# A characterisation result on a particular class of non-weighted minihypers

J. De Beule[*], A. Hallez and L. Storme[†]

**Abstract**

We present a characterisation of $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$-minihypers, $q$ square, $q = p^h$, $p > 3$ prime, $h \geq 2$, $q \geq 1217$, for $\epsilon_0+\epsilon_1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. This improves a characterisation result of S. Ferret and L. Storme [6], involving more Baer subgeometries contained in the minihyper.

**Keywords:** minihypers, Griesmer bound, blocking sets

## 1 Introduction

Let $\mathrm{PG}(n, q)$ be the $n$-dimensional projective space over the finite field $\mathbb{F}_q$ of order $q$. A *weight function* $w$ of $\mathrm{PG}(n, q)$ is a mapping from the point set of $\mathrm{PG}(n, q)$ to the set of non-negative integers. For a point $P$, the integer $w(P)$ is called the *weight* of the point $P$, and for a set $M$ of points, its *weight* is the sum of the weights of its points. The sum of the weights of all points of $\mathrm{PG}(n, q)$ is the *total weight* of $w$.

**Definition 1.1** *An* $\{f, m; n, q\}$-minihyper*,* $f \geq 1$*,* $n \geq 2$*, is a pair* $(F, w)$*, where* $F$ *is a set of points of* $\mathrm{PG}(n, q)$*,* $w$ *is a weight function of* $\mathrm{PG}(n, q)$*, and*

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$P \in F \iff w(P) > 0$*; the total weight of* $w$ *is* $f$*, and* $m$ *is the minimum weight of the hyperplanes of* $\mathrm{PG}(n,q)$*, i.e.,*

$$\sum_{P \in \pi} w(P) \geq m,$$

*for any hyperplane* $\pi$ *of* $\mathrm{PG}(n,q)$*, with equality for at least one hyperplane of* $\mathrm{PG}(n,q)$*.*

Of course, the set $F$ is determined by the weight function $w$. When the range of $w$ is $\{0,1\}$, the converse is true and then the minihyper is identified with $F$ and is called a *non-weighted minihyper*. Thus, a non-weighted $\{f,m;n,q\}$-minihyper is a set $F$ of $f$ points of $\mathrm{PG}(n,q)$ such that every hyperplane contains at least $m$ points of $F$. This is the definition of a minihyper given by Hamada and Tamari in [12] and it was generalised to the definition of a weighted minihyper in [7]. (Weighted) minihypers can also be called *(weighted) multiple blocking sets*. In the sequel, we will always clearly distinguish between weighted and non-weighted minihypers and use the term *minihyper* to denote a non-weighted minihyper.

As (weighted) minihypers are (weighted) multiple blocking sets with respect to the hyperplanes of a projective space, the study of these objects fits completely in the rich literature on the study of blocking sets and generalisations. But (weighted) minihypers are also a geometric interpretation of linear codes meeting the Griesmer bound. This connection is described in detail in several references, such as [4, 5].

Denote the number of points in an $i$-dimensional projective space $\mathrm{PG}(i,q)$ by $\theta_i$, i.e., $\theta_i = \frac{q^{i+1}-1}{q-1}$, and define $\theta_{-1} = 0$. The following characterisation theorem was shown by Hamada, Helleseth and Maekawa.

**Theorem 1.2** ([10, 11]) *A* $\{\sum_{i=0}^{s} \epsilon_i \theta_i, \sum_{i=0}^{s} \epsilon_i \theta_{i-1}; n, q\}$*-minihyper is the union of pairwise disjoint* $\epsilon_i$ *projective subspaces of dimension* $i$*, for* $i = 0, \ldots, s$*, if* $\sum_{i=0}^{s} \epsilon_i = h < \sqrt{q} + 1$*.*

In [6], Ferret and Storme proved that increasing $h$ to $2\sqrt{q} - 1$ allows one Baer subgeometry in the minihyper.

**Theorem 1.3** ([6, Theorem 5.9]) *Let* $F$ *be a* $\{\sum_{i=0}^{k-2} \epsilon_i \theta_i, \sum_{i=0}^{k-2} \epsilon_i \theta_{i-1}; k-1, q\}$*-minihyper,* $q$ *square,* $q = p^h$*,* $h$ *even,* $p$ *prime, where* $\sum_{i=0}^{k-2} \epsilon_i \leq \min\{2\sqrt{q} - 1, c_p q^{5/9}\}$*,* $c_p = 2^{-1/3}$*,* $q \geq 2^{14}$*, when* $p = 2, 3$*, and where* $\sum_{i=0}^{k-2} \epsilon_i \leq \min\{2\sqrt{q} - 1, q^{6/9}/(1 + q^{1/9})\}$*,* $q \geq 2^{12}$*, when* $p > 3$*.*

*Then* $F$ *consists of the union of pairwise disjoint*

(1) $\epsilon_{k-2}$ *spaces* $\mathrm{PG}(k-2,q), \epsilon_{k-3}$ *spaces* $\mathrm{PG}(k-3,q), \ldots, \epsilon_0$ *points, or*

(2) *one subgeometry* $\mathrm{PG}(2l+1,\sqrt{q})$*, for some integer* $l$ *with* $1 \le l \le \frac{k-2}{2}$*,* $\epsilon_{k-2}$ *spaces* $\mathrm{PG}(k-2,q), \ldots, \epsilon_{l+1}$ *spaces* $\mathrm{PG}(l+1,q), \epsilon_l - \sqrt{q} - 1$ *spaces* $\mathrm{PG}(l,q), \epsilon_{l-1}$ *spaces* $\mathrm{PG}(l-1,q), \ldots, \epsilon_0$ *points, or*

(3) *one subgeometry* $\mathrm{PG}(2l,\sqrt{q})$*, for some integer* $l$ *with* $1 \le l \le \frac{k-1}{2}$*,* $\epsilon_{k-2}$ *spaces* $\mathrm{PG}(k-2,q), \ldots, \epsilon_{l+1}$ *spaces* $\mathrm{PG}(l+1,q), \epsilon_l - 1$ *spaces* $\mathrm{PG}(l,q), \epsilon_{l-1} - \sqrt{q}$ *spaces* $\mathrm{PG}(l-1,q), \epsilon_{l-2}$ *spaces* $\mathrm{PG}(l-2,q), \ldots, \epsilon_0$ *points.*

In this article, in Theorems 3.11 and 4.2, we give a characterisation of these minihypers with $h < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ and $s = 1$. In particular, we prove the following theorem.

**Theorem 1.4** *Let* $F$ *be a non-weighted* $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$*-minihyper,* $q$ *square,* $q = p^h$*,* $p > 3$ *prime,* $h \ge 2$*,* $q \ge 1217$*,* $n \ge 4$*, where* $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$*, then* $F$ *is the union of pairwise disjoint* $A$ *lines,* $B$ *isolated Baer subplanes* $\mathrm{PG}(2,\sqrt{q})$ *and* $C$ *Baer subgeometries* $\mathrm{PG}(3,\sqrt{q})$*, with* $A + B + C(\sqrt{q}+1) = \epsilon_1$*, and* $\epsilon_0 - B\sqrt{q}$ *extra points.*

# 2 Preliminaries

We explain formally the notion of removing a point $P$ from a weighted minihyper $(F,w)$. Suppose that $(F,w)$ is a weighted minihyper. With the notation $P \in (F,w)$, we always mean $w(P) > 0$, equivalently $P \in F$.

Suppose that we have two weighted sets $(F_1,w_1)$ and $(F_2,w_2)$ in $\mathrm{PG}(n,q)$, where $w_1(P) \ge w_2(P)$ for all points $P$ of $\mathrm{PG}(n,q)$. Then we can define the new weighted set $(F,w) = (F_1,w_1) - (F_2,w_2)$ defined by the weight function $w$, with $w : \mathrm{PG}(n,q) \to \mathbb{N} : P \mapsto w(P) = w_1(P) - w_2(P)$. When the weights $w_2(P)$ of all the points $P$ of $\mathrm{PG}(n,q)$ are equal to zero or one, we simply write this difference as $(F,w) = (F_1,w_1) - F_2$.

For instance, suppose that $P \in (F,w)$ and define $w' : \mathrm{PG}(n,q) \to \mathbb{N}$: $w'(R) = w(R)$ for any point $R \in \mathrm{PG}(n,q) \setminus \{P\}$ and $w'(P) = w(P) - 1$. Then $w'$ determines a new set $F'$ and $(F,w) - \{P\}$ is by definition $(F',w')$. This is the weighted minihyper in which the point $P$ is removed once from $(F,w)$. It is clear that $F' = F \setminus \{P\}$ when $(F,w)$ is a non-weighted minihyper.

We can easily extend the notion of removing points from $(F, w)$ to removing sets $M \subseteq F$ from $(F, w)$ by defining $w' : \mathrm{PG}(n, q) \to \mathbb{N}$: $w'(R) = w(R)$ for any point $R \in \mathrm{PG}(n, q) \setminus M$ and $w'(P) = w(P) - 1$ for $P \in M$.

Removing points or sets from a minihyper $(F, w)$ can, under certain circumstances, yield a minihyper as expected, as is shown in the next lemma.

**Lemma 2.1** *Let $(F, w)$ be a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$-minihyper, $n \geq 3$, with $2\epsilon_1 + \epsilon_0 < q + 2$, containing a line $L$. Then $(F, w) - L$ is a weighted $\{(\epsilon_1 - 1)(q + 1) + \epsilon_0, \epsilon_1 - 1; n, q\}$-minihyper.*

**Proof.** We have to show that $(F, w) - L$ is a weighted $\{(\epsilon_1 - 1)(q + 1) + \epsilon_0, \epsilon_1 - 1; n, q\}$-minihyper. The essential part is to show that any hyperplane intersects $(F, w) - L$ in at least $\epsilon_1 - 1$ points. So consider the line $L$ and an arbitrary hyperplane $\pi$. Since $|\pi \cap (F, w)| \geq \epsilon_1$, it follows immediately that $|\pi \cap ((F, w) - L)| \geq \epsilon_1 - 1$ if $\pi$ intersects $L$ in exactly one point.

We are left with the case $L \subset \pi$. If $|\pi \cap (F, w)| \geq q + \epsilon_1$, then clearly, $|\pi \cap ((F, w) - L)| \geq \epsilon_1 - 1$. So suppose that $|\pi \cap (F, w)| < q + \epsilon_1$.

Consider an $(n - 3)$-dimensional space $\Omega$ in $\pi$ skew to $F$. The minihyper $(F, w)$ is projected from $\Omega$ on a weighted $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; 2, q\}$-minihyper $(F', w')$. The projection of $L$ is a line $L'$ contained in $(F', w')$. By [8, Theorem 2.2], we can reduce the weight of every point of $L'$ by one to obtain an $(\epsilon_1 - 1)$-fold blocking set $F''$ in this plane. But then $L'$ is still blocked at least $\epsilon_1 - 1$ times. So $\pi$ is blocked at least $q + \epsilon_1$ times by $F$. □

As we may consider minihypers containing no lines, the following lemma will provide information on their size. It is proved in [4] and it is a generalisation of a result from [1].

**Lemma 2.2** *A weighted $\{f, t; 2, q\}$-minihyper $(B, w)$, with $1 \leq t < q - 1$ and $q \geq 3$, contains a line or satisfies $f \geq tq + \sqrt{tq} + 1$.*

We first of all wish to characterise weighted $\{\epsilon_1(q + 1) + \epsilon_0, \epsilon_1; 3, q\}$-minihypers, $q$ square, having weighted points with total weight of the weighted points at most $\frac{2\epsilon_1^2}{q}$ and where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, as a sum of $A$ lines, $B$ isolated Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ and $C$ Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, where $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.

With an *isolated* Baer subplane $\mathrm{PG}(2\sqrt{q})$ contained in $F$, we mean a Baer subplane $\mathrm{PG}(2, \sqrt{q})$ contained in $F$, but not contained in a 3-dimensional Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$, completely contained in $F$.

We will focus on the existence of the isolated Baer subgeometries PG$(2, \sqrt{q})$ and the Baer subgeometries PG$(3, \sqrt{q})$ contained in $(F, w)$. To find Baer subgeometries completely contained in $(F, w)$, we will use a result of Barát and Storme [2]. This paper contains a lot of results on multiple (weighted) blocking sets in projective spaces, and we state a related result required in this article.

**Theorem 2.3** *Let $B$ be an $s$-fold blocking set in* PG$(2, q)$*, $q = p^h$, $p$ prime, $h \geq 1$, having points with multiplicities. Assume that $|B| \leq s(q+1)+c$ where*

(1) *$h > 1$, $c < c_p q^{2/3}$ and $s < \min(c_p q^{1/6}, q^{1/4}/2)$ where $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$,*

(2) *$q = p^2$, $s < q^{1/4}/2$ and $c < q^{3/4}/2$,*

*and assume that $B$ has at least $(s-2)(q + \sqrt{q} + 1) + 16\sqrt{q} + 8q^{1/4}$ simple points in* (1) *and at least $(s-2)(q + \sqrt{q} + 1) + 16\sqrt{q} + 16q^{1/6}$ simple points in* (2)*.*

*Then $B$ contains the sum of $s$ Baer subplanes and/or lines.*

This theorem is proved in exactly the same way as [2, Theorem 3.10], but using two new arguments. First of all, the $t \pmod p$ result on small weighted minimal $t$-fold blocking sets in PG$(2, q)$, $q = p^h$, $p$ prime, $h \geq 1$ [8, Theorem 2.13], that is now known, has to be used. Secondly, by [8, Theorem 2.2], if a line $L$ is contained in the $s$-fold blocking set $B$, then we can reduce the weight of every point of $L$ by one to obtain an $(s-1)$-fold blocking set $F''$ in this plane. By using these two new arguments, the upper bound $s(q + 1) + c - (s - 1)(s - 2)/2$ in [2, Theorem 3.10] can be replaced by the upper bound $s(q + 1) + c$.

Suppose that $(F, w)$ is a weighted minihyper in PG$(n, q)$ and consider any subspace $\pi$ of PG$(n, q)$. Then $(F, w) \cap \pi$ is the weighted minihyper $(F', w')$ induced in the subspace $\pi$, where $F' := F \cap \pi$ and $w'$ is the function $w$ restricted to the points of $\pi$. The following theorem provides useful information on intersections of weighted minihypers with subspaces.

**Theorem 2.4** ([4, Result 2.9]) *Let $(F, w)$ be a weighted $\{\sum_{i=0}^{n-1} \epsilon_i\theta_i, \sum_{i=0}^{n-1} \epsilon_i\theta_{i-1}; n, q\}$-minihyper satisfying $n \geq 1$, $\sum_{i=0}^{n-1} \epsilon_i = h \leq q$. Then every $r$-space $\pi_r$, $1 \leq r \leq n$, not contained in $(F, w)$, intersects $(F, w)$ in a weighted $\{\sum_{i=0}^{r-1} m_i\theta_i, \sum_{i=0}^{r-1} m_i\theta_{i-1}; r, q\}$-minihyper $(F, w) \cap \pi_r$, satisfying $\sum_{i=0}^{r-1} m_i \leq h$.*

**Lemma 2.5** *Let $(F, w)$ be a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$-minihyper, with $\epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, containing no lines and having at most $q^{1/6}/2$ multiple points. If a plane $\pi$ intersects $(F, w)$ in a weighted $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper, with $m_1 \geq 1$, then $(F, w) \cap \pi$ contains a sum of $m_1$ Baer subplanes.*

**Proof.** By Theorem 2.4, we know that $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. The intersection of $\pi$ with $F$ does not contain lines, since $F$ does not contain lines, so $|\pi \cap (F, w)| \geq m_1 q + \sqrt{m_1 q} + 1$ by Lemma 2.2, which implies $\sqrt{m_1 q} + 1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. Hence, $m_1 < q^{1/6}/2$. By Theorem 2.3, $\pi \cap (F, w)$ contains a sum of $m_1$ Baer subplanes. □

# 3 Minihypers in three dimensions

To obtain the desired characterisation, we will use an inductive argument on the dimension $n \geq 3$. In this inductive step, we will require a characterisation of weighted minihypers in three dimensions, which we will obtain in this section.

The following theorem, which is an improvement of [2, Theorem 3.1], also plays a crucial role.

**Theorem 3.1** ([9, Theorem 3.1]) *Let $B$ be a $t$-fold blocking set in $\mathrm{PG}(n, q)$, $q = p^h$, $p$ prime, $q \geq 661$, $n \geq 3$, of size $|B| < tq + c_p q^{2/3}$, with $c_2 = c_3 = 2^{-1/3}$, $c_p = 1$ when $p > 3$, and with $t < c_p q^{1/6}/2$. Then $B$ contains a union of $t$ pairwise disjoint lines and/or Baer subplanes.*

We assume that $(F, w)$ is a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 3, q\}$-minihyper, $q$ square, $q = p^h$, $p > 3$ prime, $h \geq 2$, $q \geq 1217$, where the total weight of the multiple points is at most $\frac{2\epsilon_1^2}{q}$ and with $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, so $\eta < \frac{q^{1/12}}{2}$, and we assume that $(F, w)$ does not contain a line of $\mathrm{PG}(3, q)$. The preceding Theorem 3.1 characterises these minihypers for $\epsilon_1 < q^{1/6}/2$, so from now on, we assume that $\epsilon_1 \geq q^{1/6}/2$.

**Remark 3.2** *As indicated in the preceding paragraph, we assume that $q = p^h$, $q$ square, $p > 3$ prime, $h \geq 2$, and that $q \geq 1217$. The condition $q \geq 1217$ follows from the inequality at the end of the proof of Lemma 3.9.*

*The other inequalities in the proofs are valid for $q \geq 1217$.*

*In the main theorems, we repeat that $q = p^h$, $p > 3$ prime, $h \geq 2$, and that $q \geq 1217$, to give the correct statements of the theorems.*

Projecting the minihyper $(F, w)$ from a point $R \notin F$ onto a plane gives a weighted $\epsilon_1$-fold blocking set $B$ in this plane. This set $B$ can contain lines, and we will distinguish two cases.

First we assume that $B$ does not contain a line.

**Lemma 3.3** *If $B$ does not contain a line, then $\epsilon_1 < \frac{q^{1/6}}{2}$ and $(F, w)$ is an $\epsilon_1$-fold blocking multiset containing a sum of $\epsilon_1$ Baer subplanes and lines.*

**Proof.** The set $B$ is a weighted $\epsilon_1$-fold blocking set in $\mathrm{PG}(2, q)$ of size $\epsilon_1(q+1)+\epsilon_0 < \epsilon_1 q + \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, containing no lines. Lemma 2.2 implies that $\epsilon_1 < \frac{q^{1/6}}{2}$. In this case, there are no multiple points since $\frac{2\epsilon_1^2}{q} < 1$.

So $(F, w)$ is an $\epsilon_1$-fold blocking set satisfying the conditions of Theorem 3.1, and by assumption not containing lines. Using Theorem 3.1, we conclude that $(F, w)$ contains a union of $\epsilon_1$ pairwise disjoint Baer subplanes. Hence, $(F, w)$ is the sum of $\epsilon_1$ Baer subplanes and points. □

An upper bound on the number of secants to $F$ through a point $R$, not in $F$, will be very useful. The next lemma yields such an upper bound.

**Lemma 3.4** *There exists in $\mathrm{PG}(3, q)$ a point not in $F$ lying on at most $\frac{\epsilon_1^2+2\eta^2}{2}$ secants to $F$, containing at least two points of $F$ of weight one.*

**Proof.** We count the number of points of $\mathrm{PG}(3, q)$, not contained in $F$, on secants to $F$ through two points of weight one. Here, $|F| \le \epsilon_1 q + \eta\sqrt{q}$, but we subtract $\frac{q^{1/6}}{2}$ from $|F|$, since there can be up to $\frac{2\epsilon_1^2}{q} < \frac{q^{1/6}}{2}$ multiple points in $(F, w)$:

$$
\begin{aligned}
& (\epsilon_1 q + \eta\sqrt{q} - \frac{q^{1/6}}{2})(\epsilon_1 q + \eta\sqrt{q} - \frac{q^{1/6}}{2} - 1)\frac{(q-1)}{2} \\
\le\ & \frac{\epsilon_1^2 q^3 + 2\eta\epsilon_1 q^2\sqrt{q} + \eta^2 q^2 - \epsilon_1 q^2 - \eta q\sqrt{q} - \epsilon_1^2 q^2 - 2\eta\epsilon_1 q\sqrt{q} - \eta^2 q + \epsilon_1 q + \eta\sqrt{q}}{2} \\
\le\ & \frac{\epsilon_1^2 q^3 + 2\eta\epsilon_1 q^2\sqrt{q} + \eta^2 q^2 - \epsilon_1^2 q^2 - 2\eta\epsilon_1 q\sqrt{q} - \eta^2 q}{2}.
\end{aligned}
$$

Since we can assume that $\epsilon_1 \ge q^{1/6}/2$, also $\epsilon_1 \ge \eta$, so the upper bound further simplifies to

$$
\frac{\epsilon_1^2 q^3 + 2\epsilon_1\eta q^2\sqrt{q}}{2} \le \frac{\epsilon_1^2 q^3 + 2\eta^2 q^3}{2},
$$

where the last inequality follows from $\epsilon_1 \leq \eta\sqrt{q}$.

There are $\theta_3 - |F| \geq q^3$ points in $\mathrm{PG}(3, q)$ not contained in $F$, hence, we find a point $R$, not in $F$, lying on at most $\frac{\epsilon_1^2+2\eta^2}{2}$ such secants to $F$. □

**Lemma 3.5** *If $B$ contains at least one line, then $\sqrt{q} - q^{1/6} \leq \epsilon_1$.*

**Proof.** Consider a point $R$ of $\mathrm{PG}(3, q) \setminus F$ lying on at most $\frac{\epsilon_1^2+2\eta^2}{2}$ secants to $F$, containing at least two simple points of $F$. The minihyper $F$ is projected from $R$ onto a weighted point set in a plane containing a line $L$. The plane $\langle R, L\rangle$ intersects $F$ in at least a 1-fold blocking set since $|\langle R, L\rangle \cap F| \geq q + 1$ (Theorem 2.4). So Lemma 2.5 implies that $\langle R, L\rangle \cap F$ contains a Baer subplane having a Baer subline on a line through $R$. This Baer subline has at most $\frac{q^{1/6}}{2}$ distinct multiple points of $F$, so is counted at least $\frac{1}{2}(\sqrt{q} - \frac{q^{1/6}}{2})^2$ times as a secant in the previous lemma. This number must be smaller than or equal to the total number of such secants to $F$ through $R$, so

$$\begin{aligned} (\sqrt{q} - \frac{q^{1/6}}{2})^2 &\leq \epsilon_1^2 + 2\eta^2 \\ \Leftrightarrow q - \sqrt{q}q^{1/6} + \frac{q^{1/3}}{4} - q^{1/6} &\leq \epsilon_1^2, \quad \text{since } \eta < \frac{q^{1/12}}{2} \\ \Rightarrow (\sqrt{q} - q^{1/6})^2 \leq q - \sqrt{q}q^{1/6} + \frac{q^{1/3}}{4} - q^{1/6} &\leq \epsilon_1^2. \end{aligned}$$

This last equation holds if $q \geq 4$ and then we have the assertion. □

From now on, we assume that $\epsilon_1 \geq \sqrt{q} - q^{1/6}$.

Since $(\epsilon_1^2 + 2\eta^2)/2 \leq \epsilon_1^2$ and since we will need in Lemma 3.8 a lot of points $R$, not in $F$, lying on a small number of secants to $F$, we will look for points $R$, not in $F$, lying on at most $\epsilon_1^2$ secants to $F$ containing at least two simple points of $F$.

**Lemma 3.6** *Let $R$ be a point of $\mathrm{PG}(3, q) \setminus F$ lying on at most $\epsilon_1^2$ secants to $F$, containing at least two simple points of $F$. Then $R$ lies on a line containing a Baer subline of $F$ which is contained in at least $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of $F$, containing at least $\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1$ points of $F$.*

**Proof.** The projection of $F$ from $R$ is a weighted $\epsilon_1$-fold blocking set $B$ in a plane, containing lines. Let $x$ be the number of lines contained in $B$, where some lines can be counted more than once in this weighted $\epsilon_1$-fold blocking set. It follows from Lemma 2.1 that the $x$ lines contained in $B$ can be removed from $B$ to obtain a new weighted $(\epsilon_1 - x)$-fold blocking set $B'$, containing no lines. Denote $\epsilon_1 - x$ by $\epsilon_1'$. By Lemma 3.3, for an $\epsilon_1'$-fold blocking set $B$ of size $\epsilon_1'(q+1)+\epsilon_0$ without lines, necessarily $\epsilon_1' < \frac{q^{1/6}}{2}$, so $B$ must contain at least $\epsilon_1 - \epsilon_1' > \epsilon_1 - \frac{q^{1/6}}{2}$ lines.

For each such line $L \subset B$, let $m_1$ be its multiplicity as a line in the weighted set $B$. Then the plane $\langle R, L\rangle$ intersects $F$ in an $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper, with $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$ (Theorem 2.4). This plane $\langle R, L\rangle$ contains $m_1$ Baer subplanes of $F$ (Lemma 2.5), and, for each such Baer subplane in $\langle R, L\rangle \cap F$, there is a line through $R$ containing a Baer subline of this Baer subplane.

A Baer subline is counted at least $\frac{1}{2}(\sqrt{q} - \frac{q^{1/6}}{2})^2$ times as a secant in Lemma 3.4. The point $R$ lies on at most $\epsilon_1^2 \leq \eta^2(q - q^{2/3} + \frac{q^{1/3}}{4})$ secants, hence $R$ lies on at most $2\eta^2$ different lines containing a Baer subline of $F$. There are at least $\epsilon_1 - \frac{q^{1/6}}{2}$ Baer sublines, in Baer subplanes of $F$, on lines through $R$. So some Baer subline lies in at least $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of $F$. These Baer subplanes of $F$ contain at least $\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1$ points of $F$. □

**Remark 3.7** *We will denote these Baer subplanes, contained in $F$, through a common Baer subline on a line through $R$ as* flags of Baer subplanes *corresponding to $R$. The next lemma shows that we can find several flags which leads to the fact that they must intersect each other in a certain minimum number of points.*

**Lemma 3.8** (1) *There are more than $8\eta^2$ points $R$ of* $\mathrm{PG}(3,q) \setminus F$*, defining different flags of Baer subplanes.*

(2) *There are two such flags intersecting each other in at least $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points of $F$.*

**Proof.** (1) We find the required points one by one. There exists at least one such point (Lemmas 3.4 and 3.6).

Suppose that we have already found $8\eta^2$ points $R$ not in $F$ with a corresponding flag of Baer subplanes, as in the previous lemma. Is there another point of $\mathrm{PG}(3,q) \setminus F$ lying on at most $\epsilon_1^2$ secants to $F$, containing at least two simple points of $F$? The number of points in these $8\eta^2$ flags, in the corresponding planes $\mathrm{PG}(2,q)$, is at most

$$8\eta^2((\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})q^2 + q + 1) = 4\epsilon_1 q^2 - 2q^{1/6}q^2 + 8\eta^2(q+1).$$

We count these points in the corresponding planes $\mathrm{PG}(2,q)$ to assure that the new flag is different from the ones we already have. There are at least $q^3+q^2+q+1-\epsilon_1(q+1)-\epsilon_0-4\epsilon_1 q^2+2q^{1/6}q^2-8\eta^2(q+1)$ points in $\mathrm{PG}(3,q)$ not in $F$ and not in the extended flags. If all these points lie on more than $\epsilon_1^2$ secants to $F$, then the number of incidences on the remaining secants is larger than $(\epsilon_1^2 q^3 + 2\eta^2 q^3)/2$, the total number of incidences on secants to $F$ we had in Lemma 3.3. So there is still another point $P \notin F$ on at most $\epsilon_1^2$ secants to $F$.

(2) Take $8\eta^2$ such points $R$ and suppose that the union of the $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes through the Baer subline of a flag corresponding to a point $R$ share for two such points $R$ and $R'$ at most $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points of $F$. Then

$$\begin{aligned}
|F| &\geq \sum_{i=1}^{8\eta^2}(\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1 - (i-1)\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}))\\
&\geq 8\eta^2(\frac{\epsilon_1 q}{2\eta^2} - \frac{q^{7/6}}{4\eta^2} + \sqrt{q} + 1) + \frac{(8\eta^2)^2}{2}\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})\\
&\geq 3\epsilon_1 q - \frac{3}{2}q^{7/6} + 8\eta^2(\sqrt{q}+1).
\end{aligned}$$

This is false since $\epsilon_1 \geq \sqrt{q} - q^{1/6}$. □

We have found different points $R$ and $R'$ with a corresponding flag of Baer subplanes. We now build with them a Baer subgeometry $\mathrm{PG}(3,\sqrt{q})$ contained in $F$.

**Lemma 3.9** *The minihyper $F$ contains a Baer subgeometry* $\mathrm{PG}(3,\sqrt{q})$ *if* $\epsilon_1 \geq \sqrt{q} - q^{1/6}$.

**Proof.** Let $R$ and $R'$ be two points of $\mathrm{PG}(3,q) \setminus F$ corresponding with a flag of $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of $F$, where these two flags share at least $\frac{q}{16\eta^2}(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$ points. Denote those two flags by $f_R$ and $f_{R'}$. Then some

Baer subplane $\pi_{R'}$ of $f_{R'}$ shares at least $\frac{q}{16\eta^2}$ points with the Baer subplanes of $f_R$. If this Baer subplane $\pi_{R'}$ shares at most two points with every Baer subplane of $f_R$, then $\frac{q}{16\eta^2} \leq 2(\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2})$, which is false since $\epsilon_1 < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$. So this Baer subplane $\pi_{R'}$ shares a Baer subline with some Baer subplane of $f_R$. Denote by $l$ the Baer subline of the flag $f_R$. This Baer subplane $\pi_{R'}$ cannot pass through $l$, since then this Baer subplane $\pi_{R'}$ only shares this subline $l$ with all these Baer subplanes of the flag $f_R$, but $\frac{q}{16\eta^2} > \sqrt{q} + 1$.

We wish to find a lower bound on the number of Baer subplanes of $f_R$, sharing a Baer subline with the Baer subplane $\pi_{R'}$. Two distinct Baer sublines share at most two points. We subtract two for every of the $\frac{\epsilon_1}{2\eta^2} - \frac{q^{1/6}}{4\eta^2}$ Baer subplanes of $f_R$ from $\frac{q}{16\eta^2}$ and then divide by $\sqrt{q} - 1$. The quotient is at least $\frac{q^{1/2}}{16\eta^2} + \frac{1}{16\eta^2} - \frac{1}{\eta}$, hence this Baer subplane $\pi_{R'}$ shares a Baer subline with at least $\frac{q^{1/2}}{16\eta^2} + \frac{1}{16\eta^2} - \frac{1}{\eta}$ Baer subplanes of $f_R$. Take this Baer subplane $\pi_{R'}$ and consider a Baer subplane $\pi_R$ of the flag $f_R$ which shares a Baer subline with $\pi_{R'}$. Together they define a Baer subgeometry $\Omega$ isomorphic to $\mathrm{PG}(3, \sqrt{q})$. Every Baer subplane of $f_R$ intersecting $\pi_{R'}$ in a Baer subline shares $l$ and this Baer subline with $\Omega$. Two intersecting Baer sublines define a Baer subplane in a unique way, so these Baer subplanes then lie completely in this Baer subgeometry $\Omega$.

Consider an arbitrary Baer subplane $\pi$ of $\Omega$ not through $l$. Then $\pi$ shares at least $\frac{q^{1/2}}{16\eta^2} + \frac{1}{16\eta^2} - \frac{1}{\eta}$ Baer sublines with $F$, so shares at least $\frac{q}{16\eta^2} + \frac{q^{1/2}}{16\eta^2} - \frac{q^{1/2}}{\eta} + 1$ points with $F$. Consider the plane over $\mathbb{F}_q$ of this Baer subplane $\pi$. This plane intersects $F$ in an $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, with $m_1 + m_0 \leq \epsilon_1 + \epsilon_0 = \eta(q^{1/2} - q^{1/6})$, which contains $m_1$ Baer subplanes (Lemma 2.5). Suppose this Baer subplane $\pi$ is not contained in $F$. It contains already at least $\frac{q}{16\eta^2} + \frac{q^{1/2}}{16\eta^2} - \frac{q^{1/2}}{\eta} + 1$ points of $F$. By [3, Lemma 4.4], we have that

$$|\pi \cap F| \leq m_0 + m_1(\sqrt{q} + 1) \leq 2\eta(q^{1/2} - q^{1/6}),$$

where the upper bound is obtained in the following way. First of all, $m_0 + m_1 \leq \eta(q^{1/2} - q^{1/6})$. Secondly, every one of the $m_1$ Baer subplanes contained in the $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper has size $q + \sqrt{q} + 1$ and contributes $\sqrt{q} + 1$ to the sum $m_1 + m_0$. Hence, also $m_1\sqrt{q} \leq \eta(q^{1/2} - q^{1/6})$.

But $\frac{q}{16\eta^2} + \frac{q^{1/2}}{16\eta^2} - \frac{q^{1/2}}{\eta} + 1 > 2\eta(q^{1/2} - q^{1/6})$ if $q \geq 1217$, so this Baer subplane $\pi$ lies completely in $F$. As a consequence, this Baer subgeometry $\Omega$ defined by $\pi_R$ and $\pi_{R'}$ lies completely in $F$. □

**Lemma 3.10** *Let $F$ be an $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 3, q\}$-minihyper, with $2\epsilon_1 + \epsilon_0 < q+2$, containing a subgeometry* $\mathrm{PG}(3, \sqrt{q})$*. Then $F \backslash \mathrm{PG}(3, \sqrt{q})$ is an $\{(\epsilon_1 - \sqrt{q} - 1)(q+1) + \epsilon_0, \epsilon_1 - \sqrt{q} - 1; 3, q\}$-minihyper.*

**Proof.** A plane $\pi$ either intersects a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ in a subline $\mathrm{PG}(1, \sqrt{q})$ or a subplane $\mathrm{PG}(2, \sqrt{q})$. We only have to discuss the case that $\pi \cap \mathrm{PG}(3, \sqrt{q})$ is a subplane $\mathrm{PG}(2, \sqrt{q})$ of size $q + \sqrt{q} + 1$.

If $\pi$ still contains $\epsilon_1 - \sqrt{q} - 1$ other points of $F$, then removing this Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ from $F$ causes no problem for the plane $\pi$. So from now on, we assume that $q + \sqrt{q} + 1 \leq |\pi \cap F| < q + \epsilon_1$.

We select a point $R$ of $\pi \backslash F$. Project $\pi$ and $F$ from $R$ onto a plane. Then we obtain an $\epsilon_1$-fold blocking multiset $B$ in this plane containing a line $L$, which is the projection of $\pi \cap F$. By [8, Theorem 2.2], we can reduce the weight of every point of $L$ by one to obtain an $(\epsilon_1 - 1)$-fold blocking set $B'$ in this plane. But then $L$ is still blocked at least $\epsilon_1 - 1$ times by $B'$. So $\pi$ is blocked at least $q + \epsilon_1$ times by $F$. □

**Theorem 3.11** *Let $F$ be a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 3, q\}$-minihyper, $q = p^h$, $q$ square, $p > 3$ prime, $h \geq 2$, having weighted points with total weight at most $\frac{2\epsilon_1^2}{q}$ and where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, then $F$ contains a sum of $A$ lines, $B$ isolated Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ and $C$ Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, where $A + B + C(\sqrt{q} + 1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.*

**Proof.** If $\epsilon_1 < q^{1/6}/2$, this is the result mentioned in Theorem 3.1, so from now on assume that $\epsilon_1 \geq q^{1/6}/2$.

If $F$ contains $A$ lines, we can remove these lines from $F$, and then apply the arguments to $F$ minus these $A$ lines (Lemma 2.1). We denote the minihyper that remains again by $F$. Let $R$ be a point not in the minihyper $F$ on at most $\frac{\epsilon_1^2 + 2\eta^2}{2}$ secants to $F$, containing at least two points of $F$ of weight one (Lemma 3.4). Projecting $F$ from $R$ onto a plane not through $R$ gives a weighted $\epsilon_1$-fold blocking set $B$ in this plane. If $B$ does not contain lines, Lemma 3.3 states that $F$ is the sum of $\epsilon_1$ lines and Baer subplanes $\mathrm{PG}(2, \sqrt{q})$, and possibly some extra points. If $B$ does contain lines, we find a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ contained in $F$, which can be removed from $F$ to obtain a new $\{(\epsilon_1 - \sqrt{q} - 1)(q+1) + \epsilon_0, \epsilon_1 - \sqrt{q} - 1; 3, q\}$-minihyper, see Lemma 3.10. Repeating the previous arguments with this minihyper gives us the assertion. □

# 4 Larger dimensions

We now characterise non-weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$-minihypers $F$, $q$ square, $q = p^h$, $p > 3$ prime, $h \geq 2$, $q \geq 1217$, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, by induction on the dimension $n$. We suppose that every $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n-1, q\}$-minihyper, with $n \geq 4$, is a pairwise disjoint union of $A$ lines, $B$ isolated Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ and $C$ Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, with $A + B + C(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points. As in the 3-dimensional case, we start by using Lemma 2.1 to remove the lines contained in $F$.

We want to project $F$ onto a hyperplane in such a way that the number of multiple points appearing in the projection is as small as possible.

**Lemma 4.1** *For $n = 4$, there is a point $R \notin F$ lying on at most $\frac{\epsilon_1^2}{q}$ secants to $F$. In larger dimensions $n$, there are points $R \notin F$ only lying on tangents to $F$.*

**Proof.** The number of points on secants to $F$ is at most

$$\frac{(\epsilon_1(q+1)+\epsilon_0)^2}{2}(q-1) = \frac{\epsilon_1(q^3+q^2-q-1) + 2\epsilon_1\epsilon_0(q^2-1) + \epsilon_0^2(q-1)}{2}.$$

Now $\epsilon_1\epsilon_0, \epsilon_0^2 < \frac{q^{7/6}}{4}$. For $n \geq 5$, this number is smaller than the number of points in $\mathrm{PG}(n, q) \backslash F$. In this case, there exists at least one point $R$ only lying on tangents to $F$.
For $n = 4$, we divide by $q^4 + q^3 \leq \theta_4 - |F|$. This gives a point $R$ lying on at most

$$\frac{\epsilon_1^2}{2q} + \frac{2q^{7/6}(q^2-1)/2 + q^{7/6}(q-1)/2}{2(q^4+q^3)} \leq \frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}}$$

secants to $F$. Either $\frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}} < 1$ and then $R$ lies on zero secants to $F$ or either $\frac{\epsilon_1^2}{2q} + \frac{1}{2q^{5/6}} \geq 1$, then $\frac{\epsilon_1^2}{2q} \geq \frac{1}{2q^{5/6}}$. In both cases, $\frac{\epsilon_1^2}{q}$ can be used as an upper bound on the number of secants to $F$ through $R$. □

In the case of $n = 4$, projecting from a point $R \notin F$ as in the previous lemma gives a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n-1, q\}$-minihyper with the total weight of the multiple points at most $\frac{2\epsilon_1^2}{q}$. This explains the bound in Theorem 3.11.

**Theorem 4.2** *Let $F$ be a non-weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$-minihyper, $q$ square, $q = p^h$, $p > 3$ prime, $h \geq 2$, $q \geq 1217$, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$, then $F$ is the union of pairwise disjoint $A$ lines, $B$ isolated Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ and $C$ Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, with $A + B + C(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points.*

**Proof.** We prove this result by induction on $n$. Theorem 3.11 proves this theorem for $n = 3$. If $F$ contains lines, these lines can be removed from $F$ (Lemma 2.1); so we assume that $F$ does not contain any lines.

Project $F$ from a point $R$, lying only on tangents to $F$ or on at most $\epsilon_1^2/q$ secants to $F$ if $n = 4$, onto a hyperplane $\pi$ not through $R$. We obtain a (weighted if $n = 4$) $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n-1, q\}$-minihyper $F'$ which is the sum of $A'$ lines, $B'$ isolated Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ and $C'$ Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, with $A' + B' + C'(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ points.

**Case I:** assume that $F'$ contains a line $L$.

The plane $\langle R, L\rangle$ intersects $F$ in at least a 1-fold blocking set, which contains a Baer subplane, since by assumption, $F$ does not contain lines (Lemma 2.5).

Since $\langle R, L\rangle$ shares a Baer subplane with $F$, $R$ lies on a Baer subline to this Baer subplane, but then $R$ lies on a $(\sqrt{q}+1)$-secant to $F$, which is false for $n > 4$. For $n = 4$, this line is projected onto a point of $F'$ with weight up to $\sqrt{q}+1 > \frac{q^{1/6}}{2} > \frac{2\epsilon_1^2}{q}$, which is false. So this case cannot occur.

**Case II:** assume that $F'$ contains an isolated Baer subplane $\mathrm{PG}(2, \sqrt{q})$.

Denote this Baer subplane $\mathrm{PG}(2, \sqrt{q})$ by $\omega$. The 3-space $\langle R, \omega\rangle$ intersects $F$ in an $\{m_1(q+1)+m_0, m_1; 3, q\}$-minihyper, with $m_1 \geq 1$ (Lemma 2.4), so $\langle R, \omega\rangle \cap F$ contains by the induction hypothesis on $n$ the union of points, isolated Baer subgeometries $\mathrm{PG}(2, \sqrt{q})$ and Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, which are all pairwise disjoint. Assume $\langle R, \omega\rangle$ contains a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ and consider the conjugate point $R^{\sqrt{q}}$ of $R$ w.r.t. $\mathrm{PG}(3, \sqrt{q})$. The line $RR^{\sqrt{q}}$ intersects $\mathrm{PG}(3, \sqrt{q})$ in a Baer subline, which is false since the projection of this Baer subline would lead to a projected point in $F'$ with weight at least $\sqrt{q}+1$. So $\langle R, \omega\rangle \cap F$ contains points and isolated Baer subplanes. One of these Baer subplanes $\mathrm{PG}(2, \sqrt{q})$ is projected from $R$ onto

$\omega$.

**Case III:** assume that $F'$ contains a Baer subgeometry PG$(3, \sqrt{q})$.

Consider two Baer subplanes $\omega_1$ and $\omega_2$ in PG$(3, \sqrt{q})$. By the arguments of case II, we find Baer subplanes $\omega_1'$ and $\omega_2'$ contained in $F$ projected onto $\omega_1$ and $\omega_2$ respectively. Since there are less than $\frac{q^{1/6}}{2}$ multiple points in the intersection line of $\omega_1$ and $\omega_2$, this projected Baer subline $\omega_1 \cap \omega_2$ must be the projection of a Baer subline contained in $F$, which must be equal to the intersection line of $\omega_1'$ and $\omega_2'$. So $\omega_1'$ and $\omega_2'$ span a Baer subgeometry PG$(3, \sqrt{q})$. The 3-space over $\mathbb{F}_q$ defined by this Baer subgeometry shares two intersecting Baer subplanes with $F$. By the induction hypothesis, this Baer subgeometry PG$(3, \sqrt{q})$ must be contained in $F$.

**Conclusion:** Let $F$ be a non-weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; n, q\}$-minihyper, $n \geq 4$, where $\epsilon_1 + \epsilon_0 = \eta(\sqrt{q} - q^{1/6}) < \frac{q^{7/12}}{2} - \frac{q^{1/4}}{2}$.

By assumption, if $F$ contains a line $L$, then this line $L$ can be removed from $F$, so that $F \setminus L$ is an $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; n, q\}$-minihyper (Lemma 2.1). From now on, assume that all the lines contained in $F$ are removed from $F$. By induction on $\epsilon_1 + \epsilon_0$, these lines contained in $F$ are pairwise disjoint among each other.

The preceding cases I, II, and III show that the projection of $F$, from a suitably selected point $R$, onto a hyperplane leads to a minihyper in this hyperplane containing $B'$ isolated Baer subplanes PG$(2, \sqrt{q})$ and $C'$ Baer subgeometries PG$(3, \sqrt{q})$, with $B' + C'(\sqrt{q}+1) = \epsilon_1$, and that these isolated Baer subplanes and 3-dimensional Baer subgeometries contained in the projected minihyper arise from $B'$ isolated Baer subplanes PG$(2, \sqrt{q})$ and $C'$ Baer subgeometries PG$(3, \sqrt{q})$ contained in the minihyper $F$.

If lines again are permitted to be contained in $F$, the preceding arguments therefore show that $F$ is the union of pairwise disjoint $A$ lines, $B$ isolated Baer subplanes PG$(2, \sqrt{q})$ and $C$ Baer subgeometries PG$(3, \sqrt{q})$, with $A + B + C(\sqrt{q}+1) = \epsilon_1$, and $\epsilon_0 - B\sqrt{q}$ extra points. $\square$

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Address of the authors:

Ghent University, Dept. of Mathematics, Krijgslaan 281-S22, 9000 Ghent, Belgium

A. Hallez: athallez@cage.ugent.be
J. De Beule: jdebeule@cage.ugent.be, `http://cage.ugent.be/~jdebeule`
L. Storme: ls@cage.ugent.be, `http://cage.ugent.be/~ls`