On the fundamental solution for higher spin Dirac operators

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Abstract

In this paper, we will determine the fundamental solution for the higher spin Dirac operator $\mathcal{Q}_\lambda$, which is a generalization of the classical Rarita-Schwinger operator to more complicated irreducible (half-integer) representations for the spin group in m dimensions. This will allow us to generalise Stokes' theorem, the Cauchy-Pompeiu theorem and Cauchy's integral formula, which lie at the very basis of the function theory behind arbitrary elliptic higher spin operators.

Keywords: Clifford analysis, fundamental solution, higher spin, Dirac operator, Stokes' theorem, Cauchy integral formula

1. Introduction

This article is to be situated in the theory of Clifford analysis, a generalisation of classical complex analysis in the plane to the case of an arbitrary dimension $m \in \mathbb{Z}$ (in case of a negative dimension, one is dealing with so-called super Clifford analysis). At the heart of the theory lies the Dirac operator ∂_x on $\mathbb{R}^m$, a conformally invariant first-order differential operator which plays the same role in classical Clifford analysis as the Cauchy-Riemann operator ∂_z does in complex analysis. Moreover, the Dirac operator satisfies the relation $\partial_x^2 = -\Delta_x$, which means that Clifford analysis can be seen as a refinement of harmonic analysis on $\mathbb{R}^m$.
The classical theory is centered around the study of functions on $\mathbb{R}^m$ which take values in the complex Clifford algebra $\mathbb{C}_m$ or a corresponding Spin(m)-subrepresentation, known as the spinor spaces (cfr. [1, 6, 10]). In recent years, several authors [2, 3, 4, 8] have been studying generalisations of classical Clifford analysis techniques to the so-called higher spin theory. This brings us to higher spin Dirac operators (or HSD-operators for short), generalised Dirac operators acting on functions on $\mathbb{R}^m$, which take values in arbitrary irreducible representations $\mathcal{S}_\lambda^\pm$ of the Spin(m)-group, with dominant half-integer highest weights. An explicit expression for these HSD-operators, which can be seen as generalised gradients in the sense of Stein and Weiss, was determined in [7]. The first generalisation appearing in Clifford analysis was the Rarita-Schwinger operator, originally inspired by equations coming from theoretical physics (see [12]). In the present context it is considered as the conformally invariant operator acting on functions taking values in $\mathcal{S}_1^\pm$ (see below for a definition).

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In classical Clifford analysis, the Cauchy integral formula has proved to be a corner stone of the function theory: it can be used to decompose arbitrary null solutions for the Dirac operator into homogeneous components and forms the basis to develop boundary value theory. This article explains how a higher spin version of this formula can be obtained. Cauchy integral formulae naturally rely upon the existence of a fundamental solution for the (higher spin) Dirac operator. That is why, in the first place, a fundamental solution for $\mathcal{Q}_\lambda$ will be constructed, hereby relying on results from distribution theory. Also, this will lead to a generalized Stokes' and Cauchy-Pompeiu theorem.

2. On Clifford analysis

The universal Clifford algebra $\mathbb{R}_m$ is the associative algebra generated by an orthonormal basis $(e_1, \ldots, e_m)$ for $\mathbb{R}^m$. The multiplication in the Clifford algebra is governed by the relations $e_ie_j + e_je_i = -2\delta_{ij}$, for all $i, j \in \{1, \ldots, m\}$. The complexification of $\mathbb{R}_m$ is defined by $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$. Let e_A be a basis element of $\mathbb{C}_m$, defined as $e_A = e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_h}$, with $i_1 < i_2 < \cdots < i_h$. The reversion or main anti-involution $a \mapsto a^*$ is defined on basis elements by means of $e_A^* = e_{i_h} \cdot \ldots \cdot e_{i_1}$, and is then linearly extended to the entire Clifford algebra $\mathbb{C}_m$:

$$(a_Ae_A + a_Be_B)^* = a_Ae_A^* + a_Be_B^*,$$

for all $a_A, a_B \in \mathbb{C}$. This anti-involution has the property that $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{C}_m$. Analogously, we can define the Hermitean conjugation $a \mapsto a^\dagger$. On the basis elements e_A, it is defined by means of $e_A^\dagger = (-1)^h e_A^* = (-1)^h e_{i_h} \cdot \ldots \cdot e_{i_1}$, and it is then extended anti-linearly:

$$(a_Ae_A + a_Be_B)^\dagger = \overline{a}_Ae_A^\dagger + \overline{a}_Be_B^\dagger,$$

for all $a_A, a_B \in \mathbb{C}$. Here, $\bar{\cdot}$ denotes the complex conjugation. An alternative basis for $\mathbb{C}_m$, which will turn out to be very convenient to introduce the spinor spaces, is the so-called Witt basis:

Definition 1. *The Witt basis in* $\mathbb{C}_{2n}$ *is defined by means of*

$$(\mathfrak{f}_j, \mathfrak{f}_j^\dagger) \quad := \quad \left(\frac{e_j - ie_{j+n}}{2}, -\frac{e_j + ie_{j+n}}{2}\right) ,$$

where $1 \leq j \leq n$.

This basis has the properties that $\mathfrak{f}_j\mathfrak{f}_k = -\mathfrak{f}_k\mathfrak{f}_j$, $\mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger = -\mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger$ and $\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}$. In terms of these basis elements, one can then define the idempotent $I = \mathfrak{f}_1\mathfrak{f}_1^\dagger \cdots \mathfrak{f}_n\mathfrak{f}_n^\dagger$, satisfying $I^2 = I$. We will now introduce the spinor space(s) $\mathbb{S}_{2n}^\pm$, vector spaces carrying the basic half-integer representations for the Spin(m)-group, which can be realised inside the Clifford algebra $\mathbb{C}_m$ by means of

$$\text{Spin}(m) := \left\{ s = \prod_{j=1}^{2k} \omega_j : k \in \mathbb{N}, \omega_j \in S^{m-1} \right\},$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$. Note that the parity sign will only play a role in case the dimension $m = 2n$ is even. First of all, we define the complex vector space $\mathbb{S}_{2n} = \mathbb{C}_{2n}I$. This space carries a canonical multiplicative action of the Clifford algebra $\mathbb{C}_{2n}$, denoted by $\gamma : \mathbb{C}_{2n} \to \text{End}(\mathbb{S}_{2n})$ and defined by means of $\gamma(a)[\psi] = a\psi$, for all $a \in \mathbb{C}_{2n}$ and $\psi \in \mathbb{S}_{2n}$. As $\text{Spin}(2n) \subset \mathbb{C}_{2n}$, we can now also restrict this representation for $\mathbb{C}_{2n}$ to the spin group. However, as each element of Spin($2n$)

belongs to the *even* subalgebra $\mathbb{C}_{2n}^+$ of $\mathbb{C}_{2n}$, the restriction of γ to $\mathrm{Spin}(2n)$ splits into 2 irreducible subrepresentations (the so-called spinor representations), respectively given by

$$\rho_\pm : \ \mathrm{Spin}(2n) \to \mathrm{Aut}(\mathbb{S}_{2n}^\pm),$$

where $\mathbb{S}_{2n} = \mathbb{S}_{2n}^+ \oplus \mathbb{S}_{2n}^-$, and $\mathbb{S}_{2n}^\pm$ are the graded subspaces of $\mathbb{S}_{2n}$. The highest weights for these representations are the basic half-integer highest weights $(\frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$. In case the dimension $m = 2n+1$ is odd, there exists a *unique* spinor representation for $\mathrm{Spin}(m)$, which amounts to saying that the parity index can be omitted. To define this representation, we first note that $\mathrm{Spin}(2n+1) \subset \mathbb{C}_{2n+1}^+ \cong \mathbb{C}_{2n}$, where the algebra isomorphism can be defined in terms of the basis vectors by means of

$$\psi : \mathbb{C}_{2n} \to \mathbb{C}_{2n+1}^+ : e_j \mapsto e_j e_m \quad (1 \leq j \leq 2n) \ .$$

Using this isomorphism, we can define the spinor representation for $\mathrm{Spin}(2n+1)$ as follows:

$$\rho : \ \mathrm{Spin}(2n+1) \to \mathrm{Aut}(\mathbb{S}_{2n}) : s \mapsto \gamma\big(\psi^{-1}(s)\big) \ .$$

Another option is to define the action of $\mathrm{Spin}(2n+1)$ on $\mathbb{S}_{2n+2}^\pm$, taking into account that both spinor spaces then become *isomorphic*. Note that we will from now on omit the dimension in the notation for spinor spaces, and write $\mathbb{S}^\pm$ instead of $\mathbb{S}_{2n}^\pm$. The parity index should then be omitted in case $m = 2n+1$ is odd.

As was mentioned in the introduction, the Dirac operator is a key operator in Clifford analysis (cfr. [1, 6, 10]). It is defined by means of

$$\partial_x := \sum_{j=1}^m e_j \partial_{x_j} \ \in \ \mathrm{Hom}_{\mathbb{C}}\big(\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^\pm), \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^\mp)\big) \ .$$

In order to define *general* HSD-operators we need to define the spaces of (homogeneous) simplicial and harmonic polynomials in several vector variables, as they provide explicite models for more complicated representations for the spin group (e.g. [5]). For notational convenience, we will use the notation ∂_i for the Dirac operator ∂_{u_i} in the vector variable u_i, and the short-hand notation $u_{(p)} = (u_1, \ldots, u_p) \in \mathbb{R}^{pm}$ (with $p \in \mathbb{N}$ the number of dummy indices).

Definition 2. *A function $P : \mathbb{R}^{pm} \to \mathbb{S}^\pm : (u_{(p)}) \mapsto P(u_{(p)})$ is called simplicial monogenic if it satisfies the system*

$$\begin{aligned} \partial_i P &= 0 \quad (i = 1, \ldots, p) \\ \langle u_i, \partial_j \rangle P &= 0 \quad (1 \leq i < j \leq p) \ . \end{aligned}$$

The space of $\mathbb{S}^\pm$-valued simplicial monogenic polynomials which are homogeneous of degree l_i in u_i, will be denoted by $\mathcal{S}^\pm_{l_1,\ldots,l_p}$, or $\mathcal{S}^\pm_\lambda$ for short ($\lambda = (l_1, \ldots, l_p)$).

Definition 3. *A function $P : \mathbb{R}^{pm} \to \mathbb{C} : (u_{(p)}) \mapsto P(u_{(p)})$ is called simplicial harmonic if it satisfies the system*

$$\begin{aligned} \langle \partial_i, \partial_j \rangle P &= 0 \quad (i = 1, \ldots, p) \\ \langle u_i, \partial_j \rangle P &= 0 \quad (1 \leq i < j \leq p) \ . \end{aligned}$$

The vector space of $\mathbb{C}$-valued simplicial harmonic polynomials which are homogeneous of degree l_i in u_i will be denoted by $\mathcal{H}_{l_1,\ldots,l_p}$ or $\mathcal{H}_\lambda$ for short. The space of polynomials $\mathcal{H}_\lambda$ can be seen as a module for the spin group under the induced regular representation, also known as the H-action, defined for all $s \in \mathrm{Spin}(m)$ by means of

$$H(s)P(u_{(p)}) = P(s^* u_{(p)} s),$$

where $s^* u_{(p)} s = (s^* u_1 s, \ldots, s^* u_p s)$. The polynomial space $\mathcal{S}_\lambda^\pm$ can also be seen as a module for the spin group under the induced representation, known as the L-action, defined for all $s \in \mathrm{Spin}(m)$ by means of $L := H \otimes \rho_\pm$. In other words:

$$L(s)P(u_{(p)}) = sP(s^* u_{(p)} s).$$

Remark 1. *We will also need the L-action for arbitrary vectors $x \in \mathbb{R}^m$ (hereby slightly abusing the notation, as $x \in \mathbb{R}^m$ is not a spin element). With $xu_{(p)}x := (xu_1x, \ldots, xu_px)$, this action is defined as follows: $L(x)P(u_{(p)}) := xP(x^* u_{(p)} x)$. This action is well-defined, since $x^* u_i x = xu_i x = -2\langle x, u_i\rangle x + |x|^2 u_i \in \mathbb{R}^m$ is still a vector.*

We can now introduce the HSD-operator $\mathcal{Q}_\lambda$ ([7]):

Definition 4. *Let λ be an arbitrary half-integer dominant highest weight for Spin(m). The HSD-operator is then defined as*

$$\mathcal{Q}_\lambda := \prod_{i=1}^{p} \left(1 + \frac{u_i \partial_i}{m + 2l_i - 2i}\right) \partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda^\pm) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda^\mp).$$

The product should be seen as an ordered product (with $i = 1$ to p from left to right) since the factors do not commute.

This first-order differential operator $\mathcal{Q}_\lambda$ is elliptic and conformally invariant. Its existence and uniqueness (up to a multiplicative constant) are guaranteed by tools coming from representation theory, see e.g. [9, 13].

3. Fundamental solution

Before turning to the fundamental solution for the operator $\mathcal{Q}_\lambda$, we will first consider a few examples to get a grip on the general idea behind its construction and properties.

3.1. HSD-operators of order ≤ 2

The fundamental solution $N(x)$ for the Laplace operator Δ_x is given by

$$N(x) = \begin{cases} \dfrac{1}{(2-m)A_m |x|^{m-2}} & m > 2 \\[2ex] \dfrac{1}{2\pi} \log|x| & m = 2, \end{cases}$$

where A_m is the surface area of the unit sphere S^{m-1}. In view of the fact that $\Delta_x = -\partial_x^2$, the fundamental solution $E(x)$ for the Dirac operator is easily obtained as

$$E(x) = -\partial_x N(x) = -\frac{1}{A_m} \frac{x}{|x|^m},$$

This expression is also called the Cauchy kernel and, as a fundamental solution for the Dirac operator, it satisfies the relation $\partial_x E(x) = \delta(x)$. Denoting $\mathbb{R}^m \backslash \{0\}$ by means of $\mathbb{R}_0^m$, we can say that

$E(x)$ is an element of the function space $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathbb{C}_m)$. Because $\mathbb{C}_m$ can be seen as the space of endomorphisms of the (total) spinor space $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, we thus have that $E(x) \in \mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathbb{S}))$.

For the Rarita-Schwinger operator $\mathcal{R}_{l_1}$, the first higher spin generalisation of the Dirac operator, the fundamental solution has been constructed in [3] as

$$E_{l_1}(x; u_1, u_1') = -\frac{1}{A_m}\frac{m+2l_1-2}{m-2}\frac{x}{|x|^{m+2l_1}} K_{l_1}(xu_1x, u_1').$$

Here, $K_{l_1}(u_1, u_1')$ denotes the so-called reproducing kernel for l_1-homogeneous monogenic polynomials, which has the property that

$$(K_{l_1}(u_1, u_1'), P_{l_1}(u_1))_{(u_1)} = P_{l_1}(u_1'),$$

where the notation $(.,.)_{(u_1)}$ refers to the Fischer inner product on $\mathcal{P}(\mathbb{R}^m, \mathbb{S}^\pm)$, given by:

$$(f(u_1), g(u_1))_{(u_1)} = \left[f(\partial_1)^\dagger g(u_1)\right]\Big|_{u_1=0}.$$

The fundamental solution then satisfies $\mathcal{R}_{l_1} E_{l_1}(x; u_1, u_1') = \delta(x) K_{l_1}(u_1, u_1')$. This time, the fundamental solution for the operator $\mathcal{R}_{l_1}$ belongs to the function space $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathcal{S}^\pm_{l_1}))$. In full generality, we can therefore expect the fundamental solution for $\mathcal{Q}_\lambda$ to belong to the function space $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathcal{S}^\pm_\lambda))$.

3.2. HSD-operators of general order.
The main result of this section is the following:

Proposition 1. *Let $C_\lambda \in \mathbb{R}$ be a constant. For every $P_\lambda(u_{(p)}) \in \mathcal{S}^\pm_\lambda$, the function*

$$E_\lambda(x; u_{(p)}) := C_\lambda |x|^{-m+1} L\left(\frac{x}{|x|}\right) P_\lambda(u_{(p)})$$

belongs to $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathcal{S}^\pm_\lambda)$. Furthermore, $E_\lambda(x; u_{(p)})$ belongs to the kernel of the operator $\mathcal{Q}_\lambda$ and has a singularity of degree $(-m+1)$ in $x = 0$.

The first step in proving Proposition 1 is showing that $E_\lambda(x; u_{(p)})$ belongs to $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathcal{S}^\pm_\lambda)$. To do so, we need the following relation:

Lemma 1. *For all indices $1 \leq a \leq p$, we have the relation*

$$\partial_a = x\partial_{xu_ax}x. \tag{1}$$

Proof. We prove this by direct calculation. Let $(u_a)_j$ be the j-th component of the vector variable u_a. We then get that

$$\begin{aligned}
\partial_a &= \sum_{j=1}^m e_j \partial_{(u_a)_j} \\
&= \sum_{j=1}^m e_j \sum_{k=1}^m \partial_{(xu_ax)_k} \frac{\partial}{\partial (u_a)_j}(-2\langle u_a, x\rangle x_k + |x|^2 (u_a)_k) \\
&= \sum_{j=1}^m e_j \sum_{k=1}^m \partial_{(xu_ax)_k} \cdot (-2x_j x_k + |x|^2 \delta_{jk}) \\
&= |x|^2 \partial_{xu_ax} - 2\langle \partial_{xu_ax}, x\rangle x \\
&= x\partial_{xu_ax}x\ ,
\end{aligned}$$

which proves the lemma. □

Lemma 2. *For all $P_\lambda(u_{(p)})$ in $\mathcal{S}_\lambda^\pm$, the polynomial $xP_\lambda(xu_{(p)}x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda^\pm)$. Moreover, we also have that $P_\lambda(xu_{(p)}x) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}^\pm)$.*

Proof. Due to Lemma 1, we have that

$$\partial_i x P_\lambda(xu_{(p)}x) = -x|x|^2 \partial_{xu_ix} P_\lambda(xu_{(p)}x) = 0,$$

since $\partial_i P_\lambda(u_{(p)}) = 0$. From (1), we have that $\partial_{xu_ix} = \frac{1}{|x|^4} x\partial_i x$, leading to

$$\langle xu_ix, \partial_{xu_jx} \rangle \;=\; \frac{1}{|x|^4} \langle xu_ix, x\partial_j x \rangle \;=\; \langle u_i, \partial_j \rangle\ .$$

From this, we can then derive that $\langle u_i, \partial_j \rangle P_\lambda(xu_{(p)}x) = \langle u_i, \partial_j \rangle xP_\lambda(xu_{(p)}x) = 0$, for all $1 \leq i < j \leq m$. Putting everything together, it then follows that $xP_\lambda(xu_{(p)}x)$ is an element of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda^\pm)$. As a result, one also has that $P_\lambda(xu_{(p)}x) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}^\pm)$. □

In [5], it was shown that the irreducible finite-dimensional representation $\mathcal{S}_\lambda^\pm$, with highest weight λ, is generated by the highest weight vector

$$\langle u_1, \mathfrak{f}_1 \rangle^{l_1 - l_2} \langle u_1 \wedge u_2, \mathfrak{f}_1 \wedge \mathfrak{f}_2 \rangle^{l_2 - l_3} \cdots \langle u_1 \wedge \cdots \wedge u_p, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_p \rangle^{l_p} I^\pm,$$

where each of these inner products is defined by means of

$$\langle u_1 \wedge \cdots \wedge u_k, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_k \rangle \;=\; \det \begin{pmatrix} \langle u_1, \mathfrak{f}_1 \rangle & \cdots & \langle u_1, \mathfrak{f}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle u_k, \mathfrak{f}_1 \rangle & \cdots & \langle u_k, \mathfrak{f}_k \rangle \end{pmatrix} \tag{2}$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \langle u_{\sigma(1)}, \mathfrak{f}_1 \rangle \cdots \langle u_{\sigma(k)}, \mathfrak{f}_k \rangle, \tag{3}$$

S_k being the symmetric group in k elements, and where $I^+ = I$ and $I^- = \mathfrak{f}_n^\dagger I$. Without loss of generality, we can now choose

$$P_\lambda(u_{(p)}) = \langle u_1, \mathfrak{f}_1 \rangle^{l_1 - l_2} \langle u_1 \wedge u_2, \mathfrak{f}_1 \wedge \mathfrak{f}_2 \rangle^{l_2 - l_3} \cdots \langle u_1 \wedge \cdots \wedge u_p, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_p \rangle^{l_p} I^\pm\ ,$$

since all operators in $\mathrm{Alg}_{\mathbb{C}}\{x, \partial_x, u_1, \ldots, u_p, \partial_1, \ldots, \partial_p\}$ are $\mathrm{Spin}(m)$-invariant, and

$$\mathcal{S}_\lambda^\pm = \mathrm{Span}_{\mathbb{C}}\{L(s)P_\lambda(u_{(p)}) : s \in \mathrm{Spin}(m)\}.$$

Defining $|\lambda| = l_1 + \cdots + l_p$, this choice for P_λ then leads to

$$|x|^{-m+1} L\left(\frac{x}{|x|}\right) P_\lambda \;=\; \frac{x\langle xu_1x, \mathfrak{f}_1 \rangle^{l_1 - l_2} \cdots \langle xu_1x \wedge \cdots \wedge xu_px, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_p \rangle^{l_p}}{|x|^{m+2|\lambda|}} I^\pm. \tag{4}$$

We will now prove the second part of Proposition 1, namely that the expression (4) indeed belongs the kernel of $\mathcal{Q}_\lambda$ for $|x| \neq 0$. Let us define π_λ as the projection operator

$$\pi_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}^\pm) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda^\pm).$$

Recalling the definition $\mathcal{Q}_\lambda = \pi_\lambda \partial_x$ for the HSD-operator on $\mathcal{S}_\lambda^\pm$-valued functions, and invoking the operator identity $\partial_x x = \Gamma_x - \mathbb{E}_x - m$ for the Dirac operator, we arrive at

$$\begin{aligned} &\mathcal{Q}_\lambda(|x|^{-m-2|\lambda|} xP_\lambda(xu_{(p)}x)) \\ &= \;(m + 2|\lambda|)|x|^{-m-2|\lambda|} \pi_\lambda P_\lambda(xu_{(p)}x) + |x|^{-m-2|\lambda|} \pi_\lambda \partial_x (xP_\lambda(xu_{(p)}x)) \\ &= \;|x|^{-m-2|\lambda|} \pi_\lambda \Gamma_x P_\lambda(xu_{(p)}x)\ . \end{aligned}$$

Here, Γ_x denotes the Gamma-operator (the tangential part of the Dirac operator, see e.g. [6]). So we are left with proving the statement

$$\pi_\lambda \Gamma_x P_\lambda(x u_{(p)} x) = 0.$$

Let us recall that, in full generality, we have the following decomposition for spinor-valued polynomials in the variables $(u_{(p)}) \in \mathbb{R}^{pm}$:

$$\mathcal{P}(\mathbb{R}^{pm}, \mathbb{S}^\pm) = \mathcal{S}_\lambda^\pm \oplus \left(u_1 \mathcal{P}(\mathbb{R}^m, \mathbb{S}^\mp) + \cdots + u_p \mathcal{P}(\mathbb{R}^m, \mathbb{S}^\mp)\right).$$

The summations between brackets are obviously not direct, but we will only use the fact that the operator π_λ is the projection operator onto the first summand $\mathcal{S}_\lambda^\pm$. So it suffices to work modulo the vector spaces $u_j \mathcal{P}(\mathbb{R}^m, \mathbb{S}^\mp)$. This means for example that

$$\Gamma_x \langle x, u_j \rangle \bmod u_j \mathcal{P} = u_j \wedge x \bmod u_j \mathcal{P} = 2\langle x, u_j \rangle \bmod u_j \mathcal{P}.$$

This, and the fact that $\mathfrak{f}_i^2 = 0$, allows us to prove the following:

$$\begin{aligned}
\left(\Gamma_x \langle x, u_j \rangle \langle x, \mathfrak{f}_i \rangle \mathfrak{f}_i \mathfrak{f}_i^\dagger\right) \bmod u_j \mathcal{P} &= \left((2\langle x, u_j \rangle \langle x, \mathfrak{f}_i \rangle + \mathfrak{f}_i \wedge x \langle x, u_j \rangle) \mathfrak{f}_i \mathfrak{f}_i^\dagger\right) \bmod u_j \mathcal{P} \\
&= \langle x, u_j \rangle \left(2\langle x, \mathfrak{f}_i \rangle + \mathfrak{f}_i x\right) \mathfrak{f}_i \mathfrak{f}_i^\dagger \bmod u_j \mathcal{P} \\
&= -\langle x, u_j \rangle x \mathfrak{f}_i \mathfrak{f}_i \mathfrak{f}_i^\dagger \bmod u_j \mathcal{P} = 0 .
\end{aligned}$$

In view of the fact that $P_\lambda(x u_{(p)} x)$ consists of factors of the form

$$\begin{aligned}
&\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \langle x u_{\sigma(1)} x, \mathfrak{f}_1 \rangle \cdots \langle x u_{\sigma(k)} x, \mathfrak{f}_k \rangle \\
&\quad = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^k \left(|x|^2 \langle u_{\sigma(i)}, \mathfrak{f}_i \rangle - 2\langle x, \mathfrak{f}_i \rangle \langle x, u_{\sigma(i)} \rangle\right).
\end{aligned}$$

it is clear that the first terms between brackets will also not contribute, since they depend on the norm of x only (on which Γ_x acts trivially). We are now ready to explain why we indeed have that

$$\Gamma_x P_\lambda(x u_{(p)} x) \bmod u_j \mathcal{P} = 0 .$$

First of all, as Γ_x is a first order differential operator, it suffices to verify that

$$\Gamma_x \langle x u_1 x \wedge \cdots \wedge x u_k x, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_k \rangle^a I^\pm \bmod u_j \mathcal{P} = 0 ,$$

for all $1 \leq k \leq p$ and $a \in \mathbb{N}$. In view of the chain rule, it suffices to prove this for $a = 1$, which amounts to showing that

$$\Gamma_x \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^k \left(|x|^2 \langle u_{\sigma(i)}, \mathfrak{f}_i \rangle - 2\langle x, \mathfrak{f}_i \rangle \langle x, u_{\sigma(i)} \rangle\right) \bmod u_j \mathcal{P} = 0.$$

But as was explained above, none of these factors will survive, which proves Proposition 1.

Note that, until now, we have excluded the pointwise singularity of $E_\lambda(x; u_{(p)})$ at $x = 0$. In order to investigate this singularity, we use results from distribution theory.

3.3. Riesz potentials

Consider the function $x \mapsto |x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x)$, for a fixed $\alpha \in \mathbb{C}$. This is obviously an element of the function space $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathcal{S}^\pm_\lambda)$. Under the action of the HSD-operator, using similar calculations as above, we get

$$\mathcal{Q}_\lambda(|x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x)) = -(\alpha+m)|x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x). \tag{5}$$

For $\alpha = -m$, we thus have that $|x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x)$ belongs to the kernel of the operator HSD-operator $\mathcal{Q}_\lambda$. Furthermore, it clearly has a pointwise singularity in the origin $x = 0$ of degree $(-m+1)$. The function defined by

$$x \mapsto |x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x)$$

is an element of the space of locally integrable functions $L_1^{loc}(\mathbb{R}^m, \mathcal{S}^\pm_\lambda)$ if $\Re(\alpha) > -m-1$, so it defines a distribution on the space $\mathcal{D}(\mathbb{R}^m, \mathcal{S}^\pm_\lambda)$ of test functions ϕ in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}^\pm_\lambda)$ with compact support. Consider, for $\Re(\gamma) > -m$, the distribution $|x|^\gamma$, whose action is defined by the integral formula

$$\langle |x|^\gamma, \phi\rangle = \int_{\mathbb{R}^m} |x|^\gamma \phi(x) dx,$$

for all test functions $\phi(x) \in \mathcal{D}(\mathbb{R}^m)$. We will use the following result, see e.g. [11]:

Lemma 3. *The mapping $\gamma \mapsto |x|^\gamma$ can be uniquely extended to a meromorphic mapping from the complex numbers to the space of tempered distributions on $\mathbb{R}^m$ (i.e. holomorphic on $\mathbb{C}$, except for a few isolated points). The poles are the points $\gamma = -m-2a$ (for all $a \in \mathbb{N}$), and they are all simple.*

Define for $\gamma \in \mathbb{C}\backslash\{m+2a, -2b : a, b \in \mathbb{N}\}$ the action of the Riesz potential I_x^γ on a rapidly decreasing test function ϕ as follows:

$$I_x^\gamma \phi := \frac{\Gamma\left(\frac{m-\gamma}{2}\right)}{2^\gamma \pi^{\frac{m}{2}} \Gamma\left(\frac{\gamma}{2}\right)} \phi * |x|^{-m+\gamma},$$

where $*$ is the convolution product on $\mathbb{R}^m$ and $I_x^0\phi = \lim_{\gamma\to 0} I_x^\gamma \phi = \phi$. Note that he poles of $|x|^{\gamma-m}$ are cancelled by the poles of $\Gamma\left(\frac{\gamma}{2}\right)$. The Riesz potential for $\gamma = 2$ can be seen as an 'inverse' of the Laplace operator Δ_x, because it satisfies the following relation in distributional sense:

$$I_x^\gamma \Delta_x \phi = \Delta_x I_x^\gamma \phi = -I_x^{\gamma-2}\phi.$$

For all $b \in \mathbb{N}$, we have that $I_x^\gamma = (-1)^b \Delta_x^b I_x^{\gamma+2b}$, so if we define

$$I_x^{-2a} := (-1)^a \Delta_x^a \delta(x),$$

where $\delta(x)$ is the Dirac-delta distribution, then this is an analytic continuation of the mapping $\gamma \mapsto I_x^\gamma$ to a holomorphic function with poles in $\{\gamma = m+2a | a \in \mathbb{N}\}$. These are the poles of $\Gamma\left(\frac{m-\gamma}{2}\right)$. If we reformulate our findings in terms of the distribution $|x|^{-m+\gamma}$, then we can analytically extend the mapping $\gamma \mapsto |x|^{-m+\gamma}$ to $\mathbb{C}\backslash\{-2a : a \in \mathbb{N}\}$, according to Lemma 3. Its singularities are simple poles, with residues

$$\begin{aligned} \mathrm{Res}[|x|^{-m-\gamma}, \gamma = -2a] &= \mathrm{Res}\left[\frac{2^\gamma \pi^{\frac{m}{2}} \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{m-\gamma}{2}\right)} I_x^\gamma, \gamma = -2a\right] \\ &= \frac{2^{-2a}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+a\right)} \mathrm{Res}\left[\Gamma\left(\frac{\gamma}{2}\right), \gamma = -2a\right] I_x^{-2a}. \end{aligned}$$

In view of the fact that

$$\mathrm{Res}\left[\Gamma\left(\frac{\gamma}{2}\right), \gamma = -2a\right] = \lim_{\gamma \to -2a} (\gamma + 2a)\Gamma\left(\frac{\gamma}{2}\right) = 2\frac{(-1)^a}{a!} \ ,$$

it then follows that

$$\mathrm{Res}[|x|^{-m-\gamma}, \gamma = -2a] = \frac{2^{-2a+1}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+a\right) a!}\Delta_x^a \delta(x).$$

Thus, the mapping $\alpha \mapsto |x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x)$ is holomorphic in $\mathbb{C}\backslash\{-m+2|\lambda|-2a, a \in \mathbb{N}\}$. Moreover, the poles at the values $\{-m+2(|\lambda|-1), \ldots, -m+2, -m\}$ are removable singularities. For instance, for the pole $\alpha = -m$, we have that

$$\mathrm{Res}[|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x), \alpha = -m] = \lim_{\alpha \to -m}(\alpha+m)|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x).$$

Putting $x = r\omega$ with $r = |x|$, this can then be rewritten as

$$\lim_{\alpha \to -m}(\alpha+m)r^{\alpha+1}\omega P_\lambda(\omega u_{(p)}\omega) = 0.$$

Similar calculations can be done for the other singularities. So we have proved the following proposition:

Proposition 2. *The mapping $\alpha \mapsto |x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x)$ can be continued holomorphically in $\mathbb{C}\backslash\{-m-2a, a \in \mathbb{N}\}$.*

This means that (5) holds in distributional sense in $\mathbb{C}$, as long as $\Re(\alpha) > -m-1$. Hence, with this restriction on α,

$$\begin{aligned}
\mathcal{Q}_\lambda\left(|x|^{-m-2|\lambda|}xP_\lambda(xu_{(p)}x)\right) &= -\lim_{\alpha \to -m}(\alpha+m)|x|^{\alpha-2|\lambda|}\pi_\lambda P_\lambda(xu_{(p)}x) \\
&= -\mathrm{Res}\left[|x|^{\alpha-2|\lambda|}, \alpha = -m\right]\pi_\lambda P_\lambda(xu_{(p)}x) \\
&= \frac{2^{-2|\lambda|+1}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+|\lambda|\right)|\lambda|!}(\Delta_x^{|\lambda|}\delta(x))\pi_\lambda P_\lambda(xu_{(p)}x). \qquad (6)
\end{aligned}$$

Moreover, in view of the fact that $\langle \delta, \varphi \rangle = \varphi(0)$, we get:

$$\begin{aligned}
\langle(\Delta_x^{|\lambda|}\delta)(\pi_\lambda P_\lambda(xu_{(p)}x)), \phi\rangle &= \langle \Delta_x^{|\lambda|}\delta, (\pi_\lambda P_\lambda(xu_{(p)}x))\phi\rangle \\
&= \langle \delta, \Delta_x^{|\lambda|}((\pi_\lambda P_\lambda(xu_{(p)}x))\phi)\rangle \\
&= \langle \delta, \Delta_x^{|\lambda|}(\pi_\lambda P_\lambda(xu_{(p)}x))\phi + \cdots\rangle \ ,
\end{aligned}$$

where the dots indicate all other terms coming from the action of $\Delta_x^{|\lambda|}$. They can safely be ignored, in view of the fact that we still need to act with the distribution $\delta(x)$, which will make all these terms disappear. We thus get that

$$\langle(\Delta_x^{|\lambda|}\delta)(\pi_\lambda P_\lambda(xu_{(p)}x)), \phi\rangle \ = \ \langle \delta, \Delta_x^{|\lambda|}(\pi_\lambda P_\lambda)\phi\rangle \ = \ \langle \Delta_x^{|\lambda|}(\pi_\lambda P_\lambda)\delta, \phi\rangle \ .$$

This means that formula (6) reduces to

$$\mathcal{Q}_\lambda\left(|x|^{-m-2|\lambda|}xP_\lambda(xu_{(p)}x)\right) = \frac{2^{-2|\lambda|+1}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+|\lambda|\right)|\lambda|!}\Delta_x^{|\lambda|}(\pi_\lambda P_\lambda(xu_{(p)}x))\delta(x).$$

In order to calculate the remaining expression $\Delta_x^{|\lambda|}(\pi_\lambda P_\lambda(xu_{(p)}x))$, we first note that Δ_x and π_λ commute. Next, we introduce the operator ξ_x by means of

$$\xi_x : \mathcal{S}_\lambda^\pm \to \mathcal{P}_{2|\lambda|}(x) \otimes \mathcal{H}_\lambda(u_{(p)}) \otimes \mathbb{S}^\pm : P_\lambda(u_{(p)}) \mapsto P_\lambda(xu_{(p)}x).$$

It can easily be calculated that the map $\Delta_x^{|\lambda|}\xi_x$ is $\mathrm{Spin}(m)$ invariant:

$$\begin{aligned} \Delta_x^{|\lambda|}\xi_x L(s)P_\lambda(u_{(p)}) &= \Delta_x^{|\lambda|} sP_\lambda(s^*xss^*u_{(p)}ss^*xs) \\ &= s\Delta_x^{|\lambda|}P_\lambda(s^*xss^*u_{(p)}ss^*xs). \end{aligned}$$

The image of the $\mathrm{Spin}(m)$-invariant map $\Delta_x^{|\lambda|}\xi_x$ equals $\mathcal{S}_\lambda^\pm$. According to Schur's lemma, there must therefore exist a constant C_λ such that

$$\Delta_x^{|\lambda|}\xi_x P_\lambda(u_{(p)}) = C_\lambda P_\lambda(u_{(p)}). \tag{7}$$

Let us then determine the constant C_λ explicitly. We do this by complexifying the variables u_j and choosing a specific value for them: $u_j := e_j + ie_{n+j}$, for all $j = 1, \ldots, n$. Then our results simplify a great deal, since $\langle u_i, \mathfrak{f}_j\rangle = \delta_{ij}$ and thus $P_\lambda(u_{(p)}) = I^\pm$. Furthermore,

$$\begin{aligned} &\langle xu_1x \wedge \cdots \wedge xu_kx, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_k\rangle \\ &= |x|^{2k}\langle u_1 \wedge \cdots \wedge u_k, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_k\rangle - 2\sum_{j=1}^k |x|^{2k-2}\langle u_j, x\rangle\langle (u_1 \wedge \cdots \wedge u_k)_j, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_k\rangle \\ &= |x|^{2k} - 2\sum_{j=1}^k |x|^{2k-2}\langle u_j, x\rangle\langle x, \mathfrak{f}_j\rangle \\ &= |x|^{2k-2}(x_{k+1}^2 + \cdots + x_n^2 + x_{n+k+1}^2 + \cdots + x_m^2). \end{aligned}$$

Putting $x^{(j)} = (0, \ldots, 0, x_j, \ldots, x_n, 0, \ldots, 0, x_{n+j}, \ldots, x_m)$, we then get that

$$P_\lambda(xu_{(p)}x) = |x|^{2|\lambda|-l_1}|x^{(2)}|^{2(l_1-l_2)} \cdots |x^{(p)}|^{2(l_{p-1}-l_p)}|x^{(p+1)}|^{2l_p} I^\pm \ .$$

Together with the relation

$$\Delta_x^a |x|^{2b} = \sum_{j=0}^{\min(a,b)} \binom{a}{j} 2^{2j} j! \binom{b}{j} \frac{\Gamma\left(\frac{m}{2} + \mathbb{E}_x - b + a + j\right)}{\Gamma\left(\frac{m}{2} + \mathbb{E}_x - b + a\right)} |x|^{2(b-j)} \Delta_x^{a-j},$$

which follows from the fact that $[\Delta_x, |x|^2] = 2m + 4\mathbb{E}_x$, we find that

$$\begin{aligned} &\Delta_x^{|\lambda|} P_\lambda(xu_{(p)}x) \\ &= 2^{2|\lambda|}|\lambda|! \frac{\Gamma\left(\frac{m}{2} + |\lambda|\right)}{\Gamma\left(\frac{m}{2} + l_1\right)} \frac{\Gamma\left(\frac{m}{2} + l_1 - 1\right)}{\Gamma\left(\frac{m}{2} + l_2 - 1\right)} \cdots \frac{\Gamma\left(\frac{m}{2} + l_{k-1} - k + 1\right)}{\Gamma\left(\frac{m}{2} + l_k - k + 1\right)} \frac{\Gamma\left(\frac{m}{2} + l_k - k\right)}{\Gamma\left(\frac{m}{2} - k\right)} P_\lambda(xu_{(p)}x). \end{aligned}$$

This then leads to the following conclusion:

$$\begin{aligned} &\mathcal{Q}_\lambda\left(|x|^{-m-2|\lambda|} x P_\lambda(xu_{(p)}x)\right) \\ &= \frac{\pi^{\frac{m}{2}} 2^{-|\lambda|+1}}{\Gamma\left(\frac{m}{2} + |\lambda|\right)|\lambda|!} \frac{2^{|\lambda|}|\lambda|!\Gamma\left(\frac{m}{2} + |\lambda|\right)}{\left(\frac{m}{2} + l_1 - 1\right) \cdots \left(\frac{m}{2} + l_k - k\right)\Gamma\left(\frac{m}{2} - k\right)} P_\lambda(u_{(p)})\delta(x) \\ &= -A_m \prod_{j=1}^p \frac{m - 2j}{m + 2l_j - 2j} P_\lambda(u_{(p)})\delta(x), \end{aligned}$$

To conclude our findings:

Theorem 1. *Defining the constant C_λ by means of*

$$C_\lambda = -\frac{1}{A_m}\prod_{j=1}^{k}\frac{m+2l_j-2j}{m-2j},$$

the distribution

$$e_\lambda(x) := C_\lambda |x|^{-m+1} L\left(\frac{x}{|x|}\right) \in \mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathcal{S}^\pm_\lambda))$$

satisfies, for every $P_\lambda \in \mathcal{S}^\pm_\lambda$, in distributional sense

$$\mathcal{Q}_\lambda e_\lambda(x) P_\lambda = \delta(x) P_\lambda.$$

Let us then introduce the notation $(.,.)_{(u_{(p)})}$ for the Fischer inner product on $\mathcal{P}_\lambda$, which is defined as follows:

$$(f(u_1,\ldots,u_p), g(u_1,\ldots,u_p))_{(u_{(p)})} = \left[f(\partial_1,\ldots,\partial_p)^\dagger g(u_1,\ldots,u_p)\right]\Big|_{u_1=\cdots=u_p=0}.$$

In order to obtain a fundamental solution for $\mathcal{Q}_\lambda$, we then let the distribution $e_\lambda(x)$ act on the reproducing kernel $K_\lambda(u_{(p)}, u'_{(p)})$ for $\mathcal{S}^\pm_\lambda$, satisfying the defining relation

$$(K_\lambda(u_{(p)}, u'_{(p)}), P_\lambda(u_{(p)}))_{(u_{(p)})} = P_\lambda(u'_{(p)}),$$

for each $P_\lambda(u_{(p)}) \in \mathcal{S}^\pm_\lambda$.

Definition 5. *The fundamental solution for the operator $\mathcal{Q}_\lambda$ is defined as*

$$E_\lambda(x; u_{(p)}, u'_{(p)}) := e_\lambda(x) K_\lambda(u_{(p)}, u'_{(p)}).$$

4. Basic integral formula

Now that we have constructed the fundamental solution, we can prove the main integral formulas in higher spin Clifford analysis. Define the volume element $dx = dx_1 \wedge \cdots \wedge dx_m$ and surface element $d\sigma_x = \sum_{j=1}^m (-1)^{j-1} e_j d\hat{x}_j$, where $d\hat{x}_j = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge dx_m$.

Theorem 2. *Let $\Omega' \subset \mathbb{R}^m$ and $\overline{\Omega} \subset \Omega'$. Then for $f(x)$ and $g(x) \in \mathcal{C}^\infty(\Omega', \mathcal{S}^\pm_\lambda)$, where we will not mention the variables $u_{(p)}$ to avoid overloaded notations, we have the following formulae (for arbitrary $y \in \Omega'$)*

(i) (Stokes' theorem)

$$\int_\Omega \left[-(\mathcal{Q}_\lambda g(x), f(x))_{(u_{(p)})} + (g(x), \mathcal{Q}_\lambda f(x))_{(u_{(p)})}\right] dx = \int_{\partial\Omega} (g(x), \pi_\lambda(d\sigma_x) f(x))_{(u_{(p)})}.$$

(ii) (Cauchy-Pompeiu)

$$-\int_{\partial\Omega}(E_\lambda(y-x), \pi_\lambda(d\sigma_x) f(x))_{(u_{(p)})} + \int_\Omega (E_\lambda(y-x), \mathcal{Q}_\lambda f(x))_{(u_{(p)})} dx = \begin{cases} f(y) & y \in \Omega \\ 0 & y \notin \overline{\Omega}. \end{cases}$$

(iii) (Cauchy integral formula) If $\mathcal{Q}_\lambda f = 0$ in Ω', one has

$$-\int_{\partial\Omega}(E_\lambda(y-x), \pi_\lambda(d\sigma_x) f(x))_{(u_{(p)})} = \begin{cases} f(y) & y \in \Omega \\ 0 & y \notin \overline{\Omega}, \end{cases}$$

where $\pi_\lambda(d\sigma_x)f(x)$ is an $\mathcal{S}_\lambda^{\mp}$-valued $(m-1)$-form.

Proof. Let $f(x), g(x) \in \mathcal{C}^\infty(\Omega', \mathcal{S}_\lambda^{\pm})$. The classical Stokes' formula for the Dirac operator (e.g. [6]) leads to

$$\int_\Omega \left[-(\partial_x g(x))^\dagger f(x) + g(x)^\dagger(\partial_x f(x))\right] dx = \int_{\partial\Omega} g(x)^\dagger d\sigma_x f(x).$$

This identity still depends on the vector variables $u_{(p)} \in \mathbb{R}^{pm}$. To obtain the generalised Stokes' theorem for the operator $\mathcal{Q}_\lambda$, it is sufficient to take the Fischer inner product with respect to $u_{(p)}$, since we have that

$$(\mathcal{Q}_\lambda g(x), f(x))_{(u_{(p)})} = (\pi_\lambda \partial_x g(x), f(x))_{(u_{(p)})} = (\partial_x g(x), f(x))_{(u_{(p)})},$$

$$(g(x), \mathcal{Q}_\lambda f(x))_{(u_{(p)})} = (\pi_\lambda \partial_x f(x), g(x))^\dagger_{(u_{(p)})} = (\partial_x f(x), g(x))^\dagger_{(u_{(p)})} = (g(x), \partial_x f(x))_{(u_{(p)})} \ ,$$

and

$$(g(x), \pi_\lambda(d\sigma_x)f(x))_{(u_{(p)})} = (g(x), (d\sigma_x)f(x))_{(u_{(p)})}.$$

The Cauchy-Pompeiu formula for the operator $\mathcal{Q}_\lambda$ is then obtained from Stokes' formula, by substituting $g(x; u_{(p)}) = E_\lambda(y - x; u_{(p)}, u'_{(p)})$. We then get

$$\begin{aligned}
&\int_\Omega \left[-(\delta(y-x)K(u_{(p)}, u'_{(p)}), f(x; u_{(p)})) + (E_\lambda(y-x; u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})}\right] dx \\
&\qquad = \int_{\partial\Omega} (E_\lambda(y-x; u_{(p)}, u'_{(p)}), \pi_\lambda(d\sigma_x)f(x; u_{(p)}))_{(u_{(p)})} \\
\Leftrightarrow &\quad \int_\Omega \left[-\delta(y-x)f(x; u'_{(p)}) + (E_\lambda(y-x; u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})}\right] dx \\
&\qquad = \int_{\partial\Omega} (E_\lambda(y-x; u_{(p)}, u'_{(p)}), \pi_\lambda(d\sigma_x)f(x; u_{(p)}))_{(u_{(p)})} \ .
\end{aligned}$$

In order to further simplify these integrals, we invoke the definition for the fundamental solution and proceed as follows:

$$\begin{aligned}
&-f(y, u'_{(p)}) + \int_\Omega (e_\lambda(y-x)K_\lambda(u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})} dx \\
&= \int_{\partial\Omega} (e_\lambda(y-x)K_\lambda(u_{(p)}, u'_{(p)}), \pi_\lambda(d\sigma_x)f(x; u_{(p)}))_{(u_{(p)})}.
\end{aligned}$$

Using the fact that

$$\left(L\left(\frac{x}{|x|}\right)P(u_{(p)}), R(u_{(p)})\right)_{(u_{(p)})} = -\left(P(u_{(p)}), L\left(\frac{x}{|x|}\right)R(u_{(p)})\right)_{(u_{(p)})},$$

for any $P(u_{(p)}), R(u_{(p)}) \in \mathcal{S}_\lambda^{\pm}$, we can now rewrite these expressions as

$$\begin{aligned}
\Leftrightarrow &\quad -f(y, u'_{(p)}) - \int_\Omega e_\lambda(y-x)(K_\lambda(u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})} dx \\
&\quad = -\int_{\partial\Omega} e_\lambda(y-x)(K_\lambda(u_{(p)}, u'_{(p)}), \pi_\lambda(d\sigma_x)f(x))_{(u_{(p)})} \\
\Leftrightarrow &\quad f(y, u'_{(p)}) + \int_\Omega e_\lambda(y-x)\mathcal{Q}_\lambda f(x; u'_{(p)}) dx \\
&\quad = +\int_{\partial\Omega} e_\lambda(y-x)\pi_\lambda(d\sigma_x)f(x; u'_{(p)}).
\end{aligned}$$

Invoking the fact that $f \in \ker \mathcal{Q}_\lambda$ immediately gives us the Cauchy integral formula. $\square$

5. Conclusion

Using the theory of Riesz distributions and techniques coming from representation theory, the fundamental solution $E_\lambda(x; u_{(p)} u'_{(p)})$ for the higher spin Dirac operator $\mathcal{Q}_\lambda$ was determined. This fundamental solution allows us to prove a generalised version of the classical integral formulae (Stokes' theorem, the Cauchy-Pompeiu theorem and the Cauchy integral formula), which are necessary in order to develop a function theory for the operator $\mathcal{Q}_\lambda$.

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