1,1])$ is a residuated lattice.*

It seems 'intuitive' that the truth values of the propositions $p\&q$ and $p\to q$ are exact if the truth values of p and q are exact. This means that in the semantics, the set of exact intervals should be closed under $*$ and $\Rightarrow$ (which model the logical connectives & (strong conjunction) and $\to$ (implication), respectively). Therefore residuated lattices on triangularizations are too general to serve as the desired semantics.

We would like that the subset of exact intervals in our interval-valued structures is closed not only under the infimum and supremum, but also under the product and implication of the residuated lattice. This leads us to the definition of interval-valued residuated lattices.

Definition 2.58

- *An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathscr{L}),\bigsqcap,\bigsqcup,*,\Rightarrow,[0,0],[1,1])$ on the triangularization $\mathbb{T}(\mathscr{L})$ of a bounded lattice $\mathscr{L}=(L,\sqcap,\sqcup)$, in which the diagonal $i(L)$ is closed under $*$ and $\Rightarrow$, i.e., $[x_1,x_1]*[y_1,y_1]\in i(L)$ (with i the injection defined in Section 2.2) and $[x_1,x_1]\Rightarrow[y_1,y_1]\in i(L)$ for x_1,y_1 in L.*

- *When we add $[0,1]$ as a constant, and pr_v and pr_h (defined by $\mathrm{pr}_v([x_1,x_2])=[x_1,x_1]$ and $\mathrm{pr}_h([x_1,x_2])=[x_2,x_2]$, for all $[x_1,x_2]$ in $Int(\mathscr{L})$) as unary operators, the structure $(Int(\mathscr{L}),\bigsqcap,\bigsqcup,*,\Rightarrow,\mathrm{pr}_v,\mathrm{pr}_h,[0,0],[0,1],[1,1])$ is called an extended IVRL.*

An IVRL in which $\mathscr{L}=([0,1],\min,\max)$ will be called a standard IVRL. *An extended IVRL in which $\mathscr{L}=([0,1],\min,\max)$ will be called a* standard extended IVRL.

We already saw that an extended triangularization is always a triangular lattice. Therefore, if $(Int(\mathscr{L}),\bigsqcap,\bigsqcup,*,\Rightarrow,\mathrm{pr}_v,\mathrm{pr}_h,[0,0],[0,1],[1,1])$ is an extended IVRL, then $(Int(\mathscr{L}),\bigsqcap,\bigsqcup,\mathrm{pr}_v,\mathrm{pr}_h,[0,0],[0,1],[1,1])$ is a triangular lattice. Now we show that the extra operations $*$ and $\Rightarrow$ satisfy the following two properties, for all x and y in $Int(\mathscr{L})$:

- $\mathrm{pr}_v(x)*\mathrm{pr}_v(y)\leqslant\mathrm{pr}_v(\mathrm{pr}_v(x)*\mathrm{pr}_v(y))$,

- $\mathrm{pr}_v(x)\Rightarrow\mathrm{pr}_v(y)\leqslant\mathrm{pr}_v(\mathrm{pr}_v(x)\Rightarrow\mathrm{pr}_v(y))$.

[23] This example is from [19] (Example 8.1), translated from intuitionistic fuzzy set theory to interval-valued fuzzy set theory.

Indeed, the first property is equivalent to $\mathrm{pr}_\nu(x) * \mathrm{pr}_\nu(y) = \mathrm{pr}_\nu(\mathrm{pr}_\nu(x) * \mathrm{pr}_\nu(y))$, which means that for any a and b in $i(L)$, $a * b = \mathrm{pr}_\nu(a * b)$, in other words $a * b \in i(L)$. So it tells us exactly that the diagonal $i(L)$ is closed under $*$. And similarly, the second property means the diagonal $i(L)$ is closed under $\Rightarrow$ too.
These two properties suggest a way to describe these IVRLs, which seem suitable as semantics for interval-valued fuzzy logic, using only identities. This leads us to the next chapter, where we will introduce this variety (called triangle algebras) and study its properties in detail.

CHAPTER 3

Triangle algebras

In Chapter 2 we introduced interval-valued residuated lattices (IVRLs) and gave some of their properties. In this chapter, we take some of these properties, use them as defining properties for a new variety of structures called triangle algebras (Section 3.1) and compare these triangle algebras with similar structures such as modal residuated lattices and rough approximation spaces (Section 3.2). In Section 3.3 we show that these properties are sufficient to describe the interval-valued structure of IVRLs: triangle algebras are isomorphic to extended IVRLs. Then, in Section 3.4, we investigate the product and implication of triangle algebras and show that these operations are determined by their action on the exact elements and by one specific product: $u * u$. This characterization is used in Section 3.5, where we uncover the connections between properties on triangle algebras and properties on their subalgebras of exact elements. In Section 3.6 we study filters of triangle algebras. A specific kind of such filters is used in Section 3.7, where we show that pseudo-prelinear triangle algebras are isomorphic to subdirect products of pseudo-linear triangle algebras. Using this result, we prove two more connections similar to those in Section 3.5 and conclude the chapter with a schematic overview of most of these connections.

3.1 Definition and elementary properties

In the definition of a triangle algebra we want to combine the structure of a residuated lattice and the structure of intervals (equipped with the order in Definition 2.6), plus the desired property that the subset of exact intervals is closed under all defined operations. This leads us to the following definition.

Definition 3.1 *A triangle algebra is an algebra* $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *of type* $(2, 2, 2, 2, 1, 1, 0, 0, 0)$ *such that* $(A, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ *is a triangular lattice,* $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *is a residuated lattice, and satisfying for all* x *and* y *in* A*,*

$$\begin{array}{ll} (T.7\dagger) & \nu x * \nu y \leqslant \nu(\nu x * \nu y), \\ (T.9) & \nu x \Rightarrow \nu y \leqslant \nu(\nu x \Rightarrow \nu y). \end{array}$$

In other words, an algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *of type* $(2, 2, 2, 2, 1, 1, 0, 0, 0)$ *such that* $(A, \sqcap, \sqcup,$

$*, \Rightarrow, 0, 1)$ *is a residuated lattice, and satisfying for all x and y in A,*

$(T.1)$	$\nu x \leqslant x,$	$(T.1')$	$x \leqslant \mu x,$
$(T.2)$	$\nu x \leqslant \nu\nu x,$	$(T.2')$	$\mu\mu x \leqslant \mu x,$
$(T.3)$	$\nu(x \sqcap y) = \nu x \sqcap \nu y,$	$(T.3')$	$\mu(x \sqcap y) = \mu x \sqcap \mu y,$
$(T.4)$	$\nu(x \sqcup y) = \nu x \sqcup \nu y,$	$(T.4')$	$\mu(x \sqcup y) = \mu x \sqcup \mu y,$
$(T.5)$	$\nu u = 0,$	$(T.5')$	$\mu u = 1,$
$(T.6)$	$\nu\mu x = \mu x,$	$(T.6')$	$\mu\nu x = \nu x,$
$(T.7\dagger)$	$\nu x * \nu y \leqslant \nu(\nu x * \nu y),$		
$(T.9)$	$\nu x \Rightarrow \nu y \leqslant \nu(\nu x \Rightarrow \nu y),$		
$(T.10)$	$x = \nu x \sqcup (\mu x \sqcap u),$	$(T.10')$	$x = \mu x \sqcap (\nu x \sqcup u).$

A triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u_A, 1_A)$ is called a standard triangle algebra iff $(A, \sqcap, \sqcup) = \mathbb{T}([0,1], \min, \max)$.

In a standard triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u_A, 1_A)$, $0_A = [0,0]$, $1_A = [1,1]$, $u = [0,1]$, $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$ for all $[x_1, x_2]$ in $Int([0,1], \min, \max)$. This is a consequence of Propositions 2.27 and 2.29.
Because a triangle algebra is an expansion of both a triangular lattice and a residuated lattice, the properties of these kinds of structures (e.g. Propositions 2.21 and 2.33) remain valid in triangle algebras, and we will also keep the same notations. For example, if $\mathscr{A}$ is a triangle algebra we will denote its set of exact elements by $E(\mathscr{A})$. Because this set is closed under all operations, $(E(\mathscr{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ (where the operations are restricted to $E(\mathscr{A})$) is a residuated lattice. We will denote this residuated lattice by $\mathscr{E}(\mathscr{A})$.
As mentioned above and at the end of Chapter 2 (p. 37), (T.7†) and (T.9) (which correspond to the two properties of IVRLs on p. 37) make sure that the set $E(\mathscr{A})$ of exact elements is closed under $*$ and $\Rightarrow$.
In our papers [75, 76, 77, 78, 79, 80], we used a different axiom instead of (T.7†), namely

$$(T.7) \quad \nu(x \Rightarrow y) \leqslant \nu x \Rightarrow \nu y,$$

which is known as the distribution axiom in modal logic [36]. This axiom is equivalent with (T.7†). Indeed, if (T.7) holds, then (using (T.1), (T.2), Proposition 2.33(4) and the monotonicity of ν) $\nu x = \nu\nu x \leqslant \nu(\nu y \Rightarrow (\nu x * \nu y)) \leqslant \nu\nu y \Rightarrow \nu(\nu x * \nu y) = \nu y \Rightarrow \nu(\nu x * \nu y)$, and $\nu x \leqslant \nu y \Rightarrow \nu(\nu x * \nu y)$ is equivalent with (T.7†) because of the residuation principle. Conversely, if (T.7†) holds, then (using (T.1), the monotonicity of ν and $*$ and Proposition 2.33(5)) $\nu x * \nu(x \Rightarrow y) \leqslant \nu(\nu x * \nu(x \Rightarrow y)) \leqslant \nu(x * (x \Rightarrow y)) \leqslant \nu y$, and $\nu x * \nu(x \Rightarrow y) \leqslant \nu y$ is equivalent with (T.7) because of the residuation principle.

Lemma 3.2 *Suppose $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, and ν is a unary operator on L satisfying $\nu 1 = 1$. Then (T.7) is equivalent to the two properties (together)*

(A) $\nu(x \sqcap y) \leqslant \nu x$, *and*

(B) $\nu x * \nu y \leqslant \nu(x * y)$,

for all x and y in L.

Proof. From Proposition 2.33(14) and (T.7), we obtain $1 = \nu 1 = \nu((x \sqcap y) \Rightarrow x) \leqslant \nu(x \sqcap y) \Rightarrow \nu x$, so applying Proposition 2.33(14) again: $\nu(x \sqcap y) \leqslant \nu x$ (which is (A)). From Proposition 2.33(4), we obtain $x = (y \Rightarrow (x * y)) \sqcap x$ and hence $\nu x = \nu((y \Rightarrow (x * y)) \sqcap x)$. Now applying (A) and (T.7): $\nu x \leqslant \nu(y \Rightarrow (x * y)) \leqslant \nu y \Rightarrow \nu(x * y)$; so $\nu x * \nu y \leqslant \nu(x * y)$ follows from the residuation principle.
Conversely, let us assume (A) and (B). We will prove that $\nu x * \nu(x \Rightarrow y) \leqslant \nu y$, which is equivalent to (T.7), due to the residuation principle. Applying (B), Proposition 2.33(5) and (A), we obtain: $\nu x * \nu(x \Rightarrow y) \leqslant \nu(x * (x \Rightarrow y)) = \nu(y \sqcap (x * (x \Rightarrow y))) \leqslant \nu y$. □

Condition (A) in Lemma 3.2 means exactly that ν is an increasing operator. Therefore it is easy to see that this is a weaker property than (T.3), or (T.4).

From (T.7) and Proposition 2.33(14) we can see that ν is an increasing operation. Together with Remark 2.16, this allows us to give an equivalent definition for triangle algebras, with less axioms: a triangle algebra is an algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ of type $(2,2,2,2,1,1,0,0,0)$ such that $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice with smallest element 0 and greatest element 1, and such that for all x and y in A

		$(T.1')$	$x \leqslant \mu x$,
$(T.2)$	$\nu x \leqslant \nu\nu x$,		
$(T.3w)$	$\nu x \sqcap \nu y \leqslant \nu(x \sqcap y)$,	$(T.3')$	$\mu(x \sqcap y) = \mu x \sqcap \mu y$,
$(T.4w)$	$\nu(x \sqcup y) \leqslant \nu x \sqcup \nu y$,	$(T.4')$	$\mu(x \sqcup y) = \mu x \sqcup \mu y$,
$(T.5)$	$\nu u = 0$,	$(T.5')$	$\mu u = 1$,
$(T.6w)$	$\mu x \leqslant \nu\mu x$,	$(T.6'w)$	$\mu\nu x \leqslant \nu x$,
$(T.7)$	$\nu(x \Rightarrow y) \leqslant \nu x \Rightarrow \nu y$,		
$(T.9)$	$\nu x \Rightarrow \nu y \leqslant \nu(\nu x \Rightarrow \nu y)$,		
$(T.10)$	$x = \nu x \sqcup (\mu x \sqcap u)$.		

Indeed, because ν is increasing, we immediately see that the reverse inequalities in (T.3w) and (T.4w) are always satisfied, hence (T.3) and (T.4) hold. From (T.10) then follows (T.1), such that also (T.6) holds because of (T.6w). Similarly from (T.1') and (T.6'w), (T.6') follows. Finally (T.2') and (T.10') also follow, similarly as for triangular lattices.

The reader may have noticed that we did not introduce an axiom (T.8). The reason is that in our papers [77, 78, 79, 80], we did not use the axiom (T.10) (nor (T.10')). Instead we used $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leqslant x \Leftrightarrow y$, which we called (T.8). Although (T.8) is valid in every triangle algebra (as we will prove later), we consider (T.10) a more natural choice for an axiom, because it involves only notions of triangular lattices, is easier to verify and can be applied more widely (recall that we have used (T.10) to show that (T.1) and (T.2') are actually superfluous in the definition of triangular lattice). Now we will prove that a number of properties ((T.8), amongst others) are valid in triangle algebras, without using (T.7†), (T.7) or (T.9). This means that these properties hold also for structures in which the set of exact elements is not closed under $*$ or $\Rightarrow$, like the structures defined in Example 2.57.

Proposition 3.3 *Let $(A, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ be a triangular lattice and $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice. Then the following properties hold for all x and y in A:*

1. $\nu(x * y) \leqslant \nu x * \nu y$,

2. $(\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) \leqslant x \Rightarrow y$,

3. $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leqslant x \Leftrightarrow y$.

Proof.

1. Observe that $\nu x \sqcup u = x \sqcup u$, because (using (T.4), (T.2), (T.4') and (T.5'))

   - $\nu(\nu x \sqcup u) = \nu\nu x \sqcup \nu u = \nu x \sqcup \nu u = \nu(x \sqcup u)$, and
   - $\mu(\nu x \sqcup u) = \mu\nu x \sqcup \mu u = 1 = \mu x \sqcup \mu u = \mu(x \sqcup u)$.

   Hence, by (2.1), $\nu x \sqcup u$ and $x \sqcup u$ must be equal.
   Furthermore $\nu(x * u) = 0$, because $x * u \leqslant u$ and $\nu u = 0$.
   Therefore, by (T.4) and Proposition 2.33(16), $\nu(x * y) = \nu(x * y) \sqcup \nu(x * u) = \nu((x * y) \sqcup (x * u)) = \nu(x * (y \sqcup u))$. By a symmetric argument we find

   $$\nu(x * y) = \nu((x \sqcup u) * (y \sqcup u)).$$

   Using this identity (twice) and $x \sqcup u = \nu x \sqcup u$, we can now conclude that $\nu(x * y) = \nu((x \sqcup u) * (y \sqcup u)) = \nu((\nu x \sqcup u) * (\nu y \sqcup u)) = \nu(\nu x * \nu y) \leqslant \nu x * \nu y$.

2. Because of the residuation principle, the property we want to prove is equivalent with $(\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) * x \leqslant y$. By Proposition 2.23 this is equivalent with $\nu((\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) * x) \leqslant \nu y$ and $\mu((\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) * x) \leqslant \mu y$.

   - To prove the first statement, we use Proposition 2.33(1 and 5), the monotonicity of ν and $*$, (T.1) and the first part of this proposition: $\nu((\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) * x) \leqslant \nu((\nu x \Rightarrow \nu y) * x) \leqslant \nu(\nu x \Rightarrow \nu y) * \nu x \leqslant (\nu x \Rightarrow \nu y) * \nu x \leqslant \nu y$.
   - To prove the second statement, we use Proposition 2.33(1 and 5), the monotonicity of μ and $*$, and (T.1'): $\mu((\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) * x) \leqslant \mu((\mu x \Rightarrow \mu y) * x) \leqslant \mu((\mu x \Rightarrow \mu y) * \mu x) \leqslant \mu\mu y \leqslant \mu y$.

3. The third statement easily follows from the second: $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leqslant (\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) \leqslant x \Rightarrow y$ and similarly $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leqslant y \Rightarrow x$. So $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leqslant (x \Rightarrow y) \sqcap (y \Rightarrow x) = x \Leftrightarrow y$.

□

Combining Lemma 3.2 and Proposition 3.3 we obtain the following corollary.

Corollary 3.4 *Let $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then, for all x and y in A,*

$$\nu(x * y) = \nu x * \nu y \tag{3.1}$$

A similar identity for μ is in general not true, but we can prove the following proposition.

Proposition 3.5 *Let $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then, for all x and y in A,*

$$\mu(x * y) \leqslant \mu x * \mu y. \tag{3.2}$$

Moreover, the following properties are equivalent:

1. *$\mu(x * y) = \mu x * \mu y$ for every x and y in A, and*

2. *$\mu(x*y)=\mu(x*z)$ if $\mu y=\mu z$, for every x, y and z in A.*

Proof. The inequality $\mu(x*y)\leqslant \mu x*\mu y$ always holds, because $\mu x*\mu y$ is in $E(\mathscr{A})$ and therefore $\mu(x*y)\leqslant \mu(\mu x*\mu y)=\mu x*\mu y$.
(1) implies (2): if $\mu y=\mu z$, $\mu(x*y)=\mu x*\mu y=\mu x*\mu z=\mu(x*z)$.
(2) implies (1): by (T.1') and (T.2') $\mu\mu y=\mu y$. Using (2), we find $\mu(x*\mu y)=\mu(x*y)$. Similarly, $\mu(x*\mu y)=\mu(\mu x*\mu y)$. Therefore $\mu x*\mu y\leqslant \mu(\mu x*\mu y)=\mu(x*y)$.

□

Example 3.14 gives triangle algebras not satisfying $\mu(x*y)=\mu x*\mu y$. An easy way to see this is by taking $t<1$ and $x=y=[0,1]$ in (2.2).

After Definition 2.15 we already noted that the operators ν and μ satisfy a lot of properties that are well-known in modal logics. In these modal logics, ν and μ are often connected by one of the following formulae: $\mu x=\neg\nu\neg x$, $\neg\mu x=\nu\neg x$, $\nu x=\neg\mu\neg x$ and/or $\neg\nu x=\mu\neg x$. In the next proposition we investigate under which conditions these properties hold in structures that expand both triangular and residuated lattices. First we prove a lemma.

Lemma 3.6 *Let $\mathscr{A}=(A,\sqcap,\sqcup,\nu,\mu,0,u,1)$ be a triangular lattice and $(A,\sqcap,\sqcup,*,\Rightarrow,0,1)$ be a residuated lattice. If the subset $E(\mathscr{A})$ of exact elements of the triangular lattice is closed under $\neg$, then $\neg u\leqslant u$.*

Proof. Because of (T.1), $\nu\neg u\leqslant\neg u$. Using Proposition 2.33(6 and 13), we find $u\leqslant\neg\neg u\leqslant\neg\nu\neg u$. But $\nu\neg u$ is an exact element, therefore also $\neg\nu\neg u$ is in $E(\mathscr{A})$. So $1=\mu u\leqslant\mu\neg\nu\neg u=\neg\nu\neg u$, which implies $\nu\neg u\leqslant\neg\neg\nu\neg u\leqslant\neg 1=0$. This is equivalent to $\neg u\leqslant u$ because of Proposition 2.23.

□

Proposition 3.7 *Let $\mathscr{A}=(A,\sqcap,\sqcup,\nu,\mu,0,u,1)$ be a triangular lattice and $(A,\sqcap,\sqcup,*,\Rightarrow,0,1)$ be a residuated lattice. Then the following equivalences and implications hold.*

1. *For all x in A, $\neg\mu x=\nu\neg x$ iff $E(\mathscr{A})$ is closed under $\neg$.*
2. *For all x in A, $\mu x=\neg\nu\neg x$ iff $E(\mathscr{A})$ is closed under $\neg$ and $\neg$ is an involution on $E(\mathscr{A})$.*
3. *For all x in A, $\neg\nu x=\mu\neg x$ iff $E(\mathscr{A})$ is closed under $\neg$ and $u*u=0$.*
4. *For all x in A, $\nu x=\neg\mu\neg x$ iff $E(\mathscr{A})$ is closed under $\neg$ and $\neg$ is an involution (on A).*

Proof.

1. If $\neg\mu x=\nu\neg x$ for all x in A, then for all y in $E(\mathscr{A})$, $\neg y=\neg\mu y=\nu\neg y\in E(\mathscr{A})$.
   Now we prove the converse. Take any x in A. On the one hand, $\neg\mu x=\nu\neg\mu x\leqslant\nu\neg x$, because $\neg\mu x\in E(\mathscr{A})$ and (T.1'). On the other hand, $\nu\neg x\leqslant\neg x$, which implies $\neg\neg x\leqslant\neg\nu\neg x$. Using Proposition 2.33(6), we find $x\leqslant\neg\nu\neg x$. This leads to $\mu x\leqslant\mu\neg\nu\neg x=\neg\nu\neg x$, because $\neg\nu\neg x\in E(A)$. We conclude $\nu\neg x\leqslant\neg\neg\nu\neg x\leqslant\neg\mu x$.
2. If $\mu x=\neg\nu\neg x$ for all x in A, then by Proposition 2.33(7), for all y in $E(\mathscr{A})$, $\neg\neg y=\neg\neg\mu y=\neg\neg\neg\nu\neg y=\neg\nu\neg y=\mu y=y$. Applying this involutivity to $\nu\neg y$ (which is in $E(\mathscr{A})$), we find for all y in $E(\mathscr{A})$: $\neg y=\neg\mu y=\neg\neg\nu\neg y=\nu\neg y\in E(\mathscr{A})$.
   Conversely, if $E(\mathscr{A})$ is closed under $\neg$, then $\neg\mu x=\nu\neg x$ for all x in A (because of the first part). Now using that $\neg$ is an involution on $E(\mathscr{A})$, we find $\mu x=\neg\neg\mu x=\neg\nu\neg x$.

3. If for all x in A $\neg\nu x = \mu\neg x$, then in particular $\mu\neg u = \neg\nu u = \neg 0 = 1$. Because of Proposition 2.23, this is equivalent with $u \leqslant \neg u$ and also (because of the residuation principle) with $u * u = 0$. Furthermore, we have for all y in $E(\mathscr{A})$, $\neg y = \neg\nu y = \mu\neg y \in E(\mathscr{A})$.
   Conversely, suppose $E(\mathscr{A})$ is closed under $\neg$ and $u * u = 0$. Then we have $u \leqslant \neg u$ and by Lemma 3.6 $\neg u = u$. Thus for all $x \leqslant u$, $u = \neg u \leqslant \neg x$ and therefore $\mu\neg x = 1 = \neg\nu x$. Also for all y in $E(\mathscr{A})$, $\mu\neg y = \mu\neg\nu y = \neg\nu y$. This allows us to conclude for all x in A that

$$\begin{aligned}
\mu\neg x &= \mu\neg(\nu x \sqcup (\mu x \sqcap u)) && \text{(T.10)}\\
&= \mu(\neg\nu x \sqcap \neg(\mu x \sqcap u)) && \text{Proposition 2.33(10)}\\
&= \mu\neg\nu x \sqcap \mu\neg(\mu x \sqcap u) && \text{(T.4')}\\
&= \neg\nu\nu x \sqcap \neg\nu(\mu x \sqcap u) && \nu x \in E(\mathscr{A}) \text{ and } \mu x \sqcap u \leqslant u\\
&= \neg(\nu\nu x \sqcup \nu(\mu x \sqcap u)) && \text{(T.10)}\\
&= \neg\nu(\nu x \sqcup (\mu x \sqcap u)) && \text{(T.4)}\\
&= \neg\nu x. && \text{(T.10)}
\end{aligned}$$

4. If $\nu x = \neg\mu\neg x$ for all x in A, then $\nu\neg\neg x = \neg\mu\neg\neg\neg x = \neg\mu\neg x = \nu x$. Moreover, for all y in $E(\mathscr{A})$, $\neg\neg y = \neg\neg\nu y = \neg\neg\neg\mu\neg y = \neg\mu\neg y = \nu y = y$ by Proposition 2.33(7). Furthermore for all y in $E(\mathscr{A})$, $\neg y = \neg\nu y = \neg\neg\mu\neg y = \mu\neg y \in E(\mathscr{A})$, which means $E(\mathscr{A})$ is closed under $\neg$. So because of the second part, for all x in A, $\mu x = \neg\nu\neg x$. Therefore $\mu\neg\neg x = \neg\nu\neg\neg\neg x = \neg\nu\neg x = \mu x$. Now we can use (2.1) to conclude from $\nu\neg\neg x = \nu x$ and $\mu\neg\neg x = \mu x$ that $\neg\neg x = x$.
   Conversely, if $E(\mathscr{A})$ is closed under $\neg$ and $\neg$ is an involution (on A), then $\neg\mu x = \nu\neg x$ (because of the first part) and we find $\neg\mu\neg x = \nu\neg\neg x = \nu x$.

□

Remark that $u * u = 0$ is equivalent with $u \leqslant \neg u$. Therefore, using Lemma 3.6, "$E(\mathscr{A})$ is closed under $\neg$ and $u * u = 0$" is equivalent with "$E(\mathscr{A})$ is closed under $\neg$ and $\neg u = u$".
Also note that the condition "$E(\mathscr{A})$ is closed under $\neg$" is always satisfied in triangle algebras, because $E(\mathscr{A})$ is closed under $\Rightarrow$ and $0 \in E(\mathscr{A})$.

Before we further investigate the properties of triangle algebras, we compare them with some other algebraic structures from the literature.

3.2 Connections with other algebraic structures

In this section, we will examine the relationship between triangle algebras and previously introduced structures:

- Belohlávek and Vychodil [6] defined a so-called "truth stresser" ν for a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ as a unary operator on L that satisfies (T.1), (T.7) and $\nu 1 = 1$.

- Ono [63] defined a modal residuated lattice as an algebra $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$, in which $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and ν a unary operator on L that satisfies (T.1), (T.2), $\nu 1 = 1$, and, for all x and y in L, $\nu(x \sqcap y) \leqslant \nu x$ and $\nu x * \nu y \leqslant \nu(x * y)$. From Lemma 3.2, it follows that the latter two properties are in this case equivalent to (T.7). Hence, in a modal residuated lattice, ν is a truth stresser additionally satisfying (T.2).

- A Hájek [42] truth stresser for a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a unary operator ν on L that satisfies (T.1), (T.2), (T.4)[1], $\nu 1 = 1$, (T.7) and $\nu x \sqcup \neg \nu x = 1$ (pseudo law of excluded middle, pseudo-LEM) for every x and y in L. Hence, $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$ is a modal residuated lattice in which (T.4) and pseudo-LEM are satisfied.

  In a triangle algebra satisfying pseudo-LEM, the residuated lattice consisting of its exact elements is a Boolean algebra (see Proposition 3.28). Triangle algebras do not maintain pseudo-LEM as, in many cases, it would imply that $\nu x = 0$ if $x \neq 1$. For example, if 1 is $\sqcup$-irreducible, then because of Propostition 2.33(14), pseudo-LEM is equivalent to "$\nu x = 0$ if $x \neq 1$". Only triangle algebras with 3 (or 1) elements can satisfy this property.

- Ciabattoni, Metcalfe and Montagna [12] defined an interior operator on a residuated lattice as a unary operator that satisfies (T.1), (T.2), (T.3), $\nu 1 = 1$ and, for all x and y in L, $\nu(\nu x * \nu y) = \nu x * \nu y$ and $\nu(x \sqcup y) = \nu(\nu x \sqcup \nu y)$. The latter property is, in this case, weaker than (T.4), because we can prove it is fulfilled whenever (T.4) is: $\nu(x \sqcup y) = \nu x \sqcup \nu y = \nu\nu x \sqcup \nu\nu y = \nu(\nu x \sqcup \nu y)$. Also in this case, $\nu(\nu x * \nu y) = \nu x * \nu y$ is equivalent to (T.7). To prove this, recall from Lemma 3.2 that (T.7) is in this case (because of (T.3)) equivalent to $\nu x * \nu y \leqslant \nu(x * y)$. Suppose this property is valid. Then we find, using (T.1) and (T.2): $\nu(\nu x * \nu y) \leqslant \nu x * \nu y = \nu\nu x * \nu\nu y \leqslant \nu(\nu x * \nu y)$. Conversely, we derive (T.7) from $\nu(\nu x * \nu y) = \nu x * \nu y$, using (T.1) and (T.3): $\nu x * \nu y = \nu(\nu x * \nu y) = \nu((\nu x * \nu y) \sqcap (x * y)) = \nu(\nu x * \nu y) \sqcap \nu(x * y) = (\nu x * \nu y) \sqcap \nu(x * y)$, and hence $\nu x * \nu y \leqslant \nu(x * y)$, which was equivalent to (T.7).
  Hence, a residuated lattice with interior operator is a modal residuated lattice additionally satisfying (T.3) and $\nu(x \sqcup y) = \nu(\nu x \sqcup \nu y)$.

- A modal operator according to Rachunek and Salounová [64] on a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a unary operator μ that satisfies (T.1'), (T.2') and $\mu(x * y) = \mu x * \mu y$ for all x and y in L. This last property is in general not true in triangle algebras (see Proposition 3.5).

We adopted the notations ν and μ from Cattaneo and Ciucci [9], who defined these operators on so-called weak Brouwer de Morgan lattices (wBD lattices). A wBD lattice $(L, \sqcap, \sqcup, ', ^{\sim}, 0, 1)$ is a bounded distributive lattice $(L, \sqcap, \sqcup)$ equipped with two complementations:

- a de Morgan complementation $'$, which is defined as an involutive unary operator on L that satisfies $(x \sqcup y)' = x' \sqcap y'$, for all x and y in L, and

- a weak Brouwer complementation $\sim$, which is defined as a unary operator satisfying $x \leqslant x^{\sim\sim}$ and $(x \sqcup y)^{\sim} = x^{\sim} \sqcap y^{\sim}$ for all x and y in L,

for which $x^{\sim\prime} = x^{\sim\sim}$ (interconnection rule).
They defined νx as $x'^{\sim}$ and μx as $x^{\sim\prime}$.

Proposition 3.8 *In a wBD lattice $(L, \sqcap, \sqcup, ', ^{\sim}, 0, 1)$, (T.1), (T.1'), (T.2), (T.2'), (T.3), (T.4'), $\nu 1 = 1$, $\mu 0 = 0$, (T.6) and (T.6') are always fulfilled, as well as $\mu x = (\nu x')'$ and the de Morgan laws for $'$.*

Proof. The validity of the properties (T.1), (T.1'), (T.2), (T.2'), (T.6) and (T.6') is a result of Cattaneo and Ciucci [9]. Since $(x \sqcup y)' = x' \sqcap y'$ holds, also $(x' \sqcup y')'' = (x'' \sqcap y'')'$. Because $'$ is

[1] Actually, Hájek imposed the condition $\nu(x \sqcup y) \leqslant \nu x \sqcup \nu y$, which is in this case equivalent to (T.4), because ν is an increasing operator (due to $\nu 1 = 1$ and (T.7)).

involutive, this means $x' \sqcup y' = (x \sqcap y)'$. So the de Morgan laws are valid. We will use them to prove the remaining properties:

- $\nu(x \sqcap y) = (x \sqcap y)'^{\sim} = (x' \sqcup y')^{\sim} = x'^{\sim} \sqcap y'^{\sim} = \nu x \sqcap \nu y$, so (T.3) holds.
- $\mu(x \sqcup y) = (x \sqcup y)^{\sim\prime} = (x^{\sim} \sqcap y^{\sim})' = x^{\sim\prime} \sqcup y^{\sim\prime} = \mu x \sqcup \mu y$, so (T.4') holds.
- The complementations $'$ and $^{\sim}$ are decreasing operators. We show this for $^{\sim}$: if $x \leqslant y$, then $y^{\sim} = (x \sqcup y)^{\sim} = x^{\sim} \sqcap y^{\sim}$, so $y^{\sim} \leqslant x^{\sim}$. Because $0 \leqslant 1^{\sim}$, we find $1 \leqslant 1^{\sim\sim} \leqslant 0^{\sim}$. Therefore $0^{\sim} = 1$. Similarly $0' = 1$ and, by involutivity $1' = 0'' = 0$. So we can now prove that $\nu 1 = 1'^{\sim} = 0^{\sim} = 1$ and $\mu 0 = 0^{\sim\prime} = 1' = 0$.
- $(\nu x')' = (x''^{\sim})' = (x^{\sim})' = \mu x$.

□

Note that (T.3') and (T.4) are not always satisfied, because $(x \sqcap y)^{\sim}$ is not necessarily equal to $x^{\sim} \sqcup y^{\sim}$. Indeed, consider the distributive bounded lattice L_2 in Figure 2.2 on p. 10. If $'$ and $^{\sim}$ are defined by $0' = 0^{\sim} = 1$, $a' = b$, $b' = a$ and $1' = 1^{\sim} = a^{\sim} = b^{\sim} = 0$, then $(L, \sqcap, \sqcup, ', ^{\sim}, 0, 1)$ is a wBD lattice in which (T.3') and (T.4) do not hold because $\mu(a \sqcap b) = 0^{\sim\prime} = 0 \neq 1 \sqcap 1 = a^{\sim\prime} \sqcap b^{\sim\prime} = \mu a \sqcap \mu b$ and $\nu(a \sqcup b) = 1'^{\sim} = 1 \neq 0 \sqcup 0 = a'^{\sim} \sqcup b'^{\sim} = \nu a \sqcup \nu b$.
Some triangle algebras can be seen as wBD lattices:

Proposition 3.9 If $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a distributive triangle algebra, if $'$ is a de Morgan complementation on $(A, \sqcap, \sqcup, 0, 1)$ such that $\mu x = (\nu x')'$ and if we define $^{\sim}$ by $x^{\sim} = (\mu x)'$, then $(A, \sqcap, \sqcup, ', ^{\sim}, 0, 1)$ is a wBD lattice.

Proof. We have to prove that $^{\sim}$ is a weak Brouwer complementation and that $x^{\sim\prime} = x^{\sim\sim}$, for all x in A.
First note that, because $\mu x = (\nu x')'$ for every x in A and $'$ is involutive, $(\mu x')' = (\nu x'')'' = \nu x$.
Furthermore, using (T.6) we find $\mu x = \nu \mu x = (\mu((\mu x)'))' = ((\mu x)')^{\sim} = x^{\sim\sim}$.

- Using (T.1'), we obtain $x \leqslant \mu x = x^{\sim\sim}$.
- Applying (T.4'), we find $(x \sqcup y)^{\sim} = (\mu(x \sqcup y))' = (\mu x \sqcup \mu y)' = (\mu x)' \sqcap (\mu y)' = x^{\sim} \sqcap y^{\sim}$.
- Finally $x^{\sim\prime} = (\mu x)'' = \mu x = x^{\sim\sim}$.

□

Finally, it can be seen that a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ induces a rough approximation space (in the sense of Cattaneo [8]).

Definition 3.10 *A rough approximation space is a structure $\mathscr{R} = (A, B, \nu, \mu)$ in which*

- *A is the set of approximable elements,*
- *B is the set of exact or 'definable' elements, a subset of A,*
- *$\nu\colon A \to B$ is the inner approximation map, satisfying*
  $(\forall x \in B)(\forall y \in A)(x \leqslant y$ iff $x \leqslant \nu y)$,

- *μ: $A \to B$ is the outer approximation map, satisfying*
  $(\forall x \in A)(\forall y \in B)(x \leqslant y$ *iff* $\mu x \leqslant y)$,

and in which for any element x in A, its rough approximation is defined by $(\nu x, \mu x)$.

Proposition 3.11 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then $(A, E(\mathscr{A}), \nu, \mu)$ is a rough approximation space. Moreover, no two different elements have the same rough approximation in this rough approximation space.*

Proof. The set $E(\mathscr{A})$ is indeed a subset of A and for all x in A, $\nu x \in E(\mathscr{A})$ and $\mu x \in E(\mathscr{A})$.
Suppose $x \in E(\mathscr{A})$ and $y \in A$. If $x \leqslant y$, then $x = \nu x \leqslant \nu y$ (since ν is increasing). Conversely, if $x \leqslant \nu y$, then $x \leqslant \nu y \leqslant y$ (using (T.1)).
Suppose $x \in A$ and $y \in E(\mathscr{A})$. If $x \leqslant y$, then $\mu x \leqslant \mu y = y$ (since μ is increasing). Conversely, if $\mu x \leqslant y$, then $x \leqslant \mu x \leqslant y$ (using (T.1')).
Equivalence (2.1) on p. 13 ensures that no two different elements have the same rough approximation. □

3.3 The connection between triangle algebras and IVRLs

In Section 2.3 we already showed that an extended triangularization of a bounded lattice is a triangular lattice (Proposition 2.14), and that a triangular lattice is isomorphic to an extended triangularization (Proposition 2.21). We also gave an alternative definition for triangular lattices in Proposition 2.24. In this section we extend these connections to triangle algebras and (extended) IVRLs.

Proposition 3.12 *Let $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, *, \Rightarrow, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ be an extended IVRL. Then $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, *, \Rightarrow, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ is a triangle algebra.*

Proof. Because by definition of IVRL $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ is an extended triangularization, Proposition 2.14 tells us $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ is a triangular lattice.
That $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, *, \Rightarrow, [0,0], [1,1])$ is a residuated lattice also follows from the definition of IVRL. So we only need to verify (T.7†) and (T.9). These axioms were already shown to hold in IVRLs in Section 2.6. □

Proposition 3.13 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then $\mathscr{A}$ is isomorphic to an extended IVRL.*

Proof. We use the same mapping $\chi\colon A \longrightarrow Int(\mathscr{E}(\mathscr{A}))$ (defined as $\chi(x) = [\nu x, \mu x]$) as in Proposition 2.21. So we already know that $(A, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ is isomorphic to $(Int(\mathscr{E}(\mathscr{L})), \bigsqcap, \bigsqcup, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$, which is an extended triangularization of a bounded lattice.
Now we define two binary operations on $Int(\mathscr{E}(\mathscr{A}))$, $\odot$ and $\Rightarrow_\odot$, in the following way: $[x_1, x_2] \odot [y_1, y_2] = [\nu((x_1 \sqcup (u \sqcap x_2)) * (y_1 \sqcup (u \sqcap y_2))), \mu((x_1 \sqcup (u \sqcap x_2)) * (y_1 \sqcup (u \sqcap y_2)))]$, and $[x_1, x_2] \Rightarrow_\odot [y_1, y_2] = [\nu((x_1 \sqcup (u \sqcap x_2)) \Rightarrow (y_1 \sqcup (u \sqcap y_2))), \mu((x_1 \sqcup (u \sqcap x_2)) \Rightarrow (y_1 \sqcup (u \sqcap y_2)))]$, for all $[x_1, x_2]$ and $[y_1, y_2]$ in $Int(\mathscr{E}(\mathscr{A}))$.
We find $\chi(x * y) = \chi((\nu x \sqcup (u \sqcap \mu x)) * (\nu y \sqcup (u \sqcap \mu y))) = [\nu((\nu x \sqcup (u \sqcap \mu x)) * (\nu y \sqcup (u \sqcap$

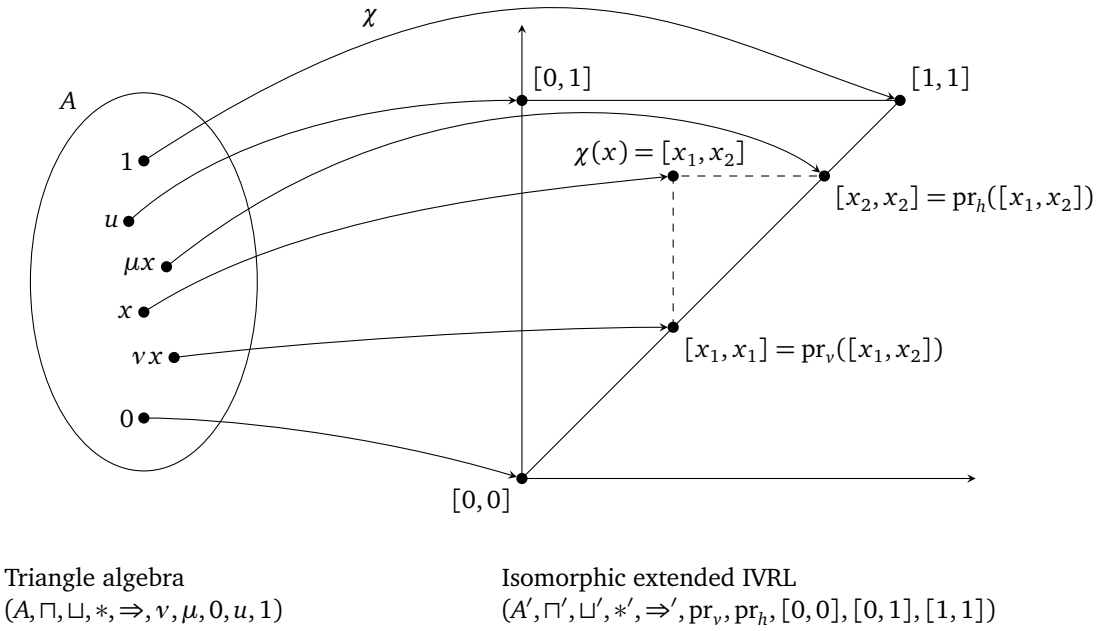

Figure 3.1: The isomorphism χ from a triangle algebra to an extended IVRL.

$\mu y))), \mu((\nu x \sqcup (u \sqcap \mu x)) * (\nu y \sqcup (u \sqcap \mu y)))] = [\nu x, \mu x] \odot [\nu y, \mu y] = \chi(x) \odot \chi(y)$ for any x and y in A, and, completely analogously, that $\chi(x \Rightarrow y) = \chi(x) \Rightarrow_\odot \chi(y)$.
So we can conclude that $(Int(\mathscr{E}(\mathscr{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is isomorphic to $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, and therefore a residuated lattice. We only need to verify that the diagonal of $Int(\mathscr{E}(\mathscr{A}))$ is closed under $\odot$ and $\Rightarrow_\odot$. Indeed, $[x,x] \odot [y,y] = [\nu((x \sqcup (u \sqcap x)) * (y \sqcup (u \sqcap y))), \mu((x \sqcup (u \sqcap x)) * (y \sqcup (u \sqcap y)))] = [\nu(x * y), \mu(x * y)] = [x * y, x * y]$ because x and y belong to $E(\mathscr{A})$. The verification for $\Rightarrow_\odot$ is completely analogous. □

The isomorphism χ of Proposition 3.13 is depicted in Figure 3.1.

Example 3.14 *Consider the t-norms $\mathscr{T}_{T,t}$ defined by (2.2) on p. 19. As the diagonal of $\mathbb{T}(\mathscr{L})$ is closed under $\mathscr{T}_{T,t}$ and $\mathscr{I}_{\mathscr{T}_{T,t}}$, Theorem 3.12 implies that $(Int(\mathscr{L}), \sqcap, \sqcup, \mathscr{T}_{T,\alpha}, \mathscr{I}_{\mathscr{T}_{T,\alpha}}, \mathrm{pr}_\nu, \mathrm{pr}_h, [0,0], [0,1], [1,1])$ is a triangle algebra if T is a residuated t-norm on the bounded lattice $\mathscr{L}$.*

In Section 2.3 (after Proposition 2.23), we characterized triangular lattices as bounded lattices in which a subset D can be found satisfying

- D is closed under $\sqcap$ and $\sqcup$,
- $\{0,1\} \subseteq D \subseteq L$,
- for all l in L, $\max\{x \in D \mid x \leqslant l\}$ and $\min\{x \in D \mid x \geqslant l\}$ exist,
- if $(d_1, d_2) \in D^2$ and $d_1 \leqslant d_2$, then there exists a unique l in L such that $\max\{x \in D \mid x \leqslant l\} = d_1$ and $\min\{x \in D \mid x \geqslant l\} = d_2$.

Now we extend this result to triangle algebras.

Proposition 3.15 *In a triangle algebra* $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$*, we can find a subset D satisfying*

- *D is closed under* $\sqcap$*,* $\sqcup$*,* $*$ *and* $\Rightarrow$*,*
- $\{0, 1\} \subseteq D \subseteq L$*,*
- *for all* l *in* L*,* $\max\{x \in D \mid x \leqslant l\}$ *and* $\min\{x \in D \mid x \geqslant l\}$ *exist,*
- *if* $(d_1, d_2) \in D^2$ *and* $d_1 \leqslant d_2$*, then there exists a unique* l *in* L *such that* $\max\{x \in D \mid x \leqslant l\} = d_1$ *and* $\min\{x \in D \mid x \geqslant l\} = d_2$*.*

Conversely, any residuated lattice in which such a subset D can be found, has the structure of a triangle algebra.

Proof. The first part follows from Proposition 2.24 and Definition 3.1.
Conversely, any residuated lattice $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which such a subset D can be found, determines a triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, in which ν, μ and u are defined as in Proposition 2.24: νl is $\max\{x \in D \mid x \leqslant l\}$ and μl is $\min\{x \in D \mid x \geqslant l\}$ for all l in L, and u is the unique element l in L for which $\nu l = 0$ and $\mu l = 1$. This follows immediately from $(A, \sqcap, \sqcup, \nu, \mu, 0, u, 1)$ being a triangular lattice, $\nu(A) = D$ and D being closed under $*$ and $\Rightarrow$. □

We already saw that residuated t-norms on bounded lattices give rise to residuated lattices, and that these t-norms can be extended to (also residuated) t-norms on the triangularizations of these bounded lattices. So residuated t-norms on bounded lattices also give rise to IVRLs, which in turn lead to triangle algebras. In the next section we show that all triangle algebras can be obtained in this way.

3.4 The relevance of the diagonal and the value $u * u$

In a triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the unary operators ν and μ are increasing. In this section, we will use the abbreviation (M) whenever we refer to the monotonicity of $*$ and $\Rightarrow$ (see Proposition 2.33), and of ν and μ.
Another property valid in triangle algebras is $x \leqslant y$ iff $x \Rightarrow y = 1$ iff $\nu x \leqslant \nu y$ and $\mu x \leqslant \mu y$ (characterization of inequality). This will be abbreviated in this section by (I).
We also already saw that $E(\mathscr{A})$ is closed under $*$ and $\Rightarrow$, i.e., $\nu(x * y) = x * y$ and $\nu(x \Rightarrow y) = x \Rightarrow y$ if $\nu x = x$ and $\nu y = y$. In this section, we will use (C) to refer to this property.

We start by recalling some additional properties of triangle algebras which will be needed later on.

Proposition 3.16 *In a triangle algebra* $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$*, the following identities hold, for every* x *in* A*:*

$$x \sqcap u = \mu x \sqcap u, \tag{3.3}$$

$$x \sqcup u = \nu x \sqcup u, \tag{3.4}$$

Proof.

- On the one hand, we have $\nu(x \sqcap u) = \nu x \sqcap \nu u = 0 = \nu\mu x \sqcap \nu u = \nu(\mu x \sqcap u)$. On the other hand, we have $\mu(x \sqcap u) = \mu x \sqcap \mu u = \mu\mu x \sqcap \mu u = \mu(\mu x \sqcap u)$. So, by (2.1), $x \sqcap u = \mu x \sqcap u$.
- The proof of $x \sqcup u = \nu x \sqcup u$ is analogous to that of $x \sqcap u = \mu x \sqcap u$ and was already proven in Proposition 3.3. □

We will prove, by a series of steps leading up to Theorem 3.20, that the operations $*$ and $\Rightarrow$ in a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ are determined by their action on $E(\mathscr{A})$ and by the value of $u * u$. To this aim, we will derive explicit representations for $\nu(x * y)$, $\mu(x * y)$, $\nu(x \Rightarrow y)$, and $\mu(x \Rightarrow y)$. To obtain these, first we prove a number of properties regarding the interaction of the operations in triangle algebras.

Proposition 3.17 *In a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following inequality and identity hold, for every x and y in A:*

$$\mu(x \Rightarrow y) \leqslant \nu x \Rightarrow \mu y, \tag{3.5}$$

$$\mu(x * \nu y) = \mu x * \nu y. \tag{3.6}$$

Proof.

1. Let x and y be in A. Because νx and μy are in $E(\mathscr{A})$, also $\nu x \Rightarrow \mu y \in E(\mathscr{A})$: $\nu x \Rightarrow \mu y = \mu(\nu x \Rightarrow \mu y)$ (C). Using $\nu x \leqslant x$ (T.1) and $y \leqslant \mu y$ (T.1'), we find $x \Rightarrow y \leqslant \nu x \Rightarrow \mu y$ (M). Thus $\mu(x \Rightarrow y) \leqslant \mu(\nu x \Rightarrow \mu y)$ (M).
2. By (3.2) and (T.6'), we already know $\mu(x * \nu y) \leqslant \mu x * \mu\nu y = \mu x * \nu y$. Furthermore, by Proposition 2.33(4), (M), (3.5) and (T.2), $\mu x \leqslant \mu(\nu y \Rightarrow (x * \nu y)) \leqslant \nu\nu y \Rightarrow \mu(x * \nu y) = \nu y \Rightarrow \mu(x * \nu y)$. Therefore also $\mu x * \nu y \leqslant \mu(x * \nu y)$ by the residuation principle.

□

Remark that (3.6) means in fact that $\mu(x * z) = \mu x * z$ if $z \in E(\mathscr{A})$ (because in this case $z = \nu z$).

The following very useful lemma will be needed in the derivation of further identities in triangle algebras.

Lemma 3.18 *In a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, it holds for all y in A and all z in $E(\mathscr{A})$ that $u * z \leqslant y$ iff $z \leqslant \mu y$.*

Proof. Remark that $\mu(u * z) = \mu u * z = z$ because of (3.6) and (T.5'). Suppose $z \leqslant \mu y$. This means $\mu(u * z) \leqslant \mu y$. As $\nu(u * z) = 0 \leqslant \nu y$ by (M) and (T.5), we conclude that $u * z \leqslant y$ (using (I)). Conversely, if $u * z \leqslant y$, then $z = \mu(u * z) \leqslant \mu y$ by (M). □

Proposition 3.19 *In a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following identities hold, for every x and y in A:*

$$\nu(\nu x \Rightarrow y) = \nu x \Rightarrow \nu y, \tag{3.7}$$

$$\mu(\nu x \Rightarrow y) = \nu x \Rightarrow \mu y, \tag{3.8}$$

$$(x \sqcap u) \Rightarrow (y \sqcap u) = (x \sqcap u) \Rightarrow y = (x \sqcap u) \Rightarrow \mu y, \tag{3.9}$$

$$\nu((x \sqcap u) \Rightarrow y) = \mu x \Rightarrow \mu y, \tag{3.10}$$

$$\mu((x \sqcap u) \Rightarrow y) = \mu x \Rightarrow \mu(u \Rightarrow y), \tag{3.11}$$

$$\mu(u \Rightarrow y) = \mu(u * u) \Rightarrow \mu y. \tag{3.12}$$

Proof.

- On the one hand, $\nu x \Rightarrow \nu y \leqslant \nu x \Rightarrow y$, by (T.1) and (M), so $\nu x \Rightarrow \nu y = \nu(\nu x \Rightarrow \nu y) \leqslant \nu(\nu x \Rightarrow y)$ by (C) and (M). On the other hand, by (T.1), $\nu(\nu x \Rightarrow y) \leqslant \nu x \Rightarrow y$. By the residuation principle, $\nu x * \nu(\nu x \Rightarrow y) \leqslant y$. Moreover, $\nu x * \nu(\nu x \Rightarrow y) = \nu(\nu x * \nu(\nu x \Rightarrow y)) \leqslant \nu y$ by (C) and (M). Applying the residuation principle again, we obtain $\nu(\nu x \Rightarrow y) \leqslant \nu x \Rightarrow \nu y$.

- We will prove (3.8) by showing that $z \leqslant \nu x \Rightarrow \mu y$ iff $z \leqslant \mu(\nu x \Rightarrow y)$ for any $z \in E(\mathscr{A})$. This implies $\nu x \Rightarrow \mu y = \mu(\nu x \Rightarrow y)$, because both elements in this identity are in $E(\mathscr{A})$. We have $z \leqslant \nu x \Rightarrow \mu y$ iff $z * \nu x \leqslant \mu y$ iff (by Lemma 3.18) $u * z * \nu x \leqslant y$ iff $u * z \leqslant \nu x \Rightarrow y$ iff (by Lemma 3.18) $z \leqslant \mu(\nu x \Rightarrow y)$.

- Because $y \leqslant \mu y$ and $\Rightarrow$ is increasing in the second argument, $(x \sqcap u) \Rightarrow y \leqslant (x \sqcap u) \Rightarrow \mu y$. Because $\nu((x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y)) \leqslant \nu(x \sqcap u) \leqslant \nu u = 0 \leqslant \nu y$ and $\mu((x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y)) \leqslant \mu\mu y = \mu y$ by Proposition 2.33(5) and (M), $(x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y) \leqslant y$, by (I). Thus $(x \sqcap u) \Rightarrow \mu y \leqslant (x \sqcap u) \Rightarrow y$. So $(x \sqcap u) \Rightarrow \mu y = (x \sqcap u) \Rightarrow y$.
  Now we apply this with $y \sqcap u$ instead of y, using $\mu y = \mu(y \sqcap u)$. We find $(x \sqcap u) \Rightarrow \mu y = (x \sqcap u) \Rightarrow \mu(y \sqcap u) = (x \sqcap u) \Rightarrow (y \sqcap u)$.

- Because $(x \sqcap u) \Rightarrow \mu y = (x \sqcap u) \Rightarrow y$ by (3.9), we know that $\nu((x \sqcap u) \Rightarrow \mu y) = \nu((x \sqcap u) \Rightarrow y)$. We now prove that $\nu((x \sqcap u) \Rightarrow \mu y) = \mu x \Rightarrow \mu y$. On the one hand, $\mu x \Rightarrow \mu y \leqslant (x \sqcap u) \Rightarrow \mu y$ because $x \sqcap u \leqslant x \leqslant \mu x$ and (M). Thus, by (C) and (M), $\mu x \Rightarrow \mu y = \nu(\mu x \Rightarrow \mu y) \leqslant \nu((x \sqcap u) \Rightarrow \mu y)$. On the other hand, we have $(x \sqcap u) * \nu((x \sqcap u) \Rightarrow \mu y) \leqslant (x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y) \leqslant \mu y$ because of (T.1), (M) and Proposition 2.33(5). Therefore, using (T.3'), (T.5'), (3.6), (M) and (T.2'), $\mu x * \nu((x \sqcap u) \Rightarrow \mu y) = \mu(x \sqcap u) * \nu((x \sqcap u) \Rightarrow \mu y) = \mu((x \sqcap u) * \nu((x \sqcap u) \Rightarrow \mu y)) \leqslant \mu\mu y = \mu y$, which implies $\nu((x \sqcap u) \Rightarrow \mu y) \leqslant \mu x \Rightarrow \mu y$.

- We will prove (3.11) by showing that $z \leqslant \mu((x \sqcap u) \Rightarrow y)$ iff $z \leqslant \mu x \Rightarrow \mu(u \Rightarrow y)$ for any $z \in E(\mathscr{A})$. This implies $\mu((x \sqcap u) \Rightarrow y) = \mu x \Rightarrow \mu(u \Rightarrow y)$, because both elements in this identity are in $E(\mathscr{A})$.
  Using Lemma 3.18, the residuation principle, (I), $\nu(z * (x \sqcap u)) = 0$, (3.6), (T.3') and (T.5'), we find $z \leqslant \mu((x \sqcap u) \Rightarrow y)$ iff $u * z \leqslant (x \sqcap u) \Rightarrow y$ iff $z * (x \sqcap u) \leqslant u \Rightarrow y$ iff $\mu(z * (x \sqcap u)) \leqslant \mu(u \Rightarrow y)$ iff $z * \mu(x \sqcap u) \leqslant \mu(u \Rightarrow y)$ iff $z * \mu x \leqslant \mu(u \Rightarrow y)$ iff $z \leqslant \mu x \Rightarrow \mu(u \Rightarrow y)$.

- We will prove (3.12) by showing that $z \leqslant \mu(u * u) \Rightarrow \mu y$ iff $z \leqslant \mu(u \Rightarrow y)$ for any $z \in E(\mathscr{A})$. This implies $\mu(u * u) \Rightarrow \mu y = \mu(u \Rightarrow y)$, because both sides in this identity are in $E(\mathscr{A})$. Applying the residuation principle, (3.6), (I), $\nu(z * u * u) = 0$ and Lemma 3.18, we find the following equivalences: $z \leqslant \mu(u * u) \Rightarrow \mu y$ iff $z * \mu(u * u) \leqslant \mu y$ iff $\mu(z * u * u) \leqslant \mu y$ iff $z * u * u \leqslant y$ iff $z * u \leqslant u \Rightarrow y$ iff $z \leqslant \mu(u \Rightarrow y)$.

□

Theorem 3.20 *In a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the implication $\Rightarrow$ and the product $*$ are completely determined by their action on $E(\mathscr{A})$ and the value of $u * u$. More specifically:*

- $\nu(x \Rightarrow y) = (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)$,

- $\mu(x \Rightarrow y) = (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y)$,

- $\nu(x * y) = \nu x * \nu y$,
- $\mu(x * y) = (\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))$

and therefore (by (T.10) and (T.10'))

$$\begin{aligned} & x \Rightarrow y \\ &= \Big(\Big(\Big(\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)\Big) \sqcap (\nu x \Rightarrow \mu y)\Big) \sqcap u\Big) \sqcup \Big((\mu x \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \nu y)\Big) \\ &= \Big(((\mu x \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \nu y)) \sqcup u\Big) \sqcap \Big(\Big(\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)\Big) \sqcap (\nu x \Rightarrow \mu y)\Big) \end{aligned}$$

and

$$\begin{aligned} x * y &= \Big(\Big((\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))\Big) \sqcap u\Big) \sqcup (\nu x * \nu y) \\ &= \Big((\nu x * \nu y) \sqcup u\Big) \sqcap \Big((\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))\Big). \end{aligned}$$

Proof.

- By means of (T.10), (3.3) and Proposition 2.33(10), we find

  $$x \Rightarrow y = ((x \sqcap u) \sqcup \nu x) \Rightarrow y = ((x \sqcap u) \Rightarrow y) \sqcap (\nu x \Rightarrow y).$$

  Therefore, by (T.3), (3.10) and (3.7),

  $$\begin{aligned} \nu(x \Rightarrow y) &= \nu(((x \sqcap u) \Rightarrow y) \sqcap (\nu x \Rightarrow y)) \\ &= \nu((x \sqcap u) \Rightarrow y) \sqcap \nu(\nu x \Rightarrow y) \\ &= (\mu x \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \nu y) \end{aligned}$$

  and, by (T.3'), (3.11), (3.12) and (3.8),

  $$\begin{aligned} \mu(x \Rightarrow y) &= \mu(((x \sqcap u) \Rightarrow y) \sqcap (\nu x \Rightarrow y)) \\ &= \mu((x \sqcap u) \Rightarrow y) \sqcap \mu(\nu x \Rightarrow y) \\ &= (\mu x \Rightarrow \mu(u \Rightarrow y)) \sqcap (\nu x \Rightarrow \mu y) \\ &= (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y). \end{aligned}$$

  On the other hand, by (T.10) and (T.10'),

  $$\begin{aligned} x \Rightarrow y &= (\mu(x \Rightarrow y) \sqcap u) \sqcup \nu(x \Rightarrow y) \\ &= (\nu(x \Rightarrow y) \sqcup u) \sqcap \mu(x \Rightarrow y). \end{aligned}$$

  In other words, $x \Rightarrow y$ is completely determined by $\mu(u * u)$ and the action of $\Rightarrow$ on $E(\mathscr{A})$.

- Using (I), the residuation principle, the first part of this proof and the definition of $\sqcap$ and $\sqcup$, we find the following equivalences, for all x, y and z in A:

$$
\begin{array}{ll}
 & \begin{cases} \nu(x*y) \leqslant \nu z \\ \mu(x*y) \leqslant \mu z \end{cases} \\
\text{iff} & x*y \leqslant z \\
\text{iff} & x \leqslant y \Rightarrow z \\
\text{iff} & \begin{cases} \nu x \leqslant \nu(y \Rightarrow z) \\ \mu x \leqslant \mu(y \Rightarrow z) \end{cases} \\
\text{iff} & \begin{cases} \nu x \leqslant (\nu y \Rightarrow \nu z) \sqcap (\mu y \Rightarrow \mu z) \\ \mu x \leqslant (\nu y \Rightarrow \mu z) \sqcap (\mu y \Rightarrow (\mu(u*u) \Rightarrow \mu z)) \end{cases} \\
\text{iff} & \begin{cases} \nu x \leqslant \nu y \Rightarrow \nu z \\ \nu x \leqslant \mu y \Rightarrow \mu z \\ \mu x \leqslant \nu y \Rightarrow \mu z \\ \mu x \leqslant \mu y \Rightarrow (\mu(u*u) \Rightarrow \mu z) \end{cases} \\
\text{iff} & \begin{cases} \nu x * \nu y \leqslant \nu z \\ \nu x * \mu y \leqslant \mu z \\ \mu x * \nu y \leqslant \mu z \\ \mu x * \mu y * \mu(u*u) \leqslant \mu z \end{cases} \\
\text{iff} & \begin{cases} \nu x * \nu y \leqslant \nu z \\ (\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u*u)) \leqslant \mu z. \end{cases}
\end{array}
$$

In this equivalence, we can take $z = x*y$. Then $\nu(x*y) \leqslant \nu z$ and $\mu(x*y) \leqslant \mu z$ are obviously both satisfied. So we can conclude that $(\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u)) \leqslant \mu(x*y)$. On the other hand, we can also take $z = (\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u))$, which is an element of $E(\mathscr{A})$ and therefore $\nu z = z = \mu z$. In this case, one can easily verify that $\nu x * \nu y \leqslant \nu z$ and $(\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u*u)) \leqslant \mu z$ are both satisfied. So we find that $\mu(x*y) \leqslant (\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u*u))$. So $\mu(x*y) = (\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u))$. Together with (3.1) and (T.10), this completes the proof.

□

Note that we can also write $\mu(x \Rightarrow y) = (\mu x \Rightarrow (\mu(u*u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y)$ as $((\mu x*\mu(u*u)) \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \mu y)$ or $((\mu x*\mu(u*u)) \sqcup \nu x) \Rightarrow \mu y$ because of Proposition 2.33(10 and 11). As a particular case of Theorem 3.20, we have $\nu\neg x = \nu(x \Rightarrow 0) = \neg\nu x \sqcap \neg\mu x = \neg\mu x$ (which we already knew from Proposition 3.7) and $\mu\neg x = \mu(x \Rightarrow 0) = \neg\nu x \sqcap \neg(\mu x * \mu(u*u))$, which can also be written as $\neg((\mu x * \mu(u*u)) \sqcup \nu x)$.
Using the correspondence between (extended) IVRLs and triangle algebras (Propositions 3.12 and 3.13), we can translate Theorem 3.20 to

Theorem 3.21 *Let* $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ *be an IVRL and* $\alpha \in L$, $* : L^2 \to L$ *and* $\Rightarrow : L^2 \to L$ *be defined by* $\alpha = \mathrm{pr}_2([0,1] \odot [0,1])$, $x*y = \mathrm{pr}_1([x,x] \odot [y,y])$ *and* $x \Rightarrow y = \mathrm{pr}_1([x,x] \Rightarrow_\odot [y,y])$, *for all* x *and* y *in* L. *Then*

$$[x_1,x_2] \Rightarrow_\odot [y_1,y_2] = [(x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2), (x_1 \Rightarrow y_2) \sqcap (x_2 \Rightarrow (\alpha \Rightarrow y_2))]$$

and

$$[x_1,x_2] \odot [y_1,y_2] = [x_1*y_1, (x_2*y_2*\alpha) \sqcup (x_1*y_2) \sqcup (x_2*y_1)],$$

for all $[x_1,x_2]$ *and* $[y_1,y_2]$ *in* $Int(\mathscr{L})$.

The negation is thus given by $\neg[x_1,x_2] = [x_1,x_2] \Rightarrow_\odot [0,0] = [(x_1 \Rightarrow 0) \sqcap (x_2 \Rightarrow 0), (x_1 \Rightarrow 0) \sqcap (x_2 \Rightarrow (\alpha \Rightarrow 0))] = [\neg x_2, \neg x_1 \sqcap \neg(x_2 * \alpha)]$ (by Proposition 2.33(11)).

Corollary 3.22 *In a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following inequalities and identities hold, for every x and y in A:*

$$\mu x \Rightarrow \nu y \leqslant \nu(x \Rightarrow y), \tag{3.13}$$
$$\nu x \leqslant \mu(u * x), \tag{3.14}$$
$$x \Rightarrow (y \sqcup u) = (\nu x \Rightarrow \nu y) \sqcup u, \tag{3.15}$$
$$\mu(x \Rightarrow (y \sqcap u)) = \mu(x \Rightarrow y) = \mu(x \Rightarrow \mu y). \tag{3.16}$$

Proof.

1. Because of (T.1), (T.1') and (M), $\mu x \Rightarrow \nu y \leqslant (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y) = \nu(x \Rightarrow y)$.

2. Using (T.5'), we find $\nu x \leqslant (\nu u * \mu x) \sqcup \nu x \sqcup (\mu(u * u) * \mu u * \mu x) = (\nu u * \mu x) \sqcup (\mu u * \nu x) \sqcup (\mu(u * u) * \mu u * \mu x) = \mu(u * x)$.

3. On the one hand, we have $\mu(x \Rightarrow (y \sqcup u)) = 1 = \mu((\nu x \Rightarrow \nu y) \sqcup u)$ because of Proposition 2.33(14), (M) and (T.5'). On the other hand, $\nu(x \Rightarrow (y \sqcup u)) = (\nu x \Rightarrow \nu(y \sqcup u)) \sqcap (\mu x \Rightarrow \mu(y \sqcup u)) = (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow 1) = \nu x \Rightarrow \nu y = \nu((\nu x \Rightarrow \nu y) \sqcup u)$, by (T.4), (T.5), (T.5'), (M) and (C). So by (2.1), $x \Rightarrow (y \sqcup u) = (\nu x \Rightarrow \nu y) \sqcup u$.

4. By (T.1') and (T.2'), we have $\mu(x \Rightarrow y) = (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y) = (\mu x \Rightarrow (\mu(u*u) \Rightarrow \mu\mu y)) \sqcap (\nu x \Rightarrow \mu\mu y) = \mu(x \Rightarrow \mu y)$. The proof of the other identity is analogous, using (T.3') and (T.5').

□

For every triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, we can consider the couple $(\mathscr{E}(\mathscr{A}), \mu(u * u))$, in which $\mathscr{E}(\mathscr{A}) = (E(\mathscr{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is an algebraic subreduct of $\mathscr{A}$ on $E(\mathscr{A})$. Remark that $\mathscr{E}(\mathscr{A})$ is a residuated lattice and that $\mu(u * u) \in E(\mathscr{A})$. From Theorem 3.20 it follows that $(\mathscr{E}(\mathscr{A}), \mu(u * u))$ completely determines the triangle algebra $\mathscr{A}$.
Conversely, given a residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ and an element $\alpha \in L$, we can construct a triangle algebra $\mathscr{A}'$ such that $\mathscr{E}(\mathscr{A}') = \mathscr{L}$ and $\mu(u * u) = \alpha$. Indeed, define on $Int(\mathscr{L})$ the unary operators ν and μ and the binary operators $\odot$ and $\Rightarrow_\odot$ as $\nu[x_1,x_2] = [x_1,x_1]$, $\mu[x_1,x_2] = [x_2,x_2]$, $[x_1,x_2] \odot [y_1,y_2] = [x_1 * y_1, (x_1 * y_2) \sqcup (x_2 * y_1) \sqcup (x_2 * y_2 * \alpha)]$ and $[x_1,x_2] \Rightarrow_\odot [y_1,y_2] = [(x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2), (x_1 \Rightarrow y_2) \sqcap (x_2 \Rightarrow (\alpha \Rightarrow y_2))]$. Then it can be verified that $\mathscr{A} = (Int(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra with $E(\mathscr{A}) = i(L)$ (with i the injection defined in Section 2.2) and $\mu([0,1] \odot [0,1]) = [\alpha, \alpha]$.
In this way $\mathscr{E}(\mathscr{A})$ is isomorphic to $\mathscr{L}$. So we already have the following proposition.

Proposition 3.23 *For any residuated lattice $\mathscr{L}$ and $\alpha \in L$, there is a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ such that (up to isomorphism) $\mathscr{E}(\mathscr{A})$ is $\mathscr{L}$ and $\mu(u * u) = \alpha$.*

To construct a triangle algebra $\mathscr{A}'$ in which $\mathscr{E}(\mathscr{A}') = \mathscr{L}$, consider the bijection φ that maps $Int(\mathscr{L})$ onto $A' = L \cup \{[x_1,x_2] | (x_1,x_2) \in L^2 \text{ and } x_1 < x_2\}$ in the following way: $\varphi([x_1,x_2]) = x_1$ if $x_1 = x_2$, $\varphi([x_1,x_2]) = [x_1,x_2]$ else. If we now define, for all a and b in A', unary

operators ν' and μ' by $\nu' a = \varphi(\nu(\varphi^{-1}(a)))$ and $\mu' a = \varphi(\mu(\varphi^{-1}(a)))$, and binary operators $\bigsqcap'$, $\bigsqcup'$, $\odot'$ and $\Rightarrow_{\odot}'$ by $a \circ' b = \varphi(\varphi^{-1}(a) \circ \varphi^{-1}(b))$ for $\circ \in \{\bigsqcap, \bigsqcup, \odot, \Rightarrow_{\odot}\}$, then $\mathscr{A}' = (A', \bigsqcap', \bigsqcup', \odot', \Rightarrow_{\odot}', \nu', \mu', 0, [0,1], 1)$ is a triangle algebra[2] (isomorphic to $\mathscr{A}$, by definition) in which $\mathscr{E}(\mathscr{A}') = \mathscr{L}$ and $\mu'([0,1] \odot' [0,1]) = \alpha$.
So we can conclude that there is a one-to-one correspondence between triangle algebras and couples $(\mathscr{L}, \alpha)$, in which α is an element in the residuated lattice $\mathscr{L}$. This characterization implies that every property that can be imposed on triangle algebras, can be formulated in terms of such couples. In the next section, for some interesting properties, we are able to find such necessary and sufficient conditions.

3.5 Connections between properties on triangle algebras and properties on their diagonal

Using the isomorphism in Figure 3.1, the set of exact elements of a triangle algebra corresponds to the diagonal of the isomorphic (extended) IVRL. We will often[3] use the term 'diagonal' for triangle algebras as well.
We already showed that $E(\mathscr{A})$ is closed under all the defined operations on a triangle algebra $\mathscr{A}$. Every property in Definition 2.34 (prelinearity, divisibility, ...) can therefore be weakened, by imposing it on $E(\mathscr{A})$ (instead of A) only. We will denote this with the prefix 'pseudo'[4]. For example, a triangle algebra is said to be pseudo-linear if its set of exact elements is linearly ordered (by the original ordering, restricted to the diagonal). Another example: a triangle algebra is pseudo-divisible if $\nu x \sqcap \nu y = \nu x * (\nu x \Rightarrow \nu y)$ for all x and y in A ($E(\mathscr{A})$ consists exactly of the elements of the form νx). Remark that pseudo-divisibility is also equivalent with $\mu x \sqcap \mu y = \mu x * (\mu x \Rightarrow \mu y)$ for all x and y in A, with $\nu x \sqcap \mu y = \nu x * (\nu x \Rightarrow \mu y)$ for all x and y in A, with $\mu x \sqcap \nu y = \mu x * (\mu x \Rightarrow \nu y)$ for all x and y in A, and with $x \sqcap y = x * (x \Rightarrow y)$ for all x and y in $E(\mathscr{A})$. Similar alternative formulations also hold for other 'pseudo-properties', of course.
Observe that 'P holds in $\mathscr{A}$' implies 'pseudo-P holds in $\mathscr{A}$' (because 'pseudo-P holds in $\mathscr{A}$' is by definition equivalent with 'P holds in $\mathscr{E}(\mathscr{A})$', for all possible properties P expressible as identities in residuated lattices. In this section we investigate the connections between properties on $\mathscr{A}$ and properties on $\mathscr{E}(\mathscr{A})$. Considering the results in Section 3.4, these connections may depend on the value of $u * u$.
For distributivity, the situation is very simple: a triangle algebra $\mathscr{A}$ is distributive iff $\mathscr{E}(\mathscr{A})$ is distributive. This follows immediately from (2.1), (T.3), (T.3'), (T.4) and (T.4'). On triangularizations this was already noted before Proposition 2.13. For other properties than distributivity, the situation is often a bit more complex.

Proposition 3.24 *The negation $\neg$ in a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is an involution if and only if $u * u = 0$ and $\neg\neg\nu x = \nu x$, for all x in A.*

[2]The construction of $\mathscr{A}'$ from $\mathscr{A}$ may look complicated at first sight, but notice that basically, we just replaced the intervals $[x, x]$ by x.

[3]In fact, we already did so in the titles of this and the previous section.

[4]Note that we have to be careful with properties that already have 'pseudo' in their name, like 'pseudocomplementation' and 'pseudo-Boolean algebra'. In the first case however, there is no problem because we do not have a property named 'complementation'. In the second case, we avoid confusion by using the more common name 'Heyting-algebra' instead of 'pseudo-Boolean algebra'

Proof. Remark that $u * u = 0$ iff $\mu(u * u) = 0$. From the definition of the negation $\neg$, Theorem 3.20 and Proposition 2.33(11), it follows that the negation is determined by $\nu\neg x = \neg\mu x$ and $\mu\neg x = \neg(\mu x * \mu(u * u)) \sqcap \neg\nu x$.
Suppose that the negation is an involution. Then obviously $\neg\neg\nu x = \nu x$ for all x in A. Furthermore, we have $0 = \nu u = \nu\neg\neg u = \neg\mu\neg u = \neg\neg\mu(u * u) = \mu(u * u)$.
Conversely, if $u * u = 0$ and $\neg\neg\nu x = \nu x$, for all x in A, then $\mu\neg x = \neg\nu x$. So $\nu\neg\neg x = \neg\mu\neg x = \neg\neg\nu x = \nu x$ and $\mu\neg\neg x = \neg\nu\neg x = \neg\neg\mu x = \neg\neg\nu\mu x = \nu\mu x = \mu x$, which implies $\neg\neg x = x$. □

Proposition 3.25 *A triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is divisible iff $\mathscr{E}(\mathscr{A})$ is a Heyting-algebra (i.e., for all x in A, $\nu x * \nu x = \nu x$) and $\mu(u * u) \sqcup \nu x \sqcup \neg\nu x = 1$ for all x in A.*
Equivalently, in terms of IVRLs, this can be rephrased as follows.
Let $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ be an IVRL and $\alpha \in L$, $* : L^2 \to L$ and $\Rightarrow : L^2 \to L$ be defined by $\alpha = \mathrm{pr}_2([0,1] \odot [0,1])$, $x * y = \mathrm{pr}_1([x,x] \odot [y,y])$ and $x \Rightarrow y = \mathrm{pr}_1([x,x] \Rightarrow_\odot [y,y])$, for all x and y in L. Then this IVRL is divisible iff $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a Heyting-algebra ($* = \sqcap$) and $\alpha \sqcup x \sqcup \neg x = 1$ for all x in L.

Proof. We give the proof in terms of IVRLs.
Suppose $(Int(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is a divisible IVRL. Now consider, for any fixed x in L, the intervals $[x,x]$ and $[x,1]$. Because of the divisibility, there must exist an interval $[y,z]$ such that $[x,x] = [x,1] \odot [y,z] = [x * y, (x * z) \sqcup (1 * y) \sqcup (1 * z * \alpha)]$. So $x = x * y$ and $y \leqslant x$, which implies $x * x = x$ (no matter what the value of α is). As this holds for any x, we find, for any y and z, that $y \sqcap z = (y \sqcap z) * (y \sqcap z) \leqslant y * z \leqslant y \sqcap z$, so $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a Heyting-algebra. Furthermore, we have $[0,1] = [x,1] \bigsqcap [0,1] = [x,1] \odot ([x,1] \Rightarrow_\odot [0,1]) = [x,1] \odot [\neg x, 1] = [\neg x * x, x \sqcup \neg x \sqcup \alpha]$.
Conversely, suppose $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a Heyting-algebra (and therefore divisible and distributive) and $\alpha \sqcup x \sqcup \neg x = 1$ for all x in L. Then $\sqcap$ and $*$ coincide. We prove that $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1, x_2] \odot ([x_1, x_2] \Rightarrow_\odot [y_1, y_2])$, or in other words: that $x_1 \sqcap y_1 = x_1 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2)$ and that $x_2 \sqcap y_2$ is the supremum of $\alpha \sqcap x_2 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2)$, $x_2 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2)$ and $x_1 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2)$, for every $[x_1, x_2]$ and $[y_1, y_2]$ in $Int(\mathscr{L})$. Indeed, $x_1 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2) = x_1 \sqcap y_1 \sqcap (x_2 \Rightarrow y_2) = x_1 \sqcap y_1$ (because $y_1 \leqslant y_2 \leqslant x_2 \Rightarrow y_2$), and the supremum of

- $\alpha \sqcap x_2 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2) = \alpha \sqcap x_2 \sqcap y_2 \sqcap (x_1 \Rightarrow y_2) = \alpha \sqcap x_2 \sqcap y_2$,
- $x_2 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2) = (x_1 \Rightarrow y_1) \sqcap x_2 \sqcap y_2$ and
- $x_1 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2) = x_1 \sqcap y_2 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) = x_1 \sqcap y_2 = x_1 \sqcap x_2 \sqcap y_2$

is, by distributivity, $(x_2 \sqcap y_2) \sqcap (\alpha \sqcup (x_1 \Rightarrow y_1) \sqcup x_1) = x_2 \sqcap y_2$ (because $1 = \alpha \sqcup \neg x_1 \sqcup x_1 \leqslant \alpha \sqcup (x_1 \Rightarrow y_1) \sqcup x_1$). □

We can distinguish two special cases in the previous proposition:

- If $\alpha = 1$ (i.e. the product $\odot$ is t-representable), then the condition $\alpha \sqcup x \sqcup \neg x = 1$ is always fulfilled. So, every Heyting algebra on $\mathscr{L}$ corresponds with a Heyting algebra on $\mathbb{T}(\mathscr{L})$ and vice versa (see also Proposition 3.27). Indeed: the t-representable extension of the infimum on $\mathscr{L}$, defined by, for $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in $Int(\mathscr{L})$,

$$\mathscr{T}_{\sqcap,1}(x, y) = [x_1 \sqcap y_1, x_2 \sqcap y_2], \tag{3.17}$$

is equal to the infimum on $\mathbb{T}(\mathscr{L})$.

- If $\alpha = 0$ (i.e. the product $\odot$ is pseudo t-representable), then the condition $\alpha \sqcup x \sqcup \neg x = 1$ becomes $x \sqcup \neg x = 1$. The only Heyting algebras in which this holds for any x, are Boolean algebras. In this case, the IVRL is not only divisible, but even an MV-algebra. This follows using Proposition 3.24, but was already proven in [44].

Remark that the condition $\alpha \sqcup x \sqcup \neg x = 1$, for any x in L, implies that $\alpha \sqcup \neg\alpha = 1$. If $\alpha \sqcup \neg\alpha = 1$, one does not need to verify the condition for $x \leqslant \alpha$ nor for $\neg\alpha \leqslant x$. Indeed, if $x \leqslant \alpha$, then $1 = \alpha \sqcup \neg\alpha \leqslant \alpha \sqcup x \sqcup \neg\alpha \leqslant \alpha \sqcup x \sqcup \neg x$; if $\neg\alpha \leqslant x$, then $1 = \alpha \sqcup \neg\alpha \leqslant \alpha \sqcup \neg\alpha \sqcup \neg x \leqslant \alpha \sqcup x \sqcup \neg x$.

Proposition 3.26 *Every $\sqcup$-definable triangle algebra that satisfies $u * u = u$ (or equivalently $\mu(u * u) = 1$), also satisfies pseudo-strong $\sqcup$-definability. In other words, if in a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, $u * u = u$ and $x \sqcup y = ((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x)$ for all x and y in A, then $\nu x \sqcup \nu y = (\nu y \Rightarrow \nu x) \Rightarrow \nu x$ for all x and y in A.*

Proof. Indeed, using the defining properties of triangle algebras, Theorem 3.20 and some properties of $\Rightarrow$ (decreasing in the first variable, increasing in the second variable, $x \Rightarrow x = 1$, $x \Rightarrow 1 = 1$, $1 \Rightarrow x = x$), we find

$$
\begin{aligned}
&\nu x \sqcup \nu y \\
&= \nu((x \sqcup u) \sqcup \nu y) && \text{(T.4),(T.5)} \\
&= \nu\Big(\big(((x \sqcup u) \Rightarrow \nu y) \Rightarrow \nu y\big) \sqcap \big((\nu y \Rightarrow (x \sqcup u)) \Rightarrow (x \sqcup u)\big)\Big) && \sqcup\text{-definability} \\
&= \Big(\nu\big((x \sqcup u) \Rightarrow \nu y\big) \Rightarrow \nu\nu y\Big) \sqcap \Big(\mu\big((x \sqcup u) \Rightarrow \nu y\big) \Rightarrow \mu\nu y\Big) \sqcap \\
&\quad \Big(\nu\big(\nu y \Rightarrow (x \sqcup u)\big) \Rightarrow \nu(x \sqcup u)\Big) \sqcap \Big(\mu\big(\nu y \Rightarrow (x \sqcup u)\big) \Rightarrow \mu(x \sqcup u)\Big) && \text{(T.3), Theorem 3.20} \\
&= \Big(\Big(\big(\nu(x \sqcup u) \Rightarrow \nu\nu y\big) \sqcap \big(\mu(x \sqcup u) \Rightarrow \mu\nu y\big)\Big) \Rightarrow \nu\nu y\Big) \sqcap \\
&\quad \Big(\Big(\big(\nu(x \sqcup u) \Rightarrow \mu\nu y\big) \sqcap \big(\mu(x \sqcup u) \Rightarrow (\mu(u * u) \Rightarrow \mu\nu y)\big)\Big) \Rightarrow \mu\nu y\Big) \sqcap \\
&\quad \Big(\Big(\big(\nu\nu y \Rightarrow \nu(x \sqcup u)\big) \sqcap \big(\mu\nu y \Rightarrow \mu(x \sqcup u)\big)\Big) \Rightarrow \nu(x \sqcup u)\Big) \sqcap \\
&\quad \big(\mu(\nu y \Rightarrow (x \sqcup u)) \Rightarrow 1\big) && \text{Theorem 3.20, (T.4')} \\
&= \Big(\big((\nu x \Rightarrow \nu y) \sqcap (1 \Rightarrow \nu y)\big) \Rightarrow \nu y\Big) \sqcap \\
&\quad \Big(\big((\nu x \Rightarrow \nu y) \sqcap (1 \Rightarrow (1 \Rightarrow \nu y))\big) \Rightarrow \nu y\Big) \sqcap && u * u = u \\
&\quad \Big(\big((\nu y \Rightarrow \nu x) \sqcap (\nu y \Rightarrow 1)\big) \Rightarrow \nu x\Big) \sqcap 1 && \text{Proposition 2.33(14)} \\
&= (\nu y \Rightarrow \nu y) \sqcap (\nu y \Rightarrow \nu y) \sqcap ((\nu y \Rightarrow \nu x) \Rightarrow \nu x) && \text{Proposition 2.33(2,12)} \\
&= (\nu y \Rightarrow \nu x) \Rightarrow \nu x. && \text{Proposition 2.33(14)}
\end{aligned}
$$

□

Remark that this does not hold if $u * u$ does not equal u, even if pseudo-prelinearity or pseudo-linearity is assumed. Indeed, consider the Heyting-algebra with three elements (say, 0, a and 1) and take on its triangularization (see Figure 3.2), the IVRL determined by $[0,1] * [0,1] =$

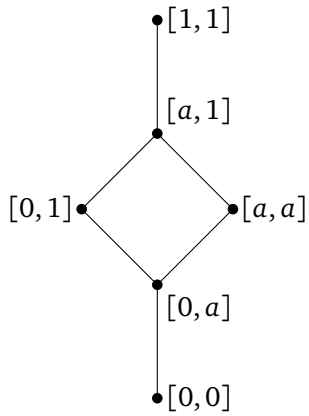

Figure 3.2: The triangularization of a lattice with three elements. The considered IVRL corresponds to example 6.1.75(16) in [50].

$[0,0]$. This IVRL is pseudo-linear, but not pseudo-strong $\sqcup$-definable (as $[a,a] \bigsqcup [0,0] = [a,a] \neq [1,1] = [0,0] \Rightarrow [0,0] = ([a,a] \Rightarrow [0,0]) \Rightarrow [0,0])$. It is $\sqcup$-definable however. We only need to check this for the two incomparable elements $[0,1]$ and $[a,a]$ (if $x \leqslant y$, then $((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x) = y \sqcap ((y \Rightarrow x) \Rightarrow x) = y = x \sqcup y$, because of Proposition 2.33(6)). We find, using Theorem 3.21 and $a \Rightarrow 0 = 0$, that $(([0,1] \Rightarrow [a,a]) \Rightarrow [a,a]) \sqcap (([a,a] \Rightarrow [0,1]) \Rightarrow [0,1]) = ([a,1] \Rightarrow [a,a]) \sqcap ([0,1] \Rightarrow [0,1]) = [a,1] \sqcap [1,1] = [a,1] = [0,1] \sqcup [a,a]$.

Proposition 3.27 *A triangle algebra is a Heyting-algebra iff its diagonal is a Heyting algebra and* $u * u = u$.

Proof. In a Heyting-algebra $x * y = x \sqcap y$, for all x and y. So $u * u = u \sqcap u = u$. As a subalgebra of a Heyting-algebra, the diagonal is also a Heyting-algebra. Conversely, if the diagonal of a triangle algebra is a Heyting-algebra, and $u * u = u$, then we find, for all x and y, that $\nu(x * y) = \nu x * \nu y = \nu x \sqcap \nu y = \nu(x \sqcap y)$ and $\mu(x * y) = (\mu(u * u) * \mu x * \mu y) \sqcup (\nu x * \mu y) \sqcup (\mu x * \nu y) = (\mu x * \mu y) \sqcup (\nu x * \mu y) \sqcup (\mu x * \nu y) = \mu x * \mu y = \mu x \sqcap \mu y = \mu(x \sqcap y)$, which implies that $x * y = x \sqcap y$ (using (2.1)). □

Proposition 3.28 *A triangle algebra* $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *is prelinear iff* $\nu x \sqcup \neg \nu x = 1$ *for all* x *in* A*. Or equivalently, iff* $x \sqcup \neg x = 1$ *for all* x *in* $E(\mathscr{A})$.

Proof. Suppose first that $\mathscr{A}$ is prelinear. In particular, we know that $(u \Rightarrow \nu x) \sqcup (\nu x \Rightarrow u) = 1$ for every x in A. So by Theorem 3.20 and the defining properties of triangle algebras, $1 = \nu 1 = \nu(u \Rightarrow \nu x) \sqcup \nu(\nu x \Rightarrow u) = ((\nu u \Rightarrow \nu\nu x) \sqcap (\mu u \Rightarrow \mu\nu x)) \sqcup ((\nu\nu x \Rightarrow \nu u) \sqcap (\mu\nu x \Rightarrow \mu u)) = ((0 \Rightarrow \nu x) \sqcap (1 \Rightarrow \nu x)) \sqcup ((\nu x \Rightarrow 0) \sqcap (\nu x \Rightarrow 1)) = (1 \sqcap \nu x) \sqcup (\neg \nu x \sqcap 1) = \nu x \sqcup \neg \nu x$.
Conversely, suppose $x \sqcup \neg x = 1$ for all x in $E(\mathscr{A})$. Then $(y \Rightarrow z) \sqcup (z \Rightarrow y) = 1$ for all y and z in $E(\mathscr{A})$, as $z \sqcup \neg z \leqslant (y \Rightarrow z) \sqcup (z \Rightarrow y)$. So $\mathscr{A}$ is pseudo-prelinear and therefore distributive. Now we prove that $\mathscr{A}$ is prelinear as well. Remark first that $\neg x \sqcup y \leqslant x \Rightarrow y$ (Proposition 2.33(2)),

and $\nu y = \nu y \sqcup (\nu y \sqcap a) = (\nu y \sqcap \mu y) \sqcup (\nu y \sqcap a)$, for all x, y and a in A. So we have

$$
\begin{aligned}
1 &= (1 \sqcap 1) \sqcup (1 \sqcap 1) \\
&= \Big(((\nu y \sqcup \neg\mu x) \sqcup \neg\nu y) \sqcap ((\nu y \sqcup \neg\mu x) \sqcup \mu x) \Big) \sqcup \\
&\quad \Big(((\nu x \sqcup \neg\mu y) \sqcup \neg\nu x) \sqcap ((\nu x \sqcup \neg\mu y) \sqcup \mu y) \Big) && \text{pseudo-LEM} \\
&= \big((\nu y \sqcup \neg\mu x) \sqcup (\neg\nu y \sqcap \mu x) \big) \sqcup \big((\nu x \sqcup \neg\mu y) \sqcup (\neg\nu x \sqcap \mu y) \big) && \text{distributivity} \\
&= \neg\mu x \sqcup (\neg\nu x \sqcap \mu y) \sqcup \nu y \sqcup \neg\mu y \sqcup (\neg\nu y \sqcap \mu x) \sqcup \nu x \\
&= (\neg\nu x \sqcap \neg\mu x) \sqcup (\neg\nu x \sqcap \mu y) \sqcup (\nu y \sqcap \neg\mu x) \sqcup (\nu y \sqcap \mu y) \sqcup \\
&\quad (\neg\nu y \sqcap \neg\mu y) \sqcup (\neg\nu y \sqcap \mu x) \sqcup (\nu x \sqcap \neg\mu y) \sqcup (\nu x \sqcap \mu x) && \text{(T.1) and (T.1')} \\
&= (\neg\nu x \sqcap (\neg\mu x \sqcup \mu y)) \sqcup (\nu y \sqcap (\neg\mu x \sqcup \mu y)) \sqcup \\
&\quad (\neg\nu y \sqcap (\neg\mu y \sqcup \mu x)) \sqcup (\nu x \sqcap (\neg\mu y \sqcup \mu x)) && \text{distributivity} \\
&= \big((\neg\nu x \sqcup \nu y) \sqcap (\neg\mu x \sqcup \mu y) \big) \sqcup \big((\neg\nu y \sqcup \nu x) \sqcap (\neg\mu y \sqcup \mu x) \big) && \text{distributivity} \\
&\leqslant ((\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)) \sqcup ((\nu y \Rightarrow \nu x) \sqcap (\mu y \Rightarrow \mu x)) && \text{Proposition 2.33(2)} \\
&= \nu((x \Rightarrow y) \sqcup (y \Rightarrow x)) && \text{(T.4) and Theorem 3.20} \\
&\leqslant (x \Rightarrow y) \sqcup (y \Rightarrow x), && \text{(T.1)}
\end{aligned}
$$

which concludes the proof. □

Remark that the property $x \sqcup \neg x = 1$ is called law of excluded middle (LEM) and that a residuated lattice satisfying LEM is a Boolean algebra.[5] Every Boolean algebra is prelinear, but not every prelinear residuated lattice is a Boolean algebra. Indeed, take any left-continuous t-norm on the unit interval, and the resulting structure will be an MTL-algebra but not a Boolean algebra. So, using Proposition 3.28, we see that not every pseudo-prelinear triangle algebra is prelinear.

Proposition 3.29 *Every weak divisible triangle algebra is pseudo-divisible.*

Proof. For all x and y in a weak divisible triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$,

$$
\begin{aligned}
&\nu x \sqcap \nu y \\
&= \nu((x \sqcup u) \sqcap \nu y) && \text{(T.1)-(T.5)} \\
&= \nu\Big(\Big((x \sqcup u) * ((x \sqcup u) \Rightarrow \nu y) \Big) \sqcup \Big(\nu y * (\nu y \Rightarrow (x \sqcup u)) \Big) \Big) && \text{weak divisibility} \\
&= \Big(\nu(x \sqcup u) * \Big((\nu(x \sqcup u) \Rightarrow \nu\nu y) \sqcap (\mu(x \sqcup u) \Rightarrow \mu\nu y) \Big) \Big) \sqcup \\
&\quad \Big(\nu\nu y * \Big((\nu\nu y \Rightarrow \nu(x \sqcup u)) \sqcap (\mu\nu y \Rightarrow \mu(x \sqcup u)) \Big) \Big) && \text{Theorem 3.20} \\
&= \Big(\nu x * ((\nu x \Rightarrow \nu y) \sqcap (1 \Rightarrow \nu y)) \Big) \sqcup \Big(\nu y * ((\nu y \Rightarrow \nu x) \sqcap (\nu y \Rightarrow 1)) \Big) \\
&= (\nu x * \nu y) \sqcup (\nu y * (\nu y \Rightarrow \nu x)) && \text{Proposition 2.33} \\
&= \nu y * (\nu y \Rightarrow \nu x). && \text{Prop. 2.33(2)} \quad \square
\end{aligned}
$$

[5] More traditionally, a Boolean algebra is defined as a distributive lattice in which $x \sqcup \neg x = 1$ and $x \sqcap \neg x = 0$ hold. Here is a short proof that a residuated lattice satisfying LEM is indeed distributive and satisfies $x \sqcap \neg x = 0$: it is a Heyting algebra (and therefore distributive) because (using Proposition 2.33(5 and 16)) $x = x * 1 = x * (x \sqcup \neg x) = (x * x) \sqcup (x * \neg x) = (x * x) \sqcup 0 = x * x$, for all x. So we have $x \sqcap \neg x = x * \neg x = 0$. Although this proof is very easy, we did not find this result anywhere in the literature. Later we saw that it follows from results in [42], [54] and [62].

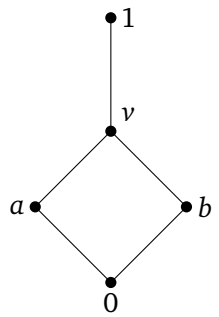

Figure 3.3: The Heyting-algebra on this lattice is divisible, but not every IVRL based on it is weak divisible.

The converse is not true: not every pseudo-divisible triangle algebra is weak divisible. For example, take the Heyting-algebra $\mathscr{L}$ on the lattice[6] in Figure 3.3 and consider the IVRL corresponding[7] to the couple $(\mathscr{L}, \alpha)$, in which α can be 0, a, b or v (but not 1). Such an IVRL is pseudo-Heyting and therefore pseudo-divisible. But it is not weak divisible, because $([a,1] * ([a,1] \Rightarrow [b,1])) \sqcup ([b,1] * ([b,1] \Rightarrow [a,1])) = ([a,1] * [b,1]) \sqcup ([b,1] * [a,1]) = [0, a \sqcup b \sqcup \alpha] = [0, v \sqcup \alpha] = [0, v] < [0,1] = [a,1] \sqcap [b,1]$.

In Section 3.7 we will give some more examples of connections between properties on triangle algebras and properties on their diagonal. In the proofs we will use a decomposition theorem for pseudo-prelinear triangle algebras, similar to a well-known decomposition theorem for MTL-algebras [28, 42]. The proof of this theorem relies on the concept of filters of triangle algebras, which brings us straight to the next section.

3.6 Filters of triangle algebras

We already introduced filters of residuated lattices in Section 2.5. Obviously, because a triangle algebra has the structure of a residuated lattice, there are connections between filters of triangle algebras and filters of residuated lattices. But to clearly distinguish both kinds of filters from each other, we refer to filters of triangle algebras as IVRL-filters.

Definition 3.30 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. An IVRL-filter (IF) of $\mathscr{A}$ is a non-empty subset F of A, satisfying:*

(F.1) if $x \in F$, $y \in A$ and $x \leqslant y$, then $y \in F$;
(F.2) if $x, y \in F$, then $x * y \in F$;
(F.3) if $x \in F$, then $\nu x \in F$.

For all x and y in A, we write $x \sim_F y$ iff $x \Rightarrow y$ and $y \Rightarrow x$ are both in F.

Remark that, because of (F.1), (F.3) and (T.1), we have

(F.3') $x \in F$ iff $\nu x \in F$.

[6]This lattice is also used in [29]. It is example 5.1.20(3) in [50].
[7]See the end of Section 3.4 for details.

Because of (F.1), an IVRL-filter always contains the element 1. Also remark that (F.2) and (F.3) together can be replaced by "If $x, y \in F$, then $\nu(x * y) \in F$".
Furthermore, from this definition we immediately derive that if F is an IVRL-filter of a triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, then F is a filter of the residuated lattice $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$.
For later use we now prove a lemma about the smallest IVRL-filter containing a fixed IVRL-filter and a given element.

Lemma 3.31 *Let F be an IVRL-filter of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, and $z \in A$. Then there exists a smallest IVRL-filter F_z containing the elements of $F \cup \{z\}$, and $F_z = \{v \in A \mid (\exists w \in F)(\exists n \in \mathbb{N})(w * (\nu z)^n \leqslant v)\}$.*

Proof.

- It is clear that F_z contains all elements of F, and z.
- Every IVRL-filter that contains z, also contains νz, $\nu z * \nu z$, $\nu z * \nu z * \nu z$, ... If such an IVRL-filter contains an element $w \in F$, it also contains $w * \nu z$, $w * (\nu z)^2$, ... So it is clear that an IVRL-filter which is a superset of $F \cup \{z\}$, is also a superset of $\{v \in A \mid (\exists w \in F)(\exists n \in \mathbb{N})(w * (\nu z)^n \leqslant v)\}$.
- The set $\{v \in A \mid (\exists w \in F)(\exists n \in \mathbb{N})(w * (\nu z)^n \leqslant v)\}$ is an IVRL-filter:
  - (F.1) If v_1 is an element and $v_1 \leqslant v_2$, then it is clear that v_2 is an element too (we can take the same w and n).
  - (F.2) If v_1 and v_2 are elements of this set, then there exist w_1 and w_2 in F and n_1 and n_2 in $\mathbb{N}$ such that $w_1 * (\nu z)^{n_1} \leqslant v_1$ and $w_2 * (\nu z)^{n_2} \leqslant v_2$. Therefore $(w_1 * w_2) * (\nu z)^{n_1+n_2} \leqslant v_1 * v_2$. As $w_1 * w_2 \in F$ and $n_1 + n_2 \in \mathbb{N}$, $v_1 * v_2$ is also an element of the set.
  - (F.3) If $w * (\nu z)^n \leqslant v$, then by Theorem 3.20, $\nu w * (\nu z)^n = \nu(w * (\nu z)^n) \leqslant \nu v$. So, νv is an element if v is, because $\nu w \in F$ if $w \in F$.

□

Similarly as for filters of residuated lattices, we have:

Proposition 3.32 *Let F be an IVRL-filter of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$. Then the relation $\sim_F$ is a congruence relation on $\mathscr{A}$.*

Proof. The fact that $\sim_F$ is an equivalence relation, is proven in a similar way as in [42] (see also Section 2.5). Indeed, reflexivity follows from $x \Rightarrow x = 1$, for every x in A; symmetry is trivial; and transitivity follows from $(x \Rightarrow y) * (y \Rightarrow z) \leqslant x \Rightarrow z$, for every x, y and z in A. Now suppose $x_1 \sim_F x_2$ and $y_1 \sim_F y_2$. We have to prove that $(x_1 \sqcap y_1) \sim_F (x_2 \sqcap y_2)$, $(x_1 \sqcup y_1) \sim_F (x_2 \sqcup y_2)$, $(x_1 * y_1) \sim_F (x_2 * y_2)$, $(x_1 \Rightarrow y_1) \sim_F (x_2 \Rightarrow y_2)$, $\nu x_1 \sim_F \nu x_2$ and $\mu x_1 \sim_F \mu x_2$. Again, the first four claims are proven similarly as in [42]. We now show the remaining two. Because F contains $x_1 \Rightarrow x_2$, it also contains $\nu(x_1 \Rightarrow x_2)$, which equals $(\nu x_1 \Rightarrow \nu x_2) \sqcap (\mu x_1 \Rightarrow \mu x_2)$ by Theorem 3.20. Therefore also $\nu x_1 \Rightarrow \nu x_2$ and $\mu x_1 \Rightarrow \mu x_2$ are in F. Similarly, also $\nu x_2 \Rightarrow \nu x_1$ and $\mu x_2 \Rightarrow \mu x_1$ are in F. So indeed $\nu x_1 \sim_F \nu x_2$ and $\mu x_1 \sim_F \mu x_2$. □

As a corollary, we can meaningfully consider the quotient algebra $\mathscr{A}_F = (A_F, \sqcap_F, \sqcup_F, *_F, \Rightarrow_F, \nu_F, \mu_F, [0]_F, [u]_F, [1]_F)$, in which A_F is the set of equivalence classes (induced by $\sim_F$) in A, and for any x and y in A, $[x]_F = \{y \in A \mid x \sim_F y\}$, $[x]_F \circ_F [y]_F = [x \circ y]_F$ (for $\circ \in \{\sqcap, \sqcup, *, \Rightarrow\}$),

$\nu_F[x]_F = [\nu x]_F$ and $\mu_F[x]_F = [\mu x]_F$. This structure is again a triangle algebra, as all defining properties are inherited from $\mathscr{A}$. For example, (T.10) follows from

$$\begin{aligned}&\nu_F[x]_F \sqcup_F (\mu_F[x]_F \sqcap_F [u]_F)\\ &= [\nu x \sqcup (\mu x \sqcap u)]_F\\ &= [x]_F.\end{aligned}$$

Note that (F.3) is a necessary condition for $\sim_F$ being a congruence relation. Indeed, if $\sim_F$ is a congruence relation on a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ and $x \in F$, then $x \sim_F 1$ and therefore $\nu x \sim_F \nu 1 = 1$, which is equivalent with $\nu x \in F$.

There is an obvious connection between the notions 'IVRL-filter of a triangle algebra' and 'filter of a residuated lattice', which is given in the next proposition.

Proposition 3.33 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra, $\mathscr{E}(\mathscr{A}) = (E(\mathscr{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be its subalgebra of exact elements and $F \subseteq A$. Then F is an IVRL-filter of the triangle algebra $\mathscr{A}$ iff (F.3') holds and $F \cap E(\mathscr{A})$ is a filter of the residuated lattice $\mathscr{E}(\mathscr{A})$.*

The proof is straightforward.
Proposition 3.33 suggests two different ways to define specific kinds of IVRL-filters of triangle algebras. The first is to impose a property on a filter of the subalgabra of exact elements and extend this filter to the whole triangle algebra, using (F.3'). We call these IVRL-extended filters. For example, an IVRL-extended prime filter of the second kind (IPF2) of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a subset F of A such that $F \cap E(\mathscr{A})$ is a prime filter of the second kind of $\mathscr{E}(\mathscr{A})$ and $x \in F$ iff $\nu x \in F \cap E(\mathscr{A})$. Similarly we define IVRL-extended prime filters[8](IPF), IVRL-extended prime filters of the third kind (IPF3), IVRL-extended Boolean filters (IBF) and IVRL-extended Boolean filters of the second kind (IBF2).
The second way is to impose a property on the whole IVRL-filter. For example, a Boolean IVRL-filter (BIF) of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is an IVRL-filter F of A such that F is a Boolean filter of $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$. Similarly, we define prime IVRL-filters (PIF), prime IVRL-filters of the second kind (PIF2), prime IVRL-filters of the third kind (PIF3) and Boolean IVRL-filters of the second kind (BIF2). However, we will show that each of these IVRL-filters is either trivial (the whole triangle algebra) or equivalent with an IVRL-extended filter.

Proposition 3.34 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and F an IVRL-filter of $\mathscr{A}$. Then F is an IVRL-extended prime filter iff $E(\mathscr{A}_F)$ is linearly ordered.*

Proof. First note that for any x and y in A, $[x]_F \leqslant_F [y]_F$ iff $[x]_F \Rightarrow_F [y]_F = [1]_F$ iff $[x \Rightarrow y]_F = [1]_F$ iff $(x \Rightarrow y) \Rightarrow 1 \in F$ and $1 \Rightarrow (x \Rightarrow y) \in F$ iff $x \Rightarrow y \in F$. Now, every element of $E(\mathscr{A}_F)$ is of the form $\nu_F[x]_F$ (with x in A), so we have:
$E(\mathscr{A}_F)$ is linearly ordered

- iff for all x and y in A, $\nu_F[x]_F \leqslant_F \nu_F[y]_F$ or $\nu_F[y]_F \leqslant_F \nu_F[x]_F$
- iff for all x and y in A, $[\nu x]_F \leqslant_F [\nu y]_F$ or $[\nu y]_F \leqslant_F [\nu x]_F$
- iff for all x and y in A, $\nu x \Rightarrow \nu y \in F$ or $\nu y \Rightarrow \nu x \in F$

[8]In [78], we called these pseudo-prime filters.

- iff F is an IVRL-extended prime filter of $\mathscr{A}$.

□

Now we summarize some properties that follow immediately from the definitions, Proposition 3.33 and the properties of filters of residuated lattices (Section 2.5).

Proposition 3.35 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and F an IVRL-filter of $\mathscr{A}$.*

1. *A and $\{1\}$ are IVRL-filters of $\mathscr{A}$.*
2. *A is IPF, IPF2, IPF3, IBF, IBF2, PIF, PIF2, PIF3, BIF and BIF2.*
3. *If F is PIF, then F is IPF.*
4. *If F is PIF2, then F is IPF2.*
5. *If F is PIF3, then F is IPF3.*
6. *If F is BIF, then F is IBF.*
7. *If F is BIF2, then F is IBF2.*
8. *The intersection property holds for IF, IPF3, IBF, PIF3 and BIF (but not for IPF, IPF2 or IBF2)*[9].
9. *The monotonicity property*[10] *holds for IF, IPF, IPF3, IBF, IBF2, PIF, PIF3, BIF and BIF2 (but not for IPF2 or PIF2).*
10. *The implications and equivalences from Section 2.5 can be 'translated' to the two kinds of IVRL-filters. For example: F is IPF iff F is IPF2 and IPF3, and F is PIF iff F is PIF2 and PIF3.*
11. *The counterexamples from Section 2.5 can be 'translated' to the IVRL-extended filters. For example: F can be IBF without being IPF2.*

Boolean IVRL-filters and Boolean IVRL-filters of the second kind are trivial. This is a consequence of the following proposition.

Proposition 3.36 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and F a Boolean IVRL-filter of $\mathscr{A}$. Then $F = A$.*

Proof. This holds because $u \sqcup \neg u$ must be in F and therefore also $\nu(u \sqcup \neg u) = \nu u \sqcup \nu \neg u = 0 \sqcup \neg \mu u = \neg 1 = 0$ (that $\nu \neg u = \neg \mu u$ is a consequence of Theorem 3.20). □

In Proposition 3.35(4) we already saw that prime IVRL-filters of the second kind are IVRL-extended prime filters of the second kind. Now we show that the converse holds as well.

Proposition 3.37 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and F an IVRL-extended prime filter of the second kind of $\mathscr{A}$. Then F is a prime IVRL-filter of the second kind of $\mathscr{A}$.*

[9] For the remaining cases (PIF, PIF2 and BIF2), we need Corollary 3.39, Proposition 3.37 and Proposition 3.36 to see that of these three only BIF2 enjoys the intersection property.

[10] Here, we mean the monotonicity property w.r.t. the set of IVRL-filters (see Definition 2.41).

Proof. Suppose $x \sqcup y \in F$. Then also $\nu x \sqcup \nu y = \nu(x \sqcup y) \in F$. Because νx and νy are exact elements and F is an IVRL-extended prime filter of the second kind of $\mathscr{A}$, νx or νy must be in F. This implies that F contains x or y. We conclude F must be a prime IVRL-filter of the second kind of $\mathscr{A}$. □

Now we show that prime IVRL-filters of the third kind are the same as IVRL-extended Boolean filters.

Proposition 3.38 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and F a subset of A. Then the following two statements are equivalent:*

- *F is a prime IVRL-filter of the third kind of $\mathscr{A}$,*
- *F is an IVRL-extended Boolean filter of $\mathscr{A}$.*

Proof. Suppose F is a prime IVRL-filter of the third kind of $\mathscr{A}$ and choose an arbitrary exact element νx. We want to show that $\nu x \sqcup \neg \nu x \in F$. Indeed, we know $(u \Rightarrow \nu x) \sqcup (\nu x \Rightarrow u) \in F$ and therefore also

$$\begin{aligned}
&\nu x \sqcup \neg \nu x \\
&= ((\nu u \Rightarrow \nu \nu x) \sqcap (\mu u \Rightarrow \mu \nu x)) \sqcup ((\nu \nu x \Rightarrow \nu u) \sqcap (\mu \nu x \Rightarrow \mu u)) && \text{(T.1),(T.2),(T.5),(T.5'),(T.6')} \\
&= \nu(u \Rightarrow \nu x) \sqcup \nu(\nu x \Rightarrow u) && \text{Theorem 3.20} \\
&= \nu((u \Rightarrow \nu x) \sqcup (\nu x \Rightarrow u)) \in F. && \text{(T.4) and (F.3)}
\end{aligned}$$

Conversely, suppose F is an IVRL-extended Boolean filter of $\mathscr{A}$. Take any two elements x and y in A. We have to show that $(x \Rightarrow y) \sqcup (y \Rightarrow x) \in F$. First note that F contains $\nu x \sqcup \neg \nu x$, $\mu x \sqcup \neg \mu x$, $\nu y \sqcup \neg \nu y$ and $\mu y \sqcup \neg \mu y$, and therefore also $S := (\nu x \sqcup \neg \nu x) * (\mu x \sqcup \neg \mu x) * (\nu y \sqcup \neg \nu y) * (\mu y \sqcup \neg \mu y)$. Using the properties $a * (b \sqcup c) = (a * b) \sqcup (a * c)$ and $d * e \leqslant d \sqcap e$ (Proposition 2.33(1 and 16)), we can easily see that $S \leqslant \nu x \sqcup \neg \mu x \sqcup \nu y \sqcup \neg \mu y \sqcup (\neg \nu x \sqcap \mu x \sqcap \neg \nu y \sqcap \mu y)$. Indeed, S is the supremum of all elements of the form $f * g * h * i$, with $f \in \{\nu x, \neg \nu x\}$, $g \in \{\mu x, \neg \mu x\}$, $h \in \{\nu y, \neg \nu y\}$ and $i \in \{\mu y, \neg \mu y\}$. Apart from $\neg \nu x * \mu x * \neg \nu y * \mu y$, any of these elements is smaller than or equal to $\nu x \sqcup \neg \mu x \sqcup \nu y \sqcup \neg \mu y$.
So, using $\neg x \sqcup y \leqslant x \Rightarrow y$, and $\nu y = \nu y \sqcup (\nu y \sqcap a) = (\nu y \sqcap \mu y) \sqcup (\nu y \sqcap a)$, for all x, y and a in A, we find

$$\begin{aligned}
S &\leqslant \neg \mu x \sqcup (\neg \nu x \sqcap \mu y) \sqcup \nu y \sqcup \neg \mu y \sqcup (\neg \nu y \sqcap \mu x) \sqcup \nu x \\
&= (\neg \nu x \sqcap \neg \mu x) \sqcup (\neg \nu x \sqcap \mu y) \sqcup (\nu y \sqcap \neg \mu x) \sqcup (\nu y \sqcap \mu y) \sqcup \\
&\quad (\neg \nu y \sqcap \neg \mu y) \sqcup (\neg \nu y \sqcap \mu x) \sqcup (\nu x \sqcap \neg \mu y) \sqcup (\nu x \sqcap \mu x) && \text{(T.1) and (T.1')} \\
&\leqslant \big((\neg \nu x \sqcup \nu y) \sqcap (\neg \mu x \sqcup \mu y)\big) \sqcup \big((\neg \nu y \sqcup \nu x) \sqcap (\neg \mu y \sqcup \mu x)\big) \\
&\leqslant ((\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)) \sqcup ((\nu y \Rightarrow \nu x) \sqcap (\mu y \Rightarrow \mu x)) && \text{Proposition 2.33(2)} \\
&= \nu((x \Rightarrow y) \sqcup (y \Rightarrow x)) && \text{Theorem 3.20 and (T.4)} \\
&\leqslant (x \Rightarrow y) \sqcup (y \Rightarrow x), && \text{(T.1)}
\end{aligned}$$

which implies $(x \Rightarrow y) \sqcup (y \Rightarrow x) \in F$. □

Remark that Proposition 3.38 generalizes Proposition 3.28, namely that a triangle algebra is prelinear iff its subalgebra of exact elements is a Boolean algebra. Indeed, this is exactly Proposition 3.38 applied to $F = \{1\}$.

Corollary 3.39 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and F a subset of A. Then the following two statements are equivalent:*

- *F is a prime IVRL-filter of $\mathscr{A}$,*
- *F is an IVRL-extended Boolean filter of the second kind of $\mathscr{A}$.*

Proof. Remark that by Corollary 2.51, F is a prime IVRL-filter of $\mathscr{A}$ iff F is a prime IVRL-filter of the second kind of $\mathscr{A}$ and a prime IVRL-filter of the third kind of $\mathscr{A}$.
Then note that F is an IVRL-extended Boolean filter of the second kind of $\mathscr{A}$ iff F is an IVRL-extended prime filter of the second kind of $\mathscr{A}$ and F an IVRL-extended Boolean filter of $\mathscr{A}$ (which was already noted after Proposition 2.52).
Therefore the result follows immediately from Propositions 3.37 and 3.38. □

Choosing $F = \{1\}$ gives: a triangle algebra is linear iff its subalgebra of exact elements has one or two elements.

Similarly as for filters of residuated lattices (cfr. Figure 2.11), we can summarize the different possibilities for an IVRL-filter F of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$:

- F is an IVRL-filter of $\mathscr{A}$, but F is not IPF, PIF, IPF2, PIF2, IPF3, PIF3, IBF, BIF, IBF2 nor BIF2;
- F is IPF2 (equivalently, PIF2), but not IPF, PIF, IPF3, PIF3, IBF, BIF, IBF2 nor BIF2;
- F is IPF3, but not IPF, PIF, IPF2, PIF2, PIF3, IBF, BIF, IBF2 nor BIF2;
- F is IPF (and therefore also IPF2/PIF2 and IPF3), but not PIF, PIF3, IBF, BIF, IBF2 nor BIF2;
- F is IBF (equivalently, PIF3) (and therefore also IPF3), but not IPF, PIF, IPF2, PIF2, IBF2, BIF nor BIF2;
- F is IBF2 (equivalently, PIF) (and therefore also IPF, IPF2, PIF2, IPF3, PIF3 and IBF), but not BIF nor BIF2;
- $F = A$, in this trivial case, F is IPF, PIF, IPF2, PIF2, IPF3, PIF3, IBF, BIF, IBF2 and BIF2.

The lattice in Figure 3.4 gives a schematic summary of these situations. Remark that the IVRL-extended filters suffice to cover all six non-trivial cases. Therefore the scheme looks the same as for residuated lattices.

Remark 3.40 *Because of Theorem 3.20 a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is completely determined by the value $\mu(u * u)$ and its subalgebra $\mathscr{E}(\mathscr{A})$ of exact elements. It is quite remarkable that for the IVRL-filters shown in Figure 3.4, the value $\mu(u * u)$ is not important. For example, if F is a prime IVRL-filter of a triangle algebra $\mathscr{A}$ with $\mathscr{E}(\mathscr{A}) = \mathscr{L}$, then F is also a prime IVRL-filter of every triangle algebra with $\mathscr{L}$ as subalgebra of exact elements.*
For other kinds of IVRL-filters this does not necessarily hold. We will show this for positive implicative IVRL-filters, pseudocomplementation IVRL-filters and involution IVRL-filters. In the proofs we will use the residuation principle, the monotonicity of the operations and Theorem 3.20 without always mentioning it. The same goes for easy properties such as $\nu x \leqslant x \leqslant \mu x$, $x \Rightarrow y = 1$ iff $x \leqslant y$, $1 \Rightarrow x = x, \ldots$

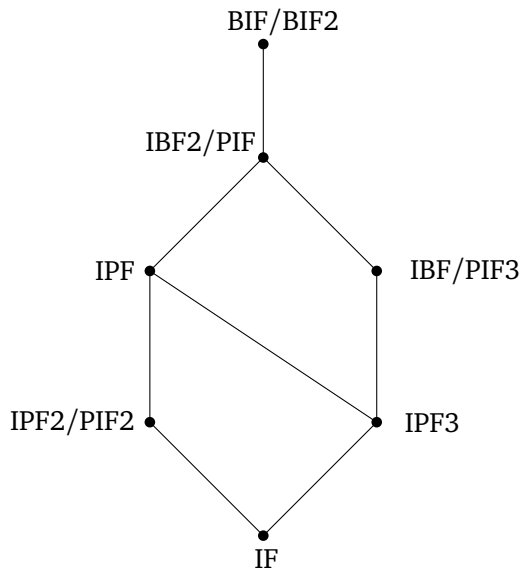

Figure 3.4: The seven possibilities for an IVRL-filter.

Proposition 3.41 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and $F \subseteq A$. Then F is a positive implicative IVRL-filter of $\mathscr{A}$ iff F is an IVRL-extended positive implicative filter of $\mathscr{A}$ and $\mu(u * u) \in F$.*

Proof. First remark that by Proposition 2.53 and (F.3'), an IVRL-filter F of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a positive implicative IVRL-filter iff it contains $\nu(x \Rightarrow (x * x))$ for all x in A. By Theorem 3.20, $\nu(x \Rightarrow (x * x)) = (\nu x \Rightarrow (\nu x * \nu x)) \sqcap (\mu x \Rightarrow ((\mu x * \nu x) \sqcup (\mu x * \mu x * \mu(u * u))))$. Taking $x = u$ and applying (T.5) and (T.5'), we see that $\mu(u * u)$ must be in F. Obviously F must also be an IVRL-extended positive implicative filter of $\mathscr{A}$.
These two conditions are also sufficient, as they imply (using Proposition 2.53) $\nu x \Rightarrow (\nu x * \nu x) \in F$, $\mu x \Rightarrow (\mu x * \mu x) \in F$ and $\mu(u * u) \in F$, such that also $\nu(x \Rightarrow (x * x)) \in F$ because (using Proposition 2.33(8 and 1)) $(\nu x \Rightarrow (\nu x * \nu x)) * (\mu x \Rightarrow (\mu x * \mu x)) * \mu(u * u) \leqslant (\nu x \Rightarrow (\nu x * \nu x)) * (\mu x \Rightarrow ((\mu x * \mu x) * \mu(u * u))) \leqslant \nu(x \Rightarrow (x * x))$. □

Taking $F = \{1\}$ as a special case in this proposition, we obtain Proposition 3.27.

Proposition 3.42 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and $F \subseteq A$. Then F is a pseudocomplementation IVRL-filter of $\mathscr{A}$ iff F is an IVRL-extended pseudocomplementation filter of $\mathscr{A}$ and $\neg\neg\mu(u * u) \in F$.*

Proof. By (F.3'), an IVRL-filter F of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a pseudocomplementation IVRL-filter iff it contains $\nu(\neg(x \sqcap \neg x))$ for all x in A. By Theorem 3.20 (see also the proof of Proposition 3.24), $\nu(\neg(x \sqcap \neg x)) = \neg(\mu x \sqcap \mu\neg x) = \neg(\mu x \sqcap \neg(\mu x * \mu(u * u)) \sqcap \neg\nu x)$. Taking $x = u$ and applying (T.5) and (T.5'), we see that $\neg\neg\mu(u * u)$ must be in F. Obviously F must also be an IVRL-extended pseudocomplementation filter of $\mathscr{A}$.
These two conditions are also sufficient. To see this, choose any x in A, denote $\mu x \sqcap \neg(\mu x * \mu(u * u))$

by X and remark that (using Proposition 2.33(11, 7, 2 and 9))

$$\begin{aligned}
X &= \mu x \sqcap \neg(\mu x * \mu(u*u)) \\
&= \mu x \sqcap (\mu x \Rightarrow \neg\mu(u*u)) \\
&= \mu x \sqcap (\mu x \Rightarrow \neg\neg\neg\mu(u*u)) \\
&= \mu x \sqcap \neg(\mu x * \neg\neg\mu(u*u)) \\
&= \mu x \sqcap (\neg\neg\mu(u*u) \Rightarrow \neg\mu x) \\
&\leqslant (\neg\neg\mu(u*u) \Rightarrow \mu x) \sqcap (\neg\neg\mu(u*u) \Rightarrow \neg\mu x) \\
&= \neg\neg\mu(u*u) \Rightarrow (\mu x \sqcap \neg\mu x).
\end{aligned}$$

This implies (using Proposition 2.33(5))

$$\begin{aligned}
&\neg\neg\mu(u*u) * X * \neg(\mu x \sqcap \neg\mu x) \\
&\leqslant \neg\neg\mu(u*u) * (\neg\neg\mu(u*u) \Rightarrow (\mu x \sqcap \neg\mu x)) * \neg(\mu x \sqcap \neg\mu x) \\
&\leqslant (\mu x \sqcap \neg\mu x) * \neg(\mu x \sqcap \neg\mu x) \\
&= 0.
\end{aligned}$$

Therefore, $\neg\neg\mu(u*u) * \neg(\mu x \sqcap \neg\mu x) \leqslant \neg X$, so $\neg X$ must be in F. This concludes the proof because $\neg X \leqslant \nu(\neg(x \sqcap \neg x))$. □

Taking $F = \{1\}$ in this proposition, we obtain 'A triangle algebra satisfies pseudocomplementation iff its diagonal satisfies pseudocomplementation and $\neg\mu(u*u) = 0$'. Note that $\neg\mu(u*u) = 0$ is equivalent with $\neg(u*u) = 0$. Indeed, if $\neg(u*u) = 0$, then $\neg\mu(u*u) \leqslant \neg(u*u) = 0$. Conversely, if $\neg\mu(u*u) = 0$, then we can use (T.1'), Theorem 3.20 and Proposition 2.33(11) and find $\neg(u*u) \leqslant \mu\neg(u*u) = \neg\nu(u*u) \sqcap \neg(\mu(u*u) * \mu(u*u)) = \neg 0 \sqcap \mu(u*u) \Rightarrow \neg\mu(u*u) = 1 \sqcap \mu(u*u) \Rightarrow 0 = \neg\mu(u*u) = 0$.

Proposition 3.43 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and $F \subseteq A$. Then F is an involution IVRL-filter of $\mathscr{A}$ iff F is an IVRL-extended involution filter of $\mathscr{A}$ and $\neg\mu(u*u) \in F$.*

Proof. Remark that by (F.3') an IVRL-filter F of a triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is an involution IVRL-filter iff it contains $\nu(\neg\neg x \Rightarrow x)$ for all x in A. By Theorem 3.20

$$\begin{aligned}
\nu(\neg\neg x \Rightarrow x) = {} & (\neg(\neg(\mu x * \mu(u*u)) \sqcap \neg\nu x) \Rightarrow \nu x) \\
& \sqcap ((\neg((\neg(\mu x * \mu(u*u)) \sqcap \neg\nu x) * \mu(u*u)) \sqcap \neg\neg\mu x) \Rightarrow \mu x).
\end{aligned}$$

Taking $x = u$ and using (T.5), (T.5') and Proposition 2.33(7), we see that $\neg\neg\neg\mu(u*u) = \neg\mu(u*u)$ must be in F. Obviously F must also be an IVRL-extended involution filter of $\mathscr{A}$.
These two conditions are also sufficient. Indeed, by the monotonicity of $\Rightarrow$, it follows that

$$(\neg((\neg(\mu x * \mu(u*u)) \sqcap \neg\nu x) * \mu(u*u)) \sqcap \neg\neg\mu x) \Rightarrow \mu x \geqslant \neg\neg\mu x \Rightarrow \mu x \in F. \tag{3.18}$$

Furthermore, using Proposition 2.33(3, 9, 14, 2 and 13), we find

$$\begin{aligned}
&\neg\mu(u*u) * \neg(\neg\mu(u*u) \sqcap \neg\nu x) \\
&\leqslant \neg(\neg\mu(u*u) \Rightarrow (\neg\mu(u*u) \sqcap \neg\nu x)) \\
&= \neg(\neg\mu(u*u) \Rightarrow \neg\nu x) \\
&\leqslant \neg\neg\nu x,
\end{aligned}$$

such that (using Proposition 2.33(5))

$$\begin{aligned}
&\neg\mu(u*u)*\neg(\neg\mu(u*u)\sqcap\neg\nu x)*(\neg\neg\nu x\Rightarrow\nu x)\\
&\leqslant\neg\neg\nu x*(\neg\neg\nu x\Rightarrow\nu x)\\
&\leqslant\nu x,
\end{aligned}$$

such that by the residuation principle, $\neg\mu(u*u)*(\neg\neg\nu x\Rightarrow\nu x)\leqslant\neg(\neg\mu(u*u)\sqcap\neg\nu x)\Rightarrow\nu x$. Therefore

$$\begin{aligned}
&\neg(\neg(\mu x*\mu(u*u))\sqcap\neg\nu x)\Rightarrow\nu x\\
&\geqslant\neg(\neg(1*\mu(u*u))\sqcap\neg\nu x)\Rightarrow\nu x\\
&\geqslant\neg\mu(u*u)*(\neg\neg\nu x\Rightarrow\nu x)\in F. \qquad (3.19)
\end{aligned}$$

From (3.18) and (3.19) it follows that $\nu(\neg\neg x\Rightarrow x)\in F$. □

Remark that this result is a generalization of Proposition 3.24. Indeed, we obtain this property by choosing $F=\{1\}$ in Proposition 3.43.

Example 3.44 *An easy example to illustrate the importance of the value $\mu(u*u)$ for positive implicative, pseudocomplementation and involution filters of triangle algebras, is the following. Consider the two triangle algebras with three elements. One of them is determined by $u*u=0$, the other by $u*u=u$. In both cases, the subalgebra of exact elements is a Boolean algebra (it has two elements). Therefore $\{1\}$ is IBF (but not BIF). In the first triangle algebra (determined by $\mu(u*u)=0$), $\{1\}$ is an involution IVRL-filter, but not a pseudocomplementation IVRL-filter (and therefore not a positive implicative IVRL-filter, see the paragraph before Proposition 2.56). In the second triangle algebra (determined by $\mu(u*u)=1$), it is the other way around: $\{1\}$ is a positive implicative IVRL-filter (and therefore also a pseudocomplementation IVRL-filter), but not an involution IVRL-filter.*

3.7 Decomposition theorem for pseudo-prelinear triangle algebras

It is well-known that MTL-algebras are isomorphic to subdirect products of linear residuated lattices [42, 28]. This is a very useful result, as it implies that identities valid in all linear residuated lattices are also valid in all MTL-algebras, which significantly simplifies several proofs, and which is also needed for the chain-completeness of the corresponding logic MTL (see Chapter 4). In this section, we show the 'interval-valued counterpart' of this result: pseudo-prelinear triangle algebras are isomorphic to subdirect products of pseudo-linear triangle algebras. In the proof we use IVRL-extended prime filters. Afterwards we give some applications of this result, two more connections between a property on a triangle algebra and a property on its diagonal (see Section 3.5).

Proposition 3.45 *Let $\mathscr{A}$ be a pseudo-prelinear triangle algebra and $a\in A\setminus\{1\}$. Then there exists an IVRL-extended prime filter F of A not containing a.*

Proof. First notice that there is an IVRL-filter of $\mathscr{A}$ not containing a: $\{1\}$. Now suppose F is an IVRL-filter of $\mathscr{A}$, not containing a and such that, for some x and y in A, neither $\nu x\Rightarrow\nu y$ nor $\nu y\Rightarrow\nu x$ are in F. We show that there exists an IVRL-filter F', for which $a\notin F'$, $F\subseteq F'$ and

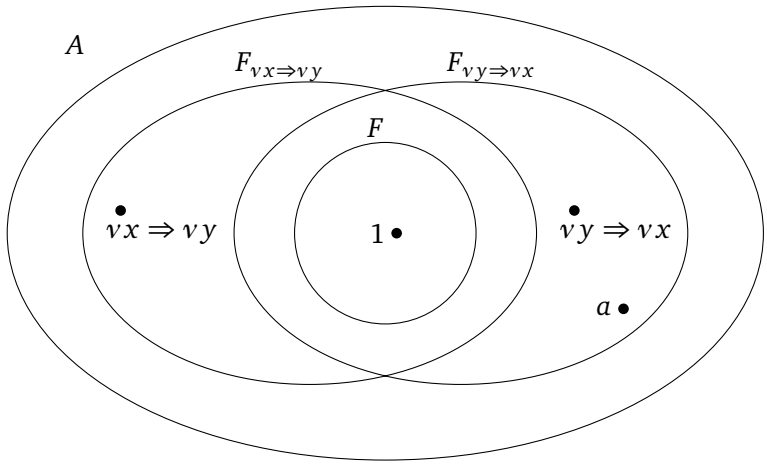

Figure 3.5: In this situation, the IVRL-filter $F_{vx\Rightarrow vy}$ is a superset of $F \cup \{vx \Rightarrow vy\}$ not containing a.

$vx \Rightarrow vy \in F'$ or $vy \Rightarrow vx \in F'$.
We will do this by showing that a is not in $F_{vx\Rightarrow vy} \cap F_{vy\Rightarrow vx}$ (which is the intersection of the smallest IVRL-filter containing F and $vx \Rightarrow vy$ and the smallest IVRL-filter containing F and $vy \Rightarrow vx$, see Lemma 3.31 for details), as this implies that at least one of the IVRL-filters $F_{vx\Rightarrow vy}$ and $F_{vy\Rightarrow vx}$ satisfies the conditions for F'. This is illustrated in Figure 3.5. If a was in $F_{vx\Rightarrow vy} \cap F_{vy\Rightarrow vx}$, then there exist w_1 and w_2 in F, and n_1 and n_2 in $\mathbb{N}$ such that $w_1 * (v(vx \Rightarrow vy))^{n_1} \leqslant a$ and $w_2 * (v(vy \Rightarrow vx))^{n_2} \leqslant a$. This would imply

$$\begin{aligned}
&w_1 * w_2 \\
&= w_1 * w_2 * 1 \\
&= (w_1 * w_2) * ((vx \Rightarrow vy)^{\max(n_1,n_2)} \sqcup (vy \Rightarrow vx)^{\max(n_1,n_2)}) \\
&= ((w_1 * w_2) * (vx \Rightarrow vy)^{\max(n_1,n_2)}) \sqcup ((w_1 * w_2) * (vy \Rightarrow vx)^{\max(n_1,n_2)}) \\
&\leqslant (w_1 * (vx \Rightarrow vy)^{n_1}) \sqcup (w_2 * (vy \Rightarrow vx)^{n_2}) \\
&\leqslant a \sqcup a \\
&= a,
\end{aligned}$$

a contradiction because this would mean that F contains a. In this calculation we used the properties $(vx \Rightarrow vy)^{\max(n_1,n_2)} \sqcup (vy \Rightarrow vx)^{\max(n_1,n_2)} = 1$ (which is valid because $E(\mathscr{A})$ is an MTL-algebra[11]), and $vx \Rightarrow vy$ and $vy \Rightarrow vx$ are in $E(\mathscr{A})$.
Now, if A is countable, we can number all the pairs of elements of A and construct a sequence of IVRL-filters $F_0 = \{1\}$, F_1, F_2, ... by defining $F_{i+1} = (F_i)_{(vx\Rightarrow vy)}$ or $F_{i+1} = (F_i)_{(vy\Rightarrow vx)}$ (the one that does not contain a), in which $\{x, y\}$ is the $(i+1)$th pair in the row. The union of these IVRL-filters is the desired IVRL-extended prime filter not containing a.
If A is uncountable, we use Zorn's Lemma: "Every non-empty partially ordered set, in which every chain has an upper bound, contains at least one maximal element." We apply this to the partially ordered set of IVRL-filters of $\mathscr{A}$ not containing a (with the inclusion as order). Every

[11]The property $(x \Rightarrow y)^n \sqcup (y \Rightarrow x)^n = 1$ obviously holds in all MTL-chains. Because MTL-algebras are subdirect products of MTL-chains, it also holds in all MTL-algebras.

chain in this set indeed has an upper bound: the union of the IVRL-filters of the chain is again an IVRL-filter not containing a. It is easy to verify that every maximal element of the set is a IVRL-extended prime filter. Indeed; suppose that such a maximal IVRL-filter F_m not containing a does not contain $vx \Rightarrow vy$, nor $vy \Rightarrow vx$ for some x and y in A. Then, according to our proof, we can construct an IVRL-filter that is a superset of F_m containing $vx \Rightarrow vy$ or $vy \Rightarrow vx$, but not a. This is a contradiction, as this would mean that F_m is not maximal in the set of IVRL-filters of $\mathscr{A}$ not containing a. This concludes the proof. □

Exactly as in [42], we can use Propositions 3.34 and 3.45 to prove the main result of this section.

Theorem 3.46 *(***Decomposition theorem for pseudo-prelinear triangle algebras***) Every pseudo-prelinear triangle algebra $\mathscr{A}$ is isomorphic to a subalgebra of the direct product of a system of pseudo-linear triangle algebras.*

Proof. Let $\mathscr{F}$ be the system of all IVRL-extended prime filters of $\mathscr{A}$. We consider the direct product of the pseudo-linear (because of Proposition 3.34) triangle algebras $\mathscr{A}^* = (\mathscr{A}_F | F \in \mathscr{F})$. Now we embed $\mathscr{A}$ into $\mathscr{A}^*$, by defining $i(x)$ as $([x]_F | F \in \mathscr{F})$, for all x in A. The mapping i clearly preserves operations. We now show that i is an injection. Suppose $x \neq y$ for x and y in A. Then, without loss of generality, we can assume that $x \not\leq y$, which is equivalent with $x \Rightarrow y \neq 1$. Because of Proposition 3.45, there exists a IVRL-extended prime filter F not containing $x \Rightarrow y$. This is equivalent with $[x]_F \not\leq [y]_F$ (see the proof of Proposition 3.34), which implies $[x]_F \neq [y]_F$ and therefore $i(x) \neq i(y)$. □

This proposition is also valid for subvarieties of pseudo-prelinear triangle algebras. For example, every $\sqcup$-definable pseudo-prelinear triangle algebra is a subalgebra of the direct product of a system of $\sqcup$-definable pseudo-linear triangle algebras. This is due to the fact that the quotient algebras of a $\sqcup$-definable pseudo-prelinear triangle algebra are also $\sqcup$-definable (and similarly for other possible properties), which is proven in an analogous way as we explained after Proposition 3.32.

Proposition 3.47 *A pseudo-divisible pseudo-linear triangle algebra is weak divisible.*

Proof. Suppose $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, v, \mu, 0, u, 1)$ is a pseudo-divisible pseudo-linear[12] triangle algebra. Because of (2.1), we have to show that for all x and y in A, $v(x \sqcap y) = v((x * (x \Rightarrow y)) \sqcup (y * (y \Rightarrow x)))$ and $\mu(x \sqcap y) = \mu((x * (x \Rightarrow y)) \sqcup (y * (y \Rightarrow x)))$. In other words, using Theorem 3.20 and abbreviating $\mu(u * u)$ by t, that

$$\min(vx, vy) = \max\Big(vx * \min(vx \Rightarrow vy, \mu x \Rightarrow \mu y), \\ vy * \min(vy \Rightarrow vx, \mu y \Rightarrow \mu x)\Big)$$

[12]Because the diagonal is linear, we use the notation $\min(x, y)$ instead of $x \sqcap y$ here, for all elements x and y of the diagonal.

and

$$\begin{aligned}\min(\mu x,\mu y) = \max\Big(\; & t*\mu x*\min\big((\mu x*t)\Rightarrow \mu y, \nu x\Rightarrow \mu y\big),\\ & t*\mu y*\min\big((\mu y*t)\Rightarrow \mu x, \nu y\Rightarrow \mu x\big),\\ & \mu x*\min\big(\nu x\Rightarrow \nu y, \mu x\Rightarrow \mu y\big),\\ & \mu y*\min\big(\nu y\Rightarrow \nu x, \mu y\Rightarrow \mu x\big),\\ & \nu x*\min\big((\mu x*t)\Rightarrow \mu y, \nu x\Rightarrow \mu y\big),\\ & \nu y*\min\big((\mu y*t)\Rightarrow \mu x, \nu y\Rightarrow \mu x\big)\Big)\end{aligned}$$

for every x and y in A and t in $E(\mathscr{A})$. Using the properties of $*$ and $\Rightarrow$ and the divisibility and linear order of the diagonal, we can reduce the right hand sides of these identities to

$$\max\Big(\min\big(\nu x,\nu y,\nu x*(\mu x\Rightarrow \mu y)\big),\min\big(\nu x,\nu y,\nu y*(\mu y\Rightarrow \mu x)\big)\Big)$$

and

$$\begin{aligned}\max\Big(\; & \min\Big(t*\mu x,\mu y,\big(t*\mu x*(\nu x\Rightarrow \mu y)\big)\Big),\\ & \min\Big(t*\mu y,\mu x,\big(t*\mu y*(\nu y\Rightarrow \mu x)\big)\Big),\\ & \min\Big(\big(\mu x*(\nu x\Rightarrow \nu y)\big),\mu x,\mu y\Big),\\ & \min\Big(\big(\mu y*(\nu y\Rightarrow \nu x)\big),\mu x,\mu y\Big),\\ & \min\Big(\big(\nu x*((\mu x*t)\Rightarrow \mu y)\big),\nu x,\mu y\Big),\\ & \min\Big(\big(\nu y*((\mu y*t)\Rightarrow \mu x)\big),\nu y,\mu x\Big)\Big).\end{aligned}$$

The first expression is clearly equal to $\min(\nu x,\nu y)$, as either $\mu x\leqslant \mu y$ or $\mu y\leqslant \mu x$ holds. In the second expression, every minimum is smaller than or equal to $\min(\mu x,\mu y)$. At least one of the two minima in the middle is equal to $\min(\mu x,\mu y)$, as either $\nu x\leqslant \nu y$ or $\nu y\leqslant \nu x$.

□

Corollary 3.48 *A pseudo-divisible pseudo-prelinear triangle algebra is weak divisible.*

Proof. Because of Theorem 3.46 a pseudo-divisible pseudo-prelinear triangle algebra is isomorphic to a subdirect product of pseudo-divisible pseudo-linear triangle algebras (which are weak divisible, by the previous proposition). A subdirect product of weak divisible triangle algebras is weak divisible. So we can conclude that pseudo-divisible pseudo-prelinear triangle algebras are weak divisible. □

Note that in Proposition 3.29 we already showed that conversely, weak divisible triangle algebras are pseudo-divisible (even if they are not pseudo-prelinear).

Proposition 3.49 *Every pseudo-strong $\sqcup$-definable*[13] *pseudo-linear triangle algebra is $\sqcup$-definable.*

Proof. Suppose $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a pseudo-linear triangle algebra satisfying $\nu x \sqcup \nu y = (\nu x \Rightarrow \nu y) \Rightarrow \nu y$. Because of (2.1), we have to show that for all x and y in A, $\nu(x \sqcup y) = \nu(((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x))$ and $\mu(x \sqcup y) = \mu(((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x))$. In other words, using Theorem 3.20 and abbreviating $\mu(u * u)$ by t, that

$$\begin{aligned}\max(\nu x, \nu y) = \min\Big(&\min(\nu x \Rightarrow \nu y, \mu x \Rightarrow \mu y) \Rightarrow \nu y,\\ &\min(\nu y \Rightarrow \nu x, \mu y \Rightarrow \mu x) \Rightarrow \nu x,\\ &\min((\mu x * t) \Rightarrow \mu y, \nu x \Rightarrow \mu y) \Rightarrow \mu y,\\ &\min((\mu y * t) \Rightarrow \mu x, \nu y \Rightarrow \mu x) \Rightarrow \mu x\Big)\end{aligned}$$

and

$$\begin{aligned}\max(\mu x, \mu y) = \min\Big(&\big(\min((\mu x * t) \Rightarrow \mu y, \nu x \Rightarrow \mu y) * t\big) \Rightarrow \mu y,\\ &\big(\min((\mu y * t) \Rightarrow \mu x, \nu y \Rightarrow \mu x) * t\big) \Rightarrow \mu x,\\ &\min(\nu x \Rightarrow \nu y, \mu x \Rightarrow \mu y) \Rightarrow \mu y,\\ &\min(\nu y \Rightarrow \nu x, \mu y \Rightarrow \mu x) \Rightarrow \mu x\Big).\end{aligned}$$

Because the diagonal is linearly ordered, we can rewrite the right hand sides of the above identities to

$$\begin{aligned}\min\Big(&\max\big((\nu x \Rightarrow \nu y) \Rightarrow \nu y, (\mu x \Rightarrow \mu y) \Rightarrow \nu y\big),\\ &\max\big((\nu y \Rightarrow \nu x) \Rightarrow \nu x, (\mu y \Rightarrow \mu x) \Rightarrow \nu x\big),\\ &\max\big(((\mu x * t) \Rightarrow \mu y) \Rightarrow \mu y, (\nu x \Rightarrow \mu y) \Rightarrow \mu y\big),\\ &\max\big(((\mu y * t) \Rightarrow \mu x) \Rightarrow \mu x, (\nu y \Rightarrow \mu x) \Rightarrow \mu x\big)\Big)\end{aligned}$$

and

$$\begin{aligned}\min\Big(&\max\big(((\mu x * t) \Rightarrow \mu y) * t) \Rightarrow \mu y, ((\nu x \Rightarrow \mu y) * t) \Rightarrow \mu y\big),\\ &\max\big(((\mu y * t) \Rightarrow \mu x) * t) \Rightarrow \mu x, ((\nu y \Rightarrow \mu x) * t) \Rightarrow \mu x\big),\\ &\max\big((\nu x \Rightarrow \nu y) \Rightarrow \mu y, (\mu x \Rightarrow \mu y) \Rightarrow \mu y\big),\\ &\max\big((\nu y \Rightarrow \nu x) \Rightarrow \mu x, (\mu y \Rightarrow \mu x) \Rightarrow \mu x\big)\Big).\end{aligned}$$

[13] In other words, satisfying $\nu x \sqcup \nu y = (\nu x \Rightarrow \nu y) \Rightarrow \nu y$.

We further reduce the previous expressions to

$$\begin{aligned}\min\Big(&\max(\nu x, \nu y, (\mu x \Rightarrow \mu y) \Rightarrow \nu y),\\ &\max(\nu x, \nu y, (\mu y \Rightarrow \mu x) \Rightarrow \nu x),\\ &\max(\mu x * t, \mu y, \nu x, \mu y),\\ &\max(\mu y * t, \mu x, \nu y, \mu x)\Big)\end{aligned}$$

and

$$\begin{aligned}\min\Bigg(&\max\Big(((\mu x * t) \Rightarrow \mu y) * t) \Rightarrow \mu y, ((\nu x \Rightarrow \mu y) * t) \Rightarrow \mu y\Big),\\ &\max\Big(((\mu y * t) \Rightarrow \mu x) * t) \Rightarrow \mu x, ((\nu y \Rightarrow \mu x) * t) \Rightarrow \mu x\Big),\\ &\max\Big((\nu x \Rightarrow \nu y) \Rightarrow \mu y, \mu x, \mu y\Big),\\ &\max\Big((\nu y \Rightarrow \nu x) \Rightarrow \mu x, \mu x, \mu y\Big)\Bigg).\end{aligned}$$

It is now easy to see that the first expression is indeed $\max(\nu x, \nu y)$: all the maxima are greater than or equal to $\max(\nu x, \nu y)$, and at least one of the first two maxima (depending on $\mu x \leqslant \mu y$ or $\mu y \leqslant \mu x$) is exactly $\max(\nu x, \nu y)$.
In the same way we see that, in the second expression, one of the two last maxima is exactly $\max(\mu x, \mu y)$. The only thing we still need to prove is that the first two maxima are greater than or equal to $\max(\mu x, \mu y)$. This follows from (using the residuation principle)

$$\begin{aligned}\mu x \leqslant \big(((\mu x * t) \Rightarrow \mu y) * t\big) \Rightarrow \mu y \quad &\text{iff} \quad (\mu x * t) * ((\mu x * t) \Rightarrow \mu y) \leqslant \mu y\\ &\text{iff} \quad (\mu x * t) \Rightarrow \mu y \leqslant (\mu x * t) \Rightarrow \mu y\end{aligned}$$

and $\mu y = 1 \Rightarrow \mu y \leqslant ((((\mu x * t) \Rightarrow \mu y) * t) \Rightarrow \mu y)$. □

Corollary 3.50 *Every pseudo-strong ⊔-definable triangle algebra is ⊔-definable.*

Proof. A strong ⊔-definable residuated lattice is always prelinear. Therefore every pseudo-strong ⊔-definable triangle triangle algebra is pseudo-prelinear. So because of Theorem 3.46 a pseudo-strong ⊔-definable triangle algebra is isomorphic to a subdirect product of pseudo-strong ⊔-definable pseudo-linear triangle algebras (which are ⊔-definable, by Proposition 3.49). A subdirect product of ⊔-definable triangle algebras is ⊔-definable. So we can conclude that pseudo-strong ⊔-definable triangle algebras are ⊔-definable. □

Note that in Proposition 3.26 we already showed that conversely, ⊔-definable triangle algebras in which $u * u = u$ are pseudo-strong ⊔-definable (and therefore automatically pseudo-prelinear).

We now summarize several[14] connections between properties on triangle algebras and properties on their diagonals in Table 3.1. Included are results and consequences of Propositions 3.24,

[14]Note that not all connections are listed. For example, we know that weak divisible triangle algebras are always pseudo-divisible, even if they are not pseudo-prelinear. From the table however we can only conclude that pseudo-prelinear weak divisible triangle algebras are pseudo-divisible.

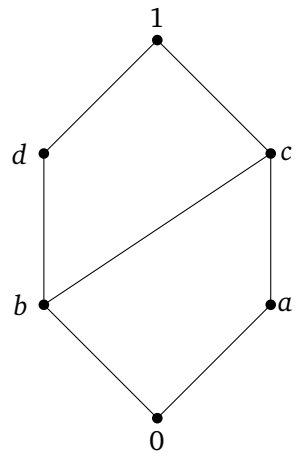

Figure 3.6: LEM is not satisfied in the Heyting-algebra on this lattice since $c \sqcup \neg c = c \sqcup 0 = c \neq 1$.

3.25, 3.26, 3.27, 3.28, 3.29, 3.42, 3.48 and 3.50.
Table 3.1 should be read like this: if the diagonal of a triangle algebra satisfies the given property (mentioned in the first column) and the value of $u * u$ is in the given range (mentioned in the first row), then the triangle algebra satisfies the property at that place in the table. If also the converse holds (for the given value of $u * u$), then the property is underlined. For example (on the fourth row of the table): if the diagonal of a triangle algebra is strong $\sqcup$-definable (in other words, an MV-algebra), then this triangle algebra is $\sqcup$-definable (no matter what the value of $u * u$ is). Conversely, if a triangle algebra is $\sqcup$-definable and $u * u = u$, then its diagonal is an MV-algebra. In this table, P stands for prelinearity, PP for pseudo-prelinearity and LEM for law

Table 3.1: Properties on triangle algebras and their diagonals.

Diagonal	$u * u = 0$	$0 < u * u < u$	$u * u = u$
distributive	<u>distributive</u>	<u>distributive</u>	<u>distributive</u>
P + div	<u>PP + weak div</u>	<u>PP + weak div</u>	<u>PP + weak div</u>
involution	<u>inv</u>	no inv	no inv
pseudocomplementation	no PC	depends on $\neg(u * u)$	<u>PC</u>
MV-algebra	$\sqcup$-def	$\sqcup$-def	<u>$\sqcup$-def</u>
Heyting	not Heyting	not Heyting	<u>Heyting</u>
Heyting + α-LEM	<u>MV</u>	<u>div</u>	<u>Heyting</u>
Boolean algebra	<u>MV</u>	<u>P</u>, not MV	<u>G-algebra</u>, not MV

of excluded middle. The property α-LEM means: for all x on the diagonal, $x \sqcup \neg x \sqcup \alpha = 1$, in which $\alpha = \mu(u * u)$. If $u * u = 0$, this is the same as LEM; if $0 < u * u$, this is weaker than LEM (for $u * u = u$ it is trivially satisfied). Remark that if 1 is $\sqcup$-irreducible (e.g., in linear residuated lattices), then α-LEM is equivalent with LEM if $\alpha < 1$. In this case the residuated lattice has only two elements, as $\neg x = 1$ iff $x = 0$ because of the residuation principle. A residuated lattice with two elements is a Boolean algebra.
An example of a residuated lattice that does not satisfy LEM, but does satisfy α-LEM for some α different from 1, is shown in Figure 3.6.

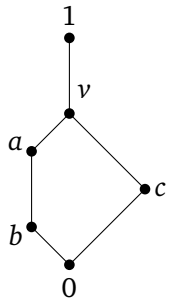

Figure 3.7: This lattice is not distributive, but all residuated lattices on it are weak divisible.

Indeed, if we take the infimum in this lattice[15] as $*$, then the negation $\neg$ is given by $\neg 0 = 1$, $\neg a = d$, $\neg b = a$, $\neg c = 0$, $\neg d = a$ and $\neg 1 = 0$. It is easy to verify that this residuated lattice satisfies d-LEM but not LEM. Moreover, it is also a prelinear Heyting-algebra (in other words, a G-algebra). Therefore the triangle algebra corresponding with this residuated lattice and $\alpha = d$, will be divisible but not prelinear.

Remark 3.51 *Using Proposition 3.29 and the properties in Table 3.1, we can see that weak divisible triangle algebras are distributive. Indeed, weak divisible triangle algebras are pseudo-divisible and therefore pseudo-distributive, which implies they are distributive. Now note that weak divisibility and distributivity are properties that can be imposed on any residuated lattice. Interestingly, not all weak divisible residuated lattices are distributive. Indeed, consider as a counterexample any residuated lattice on the lattice in Figure 3.7 (example 6.1.71 in [50]).*
This example shows that the triangular structure of a triangle algebra can have an influence on its residuated lattice structure. Moreover, it suggests a method to test if a residuated lattice might be endowed with the structure of a triangle algebra or not. If it is weak divisible, but not distributive, it is impossible to equip it with operations such that it becomes a triangle algebra.

[15]This is example 6.1.54(2) in [50].

CHAPTER 4

Interval-valued fuzzy logics

In Chapter 2 we have given the definition of interval-valued residuated lattices (IVRLs). These structures are equipped with a product and implication, and satisfy the properties of residuated lattices. Because residuated lattices constitute the semantics of formal fuzzy logics, IVRLs may be regarded as semantics for a specific kind of formal fuzzy logic. Moreover, IVRLs satisfy another desired property: their elements are intervals. So IVRLs seem the perfect candidates to serve as semantics for interval-valued fuzzy logics (of course, we defined them in this way). However, if we want to define such a logic (with axioms and deduction rules), we need to capture the structure of IVRLs by means of identities and/or inequalities. This is exactly what we did in Chapter 3. Using this characterization, we give the definition of several (propositional) interval-valued fuzzy logics in Section 4.2. But first, in Section 4.1 we give an overview of the well-studied common fuzzy logics, on which our interval-valued fuzzy logics are based, and mention their most important properties. In Section 4.3 we then investigate which of these properties hold for interval-valued fuzzy logics as well. In particular, we prove the soundness and completeness with respect to the algebraic semantics and a local deduction theorem.

4.1 Formal fuzzy logics

In this section, we give an overview of some well-known fuzzy logics. This will help us introducing the terminology. Moreover it will be useful in Section 4.3, to understand the connections between properties of fuzzy logics on the one hand and properties of interval-valued fuzzy logics on the other hand. General references (some of them more general than others) on this topic are [14, 29, 41, 42, 58, 60, 66, 71, 73]. Before we continue, we define what is meant with a formula. Because in interval-valued fuzzy logics there will be more formulae, we make a distinction between formulae for fuzzy logics (FL-formulae) and formulae for interval-valued fuzzy logics (IVFL-formulae).

Definition 4.1 *FL-formulae are built up from a countable set of propositional variables (denoted by p, q, r, p_1, p_2,...) and the constant $\overline{0}$. These symbols are FL-formulae by definition. The other FL-formulae are defined recursively: if φ and ψ are FL-formulae, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \& \psi)$ and $(\varphi \to \psi)$.*
The set of FL-formulae is denoted by $\mathscr{F}_{FL}$.

In order to avoid unnecessary brackets, we agree on the following priority rules:

- among the connectives, & has the highest priority; furthermore $\wedge$ and $\vee$ take precedence over $\rightarrow$,
- the outermost brackets are not written.

The following notations are used: $\overline{1}$ for $\overline{0} \rightarrow \overline{0}$, $\neg\varphi$ for $\varphi \rightarrow \overline{0}$, φ^2 for $\varphi\&\varphi$, φ^n (with $n \in \{3,4,5,\ldots\}$) for $(\varphi^{n-1})\&\varphi$ (moreover, φ^0 is $\overline{1}$ and φ^1 is φ), and $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, for FL-formulae φ and ψ.

The FL-formulae $\varphi\&\psi$, $\varphi \rightarrow \psi$ and $\neg\varphi$ stand for what we understand intuitively by 'φ and ψ' (strong conjunction), 'φ implies ψ' (implication) and 'not φ' (negation).

It is impossible to list all true FL-formulae of a specific fuzzy logic, because this number is not finite. Therefore axioms and deduction rules are used. In the logics we deal with, an FL-formula is true if it is provable from the axioms using the deduction rules. We will explain this in more detail. This method also allows to prove FL-formulae from a given set of FL-formulae (usually called a theory). This means that in the proof of an FL-formula not only axioms of the logic can be used but also formulae of the theory. If an FL-formula φ is provable from a theory Γ in a fuzzy logic L, this is denoted as $\Gamma \vdash_L \varphi$. The relation $\vdash$ is called provability relation or syntactic consequence.

It is often not very easy to find out if an FL-formula is true in a specific fuzzy logic. A proof might be difficult to find and such a proof can become very long. This is why soundness and completeness of (fuzzy) logics is so important. It provides a way to determine if a formula is true or provable from a theory in a purely algebraic way. Indeed, soundness and completeness of a fuzzy logic are two properties relative to a class of algebraic structures. We call such a class a semantics of the fuzzy logic. To explain this connection between formal logic and algebra in more detail, we need some terminology first.

Definition 4.2 *Let $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice, Γ a theory (i.e., a set of FL-formulae). An $\mathscr{L}$-evaluation is a mapping e from the set of FL-formulae*[1] *to L that satisfies, for each two formulae φ and ψ:*

- $e(\varphi \wedge \psi) = e(\varphi) \sqcap e(\psi)$,
- $e(\varphi \vee \psi) = e(\varphi) \sqcup e(\psi)$,
- $e(\varphi\&\psi) = e(\varphi) * e(\psi)$,
- $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$ *and*
- $e(\overline{0}) = 0$.

If an $\mathscr{L}$-evaluation e satisfies $e(\chi) = 1$ for every χ in Γ, it is called an $\mathscr{L}$-model[2] *for Γ.*
We write $\Gamma \models_{\mathscr{L}} \varphi$ if $e(\varphi) = 1$ for all $\mathscr{L}$-models e for Γ. If Γ is empty, we simply write $\models_{\mathscr{L}} \varphi$ instead of $\varnothing \models_{\mathscr{L}} \varphi$. FL-formulae φ for which $\models_{\mathscr{L}} \varphi$ are called $\mathscr{L}$-tautologies.

The relation $\models$ is called semantic consequence.

Evaluations form a connection between the connectives of the logic and the algebraic operators in residuated lattices. Note that $e(\overline{1}) = e(\overline{0} \rightarrow \overline{0}) = e(\overline{0}) \Rightarrow e(\overline{0}) = 0 \Rightarrow 0 = 1$ and $e(\neg\varphi) = e(\varphi \rightarrow \overline{0}) = e(\varphi) \Rightarrow e(\overline{0}) = \neg e(\varphi)$.

Now let $\mathscr{C}$ be a class of residuated lattices and L a fuzzy logic.

[1] Note that an $\mathscr{L}$-evaluation is completely determined by its action on the propositional variables.

[2] Note that $\mathscr{L}$-models for the empty set are just $\mathscr{L}$-evaluations.

- We say L is sound w.r.t. $\mathscr{C}$ if for all $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, $\Gamma \vdash_L \varphi$ implies $\Gamma \models_{\mathscr{L}} \varphi$ for all $\mathscr{L}$ in $\mathscr{C}$.
- We say L is complete[3] w.r.t. $\mathscr{C}$ if for all $\varphi \in \mathscr{F}_{FL}$, ($\models_{\mathscr{L}} \varphi$ for all $\mathscr{L}$ in $\mathscr{C}$) implies $\vdash_L \varphi$.
- We say L is strong complete w.r.t. $\mathscr{C}$ if for all $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, ($\Gamma \models_{\mathscr{L}} \varphi$ for all $\mathscr{L}$ in $\mathscr{C}$) implies $\Gamma \vdash_L \varphi$.

We will illustrate these definitions in the following subsections.

4.1.1 Monoidal logic

Monoidal logic (ML) was introduced by Höhle in [44]. Its axioms[4] are:

1. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
2. $\varphi \to (\varphi \vee \psi)$,
3. $\psi \to (\varphi \vee \psi)$,
4. $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$,
5. $(\varphi \wedge \psi) \to \varphi$,
6. $(\varphi \wedge \psi) \to \psi$,
7. $(\varphi \& \psi) \to \varphi$,
8. $(\varphi \& \psi) \to (\psi \& \varphi)$,
9. $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi)))$,
10. $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$,
11. $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$ and
12. $\overline{0} \to \varphi$.

This means that for all possible choices of FL-formulae for φ, ψ and χ, the above FL-formulae are provable in ML. For example, $(\overline{0} \to q) \to ((p \& q) \vee (\overline{0} \to q))$ is provable in ML, because it is an instance[5] of the third axiom, with $\psi = (\overline{0} \to q)$ and $\varphi = p \& q$.

[3]For every logic L appearing in this work, completeness w.r.t. a class $\mathscr{C}$ of residuated lattices (or, in the next sections, triangle algebras) implies that, for every finite theory $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, $\Gamma \vdash_L \varphi$ if $\Gamma \models_{\mathscr{L}} \varphi$ for all $\mathscr{L}$ in $\mathscr{C}$. In other words, completeness implies 'strong completeness for finite theories'.

[4]In [44] there are two axioms instead of the last one. In these two axioms the negation $\neg$ appears (as a unary connective), but not the constant $\overline{0}$. In this work we have chosen another way, namely to define the negation based on the constant $\overline{0}$ (instead of the other way around).
Moreover, also $(\varphi \& (\psi \& \chi)) \to ((\varphi \& \psi) \& \chi)$ was listed as an axiom. But FL-formulae of this form can be proven from the other axioms, so it can be left out.
Some of these axioms are referred to by a specific name. In [44], ML.1 is called 'syllogism law', while Hájek uses 'transitivity of implication' in [42]. Other names in [42] are 'commutativity of &-conjunction' for ML.8, 'ex falso quodlibet' for ML.12 and 'residuation' for the combination of ML.10 and ML.11 (which are called 'importation law' and 'exportation law' in [44]).

[5]An instance of an axiom of ML is any FL-formula obtained by replacing φ, ψ and χ with FL-formulae. For example, $((p_1 \to (p_2 \vee \overline{0})) \& (p_2 \wedge q)) \to (p_1 \to (p_2 \vee \overline{0}))$ is an instance of $(\varphi \& \psi) \to \varphi$.

To show that other FL-formulae are provable in ML, ML has one deduction rule. This deduction rule is called modus ponens (MP) and states that if φ and $\varphi \to \psi$ are provable then so is ψ. Now we can formally define what a proof in ML of an FL-formula φ from a theory Γ is: it is a finite sequence of FL-formulae in which every FL-formula is either an instance of an axiom of ML, an element of Γ or the result of an application of the modus ponens to two FL-formulae appearing earlier in the sequence. If such a proof exists, this is denoted as $\Gamma \vdash_{ML} \varphi$. If Γ is empty, we simply write $\vdash_{ML} \varphi$ instead of $\varnothing \vdash_{ML} \varphi$. An important result about ML is its soundness and completeness w.r.t. residuated lattices.

Theorem 4.3 *[44] Monoidal logic is sound and strong complete w.r.t. residuated lattices. In other words: for all $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, we have $\Gamma \vdash_{ML} \varphi$ iff $\Gamma \models_{\mathscr{L}} \varphi$ for all residuated lattices $\mathscr{L}$.*

As an application of this theorem, we show that the FL-formula $(p \to q) \vee (q \to p)$ is not provable in ML. Indeed, take any residuated lattice that is not prelinear, e.g. the Heyting-algebra $\mathscr{L}$ on the lattice in Figure 2.6. Consider the $\mathscr{L}$-evaluation e determined by $e(p) = a$, $e(q) = b$ and $e(s) = 0$ for all other[6] propositional variables s. Then $e((p \to q) \vee (q \to p)) = (e(p) \Rightarrow e(q)) \sqcup (e(q) \Rightarrow e(p)) = (a \Rightarrow b) \sqcup (b \Rightarrow a) = b \sqcup a = v \neq 1$. So e is an $\mathscr{L}$-model of the empty set, but not of $\{(p \to q) \vee (q \to p)\}$. Therefore $\not\models_{\mathscr{L}} (p \to q) \vee (q \to p)$. The soundness of ML now implies that $\nvdash_{ML} (p \to q) \vee (q \to p)$.
This does not mean that any FL-formula of the form $(\varphi \to \psi) \vee (\psi \to \varphi)$ is not provable in ML. We show this by verifying that any FL-formula of the form $(\varphi \to (\varphi \& \chi)) \vee ((\varphi \& \chi) \to \varphi)$ is provable in ML. Indeed, for any residuated lattice $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ and any $\mathscr{L}$-evaluation e we have $e((\varphi \to (\varphi \& \chi)) \vee ((\varphi \& \chi) \to \varphi)) = (e(\varphi) \Rightarrow (e(\varphi) * e(\chi))) \sqcup ((e(\varphi) * e(\chi)) \Rightarrow e(\varphi)) = (e(\varphi) \Rightarrow (e(\varphi) * e(\chi))) \sqcup 1 = 1$ and therefore $\models_{\mathscr{L}} (\varphi \to (\varphi \& \chi)) \vee ((\varphi \& \chi) \to \varphi)$ for all residuated lattices $\mathscr{L}$. By the standard completeness of ML, we find $\vdash_{ML} (\varphi \to (\varphi \& \chi)) \vee ((\varphi \& \chi) \to \varphi)$.
Similarly each identity or inequality that is valid in residuated lattices can easily be transformed into a scheme[7] of FL-formulae that are provable in ML. Therefore we just need to change such an identity or inequality to an equivalent 'equal to 1 identity'.

- An example with an identity: $x \Rightarrow (y \sqcap z) = (x \Rightarrow y) \sqcap (x \Rightarrow z)$ holds in all residuated lattices. This is equivalent with $(x \Rightarrow (y \sqcap z)) \Leftrightarrow ((x \Rightarrow y) \sqcap (x \Rightarrow z)) = 1$, which can be immediately transformed into the scheme $(\varphi \to (\psi \wedge \chi)) \leftrightarrow ((\varphi \to \psi) \wedge (\varphi \to \chi))$ of FL-formulae that are provable in ML.

- An example with an inequality: $x * y \leqslant x \sqcap y$ holds in all residuated lattices. This is equivalent with $(x * y) \Rightarrow (x \sqcap y) = 1$, which can be transformed into the scheme $(\varphi \& \psi) \to (\varphi \wedge \psi)$ of FL-formulae that are provable in ML.

So there is a close connection between identities (and inequalities) in residuated lattices and (schemes of) FL-formulae that are provable in ML.

ML enjoys a so-called local deduction theorem:

Theorem 4.4 *[42] Let $\Gamma \cup \{\varphi, \psi\}$ be a set of FL-formulae. Then the following are equivalent:*

- $\Gamma \cup \{\varphi\} \vdash_{ML} \psi$,

[6] Note that $e(s)$ can be chosen arbitrarily for all propositional variables s different from p or q.

[7] A scheme of FL-formulae consists of all FL-formulae of a particular form (the instances of the scheme). Note that the axioms of ML are schemes.

- *there is an integer n such that* $\Gamma \vdash_{ML} \varphi^n \to \psi$.

This local deduction theorem, as well as the soundness and completeness of ML, remain valid in axiomatic extensions of ML. An axiomatic extension of ML is a logic having the same axioms and deduction rule as ML, plus one or more other axioms. Axiomatic extensions of ML are sound and complete w.r.t. residuated lattices that satisfy the identities corresponding to the extra axioms. A first example of such an axiomatic extension of ML is MTL.

4.1.2 Monoidal t-norm based logic

Monoidal t-norm based logic (MTL) [28] is an axiomatic extension of ML. The extra axiom is $(\varphi \to \psi) \vee (\psi \to \varphi)$. This axiom corresponds with the identity $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ (prelinearity) in residuated lattices. Therefore we have the following soundness and completeness theorem.

Theorem 4.5 *[28] Monoidal t-norm based logic is sound and strong complete w.r.t. prelinear residuated lattices (MTL-algebras). In other words: for all* $\Gamma \subseteq \mathscr{F}_{FL}$ *and* $\varphi \in \mathscr{F}_{FL}$*, we have* $\Gamma \vdash_{MTL} \varphi$ *iff* $\Gamma \models_{\mathscr{L}} \varphi$ *for all MTL-algebras* $\mathscr{L}$.

The provability relation $\vdash_{MTL}$ is defined in the same way as $\vdash_{ML}$, only now also instances of the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ may appear in proofs of FL-formulae.
Because MTL-algebras are isomorphic to subdirect products of MTL-chains, the strong completeness can be strengthened to so-called (strong) chain completeness, i.e., (strong) completeness w.r.t. MTL-chains.

Theorem 4.6 *[28] Monoidal t-norm based logic is sound and strong chain complete. In other words: for all* $\Gamma \subseteq \mathscr{F}_{FL}$ *and* $\varphi \in \mathscr{F}_{FL}$*, we have* $\Gamma \vdash_{MTL} \varphi$ *iff* $\Gamma \models_{\mathscr{L}} \varphi$ *for all MTL-chains* $\mathscr{L}$.

Chain completeness and strong chain completeness are properties that remain valid for axiomatic extensions of MTL. Important theorems about MTL that do not necessarily remain valid for axiomatic extensions, are standard completeness and strong standard completeness. Recall that standard MTL-algebras are MTL-algebras on the unit interval.

Theorem 4.7 *[49] Monoidal t-norm based logic is sound and strong standard complete. In other words: for all* $\Gamma \subseteq \mathscr{F}_{FL}$ *and* $\varphi \in \mathscr{F}_{FL}$*, we have* $\Gamma \vdash_{MTL} \varphi$ *iff* $\Gamma \models_{\mathscr{L}} \varphi$ *for all standard MTL-chains* $\mathscr{L}$.

Because of this theorem, and because standard MTL-chains are induced by left-continuous t-norms, we can say that MTL is 'the logic of left-continuous t-norms'.

In MTL it is possible to restrict the set of formulae to FL-formulae that do not contain the connective $\vee$. This is due to the $\sqcup$-definability of MTL-algebras. It can be shown that the following axioms[8], together with the modus ponens, suffice to describe MTL [28]:

1. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
2. $(\varphi \wedge \psi) \to \varphi$,
3. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$,

[8] Originally also $(\varphi \& \psi) \to (\psi \& \varphi)$ was included as an axiom, but it can be proven from the other axioms. Therefore it is not necessary and can be left out.

4. $(\varphi \& \psi) \to \varphi$,
5. $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$,
6. $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$,
7. $\overline{0} \to \varphi$,
8. $(\varphi \& (\varphi \to \psi)) \to (\varphi \wedge \psi)$ and
9. $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$.

As can be seen, the connective $\vee$ does not appear in these axioms. However, if we define $\varphi \vee \psi$ as a notation for $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, then the provability relation obtained from this axiom system is exactly $\vdash_{MTL}$. So this system is equivalent with the axiom system of MTL.

MTL enjoys the same local deduction theorem as ML:

Theorem 4.8 *[42] Let $\Gamma \cup \{\varphi, \psi\}$ be a set of FL-formulae. Then the following are equivalent:*

- $\Gamma \cup \{\varphi\} \vdash_{MTL} \psi$,
- *there is an integer n such that $\Gamma \vdash_{MTL} \varphi^n \to \psi$.*

4.1.3 Basic logic

Basic logic (BL) [42] is an axiomatic extension of MTL (and therefore also of ML). The extra axiom is $(\varphi \wedge \psi) \to (\varphi \& (\varphi \to \psi))$. This axiom corresponds with the identity $(x \sqcap y) \Rightarrow (x * (x \Rightarrow y)) = 1$ in residuated lattices, in other words $x \sqcap y \leqslant x * (x \Rightarrow y)$. This is equivalent with $x \sqcap y = x * (x \Rightarrow y)$ (divisibility) because $x \sqcap y \geqslant x * (x \Rightarrow y)$ is valid in all residuated lattices. Because BL is an axiomatic extension of MTL and because divisible MTL-algebras are BL-algebras, we immediately have the following soundness and completeness result.

Theorem 4.9 *[42] Basic logic is sound and strong chain complete. In other words: for all $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, we have $\Gamma \vdash_{BL} \varphi$ iff $\Gamma \models_{\mathscr{L}} \varphi$ for all BL-chains $\mathscr{L}$.*

Note that BL is also (strong) complete w.r.t. all BL-algebras, which is a weaker property than (strong) chain completeness. BL also satisfies standard completeness (which is stronger than chain completeness), but NOT strong standard completeness. This implies that the following theorem in general does not hold for infinite theories Γ.

Theorem 4.10 *[13] For all finite $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, we have $\Gamma \vdash_{BL} \varphi$ iff $\Gamma \models_{\mathscr{L}} \varphi$ for all standard BL-chains $\mathscr{L}$.*

Because standard BL-chains are induced by continuous t-norms, we can say BL is 'the logic of continuous t-norms'.

Similarly as for MTL, we can give an axiom system [42] for BL with a restricted set of formulae (namely the set of FL-formulae not containing the connectives $\vee$ or $\wedge$) that is equivalent with the axiom system for BL given above:

1. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
2. $(\varphi \& \psi) \to \varphi$,

3. $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$,
4. $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$,
5. $\overline{0} \to \varphi$,
6. $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$ and
7. $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$.

The connective $\vee$ is then defined as in MTL ($\varphi \vee \psi$ is a notation for $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi))$ and $\varphi \wedge \psi$ is a notation for $\varphi \& (\varphi \to \psi)$.
The local deduction theorem is also valid for BL.

Theorem 4.11 *[42] Let $\Gamma \cup \{\varphi, \psi\}$ be a set of FL-formulae. Then the following are equivalent:*

- $\Gamma \cup \{\varphi\} \vdash_{BL} \psi$,
- *there is an integer n such that* $\Gamma \vdash_{BL} \varphi^n \to \psi$.

4.1.4 Łukasiewicz logic

Łukasiewicz logic (Ł) is an axiomatic extension of BL (and therefore also of MTL and ML). The extra axiom is $\neg\neg\varphi \to \varphi$, in which $\neg\varphi$ is a short notation for $\varphi \to \overline{0}$. This axiom corresponds with the identity $\neg\neg x \Rightarrow x = 1$ in residuated lattices, in other words $\neg\neg x \leqslant x$. This is equivalent with $\neg\neg x = x$ (involutive negation) because $x \leqslant \neg\neg x$ is valid in all residuated lattices. Because Ł is an axiomatic extension[9] of MTL, we immediately have the following soundness and completeness result.

Theorem 4.12 *[42] Łukasiewicz logic is sound and strong chain complete. In other words: for all $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, we have $\Gamma \vdash_{Ł} \varphi$ iff $\Gamma \models_{Ł} \varphi$ for all MV-chains $\mathscr{L}$.*

Note that Ł is also (strong) complete w.r.t. all MV-algebras, which is a weaker property than (strong) chain completeness. Ł also satisfies standard completeness (which is stronger than chain completeness), but NOT strong standard completeness. This implies that the following theorem in general does not hold for infinite theories.

Theorem 4.13 *[10, 42] For all finite $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, we have $\Gamma \vdash_{Ł} \varphi$ iff $\Gamma \models_{\mathscr{L}} \varphi$ for all standard MV-chains $\mathscr{L}$.*

Because standard MV-chains are induced by t-norms that are conjugated to the Łukasiewicz t-norm, we can say that Ł is 'the logic of the Łukasiewicz t-norm'.

Similarly as for MTL and BL, we can give an axiom system for Ł with a restricted set of formulae (namely the set of FL-formulae not containing the connectives $\vee$, $\wedge$ or &) that is equivalent with the axiom system for Ł given above:

1. $\varphi \to (\psi \to \varphi)$,
2. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

[9] Using the soundness and completeness of ML and the fact that MV-algebras are exactly strong $\sqcup$-definable residuated lattices, we can see that Łukasiewicz logic can also be obtained as ML extended with the axiom $((\varphi \to \psi) \to \psi) \to (\varphi \vee \psi)$.

3. $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$ and
4. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

in which $\neg\varphi$ is defined as usual: $\varphi \to \overline{0}$. Other notations are $\varphi \vee \psi$ for $(\varphi \to \psi) \to \psi$, $\varphi \wedge \psi$ for $\neg(\neg\varphi \vee \neg\psi)$ and $\varphi \& \psi$ for $\neg(\varphi \to \neg\psi)$.
The local deduction theorem is also valid for Ł.

Theorem 4.14 *[42] Let $\Gamma \cup \{\varphi, \psi\}$ be a set of FL-formulae. Then the following are equivalent:*

- $\Gamma \cup \{\varphi\} \vdash_{Ł} \psi$,
- *there is an integer n such that* $\Gamma \vdash_{Ł} \varphi^n \to \psi$.

4.1.5 Classical logic

Classical logic (CPC[10]) is an axiomatic extension of Ł (and therefore also of BL, MTL and ML). The extra axiom is $\varphi \to (\varphi \& \varphi)$. This axiom corresponds with the identity $x \Rightarrow (x * x) = 1$ in residuated lattices, in other words $x \leqslant x * x$. This is equivalent with $x = x * x$ (contraction) because $x * x \leqslant x$ is valid in all residuated lattices. Because CPC is an axiomatic extension[11] of MTL, we immediately have the following soundness and completeness result.

Theorem 4.15 *[69] Classical logic is sound and strong chain complete. In other words: for all $\Gamma \subseteq \mathscr{F}_{FL}$ and $\varphi \in \mathscr{F}_{FL}$, we have $\Gamma \vdash_{CPC} \varphi$ iff $\Gamma \models_{\mathscr{L}} \varphi$ for all linear Boolean algebras $\mathscr{L}$.*

Note that there is only one linear Boolean algebra (apart from the trivial one with one element), namely the Boolean algebra with two elements, 0 and 1. Therefore CPC cannot satisfy standard completeness, in the sense that CPC is not complete w.r.t. Boolean algebras on the unit interval (because there are no such Boolean algebras).

Similarly as for MTL, BL and Ł, we can give an axiom system for CPC with a restricted set of formulae (namely the set of FL-formulae not containing the connectives $\vee$, $\wedge$ or &) that is equivalent with the axiom system given above.

1. $\varphi \to (\psi \to \varphi)$,
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$, and
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

in which $\neg\varphi$ is defined as usual: $\varphi \to \overline{0}$. Other notations are $\varphi \& \psi$ and $\varphi \wedge \psi$ for $\neg(\varphi \to \neg\psi)$ and $\varphi \vee \psi$ for $\neg\varphi \to \psi$.
The local deduction theorem is also valid for CPC. But because in CPC the FL-formulae φ and φ^n (with n a strictly positive integer) are equivalent (meaning $\vdash_{CPC} \varphi \leftrightarrow \varphi^n$ holds[12]), the local deduction theorem can be strengthened to the deduction theorem.

[10] The abbreviation, taken from [14], stands for 'classical propositional calculus'.

[11] Using the soundness and completeness of ML and the fact that Boolean algebras are exactly residuated lattices satisfying the law of excluded middle, we can see that CPC can also be obtained as ML extended with the axiom $\varphi \vee \neg\varphi$ (tertium non datur).

[12] In the logics we are concerned with in this work, we have the following property. If a subformula of a formula is replaced by an equivalent subformula, the resulting formula is equivalent with the original one. This can be proven using soundness and completeness. For example, in ML and its axiomatic extensions $(p_1 \& p_2) \to q$ is equivalent with $(p_2 \& p_1) \to q$ because the subformulae $p_1 \& p_2$ and $p_2 \& p_1$ are equivalent.

Theorem 4.16 *[69] Let $\Gamma \cup \{\varphi, \psi\}$ be a set of FL-formulae. Then the following are equivalent:*

- $\Gamma \cup \{\varphi\} \vdash_{CPC} \psi$,
- $\Gamma \vdash_{CPC} \varphi \to \psi$.

4.1.6 Other fuzzy logics

Apart from the examples in the previous subsections, many other fuzzy logics can be defined by adding axioms to ML, MTL,... We list some examples (see, e.g., [14]).

- Gödel logic (G) is MTL extended with the axiom contraction $\varphi \to (\varphi \& \varphi)$.
- Weak nilpotent minimum logic (WNM) is MTL extended with the axiom weak nilpotent minimum $\neg(\varphi \& \psi) \vee ((\varphi \wedge \psi) \to (\varphi \& \psi))$.
- Involutive monoidal t-norm based logic (IMTL) is MTL extended with the axiom involution $\neg\neg\varphi \to \varphi$.
- Nilpotent minimum logic (NM) is WNM extended with the axiom $\neg\neg\varphi \to \varphi$.
- Strict monoidal t-norm based logic (SMTL) is MTL extended with the axiom pseudocomplementation $\neg(\varphi \wedge \neg\varphi)$.
- Weak cancellation monoidal t-norm based logic (WCMTL) is MTL extended with the axiom weak cancellation $\neg(\varphi \& \psi) \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$.
- ΠMTL is MTL extended with the axiom cancellation $\neg\varphi \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$.
- Product logic (Π) is BL extended with the axiom $\neg\varphi \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$.
- Strict basic logic (SBL) is BL extended with the axiom $\neg(\varphi \wedge \neg\varphi)$.

As all of these fuzzy logics are axiomatic extensions of MTL, each of them enjoys soundness and strong chain completeness (for example: Gödel logic is sound and strong complete w.r.t. Heyting-chains) and the local deduction theorem (similarly as ML, MTL, BL and Ł). Because of contraction, Gödel logic even satisfies the same deduction theorem as CPC.
It has been shown that all of the fuzzy logics listed above in this subsection, are standard complete. Moreover, all of them are finite strong standard complete[13]. However, not all of them are strong standard complete. The fuzzy logics G, WNM, IMTL, NM and SMTL are strong standard complete, but the other ones are not [27, 28, 42, 45, 46, 49, 59]. In [14] it was proven that an axiomatic extension of MTL is strong standard complete iff it has the real-chain embedding property, i.e., iff each countable MTL-chain in its semantics is embeddable in a standard MTL-chain in its semantics. For example: IMTL is strong standard complete (i.e. strong complete w.r.t. standard IMTL-algebras[14]) because each countable IMTL-chain is embeddable in a standard IMTL-chain (the semantics of IMTL is the variety of IMTL-algebras). Similarly, an axiomatic extension of MTL is finite strong standard complete iff it has the real-chain partial embedding property, i.e., iff each MTL-chain in its semantics is partially embeddable in a standard MTL-chain in its semantics. For

[13] Finite strong standard completeness means that the completeness w.r.t. standard algebras holds for finite theories, such as in Theorems 4.10 and 4.13.

[14] An IMTL-algebra is an MTL-algebra in which the negation is an involution.

example, BL is finite strong standard complete and therefore has the real-chain partial embedding property. This means that each BL-chain $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0_L, 1_L)$ is partially embeddable in a standard BL-chain, in other words, for each finite subset X of L, there is a standard BL-algebra $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$ and an injection i from X into $[0,1]$ such that $i(0_L) = 0$ (if $0_L \in X$), $i(1_L) = 1$ (if $1_L \in X$), $i(a \sqcap b) = \min(i(a), i(b))$ (if $a, b \in X$), $i(a * b) = i(a) \circ i(b)$ (if $a, b, a * b \in X$) and $i(a \Rightarrow b) = i(a) \Rightarrow_\circ i(b)$ (if $a, b, a \Rightarrow b \in X$).

4.2 Definition and elementary properties

As semantics of interval-valued fuzzy logics, we choose triangle algebras. Because triangle algebras have more operators than residuated lattices, IVFL-formulae can contain more connectives than FL-formulae.

Definition 4.17 *IVFL-formulae are built up from a countable set of propositional variables (denoted by p, q, r, p_1, p_2,...) and the constants $\overline{0}$ and $\overline{u}$. These symbols are IVFL-formulae by definition. The other IVFL-formulae are defined recursively: if φ and ψ are IVFL-formulae, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \& \psi)$, $(\varphi \to \psi)$, $\Box\varphi$ and $\Diamond\varphi$.*
The set of IVFL-formulae is denoted by $\mathscr{F}_{IVFL}$. Note that $\mathscr{F}_{FL} \subseteq \mathscr{F}_{IVFL}$.

In order to avoid unnecessary brackets, we agree on the following priority rules:

- unary operators always take precedence over binary ones, while
- among the connectives, & has the highest priority; furthermore $\wedge$ and $\vee$ take precedence over $\to$,
- the outermost brackets are not written.

The same notations ($\overline{1}$ is $\overline{0} \to \overline{0}$, ...) as for FL-formulae are used. Now we are ready to introduce interval-valued monoidal logic (IVML). Its axioms are those of ML, i.e.,

$$
\begin{array}{ll}
(ML.1) & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)), \\
(ML.2) & \varphi \to (\varphi \vee \psi), \\
(ML.3) & \psi \to (\varphi \vee \psi), \\
(ML.4) & (\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)), \\
(ML.5) & (\varphi \wedge \psi) \to \varphi, \\
(ML.6) & (\varphi \wedge \psi) \to \psi, \\
(ML.7) & (\varphi \& \psi) \to \varphi, \\
(ML.8) & (\varphi \& \psi) \to (\psi \& \varphi), \\
(ML.9) & (\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi))), \\
(ML.10) & (\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi), \\
(ML.11) & ((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)), \\
(ML.12) & \overline{0} \to \varphi,
\end{array}
$$

complemented with axioms corresponding to the 13 properties of triangle algebras that we listed in Section 3.1

		$(IVML.1')$	$\varphi \to \Diamond\varphi$,
$(IVML.2)$	$\Box\varphi \to \Box\Box\varphi$,		
$(IVML.3)$	$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$,	$(IVML.3')$	$(\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \psi)$,
$(IVML.4)$	$\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$,	$(IVML.4')$	$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$,
$(IVML.5)$	$\neg\Box\overline{u}$,	$(IVML.5')$	$\Diamond\overline{u}$,
$(IVML.6)$	$\Diamond\varphi \to \Box\Diamond\varphi$,	$(IVML.6')$	$\Diamond\Box\varphi \to \Box\varphi$,
$(IVML.7)$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,		
$(IVML.9)$	$(\Box\varphi \to \Box\psi) \to \Box(\Box\varphi \to \Box\psi)$,		
$(IVML.10)$	$\varphi \leftrightarrow (\Box\varphi \vee (\Diamond\varphi \wedge \overline{u}))$.		

All instances of these axioms are by definition provable in IVML. To determine which other IVFL-formulae are provable, there are three deduction rules: modus ponens (MP, if φ and $\varphi \to \psi$ are provable in IVML, then so is ψ), generalization[15] (G, if φ is provable in IVML, then so is $\Box\varphi$) and monotonicity of $\Diamond$ (M$\Diamond$, if $\varphi \to \psi$ is provable, then so is $\Diamond\varphi \to \Diamond\psi$). Proofs in IVML and the provability relation $\vdash_{IVML}$ are defined in the usual way, similarly as for ML (and the other fuzzy logics from Section 4.1). If Γ is a theory, i.e., a set of IVFL-formulae, then a (formal) proof of an IVFL-formula φ in Γ is a finite sequence of IVFL-formulae with φ at its end, such that every IVFL-formula in the sequence is either an instance of an axiom of IVML, an IVFL-formula of Γ, or the result of an application of a deduction rule to previous IVFL-formulae in the sequence. If a proof for φ exists in Γ, we denote this by $\Gamma \vdash_{IVML} \varphi$.
For a theory Γ, and IVFL-formulae φ and ψ, denote $\varphi \sim_{IVML,\Gamma} \psi$ iff $\Gamma \vdash_{IVML} \varphi \to \psi$ and $\Gamma \vdash_{IVML} \psi \to \varphi$ (this is also equivalent with $\Gamma \vdash_{IVML} \varphi \leftrightarrow \psi$).

Definition 4.18 *Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra, Γ a theory (i.e., a set of IVFL-formulae). An $\mathscr{A}$-evaluation is a mapping e from the set of IVFL-formulae to A that satisfies, for each two IVFL-formulae φ and ψ:*

- $e(\varphi \wedge \psi) = e(\varphi) \sqcap e(\psi)$,
- $e(\varphi \vee \psi) = e(\varphi) \sqcup e(\psi)$,
- $e(\varphi \& \psi) = e(\varphi) * e(\psi)$,
- $e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi)$,
- $e(\Box\varphi) = \nu e(\varphi)$,
- $e(\Diamond\varphi) = \mu e(\varphi)$,
- $e(\overline{0}) = 0$ *and*
- $e(\overline{u}) = u$.

If an $\mathscr{A}$-evaluation e satisfies $e(\chi) = 1$ for every χ in Γ, it is called an $\mathscr{A}$-model for Γ.
We write $\Gamma \models_{\mathscr{A}} \varphi$ if $e(\varphi) = 1$ for all $\mathscr{A}$-models e for Γ.

Soundness, completeness and strong completeness are defined similarly as for formal fuzzy logics. We just have to replace 'residuated lattice' by 'triangle algebra' and 'FL-formula' by 'IVFL-formula'.

[15] Generalization is often called necessitation, e.g. in [83]

4.3 Soundness and completeness

4.3.1 Soundness

It is easy to check that IVML is sound w.r.t. the variety of triangle algebras, i.e., that if an IVFL-formula φ can be proven from a theory Γ in IVML ($\Gamma \vdash_{IVML} \varphi$), then for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$ (in other words: for every triangle algebra $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$).

Theorem 4.19 *IVML is sound w.r.t. triangle algebras.*

Proof. We need to verify the soundness of the new axioms and deduction rules of IVML (for the axioms and rules of ML, the proofs (in ML) can be copied, because triangle algebras are expansions of residuated lattices and therefore identities that hold in residuated lattices also hold in triangle algebras). For the axioms this is easy, as they are straightforward generalizations of defining properties of triangle algebras. As an example, we show the soundness of (IVML.5): for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of a theory Γ, $e(\neg\Box\overline{u}) = \neg\nu u = \neg 0 = 1$. We will now verify the soundness of the new deduction rules.

- Generalization. We need to prove: if for all triangle algebras $\mathscr{A}$ and all $\mathscr{A}$-models e for Γ, $e(\varphi) = 1$, then for all triangle algebras $\mathscr{A}$ and all $\mathscr{A}$-models e for Γ, $e(\Box\varphi) = 1$. Take such a triangle algebra $\mathscr{A}_0$ and such an $\mathscr{A}_0$-model e_0. Then $e_0(\varphi) = 1$, and thus $e_0(\Box\varphi) = \nu e_0(\varphi) = \nu 1 = 1$.

- Monotonicity of $\Diamond$. We need to prove: if for all triangle algebras $\mathscr{A}$ and all $\mathscr{A}$-models e for Γ, $e(\varphi \to \psi) = 1$, then for all triangle algebras $\mathscr{A}$ and all $\mathscr{A}$-models e for Γ, $e(\Diamond\varphi \to \Diamond\psi) = 1$. Take such a triangle algebra $\mathscr{A}_0$ and $\mathscr{A}_0$-model e_0. Then $e_0(\varphi) \Rightarrow e_0(\psi) = e_0(\varphi \to \psi) = 1$, which means $e_0(\varphi) \leqslant e_0(\psi)$. Then $e_0(\Diamond\varphi) = \mu e_0(\varphi) \leqslant \mu e_0(\psi) = e_0(\Diamond\psi)$ because μ is increasing, and thus $e_0(\Diamond\varphi \to \Diamond\psi) = e_0(\Diamond\varphi) \Rightarrow e_0(\Diamond\psi) = 1$.

By the definition of a proof in IVML we can now see that if an IVFL-formula φ can be proven from a theory Γ in IVML, then for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$. In other words, IVML is sound. □

4.3.2 General and strong general completeness

To show that IVML is also strong complete (w.r.t. triangle algebras), i.e., that the converse of soundness also holds, we apply a general result from abstract algebraic logic (shortly AAL, see e.g. [33] for a survey). We start by showing that IVML is an implicative logic (in the sense of Rasiowa [66]). An implicative logic is a logic (defined by its logical language (the different connectives) and provability relation) in which a connective $\to$ exists in the logical language that satisfies:

- $\Gamma \vdash \varphi \to \varphi$,
- $\Gamma \vdash \varphi \to \chi$ if $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$,
- $\Gamma \vdash \psi$ if $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \varphi$,
- $\Gamma \vdash \psi \to \varphi$ if $\Gamma \vdash \varphi$, and

- $\sim_\Gamma$ is a congruence w.r.t. every connective in the logical language,

for any theory Γ and for any formulae φ, ψ and χ.
Most of these properties hold trivially for IVML as they hold for ML [44] (and the axioms of IVML include those of ML). We only need to prove that $\sim_{IVML,\Gamma}$ is also a congruence w.r.t. $\Box$ and $\Diamond$. Indeed, if $\Gamma \vdash_{IVML} \varphi \to \psi$, then by M$\Diamond$ $\Gamma \vdash_{IVML} \Diamond\varphi \to \Diamond\psi$. So if $\varphi \sim_{IVML,\Gamma} \psi$, then $\Diamond\varphi \sim_{IVML,\Gamma} \Diamond\psi$. Moreover, if $\Gamma \vdash_{IVML} \varphi \to \psi$, then also $\Gamma \vdash_{IVML} \Box(\varphi \to \psi)$ (generalization). Using (IVML.7) and MP yields $\Gamma \vdash_{IVML} \Box\varphi \to \Box\psi$. Therefore, if $\varphi \sim_{IVML,\Gamma} \psi$, then $\Box\varphi \sim_{IVML,\Gamma} \Box\psi$.
The general result (which is explained in e.g. [32]) we can apply now, is that IVML is strong complete w.r.t. the variety of triangle algebras if it is sound w.r.t. it and if in triangle algebras $x = y$ if $x \Rightarrow y = 1$ and $y \Rightarrow x = 1$. Triangle algebras indeed satisfy these conditions, so we can conclude that IVML is sound and strong complete w.r.t. triangle algebras. This result is of course very useful. But the general method (of which the proof can be found in [32], but is not given here) that is used to prove it, provides little insight in how this completeness can be obtained. Therefore we give an alternative proof for the completeness of IVML, using the relation $\sim_{IVML,\Gamma}$. This way of proving completeness, is commonly used for formal fuzzy logics, e.g. in [42].

Because $\sim_{IVML,\Gamma}$ is a congruence relation on $(\mathscr{F}_{IVFL}, \wedge, \vee, \&, \to, \Box, \Diamond)$, we can meaningfully consider the structure $(\mathscr{F}_{IVML,\Gamma}, \wedge_\Gamma, \vee_\Gamma, \&_\Gamma, \to_\Gamma, \Box_\Gamma, \Diamond_\Gamma, [\overline{0}]_\Gamma, [\overline{u}]_\Gamma, [\overline{1}]_\Gamma)$, in which

- $\mathscr{F}_{IVML,\Gamma}$ is the set of equivalence classes of $\sim_{IVML,\Gamma}$, i.e., $(\mathscr{F}_{IVFL})_{\sim_{IVML,\Gamma}}$,
- $\wedge_\Gamma$ is the binary operation on $\mathscr{F}_{IVML,\Gamma}$ that maps $([\varphi]_\Gamma, [\psi]_\Gamma)$ to $[\varphi\wedge\psi]_\Gamma$,
- $\vee_\Gamma$ is the binary operation on $\mathscr{F}_{IVML,\Gamma}$ that maps $([\varphi]_\Gamma, [\psi]_\Gamma)$ to $[\varphi\vee\psi]_\Gamma$,
- $\&_\Gamma$ is the binary operation on $\mathscr{F}_{IVML,\Gamma}$ that maps $([\varphi]_\Gamma, [\psi]_\Gamma)$ to $[\varphi\&\psi]_\Gamma$,
- $\to_\Gamma$ is the binary operation on $\mathscr{F}_{IVML,\Gamma}$ that maps $([\varphi]_\Gamma, [\psi]_\Gamma)$ to $[\varphi \to \psi]_\Gamma$,
- $\Box_\Gamma$ is the unary operation on $\mathscr{F}_{IVML,\Gamma}$ that maps $[\varphi]_\Gamma$ to $[\Box\varphi]_\Gamma$,
- $\Diamond_\Gamma$ is the unary operation on $\mathscr{F}_{IVML,\Gamma}$ that maps $[\varphi]_\Gamma$ to $[\Diamond\varphi]_\Gamma$,
- $[\overline{0}]_\Gamma$, $[\overline{u}]_\Gamma$, $[\overline{1}]_\Gamma$ are the elements of $\mathscr{F}_{IVML,\Gamma}$ that contain $\overline{0}$, $\overline{u}$ and $\overline{1}$ resp.

Proposition 4.20 The structure $(\mathscr{F}_{IVML,\Gamma}, \wedge_\Gamma, \vee_\Gamma, \&_\Gamma, \to_\Gamma, \Box_\Gamma, \Diamond_\Gamma, [\overline{0}]_\Gamma, [\overline{u}]_\Gamma, [\overline{1}]_\Gamma)$ is a triangle algebra.

Proof. The first part consists in proving that $(\mathscr{F}_{IVML,\Gamma}, \wedge_\Gamma, \vee_\Gamma, \&_\Gamma, \to_\Gamma, [\overline{0}]_\Gamma, [\overline{1}]_\Gamma)$ is a residuated lattice. This is done exactly as for monoidal logic (and residuated lattices) [44]. The following properties are still valid too:

- $\Gamma \vdash_{IVML} \varphi$ iff $[\varphi]_\Gamma = [\overline{1}]_\Gamma$,
- $\Gamma \vdash_{IVML} \neg\varphi$ iff $[\varphi]_\Gamma = [\overline{0}]_\Gamma$, and
- $\Gamma \vdash_{IVML} \varphi \to \psi$ iff $[\varphi]_\Gamma \leqslant [\psi]_\Gamma$.

Now we will prove that the other axioms are valid too.

- (T.2): $\Box_\Gamma[\varphi]_\Gamma \leqslant \Box_\Gamma\Box_\Gamma[\varphi]_\Gamma$, in other words $[\Box\varphi]_\Gamma \leqslant [\Box\Box\varphi]_\Gamma$. This is true because $\Gamma \vdash_{IVML} \Box\varphi \to \Box\Box\varphi$ (IVML.2). (T.1'), (T.3w), (T.4w), (T.6w), (T.6'w), (T.7), (T.9) and (T.10) are checked in the same way.
- (T.3'): $\Diamond_\Gamma([\varphi]_\Gamma \wedge_\Gamma [\psi]_\Gamma) = \Diamond_\Gamma[\varphi]_\Gamma \wedge_\Gamma \Diamond_\Gamma[\psi]_\Gamma$. We have to prove that $[\Diamond(\varphi\wedge\psi)]_\Gamma = [\Diamond\varphi\wedge\Diamond\psi]_\Gamma$. This means that $\Gamma \vdash_{IVML} (\Diamond(\varphi\wedge\psi)) \to (\Diamond\varphi\wedge\Diamond\psi)$ and $\Gamma \vdash_{IVML} (\Diamond\varphi\wedge\Diamond\psi) \to (\Diamond(\varphi\wedge\psi))$. The second deduction holds by (IVML.3'). The first one holds because $\Gamma \vdash_{IVML} \Diamond(\varphi\wedge\psi) \to \Diamond\varphi$ (using (ML.5) and M$\Diamond$) and $\Gamma \vdash_{IVML} \Diamond(\varphi\wedge\psi) \to \Diamond\psi$ (using (ML.6) and M$\Diamond$), by (ML.9) and MP. We can check (T.4') in the same manner.
- (T.5'): $\Diamond_\Gamma[\overline{u}]_\Gamma = [\overline{1}]_\Gamma$. We need to prove that $\Gamma \vdash_{IVML} \Diamond\overline{u}$. This is exactly (IVML.5').
- (T.5): $\Box_\Gamma[\overline{u}]_\Gamma = [\overline{0}]_\Gamma$. This follows from $\Gamma \vdash_{IVML} \Box\overline{u} \to \overline{0}$ (IVML.5).
- (T.10): $[\varphi]_\Gamma = \Box_\Gamma[\varphi]_\Gamma \vee_\Gamma (\Diamond_\Gamma[\varphi]_\Gamma \wedge_\Gamma [\overline{u}]_\Gamma)$, in other words $[\varphi]_\Gamma \leftrightarrow_\Gamma (\Box_\Gamma[\varphi]_\Gamma \vee_\Gamma (\Diamond_\Gamma[\varphi]_\Gamma \wedge_\Gamma [\overline{u}]_\Gamma)) = [\overline{1}]_\Gamma$, or $[\varphi \leftrightarrow (\Box\varphi \vee (\varphi \wedge \overline{u}))] = [\overline{1}]_\Gamma$. This is equivalent with $\Gamma \vdash_{IVML} \varphi \leftrightarrow (\Box\varphi \vee (\varphi \wedge \overline{u}))$, which follows from (IVML.10).

□

If $\Gamma = \emptyset$, the structure $(\mathscr{F}_{IVML,\emptyset}, \wedge_\emptyset, \vee_\emptyset, \&_\emptyset, \to_\emptyset, \Box_\emptyset, \Diamond_\emptyset, [\overline{0}]_\emptyset, [\overline{u}]_\emptyset, [\overline{1}]_\emptyset)$ is called the Lindenbaum-algebra [42, 44] of IVML. Using this algebra, the strong completeness of IVML can be proven.

Theorem 4.21 (**Soundness and strong completeness of IVML**) An IVFL-formula φ can be deduced from a theory Γ in IVML iff for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$.

Proof. The completeness is proven in the same way as for monoidal logic and residuated lattices (using Proposition 4.20): for any formula φ satisfying $e(\varphi) = 1$ for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, we can consider the triangle algebra $\mathscr{A}_\Gamma = (\mathscr{F}_{IVML,\Gamma}, \wedge_\Gamma, \vee_\Gamma, \&_\Gamma, \to_\Gamma, \Box_\Gamma, \Diamond_\Gamma, [\overline{0}]_\Gamma, [\overline{u}]_\Gamma, [\overline{1}]_\Gamma)$ of Proposition 4.20 and the $\mathscr{A}_\Gamma$-model e defined by $e(p_i) = [p_i]_\Gamma$ for any propositional variable p_i. Then we have $[\overline{1}]_\Gamma = e(\varphi) = [\varphi]_\Gamma$, which means $\Gamma \vdash_{IVML} \varphi$.

□

Theorem 4.21 implies similar results for axiomatic extensions (e.g. the interval-valued fuzzy logics in Definition 4.22), in the same way as the soundness and completeness of ML remains valid for axiomatic extensions. This can be seen by taking the set of all instances of the extra axioms as Γ in Theorem 4.21.

Now we introduce some axiomatic extensions of IVML, by adding well-known[16] axioms. Note that these axioms are applied to IVFL-formulae of the form $\Box\varphi$ instead of to all IVFL-formulae. As the image of a triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ under ν is the set $E(\mathscr{A})$ of exact elements[17], this means that the axioms schemes do not hold for all truth values, but only for exact truth values. This is not a drawback. On the contrary, it is precisely what we want because the exact truth values are easier to interpret and handle. Moreover, using Theorem 3.20, for all axioms equivalent axioms can be found that only involve IVFL-formulae of the form $\Box\varphi$ and $\Diamond\varphi$, and $\overline{u}$ (some examples are given after Definition 4.22).

[16] For a more detailed overview, we refer to [14] and [29].

[17] Note that the image under μ is also $E(\mathscr{A})$. All axiom schemes in Definition 4.22 can also be given in an equivalent way by changing $\Box\varphi$ to $\Diamond\varphi$ and/or $\Box\psi$ to $\Diamond\psi$.

Definition 4.22

- *Interval-valued monoidal t-norm based logic (IVMTL) is IVML extended with the axiom scheme pseudo-prelinearity*
$$(\Box\varphi \to \Box\psi) \vee (\Box\psi \to \Box\varphi).$$

- *Interval-valued Łukasiewicz logic (IVŁ) is IVML extended with the axiom scheme pseudo-strong $\vee$-definability*
$$((\Box\varphi \to \Box\psi) \to \Box\psi) \to (\Box\varphi \vee \Box\psi).$$

- *Interval-valued classical propositional calculus (IVCPC) is IVML extended with the axiom scheme pseudo-law of excluded middle*
$$\Box\varphi \vee \neg\Box\varphi.$$

- *Interval-valued Gödel logic (IVG) is IVMTL extended with the axiom scheme pseudo-contraction*
$$\Box\varphi \to (\Box\varphi \& \Box\varphi).$$

- *Interval-valued weak nilpotent minimum logic (IVWNM) is IVMTL extended with the axiom scheme pseudo-weak nilpotent minimum*
$$\neg(\Box\varphi \& \Box\psi) \vee ((\Box\varphi \wedge \Box\psi) \to (\Box\varphi \& \Box\psi)).$$

- *Interval-valued involutive monoidal t-norm based logic (IVIMTL) is IVMTL extended with the axiom scheme pseudo-involution*
$$\neg\neg\Box\varphi \to \Box\varphi.$$

- *Interval-valued nilpotent minimum logic (IVNM) is IVWNM extended with the axiom scheme*
$$\neg\neg\Box\varphi \to \Box\varphi.$$

- *Interval-valued strict monoidal t-norm based logic (IVSMTL) is IVMTL extended with the axiom scheme pseudo-pseudocomplementation*
$$\neg(\Box\varphi \wedge \neg\Box\varphi).$$

- *Interval-valued weak cancellation monoidal t-norm based logic (IVWCMTL) is IVMTL extended with the axiom scheme pseudo-weak cancellation*
$$\neg(\Box\varphi \& \Box\psi) \vee ((\Box\varphi \to (\Box\varphi \& \Box\psi)) \to \Box\psi).$$

- *Interval-valued ΠMTL (IVΠMTL) is IVMTL extended with the axiom scheme pseudo-cancellation*
$$\neg\Box\varphi \vee ((\Box\varphi \to (\Box\varphi \& \Box\psi)) \to \Box\psi).$$

- *Interval-valued basic logic (IVBL) is IVMTL extended with the axiom scheme pseudo-divisibility*
$$(\Box\varphi \wedge \Box\psi) \to (\Box\varphi \& (\Box\varphi \to \Box\psi)).$$

- *Interval-valued product logic (IVΠ) is IVBL extended with the axiom scheme*

$$\neg\Box\varphi \vee ((\Box\varphi \to (\Box\varphi \& \Box\psi)) \to \Box\psi).$$

- *Interval-valued strict basic logic (IVSBL) is IVBL extended with the axiom scheme*

$$\neg(\Box\varphi \wedge \neg\Box\varphi).$$

All these axiomatic extensions of IVML are sound and (strong) complete w.r.t. their corresponding subvariety of the variety of triangle algebras. For example, IVSBL is sound and complete w.r.t. the variety of triangle algebras satisfying $(vx \Rightarrow vy) \sqcup (vy \Rightarrow vx) = 1$, $vx \sqcap vy \leqslant vx * (vx \Rightarrow vy)$ and $(vx \sqcap \neg vx) = 0$.
For some of these logics, we can use this completeness and apply known algebraic properties of triangle algebras to derive alternative defining axiom schemes. For example, IVCPC can also be defined as IVML extended with the axiom scheme $(\varphi \to \psi) \vee (\psi \to \varphi)$ (because a triangle algebra satisfies the pseudo-law of excluded middle iff it is prelinear, see Proposition 3.28); and IVBL can also be defined as IVMTL extended with the axiom scheme $(\varphi \wedge \psi) \to ((\varphi \& (\varphi \to \psi)) \vee (\psi \& (\psi \to \varphi)))$ (because a pseudo-prelinear triangle algebra is pseudo-divisible iff it is weak divisible, see Propositions 3.29 and 3.47).
For other logics in Definition 4.22, we can work similarly, but only after adding an axiom about $\overline{u} \& \overline{u}$. For example, if we extend IVSBL with the axiom $\neg\neg(\overline{u} \& \overline{u})$, then this new interval-valued fuzzy logic is equivalent with IVMTL extended with the axioms of pseudocomplementation and weak divisibility. Indeed, using Theorem 4.21 we see that IVSBL with $\neg\neg(\overline{u} \& \overline{u})$ corresponds with pseudo-divisible, pseudo-prelinear triangle algebras satisfying pseudo-pseudocomplementation and $\neg\neg(u * u) = 1$ (which is equivalent with $\neg(u * u) = 0$). These triangle algebras are precisely weak divisible, pseudo-prelinear triangle algebras satisfying pseudocomplementation (because of (the remarks after) Propositions 3.29, 3.47 and 3.42). These triangle algebras precisely form the semantics of IVMTL extended with the axioms of pseudocomplementation and weak divisibility.
We summarize these and similar results, based on the properties that are summarized in Table 3.1 at the end of Chapter 3:

- IVCPC = IVML + $(\varphi \to \psi) \vee (\psi \to \varphi)$,
- IVBL = IVMTL + $(\varphi \wedge \psi) \to ((\varphi \& (\varphi \to \psi)) \vee (\psi \& (\psi \to \varphi)))$,
- IVG + $\Diamond(\overline{u} \& \overline{u})$ = IVMTL + $\varphi \to (\varphi \& \varphi)$,
- IVIMTL + $\neg(\overline{u} \& \overline{u})$ = IVMTL + $\neg\neg\varphi \to \varphi$,
- IVSMTL + $\neg\neg(\overline{u} \& \overline{u})$ = IVMTL + $\neg(\varphi \wedge \neg\varphi)$,
- IVSBL + $\neg\neg(\overline{u} \& \overline{u})$ = IVMTL + $\neg(\varphi \wedge \neg\varphi)$ + $(\varphi \wedge \psi) \to ((\varphi \& (\varphi \to \psi)) \vee (\psi \& (\psi \to \varphi)))$.

Another interesting axiom that might be used to extend IVML (or any of the interval-valued fuzzy logics from Definition 4.22), is $\Diamond\varphi \& \Diamond\psi \to \Diamond(\varphi \& \psi)$. Using soundness and completeness of this axiomatic extension of IVML, we see that this axiom corresponds with $\mu x * \mu y \leqslant \mu(x * y)$, or equivalently, with $\mu(x * y) = \mu x * \mu y$ (see the proof of Proposition 3.5) in a triangle algebra. This property ($\mu(x * y) = \mu x * \mu y$) implies that (in terms of IVRL) the second component of $[x_1, x_2] * [y_1, y_2]$ is independent of x_1 and y_1. This means that we can use this property to characterize IVRLs with t-representable t-norms by triangle algebras satisfying $\mu(x * y) = \mu x * \mu y$.

For IVMTL and its axiomatic extensions we can prove a stronger version of completeness, namely strong pseudo-chain completeness. This is similar to axiomatic extensions of MTL being strong chain-complete.

4.3.3 Pseudo-chain and strong pseudo-chain completeness

Together with Theorem 4.21, Theorem 3.46 implies the following result:

Theorem 4.23 *For each set of IVFL-formulae $\Gamma \cup \{\varphi\}$, the following three statements are equivalent:*

- *φ can be deduced from a theory Γ in IVMTL ($\Gamma \vdash_{IVMTL} \varphi$),*
- *for every pseudo-prelinear triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$,*
- *for every pseudo-linear triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$.*

Proof. The equivalence of the first two statements is a consequence of Theorem 4.21. It is also obvious that the second statement implies the third one; we now prove the converse. Suppose that for every pseudo-linear triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$. Now consider a pseudo-prelinear triangle algebra $\mathscr{A}_0$ and an $\mathscr{A}_0$-model e_0 of Γ. Because of Theorem 3.46, we know that $\mathscr{A}_0$ is a subdirect product of pseudo-linear triangle algebras. The mapping e_0 can therefore be identified with a system (one for each IVRL-extended prime filter F of $\mathscr{A}_0$) of mappings $(e_0)_F$ (from the set of formulae into $(\mathscr{A}_0)_F$), defined by $(e_0)_F(\chi) = [e_0(\chi)]_F$. Each $(\mathscr{A}_0)_F$ is a pseudo-linear triangle algebra, and each $(e_0)_F$ is a $(\mathscr{A}_0)_F$-model of Γ, so we can use our assumption to conclude that $(e_0)_F(\varphi) = [1]_F$ for each IVRL-extended prime filter F in $\mathscr{A}_0$. This means that $e_0(\varphi) = 1$, exactly what we wanted to prove. □

This completeness result remains valid for axiomatic extensions of IVMTL. The reason is that Theorem 3.46 also holds for subvarieties of pseudo-prelinear triangle algebras.

4.3.4 Standard and strong standard completeness

In [14] it is shown that strong standard completeness of a (propositional) formal fuzzy logic is equivalent with the real-chain embedding property of that logic, and that MTL, G, WNM, IMTL, NM and SMTL satisfy this property. We use these results in the next theorem to show that their interval-valued counterparts also satisfy strong standard completeness. Basically what we do in the proof is applying the real-chain embedding property to the diagonal of a (countable) pseudo-linear triangle algebra, which gives us an embedding of this diagonal in a standard MTL-chain. This embedding can be extended to an embedding of the whole triangle algebra in a standard triangle algebra. This interval-valued counterpart of the real-chain embedding property might be called 'pseudo-real-chain embedding property' and enables us to prove the strong standard completeness.

Theorem 4.24 *(Strong standard completeness) For each set of IVFL-formulae $\Gamma \cup \{\varphi\}$, the following four statements are equivalent:*

1. *φ can be deduced from Γ in IVMTL ($\Gamma \vdash_{IVMTL} \varphi$),*
2. *for every pseudo-prelinear triangle algebra $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$ (i.e., for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$),*

3. *for every pseudo-linear triangle algebra* $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$,

4. *for every standard triangle algebra* $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$.

Proof. The equivalence of the first three statements is already proven in the previous two subsections. We will now prove that (4) implies (3). This suffices to prove the theorem, as (3) obviously implies (4).
Suppose (3) does not hold. Thus there exists a pseudo-linear triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u, 1_A)$ and an $\mathscr{A}$-model e of Γ such that $e(\varphi) < 1_A$. Clearly, only evaluations of subformulae of $\Gamma \cup \{\varphi\}$ are relevant, therefore we can assume, without loss of generality, that $\mathscr{A}$ is at most countably generated (as the set of IVFL-formulae is countable), and therefore at most countable. Because $\mathscr{E}(\mathscr{A}) = (D, \sqcap_D, \sqcup_D, *_D, \Rightarrow_D, 0_A, 1_A)$, in which $D = E(\mathscr{A})$ and $\sqcap_D$, $\sqcup_D$, $*_D$ and $\Rightarrow_D$ are the restrictions of $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$ to D, is an MTL-chain (i.e., a linearly ordered MTL-algebra), we know from [49] that there exists[18] an embedding i from $\mathscr{E}(\mathscr{A})$ into a standard MTL-algebra $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$.
Now we define a standard triangle algebra $\mathscr{A}'$ and a mapping j from $\mathscr{A}$ to $\mathscr{A}'$ in the following way: $\mathscr{A}' := (Int([0,1], \leqslant), \inf, \sup, \odot, \rightsquigarrow, \mathrm{pr}_v, \mathrm{pr}_h, [0,0], [0,1], [1,1])$, with

- $\inf([x_1, x_2], [y_1, y_2]) = [\min(x_1, y_1), \min(x_2, y_2)]$,
- $\sup([x_1, x_2], [y_1, y_2]) = [\max(x_1, y_1), \max(x_2, y_2)]$,
- $[x_1, x_2] \odot [y_1, y_2] = [x_1 \circ y_1, \max(x_1 \circ y_2, x_2 \circ y_1, x_2 \circ y_2 \circ i(\mu(u * u)))]$,
- $[x_1, x_2] \rightsquigarrow [y_1, y_2] =$
  $[\min(x_1 \Rightarrow_\circ y_1, x_2 \Rightarrow_\circ y_2), \min(x_1 \Rightarrow_\circ y_2, (x_2 \circ i(\mu(u * u))) \Rightarrow_\circ y_2)]$,
- $\mathrm{pr}_v([x_1, x_2]) = [x_1, x_1]$,
- $\mathrm{pr}_h([x_1, x_2]) = [x_2, x_2]$ and
- $j(x) = [i(\nu x), i(\mu x)]$.

To verify that $\mathscr{A}'$ is indeed a standard triangle algebra, note that $(\{[x,x] | x \in [0,1]\}, \inf, \sup, \odot, \rightsquigarrow, [0,0], [1,1])$ is a subalgebra of $\mathscr{A}'$ isomorphic to $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$ and check Example 3.14 and Theorem 3.13. Now we show that j is an embedding from $\mathscr{A}$ into $\mathscr{A}'$:

$$j(u) = [i(\nu u), i(\mu u)] = [i(0_A), i(1_A)] = [0,1]$$

(and similarly for $j(0_A) = [0,0]$ and $j(1_A) = [1,1]$),

$$\begin{aligned} j(x \sqcap y) &= [i(\nu(x \sqcap y)), i(\mu(x \sqcap y))] \\ &= [i(\nu x \sqcap \nu y), i(\mu x \sqcap \mu y)] \\ &= [i(\nu x \sqcap_D \nu y), i(\mu x \sqcap_D \mu y)] \\ &= [\min(i(\nu x), i(\nu y)), \min(i(\mu x), i(\mu y))] \\ &= \inf([i(\nu x), i(\mu x)], [i(\nu y), i(\mu y)]) \\ &= \inf(j(x), j(y)) \end{aligned}$$

[18] In other words, in [49] the real-chain embedding property for MTL is proven.

(and similarly for $x \sqcup y$),

$$j(\nu x) = [i(\nu\nu x), i(\mu\nu x)] = [i(\nu x), i(\nu x)] = \mathrm{pr}_\nu([i(\nu x), i(\mu x)]) = \mathrm{pr}_\nu(j(x))$$

(and similarly for μx),

$$\begin{aligned}
&j(x * y)\\
&= [i(\nu(x * y)), i(\mu(x * y))]\\
&= [i(\nu x * \nu y), i((\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u)))]\\
&= [i(\nu x *_D \nu y), i((\nu x *_D \mu y) \sqcup_D (\mu x *_D \nu y) \sqcup_D (\mu x *_D \mu y *_D \mu(u * u)))]\\
&= [i(\nu x) \circ i(\nu y), \max(i(\nu x) \circ i(\mu y), i(\mu x) \circ i(\nu y), i(\mu x) \circ i(\mu y) \circ i(\mu(u * u)))]\\
&= [i(\nu x), i(\mu x)] \odot [i(\nu y), i(\mu y)]\\
&= j(x) \odot j(y),
\end{aligned}$$

$$\begin{aligned}
&j(x \Rightarrow y)\\
&= [i(\nu(x \Rightarrow y)), i(\mu(x \Rightarrow y))]\\
&= [i((\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)), i((\nu x \Rightarrow \mu y) \sqcap ((\mu x * \mu(u * u)) \Rightarrow \mu y))]\\
&= [i((\nu x \Rightarrow_D \nu y) \sqcap_D (\mu x \Rightarrow_D \mu y)), i((\nu x \Rightarrow_D \mu y) \sqcap_D ((\mu x *_D \mu(u * u)) \Rightarrow_D \mu y))]\\
&= [\min(i(\nu x) \Rightarrow_\circ i(\nu y), i(\mu x) \Rightarrow_\circ i(\mu y)),\\
&\quad \min(i(\nu x) \Rightarrow_\circ i(\mu y), (i(\mu x) \circ i(\mu(u * u))) \Rightarrow_\circ i(\mu y))]\\
&= [i(\nu x), i(\mu x)] \rightsquigarrow [i(\nu y), i(\mu y)]\\
&= j(x) \rightsquigarrow j(y)
\end{aligned}$$

and

$$\begin{aligned}
j(x) = j(y) &\text{ iff } (i(\nu x) = i(\nu y) \text{ and } i(\mu x) = i(\mu y))\\
&\text{ iff } (\nu x = \nu y \text{ and } \mu x = \mu y)\\
&\text{ iff } x = y.
\end{aligned}$$

Now remark that e', defined by $e'(\psi) = j(e(\psi))$, is an $\mathscr{A}'$-model of Γ such that $e'(\varphi) < 1$, which concludes the proof. □

This theorem can also be used, mutatis mutandis, for IVG, IVWNM, IVIMTL, IVNM and IVSMTL, because G, WNM, IMTL, NM and SMTL satisfy the real-chain embedding property, just like MTL. Because of Propositions 3.12 and 3.13, every standard triangle algebra is isomorphic to a standard extended IVRL, and every standard extended IVRL is a standard triangle algebra. These results lead to the following corollary of Theorem 4.24.

Corollary 4.25 *For each set of IVFL-formulae $\Gamma \cup \{\varphi\}$, the following two statements are equivalent:*

1. *φ can be deduced from Γ in IVMTL ($\Gamma \vdash_{IVMTL} \varphi$),*
2. *for every standard extended IVRL $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$.*

So we can truly state that IVMTL is an interval-valued fuzzy logic. It is the logic of the t-norms $\mathscr{T}_{T,t}$ in (2.2), with T a left-continuous t-norm on the unit interval. Similarly also IVG, IVWNM, IVIMTL, IVNM and IVSMTL are logics of t-norms $\mathscr{T}_{T,t}$ in (2.2), with T a t-norm on the unit interval. For example, IVG is the logic of the t-norms $\mathscr{T}_{\min,t}$ in (2.2), where min is the minimum on the unit interval.

For Ł, WCMTL, ΠMTL, BL, Π and SBL it is known [14, 42, 45, 46, 59] that they are finite strong standard complete, but not strong standard complete. Proposition 4.28 implies that their interval-valued counterparts cannot be strong standard complete either. First we show that their interval-valued counterparts are finite strong standard complete. We do this for IVBL, but the proof works for the interval-valued counterpart of any finite strong standard complete axiomatic extension of MTL.

Theorem 4.26 *(Finite strong standard completeness) For each finite set of IVFL-formulae $\Gamma \cup \{\varphi\}$, the following four statements are equivalent:*

1. *φ can be deduced from Γ in IVBL ($\Gamma \vdash_{IVBL} \varphi$),*
2. *for every pseudo-prelinear pseudo-divisible triangle algebra $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$ (i.e., for every $\mathscr{A}$-model e of Γ, $e(\varphi) = 1$),*
3. *for every pseudo-linear pseudo-divisible triangle algebra $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$,*
4. *for every standard pseudo-divisible triangle algebra $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$.*

Proof. The equivalence of the first three statements is already proven in the previous two subsections. We will now prove that (4) implies (3). This suffices to prove the theorem, as (3) obviously implies (4).
Suppose (3) does not hold. Thus there exists a pseudo-linear pseudo-divisible triangle algebra $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u, 1_A)$ and an $\mathscr{A}$-model e of Γ such that $e(\varphi) < 1_A$. Now let Φ denote the set of all subformulae of $\Gamma \cup \{\varphi\}$ (including the IVFL-formulae of $\Gamma \cup \{\varphi\}$). Remark that this set is finite. We define a finite subset X of A in four steps. First, take $X_1 = e(\Phi)$. Secondly, $X_2 = X_1 \cup \nu(X_1) \cup \mu(X_1)$. Thirdly, $X_3 = X_2 \cup \{\mu(u * u) * x \mid x \in X_2\}$. And finally, $X = X_3 \cup \{x * y \mid x, y \in X_3\} \cup \{x \Rightarrow y \mid x, y \in X_3\} \cup \{0_A, u, \mu(u * u), 1_A\}$. Note that indeed X is finite, because X_3 is finite (X_3 is finite because X_2 is, X_2 because X_1 is).
Because $\mathscr{E}(\mathscr{A}) = (D, \sqcap_D, \sqcup_D, *_D, \Rightarrow_D, 0_A, 1_A)$, in which $D = E(\mathscr{A})$ and $\sqcap_D$, $\sqcup_D$, $*_D$ and $\Rightarrow_D$ are the restrictions of $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$ to D, is a BL-chain (i.e., a linearly ordered BL-algebra), we know from the real-chain partial embedding property for BL, that there exists an injection i from $E(\mathscr{A}) \cap X$ into a standard BL-algebra $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$ that preserves the existing operations on $E(\mathscr{A}) \cap X$.
Now we define a standard pseudo-divisible triangle algebra $\mathscr{A}'$ in the following way: $\mathscr{A}' := (Int([0,1], \leqslant), \inf, \sup, \odot, \rightsquigarrow, \mathrm{pr}_\nu, \mathrm{pr}_h, [0,0], [0,1], [1,1])$, with

- $\inf([x_1, x_2], [y_1, y_2]) = [\min(x_1, y_1), \min(x_2, y_2)]$,
- $\sup([x_1, x_2], [y_1, y_2]) = [\max(x_1, y_1), \max(x_2, y_2)]$,
- $[x_1, x_2] \odot [y_1, y_2] = [x_1 \circ y_1, \max(x_1 \circ y_2, x_2 \circ y_1, x_2 \circ y_2 \circ i(\mu(u * u)))]$,
- $[x_1, x_2] \rightsquigarrow [y_1, y_2] =$
  $[\min(x_1 \Rightarrow_\circ y_1, x_2 \Rightarrow_\circ y_2), \min(x_1 \Rightarrow_\circ y_2, (x_2 \circ i(\mu(u * u))) \Rightarrow_\circ y_2)]$,

- $\mathrm{pr}_v([x_1, x_2]) = [x_1, x_1]$ and
- $\mathrm{pr}_h([x_1, x_2]) = [x_2, x_2]$.

To verify that $\mathscr{A}'$ is indeed a standard pseudo-divisible triangle algebra, note that $(\{[x,x] | x \in [0,1]\}, \inf, \sup, \odot, \rightsquigarrow, [0,0], [1,1])$ is a subalgebra of $\mathscr{A}'$ isomorphic to $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$ and check Example 3.14 and Theorem 3.13.

Now we define an $\mathscr{A}'$-evaluation e' by putting $e'(p) = [i(\nu(e(p))), i(\mu(e(p)))]$ for all propositional variables p in Φ (because $\nu(e(p))$ and $\mu(e(p))$ are in $X_2 \cap E(\mathscr{A})$, $e'(p)$ is well-defined); for the other propositional variables, the value can be chosen freely (in $Int([0,1], \leqslant)$). We show that for every IVFL-formula γ in Φ, $e'(\varphi) = [i(\nu(e(\varphi))), i(\mu(e(\varphi)))]$ (because $\nu(e(\varphi))$ and $\mu(e(\varphi))$ are in $X_2 \cap E(\mathscr{A})$, the right-hand side is well-defined). We do this by induction on the complexity of γ.
If γ is a propositional variable occurring in Φ, then $e'(p) = [i(\nu(e(p))), i(\mu(e(p)))]$ by definition of e'.
If γ is $\overline{u}$, then $e'(\overline{u}) = [0,1] = [i(0_A), i(1_A)] = [i(\nu(u)), i(\mu(u))] = [i(\nu(e(\overline{u}))), i(\mu(e(\overline{u})))]$ (because e' is an $\mathscr{A}'$-evaluation and 0_A and 1_A are elements of $X \cap E(\mathscr{A})$). Similarly, $e'(\overline{0}) = [i(\nu(e(\overline{0}))), i(\mu(e(\overline{0})))]$.
Now suppose γ is an element of Φ of the form $\alpha \wedge \beta$. Then α and β are also elements of Φ and by the induction hypothesis we know $e'(\alpha) = [i(\nu(e(\alpha))), i(\mu(e(\alpha)))]$ and $e'(\beta) = [i(\nu(e(\beta))), i(\mu(e(\beta)))]$. So we find

$$\begin{aligned} e'(\gamma) &= e'(\alpha \wedge \beta) \\ &= \inf(e'(\alpha), e'(\beta)) \\ &= \inf([i(\nu(e(\alpha))), i(\mu(e(\alpha)))], [i(\nu(e(\beta))), i(\mu(e(\beta)))]) \\ &= [\min(i(\nu(e(\alpha))), i(\nu(e(\beta)))), \min(i(\mu(e(\alpha))), i(\mu(e(\beta))))]. \end{aligned}$$

Because $\nu(e(\alpha))$ and $\nu(e(\beta))$ are elements of $X_2 \cap E(\mathscr{A})$, also $\nu(e(\alpha)) \sqcap \nu(e(\beta))$ (which is equal to $\nu(e(\alpha))$ or $\nu(e(\beta))$ because $E(\mathscr{A})$ is linearly ordered) is an element of $X_2 \cap E(\mathscr{A})$. Therefore $\min(i(\nu(e(\alpha))), i(\nu(e(\beta)))) = i(\nu(e(\alpha)) \sqcap \nu(e(\beta))) = i(\nu(e(\alpha) \sqcap e(\beta))) = i(\nu(e(\alpha \wedge \beta))) = i(\nu(e(\gamma)))$. Similarly $\min(i(\mu(e(\alpha))), i(\mu(e(\beta)))) = i(\mu(e(\gamma)))$, so we conclude that $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]$. In the same way we can also prove $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]$ for γ an element of Φ of the form $\alpha \vee \beta$.

Next suppose γ is an element of Φ of the form $\Diamond\alpha$. Then α is also an element of Φ and by the induction hypothesis we know $e'(\alpha) = [i(\nu(e(\alpha))), i(\mu(e(\alpha)))]$. Therefore

$$\begin{aligned} e'(\gamma) &= e'(\Diamond\alpha) \\ &= \mathrm{pr}_h(e'(\alpha)) \\ &= \mathrm{pr}_h([i(\nu(e(\alpha))), i(\mu(e(\alpha)))]) \\ &= [i(\mu(e(\alpha))), i(\mu(e(\alpha)))] \\ &= [i(\nu\mu(e(\alpha))), i(\mu\mu(e(\alpha)))] \\ &= [i(\nu(e(\Diamond\alpha))), i(\mu(e(\Diamond\alpha)))] \\ &= [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]. \end{aligned}$$

In the same way we can also prove $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]$ for γ an element of Φ of the form $\Box\alpha$.

If γ is an element of Φ of the form $\alpha\&\beta$. Then α and β are also elements of Φ and by the induction hypothesis we know $e'(\alpha) = [i(\nu(e(\alpha))), i(\mu(e(\alpha)))]$ and $e'(\beta) = [i(\nu(e(\beta))), i(\mu(e(\beta)))]$. Thus we obtain

$$\begin{aligned}
&e'(\gamma)\\
&= e'(\alpha\&\beta)\\
&= e'(\alpha) \odot e'(\beta)\\
&= [i(\nu(e(\alpha))), i(\mu(e(\alpha)))] \odot [i(\nu(e(\beta))), i(\mu(e(\beta)))]\\
&= [i(\nu(e(\alpha))) \circ i(\nu(e(\beta))),\\
&\quad \max(i(\nu(e(\alpha))) \circ i(\mu(e(\beta))), i(\mu(e(\alpha))) \circ i(\nu(e(\beta))), i(\mu(e(\alpha))) \circ i(\mu(e(\beta))) \circ i(\mu(u * u)))].
\end{aligned}$$

Because $\nu(e(\alpha))$, $\nu(e(\beta))$, $\mu(e(\alpha))$ and $\mu(e(\beta))$ are elements of $X_2 \cap E(\mathscr{A})$, they also belong to $X_3 \cap E(\mathscr{A})$. Also $\mu(e(\beta)) * \mu(u * u)$ is an element of $X_3 \cap E(\mathscr{A})$. Thus $\nu(e(\alpha)) * \nu(e(\beta))$, $\nu(e(\alpha)) * \mu(e(\beta))$, $\mu(e(\alpha)) * \nu(e(\beta))$ and $\mu(e(\alpha)) * \mu(e(\beta)) * \mu(u * u)$ are elements of $X \cap E(\mathscr{A})$, as well as $\mu(u * u)$. Therefore also $\nu(e(\alpha)) * \mu(e(\beta)) \sqcup \mu(e(\alpha)) * \nu(e(\beta))$ and $(\nu(e(\alpha)) * \mu(e(\beta)) \sqcup \mu(e(\alpha)) * \nu(e(\beta))) \sqcup \mu(e(\alpha)) * \mu(e(\beta)) * \mu(u * u)$ belong to $X \cap E(\mathscr{A})$ (because of the linear order). So $i(\nu(e(\alpha))) \circ i(\nu(e(\beta))) = i(\nu(e(\alpha)) * \nu(e(\beta))) = i(\nu(e(\alpha) * e(\beta))) = i(\nu(e(\alpha\&\beta))) = i(\nu(e(\gamma)))$ and

$$\begin{aligned}
&\max(i(\nu(e(\alpha))) \circ i(\mu(e(\beta))), i(\mu(e(\alpha))) \circ i(\nu(e(\beta))), i(\mu(e(\alpha))) \circ i(\mu(e(\beta))) \circ i(\mu(u * u)))\\
&= \max\Big(\max\big(i(\nu(e(\alpha))) \circ i(\mu(e(\beta))), i(\mu(e(\alpha))) \circ i(\nu(e(\beta)))\big),\\
&\qquad i(\mu(e(\alpha))) \circ \big(i(\mu(e(\beta))) \circ i(\mu(u * u))\big)\Big)\\
&= \max\Big(\max\Big(i\big(\nu(e(\alpha)) * \mu(e(\beta))\big), i\big(\mu(e(\alpha)) * \nu(e(\beta))\big)\Big), i(\mu(e(\alpha))) \circ i\big(\mu(e(\beta)) * \mu(u * u)\big)\Big)\\
&= \max\Big(i\big(\nu(e(\alpha)) * \mu(e(\beta)) \sqcup \mu(e(\alpha)) * \nu(e(\beta))\big), i\big(\mu(e(\alpha)) * \mu(e(\beta)) * \mu(u * u)\big)\Big)\\
&= i\Big(\big(\nu(e(\alpha)) * \mu(e(\beta)) \sqcup \mu(e(\alpha)) * \nu(e(\beta))\big) \sqcup \mu(e(\alpha)) * \mu(e(\beta)) * \mu(u * u)\Big)\\
&= i(\mu(e(\alpha) * e(\beta)))\\
&= i(\mu(e(\alpha\&\beta)))\\
&= i(\mu(e(\gamma))).
\end{aligned}$$

Thus we find that also for IVFL-formulae γ of Φ of the form $\alpha\&\beta$, $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]$. Similarly we can prove $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]$ for IVFL-formulae γ of Φ of the form $\alpha \to \beta$. This concludes our proof by induction that $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))]$, for all IVFL-formulae γ in Φ. In particular, for all IVFL-formulae γ in Γ, we have $e'(\gamma) = [i(\nu(e(\gamma))), i(\mu(e(\gamma)))] = [i(\nu(1_A)), i(\mu(1_A))] = [i(1_A), i(1_A)] = [1, 1]$; and $e'(\varphi) = [i(\nu(e(\varphi))), i(\mu(e(\varphi)))]$. Because i is an injection, $i(1_A) = 1$ and $\nu(e(\varphi)) \leqslant e(\varphi) < 1_A$, $i(\nu(e(\varphi))) < 1$. Therefore $e'(\varphi) \neq [1, 1]$.
Summarizing, we have found a standard pseudo-divisible triangle algebra $\mathscr{A}'$ and an $\mathscr{A}'$-model e' of Γ such that $e'(\varphi) \neq [1, 1]$. In other words, $\Gamma \not\models_{\mathscr{A}'} \varphi$, which concludes the proof. $\Box$

This theorem can also be used, mutatis mutandis, for IVŁ, IVWCMTL, IVΠMTL, IVΠ and IVSBL, because Ł, WCMTL, ΠMTL, Π and SBL satisfy the real-chain partial embedding property, just like BL.
Because of Propositions 3.12 and 3.13, every standard pseudo-divisible triangle algebra is isomorphic to a standard pseudo-divisible extended IVRL, and every standard pseudo-divisible extended IVRL is a standard pseudo-divisible triangle algebra. These results lead to the following corollary of Theorem 4.26.

Corollary 4.27 *For each finite set of IVFL-formulae $\Gamma \cup \{\varphi\}$, the following two statements are equivalent:*

1. *φ can be deduced from Γ in IVBL ($\Gamma \vdash_{IVBL} \varphi$),*

2. *for every standard pseudo-divisible extended IVRL $\mathscr{A}$, $\Gamma \models_{\mathscr{A}} \varphi$.*

Before proving Proposition 4.28 we mention some notations that will be used.

Suppose $\mathbb{K}$ is a class of residuated lattices. We define the class $TA(\mathbb{K})$ of triangle algebras as follows: a triangle algebra $\mathscr{A}$ is an element of $TA(\mathbb{K})$ iff $\mathscr{E}(\mathscr{A})$ is isomorphic to a residuated lattice in $\mathbb{K}$. Because of Proposition 3.23, $TA(\mathbb{K})$ is not empty if $\mathbb{K}$ is not empty.
Furthermore, for every FL-formula φ, we define the IVFL-formula φ' as follows: $\varphi'(p_{i_1}, \ldots, p_{i_n}) = \varphi(\Box p_{i_1}, \ldots, \Box p_{i_n})$, where $p_{i_1}, \ldots, p_{i_n}$ are the propositional variables occurring in φ. For example, if φ is the FL-formula $((p_6 \vee p_3) \to p_8)\&(p_3 \to \overline{0})$, then φ' is the IVFL-formula $((\Box p_6 \vee \Box p_3) \to \Box p_8)\&(\Box p_3 \to \overline{0})$.
Also, if χ is an FL-formula, we denote the function corresponding to χ in an expansion $\mathscr{B}$ of a residuated lattice by $f_\chi^{\mathscr{B}}$. For example, if χ is the FL-formula $(p_2 \to p_4) \wedge p_2$ (which we denote by $\chi(p_2, p_4)$) and $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra, then $f_\chi^{\mathscr{A}}$ is the binary function in A defined by $f_\chi^{\mathscr{A}}(x, y) = (x \Rightarrow y) \sqcap x$, for all x and y in A.

Now we show, roughly speaking, that a common fuzzy logic is contained in its interval-valued counterpart. Indeed, using soundness and completeness of ML and IVML (and their axiomatic extensions), we can derive from Proposition 4.28 that $\Gamma \vdash_{ML} \varphi$ is equivalent with $\Gamma' \vdash_{IVML} \varphi'$. So the provability relation $\vdash_{ML}$ can be described in terms of the provability relation $\vdash_{IVML}$. Similar results holds for axiomatic extensions of ML and IVML (see also the explanation after the proof of Proposition 4.28).

Proposition 4.28 *Suppose $\Gamma \cup \{\varphi\}$ is a set of FL-formulae and $\mathbb{K}$ is a class of residuated lattices. Then $\Gamma \models_{\mathbb{K}} \varphi$ iff $\Gamma' \models_{TA(\mathbb{K})} \varphi'$, where $\Gamma' = \{\chi' | \chi \in \Gamma\}$.*

Proof. Suppose $\Gamma' \models_{TA(\mathbb{K})} \varphi'$. Now take any residuated lattice $\mathscr{L}$ in $\mathbb{K}$ and $\mathscr{L}$-model e of Γ. We want to prove that $e(\varphi) = 1$. Take any triangle algebra $\mathscr{A}$ in $TA(\mathbb{K})$ such that $\mathscr{E}(\mathscr{A})$ is isomorphic to $\mathscr{L}$. Because of Proposition 3.23 such a triangle algebra always exists. Let i be the mapping from $\mathscr{L}$ to $\mathscr{A}$ that maps $\mathscr{L}$ isomorphically on $\mathscr{E}(\mathscr{A})$. Then the values $i(e(p_1))$, $i(e(p_2))$, $i(e(p_3))$, ... are well-defined, and we can extend this mapping of propositional variables in $\mathscr{A}$ to an $\mathscr{A}$-evaluation e' of all IVFL-formulae, in a unique way. So $e'(p_j) = i(e(p_j))$ for all propositional variables p_j. Remark now that $e'(\chi') = i(e(\chi))$ for all FL-formulae χ. Indeed, if $p_{i_1}, \ldots, p_{i_n}$ are

the propositional variables occurring in χ, then we find

$$\begin{aligned}
&e'(\chi'(p_{i_1},\dots,p_{i_n}))\\
&= e'(\chi(\Box p_{i_1},\dots,\Box p_{i_n})) && \text{by definition of } \chi'\\
&= f_\chi^{\mathscr{A}}(\nu e'(p_{i_1}),\dots,\nu e'(p_{i_n})) && e' \text{ is an evaluation}\\
&= f_\chi^{\mathscr{A}}(\nu i(e(p_{i_1})),\dots,\nu i(e(p_{i_n}))) && \text{by definition of } e'\\
&= f_\chi^{\mathscr{A}}(i(e(p_{i_1})),\dots,i(e(p_{i_n}))) && i(\dots)\in E(\mathscr{A})\\
&= i(f_\chi^{\mathscr{L}}(e(p_{i_1}),\dots,e(p_{i_n}))) && i \text{ is a morphism}\\
&= i(e(\chi(p_{i_1},\dots,p_{i_n}))). && e \text{ is an evaluation}
\end{aligned}$$

In particular, for all ψ in Γ, we have $e'(\psi') = i(e(\psi)) = i(1) = 1$. So e' is an $\mathscr{A}$-model of Γ. Our assumption $\Gamma' \models_{TA(\mathbb{K})} \varphi'$ ensures that $e'(\varphi') = 1$. We conclude $1 = e'(\varphi') = i(e(\varphi))$, which implies $e(\varphi) = 1$.

Now suppose $\Gamma \models_{\mathbb{K}} \varphi$, and take any triangle algebra $\mathscr{A}$ in $TA(\mathbb{K})$ and $\mathscr{A}$-model e' of Γ'. We want to prove that $e'(\varphi') = 1$. Therefore we consider the $\mathscr{E}(\mathscr{A})$-evaluation e determined by $e(p_i) = e'(\Box p_i)$, for all propositional variables p_i. Then for all FL-formulae χ, we have $e(\chi) = e'(\chi')$. Indeed, if $p_{i_1},\dots,p_{i_n}$ are the propositional variables occurring in χ, then we find

$$\begin{aligned}
&e(\chi(p_{i_1},\dots,p_{i_n}))\\
&= f_\chi^{\mathscr{E}(\mathscr{A})}(e(p_{i_1}),\dots,e(p_{i_n})) && e \text{ is an evaluation}\\
&= f_\chi^{\mathscr{E}(\mathscr{A})}(e'(\Box p_{i_1}),\dots,e'(\Box p_{i_n})) && \text{by definition of } e\\
&= f_\chi^{\mathscr{A}}(e'(\Box p_{i_1}),\dots,e'(\Box p_{i_n})) && e'(\Box p_{i_j})\in E(\mathscr{A})\subseteq A\\
&= e'(\chi(\Box p_{i_1},\dots,\Box p_{i_n})) && e' \text{ is an evaluation}\\
&= e'(\chi'(p_{i_1},\dots,p_{i_n})). && \text{by definition of } \chi'
\end{aligned}$$

In particular, for all ψ in Γ, we have $e(\psi) = e'(\psi') = 1$. Our assumption ensures that $e(\varphi) = 1$. We conclude $1 = e(\varphi) = e'(\varphi')$. $\Box$

This result enables us to show some negative completeness results for axiomatic extensions of IVML.

For example, if we choose $\mathbb{K}$ to be the class of all BL-algebras, then $TA(\mathbb{K})$ is the class of all triangle algebras $\mathscr{A}$ for which $\mathscr{E}(\mathscr{A})$ is a BL-algebra. In other words, $TA(\mathbb{K})$ is the class of all triangle algebras $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u, 1_A)$ satisfying $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$ and $\nu x \sqcap \nu y = \nu x * (\nu x \Rightarrow \nu y)$ for all x and y in A. So $TA(\mathbb{K})$ is the class of all IVBL-algebras. The corresponding logic is IVBL: IVML extended with the axiom schemes $(\Box\varphi \to \Box\psi) \vee (\Box\psi \to \Box\varphi)$ and $(\Box\varphi \wedge \Box\psi) \to (\Box\varphi \& (\Box\varphi \to \Box\psi))$. It is known that BL is not strong standard complete, so there exists a set of formulae $\Gamma \cup \{\varphi\}$ such that $\Gamma \models_{\mathscr{L}} \varphi$ for every standard BL-algebra $\mathscr{L}$, but not for every BL-algebra $\mathscr{L}$. Proposition 4.28 then allows us to deduce that $\Gamma' \models_{\mathscr{A}} \varphi'$ for every pseudo-divisible standard triangle algebra $\mathscr{A}$, but not for every pseudo-divisible pseudo-prelinear triangle algebra $\mathscr{A}$. Because IVBL is sound and complete w.r.t. pseudo-divisible pseudo-prelinear triangle algebras, this means exactly that this logic is not strong standard complete.

Because ML, Ł, CPC, WCMTL, ΠMTL, Π (and every axiomatic extension between ΠMTL and Π) and SBL are not strong standard complete [14], we can reason in the same way as for BL and

conclude that IVML, IVŁ, IVCPC, IVWCMTL, IVΠMTL, IVΠ and IVSBL are not strong standard complete either. We give an overview of the completeness results in Table 4.1. Most of the results for axiomatic extensions of MTL can be copied, because such an axiomatic extension is FSSC (finite strong standard complete) iff its interval-valued counterpart is FSSC, and SSC (strong standard complete) iff its interval-valued counterpart is SSC. Moreover, SSC implies FSSC and FSSC implies SC (standard completeness). Therefore this table can be extended easily with interval-valued counterparts of other axiomatic extensions of MTL. The only problem that can occur is when an axiomatic extension of MTL is SC, but not FSSC (and therefore not SSC either): in this case its interval-valued counterpart will not be FSSC (nor SSC) either, but we cannot tell if it will be standard complete or not. However, we do not know if such an axiomatic extension of MTL exists.

Table 4.1: Completeness of several axiomatic extensions of IVML.

Logic	SC	FSSC	SSC
IVML	No	No	No
IVMTL	Yes	Yes	Yes
IVŁ	Yes	Yes	No
IVCPC	No	No	No
IVG	Yes	Yes	Yes
IVWNM	Yes	Yes	Yes
IVIMTL	Yes	Yes	Yes
IVNM	Yes	Yes	Yes
IVSMTL	Yes	Yes	Yes
IVWCMTL	Yes	Yes	No
IVΠMTL	Yes	Yes	No
IVBL	Yes	Yes	No
IVΠ	Yes	Yes	No
IVSBL	Yes	Yes	No

4.4 Local deduction theorem

Now we will show a local deduction theorem for IVML and its axiomatic extensions. Let **L** be an axiomatic extension of IVML.
From the definition of a proof of $\Gamma \vdash_{\mathbf{L}} \varphi$, we immediately obtain the following property.

Proposition 4.29 *Let $\Gamma \cup \Delta \cup \{\varphi, \psi\}$ be a set of IVFL-formulae, and **L** be an axiomatic extension of IVML.*
If $\Gamma \vdash_{\boldsymbol{L}} \varphi$ and $\Delta \cup \{\varphi\} \vdash_{\boldsymbol{L}} \psi$, then $\Gamma \cup \Delta \vdash_{\boldsymbol{L}} \psi$.

Proof. Observe that putting the proof of $\Delta \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ after the proof of $\Gamma \vdash_{\mathbf{L}} \varphi$, gives a proof of $\Gamma \cup \Delta \vdash_{\mathbf{L}} \psi$. □

Proposition 4.30 *Let $\Gamma \cup \{\varphi\}$ be a set of IVFL-formulae, and **L** be an axiomatic extension of IVML. Then $\Gamma \vdash_{\mathbf{L}} \varphi$ iff $\Gamma \vdash_{\mathbf{L}} \Box\varphi$.*

Proof. On the one hand, we can apply Proposition 4.29 with $\Delta = \varnothing$ and $\psi = \Box\varphi$, because $\{\varphi\} \vdash_{\mathbf{L}} \Box\varphi$ (application of the generalization rule).
On the other hand, we can apply Proposition 4.29 to $\Gamma \vdash_{\mathbf{L}} \Box\varphi$ and $\{\Box\varphi\} \vdash_{\mathbf{L}} \varphi$ (application of the modus ponens to IVML.1). □

In a similar way we can prove the following proposition.

Proposition 4.31 *Let $\Gamma \cup \{\varphi, \psi\}$ be a set of IVFL-formulae, and **L** be an axiomatic extension of IVML. Then $\Gamma \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ iff $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \psi$.*

Proof. In one direction, apply Proposition 4.29 to $\{\varphi\} \vdash_{\mathbf{L}} \Box\varphi$ and $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \psi$. In the other direction, apply the proposition to $\{\Box\varphi\} \vdash_{\mathbf{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$. □

Now we prove the so-called local deduction theorem for IVML (and its axiomatic extensions), which gives a connection between $\vdash_{\mathbf{L}}$ and $\rightarrow$.

Proposition 4.32 *Let $\Gamma \cup \{\varphi, \psi\}$ be a set of IVFL-formulae, and **L** be an axiomatic extension of IVML.*
Then the following are equivalent:

- $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \psi$,
- *There is an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^n \rightarrow \psi$.*

Proof. Suppose $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^n \rightarrow \psi$, which is equivalent with $\Gamma \vdash_{\mathbf{L}} \Box\varphi \rightarrow ((\Box\varphi)^{n-1} \rightarrow \psi)$ because of ML.11. Then by an application of modus ponens we obtain $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} (\Box\varphi)^{n-1} \rightarrow \psi$. Proceeding like this, we get $\Gamma \cup \{\Box\varphi\} = \Gamma \cup \{\Box\varphi\} \cup \{\Box\varphi\} \vdash_{\mathbf{L}} (\Box\varphi)^{n-2} \rightarrow \psi$, ... and finally $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \Box\varphi \rightarrow \psi$ and $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \psi$.
Now suppose $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \psi$. This means that there is a proof of ψ, in which every line is an axiom, an element of $\Gamma \cup \{\Box\varphi\}$, or an application of modus ponens, generalization or monotonicity of $\Diamond$ to previous lines in the proof. We will show by induction that for all the formulae γ in the proof, there exists an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^n \rightarrow \gamma$. This will imply $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^n \rightarrow \psi$ for some integer n, as ψ is the last line of the proof. Remark that we can use soundness and completeness of IVML w.r.t. triangle algebras. So we know that $\vdash_{\mathbf{L}} \varphi$ if φ holds in every triangle algebra.
We have to consider the following possibilities:

- γ is an axiom or an element of Γ. Then we have $\Gamma \vdash_{\mathbf{L}} \gamma$, which is equivalent with $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^0 \rightarrow \gamma$.
- γ is $\Box\varphi$. In this case, we have $\Gamma \vdash_{\mathbf{L}} (\Box\varphi) \rightarrow \gamma$.
- γ is the result of an application of modus ponens. So there are two formulae α and $\alpha \rightarrow \gamma$ earlier in the proof. By induction hypothesis, we know that there are integers k and l such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^k \rightarrow \alpha$ and $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^l \rightarrow (\alpha \rightarrow \gamma)$. Combining these, we find $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^{k+l} \rightarrow (\alpha \& (\alpha \rightarrow \gamma))$. As we also have $\vdash_{\mathbf{L}} (\alpha \& (\alpha \rightarrow \gamma)) \rightarrow \gamma$, we obtain $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^{k+l} \rightarrow \gamma$.

- γ is the result of an application of generalization. This means γ is of the form $\Box\alpha$, where α is a formula occuring earlier in the proof. So by induction hypothesis, there is an integer k such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^k \to \alpha$. Applying generalization, IVML.7 and modus ponens, we get $\Gamma \vdash_{\mathbf{L}} \Box((\Box\varphi)^k) \to \Box\alpha$. This is equivalent with $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^k \to \Box\alpha$.

- γ is the result of an application of monotonicity of $\Diamond$. This means γ is of the form $\Diamond\alpha \to \Diamond\beta$, with $\alpha \to \beta$ a formula earlier in the proof. The induction hypothesis assures that there is an integer k such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^k \to (\alpha \to \beta)$. Then similarly as for generalization, we find $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^k \to \Box(\alpha \to \beta)$. Because by Theorem 3.20 we also know $\vdash_{\mathbf{L}} \Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$, $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^k \to (\Diamond\alpha \to \Diamond\beta)$.

$\Box$

Summarizing the Propositions 4.30, 4.31 and 4.32, we see that all of the following statements are equivalent.

- There is an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^n \to \psi$,
- $\Gamma \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$,
- $\Gamma \cup \{\varphi\} \vdash_{\mathbf{L}} \Box\psi$,
- there is an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\varphi)^n \to \Box\psi$,
- $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \Box\psi$,
- $\Gamma \cup \{\Box\varphi\} \vdash_{\mathbf{L}} \psi$.

Remark that in IVG $\Box\varphi$ and $(\Box\varphi)^n$ $(n \geqslant 1)$ are provably equivalent, so for IVG and its axiomatic extensions we have a stronger deduction theorem: $\Gamma \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ iff $\Gamma \vdash_{\mathbf{L}} \Box\varphi \to \psi$.

4.5 Interpretation

Interval-valued fuzzy logics as we introduced them, are truth-functional logics: the truth degree of a compound proposition is determined by the truth degree of its constituent parts. This causes some counterintuitive results, if we want to interpret the element $[0,1]$ of an IVRL as uncertainty regarding the actual truth value of a proposition. For example: suppose we don't know anything about the truth value of propositions p and q, i.e., $e(p) = e(q) = [0,1]$. Then yet the implication $p \to q$ is definitely valid: $e(p \to q) = e(p) \Rightarrow e(q) = [1,1]$. However, if $\neg[0,1] = [0,1]$ [19] (which is intuitively preferable, since the negation of an uncertain proposition is still uncertain), then we can take $q = \neg p$, and obtain that $p \to \neg p$ is true. Or, equivalently (using the residuation principle), that $p\&p$ is false. This does not seem intuitive, as one would rather expect $p\&p$ to be uncertain if p is uncertain.
Another consequence of $[0,1] \Rightarrow [0,1] = [1,1]$ is that it is impossible to interpret the intervals as a set in which the 'real' (unknown) truth value is contained, and $X \Rightarrow Y$ as the smallest closed interval containing every $x \Rightarrow y$, with x in X and y in Y (as in [26]). Indeed: $1 \in [0,1]$ and $0 \in [0,1]$, but $1 \Rightarrow 0 = 0 \notin [1,1]$.
On the other hand, for t-norms it is possible that $X * Y$ is the smallest closed interval containing

[19] This is for example the case if $\neg$ is involutive.

every $x * y$, with x in X and y in Y, but only if they are t-representable (described by the axiom $\mu(x * y) = \mu x * \mu y$). However, in this case $\neg[0,1] = [0,0]$, which does not seem intuitive ('the negation of an uncertain proposition is absolutely false').

These considerations seem to suggest that IVML and its axiomatic extensions are not suitable to reason with uncertainty. This does not mean that intervals are not a good way for representing degrees of uncertainty, only that they are not suitable as truth values in a truth-functional logical calculus when we interpret them as expressing uncertainty. It might even be impossible to model uncertainty as a truth value in any truth-functional logic. This question is discussed in [24, 23]. However, nothing prevents the intervals in interval-valued fuzzy logics from having more adequate interpretations.

CHAPTER 5

Conclusion

In this work we have constructed a propositional calculus for several interval-valued fuzzy logics. With 'interval-valued fuzzy logic', we mean a logic having intervals as truth values. More precisely, the truth values are preferably subintervals of the unit interval. The idea behind it, is that such an interval can model imprecise information. A truth value like $[0.3, 0.9]$ for example is more imprecise than $[0.4, 0.5]$, but still more precise than $[0, 1]$, which stands for '(absolute) uncertainty'. To compute the truth values of 'p implies q' and 'p and q', given the truth values of p and q, we have used operations from residuated lattices. This truth-functional approach is similar to the methods developed for the well-studied fuzzy logics. Although the interpretation of the intervals as truth values expressing some kind of imprecision is a bit problematic (see Section 4.5), the purely mathematical study of the properties of interval-valued fuzzy logics and their algebraic semantics can be done without any problem. This study has been the focus of this work.

Because the truth values had to be intervals and the algebraic structures residuated lattices, we worked towards a definition of interval-valued residuated lattices (IVRLs). This was done in Chapter 2. We did this in several steps, going from very general structures (posets) to more specific ones, and gradually adding more operations. In every step we had a closer look at the corresponding interval-valued structures. For interval-valued bounded lattices we found a variety of equivalent structures: triangular lattices. These structures have the two binary operations of lattices (infimum and supremum), but also two unary operations called necessity and possibility, playing the role of the lower and upper bound of an interval, and a constant playing the role of the interval $[0, 1]$. In Proposition 2.24 we gave an alternative definition for triangular lattices. One problem about triangular lattices is still open:

Open problem 5.1 *Suppose two triangular lattices exist on the same bounded lattice. Are the two sublattices of their exact elements necessarily isomorphic?*

As a consequence of Propositions 2.21, 2.27 and 2.29, at most one triangular lattice exists on many bounded lattices. The open problem is about the remaining cases, such as Example 2.26 and Figures 2.4 and 2.5.
Then we recalled the definition and terminology of residuated lattices and summarized their most important properties. More general than the product of a residuated lattice, is the concept of a t-norm (which can be defined on any bounded poset). Similarly, more general than the implication of a residuated lattice, is the concept of an implicator. We characterized the t-norms and implicators in IVRLs that are increasing w.r.t. the imprecision ordering (Proposition 2.8). For a large class of interval-valued t-norms (defined in (2.2)), we proved the validity of an expression

for the residual implicator (Proposition 2.31) and determined which of them satisfy prelinearity (Proposition 2.36). All of these interval-valued t-norms and their residual implicators map couples of exact intervals onto exact intervals, a property we would like to have for the semantics of interval-valued fuzzy logics. Therefore we defined IVRLs as residuated lattices on interval-valued lattices in which the diagonal is closed under the product and implication.
In Chapter 2 we also introduced and investigated filters of residuated lattices (Section 2.5). Although strictly speaking we only needed prime filters (more precisely, their interval-valued counterparts IVRL-extended prime filters) in the rest of the work (namely in Section 3.7), we also had a closer look at other kinds of filters. For each identity or inequality in residuated lattices, one can define a corresponding kind of filter in such a way that the identity of inequality holds in a specific residuated lattice iff the singleton $\{1\}$ is a filter of that specific kind. Using this equivalence we can see that filters enable us to generalize connections between identities. For example, that Boolean algebras are Heyting-algebras is a special case of Boolean filters being positive implicative filters. We compared these new kinds of filters with existing ones, and summarized some of the results in Figures 2.11 and 2.12. While doing this we found out that Heyting-algebras are precisely residuated lattices satisfying generalized pseudocomplementation (Proposition 2.53). We also found a bounded lattice on which both a prelinear and a non-prelinear residuated lattice exist (Example 2.47). Furthermore we characterized finite MTL-algebras as residuated lattices in which prime filters and prime filters of the second kind coincide. In general (not assuming finiteness) this remains an open problem.

Open problem 5.2 *Let $\mathscr{L}$ be a residuated lattice in which every prime filter of the second kind is a prime filter. Is $\mathscr{L}$ prelinear?*

In Chapter 3 we introduced triangle algebras. These structures form a variety and are therefore suitable as general semantics for interval-valued fuzzy logics. Triangle algebras are a combination of triangular lattices and residuated lattices, additionally satisfying two properties that ensure that the subset of exact elements of a triangle algebra is closed under the product and implication. We gave a few alternative defining properties and derived an alternative definition, with a smaller set of (weaker) defining properties. Before we compared triangle algebras with other algebraic structures in Section 3.2, we first showed some basic properties of triangle algebras, sometimes in a slightly more general setting. A few interesting equivalences were proven in Proposition 3.7, in which we gave necessary and sufficient conditions for four properties known from modal logic. In Section 3.3 we showed the connection between triangle algebras and IVRLs and proved an alternative characterization of triangle algebras. These results are comparable with similar properties of triangular lattices shown in Section 2.3. In Section 3.4 we proved the very important Theorem 3.20: in a triangle algebra it suffices to know the product (or the implication) of each two exact elements, and the product $u * u$, to know the product and the implication of each two elements of the triangle algebra. As a consequence of this theorem, triangle algebras correspond with couples $(\mathscr{L}, \alpha)$, in which $\mathscr{L}$ is a residuated lattice and α an element in that residuated lattice. The theorem also implies that the large class of interval-valued t-norms from Chapter 2 contains all products of IVRLs. In Section 3.5, the theorem was used as well. In this section we revealed connections between properties on a triangle algebra and properties of the corresponding couple $(\mathscr{L}, \alpha)$. For some properties, such as distributivity and (pseudo/weak) divisibility, the parameter α is not important. For other properties, α does play an important role. For example: a triangle algebra can only have an involutive negation if $\alpha = 0$, while it can only satisfy contraction if $\alpha = 1$. We examined these connections for several other properties as well, and summarized them in Table 3.1. For some of them we applied a decomposition theorem for

pseudo-prelinear triangle algebras that was proven in Section 3.7. It states that each pseudo-prelinear triangle algebra is isomorphic to a subdirect product of pseudo-linear triangle algebras. This theorem is similar to a comparable decomposition theorem for MTL-algebras and implies that identities valid in all pseudo-linear triangle algebras are also valid in all pseudo-prelinear triangle algebras. The proof relied on IVRL-extended prime filters, which were introduced in Section 3.6. We distinguished two ways to define different kinds of filters of triangle algebras (IVRL-filters): one in which a kind of filter is taken on the exact elements and then extended to the whole triangle algebra (the IVRL-extended filters), and one in which the specific property is required for the whole triangle algebra. We investigated the relations between these two methods and showed that for most kinds of filters of triangle algebras, the IVRL-extended filters suffice: the IVRL-filters defined in the other way can also be constructed as IVRL-extended filters. Some cases are schematically depicted in Figure 3.4. Similarly as for filters of residuated lattices, many of the results on IVRL-filters are generalizations of properties of triangle algebras. These specific cases can be obtained by applying the propositions to the IVRL-filter $\{1\}$.

Finally, in Chapter 4, we introduced interval-valued fuzzy logics. As these logics are based on the common formal fuzzy logics, we started the chapter with an overview about formal fuzzy logics (Section 4.1). After a preliminary introduction, we had a closer look at monoidal logic (ML), monoidal t-norm based logic (MTL), basic logic (BL), Łukasiewicz logic (Ł) and classical logic (CPC). In particular, for each of them we mentioned the local deduction theorem, which soundness and completeness theorems are valid, and we gave a simplified set of axioms. We concluded Section 4.1 with a summarization of other formal fuzzy logics and mentioned the properties that an axiomatic extension of MTL is strong standard complete iff it has the real-chain embedding property, and finite strong standard complete iff it has the real-chain partial embedding property. Then we introduced interval-valued monoidal logic (IVML) – the most general interval-valued fuzzy logic in our framework – by defining its language and giving its axioms and deduction rules (Section 4.2). As all axioms and the deduction rule of ML are included in the axioms and deduction rules of IVML, all provable FL-formulae in ML are also provable in IVML. Most of the new axioms correspond with the defining properties of triangular lattices, to ensure that the truth values can be regarded as intervals. In Section 4.3, we then showed soundness and strong completeness of IVML and its axiomatic extensions, and pseudo-chain strong completeness of IVMTL and its axiomatic extensions. These results were obtained in the usual way, but paying extra attention to the new connectives, axioms and deduction rules. For the pseudo-chain completeness, we relied on the decomposition theorem for pseudo-prelinear triangle algebras from Section 3.7. Apart from IVML, we defined other interval-valued counterparts of formal fuzzy logics: IVMTL, IVBL, ... By means of the soundness and completeness result we were able to immediately give a different but equivalent axiom system for some of them. We proved that if a fuzzy logic is strong standard complete, then its interval-valued counterpart is also strong standard complete. To prove this theorem, we used the real-chain embedding property of that fuzzy logic and showed that it implies what we might call the 'pseudo-real-chain embedding property' of its interval-valued counterpart, from which the strong standard completeness (and therefore also the standard completeness) easily follows. More or less similarly, we proved that if a fuzzy logic is finite strong standard complete, then its interval-valued counterpart is also finite strong standard complete.For fuzzy logics that are not strong standard complete, we were able to show that their interval-valued counterparts are not strong standard complete either. In fact, all 'negative completeness results' of a fuzzy logic can be transferred to its interval-valued counterpart. This is due to Proposition 4.28, which states, roughly speaking, that a fuzzy logic is contained

in its interval-valued counterpart. The results about standard completeness and strong standard completeness of interval-valued fuzzy logics are summarized in Table 4.1.

Open problem 5.3 *Let* $\boldsymbol{L}$ *be an axiomatic extension of MTL that is standard complete but not finite strong standard complete. Is its interval-valued counterpart IV*$\boldsymbol{L}$ *(defined as in Definition 4.22) standard complete?*
If so, does this hold for all standard but not finite strong standard complete axiomatic extensions of MTL?
If not so, does this hold for all standard but not finite strong standard complete axiomatic extensions of MTL?

In Section 4.4 we concluded the results about interval-valued fuzzy logic with a local deduction theorem. Similarly as in formal fuzzy logic, this theorem provides a connection between the provability relation $\vdash$ and the implication symbol $\rightarrow$ appearing in formulae.

CHAPTER 6

Samenvatting

In dit werk hebben we een propositionele calculus opgebouwd voor verscheidene intervalwaardige vaaglogica's. Met 'intervalwaardige vaaglogica' bedoelen we een logica met intervallen als waarheidswaarden. Om precies te zijn, zijn de waarheidswaarden bij voorkeur deelintervallen van het eenheidsinterval. Het achterliggende idee is dat zo'n interval imprecieze informatie kan modelleren. Een waarheidswaarde als $[0.3, 0.9]$ bijvoorbeeld, is minder precies dan $[0.4, 0.5]$, maar wel preciezer dan $[0, 1]$, dat staat voor '(absolute) onzekerheid'. Om, gegeven de waarheidswaarden van p en q, de waarheidswaarden van 'p impliceert q' en 'p en q' te berekenen, gebruikten we bewerkingen van geresidueerde tralies. Deze zogenaamde waarheidsfunctionele aanpak is gelijkaardig aan de methodes ontwikkeld voor de reeds uitvoerig bestudeerde vaaglogica's. Hoewel de interpretatie van de intervallen als waarheidswaarden die een soort imprecisie uitdrukken, enigszins problematisch is (zie Paragraaf 4.5), kan het zuiver wiskundig onderzoek naar de eigenschappen van intervalwaardige vaaglogica's en hun algebraïsche semantiek probleemloos uitgevoerd worden. Het is dan ook deze studie die in het middelpunt stond van dit werk.

Omdat de waarheidswaarden intervallen moesten zijn en de algebraïsche structuren geresidueerde tralies, werkten we in Hoofdstuk 2 toe naar een definitie van intervalwaardige geresidueerde tralies (IVRLs). We deden dit stap voor stap, gaande van zeer algemene structuren (posets) tot specifiekere, waarbij geleidelijk aan meer bewerkingen werden toegevoegd. In elke stap bekeken we telkens de intervalwaardige structuren in meer detail. Voor intervalwaardige begrensde tralies vonden we een variëteit van equivalente structuren: driehoekige tralies. Deze structuren zijn niet enkel uitgerust met de twee binaire bewerkingen van tralies (infimum en supremum), maar ook met twee unaire bewerkingen genaamd necessiteit en possibiliteit, die de rol spelen van de onder- en bovengrens van een interval, en een constante die de rol speelt van het interval $[0, 1]$. In Propositie 2.24 gaven we een alternatieve definitie voor driehoekige tralies. Een probleem in verband met driehoekige tralies is nog open:

Open probleem 6.1 *Veronderstel dat er twee driehoekige tralies bestaan op eenzelfde begrensde tralie. Zijn dan de twee deeltralies bestaande uit hun exacte elementen isomorf?*

Als gevolg van Proposities 2.21, 2.27 en 2.29, bestaat er op vele begrensde tralies slechts één driehoekige tralie. Het open probleem betreft de overige gevallen, zoals Voorbeeld 2.26 en Figuren 2.4 en 2.5.
Daarna gaven wij de definitie van geresidueerde tralies, overliepen we de nodige terminologie hieromtrent en somden we de belangrijkste eigenschappen ervan op. Algemener dan het product

in geresidueerde tralies is het concept van een t-norm (die op elke begrensde poset gedefinieerd kan worden). Gelijkaardig is het concept van implicator, dat algemener is dan de implicatie in een geresidueerde tralie. We karakteriseerden de t-normen en implicatoren in IVRLs die stijgend zijn m.b.t. de imprecisie-ordening (Propositie 2.8). Voor een grote klasse van intervalwaardige t-normen (gedefinieerd in (2.2)), bewezen we de geldigheid van een uitdrukking voor de residuele implicator (Propositie 2.31) en gingen we na welke er voldoen aan prelineariteit (Propositie 2.36). Al deze intervalwaardige t-normen en hun residuele implicatoren beelden koppels exacte intervallen af op exacte intervallen, een wenselijke eigenschap voor de semantiek van intervalwaardige vaaglogica's. Daarom definieerden we IVRLs als geresidueerde tralies op intervalwaardige tralies waarin de diagonaal gesloten is onder het product en de implicatie.
In Hoofdstuk 2 introduceerden en onderzochten we ook filters van geresidueerde tralies (Paragraaf 2.5). Hoewel we strikt genomen enkel priemfilters (of meer precies, hun intervalwaardige tegenhangers IVRL-uitgebreide priemfilters) nodig hadden in de rest van het werk (namelijk in Paragraaf 3.7), traden we toch ook voor andere soorten filters in meer detail. Voor elke identiteit of ongelijkheid in geresidueerde tralies, kan men een corresponderende soort filter definiëren, op zo'n manier dat de identiteit of ongelijkheid geldig is in een bepaalde geresidueerde tralie als en slechts als het singleton $\{1\}$ een filter van die specifieke soort is. Gebruik makende van deze equivalentie kunnen we inzien dat filters ons in staat stellen om verbanden tussen identiteiten te veralgemenen. Dat Boole-algebra's Heyting-algebra's zijn bijvoorbeeld, is een speciaal geval van het feit dat Boole-filters positief implicatieve filters zijn. We vergeleken deze nieuwe soorten filters met de bestaande, en vatten enkele van de resultaten samen in Figuur 2.11 en Figuur 2.12. Bij dit onderzoek ontdekten we dat Heyting-algebra's precies de geresidueerde tralies zijn die voldoen aan veralgemeende pseudocomplementatie (Propositie 2.53). Tevens vonden we een begrensde tralie waarop zowel de structuur van een prelineaire als die van een niet-prelineaire geresidueerde tralie gelegd kon worden (Voorbeeld 2.47). Bovendien karakteriseerden we eindige MTL-algebra's als geresidueerde tralies waarin er geen onderscheid is tussen priemfilters en priemfilters van de tweede soort. In het algemeen (zonder eindigheid te veronderstellen) is dit nog een open probleem.

Open probleem 6.2 *Veronderstel dat $\mathscr{L}$ een geresidueerde tralie is waarin elke priemfilter van de tweede soort een priemfilter is. Is $\mathscr{L}$ dan noodzakelijk prelineair?*

In Hoofdstuk 3 introduceerden we driehoeksalgebra's. Deze structuren vormen een variëteit en zijn daardoor geschikt als algemene semantiek voor intervalwaardige vaaglogica's. Driehoeksalgebra's zijn een combinatie van driehoekige tralies en geresidueerde tralies, die bovendien voldoen aan twee bijkomende eigenschappen die ervoor zorgen dat de deelverzameling van exacte elementen in een driehoeksalgebra gesloten is onder het product en de implicatie. We gaven alternatieve definiërende eigenschappen en leidden een alternatieve definitie af, met minder en zwakkere definiërende eigenschappen. Voordat we driehoeksalgebra's vergeleken met andere algebraïsche structuren in Paragraaf 3.2, toonden we eerst nog enkele basiseigenschappen van driehoeksalgebra's aan, af en toe zelfs in een iets algemener kader. Enkele interessante equivalenties werden bewezen in Propositie 3.7, waarin we nodige en voldoende voorwaarden vonden voor vier eigenschappen uit modale logica. In Paragraaf 3.3 toonden we het verband aan tussen driehoeksalgebra's en IVRLs, en bewezen we een alternatieve manier om driehoeksalgebra's te definiëren. Deze resultaten zijn vergelijkbaar met eigenschappen van driehoekige tralies die werden aangetoond in Paragraaf 2.3. In Paragraaf 3.4 bewezen we het zeer belangrijke Stelling 3.20: in een driehoeksalgebra volstaat het om het product (of de implicatie) te kennen van elke twee exacte elementen, en het product $u * u$, om het product en de implicatie te kennen van

elke twee elementen van de driehoeksalgebra. Een gevolg van deze stelling is dat driehoeksalgebra's corresponderen met koppels $(\mathscr{L}, \alpha)$, waarin $\mathscr{L}$ een geresidueerde tralie is en α een element in die geresidueerde tralie. De stelling impliceert ook dat de grote klasse van intervalwaardige t-normen uit Hoofdstuk 2 alle productbewerkingen van IVRLs bevat. In Paragraaf 3.5 werd de stelling eveneens gebruikt. In deze paragraaf ontrafelden we de verbanden tussen eigenschappen in een driehoeksalgebra en eigenschappen van het corresponderende koppel $(\mathscr{L}, \alpha)$. Voor sommige eigenschappen, zoals distributiviteit en (pseudo-/zwakke) deelbaarheid, had de parameter α geen belang. Voor andere eigenschappen speelde α wel een belangrijke rol. Bijvoorbeeld: een driehoeksalgebra kan enkel een involutieve negatie hebben indien $\alpha = 0$, terwijl ze daarentegen enkel kan voldoen aan de contractie-eigenschap als $\alpha = 1$. We onderzochten deze verbanden ook voor verscheidene andere eigenschappen, en vatten deze samen in Tabel 3.1. Voor het bewijs van enkele ervan gebruikten we een decompositiestelling voor pseudo-prelineaire driehoeksalgebra's die werd aangetoond in Paragraaf 3.7. Deze zegt dat elke pseudo-prelineaire driehoeksalgebra isomorf is met een subdirect product van pseudo-lineaire driehoeksalgebra's. Deze stelling is analoog aan een vergelijkbare decompositiestelling voor MTL-algebra's en impliceert dat identiteiten die geldig zijn in alle pseudo-lineaire driehoeksalgebra's ook geldig zijn in alle pseudo-prelineaire driehoeksalgebra's. Het bewijs steunde op IVRL-uitgebreide priemfilters, die werden geïntroduceerd in Paragraaf 3.6. We onderscheidden twee manieren om verschillende soorten filters van driehoeksalgebra's (IVRL-filters) te definiëren: één waarin een filter genomen wordt van de exacte elementen en dan uitgebreid naar de volledige driehoeksalgebra (de IVRL-uitgebreide filters), en één waarin de specifieke eigenschap gewoon aan de volledige driehoeksalgebra wordt opgelegd. We onderzochten de verwantschappen tussen deze twee methodes en toonden aan dat voor de meeste soorten filters van driehoeksalgebra's, de IVRL-uitgebreide filters volstaan: de IVRL-filters die op de andere manier werden gedefinieerd kunnen ook geconstrueerd worden als IVRL-uitgebreide filters. Enkele gevallen zijn schematisch samengevat in Figuur 3.4. Net zoals bij filters van geresidueerde tralies zijn verschillende resultaten over IVRL-filters veralgemeningen van eigenschappen van driehoeksalgebras. Deze specifieke gevallen kunnen bekomen worden door de proposities toe te passen op de IVRL-filter $\{1\}$.

Tenslotte introduceerden we intervalwaardige vaaglogica's in Hoofdstuk 4. Aangezien deze logica's gebaseerd zijn op de gewone formele vaaglogica's, vingen we het hoofdstuk aan met een overzicht van formele vaaglogica's (Paragraaf 4.1). Na een inleidend gedeelte gingen we dieper in op monoïdale logica (ML), monoïdale t-norm gebaseerde logica (MTL), basis logica (BL), Łukasiewicz logica (Ł) en klassieke logica (CPC). In het bijzonder vermeldden we voor elk van deze logica's de locale deductiestelling, welke volledigheidsstellingen er geldig zijn, en gaven we een vereenvoudigde lijst axioma's. We beëindigden Paragraaf 4.1 met een opsomming van nog niet aan bod gekomen formele vaaglogica's en met de eigenschappen dat een axiomatische uitbreiding van MTL sterk standaard volledig is als en slechts als ze voldoet aan de reële-ketting inbeddingseigenschap, en eindig sterk standaard volledig als en slechts als ze voldoet aan de reële-ketting partiële inbeddingseigenschap. Daarna introduceerden we intervalwaardige monoïdale logica (IVML) – de meest algemene intervalwaardige vaaglogica in ons raamwerk – door de taal ervan vast te leggen en de axioma's en deductieregels op te geven (Paragraaf 4.2). Aangezien alle axioma's en de deductieregel van ML bevat zijn in de axioma's en deductieregels van IVML, zijn alle in ML bewijsbare FL-formules ook bewijsbaar in IVML. De meeste nieuwe axioma's komen overeen met de definiërende eigenschappen van driehoekige tralies, om te verzekeren dat de waarheidswaarden beschouwd kunnen worden als intervallen. In Paragraaf 4.3 toonden we vervolgens de betrouwbaarheid en sterke volledigheid van IVML en zijn axiomatische extensies aan, en pseudo-ketting sterke volledigheid van IVMTL en zijn axiomatische extensies. Deze resultaten werden

bekomen op de gebruikelijke manier, maar met extra aandacht voor de nieuwe connectieven, axioma's en deductieregels. Voor de pseudo-ketting volledigheid steunden we op de decompositiestelling voor pseudo-prelineaire driehoeksalgebra's uit Paragraaf 3.7. Naast IVML definieerden we nog andere intervalwaardige tegenhangers van formele vaaglogica's: IVMTL, IVBL, . . . Met behulp van de betrouwbaarheid en volledigheid waren we in staat om voor enkele van deze logica's een andere, maar equivalente lijst axioma's te geven. We toonden aan dat als een vaaglogica sterk standaard volledig is, dat dan ook zijn intervalwaardige tegenhanger sterk standaard volledig is. Om dit aan te tonen, gebruikten we de reële-ketting inbeddingseigenschap van de vaaglogica en toonden aan dat dit een eigenschap impliceert van de intervalwaardige tegenhanger die we 'pseudo-reële-ketting inbeddingseigenschap' zouden kunnen noemen. Uit deze eigenschap volgt dan eenvoudig de sterke standaard volledigheid (en dus ook de standaard volledigheid). Min of meer analoog toonden we aan dat als een vaaglogica eindig sterk standaard volledig is, dat dan ook zijn intervalwaardige tegenhanger eindig sterk standaard volledig is. Voor vaaglogica's die niet sterk standaard volledig zijn, konden we aantonen dat hun intervalwaardige tegenhangers ook niet sterk standaard volledig konden zijn. In feite kunnen alle 'negatieve volledigheidsresultaten' van een vaaglogica overgedragen worden op zijn intervalwaardige tegenhanger. Dit volgt uit Propositie 4.28, die enigszins vereenvoudigd stelt dat een vaaglogica bevat is in zijn intervalwaardige tegenhanger. De resultaten over standaard volledigheid en sterke standaard volledigheid van intervalwaardige vaaglogica's zijn samengevat in Tabel 4.1.

Open probleem 6.3 *Veronderstel dat* $\mathbf{L}$ *een axiomatische extensie is van MTL die standaard volledig is maar niet eindig sterk standaard volledig. Is de intervalwaardige tegenhanger IV*$\mathbf{L}$ *(gedefinieerd zoals in Definition 4.22) standaard volledig?*
Indien ja, is dit geldig voor alle standaard maar niet eindig sterk standaard volledige axiomatische extensies van MTL?
Indien nee, is dit geldig voor alle standaard maar niet eindig sterk standaard volledige axiomatische extensies van MTL?

In Paragraaf 4.4 sloten we de resultaten over intervalwaardige vaaglogica's af met een locale deductiestelling. Net zoals in formele vaaglogica legt deze stelling een verband tussen de bewijsbaarheidsrelatie $\vdash$ en het implicatiesymbool $\rightarrow$ dat in formules voorkomt.

Bibliography

[1] C. Alcalde, A. Burusco, R. Fuentes-González, **A constructive method for the definition of interval-valued fuzzy implication operators**, Fuzzy Sets and Systems 153, (2005) 211–227

[2] K. Atanassov, **Intuitionistic fuzzy sets**, Physica-Verlag, Heidelberg, (1999)

[3] B.R.C. Bedregal, A. Takahashi, **The best interval representations of t-norms and automorphisms**, Fuzzy Sets and Systems 157, (2006), 3220–3230

[4] L. Běhounek, P. Cintula, **Fuzzy logics as the logics of chains**, Fuzzy Sets and Systems 157(5), (2006), 604–610

[5] R. Belohlávek, **Some properties of residuated lattices**, Czechoslovak Mathematical Journal 53 (128), (2003), 161–171

[6] R. Belohlávek, V. Vychodil, **Fuzzy Equational Logic**, Studies in Fuzziness and Soft Computing 186, (2005)

[7] H. Bustince, **Indicator of inclusion grade for interval-valued fuzzy sets. Application to approximate reasoning based on interval-valued fuzzy sets**, International Journal of Approximate Reasoning 23, (2000), 137–209

[8] G. Cattaneo, **Abstract approximation spaces for rough theories**, in: Rough Sets in Knowledge Discovery 1: Methodology and Applications (L. Polkowski, A. Skowron, eds.), Physica-Verlag, (1998), 59–98

[9] G. Cattaneo, D. Ciucci, **Intuitionistic fuzzy sets or orthopair fuzzy sets?**, Proceedings of the third EUSFLAT Conference, Zittau, Germany, (2003), 153–158

[10] C.C. Chang, **Algebraic analysis of many valued logics**, Transactions of the American Mathematical Society 88(2), (1958), 467–490

[11] C.C. Chang, **A new proof of the completeness of the Lukasiewicz axioms**, Transactions of the American Mathematical Society 93(1), (1959), 74–80

[12] A. Ciabattoni, G. Metcalfe, F. Montagna, **Adding modalities to fuzzy logics**, Proceedings of the 26th Linz Seminar on Fuzzy Set Theory, Austria, (2005), 27–33

[13] R. Cignoli, F. Esteva, L. Godo, A. Torrens, **Basic fuzzy logic is the logic of continuous t-norms and their residua**, Soft Computing 4, (2000), 106–112

[14] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna, C. Noguera, **Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies**, Annals of Pure and Applied Logic 160(1), (2009), 53–81

[15] C. Cornelis, G. Deschrijver, E.E. Kerre, **Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application**, International Journal of Approximate Reasoning 35, (2004), 55–95

[16] C. Cornelis, G. Deschrijver, E.E. Kerre, **Advances and challenges in interval-valued fuzzy logic**, Fuzzy Sets and Systems 157(5), (2006), 622–627

[17] G. Deschrijver, **The Łukasiewicz t-norm in interval-valued fuzzy and intuitionistic fuzzy set theory**, in: Issues in the Representation and Processing of Uncertain and Imprecise Information. Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets, and Related Topics (K. T. Atanassov, J. Kacprzyk, M. Krawczak, E. Szmidt, eds.), Akademicka Oficyna Wydawnicza EXIT, (2005), 83–101

[18] G. Deschrijver, **A representation of t-norms in interval-valued L-fuzzy set theory**, Fuzzy Sets and Systems 159(13), (2008), 1597–1618

[19] G. Deschrijver, C. Cornelis, E.E. Kerre, **On the representation of intuitionistic fuzzy t-norms and t-conorms**, IEEE Transactions on Fuzzy Systems 12, (2004), 45–61

[20] G. Deschrijver, E.E. Kerre, **On the relationship between some extensions of fuzzy set theory**, Fuzzy Sets and Systems 133(2), (2003), 227–235

[21] G. Deschrijver, E.E. Kerre, **Classes of intuitionistic fuzzy t-norms satisfying the residuation principle**, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 11, (2003), 691–709

[22] R.P. Dilworth, M. Ward, **Residuated latties**, Transactions of the American Mathematical Society 45, (1939), 335–354

[23] D. Dubois, **On ignorance and contradiction considered as truth values**, personal communication, (2006)

[24] D. Dubois, H. Prade, **Can we enforce full compositionality in uncertainty calculi?**, in: Proceedings of the 11th National Conference on Artificial Intelligence (AAAI'94), Seattle, Washington, (1994), 149–154

[25] M. Dummett, **A propositional calculus with denumerable matrix**, The Journal of Symbolic Logic 24(2), (1959), 97–106

[26] F. Esteva, P. Garcia-Calvés, L. Godo, **Enriched interval bilattices and partial many-valued logics: an approach to deal with graded truth and imprecision**, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 2(1), (1994), 37–54

[27] F. Esteva, J. Gispert, L. Godo, F. Montagna, **On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic**, Studia Logica 71(2), (2002), 199–226

[28] F. Esteva, L. Godo, **Monoidal t-norm based logic: towards a logic for left-continuous t-norms**, Fuzzy Sets and Systems 124, (2001), 271–288

[29] F. Esteva, L. Godo, A. Garcia-Cerdaña, **On the hierarchy of t-norm based residuated fuzzy logics**, in: Beyond Two: Theory and Applications of Multiple Valued Logic (M. Fitting, E. Orlowska, eds.), Physica-Verlag, (2003), 251–272

[30] L. Ferrari, G. Nicoletti, **A combinatorial representation for a special class of complete distributive lattices**, Annals of Combinatorics 5, (2001), 285–304

[31] J. Fodor, R. Yager, **Fuzzy set-theoretic operators and quantifiers**, in: Fundamentals of Fuzzy Sets (D. Dubois, H. Prade, eds.), Kluwer Academic Publishers, (2000), 125–193

[32] J.M. Font, **Beyond Rasiowa's algebraic approach to non-classical logics**, Studia Logica 82(2), (2006), 172–209

[33] J.M. Font, R. Jansana, D. Pigozzi, **A survey of abstract algebraic logic**, Studia Logica 74, (2003), 13–79

[34] J.M. Font, A.J. Rodriguez, A. Torrens, **Wajsberg algebras**, Stochastica 8, (1984), 5–31

[35] À. García-Cerdaña, C. Noguera, F. Esteva, **On the scope of some formulas defining additive connectives in fuzzy logics**, Fuzzy Sets and Systems 154(1), (2005), 56–75

[36] J. Garson, **Modal Logic**, The Stanford Encyclopedia of Philosophy (Edward N. Zalta, ed.), (2003)

[37] M. Gehrke, C. Walker, E. Walker, **Some comments on interval-valued fuzzy sets**, International Journal of Intelligent Systems 11, (1996), 751–759

[38] K. Gödel, **Zum intuitionistischen Aussagenkalkül**, Anzeiger der Akademie der Wissenschaften in Wien, (1932), 65–66

[39] J.A. Goguen, **L-fuzzy sets**, Journal of Mathematical Analysis and Applications 18, (1967), 145Ű-174

[40] S. Gottwald, **Mathematical fuzzy logic as a tool for the treatment of vague information**, Information Sciences 172, (2005), 41–71

[41] S. Gottwald, P. Hájek, **Triangular norm-based mathematical fuzzy logics**, in: Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (E.P. Klement, R. Mesiar, eds.), Elsevier, (2005), 275–300

[42] P. Hájek, **Metamathematics of Fuzzy Logic**, Trends in Logic—Studia Logica Library, Kluwer Academic Publishers, (1998)

[43] A. Heyting, **Die formalen Regeln der intuitionistischen Logik**, Sitzungsberichte der preuszischen Akademie der Wissenschaften, physikalisch-mathematische Klasse, (1930), 42–56 57–71 158–169 in three parts, Sitzungsber. preuss. Akad. Wiss.: 42–71, 158–169. English translation of Part I in Mancosu 1998: 311–327.

[44] U. Höhle, **Commutative, residuated l-monoids**, in: Non-classical Logics and their Applications to Fuzzy Subsets: a Handbook of the Mathematical Foundations of Fuzzy Set Theory (U. Höhle, E.P. Klement, eds.), Kluwer Academic Publishers, (1995), 53–106

[45] R. Horčík, **Standard completeness theorem for ΠMTL**, Archive for Mathematical Logic 44(4), (2005), 413–424

[46] R. Horčík, **On the failure of standard completeness in ΠMTL for infinite theories**, Fuzzy Sets and Systems 158, (2007), 619–624

[47] E.V. Huntington, **Sets of independent postulates for the algebra of logic**, Transactions of the American Mathematical Society 5, (1904), 288–309

[48] S. Jenei, **A more efficient method for defining fuzzy connectives**, Fuzzy Sets and Systems 90, (1997), 25–35

[49] S. Jenei, F. Montagna, **A proof of standard completeness for Esteva and Godo's logic MTL**, Studia Logica 70, (2002), 1–10

[50] P. Jipsen, **Some lists of finite structures**, at `http://www1.chapman.edu/~jipsen/gap/rl.html`,

[51] Y.B. Jun, Y. Xu, X.H. Zhang, **Fuzzy filters of MTL-algebras**, Information Sciences 175, (2005), 120–138

[52] E.P. Klement, R. Mesiar, E. Pap, **Triangular Norms**, Trends in Logic—Studia Logica Library, Kluwer Academic Publishers, (2000)

[53] L. Liu, K. Li, **Fuzzy filters of BL-algebras**, Information Sciences 173, (2005), 141–154

[54] L. Liu, K. Li, **Boolean filters and positive implicative filters of residuated lattices**, Information Sciences 177(24), (2007), 5725–5738

[55] J. Łukasiewicz, A. Tarski, **Untersuchungen über den Aussagenkalkül**, Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, (1930), 1–21

[56] J.M. Mendel, **Uncertain Rule-Based Fuzzy Logic Systems**, Prentice Hall PTR, Upper Saddle River, New Jersey, (2001)

[57] J.M. Mendel, **Advances in type-2 fuzzy sets and systems**, Information Sciences 177, (2007), 84–110

[58] G. Metcalfe, N. Olivetti, D. Gabbay, **Proof Theory for Fuzzy Logics**, Springer Series in Applied Logic 36, (2008)

[59] F. Montagna, C. Noguera, R. Horčík, **On weakly cancellative fuzzy logics**, Journal of Logic and Computation 16(4), (2006), 423–450

[60] V. Novák, I. Perfilieva, J. Mockor, **Mathematical Principles of Fuzzy Logic**, Kluwer Academic Publishers, (1999)

[61] M. Ohnishi, K. Matsumoto, **Gentzen method in modal calculi, parts I and II**, Osaka Mathematical Journal 9 and 11, (1957), 113–130, (1959), 115–120

[62] H. Ono, **Logics without contraction rule and residuated lattices I**, personal communication, (2001)

[63] H. Ono, **Modalities in substructural logic – a preliminary report**, Proceedings of the 39th MLG meeting at Gamagori, Japan, (2005), 36–38

[64] J. Rachunek, D. Salounová, **Modal operators on bounded commutative residuated l-monoids**, Mathematica Slovaca 57(4), (2007), 321–332

[65] J. Rachunek, D. Salounová, **Fuzzy filters and fuzzy prime filters of bounded Rl-monoids and pseudo BL-algebras**, Information Sciences 178(17), (2008), 3474–3481

[66] H. Rasiowa, **An algebraic approach to non-classical logics**, Studies in Logic and the Foundations of Mathematics 78, (1974)

[67] H. Rasiowa, R. Sikorski, **The mathematics of metamathematics**, Polska Akad. Nauk, (1963)

[68] B. Schweizer, A. Sklar, **Statistical metric spaces**, Pacific J. Math. 10, (1960), 313–334

[69] J.R. Shoenfield, **Mathematical Logic**, Addison-Wesley, (1967)

[70] P. Smets, P. Magrez, **Implication in fuzzy logic**, International Journal of Approximate Reasoning 1, (1987), 327–347

[71] E. Turunen, **Mathematics behind Fuzzy Logic**, Physica-Verlag, (1999)

[72] E. Turunen, **Boolean deductive systems of BL-algebras**, Archive for Mathematical Logic 40, (2001), 467–473

[73] B. Van Gasse, **Algebraïsche ordestructuren voor vaaglogische calculi**, Licentiaatsproefschrift (in Dutch), Universiteit Gent, (2005)

[74] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **On the properties of a generalized class of t-norms in interval-valued fuzzy logics**, New Mathematics and Natural Computation 2(1), (2006), 29–42

[75] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **Triangle algebras: towards an axiomatization of interval-valued residuated lattices**, in: Lecture Notes in Computer Science, Proceedings of the 5th International Conference on Rough Sets and Current Trends in Computing (RSCTC 2006), (2006), 117–126

[76] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **Triangle algebras: A formal logic approach to interval-valued residuated lattices**, Fuzzy Sets and Systems 159, (2008), 1042–1060

[77] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **A characterization of interval-valued residuated lattices**, International Journal of Approximate Reasoning 49, (2008), 478–487

[78] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **The pseudo-linear semantics of interval-valued fuzzy logics**, Information Sciences 179, (2009), 717–728

[79] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **Filters of residuated lattices and triangle algebras**, Information Sciences, accepted

[80] B. Van Gasse, C. Cornelis, G. Deschrijver, E.E. Kerre, **The standard completeness of interval-valued monoidal t-norm based logic**, Information Sciences, under review

[81] L.A. Zadeh, **Fuzzy sets**, Information and Control 8, (1965), 338–353

[82] L.A. Zadeh, **The concept of a linguistic variable and its application to approximate reasoning - I**, Information Sciences 8, (1975), 199–249

[83] E.N. Zalta, **Basic concepts in modal logic**, `http://mally.stanford.edu/notes.pdf`, (1995)

Index

— *H* —

— *I* —

— *L* —

— *M* —

— *N* —

— *P* —

— *R* —

— *S* —

— *T* —

— *U* —

— *V* —

— *W* —