Nonlinear diffusion problem with dynamical boundary value

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Abstract

A nonlinear degenerate convection-diffusion initial boundary value problem is studied in a bounded domain. A dynamical boundary condition (containing the time derivative of a solution) is prescribed on the one part of the boundary. This models a non-perfect contact on the boundary. Existence and uniqueness of a weak solution in corresponding function spaces is proved using the backward Euler method for the time discretization. Error estimates for time-discrete approximations are derived.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ be a bounded domain with the Lipschitz continuous boundary Γ which consists of the three non-overlapping parts Γ_N, Γ_D and Γ_T with $meas\{\Gamma_D\} > 0$. In a one dimensional case we just skip the part Γ_N.

In this article we deal with the following nonlinear degenerate convection-diffusion equation with dynamical boundary condition

$$
\begin{aligned}
\partial_t \theta(u) - \nabla \cdot (\nabla u + \boldsymbol{b}(u)) &= f &&\text{in} && Q_T = \Omega \times (0, T) \\
\partial_t \beta(u) + (\nabla u + \boldsymbol{b}(u)) \cdot \boldsymbol{\nu} &= g &&\text{on} && \Gamma_T \times (0, T) \\
u &= 0 &&\text{on} && \Gamma_D \times (0, T) \\
(\nabla u + \boldsymbol{b}(u)) \cdot \boldsymbol{\nu} &= 0 &&\text{on} && \Gamma_N \times (0, T) \\
u(0) &= u_0 &&\text{in} && \Omega.
\end{aligned}
\tag{1}
$$

Here, the nonlinear functions θ and β are monotonically increasing. Their derivatives can vanish or they can be unbounded, both cases are allowed. A physical motivation comes from the modelling of rainfall infiltration through soil. Fluid flow through a porous medium is described by Darcy's law. On the ground surface the rainfall rate q_0 equals to the seepage velocity of the water $\boldsymbol{q} \cdot \boldsymbol{\nu} = \boldsymbol{a}(u, \nabla u) \cdot \boldsymbol{\nu}$. This standard Neumann boundary condition (BC) holds when the rain totally infiltrates. Nevertheless in practice, the ground surface can become saturated. Then water doesn't infiltrate into the soil, but accumulates on its surface. We denote the ponding rate $\partial_t u$ and impose the following refined BC on the ground surface

$$
H(u)\partial_t u - \boldsymbol{q} \cdot \boldsymbol{\nu} = \partial_t \int_0^u H(s) \, \mathrm{d}s - \boldsymbol{q} \cdot \boldsymbol{\nu} = q_0
$$

where $H(v) = 1$ if $v > 0$ and $H(v) = 0$ if $v < 0$. Setting $\beta(v) = \max\{v, 0\}$ we arrive at (1)b.

The Dirichlet and Neumann conditions come naturally from limiting the problem to a bounded domain.

Settings with dynamical BCs have already been studied in the literature. Similar problem but with a linear differential operator was studied in [1]. Paper [2] deals with a nonlinear differential operator with $\beta = \theta$, but the case

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when β' can vanish is not considered there. In fact, the nonlinearity there is hidden under the differential operator. Problem (1) was discussed in [3] with restriction to a Lipschitz-type nonlinearity. The boundary condition with a time derivative is sometimes called Wentzel boundary condition after A.D. Wentzel (see [4]). The authors of [5] have extensively studied similar BCs using semigroup theory. A derivation of a physical interpretation has been performed in [6]. We would like to recall [7], where a general setting for parabolic degenerate equations has been studied subjected to standard BCs.

The aim of this paper is to prove the well-possedness of the problem (1). This will be done using the theory of monotone operators and the Rothe method for the time discretization. We recall that the derivatives of θ and β are allowed to vanish and they can be unbounded. We also derive the error estimates for time-discrete approximations. The last section is devoted to numerical experiments to support obtained convergence results.

2. Stability

Let us start with our assumptions on the data. The real functions β and θ are continuous. Furthermore

$$
\begin{aligned}
&\beta(0) = 0, \quad 0 \le \beta', \quad |\beta(s)| \le C(1 + |s|) \\
&\theta(0) = 0, \quad 0 < \lambda \le \theta', \quad |\theta(s)| \le C(1 + |s|), \\
&|\boldsymbol{b}(u) = (b_1(u), \dots, b_d(u))| \le C, \quad |\boldsymbol{b}'(u)| = |(b'_1(u), \dots, b'_d(u))| \le C, \\
&f \in C((0, T), L_2(\Omega)), \\
&g \in C\left((0, T); L^2(\Gamma_T)\right).
\end{aligned}
\tag{2}
$$

We assume the function θ to be strictly monotone, but neither θ nor β have upper bound for its first derivative, hence they can degenerate in this sense.

We denote by $(w, z)_M$ the standard L^2-scalar product of the functions w and z on a measurable set M, i.e.

$$
(w, z)_M = \int_M w(x)z(x)\, \mathrm{d}x
$$

and the corresponding norm

$$
\|w\|_M^2 = (w, w)_M.
$$

The subscript will be suppressed if $M = \Omega$. A natural choice of the test space is

$$
V = \{\varphi \in H^1(\Omega)| \quad \varphi|_{\Gamma_D} = 0\}.
\tag{3}
$$

Its dual space is denoted by V^*. The variational formulation of (1) then reads as follows:
Find $u \in L^2((0, T); V)$ and $\partial_t \theta(u), \partial_t \beta(u) \in L^2((0, T); V^*)$ such that the following identity

$$
(\partial_t \theta(u), \varphi) + (\nabla u + \boldsymbol{b}(u), \nabla \varphi) + (\partial_t \beta(u), \varphi)_{\Gamma_T} = (f, \varphi) + (g, \varphi)_{\Gamma_T}
\tag{4}
$$

holds true for all $\varphi \in V$ and a.e in $(0, T)$. Let us here recall Friedrich's inequality for a function $\varphi \in V$

$$
C\|\varphi\|_{H^1(\Omega)} \le \|\nabla \varphi\|.
\tag{5}
$$

As it is usual in papers of this sort, C, ε and C_ε will denote generic positive constants depending only on a priori known quantities, where ε is sufficiently small and C_ε is large.

The proof of existence of a solution to (1) will be based on the Rothe method of time discretization, cf. [8]. We divide the time interval $[0, T]$ into $n \in \mathbb{N}$ equidistant subintervals $[t_{i-1}, t_i]$ for $t_i = i\tau$, where $\tau = \frac{T}{n}$. We introduce the following notation

$$
w_i = w(t_i), \qquad \delta w_i = \frac{w_i - w_{i-1}}{\tau}
$$

for any function w.

The time discretised variational formulation reads as

$$
(\delta\theta(u_i), \varphi) + (\nabla u_i + \boldsymbol{b}(u_i), \nabla \varphi) + (\delta\beta(u_i), \varphi)_{\Gamma_T} = (f_i, \varphi) + (g_i, \varphi)_{\Gamma_T}
\tag{6}
$$

for all $\varphi \in V$ and $i = 1, \dots, n$. The first lemma asserts existence and uniqueness of the solution on every time step.

Lemma 1. *Suppose (2), $u_0 \in V$. Then there exist $\tau_0 > 0$ and a unique $u_i \in V$ solving the variational problem (6) for any $i = 1, \ldots, n$ and $\tau < \tau_0$.*

Proof. We apply the theory of monotone operators - see [9]. Let the operator $A : V \to V^*$ be defined as

$$\langle A(u), \varphi \rangle := (\theta(u), \varphi) + \tau \left(\nabla u + \boldsymbol{b}(u), \nabla \varphi \right) + (\beta(u), \varphi)_{\Gamma_T} \ .$$

This operator is obviously hemicontinuous.

We employ the monotonicity of β and θ to get the strong monotonicity of A. For a sufficiently small time step τ, it holds that

$$
\begin{aligned}
\langle A(u) - A(v), u - v \rangle &= (\theta(u) - \theta(v), u - v) + \tau \left\| \nabla(u - v) \right\|^2 \\
&\quad + \tau \left(\boldsymbol{b}(u) - \boldsymbol{b}(v), \nabla(u - v) \right) + (\beta(u) - \beta(v), u - v)_{\Gamma_T} \\
&\geq \lambda \left\| u - v \right\|^2 + \tau \left\| \nabla(u - v) \right\|^2 + \tau (\boldsymbol{b}(u) - \boldsymbol{b}(v), \nabla(u - v)) \\
&\geq (\lambda - \tau C) \| u - v \|^2 + \frac{\tau}{2} \| \nabla(u - v) \|^2 \\
&\geq \frac{\tau}{2} \| \nabla(u - v) \|^2.
\end{aligned}
$$

The strong monotonicity implies the uniqueness of the solution. $\qquad\square$

Let us note that the problem (6) at each time step is nonlinear. For suitable linearizations we refer the reader e.g. to [10, 11].

Let γ be a monotone increasing real function with $\gamma(0) = 0$. We introduce $\Phi_\gamma(z) := \int_0^z \gamma(s) \, ds$. One can easily check

$$\gamma(z_1)(z_2 - z_1) \leq \Phi_\gamma(z_2) - \Phi_\gamma(z_1) \leq \gamma(z_2)(z_2 - z_1), \tag{7}$$

which holds for any $z_1, z_2 \in \mathbb{R}$. One can generalize this assertion to monotone increasing graphs like in Fig. (1), cf [12]. Now, we derive a priori estimates for u_i.
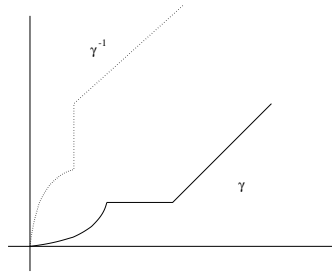

Figure 1: γ function and its inverse "graph" γ^{-1}

Lemma 2. *Suppose (2), $u_0 \in V$. Then there exists a positive constant C such that*

1. $\displaystyle\sum_{i=1}^{n} \| \nabla u_i \|^2 \tau \leq C.$

2. $\sum_{i=1}^{n} \|\delta u_i\|^2 \tau + \max_{1 \le j \le n} \|\nabla u_j\|^2 + \sum_{i=1}^{n} \|\nabla(u_i - u_{i-1})\|^2 \le C.$

Proof. (*i*) We set $\varphi = \tau u_i$ in (6) and sum up over $i = 1, \dots, n$. One can get

$$\sum_{i=1}^{n} (\delta\theta(u_i), u_i)\,\tau + \sum_{i=1}^{n} \|\nabla u_i\|^2\,\tau + \sum_{i=1}^{n} (\boldsymbol{b}(u_i), \nabla u_i)\,\tau + \sum_{i=1}^{n} (\delta\beta(u_i), u_i)_{\Gamma_T}\,\tau$$
$$= \sum_{i=1}^{n} (f_i, u_i)\,\tau + \sum_{i=1}^{n} (g_i, u_i)_{\Gamma_T}\,\tau.$$

According to (7), one can write for the first term ($\tilde{\Phi}_\theta(z) := z\theta(z) - \Phi_\theta(z) \ge 0$)

$$\begin{aligned}
\sum_{i=1}^{n} (\theta(u_i) - \theta(u_{i-1}), u_i) &= (\theta(u_n), u_n) - (\theta(u_0), u_0) - \sum_{i=1}^{n} (u_i - u_{i-1}, \theta(u_{i-1})) \\
&\ge (\theta(u_n), u_n) - (\theta(u_0), u_0) - \sum_{i=1}^{n} \int_\Omega [\Phi_\theta(u_i) - \Phi_\theta(u_{i-1})] \\
&= \left[(\theta(u_n), u_n) - \int_\Omega \Phi_\theta(u_n)\right] - \left[(\theta(u_0), u_0) - \int_\Omega \Phi_\theta(u_0)\right] \\
&= \int_\Omega \left[\tilde{\Phi}_\theta(u_n) - \tilde{\Phi}_\theta(u_0)\right] \\
&\ge -C.
\end{aligned}$$

The term containing β can be estimated in the same way. For the terms containing $\boldsymbol{b}, f$ and g we make use of the Cauchy, Young and Friedrichs inequalities and the trace theorem to conclude the proof.

(*ii*) We set $\varphi = \tau\delta u_i$ and sum it up for $i = 1, \dots, j$

$$\sum_{i=1}^{j} (\delta\theta(u_i), \delta u_i)\,\tau + \sum_{i=1}^{j} (\nabla u_i, \nabla\delta u_i)\,\tau + \sum_{i=1}^{j} (\boldsymbol{b}(u_i), \nabla\delta u_i)\,\tau + \sum_{i=1}^{j} (\delta\beta(u_i), \delta u_i)_{\Gamma_T}\,\tau \tag{8}$$
$$= \sum_{i=1}^{j} (f_i, \delta u_i)\,\tau + \sum_{i=1}^{j} (g_i, \delta u_i)_{\Gamma_T}\,\tau.$$

For the terms in the right hand side (RHS) we apply the Abel summation; Cauchy's, Young's inequalities; the trace theorem and the Friedrichs inequality. We demonstrate this for the term containing g. We deduce that

$$\begin{aligned}
\sum_{i=1}^{j} (g_i, \delta u_i)_{\Gamma_T}\,\tau &= \left(g_j, u_j\right)_{\Gamma_T} - (g_0, u_0)_{\Gamma_T} - \sum_{i=1}^{j} (\delta g_i, u_{i-1})_{\Gamma_T}\,\tau \\
&\le \|g_j\|_{\Gamma_T} \|u_j\|_{\Gamma_T} + \|g_0\|_{\Gamma_T} \|u_0\|_{\Gamma_T} + \sum_{i=1}^{j} \|\delta g_i\|_{\Gamma_T} \|u_i\|_{\Gamma_T}\,\tau \\
&\le \varepsilon \|\nabla u_j\|^2 + C_\varepsilon.
\end{aligned}$$

Further we can write

$$\sum_{i=1}^{j} (\boldsymbol{b}(u_i), \delta\nabla u_i)\,\tau = \left(\boldsymbol{b}(u_j), \nabla u_j\right) - (\boldsymbol{b}(u_0), \nabla u_0) - \sum_{i=1}^{j} (\delta\boldsymbol{b}(u_i), \nabla u_{i-1})\,\tau. \tag{9}$$

From this we see that

$$\sum_{i=1}^{j} (\boldsymbol{b}(u_i), \delta\nabla u_i)\,\tau \le \varepsilon \|\nabla u_j\|^2 + C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\,\tau.$$

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According to the monotonicity of θ and β we get from (8)

$$\lambda \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \frac{1}{2} \left[\|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 - \|\nabla u_0\|^2 \right] \leq \varepsilon \|\nabla u_j\|^2 + C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.$$

We conclude the proof by choosing a sufficiently small positive ε.

$\square$

Remark 1. *If the rainfall water is cumulated over the porous medium, then fully saturated zone at the top of the porous medium can appear. The derivative θ' equals zero and an elliptic equation takes place. The assumption of Lemma 2 (ii) on strict monotonicity of θ is no longer satisfied. Even in this case one can establish a priori estimates for time derivatives using dual norms. Existence and uniqueness can be still showed, however no error estimates can be obtained.*

3. Existence and uniqueness

We introduce the following notation for piecewise-linear (constant) in time functions u_n ($\bar{u}_n$) for $i = 1, \ldots, n$

$$u_n(0) = u(0) \qquad\qquad\qquad \bar{u}_n(0) = u(0)$$
$$u_n(t) = u_{i-1} + \delta u_i(t - t_{i-1}) \quad \text{for } t \in (t_{i-1}, t_i] \qquad \bar{u}_n(t) = u_i \quad \text{for } t \in (t_{i-1}, t_i]$$

and θ_n, β_n:

$$\theta_n(0) = \theta(u(0)) \qquad\qquad\qquad \beta_n(0) = \beta(u(0))$$
$$\theta_n(t) = \theta(u_{i-1}) + \delta\theta(u_i)(t - t_{i-1}) \quad \text{for } t \in (t_{i-1}, t_i] \qquad \beta_n(t) = \beta(u_{i-1}) + \delta\beta(u_i)(t - t_{i-1})$$
$$\text{for } t \in (t_{i-1}, t_i].$$

The functions $\bar{f}_n$ and $\bar{g}_n$ are defined in a similar way. We can rewrite (6) in terms of the new notation as

$$(\partial_t \theta_n, \varphi) + (\nabla \bar{u}_n + \boldsymbol{b}(\bar{u}_n), \nabla \varphi) + (\partial_t \beta_n, \varphi)_{\Gamma_T} = \left(\bar{f}_n, \varphi \right) + (\bar{g}_n, \varphi)_{\Gamma_T}. \tag{10}$$

Now, we are in a position to address the existence of a weak solution to (1).

Theorem 1 (Existence). *Suppose (2), $u_0 \in V$. Then there exists a solution to (4).*

Proof. Using the results of Lemma (2) and applying [13, Thm. 2.13.1] we get the existence of a subsequence of $\bar{u}_n$ (denoted by the same symbol again) such that

$$\lim_{n \to \infty} \bar{u}_n \to u \qquad \text{in } L^2(Q_T).$$

Therefore we also get

$$\bar{u}_n \to u \qquad \text{a.e. in } Q_T. \tag{11}$$

The following inequality (see [14]) holds true

$$\|z\|_\Gamma^2 \leq \varepsilon \|\nabla z\|^2 + C_\varepsilon \|z\|^2, \qquad \forall z \in H^1(\Omega), \qquad 0 < \varepsilon < \varepsilon_0. \tag{12}$$

Lemma (2) and the reflexivity of $L^2((0, T); V)$ give (for a subsequence)

$$\bar{u}_n \rightharpoonup u \qquad \text{in } L^2((0, T); V).$$

This together with (12) implies

$$\int_0^T \|\bar{u}_n - u\|_\Gamma^2 \leq \varepsilon \int_0^T \|\nabla(\bar{u}_n - u)\|^2 + C_\varepsilon \int_0^T \|\bar{u}_n - u\|^2 \leq C\varepsilon + C_\varepsilon \int_0^T \|\bar{u}_n - u\|^2.$$

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Passing to the limit for $\tau \to 0$ and applying (11) we obtain

$$\lim_{\tau \to 0} \int_0^T \|\bar{u}_n - u\|_\Gamma^2 \leq C\varepsilon,$$

which is valid for any small $\varepsilon > 0$. Hence

$$\lim_{\tau \to 0} \int_0^T \|\bar{u}_n - u\|_\Gamma^2 = 0 \qquad \text{and} \qquad \bar{u}_n \to u \text{ a.e. in } \Gamma \times (0, T). \tag{13}$$

A simple calculation gives for $t \in (t_{i-1}, t_i]$

$$
\begin{aligned}
\langle G_n(t), \varphi \rangle &:= (\theta_n(t) - \theta(\bar{u}_n(t)), \varphi) + (\beta_n(t) - \beta(\bar{u}_n(t)), \varphi)_{\Gamma_T} \\
&= \frac{t - t_i}{\tau} \left[(\theta(\bar{u}_n(t)) - \theta(\bar{u}_n(t - \tau)), \varphi) + (\beta(\bar{u}_n(t)) - \beta(\bar{u}_n(t - \tau)), \varphi)_{\Gamma_T} \right] \\
&= (t - t_i) \left[\left(\bar{f}_n(t), \varphi \right) + (\bar{g}_n(t), \varphi)_{\Gamma_T} - (\nabla \bar{u}_n(t) + \boldsymbol{b}(\bar{u}_n(t)), \nabla \varphi) \right].
\end{aligned} \tag{14}
$$

Therefore we can see that

$$\lim_{n \to \infty} \langle G_n(t), \varphi \rangle = 0.$$

Let us introduce

$$
\begin{aligned}
\langle F_n(t), \varphi \rangle &:= (\theta_n(t), \varphi) + (\beta_n(t), \varphi)_{\Gamma_T} \\
&= (\theta(\bar{u}_n(t)), \varphi) + (\beta(\bar{u}_n(t)), \varphi)_{\Gamma_T} + \langle G_n(t), \varphi \rangle.
\end{aligned}
$$

We may write

$$\langle F_n(t), \varphi \rangle - \langle F_n(0), \varphi \rangle = \int_0^t \langle \partial_t F_n(s), \varphi \rangle \, \mathrm{d}s$$

and

$$
\begin{aligned}
&(\theta(\bar{u}_n(t)), \varphi) + (\beta(\bar{u}_n(t)), \varphi)_{\Gamma_T} + \langle G_n(t), \varphi \rangle - (\theta(u_0), \varphi) - (\beta(u_0), \varphi)_{\Gamma_T} \\
&= \int_0^t \left[\left(\bar{f}_n(s), \varphi \right) + (\bar{g}_n(s), \varphi)_{\Gamma_T} - (\nabla \bar{u}_n(s) + \boldsymbol{b}(\bar{u}_n(s)), \nabla \varphi) \right] \, \mathrm{d}s.
\end{aligned}
$$

According to the previous considerations, we may apply the Lebesgue dominated theorem to pass to the limit for $n \to \infty$ in the last relation. We arrive at

$$
\begin{aligned}
&(\theta(u(t)), \varphi) + (\beta(u(t)), \varphi)_{\Gamma_T} - (\theta(u_0), \varphi) + (\beta(u_0), \varphi)_{\Gamma_T} \\
&= \int_0^t \left[(f(s), \varphi) + (g(s), \varphi)_{\Gamma_T} - (\nabla u(s) + \boldsymbol{b}(u(s)), \nabla \varphi) \right] \, \mathrm{d}s.
\end{aligned}
$$

This is valid for any $t \in (0, T)$. Hence, differentiating with respect to the time variable we see that u is a solution to (4). $\qquad \square$

Theorem 2 (Uniqueness). *Suppose (2), $u_0 \in V$. Then (4) admits at most one solution.*

Proof. Assume that u and v solve (4). We subtract the corresponding variational formulations from each other, integrate in time, set the difference of both solutions as a test function and again integrate in time. We get

$$
\begin{aligned}
&\int_0^t (\theta(u) - \theta(v), u - v) + \int_0^t (\beta(u) - \beta(v), u - v)_{\Gamma_T} + \int_0^t \left(\int_0^s \nabla[u - v], \nabla[u(s) - v(s)] \right) \\
&= \int_0^t \left(\int_0^s \boldsymbol{b}(v) - \boldsymbol{b}(u), \nabla[u(s) - v(s)] \right).
\end{aligned} \tag{15}
$$

Let us note that the first two terms are non-negative and

$$\int_0^t \left(\int_0^s \nabla[u - v], \nabla[u(s) - v(s)] \right) = \frac{1}{2} \left\| \int_0^t \nabla[u - v] \right\|^2.$$

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Next, we deduce that

$$\lambda \int_0^t \|u - v\|^2 + \frac{1}{2} \left\| \int_0^t \nabla[u - v] \right\|^2 \leq \left(\int_0^t \boldsymbol{b}(v) - \boldsymbol{b}(u), \int_0^t \nabla[u(s) - v(s)] \right)$$
$$- \int_0^t \left(\boldsymbol{b}(v) - \boldsymbol{b}(u), \int_0^s \nabla[u(s) - v(s)] \right)$$
$$\leq \varepsilon \int_0^t \|u - v\|^2 + C_\varepsilon \int_0^t \left\| \int_0^t \nabla[u - v] \right\|^2.$$

Fixing a sufficiently small positive ε and applying Gronwall's argument we arrive at

$$\left\| \int_0^t \nabla[u - v] \right\|^2 = 0.$$

Hence $\int_0^t [u - v]$ is constant in space for any $t \in (0, T)$. Taking into account the fact that $u(t) = v(t)$ on Γ_D we conclude that $u(t) = v(t)$ in Ω. $\qquad\square$

4. Error estimates

The following theorem concerns the error estimates for the time discretization.

Theorem 3 (Error). *Suppose (2), $u_0 \in V$. Then there exists a positive constant C such that*

$$\int_0^T \|\bar{u}_n - u\|^2 + \left\| \int_0^T \nabla(\bar{u}_n - u) \right\|^2 \leq C\tau.$$

Proof. First we subtract (4) from (10). We put $\varphi(t) = -\int_s^t (\bar{u}_n(\sigma) - u(\sigma)) \, d\sigma$ and integrate in time:

$$- \int_0^s \left(\partial_t \theta_n - \partial_t \theta(u), \int_s^t (\bar{u}_n - u) \right) - \int_0^s \left(\nabla(\bar{u}_n - u), \int_s^t \nabla(\bar{u}_n - u) \right)$$
$$- \int_0^s \left(\partial_t \beta_n - \partial_t \beta(u), \int_s^t (\bar{u}_n - u) \right)_{\Gamma_T} = \int_0^s \left((\boldsymbol{b}(\bar{u}_n) - \boldsymbol{b}(u), \int_s^t \nabla(\bar{u}_n - u) \right). \qquad (16)$$

For the first term in (16), integration by parts gives us

$$- \int_0^s \left(\partial_t (\theta_n - \theta(u)), \int_s^t (\bar{u}_n - u) \right) = \int_0^s (\theta_n - \theta(u), \bar{u}_n - u) - \left(\theta_n - \theta(u), \int_s^t (\bar{u}_n - u) \right) \Big|_0^s$$
$$= \int_0^s (\theta_n \pm \theta(\bar{u}_n) - \theta(u), \bar{u}_n - u) \geq \int_0^s \lambda \|\bar{u}_n - u\|^2 - \int_0^s (\theta(\bar{u}_n) - \theta_n, \bar{u}_n - u).$$

For the second term in (16), it easy to check that

$$- \int_0^s \left(\nabla(\bar{u}_n - u), \int_s^t \nabla(\bar{u}_n - u) \right) = \frac{1}{2} \left\| \int_0^s \nabla(\bar{u}_n - u) \right\|^2.$$

For the third term in (16) we have a similar result as for the first one

$$- \int_0^s \left(\partial_t (\beta_n - \beta(u)), \int_s^t (\bar{u}_n - u) \right)_{\Gamma_T} \geq - \int_0^s (\beta(\bar{u}_n) - \beta_n, \bar{u}_n - u)_{\Gamma_T}.$$

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For the RHS of (16) we use Young's inequality

$$\int_0^s \left((\boldsymbol{b}(\bar{u}_n) - \boldsymbol{b}(u)), \int_s^t \nabla(u - \bar{u}_n) \right) \le \varepsilon \int_0^s \|(\boldsymbol{b}(\bar{u}_n) - \boldsymbol{b}(u))\|^2 + C_\varepsilon \int_0^s \left\| \int_s^t \nabla(u - \bar{u}_n) \right\|^2$$

$$\le \varepsilon \int_0^s \|\bar{u}_n - u\|^2 + C_\varepsilon \int_0^s \left\| \int_s^t \nabla(u - \bar{u}_n) \right\|^2.$$

Collecting the estimates and choosing ε sufficiently small we obtain

$$\int_0^s \|\bar{u}_n - u\|^2 + \left\| \int_0^s \nabla(\bar{u}_n - u) \right\|^2$$

$$\le C \int_0^s \left\| \int_s^t \nabla(u - \bar{u}_n) \right\|^2 + C \int_0^s (\theta(\bar{u}_n) - \theta_n, \bar{u}_n - u) + C \int_0^s (\beta(\bar{u}_n) - \beta_n, \bar{u}_n - u)_{\Gamma_T}$$

$$\le C \int_0^s \left\| \int_t^s \nabla(u - \bar{u}_n) \right\|^2 + C\tau,$$

where we used (14) and Lemma (2) to get

$$\int_0^s (\theta(\bar{u}_n) - \theta_n, \bar{u}_n - u) + \int_0^s (\beta(\bar{u}_n) - \beta_n, \bar{u}_n - u)_{\Gamma_T} \le C\tau.$$

Finally Gronwall's argument implies the statement. $\qquad\square$

5. Numerical experiments

This part is devoted to some numerical experiments to support the theoretical part. In the first two examples we study an absolute error in the $L^2(Q_T)$ sense. The third example concerns a qualitative behaviour of a solution with the dynamical BC. We make a comparision to the model with Neumann BC (see Introduction for the derivation).

Our computational scheme follows the theoretical analysis. For the space discretization we use the finite element method with linear Lagrange finite elements and a sufficiently fine grid to make space error lower order than the time error. We apply fixed-point method for solving nonlinear algebraic systems. In experiments the domain Ω is the unit square $(0, 1)^2$ in xy-plane. Its triangularization consists of 5000 triangles.

Experiment 1

If we take the advection term $\boldsymbol{b}(u)$ to be zero and consider the nonlinearity $\theta(u) = u^{1/\gamma}$, where $\gamma > 1$, then the equation in the domain simplifies to the well-known porous medium equation. Providing appropriate initial and boundary conditions a solution to this equation is Barenblatt solution (see for instance [15, Chapter 4.2.2])

$$u(\mathbf{x}, t) = \frac{1}{t^a} \left(c - \frac{\gamma - 1}{2\gamma} b \frac{|\mathbf{x}|^2}{t^{2b}} \right)_+^{\gamma/(\gamma-1)} \qquad (\mathbf{x} \in \mathbb{R}^d, t > 0)$$

where $a = \frac{d}{d(\gamma-1)+2}$, $b = a/d$, $c > 0$ and $u_+ = \max\{u, 0\}$. Herein we take $d = 2$ and $\gamma = 2$. For this particular value of the exponent γ the equation is also known as Boussinesq's equation . The function $\beta(u)$ will equal $u^{1/2}$. Assuming that this is the exact solution of our problem, we compute the function g and modify the Dirichlet and the Neumann conditions in accordance with it. We solve the numerical problem on the time interval $(0.1, 1.1)$. The Fig. 2(a) shows the dependence of the absolute error between the exact and numerical solution with respect to the time step τ in log-log scale. The dotted line has the slope 1.

Experiment 2

In this experiment we assume a linear advective vector field $\boldsymbol{b}(u) = (u, u, u)^T$. The nonlinear functions are $\theta(u) = u^{1/2}$ and $\beta(u) = u^2$. We test our scheme with the following exact solution.

$$u(\mathbf{x}, t) = (y + 0.1x - 1 + t)_+^2$$

for which we compute f and g. We solve the numerical problem on the time interval $(0.0, 1.0)$. The Fig. 2(b) shows the dependence of the absolute with respect to the time step τ in the log-log scale. We see that the behaviour of the error is similar as the one in the previous experiment.
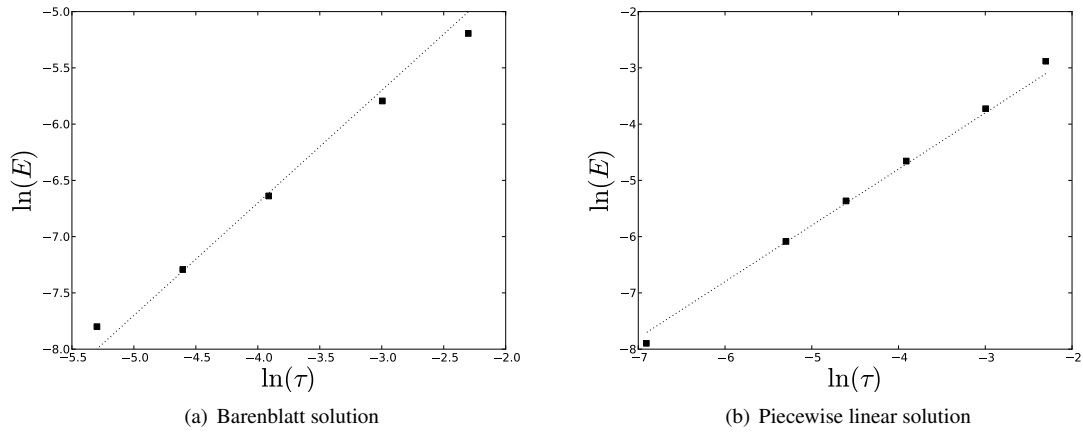
(a) Barenblatt solution

(b) Piecewise linear solution

Figure 2: Absolute errors with respect to time step

Experiment 3

We consider $\theta(u) = u^{1/2}$, $\beta(u) = u^{3/2}$, $\boldsymbol{b}(u) = \boldsymbol{0}$ and the zero source inside the domain ($f = 0$). The dynamical BC prescribed on the top of domain: $\Gamma_T = \{0 \le x \le 1 , y = 1\}$. The starting time $T_0 = 0$ and the end time $T_{\text{end}} = 1.0$. We compare the numerical solution of (1) ("dynamical problem") to the one just with Neumann BC on Γ_T, i.e. the case when $\beta(u) \equiv 0$ ("Neumann problem"). Let us first consider time-constant boundary source $g(\mathbf{x}) = 0.5 + 0.2 \sin(2\pi x)$. Figure 3 shows contour plots of the solution with dynamical and Neumann condition BC in different in two time steps. The dynamical condition causes a certain delay of the flow from the boundary. The time-dependent source case is more interesting. Figure 4 shows the results for $g(\mathbf{x}, t) = \big(0.03 + 0.012 \sin(2\pi x)\big)/\big(0.01 + (t - 0.5)^2\big)$ which has stationary point in $t = 0.5$ as a function of time. Here we can see that a saturation effect takes place and slows down the diffusion through the upper boundary. When g starts decreasing in time, the delay is compensated and both solutions line up.

Figure 3: Contour plot of the numerical solutions for time-constant g at $t = 1.0$

(a) Solution of "dynamical problem"
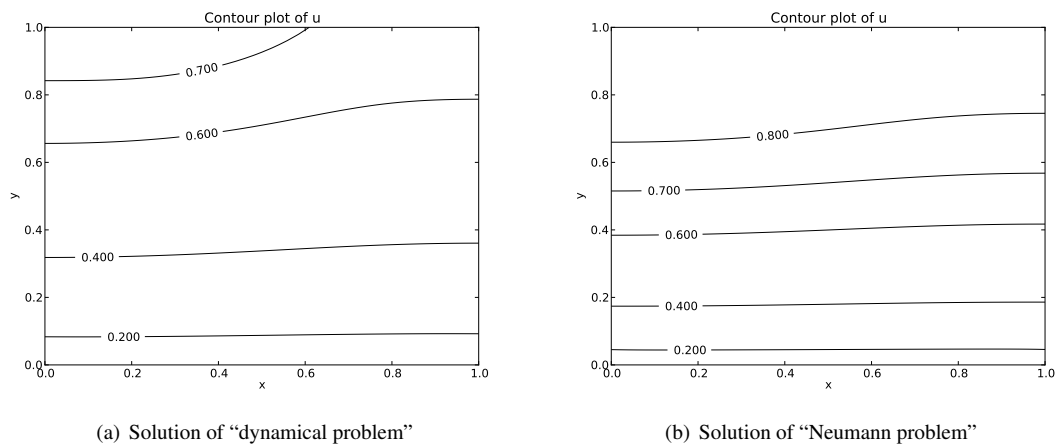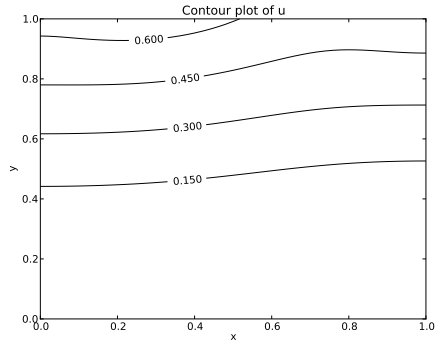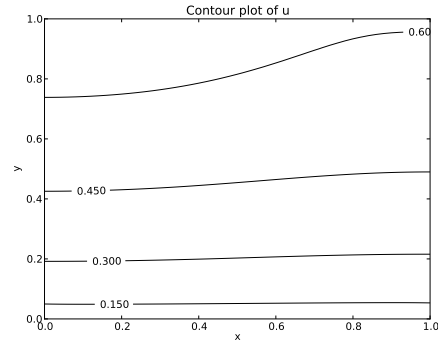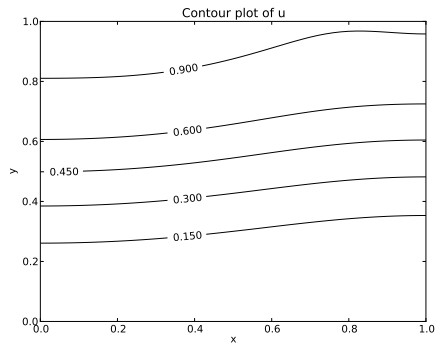

(b) Solution of "Neumann problem"

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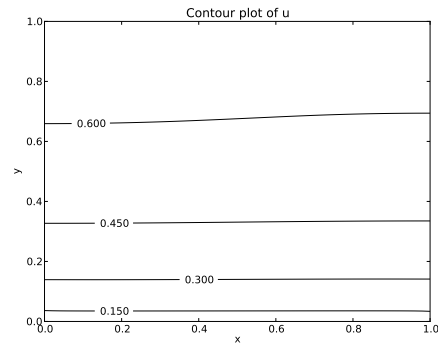Figure 4: Contour plot of the numerical solutions for time-dependant g

(a) Solution of "dynamical problem" at time $t = 0.5$

(b) Solution of "dynamical problem" at time $t = 1.0$

(c) Solution of "Neumann problem" at time $t = 0.5$

(d) Solution of "Neumann problem" at time $t = 1.0$

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