Implicit Lagrange-Routh Equations and Dirac Reduction

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Abstract

In this paper, we make a generalization of Routh's reduction method for Lagrangian systems with symmetry to the case where not any regularity condition is imposed on the Lagrangian. First, we show how implicit Lagrange-Routh equations can be obtained from the Hamilton-Pontryagin principle, by making use of an anholonomic frame, and how these equations can be reduced. To do this, we keep the momentum constraint implicit throughout and we make use of a Routhian function defined on a certain submanifold of the Pontryagin bundle. Then, we show how the reduced implicit Lagrange-Routh equations can be described in the context of dynamical systems associated to Dirac structures, in which we fully utilize a symmetry reduction procedure for implicit Hamiltonian systems with symmetry.

Keywords. Routh reduction, implicit Lagrange-Routh equations, Hamilton-Pontryagin principle, Dirac structures.
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1 Introduction

There is no doubt that there exists a close relation between symmetries and conservation laws, which has been one of the fundamental motivations for many geometric approaches to mechanical systems. The symmetry group of a dynamical system can always be used to reduce the system to one with fewer variables. When symmetry, besides, leads to conserved quantities, it can be very advantageous to incorporate that property into the reduction process. For example, when the system is Hamiltonian on a symplectic manifold, one first restricts the attention to the submanifold determined by the conserved momenta, and only later one takes the quotient of this submanifold by the remaining symmetry (which in general happens to be only a subgroup of the original symmetry group). This, in a few words, is the so-called *symplectic reduction theorem* (see, Marsden and Weinstein [1974]; Marsden [1992]).

Symplectic reduction may be applied to the standard case of classical Hamiltonian systems defined on the cotangent bundle. While this procedure has been thoroughly studied in the literature, its Lagrangian counterpart, the so-called Routh or tangent bundle reduction, has traditionally received much less attention, even though since its conception in Routh [1877, 1884] it has proven to be a valuable tool to obtain and discuss, e.g., the stability of steady motions or relative equilibria. A few modern approaches to the topic can be found in the papers Crampin and Mestdag [2008]; Langerock, Cantrijn and Vankerschaver [2010]; Marsden, Ratiu and Scheurle [2000]. One of the drawbacks of these papers is that a regularity condition needs to be assumed. In this paper, we will focus upon Routh reduction within the context of Dirac structures, *without* assuming any regularity hypotheses.

For simplicity, let us consider for a moment the case of a Lagrangian $L(x, \dot{x}, \dot{\theta})$ with a single cyclic coordinate θ. The first step in Routh's procedure is to write the corresponding velocity $\dot{\theta}$ in terms of the remaining coordinates and velocities $(x, \dot{x})$ by making use of the conservation law $\partial L/\partial\dot{\theta} = \mu$, which follows from Noether's theorem. One then introduces the restriction $R^\mu(x, \dot{x})$ of the function $L - \dot{\theta}(\partial L/\partial\dot{\theta})$ to the level set where the momentum is μ, the so-called *Routhian* (see, e.g., Marsden [1992]). With this function, one can observe that the remaining Euler-Lagrange equations of the coordinates x, again when constrained to the level set associated to μ, are in fact Euler-Lagrange equations for the Routhian R^μ. The end result of Routh's reduction method is therefore that it reduces the Euler-Lagrange equations of the Lagrangian $L(x, \dot{x}, \dot{\theta})$ to those of $R^\mu(x, \dot{x})$ on a reduced configuration space. A crucial ingredient in the above process, however, is that the Lagrangian satisfies the regularity condition $(\partial^2 L/\partial^2\dot{\theta}) \neq 0$, which is necessary for carrying out the first step. Routh's procedure and the regularity condition for it to be applicable can be generalized to arbitrary Lagrangians with a (possibly) non-Abelian symmetry group G. In this situation, the condition is often referred to as *G-regularity*.

It is easy to construct a Lagrangian which fails to be G-regular. The following example in $\mathbb{R}^2$ is taken from Langerock and Castrillón [2010]:

$$L(x, y, v_x, v_y) = (v_x)^2 + v_x v_y - V(x).$$

Note that it has a cyclic coordinate y (and therefore an Abelian symmetry group $G = \mathbb{R}$), but that it is not G-regular. Also linear G-invariant Lagrangians will always fail to be G-regular.

For example, the dynamics of N vortices in the plane admit the following Lagrangian:

$$L(z_l, \dot{z}_l) = \frac{1}{2i} \sum_k \gamma_k \left(\bar{z}_k \dot{z}_k - z_k \dot{\bar{z}}_k \right) - \frac{1}{2} \sum_n \sum_{k \neq n} \gamma_n \gamma_k \ln |z_n - z_k|, \qquad z_l \in \mathbb{C},$$

where $\gamma_k \in \mathbb{R}$ are parameters of the model; see Chapman [1978] for more details. This Lagrangian is clearly linear in its velocities and invariant under rotations of the vortices in the plane but not G-regular. Also in the context of plasma physics, linear Lagrangians often appear (see, e.g., Littlejohn [1983]).

The aim of this paper is to extend Routh's method to the most general case where not any regularity condition is imposed on the Lagrangian. Our approach is based on the *Hamilton-Pontryagin principle* (as it is called in Yoshimura and Marsden [2006b]) which leads to an implicit formulation of the Euler-Lagrange equations on the so-called Pontryagin bundle $TQ \oplus T^*Q$. We will show that under the assumption of symmetry, we can reduce these implicit equations to a set of *reduced implicit Lagrange-Routh equations*. The key ingredient is that we can circumvent the hypotheses on regularity by keeping the momentum constraint implicit throughout. Our method involves a generalized Routhian function which is defined on a certain submanifold of the Pontryagin bundle rather than on a submanifold of the tangent bundle, as is commonly the case for G-regular systems. Implicit Lagrangian systems can be geometrically described in the framework of Dirac structures (see Yoshimura and Marsden [2006a]). The definition of Dirac structure in Courant [1990]; Dorfman [1993] was originally inspired by the notion of Dirac brackets, which was coined by Paul Dirac (in the 1950s) for dealing with constraints in the Hamiltonian setting when the given Lagrangian is singular (see e.g. Courant and Weinstein [1988]; van der Schaft and Jeltsema [2014]). So from the very start, there has been a strong relation with singular Lagrangians and constraints. In the second part of the paper, we will show that the reduced implicit Lagrange-Routh equations may be also formulated in terms of a Dirac structure, by considering a reduction method known for implicit Hamiltonian systems from van der Schaft [1998]; Blankenstein and van der Schaft [2001].

For completeness, we mention that the paper by Langerock and Castrillón [2010] also deals with the general case. However, these authors use a variational approach which is based on Hamilton's principle rather than on the Hamilton-Pontryagin principle. Therefore, it focusses on different aspects of the theory.

This paper is organized as follows. In §2, we review the derivation of the standard implicit Euler-Lagrange equations for a possibly degenerate (or singular) Lagrangian L via the Hamilton-Pontryagin principle. We use a technique that is similar to the one that has been used in, for instance, Crampin and Mestdag [2008, 2010] to rewrite the implicit Euler-Lagrange equations in terms of an *anholonomic* frame. Once the implicit equations on a general frame are obtained, we specialize these expressions to a particular frame adapted to a given symmetry of the Lagrangian (§3). For a prescribed value of momenta, we find the implicit Lagrange-Routh equations, and express them in an invariant form. In §4 we reduce them to obtain the reduced implicit Lagrange-Routh equations. The regular cases are discussed in §5, where we illustrate our theory by showing how the reduced implicit Lagrange-Routh equations agree with those developed in the literature. §6 rephrases the previous results in terms of reduction of Dirac structures. We show how the reduced implicit Lagrange-Routh equations correspond to a certain reduced implicit Hamiltonian system. Finally, in §7, some examples are shown.

2 A version of the Hamilton-Pontryagin principle using anholonomic frames

Hamilton-Pontryagin principles. Let Q be a configuration manifold of a mechanical system with $\dim Q = n$. Coordinates on Q are given by q^α, fiber coordinates on TQ and T^*Q will be denoted by v^α and p_α, respectively. In the following, the index α runs from 1 to n unless otherwise noted. The notations are chosen in such a way that we can make a notational difference between a general curve $(q(t), v(t))$ in TQ, and the lifted curve $(q(t), \dot{q}(t))$ in TQ of a curve $q(t)$ in Q, where t denotes the time in $I = \{t \in \mathbb{R} \mid a \leq t \leq b\}$.

The Hamilton-Pontryagin principle leads to an implicit form of the Euler-Lagrange equations of a possibly degenerate Lagrangian L. These equations follow from considering the following variational principle on $TQ \oplus T^*Q$:

$$\delta \int_b^a \big[L(q,v) + \langle p, (\dot{q} - v)\rangle\big] dt = \delta \int_b^a \big[L(q^\alpha, v^\alpha) + p_\alpha(\dot{q}^\alpha - v^\alpha)\big] dt = 0,$$

for variations of $(q(t), v(t), p(t))$ where $q(t)$ has fixed endpoints and $v(t)$ and $p(t)$ are arbitrary. From this, we can easily conclude that a solution $(q(t), v(t), p(t))$ of the implicit Euler-Lagrange equations must satisfy

$$\dot{q}^\alpha = v^\alpha, \qquad p_\alpha - \frac{\partial L}{\partial v^\alpha} = 0, \qquad \dot{p}_\alpha = \frac{\partial L}{\partial q^\alpha}. \tag{2.1}$$

See Yoshimura and Marsden [2006b] for more details.

Anholonomic frames and quasi-velocities. In this section, we shall rewrite the implicit Euler-Lagrange equations in terms of the so-called *quasi-velocities*. Lagrangian equations which involve quasi-velocities are often called *Hamel equations* in the literature (see, for instance, Marsden and Scheurle [1993]; Bloch, Marsden and Zenkov [2009]; Crampin and Mestdag [2010]). We will need these expressions when we consider the Routhian in the following sections.

In the next paragraphs, we will need the natural lifts of vector fields on Q to its tangent manifold and Pontryagin bundle, respectively. Let (q^α, v^α) be the natural tangent bundle coordinates on TQ. If $X = X^\alpha(\partial/\partial q^\alpha)$ is a vector field on Q, then its complete lift X^{C} and vertical lift X^{V} are the vector fields on TQ, given by

$$X^{\mathrm{C}} = X^\beta \frac{\partial}{\partial q^\beta} + \frac{\partial X^\beta}{\partial q^\gamma} v^\gamma \frac{\partial}{\partial v^\beta}, \qquad X^{\mathrm{V}} = X^\beta \frac{\partial}{\partial v^\beta}.$$

Likewise, its complete lift to $M = TQ \oplus T^*Q$ is the vector field

$$X^M = X^\alpha \frac{\partial}{\partial q^\alpha} + \frac{\partial X^\beta}{\partial q^\alpha} v^\alpha \frac{\partial}{\partial v^\beta} - \frac{\partial X^\beta}{\partial q^\alpha} p_\beta \frac{\partial}{\partial p_\alpha}.$$

A standard reference for the properties of these vector fields is the book Yano and Ishihara [1973]. Given a vector field $Y = Y^\alpha(\partial/\partial q^\alpha)$ on Q, we can form the linear function $\bar{Y} = Y^\alpha p_\alpha$ on $T^*Q \subset M$. Likewise, for a 1-form $\theta = \theta_\alpha dq^\alpha$ we can define a linear function on $\bar{\theta} = \theta_\alpha v^\alpha$ on $TQ \subset M$. The following properties can then easily be verified:

$$X^M(\bar{Y}) = \overline{[X, Y]}, \qquad X^M(\bar{\theta}) = \overline{\mathcal{L}_X \theta}. \tag{2.2}$$

Quasi-velocities are fiber coordinates in T_qQ, defined with respect to a non-coordinate or ***anholonomic frame***. Let $Z_\alpha = Z_\alpha^\beta(\partial/\partial q^\beta)$ be a new basis for the set of vector fields on Q. This means that at each point q the matrix $(Z^\alpha_\beta(q))$ has an inverse matrix, smoothly defined, which we will denote by $(W^\alpha_\beta(q))$. Each vector $v_q \in T_qQ$ can be expressed by $v_q = \mathrm{v}^\alpha Z_\alpha(q)$. The fiber coordinates (v^α) are then the ***quasi-velocities*** of v_q with respect to the frame $\{Z_\alpha\}$. Their relation to the natural fiber coordinates is simply $\mathrm{v}^\alpha = W^\alpha_\beta v^\beta$.

The coordinate frame $\{\partial/\partial q^\alpha\}$ is an example of a frame, whose corresponding quasi-velocities are simply the natural fiber coordinates v^α. A measure for the deviation of a given frame $\{Z_\alpha\}$ from being a coordinate frame, is given by its ***object of anholonomity*** (see, e.g., Schouten [1954]), which is defined by the relation

$$[Z_\beta, Z_\gamma] = R^\alpha_{\beta\gamma} Z_\alpha.$$

The $R^\alpha_{\beta\gamma}$ are given in coordinates by the following expressions:

$$R^\alpha_{\beta\gamma} = \left(Z^\tau_\beta W^\alpha_\delta \frac{\partial Z^\delta_\gamma}{\partial q^\tau} - Z^\tau_\gamma W^\alpha_\delta \frac{\partial Z^\delta_\beta}{\partial q^\tau} \right) = -\left(Z^\tau_\beta \frac{\partial W^\alpha_\delta}{\partial q^\tau} Z^\delta_\gamma - Z^\tau_\gamma \frac{\partial W^\alpha_\delta}{\partial q^\tau} Z^\delta_\beta \right). \tag{2.3}$$

We can lift the frame $\{Z_\alpha\}$ on Q to the frame $\{Z^{\mathrm{C}}_\alpha, Z^{\mathrm{V}}_\alpha\}$ on TQ. In what follows, we will often make use of the following, easily verifiable, properties:

$$\begin{aligned} Z^{\mathrm{C}}_\alpha(q^\beta) &= Z^\beta_\alpha, & Z^{\mathrm{V}}_\alpha(q^\beta) &= 0, \\ Z^{\mathrm{C}}_\alpha(\mathrm{v}^\beta) &= -R^\beta_{\alpha\gamma}\mathrm{v}^\gamma, & Z^{\mathrm{V}}_\alpha(\mathrm{v}^\beta) &= \delta^\beta_\alpha. \end{aligned} \tag{2.4}$$

The 1-forms $W^\alpha = W^\alpha_\beta dq^\beta$ form a basis for 1-forms on Q. If (q^α, p_α) denote the natural coordinates on T^*Q, we can also introduce ***quasi-momenta*** by means of $\mathrm{p}_\beta = Z^\alpha_\beta p_\alpha$, and we note that the natural pairing is preserved: $\langle p, v\rangle = p_\alpha v^\alpha = \mathrm{p}_\alpha \mathrm{v}^\alpha$.

The independent variations $\{\delta q^\alpha, \delta v^\alpha, \delta p_\alpha\}$ form a basis for all variations on $M = TQ \oplus T^*Q$. We can change this to a new basis, adjusted to the new frame $\{Z_\alpha, W^\alpha\}$ on M. If we denote

$$w^\alpha = W^\alpha_\beta \delta q^\beta,$$

then $\{w^\alpha, \delta\mathrm{v}^\alpha, \delta\mathrm{p}_\alpha\}$ will be all independent variations and hence this set will form a new basis for all variations on M. Here one should think of the quasi-velocities v^α as functions on TQ (or on M), and therefore

$$\delta\mathrm{v}^\alpha = \frac{\partial W^\alpha_\beta}{\partial q^\gamma} v^\beta \delta q^\gamma + W^\alpha_\beta \delta v^\beta.$$

The implicit Lagrangian systems with quasi-velocities and quasi-momenta. The direct computation using (2.3) yields the variation of a Lagrangian L on TQ as

$$\delta L = \frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L}{\partial v^\alpha}\delta v^\alpha = Z^{\mathrm{C}}_\alpha(L) w^\alpha + Z^{\mathrm{V}}_\alpha(L)(\delta\mathrm{v}^\alpha + R^\alpha_{\beta\gamma}\mathrm{v}^\gamma w^\beta). \tag{2.5}$$

We will now give a version of the Hamilton-Pontryagin principle that makes use of the anholonomic frame. We are looking for a curve $(q^\alpha(t), \mathrm{v}^\alpha(t), \mathrm{p}_\alpha(t))$ in $M = TQ \oplus T^*Q$, namely the one whose base curve is $q(t)$ and whose fiber coordinates are given by the curve $(\mathrm{v}^\alpha(t), \mathrm{p}_\alpha(t))$ in quasi-velocities and quasi-momenta which satisfies the following variational principle

$$0 = \delta \int_b^a \big[L(q, v) + \langle p, \dot q - v\rangle\big] dt = \delta \int_b^a \big[L(q, \mathrm{v}) + \mathrm{p}_\alpha(\mathrm{u}^\alpha - \mathrm{v}^\alpha)\big] dt.$$

Here $\mathrm{u}^\alpha(t)$ stand for the quasi-velocities of the lifted curve $\dot q(t)$ in TQ, namely, $\mathrm{u}^\alpha(t) = W^\alpha_\beta(q(t))\dot q^\beta(t)$.

If we take the above expression (2.5) for δL into account, we obtain

$$0=\int_b^a \left[Z^{\rm C}_\alpha(L)w^\alpha + Z^{\rm V}_\alpha(L)(\delta {\rm v}^\alpha + R^\alpha_{\beta\gamma}{\rm v}^\gamma w^\beta) + \delta{\rm p}_\alpha({\rm u}^\alpha - {\rm v}^\alpha) + {\rm p}_\alpha\delta{\rm u}^\alpha - {\rm p}_\alpha\delta{\rm v}^\alpha\right] dt.$$

First, we compute

$$\begin{aligned}\int_b^a {\rm p}_\alpha\delta{\rm u}^\alpha dt &= \int_b^a \left[{\rm p}_\alpha\frac{\partial W^\alpha_\beta}{\partial q^\gamma}\dot q^\beta\delta q^\gamma + {\rm p}_\alpha W^\alpha_\beta\delta\dot q^\beta\right] dt\\ &= \int_b^a \left[{\rm p}_\alpha\left(\frac{\partial W^\alpha_\beta}{\partial q^\gamma} - \frac{\partial W^\alpha_\gamma}{\partial q^\beta}\right)\dot q^\beta\delta q^\gamma - \dot{\rm p}_\alpha w^\alpha\right] dt\\ &= \int_b^a \left[({\rm p}_\alpha R^\alpha_{\mu\tau}{\rm u}^\mu - \dot{\rm p}_\tau)w^\tau\right] dt.\end{aligned}$$

In the above, we have used integration by parts and the fact that $\delta q^\alpha(a) = \delta q^\alpha(b) = 0$. We have also made use of the expression (2.3) for $R^\alpha_{\mu\tau}$. Then, by substituting this into our variational principle, it follows that

$$\begin{aligned}0=\int_b^a \Big[\Big(Z^{\rm C}_\tau(L) + {\rm p}_\alpha R^\alpha_{\mu\tau}{\rm u}^\mu + Z^{\rm V}_\alpha(L)R^\alpha_{\tau\gamma}{\rm v}^\gamma - \dot{\rm p}_\tau\Big)w^\tau\\ +\Big(Z^{\rm V}_\alpha(L) - {\rm p}_\alpha\Big)\delta{\rm v}^\alpha - \Big({\rm v}^\alpha - {\rm u}^\alpha\Big)\delta{\rm p}_\alpha\Big] dt\end{aligned}$$

holds for all variations w^τ, $\delta{\rm v}^\alpha$ and $\delta{\rm p}_\alpha$. Since these are all independent, we conclude that the curve $(q^\alpha(t), {\rm v}^\alpha(t), {\rm p}_\alpha(t))$ should satisfy

$$ {\rm v}^\alpha = {\rm u}^\alpha, \qquad {\rm p}_\alpha = Z^{\rm V}_\alpha(L), \qquad \dot{\rm p}_\alpha = Z^{\rm C}_\alpha(L). \tag{2.6}$$

These are the ***implicit Euler-Lagrange equations in quasi-velocities and quasi-momenta***. Remark that the last equation takes such a simple form, only because we have already made use of the first two equations.

3 The implicit Lagrange-Routh equations

The Lie group action on Q. Assume that $G\times Q\to Q$ is a free and proper action of a possibly non-Abelian Lie group G on Q, so that we can regard $Q\to Q/G$ as a principal G-bundle. We will assume in this section that the Lagrangian $L: TQ\to\mathbb{R}$ is invariant under the symmetry group G. We use x^i for coordinates on the *shape space* Q/G and $(q^\alpha) = (x^i, \theta^a)$ for coordinates on Q. We denote the corresponding coordinates on the tangent bundle by (v^i, v^a). As before, we use the notation $(x(t), \dot x(t))$ to denote the lifted curve on $T(Q/G)$ of a curve $x(t)$ in Q/G.

We now choose a principal connection on $Q\to Q/G$ and let X_i denote the horizontal lift of the coordinate vector fields $\partial/\partial x^i$ on the shape space Q/G with respect to this connection. In terms of the coordinates introduced above, we have

$$X_i = \frac{\partial}{\partial x^i} - \Lambda^a_i\frac{\partial}{\partial\theta^a},$$

where Λ^a_i are functions on Q (the connection coefficients) and we employ the symbol Λ to denote the *connection one-form* on TQ which takes values in $\mathfrak{g}$.

Let $\{E_a\}$ be a basis of the Lie algebra $\mathfrak{g}$, and $\{E^a\}$ the corresponding dual basis of $\mathfrak{g}^*$. We denote by C^c_{ab} the structure constants of $\mathfrak{g}$, $[E_a, E_b] = C^c_{ab}E_c$. The fundamental vector fields associated to the action will be denoted by $\{\tilde{E}_a\}$. We can express them as

$$\tilde{E}_a = K^b_a \frac{\partial}{\partial \theta^b},$$

for some functions K^b_a on Q, often called the coefficients of the infinitesimal generator map.

We suppose throughout the paper that G is connected. This has the advantage that invariance of functions and tensor fields can be checked by the vanishing of the Lie derivatives of these functions and tensor fields, in the direction of the fundamental vector fields of the action.

The action on Q lifts to actions on TQ, T^*Q and $M = TQ \oplus T^*Q$. In each case, it is well-known that the fundamental vector fields of the lifted actions are given by the complete lifts (to TQ, T^*Q and M, respectively) of the fundamental vector fields on Q. In the case of the action on M the fundamental vector fields are therefore (linear combinations of) the vector fields $\tilde{E}^M_a$. Since G is supposed to be connected, a function F on M is invariant if, and only if, $\tilde{E}^M_a(F) = 0$.

The implicit Lagrange-Routh equations. Let us rewrite the implicit Lagrange equations (2.6) by making use of a specific anholonomic frame. First we consider the frame $\{Z_\alpha\} = \{X_i, \tilde{E}_a\}$ on Q. This corresponds to the so-called ***moving frame*** in literature. We denote the corresponding quasi-velocities and quasi-momenta by $(\mathrm{v}^i, \tilde{\mathrm{v}}^a)$ and $(\mathrm{p}_i, \tilde{\mathrm{p}}_a)$. In fact, since the vector fields X_i are assumed to project onto the coordinate fields on Q/G, the quasi-velocities v^i can be naturally identified with the natural fiber coordinates v^i on $T(Q/G)$. The curve $\mathrm{u}^i(t)$ that appears in the equations (2.6) is then simply the lifted curve $\dot{x}^i(t)$ of the curve $x^i(t)$ in Q/G. Similarly, the quasi-momenta p_i can be identified with the momenta p_i of $T^*(Q/G)$, but this identification is not canonical since it depends on the choice of the connection Λ. From now on, for simplicity, we will use the notation v^i and p_i to denote the corresponding quasimomenta.

The brackets of the frame are given by:

$$[X_i, X_j] = B^a_{ij}\tilde{E}_a, \qquad [X_i, \tilde{E}_a] = 0, \qquad [\tilde{E}_a, \tilde{E}_b] = -C^c_{ab}\tilde{E}_c,$$

where the B^a_{ij} stand, up to a sign, for the curvature coefficients of the principal connection Λ (this is the convention in Crampin and Mestdag [2008], but differs from e.g. Marsden, Ratiu and Scheurle [2000]). The relation $[X_i, \tilde{E}_a]$ is a consequence of the invariance of the X_i.

The Lagrangian is invariant if, and only if, $\tilde{E}^M_a(L) = \tilde{E}^{\mathrm{C}}_a(L) = 0$. The implicit Euler-Lagrange equations (2.6) are therefore

$$\begin{aligned} \tilde{\mathrm{v}}^a &= \tilde{\mathrm{u}}^a, \qquad & \tilde{\mathrm{p}}_a &= \tilde{E}^{\mathrm{V}}_a(L), \qquad & \dot{\tilde{\mathrm{p}}}_a &= 0, \\ v^i &= \dot{x}^i, & p_i &= X^{\mathrm{V}}_i(L), & \dot{p}_i &= X^{\mathrm{C}}_i(L). \end{aligned}$$

From the top row, we see that $\tilde{\mathrm{p}}_a$ is constant along solutions, say $\tilde{\mathrm{p}}_a = \mu_a$, with $\mu = \mu_a E^a \in \mathfrak{g}^*$. We can define a ***generalized Routhian***, as the function on TQ given by

$$R^\mu(q, v) = L(q, v) - \mu_a \tilde{\mathrm{v}}^a. \tag{3.1}$$

It is, however, not the standard definition of the Routhian function as one may find in, for instance, Marsden, Ratiu and Scheurle [2000]; Crampin and Mestdag [2008]; Langerock, Cantrijn and Vankerschaver [2010]. We will clarify the relation between these two definitions later in in §5. The advantage of the current definition is that it allows us to keep the momentum constraint implicit.

From the relations (2.4) of the previous section, we obtain

$$\begin{aligned} X_i^{\rm C}(R^\mu) &= X_i^{\rm C}(L) + \mu_a B^a_{ij} v^j, & X_i^{\rm V}(R^\mu) &= X_i^{\rm V}(L), \\ \tilde{E}_a^{\rm C}(R^\mu) &= -\mu_c C^c_{ab}\tilde{\rm v}^b, & \tilde{E}_a^{\rm V}(R^\mu) &= \tilde{E}_a^{\rm V}(L) - \mu_a. \end{aligned} \tag{3.2}$$

Therefore, a solution of the implicit Euler-Lagrange equations (2.6) is a curve

$$(x^i(t), \theta^a(t), v^i(t), \tilde{\rm v}^a(t), p_i(t), \tilde{\rm p}_a(t)) : I \subset \mathbb{R} \to M = TQ \oplus T^*Q$$

satisfying

$$\begin{aligned} \tilde{\rm v}^a &= \tilde{\rm u}^a, & \tilde{E}_a^{\rm V}(R^\mu) &= 0, & \tilde{\rm p}_a &= \mu_a, \\ v^i &= \dot{x}^i, & X_i^{\rm V}(R^\mu) &= p_i, & \dot{p}_i &= X_i^{\rm C}(R^\mu) - \mu_a B^a_{ij} v^j. \end{aligned} \tag{3.3}$$

We will call these equations the ***implicit Lagrange-Routh equations***. The terminology Lagrange-Routh equations is adopted from Marsden, Ratiu and Scheurle [2000].

The implicit Lagrange-Routh equations in invariant form. We will restrict our attention to one specific level set of momentum. Consider the canonical momentum map $J : T^*Q \to \mathfrak{g}^*$ of the G-action on Q, and fix a value $\mu \in \mathfrak{g}^*$. Let M_μ denote the submanifold $TQ \oplus J^{-1}(\mu)$ in $M = TQ \oplus T^*Q$. Local coordinates on M_μ are then $(q, v^i, \tilde{\rm v}^a, p_i)$, the $\tilde{\rm p}_a$ being fixed by the value $\mu \in \mathfrak{g}^*$.

The G-action on M restricts to a G_μ-action on M_μ, where G_μ stands for the isotropy group. We will describe how solutions of the implicit Lagrange-Routh equations (3.3) which happen to lie on M_μ can be projected to curves in M_μ/G_μ, satisfying some reduced equations. In order to do that, we need to rewrite them in such a way that all involved terms are given by G_μ-invariant functions. When that is the case, these G_μ-invariant equations will project to equations on M_μ/G_μ.

Let $\xi = \xi^a E_a \in \mathfrak{g}$ be an arbitrary element of $\mathfrak{g}$, then it follows from (3.2) that $\tilde{\xi}^{\rm C}(R^\mu) = -\xi^a \mu_c C^c_{ab} \tilde{\rm v}^b$. From this we see that $\tilde{\xi}^{\rm C}(R^\mu) = 0$, if and only if, $\xi \in \mathfrak{g}_\mu$. Therefore, we see that R^μ is (only) G_μ-invariant. The Routhian R^μ can thus be identified with a reduced function on TQ/G_μ (for which we shall use the same notation).

We next define local coordinates on M_μ/G_μ. It is easy to see that the quasi-velocities v^i and the quasi-momenta p_i are G_μ-invariant functions on M_μ. Indeed, from (2.2) and (2.4), we have

$$\tilde{E}_a^M(v^i) = 0, \qquad \tilde{E}_a^M(p_i) = \overline{[\tilde{E}_a, X_i]} = 0.$$

The last property is based on the observation that $\overline{X_i} = p_i$. The above expressions show that v^i and p_i (thought of as coordinate functions on M) are G-invariant, and thus also G_μ-invariant functions on M (and therefore also on M_μ). On the other hand, the quasi-velocities $\tilde{\rm v}^a$ are not G_μ-invariant functions on M_μ since

$$\tilde{E}_a^M(\tilde{\rm v}^b) = -C^b_{ac}\tilde{\rm v}^c.$$

To overcome this issue we introduce a new frame that is completely G-invariant (and therefore also G_μ-invariant), see also Crampin and Mestdag [2008]. This coincides in the literature with the so-called ***body-fixed frame***. Consider a new set of vector fields, given by $\hat{E}_a = A^b_a \tilde{E}_b$. The following reasoning shows that there exists a matrix (A^b_a) of functions on Q for which these vector fields are all invariant. They may be invariant if and only if $0 = [\tilde{E}_a, \hat{E}_b] = \Big(\tilde{E}_a(A^c_b) - C^c_{ad}A^d_b\Big)\tilde{E}_c$. The *integrability condition* that is needed for the PDE equation

$$\tilde{E}_a(A^c_b) - C^c_{ad}A^d_b = 0$$

to have a solution A^a_b is satisfied by virtue of the Jacobi identity of the Lie bracket on $\mathfrak{g}$. We can therefore claim that, at least locally, the above PDE has a solution for which $A = (A^b_a)$ is non-singular, and for which A is the identity on some specified local section of $\pi : Q \to Q/G$.

An explicit way to define the vector fields $\hat{E}_a$, and the one we will use henceforth, is as follows. Let $U \subset Q/G$ be an open set over which Q is locally trivial. Then the fibration is $\pi : U \times G \to U$, and the action is given by $\psi_g(x,h) = (x, gh)$. We can define $\hat{E}_a : (x,g) \mapsto \widetilde{\mathrm{Ad}_g E_a}(x,g) = T\psi_g\big(\tilde{E}_a(x,e)\big)$, where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action. In coordinates, we write

$$\hat{E}_a = L^b_a \frac{\partial}{\partial \theta^b},$$

where L^b_a are functions on Q.

Another way to think of these two frames is the following: If Q is the Lie group G, and the action is given by left multiplication then the tilde-vector fields coincide with a basis of right-invariant vector fields, while the hat-vector fields are all left-invariant.

The quasi-velocities $(v^i, \hat{\mathrm{v}}^a)$ with respect to $\{X_i, \hat{E}_a\}$ are all invariant functions on M, since now

$$\tilde{E}^M_a(\hat{\mathrm{v}}^b) = 0.$$

We can therefore take $([q]_{G_\mu}, v^i, \hat{\mathrm{v}}^a, p_i)$ for our local coordinates on M_μ/G_μ, where $[q]_{G_\mu}$ stands for the coordinates of the orbit of q under the G_μ-action. In a more global interpretation, we have split the quotient M_μ/G_μ by making use of the principal connection Λ as

$$M_\mu/G_\mu \simeq (Q/G_\mu) \times_{Q/G} \big(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus T^*(Q/G)\big),$$

where $\tilde{\mathfrak{g}}$ is the adjoint bundle. In fact, the coordinates above correspond to fiber coordinates with respect to this identification (see Langerock and Castrillón [2010] for more details).

We now check whether all the terms that appear in the implicit Lagrange-Routh equations (3.3) are G_μ-invariant. For example, the function $\tilde{E}^{\mathrm{V}}_a(R^\mu)$ is not G_μ-invariant, but the function $\hat{E}^{\mathrm{V}}_b(R^\mu) = A^a_b \tilde{E}^{\mathrm{V}}_a(R^\mu)$ is. Indeed, for $\xi \in \mathfrak{g}_\mu$,

$$\xi^c \tilde{E}^{\mathrm{C}}_c \left(\hat{E}^{\mathrm{V}}_b(R^\mu)\right) = \hat{E}^{\mathrm{V}}_b \left(\xi^c \tilde{E}^{\mathrm{C}}_c(R^\mu)\right) = 0,$$

where we have used that $[\hat{E}^{\mathrm{V}}_b, \tilde{E}^{\mathrm{C}}_c] = [\hat{E}_b, \tilde{E}_c]^{\mathrm{C}} = 0$, and where we have also used that the Routhian is G_μ-invariant. We conclude that we need to replace the equation $\tilde{E}^{\mathrm{V}}_a(R^\mu) = 0$ in (3.3) by the equivalent equation $\hat{E}^{\mathrm{V}}_b(R^\mu) = A^a_b \tilde{E}^{\mathrm{V}}_a(R^\mu) = 0$ to obtain a G_μ-invariant equation. With a similar argument we can show that the functions $X^{\mathrm{C}}_i(R^\mu)$ and $X^{\mathrm{V}}_i(R^\mu)$ are all G_μ-invariant.

Remark that, since $[X_i, X_j]$ is an invariant vector field, we must have that $[\tilde{E}^{\mathrm{C}}_c, B^a_{ij}\tilde{E}^{\mathrm{C}}_a] = 0$, or equivalently

$$\tilde{E}^{\mathrm{C}}_c(B^b_{ij}) - B^a_{ij} C^b_{ca} = 0.$$

Using this, we can now check that the function $\mu_a B^a_{ij} v^j$ is G_μ-invariant; namely, for $\xi \in \mathfrak{g}_\mu$, we find

$$\xi^c \tilde{E}^{\mathrm{C}}_c(\mu_a B^a_{ij} v^j) = \xi^c \mu_a C^a_{cd} B^d_{ij} v^j = 0.$$

To conclude, apart from the defining relation $\tilde{\mathrm{p}}_a = \mu_a$ for M_μ, the following system of equations is equivalent to the implicit Lagrange-Routh equations (3.3) and consists only of G_μ-invariant equations:

$$\begin{aligned} &\hat{\mathrm{v}}^a = \hat{\mathrm{u}}^a, \qquad \hat{E}^{\mathrm{V}}_a(R^\mu) = 0, \\ &v^i = \dot{x}^i, \qquad X^{\mathrm{V}}_i(R^\mu) = p_i, \qquad \dot{p}_i = X^{\mathrm{C}}_i(R^\mu) - \mu_a B^a_{ij} v^j. \end{aligned} \tag{3.4}$$

All these equations correspond to equations on M_μ/G_μ, whose coordinate expressions will be obtained in the next section.

We finish this paragraph with a coordinate expression for the equations $\hat{\mathrm{v}}^a = \hat{\mathrm{u}}^a$. Recall first that $\hat{\mathrm{u}}^a$ is the quasi-velocity corresponding to the lifted curve $(q, \dot{q})$. If we write

$$\dot{x}^i \frac{\partial}{\partial x^i} + \dot{\theta}^a \frac{\partial}{\partial \theta^a} = \dot{x}^i X_i + (L^{-1})^a_b(\dot{\theta}^b + \Lambda^b_i \dot{x}^i)\hat{E}_a,$$

we can conclude that the equation $\hat{\mathrm{v}}^a = \hat{\mathrm{u}}^a$ is equivalent with the equation $\hat{\mathrm{v}}^b L^a_b = \dot{\theta}^a + \dot{x}^i \Lambda^a_i$.

4 Reduction of the implicit Lagrange-Routh equations

We shall use the residual G_μ-symmetry of the equations (3.4) to drop them to M_μ/G_μ. To do this, we will make use of an invariant decomposition (with respect to the adjoint action of G_μ) of the Lie algebra $\mathfrak{g} = \mathfrak{g}_\mu \oplus (\mathfrak{g}/\mathfrak{g}_\mu)$ meaning that, for all $g \in G_\mu$, we have $\mathrm{Ad}_g(\mathfrak{g}/\mathfrak{g}_\mu) \subset (\mathfrak{g}/\mathfrak{g}_\mu)$. In terms of coordinates, this splitting will be denoted as follows: we choose a basis $\{E_a\} = \{E_A, E_I\}$ of $\mathfrak{g}$ in such a way that $\{E_A\}$ represents a basis of $\mathfrak{g}_\mu$ and $\{E_I\}$ a basis of $(\mathfrak{g}/\mathfrak{g}_\mu)$.

An invariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus (\mathfrak{g}/\mathfrak{g}_\mu)$ of the Lie algebra as above, together with a principal connection on the bundle $Q \to Q/G$, induces a principal connection on the bundle $Q \to Q/G_\mu$. Indeed, let $\pi_{\mathfrak{g}_\mu}$ denote the projector $\pi_{\mathfrak{g}_\mu} : \mathfrak{g} \to \mathfrak{g}_\mu$ with respect to the previous decomposition. Define the principal connection 1-form $\Lambda^\mu := \pi_{\mathfrak{g}_\mu} \circ \Lambda : TQ \to \mathfrak{g}_\mu$, where $\Lambda : Q \to \mathfrak{g}$ is the principal connection 1-form on the bundle $Q \to Q/G$. This is a well-defined connection 1-form because

(1) If $\xi \in \mathfrak{g}_\mu$, then certainly $\Lambda^\mu(\tilde{\xi}) = \pi_{\mathfrak{g}_\mu}(\xi) = \xi$;

(2) If $g \in G_\mu$, then since Ad_g is linear and $\mathrm{Ad}_g(\mathfrak{g}_\mu) = \mathfrak{g}_\mu$, it follows that $\pi_{\mathfrak{g}_\mu} \circ \mathrm{Ad}_g = \mathrm{Ad}_g \circ \pi_{\mathfrak{g}_\mu}$.

Therefore, Λ^μ is G_μ-equivariant. Although we will not explicitly make use of it, we mention that an (Ehresmann) connection on $Q/G_\mu \to Q/G$ can be directly obtained by projecting the connection Λ (note that it is well-defined due to the equivariance of Λ). This connection, together with Λ and Λ_μ, plays a role when looking at variational principles in the context of Routh reduction (see also Langerock and Castrillón [2010] and Marsden, Ratiu and Scheurle [2000]).

Using that $\mathfrak{g}_\mu$ is a Lie subalgebra, we have $C^J_{AB} = 0$. On the other hand, from the definition of $\mathfrak{g}_\mu$, we also get $C^c_{Ab}\mu_c = 0$. Finally the fact that the term $(\mathfrak{g}/\mathfrak{g}_\mu)$ is invariant leads to the relation $C^C_{ab} = 0$. We will need these relations later in the paper.

We will use coordinates $(\theta^a) = (\theta^A, \theta^I)$ such that the fibers of $G \to G/G_\mu$ are given by $\theta^I =$ constant. Then, there are functions K^a_b on Q such that

$$\tilde{E}_A = K^B_A \frac{\partial}{\partial \theta^B}, \qquad \tilde{E}_I = K^B_I \frac{\partial}{\partial \theta^B} + K^J_I \frac{\partial}{\partial \theta^J}.$$

We remark that, by construction, $K^J_A = 0$. Likewise, we can set

$$\hat{E}_A = L^B_A \frac{\partial}{\partial \theta^B}, \qquad \hat{E}_I = L^B_I \frac{\partial}{\partial \theta^B} + L^J_I \frac{\partial}{\partial \theta^J}.$$

Since X^{V}_j is a G_μ-invariant vector field on TQ, it can be thought of as a vector field on TQ/G_μ. This reduced vector field will be completely determined by its action on the coordinate functions $(x^i, \theta^I, v^i, \hat{\mathrm{v}}^a)$, which define the coordinates on TQ/G_μ. We have, again using (2.2) and (2.4),

$$X^{\mathrm{V}}_j(x^i) = 0, \quad X^{\mathrm{V}}_j(\theta^I) = 0, \quad X^{\mathrm{V}}_j(v^i) = \delta^i_j, \quad X^{\mathrm{V}}_j(\hat{\mathrm{v}}^a) = 0.$$

The reduced vector field of X_j^{V} is therefore the vector field $\partial/\partial v^j$ on M_μ/G_μ. Likewise, the reduced vector fields of $\hat{E}_a^{\mathrm{V}}$ are $\partial/\partial\hat{\mathrm{v}}^a$. It also follows that

$$X_j^{\mathrm{C}}(x^i) = \delta_j^i, \quad X_j^{\mathrm{C}}(\theta^I) = -\Lambda_j^I, \quad X_j^{\mathrm{C}}(v^i) = 0, \quad X_j^{\mathrm{C}}(\hat{\mathrm{v}}^a) = -\hat{B}_{jk}^a v^k,$$

where $[X_i, X_j] = \hat{B}_{ij}^a \hat{E}_a$ (by construction, we have $\hat{B}_{ij}^a L_a^b = B_{ij}^a K_a^b$, or equivalently $B_{ij}^a = A_b^a \hat{B}_{ij}^b$). Therefore the reduction of X_j^{C} to a vector field on TQ/G_μ is

$$\frac{\partial}{\partial x^j} - \Lambda_j^I \frac{\partial}{\partial\theta^I} - \hat{B}_{jk}^a v^k \frac{\partial}{\partial\hat{\mathrm{v}}^a}.$$

Summarizing, we have proved the following:

Proposition 4.1. *A curve $(x^i(t), \theta^I(t), v^i(t), \hat{\mathrm{v}}^a(t), p_i(t))$ in M_μ/G_μ is a solution of the* ***reduced implicit Lagrange-Routh equations*** *if it satisfies:*

$$\begin{aligned} &\dot{x}^i = v^i, \qquad \dot{\theta}^I = \hat{\mathrm{v}}^J L_J^I - \dot{x}^i \Lambda_i^I, \qquad \dot{p}_i = \frac{\partial R^\mu}{\partial x^i} - \Lambda_i^I \frac{\partial R^\mu}{\partial\theta^I} - \mu_a B_{ij}^a \dot{x}^j, \\ &p_i = \frac{\partial R^\mu}{\partial v^i}, \qquad \frac{\partial R^\mu}{\partial\hat{\mathrm{v}}^a} = 0. \end{aligned} \tag{4.1}$$

None of the above equations depends explicitly on θ^A. Therefore, the above equations determine the reduced curve on M_μ/G_μ.

Once we have solved these reduced equations for $(x^i(t), \theta^I(t), v^i(t), \hat{\mathrm{v}}^a(t), p_i(t))$ on M_μ/G_μ, we can recover the complete solution $(x^i(t), \theta^I(t), \theta^A(t), v^i(t), \hat{\mathrm{v}}^a(t), p_i(t), \tilde{\mathrm{p}}_a(t))$ on M by solving the reconstruction equations

$$\dot{\theta}^A = \hat{\mathrm{v}}^a L_a^A - \dot{x}^i \Lambda_i^A, \quad \tilde{\mathrm{p}}_a = \mu_a.$$

The first equation can only be given a geometrically concise interpretation if we make use of a principal connection on the principal bundle $TQ \to TQ/G_\mu$, but we will not go into the details of this procedure (we refer the interested reader to Crampin and Mestdag [2008] for a similar situation in the case of standard Routh reduction).

5 Special cases

In this section, we shall obtain the implicit Lagrange-Routh equations in some particular cases and relate the resultant equations, when possible, to the results derived elsewhere in the literature on Routh reduction.

The regular case. Let us consider the case where the Lagrangian is regular with respect to the group variables. More precisely, a Lagrangian is said to be *G-regular* if the Hessian $(\tilde{E}_a^{\mathrm{V}}(\tilde{E}_b^{\mathrm{V}}(L)))$ is non-singular everywhere on TQ, which in coordinates can be expressed as

$$\det\left(\frac{\partial^2 L}{\partial\theta^a \partial\theta^b}\right) \neq 0.$$

G-regularity is probably the weakest condition that the Lagrangian should satisfy to have an analogy with the classical procedure of Routh. In this case, one of the implicit Euler-Lagrange equations (3.4), namely the relation $\hat{E}_b^{\mathrm{V}}(R^\mu) = 0$ (or, equivalently, $\tilde{E}_a^{\mathrm{V}}(L) = \mu_a$) can locally be rewritten in either one of the following explicit forms $\tilde{\mathrm{v}}^a = \tilde{\iota}_\mu^a(q, v^i)$ or $\hat{\mathrm{v}}^a = \hat{\iota}_\mu^a(q, v^i)$, where $\tilde{\iota}_\mu^a, \hat{\iota}_\mu^a$ are smooth functions of (q, v^i). This defines a submanifold N_μ of TQ, with inclusion ι_μ. We can introduce the new function $\bar{R}^\mu = R^\mu \circ \iota_\mu$ on N_μ. This is the function which is commonly called the ***Routhian*** (see, e.g., the papers Crampin and Mestdag [2008]; Langerock, Cantrijn and Vankerschaver [2010]; Marsden, Ratiu and Scheurle [2000]). It is easy to see that, for its reduced version on N_μ/G_μ, we obtain

$$\frac{\partial \bar{R}^\mu}{\partial x^i} = \left(\frac{\partial R^\mu}{\partial x^i} \circ \iota_\mu\right) + \left(\frac{\partial R^\mu}{\partial \hat{\mathrm{v}}^a} \circ \iota_\mu\right)\frac{\partial \hat{\iota}^a_\mu}{\partial x^i}.$$

In view of the fact that $\partial R^\mu/\partial \hat{\mathrm{v}}^a = 0$ is part of the reduced implicit Lagrange-Routh equations (4.1), every instance of $\partial R^\mu/\partial x^i$ in (4.1) can be replaced by $\partial \bar{R}^\mu/\partial x^i$, and similarly for $\partial \bar{R}^\mu/\partial v^i$. The remaining reduced equations are then simply given by

$$\dot{x}^i = v^i, \quad \dot{\theta}^I = \tilde{\iota}^J_\mu(q, v^i) K^I_J - \dot{x}^i \Lambda^I_i, \quad \frac{d}{dt}\left(\frac{\partial \bar{R}^\mu}{\partial v^i}\right) = \frac{\partial \bar{R}^\mu}{\partial x^i} - \Lambda^I_i \frac{\partial \bar{R}^\mu}{\partial \theta^I} - \mu_a B^a_{ij}\dot{x}^j. \tag{5.1}$$

The above equations can be found in Crampin and Mestdag [2008].

An even stronger regularity condition is the one used in Langerock, Cantrijn and Vankerschaver [2010]. Define for each $v_q \in T_qQ$ a map $\mathfrak{J}^{v_q}_L : \mathfrak{g} \to \mathfrak{g}^*$ as follows:

$$\mathfrak{J}^{v_q}_L : \mathfrak{g} \to \mathfrak{g}^* \,;\; \xi \mapsto J_L\left(v_q + \tilde{\xi}(q)\right).$$

In the above, $J_L : TQ \to \mathfrak{g}^*$ is a standard momentum map on TQ defined by $J_L = J \circ \mathbb{F}L$, where $J : T^*Q \to \mathfrak{g}^*$ is the standard momentum map. If we assume that $\mathfrak{J}^{v_q}_L$ is a diffeomorphism for every $v_q \in TQ$, it has been shown in Langerock, Cantrijn and Vankerschaver [2010] that it is possible to realize the previous equations as the symplectic reduction of the original Lagrangian system. Simple mechanical systems, for which the Lagrangian is of the form $L = T - V$ with T given by a Riemannian metric on Q, satisfy automatically this stronger form of G-regularity (this follows easily from positive-definiteness of the metric). For a detailed study of the Routh reduction for simple mechanical systems, see Marsden, Ratiu and Scheurle [2000].

The Abelian case. When the group of symmetries is Abelian we have $G_\mu = G$. In this case, there are no coordinates θ^I and we can just write θ^a everywhere. The equations for the curve $(x^i(t), v^i(t), \hat{\mathrm{v}}^a(t), p_i(t))$ become

$$\dot{x}^i = v^i, \quad \dot{p}_i = \frac{\partial R^\mu}{\partial x^i} - \mu_a B^a_{ij}\dot{x}^j, \quad p_i = \frac{\partial R^\mu}{\partial v^i}, \quad \frac{\partial R^\mu}{\partial \hat{\mathrm{v}}^a} = 0.$$

Note that even when the group is Abelian, there remains a curvature term in the equations. From the reconstruction equations

$$\dot{\theta}^a = \hat{\mathrm{v}}^b L^a_b - \dot{x}^i \Lambda^a_i, \quad \tilde{\mathrm{p}}_a = \mu_a,$$

we can determine $\theta^a(t)$ and $\tilde{\mathrm{p}}_a(t)$. In the case where L is G-regular, the manifold N_μ can be identified with $T(Q/G)$ and the reduced equations (5.1) can be regarded as the Euler-Lagrange equations of the Routhian $\bar{R}^\mu$ on $T(Q/G)$ subjected to a *gyroscopic term* arising from the curvature of Λ.

An important case of an Abelian symmetry group is that when the Lagrangian has *cyclic coordinates*. In this case we have a configuration manifold that is a product $Q = S \times G$, and the Lagrangian is assumed to be invariant under the action of G on the second factor. The equations for $(x^i(t), v^i(t), \hat{\mathrm{v}}^a(t), p_i(t))$ become then

$$\dot{x}^i = v^i, \quad \dot{p}_i = \frac{\partial R^\mu}{\partial x^i}, \quad p_i = \frac{\partial R^\mu}{\partial v^i}, \quad \frac{\partial R^\mu}{\partial \hat{\mathrm{v}}^a} = 0.$$

Again, if we have G-regularity, these might be further simplified to yield the Euler-Lagrange equations of $\bar{R}^\mu$ (with no gyroscopic term).

6 Routh-Dirac reduction

In this section, we shall show how the reduced implicit Lagrange-Routh equations can be obtained as a reduced Lagrange-Dirac dynamical system. We will call this type of reduction ***Routh-Dirac reduction***. The procedure we follow relies on some results known for implicit Hamiltonian systems. Again, we assume the same setting as before: a free and proper action of a Lie group G on Q, which leaves the Lagrangian L invariant.

Dirac dynamical system. We refer to Courant [1990]; Dorfman [1993] for the original works on the notion of a Dirac structure and to Dalsmo and van der Schaft [1999]; Yoshimura and Marsden [2006a]; Cendra, Ratiu and Yoshimura [2015] for more details on Dirac structures and the dynamics associated to them. A linear Dirac structure on a vector space V is a subspace D_V of $V \oplus V^*$ which is a Lagrangian subspace with respect to the pairing

$$\langle\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle\rangle = \langle \alpha_2, v_1 \rangle + \langle \alpha_1, v_2 \rangle.$$

A Dirac structure on a manifold M is a subbundle $D_M \subset TM \oplus T^*M$ such that $D_M(m) \subset T_mM \times T_m^*M$ is a linear Dirac structure on T_mM at each $m \in M$. In what follows, we will use the terminology *Dirac dynamical system*, as in Cendra, Ratiu and Yoshimura [2015], to refer to a wide class of implicit Lagrangian or Hamiltonian systems that can be defined in the context of Dirac structures. Given an energy form $\varphi \in \Gamma(T^*M)$, the dynamics of the ***Dirac dynamical system*** (φ, D_M) is given by the following condition on a curve $c : I \subset \mathbb{R} \to M$:

$$\dot{c}(t) \oplus \varphi\,(c(t)) \in D_M(c(t)).$$

Implicit Euler-Lagrange equations. Using the above, we can obtain the implicit Euler-Lagrange equations, by taking M to be the Pontryagin bundle $M = TQ \oplus T^*Q$ over Q. In order to construct D_M we consider the pullback $\Omega_M = \pi_{T^*Q}^*\Omega_{T^*Q}$ of the canonical symplectic form $\Omega_{T^*Q} = -\mathbf{d}\theta_{T^*Q}$ on T^*Q to M by using the canonical projection $\pi_{T^*Q} : M \to T^*Q$. Then, Ω_M is a presymplectic form on M, which naturally defines a Dirac structure D_M on M by means of the graph of Ω_M: for each $(q, v, p) \in M$, set

$$D_M(q, v, p) = \{((\delta q, \delta v, \delta p), (\alpha, \beta, \gamma)) \in TM \oplus T^*M \mid \alpha + \delta p = 0,\, \gamma = \delta q,\, \beta = 0\}.$$

The *generalized energy function* $\mathcal{E}_L(q, v, p) = \langle p, v \rangle - L(q, v)$ on M defines the energy form $\mathbf{d}\mathcal{E}_L$. When we express the two-form Ω_{T^*Q} in natural coordinates as $dq^\alpha \wedge dp_\alpha$ it is easy to check that a curve $c(t) = (q(t), v(t), p(t))$ in M is a solution of the implicit Euler-Lagrange equations (2.1) if $(\mathbf{d}\mathcal{E}_L, D_M)$ satisfies the so-called Lagrange-Dirac dynamical system

$$\dot{c}(t) \oplus \mathbf{d}\mathcal{E}_L(c(t)) \in D_M(c(t)), \quad \text{for all } t. \tag{6.1}$$

On the other hand, when we write the generalized energy in quasi-velocities and quasi-momenta as $\mathcal{E}_L = \mathrm{p}_\alpha \mathrm{v}^\alpha - L$ and $\mathbf{d}\mathcal{E}_L = \mathrm{v}^\alpha d\mathrm{p}_\alpha + \mathrm{p}_\alpha d\mathrm{v}^\alpha - \mathbf{d}L$, and likewise the two-form Ω_{T^*Q} as

$$\Omega_{T^*Q} = W^\alpha \wedge d\mathrm{p}_\alpha + \frac{1}{2} R^\alpha_{\beta\gamma} \mathrm{p}_\alpha W^\beta \wedge W^\gamma \tag{6.2}$$

we may easily obtain from (6.1) the implicit Euler-Lagrange equations with quasi-velocities and quasi-momenta (2.6). Note that W^α has been introduced before as a 1-form on Q, while in the above we think of W^α as a (semi-basic) 1-form on T^*Q. We will use this slight abuse of notation from now on.

Reduction of Dirac structures. In this paragraph we assume that M is an arbitrary manifold, endowed with a Dirac structure D_M. We first recall some generalities on the reduced Dirac structure induced by a Lie group action. The results here were originally developed in van der Schaft [1998]; Blankenstein and van der Schaft [2001].

Assume that a Lie group G acts (freely and properly) on the manifold M, and denote by $\phi_g(m) = g \cdot m$ the action of $g \in G$ on a point $m \in M$. This action lifts naturally by tangent and cotangent lifts to $TM \oplus T^*M$. Assume also that the action admits an equivariant momentum map relative to D_M, that is, assume that there exists a G-equivariant map $J^M : M \to \mathfrak{g}^*$ such that $\tilde{\xi} \oplus \mathbf{d}J^M_\xi \in D_M$, for all $\xi \in \mathfrak{g}$, where we recall that $\tilde{\xi}$ stands for the infinitesimal generator of the action associated with $\xi \in \mathfrak{g}$ and J^M_ξ is the smooth function on M defined by $J^M_\xi(m) = \left\langle J^M(m), \xi \right\rangle,\ m \in M$.

We assume now that D is invariant under G. The Dirac reduction procedure will be carried out in the following two steps. First, if $\mu \in \mathfrak{g}^*$ is a regular value of J^M, then $M_\mu = (J^M)^{-1}(\mu) \subset M$ is a submanifold. If the vector subspace $D_M(m) \cap (T_m M_\mu \times T^*_m M|_{M_\mu}) \subset T_m M_\mu \times T^*_m M|_{M_\mu}$ has constant dimension at each $m \in M_\mu$, then these vector spaces naturally induces a ***restriction of the Dirac structure*** $D_{M_\mu} \subset TM_\mu \oplus T^*M_\mu$. Second, one observes that the Dirac structure D_{M_μ} is G_μ-invariant since

$$D_{M_\mu}(g \cdot m) = g \cdot D_{M_\mu}(m),$$

where $G_\mu = \{g \in G \mid \mathrm{Ad}^*_g \mu = \mu\}$ is the coadjoint isotropy subgroup of μ. This leads to a ***reduced Dirac structure*** $D_{M_\mu/G_\mu} \subset T(M_\mu/G_\mu) \oplus T^*(M_\mu/G_\mu)$ on the reduced space $M_\mu/G_\mu = J^{-1}(\mu)/G_\mu$, which is given by

$$\begin{aligned} D_{M_\mu/G_\mu} := \{(X, \alpha) \in \mathfrak{X}(M_\mu/G_\mu) \times \Omega^1(M_\mu/G_\mu) \mid \exists\, (Y, \beta) \in D_M, \\ \text{such that } T\pi_\mu \circ Y = X \circ \pi_\mu,\ \pi^*_\mu \alpha = \beta\}, \end{aligned} \tag{6.3}$$

where $\pi_\mu : M_\mu \to M_\mu/G_\mu$ is the canonical projection, which is a surjective submersion.

Symmetry reduction of implicit Hamiltonian systems. An important case of Dirac dynamical systems is that when the energy section is given by the differential of a Hamiltonian function H on M. Let H be a G-invariant Hamiltonian on M and let $c(t)$ be a solution curve for the ***implicit Hamiltonian system*** $(\mathbf{d}H, D_M)$, i.e. a curve that satisfies

$$\dot{c}(t) \oplus \mathbf{d}H(c(t)) \in D_M(c(t)).$$

Then,

$$\frac{dJ^M_\xi}{dt}(c(t)) = \left\langle \mathbf{d}J^M_\xi(c(t)), \dot{c}(t) \right\rangle = -\left\langle \mathbf{d}H, \tilde{\xi} \right\rangle(c(t)) = 0, \ \text{ for all } t \text{ and } \xi \in \mathfrak{g}.$$

So, J^M_ξ is a *first integral* of the implicit Hamiltonian system and we can restrict the Dirac dynamical system $(\mathbf{d}H, D_M)$ to $(\mathbf{d}H_\mu, D_{M_\mu})$, where $H_\mu = H|_{M_\mu}$. Next, we can reduce the restricted implicit Hamiltonian system $(\mathbf{d}H_\mu, D_{M_\mu})$ to obtain the ***reduced implicit Hamiltonian system*** $(\mathbf{d}\mathfrak{H}_\mu, D_{M_\mu/G_\mu})$, which satisfies

$$\dot{\mathfrak{c}}(t) \oplus \mathbf{d}\mathfrak{H}_\mu(\mathfrak{c}(t))) \in D_{M_\mu/G_\mu}(\mathfrak{c}(t), \tag{6.4}$$

for each $\mathfrak{c}(t) = \pi_\mu(c(t))$ in M_μ/G_μ, where $\mathfrak{H}_\mu \circ \pi = H_\mu$ is the reduced Hamiltonian on M_μ/G_μ.

More details on this reduction of Dirac structures and its associated reduced implicit Hamiltonian systems can also be found in Blankenstein and Ratiu [2004].

The reduced implicit Lagrange-Routh equations. We consider again the Dirac structure D_M on the Pontryagin bundle $M = TQ \oplus T^*Q$ given by the graph of the presymplectic form $\Omega_M = \pi^*_{T^*Q}\Omega_{T^*Q}$. For $J^M : M \to \mathfrak{g}^*$, we take $J^M = J \circ \pi_{T^*Q}$, where $J : T^*Q \to \mathfrak{g}^*$ stands for the standard momentum map on Q. The goal of this paragraph is to demonstrate that, in this particular setting, the reduced implicit Hamiltonian system (6.4), with Hamiltonian $H = \mathcal{E}_L$, is nothing but the system given by the reduced implicit Lagrange-Routh equations (4.1).

It is well-known that if D_M is a Dirac structure given by the graph of a symplectic form Ω_M the reduced Dirac structure D_{M_μ/G_μ} may be given by the graph of a reduced symplectic form Ω_{M_μ/G_μ}. The following observations may be obtained:

i) The action of G on $M = TQ \oplus T^*Q$ restricts to a G_μ-action on M_μ by tangent and cotangent lifts, and this action leaves the presymplectic form Ω_{M_μ} invariant. Indeed, since $\Omega_{M_\mu} = \imath^*\Omega_M$ (where $\imath : M_\mu \to M$ is the inclusion), its invariance follows directly from the G_μ-invariance of Ω_M.

ii) Moreover, one can show that the form Ω_{M_μ} drops to M_μ/G_μ. Indeed, it suffices to check that Ω_{M_μ} annihilates vectors which are vertical to the fibration $\pi_\mu : M_\mu \to M_\mu/G_\mu$. Thus the *reduced presymplectic form* Ω_{M_μ/G_μ} is defined on M_μ/G_μ from the invariance of Ω_{M_μ}.

iii) It follows immediately that D_{M_μ} is G_μ-invariant. In particular, we can define a ***reduced Dirac structure*** D_{M_μ/G_μ} on M_μ/G_μ,

$$D_{M_\mu/G_\mu} \subset T(M_\mu/G_\mu) \oplus T^*(M_\mu/G_\mu)$$

such that

$$\pi_\mu^* D_{M_\mu/G_\mu} = \imath^* D_M.$$

Note that the above characterization of D_{M_μ/G_μ} can also be obtained in the standard cases of symplectic and presymplectic reduction by the well-known characterization of the reduced (pre)symplectic form, to be found in Marsden and Weinstein [1974] and Echeverría-Enríquez, Muñoz-Lecanda and Román-Roy [1999]). We plan to investigate whether the same result may also hold for almost Dirac structures with regular distributions.

To write down the reduced system of the implicit Hamilton system (6.1), we first need to give a coordinate expression of the two 2-forms Ω_{M_μ} and Ω_{M_μ/G_μ} whose graph define the Dirac structures D_{M_μ} and D_{M_μ/G_μ}.

Using a principal connection A on the bundle $\pi : Q \to Q/G$, one identifies $J^{-1}(\mu) \simeq T^*(Q/G) \times_{Q/G} Q$. Under this identification the presymplectic form reads

$$\Omega_{M_\mu} = \Omega_{Q/G} - \mathbf{d}A_\mu,$$

where we have used a slight abuse of notations and omitted the pullbacks from the spaces $T^*(Q/G)$ and Q where the forms $\Omega_{Q/G}$ and $\mathbf{d}A_\mu$ are defined, respectively. From Cartan's structure equation, it follows $\mathbf{d}A_\mu = \langle \mu, [A, A] \rangle - B_\mu$. Therefore, in coordinates, we have

$$\Omega_{M_\mu} = dx^i \wedge dp_i + \frac{1}{2}\mu_a \left(B^a_{ij} dx^i \wedge dx^j - C^a_{bc} \tilde{E}^b \wedge \tilde{E}^c \right), \tag{6.5}$$

where $\{dx^i, \tilde{E}^a\}$ stand for the dual of the basis $\{X_i, \tilde{E}_a\}$ and where we have identified $X^i = dx^i$ in the notations of the previous sections. We point out that, again, we do not write explicitly the pullbacks: one should think of the forms $\tilde{E}^a$ in (6.5) as semi-basic forms on T^*Q or, equivalently, as the vertical lifts $(\tilde{E}^a)^{\mathrm{V}}$ to T^*Q of the corresponding forms in Q (see Yano and Ishihara [1973] for more details). To ease the notation we will keep this convention from now on since the coordinate expressions agree, unless there is risk of confusion. The expression (6.5) is a particular instance of the expression (6.2) in the current frame.

Let R^μ be the generalized Routhian given in (3.1) and consider the energy section determined by the differential of the energy $\mathcal{E}_L$ restricted to M_μ. Writing $\mathcal{E}_\mu = \imath_\mu^*\mathcal{E}_L$ in terms of the Routhian as $\mathcal{E}_\mu = p_i v^i - R^\mu$, and using a formula similar to the one we had for calculating variations of L in terms of the anholonomic frame (see (2.5)), it follows

$$\begin{aligned}\mathbf{d}\mathcal{E}_\mu = &-\Big(X_i^{\mathrm{C}}(R^\mu) + \tilde{E}_a^{\mathrm{V}}(R^\mu)B_{ij}^a v^j\Big)\,dx^i - \big(X_i^{\mathrm{V}}(R^\mu) - p_i\big)\,dv^i \\ &+ \Big(\mu_a + \tilde{E}_a^{\mathrm{V}}(R^\mu)\Big)\,C_{bc}^a \tilde{\mathrm{v}}^c \tilde{E}^b - \tilde{E}_a^{\mathrm{V}}(R^\mu) d\tilde{\mathrm{v}}^a + v^i dp_i,\end{aligned} \tag{6.6}$$

where we have used the relation $\tilde{E}_a^{\mathrm{C}}(R^\mu) = -\mu_c C_{ab}^c \tilde{\mathrm{v}}^b$ from (3.2).

Recall that $D_{M_\mu} \subset TM_\mu \oplus T^*M_\mu$ is the Dirac structure induced from Ω_{M_μ}. Before we continue with the expression for Ω_{M_μ/G_μ}, it is instructive to have a look at the first step in the reduction, namely the restriction of the dynamics to M_μ. Consider for that reason the Lagrange-Dirac dynamical system $(\mathbf{d}\mathcal{E}_\mu, D_{M_\mu})$, which satisfies, for each $c(t) = (x^i(t), \theta^a(t), v^i(t), \tilde{\mathrm{v}}^a(t), p_i(t))$ in M_μ,

$$\dot{c}(t) \oplus \mathbf{d}\mathcal{E}_\mu\,(c(t)) \in \big(D_{M_\mu}\big)_{(c(t))}\,.$$

It leads to the following set of equations:

$$\begin{aligned}&\dot{x}^i = v^i, \qquad \tilde{E}_a^{\mathrm{V}}(R^\mu) = 0, \qquad X_i^{\mathrm{V}}(R^\mu) - p_i = 0,\\ &\mu_a B_{ij}^a \dot{x}^j + \dot{p}_i = X_i^{\mathrm{C}}(R^\mu) + \tilde{E}_a^{\mathrm{V}}(R^\mu) B_{ij}^a v^j, \qquad \Big(\mu_a + \tilde{E}_a^{\mathrm{V}}(R^\mu)\Big)\,C_{bc}^a \tilde{\mathrm{v}}^c = \mu_a C_{bc}^a \tilde{\mathrm{u}}^c,\end{aligned}$$

where $\tilde{\mathrm{u}}^b = (K^{-1})_a^b(\dot{\theta}^a + \dot{x}^i \Lambda_i^a)$. The above equations might be further simplified as

$$\begin{aligned}&\mu_a C_{bc}^a \tilde{\mathrm{v}}^c = \mu_a C_{bc}^a \tilde{\mathrm{u}}^c, \qquad \tilde{E}_a^{\mathrm{V}}(R^\mu) = 0,\\ &v^i = \dot{x}^i, \qquad X_i^{\mathrm{V}}(R^\mu) = p_i, \qquad \dot{p}_i = X_i^{\mathrm{C}}(R^\mu) - \mu_a B_{ij}^a v^j.\end{aligned} \tag{6.7}$$

The similarity with the implicit Lagrange-Routh equations (3.3) is obvious, with the only difference that it is not possible to conclude from equations (6.7) that $\tilde{\mathrm{v}}^c = \tilde{\mathrm{u}}^c$. This is a consequence of the fact that solutions of a presymplectic equation are only determined up to elements in the kernel of the presymplectic form (see Gotay and Nester [1979]). Indeed, considering the splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus (\mathfrak{g}/\mathfrak{g}_\mu)$ introduced earlier in §4, the relation $\mu_a C_{bc}^a(\tilde{\mathrm{v}}^c - \tilde{\mathrm{u}}^c) = 0$ implies that $\tilde{\mathrm{v}}^I = \tilde{\mathrm{u}}^I$, but it is not true in general that also $\tilde{\mathrm{v}}^A = \tilde{\mathrm{u}}^A$. This is reminiscent of the standard case of symplectic reduction on T^*Q; if $\imath_\mu : J^{-1}(\mu) \to T^*Q$ denotes the inclusion, then the kernel of the presymplectic form $\Omega_\mu = \imath_\mu^*\Omega_Q$ is given by $\ker \Omega_\mu = \{\tilde{E} \mid E \in \mathfrak{g}_\mu\}$.

The equations (6.7) only refer to the restriction step in the reduction process of an implicit Hamilton system. As explained before, in the second step, we need to reduce that system by G_μ. This boils down to looking at the Dirac structure defined by the graph of Ω_{M_μ/G_μ}.

Proposition 6.1. *Let $\mathfrak{E}_\mu$ be the reduced energy function on M_μ/G_μ defined by $\mathfrak{E}_\mu \circ \pi_\mu = \mathcal{E}_\mu$. Given the Dirac structure D_{M_μ/G_μ}, a curve $\mathfrak{c}(t) = \pi_\mu(c(t))$ in M_μ/G_μ is a solution curve of the* ***Routh-Dirac dynamical system*** *$(\mathbf{d}\mathfrak{E}_\mu, D_{M_\mu/G_\mu})$, which satisfies*

$$\dot{\mathfrak{c}}(t) \oplus \mathbf{d}\mathfrak{E}_\mu\,(\mathfrak{c}(t)) \in \big(D_{M_\mu/G_\mu}\big)\,(\mathfrak{c}(t))\,,$$

if and only if, the reduced implicit Lagrange-Routh equations (4.1) hold.

Proof. Using the splitting $(E_a) = (E_I, E_A)$ of $\mathfrak{g}$, we write the presymplectic form Ω_{M_μ} in M_μ as

$$\Omega_{M_\mu} = dx^i \wedge dp_i + \frac{1}{2}\mu_a \left(B^a_{ij} dx^i \wedge dx^j - C^a_{IJ} \tilde{E}^I \wedge \tilde{E}^J \right).$$

Recall that, with the notations in §3, we have $\hat{E}_a = A^b_a \tilde{E}_b$. Then we have, for duals, $\tilde{E}^I = A^I_J \hat{E}^J$. With this we can rewrite the previous expression of Ω_{M_μ} in terms of the forms $\hat{E}^I$:

$$\Omega_{M_\mu} = dx^i \wedge dp_i + \frac{1}{2}\mu_a \left(B^a_{ij} dx^i \wedge dx^j - A^I_K A^J_L C^a_{IJ} \hat{E}^K \wedge \hat{E}^L \right).$$

In this frame, the reduced two-form Ω_{M_μ/G_μ} has formally the same expression.

The proof will follow from the expression (6.6) for $\mathbf{d}\mathcal{E}_\mu$ in invariant (hat) coordinates, and from its reduction to $\mathbf{d}\mathfrak{E}_\mu$. This computation is completely analogous to the one carried out before in §4. If we take into account that $B^a_{ij} = A^a_b \hat{B}^b_{ij}$, $\tilde{\mathrm{v}}^I = A^I_J \hat{\mathrm{v}}^J$, etc., it follows that

$$\begin{aligned}\mathbf{d}\mathcal{E}_\mu = -\left(\frac{\partial R^\mu}{\partial x^i} - \Lambda^I_i \frac{\partial R^\mu}{\partial \theta^I} \right) dx^i - \left(\frac{\partial R^\mu}{\partial v^i} - p_i \right) dv^i - \mu_a A^I_K A^J_L C^a_{cd} \hat{\mathrm{v}}^L \hat{E}^K \\ - \frac{\partial R^\mu}{\partial \hat{\mathrm{v}}^b} C^a_{cd} (A^{-1})^b_a A^c_K A^d_L \hat{\mathrm{v}}^L \hat{E}^K - \frac{\partial R^\mu}{\partial \hat{\mathrm{v}}^a} d\hat{\mathrm{v}}^a + v^i dp_i.\end{aligned}$$

Using the relations for the structure constants derived in §4, it is then straightforward to check that a curve

$$\mathfrak{c}(t) = (x^i(t), \theta^I(t), v^i(t), \hat{\mathrm{v}}^a(t), p_i(t)) : I \subset \mathbb{R} \to M_\mu / G_\mu,$$

satisfies (6.4), if and only if the implicit Lagrange-Routh equations (4.1) hold. One of the equations will appear rather as $\mu_a C^a_{IJ}(\hat{\mathrm{v}}^I - \hat{\mathrm{u}}^I) = 0$, but, as we mentioned before, this implies $\hat{\mathrm{v}}^I = \hat{\mathrm{u}}^I$. ■

Remarks. We will call the reduction process that we have just described ***Routh-Dirac reduction***. It is a particular instance of a reduced Dirac dynamical system. One can see a similar discussion on the Dirac reduction associated with the symplectic reduction in the more general context of Dirac anchored vector bundle reduction in Cendra, Ratiu and Yoshimura [2015].

It should be remarked that the above Dirac structure D_{M_μ/G_μ} is different from the one used in Yoshimura and Marsden [2007, 2009], where the Dirac structure is rather defined as a subbundle of the vector bundle $(TM \oplus T^*M)/G$. This last definition may be advantageous when one wants to give a geometric interpretation of the so-called implicit Lagrange-Poincaré or Hamilton-Poincaré equations in the variational link with Dirac structures. These two sets of equations are the result of a reduction process that takes the full symmetry group G of a mechanical system into account, but it does not produce *conserved quantities* μ. Our aim, in the previous sections, was a *Routh-type reduction*; so we wanted to take full advantage of the conserved quantities, at the price of reducing by a possibly smaller symmetry group G_μ.

7 Examples

To conclude, we will now discuss two illustrative examples where the regularity conditions on the Lagrangian fail.

Linear Lagrangians. A linear Lagrangian is probably the easiest case where L is not locally G-regular. In general, a linear Lagrangian L on TQ conveys to the form $L = \langle \alpha(q), v_q \rangle - f(q)$ for some 1-form α on Q and function f on Q. We say a few words about the two examples we mentioned in the Introduction.

The Lagrangian

$$L(x, y, v_x, v_y) = (v_x)^2 + v_x v_y - V(x)$$

of the first example has a cyclic coordinate y. Fix a value $\mu \in \mathbb{R}$ for the momentum p_y. The Routhian, constructed with respect to the trivial connection on $\mathbb{R}^2 \to \mathbb{R}$, reads

$$R^\mu(x, y, v_x, v_y) = (v_x)^2 + v_x v_y - \mu v_y - V(x),$$

and the reduced implicit Routh equations are then

$$\dot{x} = v_x, \qquad \dot{p}_x = \frac{\partial R^\mu}{\partial x} = -V'(x), \qquad p_x = 2v_x + v_y, \qquad v_x - \mu = 0.$$

Together with the reconstructions equations $\dot{y} = v_y$ and $p_y = \mu$, we get a solution of the original implicit Euler-Lagrange equations for L with the prescribed value of the momentum p_y.

Consider again the Lagrangian for the dynamics of point vortices in the plane from Chapman [1978]:

$$L(z_l, \dot{z}_l) = \frac{1}{2i} \sum_k \gamma_k \left(\bar{z}_k \dot{z}_k - z_k \overline{\dot{z}_k} \right) - \frac{1}{2} \sum_n \sum_{k \neq n} \gamma_n \gamma_k \ln|z_n - z_k|.$$

Although it is S^1 invariant under rotations on $\mathbb{C}$, the coordinates are not adapted. A possible way to proceed is to take polar coordinates for each position $z_k(t) \in \mathbb{C}$, say $z_k = \rho_k e^{i\theta_k}$, and then to consider the relative angle with respect to θ_1. Defining $\phi_k = \theta_k - \theta_1$ for $k \geq 2$ and $\phi_1 = \theta_1$, we have an S^1-action along ϕ_1 with associated momentum $\partial \bar{L} / \partial \dot{\phi}_1$, where $\bar{L}$ is the Lagrangian in the new coordinates. A computation then shows that this conserved quantity is precisely the moment of circulation (also called angular impulse in Newton [2001])

$$I = \sum_k \gamma_k \rho_k^2 = \sum_k \gamma_k |z_k|^2.$$

From here the implicit Lagrange-Routh equations follow without further difficulty.

A degenerate Lagrangian. We will now discuss a mechanical model for field theories to be found in Capri and Kobayashi [1982, 1987], although we will follow the exposition in Echeverría-Enríquez, Muñoz-Lecanda and Román-Roy [1999], where it appears in the context of presymplectic reduction. The Lagrangian is

$$L = (\dot{\bar{\psi}})^i m_{ij} (\dot{\psi})^j + (\dot{\bar{\psi}})^i c_{ij} (\psi)^j - (\bar{\psi})^i \tilde{c}_{ij} (\dot{\psi})^j - (\bar{\psi})^i \tilde{r}_{ij} (\psi)^j,$$

where $\psi^i, \bar{\psi}^i$ represent the scalar complex fields (which are regarded as coordinates in the model), and where the matrices $m_{ij}, c_{ij}, \tilde{c}_{ij}, r_{ij}$ satisfy: m_{ij}, r_{ij} are hermitian and $\overline{(\tilde{c}_{ij})} = -c_{ji}$. This guarantees that L is real.

We will consider the same values for these matrices that appear in Echeverría-Enríquez, Muñoz-Lecanda and Román-Roy [1999]:

$$m_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \qquad c_{ij} = \tilde{c}_{ij} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad r_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and which lead to a degenerate Lagrangian. With that choice, it is apparent that L becomes invariant under the actions of S^1 by rotation in the second and third scalar fields. Thus, we have a $\mathbb{T}^2$-action on Q. Writing $\psi^k = x^k + iy^k$ and $\dot\psi^k = u^k + iv^k$ the Lagrangian becomes:

$$L = m_2\left((u^2)^2 + (v^2)^2\right) + m_3\left((u^3)^2 + (v^3)^2\right) + v^2x^2 + v^3x^3 - u^2y^2 - u^3y^3 \\ - \left((x^1)^2 + (y^1)^2\right) - \left((x^2)^2 + (y^2)^2\right) - \left((x^3)^2 + (y^3)^2\right).$$

In this case, coordinates adapted to the $\mathbb{T}^2$-action are simply the usual polar coordinates on both (x^2, y^2) and (x^3, y^3). We will denote these sets of polar coordinates by (r, θ) and (ρ, ϕ) respectively. Then, it follows

$$L = m_2\left(r^2v_\theta^2 + v_r^2\right) + m_3\left(\rho^2v_\theta^2 + v_\rho^2\right) + r^2v_\theta + \rho^2v_\phi - r^2 - \rho^2 - \left((x^1)^2 + (y^1)^2\right),$$

and the equations for p_θ and p_ϕ are of the form

$$p_\theta = 2m_2r^2v_\theta + r^2, \qquad p_\phi = 2m_3\rho^2v_\theta + \rho^2.$$

The Routhian reads, for a given choice $(p_\theta, p_\phi) = (\mu_\theta, \mu_\phi)$, and with the trivial connection, as follows:

$$R^\mu = \left(r^2v_\theta^2 + v_r^2\right) + m_3\left(\rho^2v_\theta^2 + v_\rho^2\right) + r^2v_\theta + \rho^2v_\phi - r^2 - \rho^2 - \left((x^1)^2 + (y^1)^2\right) - \mu_\theta v_\theta - \mu_\phi v_\phi.$$

The reduced implicit Lagrange-Routh equations in this case are

$$\begin{aligned} \dot x^1 &= u^1, & \dot p_{x^1} &= -2x^1, & p_{x^1} &= 0,\\ \dot y^1 &= v^1, & \dot p_{y^1} &= -2y^1, & p_{y^1} &= 0,\\ \dot r &= v_r, & \dot p_r &= 2rv_\theta^2 + 2rv_\theta, & p_r &= 2v_r,\\ \dot\rho &= v_\rho, & \dot p_\rho &= 2rv_\phi^2 + 2\rho v_\phi, & p_\rho &= 2v_\rho,\\ 2r^2v_\theta + r^2 &= \mu_\theta, & 2\rho^2v_\phi + \rho^2 &= \mu_\phi, \end{aligned}$$

and the reconstruction equations are given by

$$\dot\theta = v_\theta, \qquad \dot\phi = v_\phi, \qquad p_\theta = \mu_\theta, \qquad p_\phi = \mu_\phi.$$

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References

Blankenstein, G. and A. J. van der Schaft [2001], Symmetry and reduction in implicit generalized Hamiltonian systems, *Reports on Mathematical Physics*, **47**, 57–100.

Blankenstein, G. and T. S. Ratiu [2004], Singular reduction of implicit Hamiltonian systems, *Reports on Mathematical Physics*, **53**, 211–260.

Bloch, A. M., J. E. Marsden and D. V. Zenkov [2009], Quasi-velocities and symmetries in nonholonomic systems, *Dynamical Systems: An International Journal*, **24**, 187–222.

Capri, A. Z. and M. Kobayashi [1982], A mechanical model with constraints, *Journal of Mathematical Physics*, **23**, 736–741.

Capri, A. Z. and M. Kobayashi [1987] The first-rank tensor field coupled to an electromagnetic field, *Journal of Physics A: Mathematical and General*, **20**, 6101–6112.

Cendra, H., T. S. Ratiu and H. Yoshimura [2015], Dirac-Weinstein reduction, *Preprint.*

Chapman, D. M. F. [1978], Ideal vortex motion in two dimensions: Symmetries and conservation laws, *Journal of Mathematical Physics*, **19**, 1988–1992.

Courant, T. J. [1990], Dirac manifolds, *Transactions of the American Mathematical Society*, **319**, 631–661.

Courant, T. and A. Weinstein [1988], Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume 27 of *Travaux en Cours*, pages 39–49. Hermann, Paris.

Crampin, M. and T. Mestdag [2008], Routh's procedure for non-Abelian symmetry groups, *Journal of Mathematical Physics*, **49**, 032901 (28 pages).

Crampin, M. and T. Mestdag [2010], Anholonomic frames in constrained dynamics, *Dynamical Systems: An International Journal*, **25**, 159–187.

Dalsmo, M and A.J. van der Schaft [1999], On representations and integrability of mathematical structures in energy-conserving physical systems, *S*IAM J. Control and Optimization, **37**, 54–91.

I. Dorfman, I [1993], Dirac Structures and Integrability of Nonlinear Evolution Equations. Chichester:John Wiley.

Echeverría-Enríquez, A., M. C. Muñoz-Lecanda and N. Román-Roy [1999], Reduction of presymplectic manifolds with symmetry, *Reviews on Mathematical Physics*, **11**, 1209–1247.

Gotay, M. J. and J. M. Nester [1979], Presymplectic Lagrangian systems. I. The constraint algorithm and the equivalence theorem, *Annales de I'Institut Henri Poincaré, Section A (N.S.)* **30**, 129–142.

Langerock, B., F. Cantrijn and J. Vankerschaver [2010], Routhian reduction for quasi-invariant Lagrangians, *Journal of Mathematical Physics*, **51**, 022902 (20 pages).

Langerock, B. and M. Castrillón [2010], Routh reduction for singular Lagrangians, *International Journal of Geometric Methods in Modern Physics*, **7**, 1451–1489.

Littlejohn, R. G. [1983], Variational principles of guiding centre motion, *Journal of Plasma Physics*, **29**, 111–125.

Marsden, J. E., T. S. Ratiu and J. Scheurle, Reduction theory and the Lagrange-Routh equations, *Journal of Mathematical Physics*, **41**, 3379–3429.

Marsden, J. E. and A. Weinstein[1974], Reduction of symplectic manifolds with symmetry, *Reports on Mathematical Physics*, **5**, 121–130.

Marsden, J. E. and J. Scheurle [1993], The reduced Euler-Lagrange equations, *Fields Inst. Comm.,* **1**, 139–164.

Marsden, J.E. [1992], *Lectures on Mechanics*, London Mathematical Society Lecture Note Series, Cambridge University Press.

Newton, P. K.[2001], *The N-Vortex Problem: Analytical Techniques*, Applied Mathematical Sciences, **145**, Springer.

Routh, E. J. [1877], *A Treatise on the Stability of a Given State of Motion: Particularly Steady Motion*, Macmillan and Company.

Routh, E. J. [1884], *A Treatise on the Dynamics of a System of Rigid Bodies. With Numerous Examples: The advanced part*, Macmillan and Company.

Schouten, J. A. [1954], *Ricci-Calculus*, Springer (first published 1923).

van der Schaft, A.J. [1998], Implicit Hamiltonian systems with symmetry, *Reports on Mathematical Physics*, **41**, 203–221.

van der Schaft, A.J. and D. Jeltsema[2014], Port-Hamiltonian Systems Theory: An Introductory Overview, *F*oundations and Trends in Systems and Control **1**, 173–378.

Yano, K. and S. Ishihara [1973], *Tangent and cotangent bundles: differential geometry*, Marcel Dekker, Inc.

Yoshimura, H. and J. E. Marsden [2006a], Dirac structures in Lagrangian mechanics. Part I: Implicit Lagrangian systems, *Journal of Geometry and Physics,* **57**, 133–156.

Yoshimura, H. and J. E. Marsden [2006b], Dirac structures in Lagrangian mechanics. Part II: Variational structures, *Journal of Geometry and Physics,* **57**, 209–250.

Yoshimura, H. and J. E. Marsden [2007], Reduction of Dirac structures and the Hamilton-Pontryagin principle, *Reports on Mathematical Physics,* **60**, 381–426.

Yoshimura, H. and J. E. Marsden [2009], Dirac cotangent bundle reduction, *Journal of Geometric Mechanics*, **1**(1), 87–158.