Isometric embeddings between the near polygons $\mathbb{H}_n$ and $\mathbb{G}_n$

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Abstract

Let $n \in \mathbb{N} \setminus \{0,1,2\}$. We prove that there exists up to equivalence one and up to isomorphism $(n+1)(2n+1)$ isometric embeddings of the near $2n$-gon $\mathbb{H}_n$ into the near $2n$-gon $\mathbb{G}_n$.

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1 Basic definitions and main result

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, (possibly empty) line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *partial linear space* if every two distinct points are incident with at most one line. If x_1 and x_2 are two points of a partial linear space $\mathcal{S}$, then the distance $\mathrm{d}(x_1, x_2)$ between x_1 and x_2 will always be measured in the collinearity graph of $\mathcal{S}$. A set X of points of $\mathcal{S}$ is called a *subspace* if every line having two of its points in X has all its points in X. If X is a nonempty subspace of $\mathcal{S}$, then we denote by $\widetilde{X}$ the subgeometry of $\mathcal{S}$ induced on X by those lines of $\mathcal{S}$ that have all their points in X.

A *near polygon* is a partial linear space $\mathcal{S}$ with the property that for every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x. If d is the maximal distance between two points of a near polygon $\mathcal{S}$, then $\mathcal{S}$ is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles. We now define two classes of near polygons.

Let $n \in \mathbb{N}$. With every set X of size $2n+2$, there is associated a point-line geometry $\mathbb{H}_n(X)$: the points of $\mathbb{H}_n(X)$ are the partitions of X in $n+1$ subsets of size 2; the lines of $\mathbb{H}_n(X)$ are the partitions of X in $n-1$ subsets of size 2 and 1 subset of size 4; a point p of $\mathbb{H}_n(X)$ is incident with a line L of $\mathbb{H}_n(X)$ if and only if the partition corresponding to p is a refinement of the partition corresponding to L. By Brouwer et al. [1], $\mathbb{H}_n(X)$ is a near $2n$-gon with three points on each line. The isomorphism class of the geometry $\mathbb{H}_n(X)$ is obviously independent of the set X of size $2n+2$. We will denote by $\mathbb{H}_n$ any suitable representative of this isomorphism class. The near polygon $\mathbb{H}_0$ consists of a unique point, the near polygon $\mathbb{H}_1$ is the line of size 3 and the generalized quadrangle $\mathbb{H}_2$ is isomorphic to the generalized quadrangle $W(2)$ described in Payne and Thas [8, Section 3.1].

Let $n \in \mathbb{N} \setminus \{0,1\}$, let V be a $2n$-dimensional vector space over $\mathbb{F}_4$ and let $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ be an ordered basis of V. The set of all points $\langle \sum_{i=1}^{2n} X_i \bar{e}_i \rangle$ of $\mathrm{PG}(V)$ satisfying the equation $\sum_{i=1}^{2n} X_i^3 = 0$ is a nonsingular Hermitian variety $\mathcal{H}$ of $\mathrm{PG}(V)$. We denote the dual polar space (isomorphic to $DH(2n-1,4)$) associated with $\mathcal{H}$ by $DH(V,B)$. So, the points of $DH(V,B)$ are the $(n-1)$-dimensional subspaces of $\mathrm{PG}(V)$ contained in $\mathcal{H}$, the lines of $DH(V,B)$ are the $(n-2)$-dimensional subspaces of $\mathrm{PG}(V)$ contained in $\mathcal{H}$ and incidence is reverse containment. The *support* S_p of a point $p = \langle \sum_{i=1}^{2n} X_i \bar{e}_i \rangle$ of $\mathrm{PG}(V)$ (*with respect to* B) is the set of all $i \in \{1, 2, \ldots, 2n\}$ for which $X_i \neq 0$. The number of elements of S_p is called the *weight* of p (*with respect to* B). Observe that a point of $\mathrm{PG}(V)$ belongs to $\mathcal{H}$ if and only if its weight is even. Let X denote the set of all $(n-1)$-dimensional subspaces contained in $\mathcal{H}$ generated by n points of weight 2 (whose supports are mutually disjoint). By De Bruyn [3], X is a subspace of $DH(V,B)$ and $\mathbb{G}_n(V,B) := \widetilde{X}$ is a near $2n$-gon with three points on each line. The isomorphism class of the geometry $\mathbb{G}_n(V,B)$ is obviously independent of the $2n$-dimensional vector space V and the ordered basis B of V. We will denote by $\mathbb{G}_n$ any suitable representative of this isomorphism class. By [3], the generalized quadrangle $\mathbb{G}_2$ is isomorphic to the generalized quadrangle $Q^-(5,2)$ described in Payne and Thas [8, Section 3.1]. By convention, $\mathbb{G}_1$ is the line with three points and $\mathbb{G}_0$ is the near 0-gon.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two partial linear spaces. An *embedding* of $\mathcal{S}_1$ into $\mathcal{S}_2$ is an injective mapping e from the point set of $\mathcal{S}_1$ to the point set of $\mathcal{S}_2$ satisfying the following two properties:

- e maps every line of $\mathcal{S}_1$ into a line of $\mathcal{S}_2$;
- e maps distinct lines of $\mathcal{S}_1$ into distinct lines of $\mathcal{S}_2$.

An embedding e of $\mathcal{S}_1$ into $\mathcal{S}_2$ will be denoted by $e : \mathcal{S}_1 \to \mathcal{S}_2$. An embedding $e : \mathcal{S}_1 \to \mathcal{S}_2$ is called *full* if it maps lines of $\mathcal{S}_1$ to full lines of $\mathcal{S}_2$. The embedding e is called *isometric* if it preserves the distances between points.

Suppose e is an embedding of the partial linear space $\mathcal{S}_1$ into the partial linear space $\mathcal{S}_2$. If θ_1 and θ_2 are automorphisms of respectively $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\theta_2 \circ e = e \circ \theta_1$, then we say that θ_1 *lifts (through* e*) to* θ_2. If the automorphism θ_1 of $\mathcal{S}_1$ lifts through e to an automorphism of $\mathcal{S}_2$, then we will denote by $\widetilde{\theta_1}$ any of the automorphisms of $\mathcal{S}_2$ to which θ_1 lifts. The set G of all automorphisms of $\mathcal{S}_1$ which lift through e to an automorphism of $\mathcal{S}_2$ is clearly a subgroup of the full automorphism group $\mathrm{Aut}(\mathcal{S}_1)$ of $\mathcal{S}_1$. If $G = \mathrm{Aut}(\mathcal{S}_1)$, then e is called a *homogeneous embedding.*

Two embeddings $e : \mathcal{S}_1 \to \mathcal{S}_2$ and $e' : \mathcal{S}_1' \to \mathcal{S}_2'$ are called *equivalent* if there exists an isomorphism θ_1 from $\mathcal{S}_1$ to $\mathcal{S}_1'$ and an isomorphism θ_2 from $\mathcal{S}_2$ to $\mathcal{S}_2'$ such that $e' \circ \theta_1 = \theta_2 \circ e$. Two embeddings $e : \mathcal{S}_1 \to \mathcal{S}_2$ and $e' : \mathcal{S}_1 \to \mathcal{S}_2'$ of the same partial linear space $\mathcal{S}_1$ are called *isomorphic* if there exists an isomorphism θ from $\mathcal{S}_2$ to $\mathcal{S}_2'$ such that $e' = \theta \circ e$. If e and e' are isomorphic, then they are also equivalent.

The following is the main result of this paper.

Theorem 1.1 *Suppose $n \geq 3$. Then there exists up to equivalence a unique isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$. Up to isomorphism, there are $(n+1)(2n+1)$ isometric embeddings of $\mathbb{H}_n$ into $\mathbb{G}_n$.*

The conclusion of the theorem is false if $n = 2$. There is up to isomorphism a unique isometric embedding of the generalized quadrangle $\mathbb{H}_2 \cong W(2)$ into the generalized quadrangle $\mathbb{G}_2 \cong Q^-(5,2)$.

This paper is part of a project of the author to study isometric embeddings between dense near polygons. In the paper [5], we studied isometric embeddings of the near polygon $\mathbb{H}_n$, $n \geq 2$, into the symplectic dual polar space $DW(2n-1,2)$ and isometric embeddings of the near polygon $\mathbb{G}_n$, $n \geq 2$, into the Hermitian dual polar space $DH(2n-1,4)$. Isometric embeddings between near polygons are important for the study of valuations (De Bruyn and Vandecasteele [6]) which themselves are very important for obtaining classification results about near polygons.

We will prove Theorem 1.1 in Section 4. In Section 3, we will give an explicit description of an isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$. The background on near polygons and some of the machinery that are necessary for the proof of Theorem 1.1 will be discussed in Section 2.

2 Preliminaries

2.1 Embeddings between partial linear spaces

In this subsection, we collect some facts regarding embeddings between general partial linear spaces.

Suppose $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ are two partial linear spaces. We call an embedding $e : \mathcal{S}_1 \to \mathcal{S}_2$ *regular* if for all points x_1 and x_2 of $\mathcal{S}_1$, we have that $e(x_1)$ and $e(x_2)$ are collinear if and only if x_1 and x_2 are collinear. Every isometric embedding is also regular. If $e : \mathcal{S}_1 \to \mathcal{S}_2$ is an embedding, then we denote by $\mathcal{S}_1^e$ the subgeometry of $\mathcal{S}_2$ induced on the point set $e(\mathcal{P}_1)$ by those lines of $\mathcal{S}_2$ that have at least two points in $e(\mathcal{P}_1)$.

Lemma 2.1 *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two partial linear spaces and suppose $e : \mathcal{S}_1 \to \mathcal{S}_2$ is a regular embedding. Then e defines an isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_1^e$.*

Proof. Suppose $L_1 = \{x_i \,|\, i \in I\}$ is a line of $\mathcal{S}_1$ for some index set I of size at least 2. Then $e(L_1) = \{e(x_i) \,|\, i \in I\}$ is contained in some line L_2 of $\mathcal{S}_2$. We claim that $e(L_1)$ is a line of $\mathcal{S}_1^e$. If this were not the case, then L_2 would contain a point $e(y)$ where $y \in \mathcal{P}_1 \setminus L_1$. If z denotes an arbitrary point of L_1, then the points y and z must be incident with a unique line L_1' of $\mathcal{S}_1$ as the points $e(y)$ and $e(z)$ are collinear in $\mathcal{S}_2$. Since each of the sets $e(L_1)$ and $e(L_1')$ are contained in the same line L_2 of $\mathcal{S}_2$, we should have $L_1 = L_1'$, in contradiction with $y \in L_1' \setminus L_1$. So, $e(L_1)$ is indeed a line of $\mathcal{S}_1^e$.

Conversely, suppose that L_1' is a line of $\mathcal{S}_1^e$, let x and y be two distinct points of $e^{-1}(L_1')$ and let L_1 denote denote the unique line of $\mathcal{S}_1$ containing x and y. By the previous paragraph, $e(L_1)$ should be a line of $\mathcal{S}_1^e$ which necessarily equals L_1'.

So, the bijection e between the point sets of $\mathcal{S}_1$ and $\mathcal{S}_1^e$ defines a bijection between the line sets of $\mathcal{S}_1$ and $\mathcal{S}_1^e$ preserving incidence. ■

Lemma 2.2 *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two partial linear spaces and let e_1 and e_2 be two embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$. Then:*

(1) *If e_1 and e_2 are regular embeddings and if there exists an automorphism θ_2 of $\mathcal{S}_2$ mapping $e_1(\mathcal{P}_1)$ to $e_2(\mathcal{P}_1)$, then e_1 and e_2 are equivalent embeddings.*

(2) *If e_1 is a homogeneous embedding and e_2 is equivalent to e_1, then e_1 and e_2 are isomorphic.*

(3) *The following are equivalent for an automorphism θ_1 of $\mathcal{S}_1$: (i) θ_1 lifts through e_1 to an automorphism of $\mathcal{S}_2$; (ii) the embeddings e_1 and $e_1 \circ \theta_1$ are isomorphic.*

Proof. (1) The automorphism θ_2 of $\mathcal{S}_2$ induces an isomorphism between $\mathcal{S}_1^{e_1} \cong \mathcal{S}_1$ and $\mathcal{S}_1^{e_2} \cong \mathcal{S}_1$. Hence, $\theta_1 := e_1^{-1} \circ \theta_2^{-1} \circ e_2$ is an automorphism of $\mathcal{S}_1$ and $e_2 = \theta_2 \circ e_1 \circ \theta_1$.

(2) Let θ_1 be an automorphism of $\mathcal{S}_1$ and θ_2 be an automorphism of $\mathcal{S}_2$ such that $e_2 = \theta_2 \circ e_1 \circ \theta_1$. Then $e_2 = \theta_2 \circ \widetilde{\theta_1} \circ e_1$. Hence, e_1 and e_2 are isomorphic.

(3) The embeddings e_1 and $e_1 \circ \theta_1$ are isomorphic if and only if there exists an automorphism θ_2 of $\mathcal{S}_2$ such that $\theta_2 \circ e_1 = e_1 \circ \theta_1$, i.e. if and only if θ_1 lifts through e_1 to an automorphism of $\mathcal{S}_2$. ■

By Lemma 2.2(1)+(2), we immediately have:

Corollary 2.3 *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two partial linear spaces and let e^* be a given regular homogeneous embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$. If e is a regular embedding for which there exists an automorphism θ_2 of $\mathcal{S}_2$ such that $e(\mathcal{P}_1) = \theta_2 \circ e^*(\mathcal{P}_1)$, then e is isomorphic to e^*.*

Corollary 2.3 can be improved as follows:

Proposition 2.4 *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two partial linear spaces and let e^* be a given regular embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$. Let G denote the full automorphism group of $\mathcal{S}_1$ and let H denote the subgroup of G consisting of all automorphisms of $\mathcal{S}_1$ which lift through e^* to an automorphism of $\mathcal{S}_2$. Choose an index set I of size $|G:H|$ and automorphisms $\theta_1^{(i)}$, $i \in I$, of $\mathcal{S}_1$ such that $\{H \circ \theta_1^{(i)} \,|\, i \in I\}$ is the set of all right cosets of H in G. Then the following holds:*

(1) *Any two embeddings of the set $\{e^* \circ \theta_1^{(i)} \,|\, i \in I\}$ are nonisomorphic.*

(2) *If e is a regular embedding for which there exists an automorphism θ_2 of $\mathcal{S}_2$ such that $e(\mathcal{P}_1) = \theta_2 \circ e^*(\mathcal{P}_1)$, then e is isomorphic to (precisely) one of the embeddings of the set $\{e^* \circ \theta_1^{(i)} \,|\, i \in I\}$.*

Proof. (1) Suppose i_1 and i_2 are two distinct elements of I such that the embeddings $e^* \circ \theta_1^{(i_1)}$ and $e^* \circ \theta_1^{(i_2)}$ of $\mathcal{S}_1$ into $\mathcal{S}_2$ are isomorphic. Then also the embeddings e^* and $e^* \circ \theta_1^{(i_1)} \circ [\theta_1^{(i_2)}]^{-1}$ of $\mathcal{S}_1$ into $\mathcal{S}_2$ are isomorphic. By Lemma 2.2(3), this would imply that $\theta_1^{(i_1)} \circ [\theta_1^{(i_2)}]^{-1} \in H$, clearly a contradiction.

(2) By Lemma 2.2(1), e^* and e are equivalent embeddings. Hence, there exists an automorphism θ_1 of $\mathcal{S}_1$ and an automorphism θ_2 of $\mathcal{S}_2$ such that $e = \theta_2 \circ e^* \circ \theta_1$. Now, there exists a unique $\theta_1' \in H$ and a unique $i \in I$ such that $\theta_1 = \theta_1' \circ \theta_1^{(i)}$. Then $e = \theta_2 \circ e^* \circ \theta_1' \circ \theta_1^{(i)} = \theta_2 \circ \widetilde{\theta_1'} \circ e^* \circ \theta_1^{(i)}$. Hence, e and $e^* \circ \theta_1^{(i)}$ are isomorphic. ■

2.2 Near polygons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. If x is a point of $\mathcal{S}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x. A subspace X of $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of X is also contained in X. Clearly, the whole point set $\mathcal{P}$ is an example of a convex subspace and the intersection of any number of (convex) subspaces of $\mathcal{S}$ is again a (convex) subspace of $\mathcal{S}$. If X is a nonempty convex subspace of $\mathcal{S}$, then $\widetilde{X}$ itself is also a near polygon. If $*_1, *_2, \ldots, *_k$ are $k \geq 1$ objects of $\mathcal{S}$, each being a point or a nonempty set of points, then $\langle *_1, *_2, \ldots, *_k \rangle$ denotes the smallest convex subspace of $\mathcal{S}$ containing $*_1, *_2, \ldots, *_k$. The set $\langle *_1, *_2, \ldots, *_k \rangle$ is well-defined since it equals the intersection of all convex subspaces containing $*_1, *_2, \ldots, *_k$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. Suppose now that $\mathcal{S}$ is a dense near $2n$-gon. If x and y are two points of $\mathcal{S}$ at distance δ from each other, then by Shult and Yanushka [9, Proposition 2.5] and Brouwer and Wilbrink [2, Theorem 4], $\langle x, y \rangle$ is the unique convex subspace of diameter δ containing x and y. The convex subspace $\langle x, y \rangle$ is called a *quad* if $\delta = 2$ and a *max* if $\delta = n - 1$. For every point x of $\mathcal{S}$, let $\mathcal{L}(\mathcal{S}, x)$ be the point-line geometry whose points are the lines through x, whose lines are the quads through x, and whose incidence relation is containment. $\mathcal{L}(\mathcal{S}, x)$ is called the *local space at* x. The *modified local space* $\mathcal{ML}(\mathcal{S}, x)$ *at* x is obtained from $\mathcal{L}(\mathcal{S}, x)$ by removing all lines of size 2. A proof of the following proposition is essentially contained in Brouwer and Wilbrink [2].

Proposition 2.5 *Let x and y be two distinct points of a dense near polygon $\mathcal{S}$ and let $\mathcal{L}_{x,y}$ denote the set of lines through x containing a point at distance $\mathrm{d}(x, y) - 1$ from y. Then: (i) a line through x is contained in $\langle x, y \rangle$ if and only if $L \in \mathcal{L}_{x,y}$; (ii) $\langle L \mid L \in \mathcal{L}_{x,y} \rangle = \langle x, y \rangle$.*

If x is a point of a dense near polygon $\mathcal{S}$ at distance 1 from a nonempty convex subspace F, then F contains a unique point x' collinear with x and $\mathrm{d}(x, y) = 1 + \mathrm{d}(x', y)$ for every point y of F. A convex subspace F of $\mathcal{S}$ is called *classical in* $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a (necessarily unique) point $\pi_F(x) \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. The map $\mathcal{P} \to F; x \mapsto \pi_F(x)$ is called the *projection* on F. A max M of $\mathcal{S}$ is called *big in* $\mathcal{S}$ if every point outside M is collinear with a (necessarily unique) point of M. Every big max M of $\mathcal{S}$ is classical in $\mathcal{S}$ and for every point x outside M, $\pi_M(x)$ is the unique point of M collinear with x. If M is a big max and F is a convex subspace meeting M, then either $F \subseteq M$ or $F \cap M$ is a big max of $\widetilde{F}$. If M_1 and M_2 are two disjoint big maxes of $\mathcal{S}$, then the map $M_1 \to M_2; x \mapsto \pi_{M_2}(x)$ is an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$, and its inverse map is given by $M_2 \to M_1; x \mapsto \pi_{M_1}(x)$.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near polygon with three points on each line, and that M is a big max of $\mathcal{S}$. For every point x of M, we define $\mathcal{R}_M(x) := x$. For every point x outside M, let $\mathcal{R}_M(x)$ denote the third point on the line through x and $\pi_M(x)$. The map $\mathcal{R}_M : \mathcal{P} \to \mathcal{P}$ defines an automorphism of $\mathcal{S}$ and is called the *reflection about* $\mathcal{S}$. So, if F is a convex subspace of $\mathcal{S}$, then $\mathcal{R}_M(F)$ is a convex subspace of the same diameter as F. If F is a big max, then also $\mathcal{R}_M(F)$ is a big max.

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two near polygons. We suppose that the sets $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are mutually disjoint. We define the following point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$:

- $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$;
- $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$;
- the point (x, y) of $\mathcal{S}$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $(y, L) \in \mathrm{I}_2$; the point (x, y) of $\mathcal{S}$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $(x, M) \in \mathrm{I}_1$ and $y = u$.

The point-line geometry $\mathcal{S}$ is also denoted by $\mathcal{S}_1 \times \mathcal{S}_2$ and is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a (dense) near $2d_i$-gon, then $\mathcal{S}_1 \times \mathcal{S}_2$ is a (dense) near $2(d_1 + d_2)$-gon. A generalized quadrangle is called an $(s_1 + 1) \times (s_2 + 1)$-grid if it is isomorphic to the direct product of a line of size $s_1 + 1$ and a line of size $s_2 + 1$. If p is a point of $\mathcal{S}_1 \times \mathcal{S}_2$, then there exist convex subspaces F_1 and F_2 through p such that the following properties are satisfied: (i) $\widetilde{F_1} \cong \mathcal{S}_1$ and $\widetilde{F_2} \cong \mathcal{S}_2$; (ii) every line through p is contained in either F_1 or F_2; (iii) if L_i, $i \in \{1, 2\}$, is a line contained in F_i, then $\widetilde{\langle L_1, L_2 \rangle}$ is a grid.

More information on (dense) near polygons and proofs of the above-mentioned facts can be found in the book [4]. Regarding isometric embeddings between dense near polygons, we can say the following.

Proposition 2.6 (Huang [7, Corollary 3.3]) *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two dense near polygons with respective distance functions $d_1(\cdot, \cdot)$ and $d_2(\cdot, \cdot)$ and respective diameters n_1 and n_2. Let e be a map from $\mathcal{P}_1$ to $\mathcal{P}_2$ satisfying the following for any two points x and y of $\mathcal{P}_1$: if $d_1(x, y) = 1$, then also $d_2(x, y) = 1$. Then e is an isometric embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$ if and only if there exist points x^* and y^* in $\mathcal{S}_1$ satisfying $d_1(x^*, y^*) = d_2(e(x^*), e(y^*)) = n_1$.*

Proposition 2.7 (De Bruyn [5, Proposition 2.5]) *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two dense near polygons and let e be an isometric embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$. Then for every nonempty convex subspace F of $\mathcal{S}_1$, there exists a unique nonempty convex subspace $\overline{F}$ of $\mathcal{S}_2$ satisfying:*

- *$\overline{F}$ and F have the same diameter;*

• $\overline{F} \cap e(\mathcal{P}_1) = e(F)$.
If F_1 and F_2 are two distinct nonempty convex subspaces of $\mathcal{S}_1$, then $\overline{F_1}$ and $\overline{F_2}$ are distinct.

2.3 Properties of the near $2n$-gons $\mathbb{H}_n$ and $\mathbb{G}_n$

In this subsection, we collect some known properties of the near $2n$-gons $\mathbb{H}_n$ and $\mathbb{G}_n$ which can be found in the book [4].

Let $n \geq 2$, let X be a set of size $2n+2$ and put $\mathbb{H}_n := \mathbb{H}_n(X)$.

Let P_1 and P_2 be two points of $\mathbb{H}_n$, i.e. P_1 and P_2 are two partitions of X in $n+1$ subsets of size 2. Let Γ_{P_1,P_2} denote the graph with vertices the elements of X, with two distinct vertices i and j adjacent whenever $\{i,j\}$ is contained in $P_1 \cup P_2$. Then the distance between P_1 and P_2 in the near polygon $\mathbb{H}_n$ is equal to $n+1-C$ where C is the number of connected components of Γ_{P_1,P_2}.

Every quad of $\mathbb{H}_n$ is either a grid-quad or a $W(2)$-quad, with both types of quads occurring if $n \geq 3$. Every line of $\mathbb{H}_n$ is contained in $n-1$ $W(2)$-quads and $\frac{(n-1)(n-2)}{2}$ grid-quads. Every $W(2)$-quad of $\mathbb{H}_n$ is classical in $\mathbb{H}_n$. There exists a bijective correspondence between the nonempty convex subspaces F of $\mathbb{H}_n$ and the partitions $\mathcal{P}_F$ of X in subsets of even size. This correspondence is as follows: F consists of all partitions of X in $n+1$ subsets of size 2 that are refinements of the partition $\mathcal{P}_F$. In the sequel, we will often say that $\mathcal{P}_F$ is the *partition of X corresponding to F*, or that F is *the convex subspace of $\mathbb{H}_n$ corresponding to $\mathcal{P}_F$*. The grid-quads of $\mathbb{H}_n$ are precisely the convex subspaces of F that correspond to the partitions of X in $n-3$ subsets of size 2 and two subsets of size 4. The $W(2)$-quads of $\mathbb{H}_n$ are precisely the convex subspaces of F that correspond to the partitions of X in $n-2$ subsets of size 2 and one subset of size 6. The maxes of $\mathbb{H}_n$ are the convex subspaces of $\mathbb{H}_n$ that correspond to the partitions of X in 2 subsets of even size. If $\mathcal{P}$ is a partition of X in a subset of size $2m$ and a subset of size $2n+2-2m$ ($m \in \{1,2,\ldots,n\}$), then the convex subspace corresponding to $\mathcal{P}$ is isomorphic to $\mathbb{H}_{m-1} \times \mathbb{H}_{n-m}$. The big maxes of $\mathbb{H}_n$ are precisely the convex subspaces of $\mathbb{H}_n$ that correspond to the partitions of X in a subset of size 2 and a subset of size $2n$. If M is a big max, then $\widetilde{M} \cong \mathbb{H}_{n-1}$. There exists a bijective correspondence between the big maxes M of $\mathbb{H}_n$ and the subsets Y_M of size 2 of X. This correspondence is as follows: M consists of all partitions $\mathcal{P}$ of X in $n+1$ subsets of size 2 such that $Y_M \in \mathcal{P}$. In the sequel, we will often say that Y_M is *the subset of size two corresponding to M*, or that M is *the big max corresponding to Y_M*.

Let $\mathcal{L}(\mathbb{H}_n)$ be the following point-line geometry:

• The points of $\mathcal{L}(\mathbb{H}_n)$ are the subsets of size 2 of $\{1,2,\ldots,n+1\}$.

• The lines of $\mathcal{L}(\mathbb{H}_n)$ are of two types: (i) lines of the form $\{\{a,b\},\{c,d\}\}$ where a, b, c and d are four distinct elements of $\{1,2,\ldots,n+1\}$; (ii) lines of the form $\{\{a,b\},\{a,c\},\{b,c\}\}$ where a, b and c are three distinct elements of $\{1,2,\ldots,n+1\}$.

• Incidence is containment.

Then every local space of $\mathbb{H}_n$ is isomorphic to $\mathcal{L}(\mathbb{H}_n)$.

Suppose M_1 and M_2 are two distinct big maxes of $\mathbb{H}_n$. Let $\{x_i,y_i\}$, $i \in \{1,2\}$, be the subset of size 2 corresponding to M_i. If $|\{x_1,y_1\} \cap \{x_2,y_2\}| = 1$, say $x_1 = x_2$ and $y_1 \neq y_2$, then M_1 and M_2 are disjoint and the subset of size 2 of X corresponding to the big max $\mathcal{R}_{M_1}(M_2)$ is equal to $\{y_1, y_2\}$. If $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$, then $M_1 \cap M_2 \neq \emptyset$.

Every permutation of X determines in a natural way a permutation of the point set of $\mathbb{H}_n = \mathbb{H}_n(X)$ defining an automorphism of $\mathbb{H}_n$. Every automorphism of $\mathbb{H}_n$ is obtained in this way. The correspondence between the permutations of X and the automorphisms of $\mathbb{H}_n$ is bijective for $n \geq 2$. So, the automorphism group of $\mathbb{H}_n$, $n \geq 2$, is isomorphic to the symmetric group S_{2n+2}.

Let $n \geq 3$, let V be a $2n$-dimensional vector space over $\mathbb{F}_4$ and let $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ be an ordered basis of V. Put $\mathbb{G}_n = \mathbb{G}_n(V, B)$.

Suppose P_1 and P_2 are two points of $\mathbb{G}_n$, i.e. two $(n-1)$-dimensional subspaces of $\mathrm{PG}(V)$ that are generated by n points of weight 2 (w.r.t. B) whose supports (w.r.t. B) are mutually disjoint. Then the distance between P_1 and P_2 in the near $2n$-gon $\mathbb{G}_n$ is equal to $n - 1 - \dim(P_1 \cap P_2)$.

The lines of $\mathbb{G}_n(V, B)$ are of two types, the special lines and the ordinary lines. The *special lines* of $\mathbb{G}_n(V, B)$ are in bijective correspondence with the $(n-2)$-dimensional subspaces of $\mathrm{PG}(V)$ which are generated by $(n-1)$ points of weight 2 (whose supports are mutually disjoint): if α is such an $(n-2)$-dimensional subspace, then the set of all $(n-1)$-dimensional subspaces of $\mathrm{PG}(V)$ through α which are generated by n points of weight 2 is a special line. The *ordinary lines* of $\mathbb{G}_n(V, B)$ are in bijective correspondence with the $(n-2)$-dimensional subspaces of $\mathrm{PG}(V)$ which are generated by one point of weight 4 and $n-2$ points of weight 2 such that the $n-1$ associated supports are mutually disjoint: if α is such an $(n-2)$-dimensional subspace, then the set of all $(n-1)$-dimensional subspaces of $\mathrm{PG}(V)$ through α which are generated by n points of weight 2 is an ordinary line.

If σ is a permutation of $\{1, 2, \ldots, 2n\}$, if ψ is an automorphism of $\mathbb{F}_4$ and if $\lambda_i \in \mathbb{F}_4^*$ for every $i \in \{1, 2, \ldots, 2n\}$, then the unique semi-linear map of V with associated field automorphism ψ that maps $\bar{e}_i$ to $\lambda_i \cdot \bar{e}_{\sigma(i)}$ for every $i \in \{1, 2, \ldots, 2n\}$ induces an automorphism of $\mathbb{G}_n = \mathbb{G}_n(V, B)$. Conversely, every automorphism of $\mathbb{G}_n$, $n \geq 3$, is obtained in this way. (This latter statement would not be true if n were equal to 2.)

Every quad of $\mathbb{G}_n$ is either a grid-quad, a $W(2)$-quad or a $Q^-(5, 2)$-quad. Every special line of $\mathbb{G}_n$ is contained in exactly $n - 1$ $Q^-(5, 2)$-quads, 0 $W(2)$-quads and $3\frac{(n-1)(n-2)}{2}$ grid-quads. Every ordinary line of $\mathbb{G}_n$ is contained in a unique $Q^-(5, 2)$-quad, $3(n-2)$ $W(2)$-quads and $\frac{(n-2)(3n-7)}{2}$ grid-quads. There exists a bijective correspondence between the convex subspaces F of diameter $k \in \{0, 1, \ldots, n\}$ of $\mathbb{G}_n$ and the $(n-1-k)$-dimensional subspaces α_F of $\mathrm{PG}(V)$ which are generated by $n - k$ points of even weight (w.r.t. B) whose supports (w.r.t. B) are mutually disjoint. This correspondence is as follows: F consists of all points of $\mathbb{G}_n$ which regarded as subspaces of $\mathrm{PG}(V)$ contain α_F. If M is a max of $\mathbb{G}_n$, then α_M consists of a unique point x_M of even weight. If the weight of x_M is equal to $2m$, then $\widetilde{M} \cong \mathbb{H}_{m-1} \times \mathbb{G}_{n-m}$. In particular, if x_M has weight $2n$, then $\widetilde{M} \cong \mathbb{H}_{n-1}$. The big maxes of $\mathbb{G}_n$ are precisely the convex subspaces F for which α_F consists of a unique point of weight 2. If M is a big max, then $\widetilde{M} \cong \mathbb{G}_{m-1}$. If M is a big max and Q is a $Q^-(5, 2)$-quad meeting M, then either $Q \subseteq M$ or $Q \cap M$ is a special line.

Now, suppose that M_1 and M_2 are two distinct big maxes of $\mathbb{G}_n$ and let x_i, $i \in \{1,2\}$, denote the unique point of weight 2 contained in α_{M_i}. We can distinguish the following three cases.

(i) $x_1 = \langle \bar{e}_{i_1} + \alpha \cdot \bar{e}_{i_2} \rangle$ and $x_2 = \langle \bar{e}_{i_1} + \beta \cdot \bar{e}_{i_2} \rangle$ for some $i_1, i_2 \in \{1,2,\ldots,2n\}$ with $i_1 \neq i_2$ and some $\alpha, \beta \in \mathbb{F}_4^*$ with $\alpha \neq \beta$. Then M_1 and M_2 are disjoint and every line meeting M_1 and M_2 is a special line. Moreover, if γ is the unique element in $\mathbb{F}_4 \setminus \{0, \alpha, \beta\}$ and $M_3 = \mathcal{R}_{M_1}(M_2)$, then $x_{M_3} = \langle \bar{e}_{i_1} + \gamma \cdot \bar{e}_{i_2} \rangle$.

(ii) $x_1 = \langle \bar{e}_{i_1} + \alpha \cdot \bar{e}_{i_2} \rangle$ and $x_2 = \langle \bar{e}_{i_2} + \beta \cdot \bar{e}_{i_3} \rangle$ for some mutually distinct $i_1, i_2, i_3 \in \{1,2,\ldots,2n\}$ and some $\alpha, \beta \in \mathbb{F}_4^*$. Then M_1 and M_2 are disjoint and every line meeting M_1 and M_2 is an ordinary line. Moreover, if $M_3 = \mathcal{R}_{M_1}(M_2)$, then $x_{M_3} = \langle \bar{e}_{i_1} + \alpha\beta \cdot \bar{e}_{i_3} \rangle$.

(iii) $x_1 = \langle \bar{e}_{i_1} + \alpha \cdot \bar{e}_{i_2} \rangle$ and $x_2 = \langle \bar{e}_{i_3} + \beta \cdot \bar{e}_{i_4} \rangle$ for some mutually distinct $i_1, i_2, i_3, i_4 \in \{1,2,\ldots,2n\}$ and some $\alpha, \beta \in \mathbb{F}_4^*$. Then $M_1 \cap M_2 \neq \emptyset$.

Let $\mathcal{L}(\mathbb{G}_n)$ be the following point-line geometry:

- the points of $\mathcal{L}(\mathbb{G}_n)$ are the points of $PG(n-1,4)$ whose weight is either 1 or 2 with respect to a given reference system of $PG(n-1,4)$;
- the lines of $\mathcal{L}(\mathbb{G}_n)$ are the lines of $PG(n-1,4)$ which contain at least two points of $\mathcal{L}(\mathbb{G}_n)$;
- incidence is derived from $PG(n-1,4)$.

Then every local space of $\mathbb{G}_n$ is isomorphic to $\mathcal{L}(\mathbb{G}_n)$.

3 A full isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$

Let $n \geq 2$ and let V be a $2n$-dimensional vector space over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ with ordered basis $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$. Put $X = \{1, 2, \ldots, 2n+2\}$. Define $\mathbb{H}_n := \mathbb{H}_n(X)$ and $\mathbb{G}_n := \mathbb{G}_n(V, B)$. We now define a map e^* from the point set of $\mathbb{H}_n$ to the point set of $\mathbb{G}_n$. We consider the following two possibilities for a point p of $\mathbb{H}_n$:

(1) p is of the form $\{\{a_1,b_1\},\{a_2,b_2\},\ldots,\{a_n,b_n\},\{2n+1,2n+2\}\}$. Then we define $e^*(p) := \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \bar{e}_{a_2} + \bar{e}_{b_2}, \ldots, \bar{e}_{a_n} + \bar{e}_{b_n} \rangle$.

(2) p is of the form $\{\{a_1,b_1\},\{a_2,b_2\},\ldots,\{a_{n-1},b_{n-1}\},\{a_n,2n+1\},\{b_n,2n+2\}\}$. Then we define $e^*(p) := \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \bar{e}_{a_2} + \bar{e}_{b_2}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle$.

It is easily seen that e^* is injective.

Proposition 3.1 *The map e^* defines a full isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$.*

Proof. (a) We first prove that e^* is full. Let $\{p_1, p_2, p_3\}$ be an arbitrary line of $\mathbb{H}_n$. Without loss of generality, we may suppose that one of the following five cases occurs (in each of these cases, we have $\{a_1, b_1, \ldots, a_n, b_n\} = \{1, 2, \ldots, 2n\}$):

<u>Case I:</u>

$$\begin{aligned} p_1 &= \{\{a_1,b_1\},\ldots,\{a_{n-1},b_{n-1}\},\{a_n,b_n\},\{2n+1,2n+2\}\}, \\ p_2 &= \{\{a_1,b_1\},\ldots,\{a_{n-1},b_{n-1}\},\{a_n,2n+1\},\{b_n,2n+2\}\}, \\ p_3 &= \{\{a_1,b_1\},\ldots,\{a_{n-1},b_{n-1}\},\{a_n,2n+2\},\{b_n,2n+1\}\}. \end{aligned}$$

Then the points

$$\begin{aligned} e^*(p_1) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \bar{e}_{b_n} \rangle, \\ e^*(p_2) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle, \\ e^*(p_3) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{b_n} + \omega \cdot \bar{e}_{a_n} \rangle \end{aligned}$$

of $\mathbb{G}_n$ are incident with the special line of $\mathbb{G}_n$ corresponding to the subspace $\langle \bar{e}_{a_1} + \bar{e}_{b_1}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}} \rangle$.

<u>Case II:</u>

$$\begin{aligned} p_1 &= \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \ldots, \{a_n, b_n\}, \{2n+1, 2n+2\}\}, \\ p_2 &= \{\{a_1, a_2\}, \{b_1, b_2\}, \{a_3, b_3\}, \ldots, \{a_n, b_n\}, \{2n+1, 2n+2\}\}, \\ p_3 &= \{\{a_1, b_2\}, \{a_2, b_1\}, \{a_3, b_3\}, \ldots, \{a_n, b_n\}, \{2n+1, 2n+2\}\}. \end{aligned}$$

Then the points

$$\begin{aligned} e^*(p_1) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \bar{e}_{a_2} + \bar{e}_{b_2}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_n} + \bar{e}_{b_n} \rangle, \\ e^*(p_2) &= \langle \bar{e}_{a_1} + \bar{e}_{a_2}, \bar{e}_{b_1} + \bar{e}_{b_2}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_n} + \bar{e}_{b_n} \rangle, \\ e^*(p_3) &= \langle \bar{e}_{a_1} + \bar{e}_{b_2}, \bar{e}_{a_2} + \bar{e}_{b_1}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_n} + \bar{e}_{b_n} \rangle \end{aligned}$$

of $\mathbb{G}_n$ are incident with the ordinary line of $\mathbb{G}_n$ corresponding to the subspace $\langle \bar{e}_{a_1} + \bar{e}_{b_1} + \bar{e}_{a_2} + \bar{e}_{b_2}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_n} + \bar{e}_{b_n} \rangle$.

<u>Case III:</u>

$$\begin{aligned} p_1 &= \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \ldots, \{a_{n-1}, b_{n-1}\}, \{a_n, 2n+1\}, \{b_n, 2n+2\}\}, \\ p_2 &= \{\{a_1, a_2\}, \{b_1, b_2\}, \{a_3, b_3\}, \ldots, \{a_{n-1}, b_{n-1}\}, \{a_n, 2n+1\}, \{b_n, 2n+2\}\}, \\ p_3 &= \{\{a_1, b_2\}, \{a_2, b_1\}, \{a_3, b_3\}, \ldots, \{a_{n-1}, b_{n-1}\}, \{a_n, 2n+1\}, \{b_n, 2n+2\}\}. \end{aligned}$$

Then the points

$$\begin{aligned} e^*(p_1) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \bar{e}_{a_2} + \bar{e}_{b_2}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle, \\ e^*(p_2) &= \langle \bar{e}_{a_1} + \bar{e}_{a_2}, \bar{e}_{b_1} + \bar{e}_{b_2}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle, \\ e^*(p_3) &= \langle \bar{e}_{a_1} + \bar{e}_{b_2}, \bar{e}_{a_2} + \bar{e}_{b_1}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle \end{aligned}$$

of $\mathbb{G}_n$ are incident with the ordinary line of $\mathbb{G}_n$ corresponding to the subspace $\langle \bar{e}_{a_1} + \bar{e}_{a_2} + \bar{e}_{b_1} + \bar{e}_{b_2}, \bar{e}_{a_3} + \bar{e}_{b_3}, \ldots, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle$.

<u>Case IV:</u>

$$\begin{aligned} p_1 &= \{\{a_1, b_1\}, \ldots, \{a_{n-2}, b_{n-2}\}, \{a_{n-1}, b_{n-1}\}, \{a_n, 2n+1\}, \{b_n, 2n+2\}\}, \\ p_2 &= \{\{a_1, b_1\}, \ldots, \{a_{n-2}, b_{n-2}\}, \{a_{n-1}, a_n\}, \{b_{n-1}, 2n+1\}, \{b_n, 2n+2\}\}, \\ p_3 &= \{\{a_1, b_1\}, \ldots, \{a_{n-2}, b_{n-2}\}, \{a_{n-1}, 2n+1\}, \{b_{n-1}, a_n\}, \{b_n, 2n+2\}\}. \end{aligned}$$

Then the points

$$\begin{aligned} e^*(p_1) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle, \\ e^*(p_2) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{a_{n-1}} + \bar{e}_{a_n}, \bar{e}_{b_{n-1}} + \omega \cdot \bar{e}_{b_n} \rangle, \\ e^*(p_3) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{b_{n-1}} + \bar{e}_{a_n}, \bar{e}_{a_{n-1}} + \omega \cdot \bar{e}_{b_n} \rangle \end{aligned}$$

of $\mathbb{G}_n$ are incident with the ordinary line of $\mathbb{G}_n$ corresponding to the subspace $\langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}} + \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle$.

<u>Case V:</u>

$$\begin{aligned} p_1 &= \{\{a_1, b_1\}, \dots, \{a_{n-2}, b_{n-2}\}, \{a_{n-1}, b_{n-1}\}, \{a_n, 2n+1\}, \{b_n, 2n+2\}\}, \\ p_2 &= \{\{a_1, b_1\}, \dots, \{a_{n-2}, b_{n-2}\}, \{a_{n-1}, b_n\}, \{a_n, 2n+1\}, \{b_{n-1}, 2n+2\}\}, \\ p_3 &= \{\{a_1, b_1\}, \dots, \{a_{n-2}, b_{n-2}\}, \{a_{n-1}, 2n+2\}, \{a_n, 2n+1\}, \{b_{n-1}, b_n\}\}. \end{aligned}$$

Then the points

$$\begin{aligned} e^*(p_1) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_n} \rangle, \\ e^*(p_2) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{a_{n-1}} + \bar{e}_{b_n}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{b_{n-1}} \rangle, \\ e^*(p_3) &= \langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{b_{n-1}} + \bar{e}_{b_n}, \bar{e}_{a_n} + \omega \cdot \bar{e}_{a_{n-1}} \rangle \end{aligned}$$

of $\mathbb{G}_n$ are incident with the ordinary line of $\mathbb{G}_n$ corresponding to the subspace $\langle \bar{e}_{a_1} + \bar{e}_{b_1}, \dots, \bar{e}_{a_{n-2}} + \bar{e}_{b_{n-2}}, \bar{e}_{a_{n-1}} + \bar{e}_{b_{n-1}} + \bar{e}_{b_n} + \omega^{-1} \cdot \bar{e}_{a_n} \rangle$.

(b) We now prove that e^* is isometric. By Proposition 2.6, it suffices to prove that there exist opposite points p_1 and p_2 in $\mathbb{H}_n$ such that $e^*(p_1)$ and $e^*(p_2)$ are opposite points in $\mathbb{G}_n$. Take

$$\begin{aligned} p_1 &= \{\{1, 2\}, \dots, \{2n-1, 2n\}, \{2n+1, 2n+2\}\}, \\ p_2 &= \{\{2, 3\}, \dots, \{2n, 2n+1\}, \{2n+2, 1\}\}. \end{aligned}$$

Then p_1 and p_2 are opposite points of $\mathbb{H}_n$ and

$$\begin{aligned} e^*(p_1) &= \langle \bar{e}_1 + \bar{e}_2, \dots, \bar{e}_{2n-1} + \bar{e}_{2n} \rangle, \\ e^*(p_2) &= \langle \bar{e}_2 + \bar{e}_3, \dots, \bar{e}_{2n} + \omega \cdot \bar{e}_1 \rangle. \end{aligned}$$

Since $(\bar{e}_2 + \bar{e}_3) + \cdots + (\bar{e}_{2n} + \omega \cdot \bar{e}_1) - (\bar{e}_1 + \bar{e}_2) - \cdots - (\bar{e}_{2n-1} + \bar{e}_{2n}) = (\omega - 1)\bar{e}_1$, we have $\bar{e}_1 \in \langle \bar{e}_1 + \bar{e}_2, \dots, \bar{e}_{2n-1} + \bar{e}_{2n}, \bar{e}_2 + \bar{e}_3, \dots, \bar{e}_{2n} + \omega \cdot \bar{e}_1 \rangle$. If $i \in \{2, 3, \dots, 2n\}$, then $\bar{e}_i = \bar{e}_1 + (\bar{e}_1 + \bar{e}_2) + \cdots + (\bar{e}_{i-1} + \bar{e}_i) \in \langle \bar{e}_1 + \bar{e}_2, \dots, \bar{e}_{2n-1} + \bar{e}_{2n}, \bar{e}_2 + \bar{e}_3, \dots, \bar{e}_{2n} + \omega \cdot \bar{e}_1 \rangle$. Hence, $\langle \bar{e}_1 + \bar{e}_2, \dots, \bar{e}_{2n-1} + \bar{e}_{2n}, \bar{e}_2 + \bar{e}_3, \dots, \bar{e}_{2n} + \omega \cdot \bar{e}_1 \rangle = \langle \bar{e}_1, \bar{e}_2, \dots, \bar{e}_{2n-1}, \bar{e}_{2n} \rangle$ and $e^*(p_1)$ and $e^*(p_2)$ are opposite points of $\mathbb{G}_n$. ■

Lemma 3.2 *Let $i, j \in \{1, 2, \dots, 2n\}$ with $i \neq j$. Let M be a big max of $\mathbb{H}_n$ and let $\overline{M}$ be the max of $\mathbb{G}_n$ corresponding to M in the sense of Proposition 2.7.*

(1) *If M is the big max of $\mathbb{H}_n$ corresponding to $\{i,j\}$, then $\overline{M}$ is the max of $\mathbb{G}_n$ corresponding to $\langle \bar{e}_i + \bar{e}_j \rangle$.*

(2) *If M is the big max of $\mathbb{H}_n$ corresponding to $\{i, 2n+1\}$, then $\overline{M}$ is the max of $\mathbb{G}_n$ corresponding to $\langle \omega \cdot \bar{e}_i + (\bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n}) \rangle$.*

(3) *If M is the big max of $\mathbb{H}_n$ corresponding to $\{i, 2n+2\}$, then $\overline{M}$ is the max of $\mathbb{G}_n$ corresponding to $\langle \omega^2 \cdot \bar{e}_i + (\bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n}) \rangle$.*

(4) *If M is the big max of $\mathbb{H}_n$ corresponding to $\{2n+1, 2n+2\}$, then $\overline{M}$ is the max of $\mathbb{G}_n$ corresponding to $\langle \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$.*

Proof. In order to show that $\langle \bar{x} \rangle$ is the point of $PG(V)$ corresponding to $\overline{M}$, it suffices to show that $\langle \bar{x} \rangle$ is contained in all subspaces $e^*(p)$, $p \in M$. The verification of this claim is rather straightforward in each of the four cases. ∎

Lemma 3.2 has the following corollary.

Corollary 3.3 *If $n \geq 3$, then for every big max M of $\mathbb{H}_n$, $\widetilde{\overline{M}}$ is isomorphic to either $\mathbb{H}_{n-1}$ or $\mathbb{G}_{n-1}$. Moreover, $\widetilde{\overline{M}}$ is isomorphic to $\mathbb{G}_{n-1}$ if and only if the pair $\{i,j\}$ corresponding to M is contained in $\{1, 2, \ldots, 2n\}$.*

Lemma 3.4 *Let $n \geq 3$. Let θ be an automorphism of $\mathbb{H}_n$ and let σ be the permutation of $\{1, 2, \ldots, 2n+2\}$ corresponding to θ. Then θ lifts through e^* to an automorphism of $\mathbb{G}_n$ if and only if σ leaves the partition $\{\{1, 2, \ldots, 2n\}, \{2n+1, 2n+2\}\}$ invariant.*

Proof. (1) Suppose θ lifts through e^* to an automorphism of $\mathbb{G}_n$. Let i, j be two arbitrary distinct elements of $\{1, 2, \ldots, 2n\}$, let M_1 denote the big max of $\mathbb{H}_n$ corresponding to $\{i,j\}$ and let M_2 denote the big max of $\mathbb{H}_n$ corresponding to $\{\sigma(i), \sigma(j)\}$. We will now apply Corollary 3.3. Since $\widetilde{\overline{M_1}}$ is isomorphic to $\mathbb{G}_{n-1}$ also $\widetilde{\overline{M_2}}$ must be isomorphic to $\mathbb{G}_{n-1}$ and hence $\{\sigma(i), \sigma(j)\} \subseteq \{1, 2, \ldots, 2n\}$. Since i and j were arbitrary distinct elements of $\{1, 2, \ldots, 2n\}$, σ leaves the partition $\{\{1, 2, \ldots, 2n\}, \{2n+1, 2n+2\}\}$ invariant.

(2) Suppose σ leaves the partition $\{\{1, 2, \ldots, 2n\}, \{2n+1, 2n+2\}\}$ invariant. Then $\sigma = \sigma_1 \circ \sigma_2$, where σ_1 is a permutation of $\{1, 2, \ldots, 2n+2\}$ fixing $2n+1$ and $2n+2$ and σ_2 is either the identity or the transposition $(2n+1\ 2n+2)$. Let θ_i, $i \in \{1, 2\}$, denote the automorphism of $\mathbb{H}_n$ corresponding to σ_i. It suffices to show that θ_1 and θ_2 lift to automorphisms of $\mathbb{G}_n$.

(a) Consider the automorphism θ_1 of $\mathbb{H}_n$. Let $\widetilde{\theta_1}$ be the automorphism of $\mathbb{G}_n$ induced by the following linear map of V: $\bar{e}_j \mapsto \bar{e}_{\sigma_1(j)}$, $j \in \{1, 2, \ldots, 2n\}$. Then $\widetilde{\theta_1} \circ e^* = e^* \circ \theta_1$.

(b) Consider the automorphism θ_2 of $\mathbb{H}_n$. If σ_2 is the identity, let $\widetilde{\theta_2}$ be the trivial automorphism of $\mathbb{G}_n$. If $\sigma_2 = (2n+1\ 2n+2)$, let $\widetilde{\theta_2}$ denote the automorphism of $\mathbb{G}_n$ corresponding to the following semi-linear map of V: $\sum_{i=1}^{2n} X_i \bar{e}_i \mapsto \sum_{i=1}^{2n} X_i^2 \bar{e}_i$. In either case, we have $\widetilde{\theta_2} \circ e^* = e^* \circ \theta_2$. ∎

Corollary 3.5 *Suppose $n \geq 3$. Then:*

(1) *The embedding e^* is not homogeneous.*

(2) *If* G *is the full automorphism group of* $\mathbb{H}_n$ *and if* H *is the group of all automorphisms of* $\mathbb{H}_n$ *which lift through* e^* *to an automorphism of* $\mathbb{G}_n$*, then* $|G : H| = (n+1)(2n+1)$.

4 Proof of Theorem 1.1

Let $n \geq 3$. Let X be a set of size $2n+2$ and let V be a $2n$-dimensional vector space over $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ with basis $B^* = (\bar{e}_1^*, \bar{e}_2^*, \ldots, \bar{e}_{2n-1}^*, \bar{e}_{2n}^*)$. The set of all points $\langle \sum_{i=1}^{2n} X_i \bar{e}_i^* \rangle$ of $\mathrm{PG}(V)$ satisfying $\sum_{i=1}^{2n} X_i^3 = 0$ is a nonsingular Hermitian variety $H(2n-1, 4)$ of $\mathrm{PG}(V)$.

The dual polar space $DH(V, B^*)$ associated with $H(2n-1, 4)$ is a dense near $2n$-gon. If p is a point of $H(2n-1, 4)$, then the set of all maximal singular subspaces of $H(2n-1, 4)$ containing p is a max M_p of $DH(V, B^*)$, and every max of $DH(V, B^*)$ is obtained in this way. We call M_p *the max of* $DH(V, B^*)$ *corresponding to* p and p *the point of* $H(2n-1, 4)$ *corresponding to* M_p. Every max of $DH(V, B^*)$ is big. If L is a line of $\mathrm{PG}(V)$ intersecting $H(2n-1, 4)$ in three points p_1, p_2 and p_3, then $\{p_1, p_2, p_3\}$ is called a *hyperbolic line* of $H(2n-1, 4)$. The set $\{M_{p_1}, M_{p_2}, M_{p_3}\}$ of maxes of $DH(V, B^*)$ corresponding to this hyperbolic line $\{p_1, p_2, p_3\}$ satisfies the following properties.

(1) The maxes M_{p_1}, M_{p_2} and M_{p_3} are mutually disjoint.

(2) Every point of M_{p_1} is contained in a unique line meeting M_{p_2} and this line also meets M_{p_3}.

Now, put $\mathbb{H}_n := \mathbb{H}_n(X)$ and $\mathbb{G}_n := \mathbb{G}_n(V, B^*)$. Before we will study isometric embeddings of $\mathbb{H}_n$ into $\mathbb{G}_n$, we need to give some introductory lemmas (4.1 till 4.5)

Lemma 4.1 *Let* $k \in \{1, 2, \ldots, 2n\}$ *and let* n_i, $i \in \{1, 2, \ldots, k\}$ *be strictly positive integers such that* $n_1 + n_2 + \ldots + n_k = 2n$. *Then* $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} \geq (2n-1)(n-2)$ *if and only if one of the following two conditions hold:*

- *there exists a* $j \in \{1, 2, \ldots, k\}$ *such that* $n_j \in \{2n, 2n-1, 2n-2\}$;
- $n = 3$, $k = 2$ *and* $n_1 = n_2 = 3$.

Proof. If there exists a $j \in \{1, 2, \ldots, k\}$ such that $n_j \in \{2n, 2n-1, 2n-2\}$, then $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} \geq \frac{n_j(n_j-1)}{2} \geq \frac{(2n-2)(2n-3)}{2} > \frac{(2n-1)(2n-4)}{2} = (2n-1)(n-2)$. If $n = 3$, $k = 2$ and $n_1 = n_2 = 3$, then $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} = 6 > 5 = (2n-1)(n-2)$.

Conversely, suppose that $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} \geq (2n-1)(n-2)$. Together with $\sum_{i=1}^k n_i = 2n$, this implies that $\sum_{i=1}^k n_i^2 \geq (2n-1)(2n-4) + 2n = 4n^2 - 8n + 4$. On the other hand, we also know that $\sum_{i=1}^k n_i^2 + 2 \cdot \sum_{i<j} n_i n_j = \left(\sum_{i=1}^k n_i \right)^2 = 4n^2$. Hence, $\sum_{i<j} n_i n_j \leq 2(2n-1)$. Let $j^* \in \{1, 2, \ldots, k\}$ such that n_{j^*} is as big as possible. If $n \geq 4$ and $n_{j^*} \in \{3, 4, \ldots, 2n-3\}$, then $\sum_{i<j} n_i n_j \geq \sum_{i \neq j^*} n_i n_{j^*} = (2n - n_{j^*}) n_{j^*} \geq (2n-3) \cdot 3 > 2(2n-1)$, a contradiction. If $n \geq 4$ and $n_{j^*} \in \{1, 2\}$, then $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} \leq k \leq 2n < (2n-1)(n-2)$, again a contradiction. Hence, $n_{j^*} \in \{2n, 2n-1, 2n-2\}$ if $n \geq 4$. If $n = 3$, $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} \geq$

$(2n-1)(n-2) = 5$ and $n_{j^*} \not\in \{2n, 2n-1, 2n-2\} = \{6, 5, 4\}$, then one readily verifies that $k = 2$ and $n_1 = n_2 = 3$. ■

Lemma 4.2 *Every modified local space of $\mathbb{H}_m$, $m \geq 2$, is connected.*

Proof. Every modified local space of $\mathbb{H}_m$ is isomorphic to the subgeometry of $\mathcal{L}(\mathbb{H}_m)$ obtained from $\mathcal{L}(\mathbb{H}_m)$ by removing all lines of size 2. So, every modified local space is isomorphic to the point-line geometry $\mathcal{A}$ whose points are the subsets of size 2 of $\{1, 2, \ldots, m+1\}$ and whose lines are all the sets of the form $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ where a, b and c are three distinct elements of $\{1, 2, \ldots, m+1\}$ (natural incidence). The point-line geometry $\mathcal{A}$ is easily seen to be connected of diameter 2. Indeed, if a, b, c and d are four distinct elements of $\{1, 2, \ldots, m+1\}$, then $\{a, c\}$ is a common neighbor of $\{a, b\}$ and $\{c, d\}$. ■

Lemma 4.3 *Let $\mathbb{G}'_n$ be the subgeometry of $\mathbb{G}_n$ having the same point set but obtained from $\mathbb{G}_n$ by considering only those lines of $\mathbb{G}_n$ that are special. Then $\mathbb{G}'_n$ has precisely $\frac{(2n)!}{n!\cdot 2^n}$ connected components, each of size 3^n.*

Proof. Let $\mathcal{P}'$ denote the point set of $\mathbb{G}'_n$ and let $\mathcal{A}$ be the set of all partitions of $\{1, 2, \ldots, 2n\}$ in n subsets of size 2. Then $|\mathcal{A}| = \frac{(2n)!}{n!\cdot 2^n}$. Let κ be the map from $\mathcal{P}'$ to $\mathcal{A}$ mapping the point $p = \langle \bar{e}^*_{i_1} + a_1 \bar{e}^*_{j_1}, \bar{e}^*_{i_2} + a_2 \bar{e}^*_{j_2}, \ldots, \bar{e}^*_{i_n} + a_n \bar{e}^*_{j_n} \rangle$ of $\mathbb{G}'_n$ to the element $\kappa(p) := \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\}$ of $\mathcal{A}$. Here, $a_1, a_2, \ldots, a_n \in \mathbb{F}_4^*$ and $\{i_1, j_1, i_2, j_2, \ldots, i_n, j_n\} = \{1, 2, \ldots, 2n\}$. Without loss of generality, we may suppose that $i_1 < i_2 < \cdots < i_n$ and $i_l < j_l$ for every $l \in \{1, 2, \ldots, n\}$. Let $p' = \langle \bar{e}^*_{i'_1} + a'_1 \bar{e}^*_{j'_1}, \bar{e}^*_{i'_2} + a'_2 \bar{e}^*_{j'_2}, \ldots, \bar{e}^*_{i'_n} + a'_n \bar{e}^*_{j'_n} \rangle$ be another point of $\mathbb{G}'_n$, where we again suppose that $a'_1, a'_2, \ldots, a'_n \in \mathbb{F}_4^*$ and $\{i'_1, j'_1, i'_2, j'_2, \ldots, i'_n, j'_n\} = \{1, 2, \ldots, n\}$ such that $i'_1 < i'_2 < \cdots < i'_n$ and $i'_l < j'_l$ for every $l \in \{1, 2, \ldots, n\}$. Now, the points p and p' are contained in a special line of $\mathbb{G}_n$ if and only if $(i_1, j_1, i_2, j_2, \ldots, i_n, j_n) = (i'_1, j'_1, i'_2, j'_2, \ldots, i'_n, j'_n)$ and there exists an $l \in \{1, 2, \ldots, n\}$ such that $a_k = a'_k$ for every $k \in \{1, 2, \ldots, n\} \setminus \{l\}$. It now easily follows that the connected components of $\mathbb{G}'_n$ are the sets $\kappa^{-1}(A)$ where $A \in \mathcal{A}$. So, there are $|\mathcal{A}| = \frac{(2n)!}{n!\cdot 2^n}$ connected components, each of size 3^n. ■

Lemma 4.4 *Let Y be a set of size $2m+2$, $m \geq 1$, and let s be an arbitrary element of Y. Let $\mathbb{H}_m(Y, s)$ denote the subgeometry of $\mathbb{H}_m(Y)$ defined on the point set of $\mathbb{H}_m(Y)$ by those lines of $\mathbb{H}_m(Y)$ whose corresponding partition of Y has a 4-subset containing the element s. Then $\mathbb{H}_m(Y, s)$ is connected.*

Proof. We will prove this by induction on m. Clearly, the lemma holds if $m = 1$, since $\mathbb{H}_m(Y, s) = \mathbb{H}_m(Y)$ is this case. So, suppose $m \geq 2$. Let $P_1 = \{\{s, x_2\}, \{x_3, x_4\}, \ldots, \{x_{2m+1}, x_{2m+2}\}\}$ and $P_2 = \{\{s, y_2\}, \{y_3, y_4\}, \ldots, \{y_{2m+1}, y_{2m+2}\}\}$ be two arbitrary points of $\mathbb{H}_m(Y, s)$. Then there exists a point P_3 in $\mathbb{H}_m(Y, s)$ which is collinear with or equal to P_1 and which has the form $\{\{s, y_3\}, \{y_4, *\}, \ldots\}$. There exists a point P_4 in $\mathbb{H}_m(Y, s)$ which is collinear with P_3 and which has the form $\{\{s, *\}, \{y_3, y_4\}, \ldots\}$. Now, the map $P \mapsto P \cup \{\{y_3, y_4\}\}$ between the point set of $\mathbb{H}_{m-1}(Y \setminus \{y_3, y_4\}, s)$ and the point set of $\mathbb{H}_m(Y, s)$ defines a full embedding of the geometry $\mathbb{H}_{m-1}(Y \setminus \{y_3, y_4\}, s)$ into the geometry

$\mathbb{H}_m(Y,s)$. By the induction hypothesis, the points $P_4 \setminus \{\{y_3,y_4\}\}$ and $P_2 \setminus \{\{y_3,y_4\}\}$ of $\mathbb{H}_{m-1}(Y \setminus \{y_3,y_4\},s)$ are connected by a path of $\mathbb{H}_{m-1}(Y \setminus \{y_3,y_4\},s)$. Hence, the points P_4 and P_2 of $\mathbb{H}_m(Y,s)$ are connected by a path of $\mathbb{H}_m(Y,s)$. This implies that also the points P_1 and P_2 of $\mathbb{H}_m(Y,s)$ are connected by a path of $\mathbb{H}_m(Y,s)$. ■

Definition. We will use the notation $\mathbb{H}'_m$, $m \geq 1$, to denote a suitable geometry belonging to the isomorphism class of $\mathbb{H}_m(Y,s)$, where Y is some arbitrary set of size $2m+2$ and $s \in Y$.

Lemma 4.5 *Let M_1 and M_2 be two disjoint big maxes of $\mathbb{H}_n$. Let $\mathcal{L}$ denote the set of lines L contained in M_1 such that $\langle L, \pi_{M_2}(L)\rangle$ is a $W(2)$-quad of $\mathbb{H}_n$. Then the subgeometry of $\widetilde{M_1}$ induced on M_1 by the lines of $\mathcal{L}$ is isomorphic to $\mathbb{H}'_{n-1}$.*

Proof. Suppose the big max M_1 of $\mathbb{H}_n$ corresponds to the pair $\{x_1,x_2\} \subseteq X$ and that the big max M_2 of $\mathbb{H}_n$ corresponds to the pair $\{x_1,x_3\} \subseteq X$. The map $P \mapsto P \cup \{\{x_1,x_2\}\}$ defines an isomorphism between $\mathbb{H}_{n-1}(X \setminus \{x_1,x_2\})$ and $\widetilde{M_1}$. We will now prove that this isomorphism defines an isomorphism between $\mathbb{H}_{n-1}(X \setminus \{x_1,x_2\},x_3) \cong \mathbb{H}'_{n-1}$ and the subgeometry of $\widetilde{M_1}$ induced on M_1 by the lines of $\mathcal{L}$. Let $P_1 = \{\{x_1,x_2\},\{x_3,x_4\},\{x_5,x_6\},\dots,\{x_{2n+1},x_{2n+2}\}\}$ be an arbitrary point of M_1. Then $P_2 = \{\{x_1,x_3\},\{x_2,x_4\},\{x_5,x_6\},\dots,\{x_{2n+1},x_{2n+2}\}\}$ is the unique point of M_2 collinear with P_1. The lines L of M_1 through P_1 such that $\langle L,\pi_{M_2}(L)\rangle$ is a $W(2)$-quad of $\mathbb{H}_n$ are precisely the lines $Q \cap M_1$ where Q is a $W(2)$-quad through P_1P_2. Now, the line P_1P_2 corresponds to the partition

$$\{\{x_1,x_2,x_3,x_4\},\{x_5,x_6\},\dots,\{x_{2n+1},x_{2n+2}\}\}$$

of X, and the $W(2)$-quads through the line P_1P_2 correspond to the partitions

$$\{\{x_1,x_2,x_3,x_4,x_{2i+1},x_{2i+2}\}\} \cup \Big(\bigcup_{2\leq j\leq n,\ j\neq i} \{\{x_{2j+1},x_{2j+2}\}\}\Big).$$

These $W(2)$-quads intersect the max M_1 in lines whose corresponding partitions have the form

$$\{\{x_1,x_2\},\{x_3,x_4,x_{2i+1},x_{2i+2}\}\} \cup \Big(\bigcup_{2\leq j\leq n,\ j\neq i} \{\{x_{2j+1},x_{2j+2}\}\}\Big).$$

The lemma now readily follows. ■

In the sequel, let e be an isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$. For every convex subspace F of $\mathbb{H}_n$, let $\overline{F}$ denote the convex subspace of $\mathbb{G}_n$ having the same diameter as F and containing $e(F)$ (see Proposition 2.7).

Lemma 4.6 *For every big max M of $\mathbb{H}_n$, either $\widetilde{\overline{M}} \cong \mathbb{H}_{n-1}$ or $\widetilde{\overline{M}} \cong \mathbb{G}_{n-1}$.*

Proof. Since $\overline{M}$ is a max of $\mathbb{G}_n$, $\widetilde{\overline{M}} \cong \mathbb{H}_{k-1} \times \mathbb{G}_{n-k}$ for some $k \in \{1,2,\dots,n\}$. If $k \in \{1,n\}$, then we are done. So, we may suppose that $k \in \{2,3,\dots,n-1\}$.

Let x be an arbitrary point of M. Since $\widetilde{\overline{M}} \cong \mathbb{H}_{k-1} \times \mathbb{G}_{n-k}$, there exist convex subspaces F_1 and F_2 of $\widetilde{\overline{M}}$ through $e(x)$ such that (i) $\widetilde{F_1} \cong \mathbb{H}_{k-1}$; (ii) $\widetilde{F_2} \cong \mathbb{G}_{n-k}$; (iii) every line through $e(x)$ is contained in either F_1 or F_2; (iv) if L_i, $i \in \{1,2\}$, is a line of F_i through $e(x)$, then $\langle L_1, L_2 \rangle$ is a grid-quad of $\mathbb{G}_n$. For every $i \in \{1,2\}$, $e^{-1}(e(M) \cap F_i)$ is a convex subspace of $\mathbb{H}_n$ through x contained in M having a diameter which is at most the diameter of F_i, i.e. smaller than the diameter $n-1$ of M. Proposition 2.5 then implies that there exists a line in M through x not contained in $e^{-1}(e(M) \cap F_i)$. So, there exist lines $L, L' \subseteq M$ through x such that $e(L) \subseteq F_1$ and $e(L') \subseteq F_2$. By Lemma 4.2, there exists a path $L = L_0, L_1, \ldots, L_k = L'$ in the modified local space of $\widetilde{M} \cong \mathbb{H}_{n-1}$ at the point x. Then there exists an $i \in \{1, 2, \ldots, k\}$ such that $e(L_{i-1}) \subseteq F_1$ and $e(L_i) \subseteq F_2$. But then the $W(2)$-quad $\langle L_{i-1}, L_i \rangle$ of $\widetilde{M}$ is mapped by e into the grid-quad $\langle e(L_{i-1}), e(L_i) \rangle$ of $\widetilde{\overline{M}}$, which is clearly impossible. So, the case $k \in \{2, 3, \ldots, n-1\}$ cannot occur. ■

Lemma 4.7 *Let M_1 and M_2 be two disjoint big maxes of $\mathbb{H}_n$ and put $M_3 := \mathcal{R}_{M_1}(M_2)$. Then:*

(1) *Also $\overline{M_1}$ and $\overline{M_2}$ are two disjoint maxes of $\mathbb{G}_n$.*

(2) *If $\overline{M_1}$ is big, then $\overline{M_3} = \mathcal{R}_{\overline{M_1}}(\overline{M_2})$.*

(3) *The points of $H(2n-1, 4)$ corresponding to $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$ form a hyperbolic line of $H(2n-1, 4)$.*

Proof. (1) This is a special case of Proposition 2.7 of De Bruyn [5].

(2) Suppose $\overline{M_1}$ is a big max of $\mathbb{G}_n$. Since every point of M_3 is contained in a line meeting M_1 and M_2, we have $e(M_3) \subseteq \mathcal{R}_{\overline{M_1}}(\overline{M_2})$. Hence, we have $\overline{M_3} = \mathcal{R}_{\overline{M_1}}(\overline{M_2})$ since both have the same diameter.

(3) In view of the natural embedding of $\mathbb{G}_n(V, B)$ into $DH(V, B)$, the map e induces an embedding e' of $\mathbb{H}_n$ into $DH(V, B)$. For every $i \in \{1, 2, 3\}$, let $\overline{\overline{M_i}}$ denote the max of $DH(V, B)$ containing $e'(M_i)$. Similarly as in part (1), Proposition 2.7 of [5] implies that $\overline{\overline{M_1}}$ and $\overline{\overline{M_2}}$ are disjoint maxes of $DH(V, B)$. With a completely similar reasoning as in part (2), we know that $\mathcal{R}_{\overline{\overline{M_1}}}(\overline{\overline{M_2}}) = \overline{\overline{M_3}}$. So, the three points of $H(2n-1, 4)$ corresponding to $\overline{\overline{M_1}}$, $\overline{\overline{M_2}}$ and $\overline{\overline{M_3}}$ form a hyperbolic line of $H(2n-1, 4)$. These three points coincide with the three points of $H(2n-1, 4)$ corresponding to $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$. ■

Lemma 4.8 *Let M_1 and M_2 be two disjoint big maxes of $\mathbb{H}_n$ such that $\widetilde{\overline{M_1}} \cong \widetilde{\overline{M_2}} \cong \mathbb{H}_{n-1}$. Let p_i, $i \in \{1, 2\}$, denote the point of $H(2n-1, 4)$ corresponding to $\overline{M_i}$. Then the line p_1p_2 of $\mathrm{PG}(V)$ contains a point of weight 1 or 2 (with respect to B^*).*

Proof. Let m_1, respectively m_2, denote the number of ordinary lines, respectively special lines, meeting $\overline{M_1}$ and $\overline{M_2}$. Then $m_1 + m_2 = |\overline{M_1}| = \frac{(2n)!}{2^n \cdot n!}$. We count the number of big maxes of $\mathbb{G}_n$ meeting $\overline{M_1}$ and $\overline{M_2}$. Each ordinary line (meeting $\overline{M_1}$ and $\overline{M_2}$) is contained in precisely $n-2$ big maxes and each special line (meeting $\overline{M_1}$ and $\overline{M_2}$) is contained in precisely $n-1$ big maxes. If M is a big max meeting $\overline{M_1}$ and $\overline{M_2}$, then $M \cap \overline{M_1}$ is a big max of $\widetilde{\overline{M_1}}$ and hence $\widetilde{M \cap \overline{M_1}} \cong \mathbb{H}_{n-2}$. So, the total number N of big maxes meeting $\overline{M_1}$

and $\overline{M_2}$ is equal to

$$N = \frac{m_1 \cdot (n-2) + m_2 \cdot (n-1)}{\frac{(2n-2)!}{2^{n-1} \cdot (n-1)!}}. \quad (1)$$

Clearly,

$$N \geq \frac{(m_1 + m_2) \cdot (n-2)}{\frac{(2n-2)!}{2^{n-1} \cdot (n-1)!}} = \frac{\frac{(2n)!}{2^n \cdot n!} \cdot (n-2)}{\frac{(2n-2)!}{2^{n-1} \cdot (n-1)!}} = (2n-1)(n-2). \quad (2)$$

Now, let $p_1 = \langle a_1 \bar{e}_1^* + a_2 \bar{e}_2^* + \cdots + a_{2n} \bar{e}_{2n}^* \rangle$ and $p_2 = \langle b_1 \bar{e}_1^* + b_2 \bar{e}_2^* + \cdots + b_{2n} \bar{e}_{2n}^* \rangle$. Define the following equivalence relation R_{p_1,p_2} on the set $\{1, 2, \ldots, 2n\}$. If $i, j \in \{1, 2, \ldots, 2n\}$, then we say that $(i, j) \in R_{p_1,p_2}$ if $\frac{a_i}{a_j} = \frac{b_i}{b_j}$. Let $C_1, C_2, \ldots, C_k$ denote the equivalence classes of R_{p_1,p_2} and put $n_i := |C_i|$, $i \in \{1, 2, \ldots, k\}$. Suppose M is a big max of $\mathbb{G}_n$ and let $\langle c_i \bar{e}_i^* + c_j \bar{e}_j^* \rangle$ denote the point of $H(2n-1, 4)$ corresponding to M. Then M meets $\overline{M_1}$ if and only if $\frac{a_i}{a_j} = \frac{c_i}{c_j}$. Similarly, M meets $\overline{M_2}$ if and only if $\frac{b_i}{b_j} = \frac{c_i}{c_j}$. So, if M meets $\overline{M_1}$ and $\overline{M_2}$, then $(i, j) \in R_{p_1,p_2}$. It follows that the number of big maxes meeting $\overline{M_1}$ and $\overline{M_2}$ is equal to

$$N = \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2}. \quad (3)$$

By equations (2), (3) and Lemma 4.1, one of the following two cases occurs:

(a) there exists a $j^* \in \{1, 2, \ldots, k\}$ such that $n_{j^*} \in \{2n, 2n-1, 2n-2\}$;

(b) $n = 3$, $k = 2$ and $n_1 = n_2 = 3$.

If case (a) occurs, then $n_{j^*} \in \{2n-1, 2n-2\}$ since p_1 and p_2 are distinct. If $n_{j^*} = 2n-1$, then $p_1 p_2$ contains a point of weight 1. If $n_{j^*} = 2n-2$, then $p_1 p_2$ contains a point of weight 2.

Suppose now that case (b) occurs. So, $n = 3$, $k = 2$ and $n_1 = n_2 = 3$. Our intention is to derive a contradiction. Notice that by equations (1), (3) and the fact that $m_1 + m_2 = 15$, we would have that $m_1 = 12$ and $m_2 = 3$, but we will not use these facts. From the fact that $k = 2$ and $n_1 = n_2 = 3$, we may without loss of generality suppose that $p_2 = \langle a_1 \bar{e}_1^* + a_2 \bar{e}_2^* + a_3 \bar{e}_3^* + t \cdot (a_4 \bar{e}_4^* + a_5 \bar{e}_5^* + a_6 \bar{e}_6^*) \rangle$, where $t \notin \{0, 1\}$. Consider the point $x_1 = \langle a_1 \bar{e}_1^* + a_4 \bar{e}_4^*, a_2 \bar{e}_2^* + a_5 \bar{e}_5^*, a_3 \bar{e}_3^* + a_6 \bar{e}_6^* \rangle$ of $\overline{M_1}$. The point x_1 is contained in three big $Q^-(5, 2)$-quads which cover all the lines through x_1. The points of $H(5, 4)$ corresponding to these three big quads are $\langle a_1 \bar{e}_1^* + a_4 \bar{e}_4^* \rangle$, $\langle a_2 \bar{e}_2^* + a_5 \bar{e}_5^* \rangle$ and $\langle a_3 \bar{e}_3^* + a_6 \bar{e}_6^* \rangle$. Since these three points are noncollinear with p_2 on $H(5, 4)$, the three deep quads through x_1 are disjoint from $\overline{M_2}$. Hence, no point collinear with x_1 is contained in $\overline{M_2}$, clearly a contradiction. So, case (b) cannot occur. ■

Lemma 4.9 *Let M_1 and M_2 be two disjoint big maxes of $\mathbb{H}_n$ such that $\widetilde{\overline{M_1}} \cong \widetilde{\overline{M_2}} \cong \mathbb{H}_{n-1}$. Put $M_3 := \mathcal{R}_{M_1}(M_2)$. Let p_i, $i \in \{1, 2\}$, denote the point of $H(2n-1, 4)$ corresponding to $\overline{M_i}$. If the line $p_1 p_2$ contains a point of weight 1, then $\widetilde{\overline{M_3}} \cong \mathbb{H}_{n-1}$. If the line $p_1 p_2$ contains a point of weight 2, then $\widetilde{\overline{M_3}} \cong \mathbb{G}_{n-1}$.*

Proof. Since $\widetilde{\overline{M_1}}$ and $\widetilde{\overline{M_2}}$ are disjoint, $\{p_1, p_2, p_3\} := p_1p_2 \cap H(2n-1, 4)$ is a hyperbolic line of $H(2n-1, 4)$. By Lemma 4.7, p_3 is the point of $H(2n-1, 4)$ corresponding to $\overline{M_3}$. It is also clear that p_3 is the unique point of even weight on the line p_1p_2 that is distinct from p_1 and p_2. If p_1p_2 contains a point of weight 1, then p_3 necessarily has weight $2n$ and hence $\widetilde{\overline{M_3}} \cong \mathbb{H}_{n-1}$. If p_1p_2 contains a point of weight 2, then this point coincides with p_3 and we have $\widetilde{\overline{M_3}} \cong \mathbb{G}_{n-1}$. ∎

Lemma 4.10 *There exists a partition $\mathcal{U}$ of the set X such that the following holds for every big max M of $\mathbb{H}_n$:*

(1) *If the pair $\{i, j\}$ corresponding to M is contained in an element of $\mathcal{U}$, then $\widetilde{\overline{M}}$ is isomorphic to $\mathbb{G}_{n-1}$.*

(2) *If the pair $\{i, j\}$ corresponding to M is not contained in an element of $\mathcal{U}$, then $\widetilde{\overline{M}}$ is isomorphic to $\mathbb{H}_{n-1}$.*

Proof. Consider the following relation R on the set X. If $i, j \in X$, then $(i, j) \in R$ if and only if precisely one of the following two conditions is satisfied: (i) $i = j$; (ii) $i \neq j$ and $\widetilde{\overline{M}} \cong \mathbb{G}_{n-1}$ where M is the big max of $\mathbb{H}_n$ corresponding to $\{i, j\}$. Clearly, R is reflexive and symmetric. In order to prove that R is transitive, we must show that if i, j and k are three mutually distinct elements of X such that $(i, j) \in R$ and $(j, k) \in R$, then also $(i, k) \in R$. Let M_1, M_2 and M_3 denote the big maxes of $\mathbb{H}_n$ corresponding to the respective pairs $\{i, j\}$, $\{j, k\}$ and $\{i, k\}$. Then M_1, M_2 and M_3 are mutually disjoint and $M_3 = \mathcal{R}_{M_1}(M_2)$. By Lemma 4.7, $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$ are mutually disjoint and $\overline{M_3} = \mathcal{R}_{\overline{M_1}}(\overline{M_2})$. Since $\mathcal{R}_{\overline{M_1}}$ is an automorphism of $\mathbb{G}_n$ and $\widetilde{\overline{M_2}} \cong \mathbb{G}_{n-1}$, we must have $\widetilde{\overline{M_3}} \cong \mathbb{G}_{n-1}$. So, $\{i, k\} \in R$.

So, R is an equivalence relation. The partition $\mathcal{U}$ mentioned in the statement of the lemma is just the set of equivalence classes of R. ∎

Let $\mathcal{U}$ be as in Lemma 4.10. Let $\{i, j\}$ be a subset of size 2 of X which is contained in an element of $\mathcal{U}$. So, if M is the big max of $\mathbb{H}_n$ corresponding to $\{i, j\}$, then $\widetilde{\overline{M}} \cong \mathbb{G}_{n-1}$. The point p of $H(2n-1, 4)$ corresponding to $\overline{M}$ has weight 2 with respect to B^*. We define $\Omega(\{i, j\}) := \{k, l\}$, where k and l are the unique (up to transposition) elements of $\{1, 2, \ldots, 2n\}$ such that $p \in \langle \bar{e}_k^*, \bar{e}_l^* \rangle$.

Lemma 4.11 *Let $i_1, i_2, i_3, i_4 \in X$ with $i_1 \neq i_2$ and $i_3 \neq i_4$ such that each of $\{i_1, i_2\}$, $\{i_3, i_4\}$ is contained in some element of $\mathcal{U}$. If $\{i_1, i_2\} \neq \{i_3, i_4\}$, then $\Omega(\{i_1, i_2\}) \neq \Omega(\{i_3, i_4\})$.*

Proof. Let M_1 be the big max of $\mathbb{H}_n$ corresponding to $\{i_1, i_2\}$ and let M_2 be the big max of $\mathbb{H}_n$ corresponding to $\{i_3, i_4\}$. Since $\{i_1, i_2\} \neq \{i_3, i_4\}$, we have $M_1 \neq M_2$ and hence $\overline{M_1} \neq \overline{M_2}$ by Proposition 2.7. Suppose that $\Omega(\{i_1, i_2\}) = \Omega(\{i_3, i_4\})$. Then every line meeting $\overline{M_1}$ and $\overline{M_2}$ is a special line. We will prove that $e(M_1)$ is a connected component of the geometry of type $\mathbb{G}'_{n-1}$ defined on $\overline{M_1}$ by the special lines of $\widetilde{\overline{M_1}}$. Let x be an arbitrary point of M_1. By Lemmas 4.4 and 4.5, it suffices to prove the following:

(∗) e defines a bijection between the set of lines L of M_1 through x such that $\langle L, \pi_{M_2}(L) \rangle$ is a $W(2)$-quad and the set of special lines of $\widetilde{\overline{M_1}}$ through $e(x)$.

Let L^* be the unique line through x meeting M_2. Then $e(L^*)$ is the unique line through $e(x)$ meeting $\overline{M_2}$. Recall that $e(L^*)$ is a special line. So, every quad of $\mathbb{G}_n$ through $e(L^*)$ is either a (3×3)-grid or a $Q^-(5,2)$-quad. Hence, each of the $(n-1)$ $W(2)$-quads of $\mathbb{H}_n$ through L^* is mapped into a $Q^-(5,2)$-quad through $e(L^*)$. Notice that there are also $n-1$ $Q^-(5,2)$-quads through $e(L^*)$ and that these $n-1$ $Q^-(5,2)$-quads intersect $\overline{M_1}$ in the $n-1$ special lines of $\widetilde{\overline{M_1}}$ through $e(x)$. Hence, (∗) holds.

So, $e(M_1)$ is a connected component of the geometry $\mathbb{G}'_{n-1}$. Hence, $|M_1| = 3^n$ by Lemma 4.3. On the other hand, we know that $|M_1| = \frac{(2n)!}{n! \cdot 2^n}$ since this is the total number of points of $\mathbb{H}_{n-1}$. Now, one can readily verify that $\frac{(2n)!}{n! \cdot 2^n} < 3^n$ if $n \leq 3$ and $\frac{(2n)!}{n! \cdot 2^n} > 3^n$ if $n \geq 4$. So, a contradiction has been obtained. Hence, $\Omega(\{i_1, i_2\}) \neq \Omega(\{i_3, i_4\})$. ■

Lemma 4.12 *Let i_1, i_2 and i_3 be three distinct elements of X which are contained in the same element of $\mathcal{U}$. Then $\Omega(\{i_1, i_2\}) \cap \Omega(\{i_1, i_3\})$ is a singleton.*

Proof. Let M_1 denote the big max of $\mathbb{H}_n$ corresponding to $\{i_1, i_2\}$ and let M_2 denote the big max of $\mathbb{H}_n$ corresponding to $\{i_1, i_3\}$. Then $M_1 \cap M_2 = \emptyset$. Hence, also $\overline{M_1}$ and $\overline{M_2}$ are disjoint by Lemma 4.7. This implies that $\Omega(\{i_1, i_2\}) \cap \Omega(\{i_1, i_3\}) \neq \emptyset$. By Lemma 4.11, we must then have that $\Omega(\{i_1, i_2\}) \cap \Omega(\{i_1, i_3\})$ is a singleton. ■

Lemma 4.13 *Let i_1, i_2, i_3 and i_4 be four distinct elements of X such that each of $\{i_1, i_2\}, \{i_3, i_4\}$ is contained in some element of $\mathcal{U}$. Then $\Omega(\{i_1, i_2\})$ and $\Omega(\{i_3, i_4\})$ are disjoint.*

Proof. Let M_1 denote the big max of $\mathbb{H}_n$ corresponding to $\{i_1, i_2\}$ and let M_2 denote the big max of $\mathbb{H}_n$ corresponding to $\{i_3, i_4\}$. Since M_1 and M_2 meet, also $\overline{M_1}$ and $\overline{M_2}$ meet. Hence, $\Omega(\{i_1, i_2\})$ and $\Omega(\{i_3, i_4\})$ must be disjoint. ■

Lemma 4.14 *For every $U \in \mathcal{U}$ with $|U| \geq 2$, there exists a bijection ϕ_U between U and a subset U' of $\{1, 2, \ldots, 2n\}$ such that $\Omega(\{i, j\}) = \{\phi_U(i), \phi_U(y)\}$ for all $i, j \in U$ with $i \neq j$. If $|U| \geq 3$, then ϕ_U is uniquely determined.*

Proof. For all $i, j \in U$ with $i \neq j$, let $M_{i,j}$ denote the big max of $\mathbb{H}_n$ corresponding to $\{i, j\}$.

Suppose first that $|U| = 2$, say $U = \{i, j\}$. If $\langle \bar{e}_{i'} + \alpha \cdot \bar{e}_{j'} \rangle$ is the point of $H(2n-1, 4)$ corresponding to the big max $\overline{M_{i,j}}$ of $\mathbb{G}_n$, then there are two possibilities for ϕ_U. Either $(\phi_U(i) = i'$ and $\phi_U(j) = j')$ or $(\phi_U(i) = j'$ and $\phi_U(j) = i')$.

Suppose next that $|U| \geq 3$. We claim that for every $i \in U$, there exists a unique $\phi_U(i) \in \{1, 2, \ldots, 2n\}$ such that all sets $\Omega(\{i, j\})$, $j \in U \setminus \{i\}$, contain $\phi_U(i)$. Let j_1 and j_2 be two distinct elements of $U \setminus \{i\}$. By Lemma 4.12, $\Omega(\{i, j_1\})$ and $\Omega(\{i, j_2\})$ have a unique element $\phi_U(i)$ in common. If $|U| = 3$, then all sets $\Omega(\{i, j\})$, $j \in U \setminus \{i\}$, contain $\phi_U(i)$. Suppose therefore that $|U| \geq 4$ and let j_3 be an arbitrary element of $U \setminus \{i, j_1, j_2\}$.

Put $\Omega(\{i,j_1\}) = \{\phi_U(i), j'_1\}$ and $\Omega(\{i,j_2\}) = \{\phi_U(i), j'_2\}$. Then $\Omega(\{j_1,j_2\}) = \{j'_1, j'_2\}$ by Lemma 4.7. By Lemma 4.13, $\Omega(\{i,j_3\})$ and $\Omega(\{j_1,j_2\}) = \{j'_1,j'_2\}$ are disjoint, and by Lemma 4.12 $\Omega(\{i,j_3\})$ intersects $\Omega(\{i,j_1\}) = \{\phi_U(i), j'_1\}$ in a singleton. It follows that $\phi_U(i) \in \Omega(\{i,j_3\})$. We can conclude that $\phi_U(i)$ is contained in all sets $\Omega(\{i,j\})$, $j \in U \setminus \{i\}$.

In order to prove that the lemma holds, it still remains to show that ϕ_U is injective. Suppose i_1 and i_2 are two distinct elements of U such that $\phi_U(i_1) = \phi_U(i_2) = i'$. Let j be an element of $U \setminus \{i_1, i_2\}$. Put $\Omega(\{i_1,i_2\}) = \{i', j'_1\}$ and $\Omega(\{i_1,j\}) = \{i', j'_2\}$. Then $j'_1 \neq j'_2$ by Lemma 4.11. By Lemma 4.7, $\Omega(\{i_2,j\}) = \{j'_1,j'_2\}$, but this is impossible since $\phi_U(i_2) = i' \notin \{j'_1,j'_2\}$. So, ϕ_U is indeed injective. ■

For every $U \in \mathcal{U}$ with $|U| \geq 2$, let $\phi_U : U \to U'$ be a bijection satisfying the conditions of Lemma 4.14.

Lemma 4.15 *If $U_1, U_2 \in \mathcal{U}$ with $|U_1|, |U_2| \geq 2$ and $U_1 \cap U_2 = \emptyset$, then also $U'_1 \cap U'_2 = \emptyset$.*

Proof. Suppose $|U'_1 \cap U'_2| \geq 2$. Then there exists a subset $\{i,j\}$ of size 2 of U_1 and a subset $\{k,l\}$ of size 2 of U_2 such that $\Omega(\{i,j\}) = \Omega(\{k,l\})$, in contradiction with Lemma 4.11.

Suppose $|U'_1 \cap U'_2| = 1$. Then there exists a subset $\{i,j\}$ of size 2 of U_1 and a subset $\{k,l\}$ of size 2 of U_2 such that $\Omega(\{i,j\})$ has the form $\{i', j'_1\}$ and $\Omega(\{k,l\})$ has the form $\{i', j'_2\}$ where i' is the unique element in $U'_1 \cap U'_2$, $j'_1 \in U'_1 \setminus U'_2$ and $j'_2 \in U'_2 \setminus U'_1$. This situation is impossible by Lemma 4.13.

Hence, $U'_1 \cap U'_2 = \emptyset$. ■

Lemma 4.16 *We have $|\mathcal{U}| \leq 3$.*

Proof. Suppose to the contrary that U_1, U_2, U_3 and U_4 are four distinct elements of $\mathcal{U}$. Without loss of generality, we may suppose that $X = \{1, 2, \ldots, 2n+2\}$ and that $1 \in U_1$, $2 \in U_2$, $3 \in U_3$ and $4 \in U_4$.

For all $i, j \in \{1,2,3,4\}$ with $i \neq j$, let $p_{i,j}$ denote the point of $H(2n-1,4)$ corresponding to the max $\widetilde{M_{i,j}}$ where $M_{i,j}$ is the big max of $\mathbb{H}_n$ corresponding to the pair $\{i,j\}$. Since $\widetilde{M_{i,j}} \cong \mathbb{H}_{n-1}$, $p_{i,j}$ is a point of weight $2n$.

Let i, j and k be three distinct elements of $\{1,2,3,4\}$. Consider the three big maxes $M_{i,j}$, $M_{i,k}$ and $M_{j,k}$ of $\mathbb{H}_n$. Then $\mathcal{R}_{M_{i,j}}(M_{i,k}) = M_{j,k}$. By Lemma 4.7, the points $p_{i,j}$, $p_{i,k}$ and $p_{j,k}$ form a hyperbolic line of $H(2n-1,4)$. By Lemma 4.9, the unique line of $\mathrm{PG}(V)$ containing $\{p_{i,j}, p_{i,k}, p_{j,k}\}$ has a unique point $p_{i,j,k}$ of weight 1.

Since $\{p_{1,2}, p_{1,3}, p_{2,3}\}$ and $\{p_{1,2}, p_{1,4}, p_{2,4}\}$ are two distinct hyperbolic lines through $p_{1,2}$, we have $p_{1,2,3} \neq p_{1,2,4}$. The points $p_{1,3}$, $p_{1,4}$, $p_{1,2,3}$ and $p_{1,2,4}$ are contained in the plane $\langle p_{1,2}, p_{1,2,3}, p_{1,2,4} \rangle$. Hence, the lines $p_{1,3}p_{1,4}$ and $p_{1,2,3}p_{1,2,4}$ meet in a point distinct from $p_{1,2,3}$ and $p_{1,2,4}$. So, the line $p_{1,3}p_{1,4}$ contains a point of weight 2. But then $\widetilde{M_{3,4}} \cong \mathbb{G}_{n-1}$ by Lemma 4.9 applied to the maxes $M_{1,3}$ and $M_{1,4}$, clearly a contradiction. ■

Lemma 4.17 *$\mathcal{U}$ consists of one subset of size $2n$ and two singletons.*

Proof. Let Y denote the union of all elements $U \in \mathcal{U}$ satisfying $|U| \geq 2$. By Lemma 4.15, $|Y| \leq 2n$. So, there must be at least two singletons in $\mathcal{U}$. Hence by Lemma 4.16, $\mathcal{U}$ consists of one subset of size $2n$, namely Y, and two singletons. ∎

Lemma 4.18 *There exists an ordered basis $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ of V such that the following hold:*

(1) $\{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \ldots, \langle \bar{e}_{2n} \rangle\} = \{\langle \bar{e}_1^* \rangle, \langle \bar{e}_2^* \rangle, \ldots, \langle \bar{e}_{2n}^* \rangle\}$;

(2) *If $\mathcal{P}$ is the point set of $\mathbb{H}_n$, then $e(\mathcal{P})$ consists of all points of $\mathbb{G}_n$ of the form $\langle \bar{e}_{\sigma(1)} + \bar{e}_{\sigma(2)}, \bar{e}_{\sigma(3)} + \bar{e}_{\sigma(4)}, \ldots, \bar{e}_{\sigma(2n-1)} + \bar{e}_{\sigma(2n)} \rangle$ for some permutation σ of $\{1, 2, \ldots, 2n\}$ or of the form $\langle \bar{e}_{\sigma(1)} + \bar{e}_{\sigma(2)}, \bar{e}_{\sigma(3)} + \bar{e}_{\sigma(4)}, \ldots, \bar{e}_{\sigma(2n-3)} + \bar{e}_{\sigma(2n-2)}, \bar{e}_{\sigma(2n-1)} + \omega \cdot \bar{e}_{\sigma(2n)} \rangle$ for some permutation σ of $\{1, 2, \ldots, 2n\}$.*

Proof. By Lemma 4.17, $\mathcal{U}$ consists of a set $X^* = \{x_1^*, x_2^*, \ldots, x_{2n}^*\}$ of size $2n$ and two singletons $\{x_{2n+1}^*\}$ and $\{x_{2n+2}^*\}$. For every $i \in \{1, 2, \ldots, 2n\}$, we define $\phi(i) := \phi_{X^*}(x_i^*)$. Then ϕ is a permutation of the set $\{1, 2, \ldots, 2n\}$.

Let M denote the big max of $\mathbb{H}_n$ corresponding to $\{x_{2n+1}^*, x_{2n+2}^*\}$. Then $\widetilde{\overline{M}} \cong \mathbb{H}_{n-1}$. So, if u denotes the point of $H(2n-1, 4)$ corresponding to $\overline{M}$, then u has weight $2n$ with respect to B^*. Now, choose an ordered basis $B = (\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n})$ of V such that: (i) $\bar{e}_{\phi(i)}^*$ and $\bar{e}_i$ are parallel vectors; (ii) $u = \langle \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$. Note that Claim (1) of the lemma holds for this ordered basis B.

Let $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$. Let $M_{i,j}$ denote the big max of $\mathbb{H}_n$ corresponding to $\{x_i^*, x_j^*\}$ and let $u_{i,j}$ denote the point of $H(2n-1, 4)$ corresponding to $\overline{M_{i,j}}$. Since $\widetilde{\overline{M_{i,j}}} \cong \mathbb{G}_{n-1}$, there exist $a, b \in \mathbb{F}_4^*$ such that $u_{i,j} = \langle \bar{e}_{\phi(i)}^* + b\bar{e}_{\phi(j)}^* \rangle = \langle \bar{e}_i + a\bar{e}_j \rangle$. Since the maxes M and $M_{i,j}$ meet, also the maxes $\overline{M}$ and $\overline{M_{i,j}}$ should meet. So, the points u and $u_{i,j}$ are collinear on $H(2n-1, 4)$, implying that $a = 1$ and $u_{i,j} = \langle \bar{e}_i + \bar{e}_j \rangle$.

Let $i \in \{1, 2, \ldots, 2n\}$. Let M_i denote the big max of $\mathbb{H}_n$ corresponding to $\{x_i^*, x_{2n+1}^*\}$ and let M_i' denote the big max of $\mathbb{H}_n$ corresponding to $\{x_i^*, x_{2n+2}^*\}$. Then $\widetilde{\overline{M_i}} \cong \widetilde{\overline{M_i'}} \cong \mathbb{H}_{n-1}$. Let u_i and u_i' denote the points of $H(2n-1, 4)$ corresponding to respectively $\overline{M_i}$ and $\overline{M_i'}$. Since $\mathcal{R}_M(M_i) = M_i'$, the points u, u_i and u_i' are contained in a line of $\mathrm{PG}(V)$ by Lemma 4.7. By Lemma 4.9, this line contains a unique point of weight 1 (w.r.t. B^* and hence also w.r.t. B). We denote this point by $\langle \bar{e}_{\phi'(i)} \rangle$. We conclude that u_i is of the form $\langle \omega_i \cdot \bar{e}_{\phi'(i)} + \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$ and that u_i' is of the form $\langle \omega_i' \cdot \bar{e}_{\phi'(i)} + \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$. Since $u_i, u_i' \in H(2n-1, 4)$, we have $\omega_i \neq 1 \neq \omega_i'$. Since u, u_i and u_i' are mutually distinct, we have $\{\omega_i, \omega_i'\} = \{\omega, \omega^2\}$.

We prove that $\phi'(i) = i$ and $\omega_i = \omega_j$ for all $i, j \in \{1, 2, \ldots, 2n\}$ with $i \neq j$. Consider the maxes M_i, M_j and $M_{i,j}$. Since $\mathcal{R}_{M_i}(M_j) = M_{i,j}$, the points $u_i = \langle \omega_i \cdot \bar{e}_{\phi'(i)} + \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$, $u_j = \langle \omega_j \cdot \bar{e}_{\phi'(j)} + \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$ and $u_{i,j} = \langle \bar{e}_i + \bar{e}_j \rangle$ are contained in a line by Lemma 4.7. Hence, $\{\phi'(i), \phi'(j)\} = \{i, j\}$ and $\omega_i = \omega_j$. So, if j_1 and j_2 are two distinct elements of $\{1, 2, \ldots, 2n\} \setminus \{i\}$, then $\phi'(i)$ must be contained in $\{i, j_1\}$ and $\{i, j_2\}$, showing that $\phi'(i) = i$.

By the previous two paragraphs, we know that $u_i = \langle \omega_i \cdot \bar{e}_i + \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$ and $u_i' = \langle \omega_i' \cdot \bar{e}_i + \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n} \rangle$ for every $i \in \{1, 2, \ldots, 2n\}$.

Now, let p be a point of $\mathbb{H}_n$. Then there exists a permutation σ of $\{1,2,\ldots,2n\}$ such that p is equal to either $\{\{x^*_{\sigma(1)},x^*_{\sigma(2)}\},\{x^*_{\sigma(3)},x^*_{\sigma(4)}\},\ldots,\{x^*_{\sigma(2n-1)},x^*_{\sigma(2n)}\},\{x^*_{2n+1},x^*_{2n+2}\}\}$ or $\{\{x^*_{\sigma(1)},x^*_{\sigma(2)}\},\{x^*_{\sigma(3)},x^*_{\sigma(4)}\},\ldots,\{x^*_{\sigma(2n-3)},x^*_{\sigma(2n-2)}\},\{x^*_{\sigma(2n-1)},x^*_{2n+1}\},\{x^*_{\sigma(2n)},x^*_{2n+2}\}\}$.

Suppose $p=\{\{x^*_{\sigma(1)},x^*_{\sigma(2)}\},\{x^*_{\sigma(3)},x^*_{\sigma(4)}\},\ldots,\{x^*_{\sigma(2n-1)},x^*_{\sigma(2n)}\},\{x^*_{2n+1},x^*_{2n+2}\}\}$. Since p is contained in the maxes $M_{\sigma(1),\sigma(2)},M_{\sigma(3),\sigma(4)},\ldots,M_{\sigma(2n-1),\sigma(2n)}$, the point $e(p)$ is contained in the maxes $\overline{M_{\sigma(1),\sigma(2)}},\overline{M_{\sigma(3),\sigma(4)}},\ldots,\overline{M_{\sigma(2n-1),\sigma(2n)}}$. So, $e(p)$ regarded as a maximal singular subspace of $H(2n-1,4)$ contains the points $\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)}\rangle,\langle\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)}\rangle,\ldots,\langle\bar{e}_{\sigma(2n-1)}+\bar{e}_{\sigma(2n)}\rangle$. Hence, $e(p)=\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)},\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)},\ldots,\bar{e}_{\sigma(2n-1)}+\bar{e}_{\sigma(2n)}\rangle$.

Suppose $p=\{\{x^*_{\sigma(1)},x^*_{\sigma(2)}\},\{x^*_{\sigma(3)},x^*_{\sigma(4)}\},\ \ldots\ ,\{x^*_{\sigma(2n-3)},x^*_{\sigma(2n-2)}\},\{x^*_{\sigma(2n-1)},x^*_{2n+1}\},\{x^*_{\sigma(2n)},x^*_{2n+2}\}\}$. Since the point p of $\mathbb{H}_n$ is contained in the maxes $M_{\sigma(1),\sigma(2)},M_{\sigma(3),\sigma(4)},\ldots,M_{\sigma(2n-3),\sigma(2n-2)},M_{\sigma(2n-1)}$ of $\mathbb{H}_n$, the point $e(p)$ of $\mathbb{G}_n$ is contained in the maxes $\overline{M_{\sigma(1),\sigma(2)}},\overline{M_{\sigma(3),\sigma(4)}},\ldots,\overline{M_{\sigma(2n-3),\sigma(2n-2)}},\overline{M_{\sigma(2n-1)}}$ of $\mathbb{G}_n$. So, $e(p)$ regarded as a maximal singular subspace of $H(2n-1,4)$ contains the points $\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)}\rangle,\langle\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)}\rangle,\ldots,\langle\bar{e}_{\sigma(2n-3)}+\bar{e}_{\sigma(2n-2)}\rangle,\langle\omega_{\sigma(2n-1)}\cdot\bar{e}_{\sigma(2n-1)}+\bar{e}_1+\bar{e}_2+\cdots+\bar{e}_{2n}\rangle$. So, $e(p)=\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)},\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)},\ldots,\bar{e}_{\sigma(2n-3)}+\bar{e}_{\sigma(2n-2)},\omega_{\sigma(2n-1)}\cdot\bar{e}_{\sigma(2n-1)}+\bar{e}_1+\bar{e}_2+\cdots+\bar{e}_{2n}\rangle=\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)},\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)},\ldots,\bar{e}_{\sigma(2n-3)}+\bar{e}_{\sigma(2n-2)},\omega_{\sigma(2n-1)}\cdot\bar{e}_{\sigma(2n-1)}+\bar{e}_{\sigma(2n)}\rangle=\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)},\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)},\ldots,\bar{e}_{\sigma(2n-3)}+\bar{e}_{\sigma(2n-2)},\bar{e}_{\sigma(2n-1)}+(\omega_{\sigma(2n-1)})^2\cdot\bar{e}_{\sigma(2n)}\rangle$.

So, $e(\mathcal{P})$ consists of all points of $\mathbb{G}_n$ of the form $\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)},\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)},\ldots,\bar{e}_{\sigma(2n-1)}+\bar{e}_{\sigma(2n)}\rangle$ for some permutation σ of $\{1,2,\ldots,2n\}$ or of the form $\langle\bar{e}_{\sigma(1)}+\bar{e}_{\sigma(2)},\bar{e}_{\sigma(3)}+\bar{e}_{\sigma(4)},\ldots,\bar{e}_{\sigma(2n-3)}+\bar{e}_{\sigma(2n-2)},\bar{e}_{\sigma(2n-1)}+\omega\cdot\bar{e}_{\sigma(2n)}\rangle$ for some permutation σ of $\{1,2,\ldots,2n\}$. So, also Claim (2) of the lemma holds. ■

Proposition 4.19 *Let $\mathcal{P}$ denote the point set of $\mathbb{H}_n$. If e_1 and e_2 are two full isometric embeddings of $\mathbb{H}_n$ into $\mathbb{G}_n$, then there exists an automorphism θ of $\mathbb{G}_n$ such that $e_2(\mathcal{P})=\theta\circ e_1(\mathcal{P})$.*

Proof. By Lemma 4.18, there exists for every $i\in\{1,2\}$ an ordered basis $B_i=(\bar{e}^{(i)}_1,\bar{e}^{(i)}_2,\ldots,\bar{e}^{(i)}_{2n})$ of V such that the following hold:

(1) $\{\langle\bar{e}^{(i)}_1\rangle,\langle\bar{e}^{(i)}_2\rangle,\ldots,\langle\bar{e}^{(i)}_{2n}\rangle\}=\{\langle\bar{e}^*_1\rangle,\langle\bar{e}^*_2\rangle,\ldots,\langle\bar{e}^*_{2n}\rangle\}$;

(2) If $\mathcal{P}$ is the point set of $\mathbb{H}_n$, then $e_i(\mathcal{P})$ consists of all points of $\mathbb{G}_n$ of the form $\langle\bar{e}^{(i)}_{\sigma(1)}+\bar{e}^{(i)}_{\sigma(2)},\bar{e}^{(i)}_{\sigma(3)}+\bar{e}^{(i)}_{\sigma(4)},\ldots,\bar{e}^{(i)}_{\sigma(2n-1)}+\bar{e}^{(i)}_{\sigma(2n)}\rangle$ for some permutation σ of $\{1,2,\ldots,2n\}$ or of the form $\langle\bar{e}^{(i)}_{\sigma(1)}+\bar{e}^{(i)}_{\sigma(2)},\bar{e}^{(i)}_{\sigma(3)}+\bar{e}^{(i)}_{\sigma(4)},\ldots,\bar{e}^{(i)}_{\sigma(2n-3)}+\bar{e}^{(i)}_{\sigma(2n-2)},\bar{e}^{(i)}_{\sigma(2n-1)}+\omega\cdot\bar{e}^{(i)}_{\sigma(2n)}\rangle$ for some permutation σ of $\{1,2,\ldots,2n\}$.

Clearly, the unique element of $GL(V)$ mapping B_1 to B_2 determines an automorphism of $\mathbb{G}_n$ mapping $e_1(\mathcal{P})$ to $e_2(\mathcal{P})$. ■

The following corollary is precisely Theorem 1.1.

Corollary 4.20 *Up to equivalence, there exists a unique isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$. Up to isomorphism, there are $(n+1)(2n+1)$ isometric embeddings of $\mathbb{H}_n$ into $\mathbb{G}_n$. No isometric embedding of $\mathbb{H}_n$ into $\mathbb{G}_n$ is homogeneous.*

Proof. This is a consequence of Lemma 2.2, Proposition 2.4, Proposition 3.1, Corollary 3.5 and Proposition 4.19. ■

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