SATURATION AND Σ_2-TRANSFER FOR ERNA

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Abstract. Elementary Recursive Nonstandard Analysis, in short ERNA, is a constructive system of nonstandard analysis with a PRA consistency proof, proposed around 1995 by Patrick Suppes and Richard Sommer. It is built on a previous system by Rolando Chuaqui and Patrick Suppes, which was recently reconsidered by Michal Rössler and Emil Jeřábek. A Π_1-transfer principle has already been added to ERNA and the consistency of the resulting theory proved in PRA. Here, we equip ERNA with Σ_2-transfer and a saturation principle, while keeping the consistency proof inside PRA. We show that the extended theory allows for generalized transfer, a basic tool of nonstandard analysis, and interprets several strong theories, like $B\Sigma_2$ and $I\Sigma_2$.

The theory ERNA (short for Elementary Recursive Nonstandard Analysis) was introduced around 1995 by Patrick Suppes and Richard Sommer ([9] and [10]), who also proved its consistency inside PRA. ERNA's predecessor, developed by Rolando Chuaqui and Patrick Suppes ([3] and [12]), has recently been reconsidered in the systems $\text{NQA}^{\pm}$ of Michal Rössler and Emil Jeřábek ([7]). In [5], we added a Π_1-transfer principle to ERNA and provided a PRA proof for the consistency of the resulting theory ERNA + Π_1-TRANS. Among the results obtained in this theory is a Σ_1-supremum principle 'up-to-infinitesimals'. In this paper, we will further extend ERNA with Σ_2-transfer and a saturation principle, both being powerful tools of nonstandard analysis. The consistency proof will still remain inside PRA.

Though the two principles are very different in nature and scope, the proofs of their consistency with ERNA are very similar, both relying on the same compactness argument. Both could easily be added to $\text{NQA}^{\pm}$ with identical consistency proofs as below. For ERNA, we refer to [5], from which we adopt notations.

1. ERNA + Σ_2-Transfer

In nonstandard mathematics, Transfer expresses Leibniz's principle that the 'same' laws hold for standard and nonstandard objects alike. For Π_1-formulas, Leibniz's equivalence includes a trivial direction. Therefore, Π_1-transfer was given in [5] in the form of an implication

$$(\forall^{st} n)\varphi(n) \to (\forall n)\varphi(n),$$

and transfer for Σ_1-formulas followed by contraposition. For Σ_2-formulas, there is no trivial implication involved, and we have to state transfer in the form of a biconditional, as follows.

Σ_2-Transfer Principle. *For every quantifier-free formula $\varphi(n,m)$ of L^{st} not involving* min*, we have the equivalence*

$$(1)\qquad (\exists n)(\forall m)\varphi(n,m) \leftrightarrow (\exists^{st} n)(\forall^{st} m)\varphi(n,m).$$

By contraposition, it is equivalent to the Π_2-transfer principle

$$(2)\qquad (\forall n)(\exists m)\varphi(n,m) \leftrightarrow (\forall^{st} n)(\exists^{st} m)\varphi(n,m).$$

In view of the equivalence between (1) and (2), we will not explicitly mention Π_2-transfer in the sequel. We will add certain axioms to ERNA+Π_1-TRANS and prove the consistency of the resulting theory in PRA. Then we show that the extended theory proves the above Σ_2-transfer principle.

The method used for ERNA + Π_1-TRANS in [5] could not be applied here: Herbrand's theorem ([4]) reduces proving the consistency of a quantifier-free theory T to proving the consistency of an arbitrary finite subset of instantiated axioms T_0, which can be achieved by using a large enough initial segment of the natural numbers. For ERNA, modeling T_0 requires an interval for the interpretation of the finite numbers and one for the infinite numbers, separated by a gap which is so large that no finite operation of T_0 can take a number from the first interval to the second. The model gives no information on the behaviour of the numbers in the gap. In [5], we used a parameter B, estimating the growth of the functions in the model. Increasing it, as we did in [5], would result in some numbers having an interpretation in the gap. But, because of the increased quantifier complexity (Π_2 and Σ_2 instead of Π_1 and Σ_1), we need information on the numbers in both intervals *and* in the gap. At this point the method of [5] breaks down and there seems to be no immediate way of fixing it. Hence the need for a different approach. Recall that $n, m, k, l, \ldots$, both lower and upper-case, invariably refer to hypernatural variables or constants. See [5] for additional definitions and notational conventions.

First, the downward Σ_2-transfer principle

$$(\exists n)(\forall m)\varphi(n,m) \to (\exists^{st} n)(\forall^{st} m)\varphi(n,m), \tag{3}$$

with φ as in (1), needs to be reformulated as in formula (6) below. The following theorem of ERNA + Π_1-TRANS, required for (3), suggested the reformulation (6).

1. **Theorem.** *In* ERNA + Π_1-TRANS *we have, for every quantifier-free formula* $\varphi(n,m)$ *of* L^{st} *not involving* min*, the implication*

$$(\exists n)(\forall m)\varphi(n,m) \to (\forall^{st} k)(\exists^{st} n)(\forall m \le k)\varphi(n,m). \tag{4}$$

Proof. If the antecedent holds, we have $(\exists n)(\forall m \le k)\varphi(n,m)$ for every finite k. By Σ_1-transfer, $(\exists^{st} n)(\forall m \le k)\varphi(n,m)$, hence the consequent of (4). □

To deduce (3) from (4), we need the implication

$$(\forall^{st} k)(\exists^{st} n)(\forall m \le k)\varphi(n,m) \to (\exists^{st} n)(\forall^{st} m)\varphi(n,m), \tag{5}$$

which is contained, if rather indirectly, in (6). In spite of its complicated form, the latter will turn out to be a tautology in the finite setting of the model for an arbitrary finite subset of instantiated ERNA + Π_1-TRANS-axioms. The reason is, that it inherits from (5) the typical 'compactness' form, stating that 'if something holds for all initial segments, it holds for the entire set'.

2. **Axiom schema** (TRANS^+). *For every quantifier-free formula* $\varphi(n,m)$ *of* L^{st} *not involving* min*, we have*

$$(\exists n)(\forall m)\varphi(n,m) \to \begin{pmatrix} (\forall^{st} k)(\exists^{st} n)(\forall m \le k)\varphi(n,m) \\ \downarrow \\ (\exists^{st} n)(\forall^{st} m)\varphi(n,m) \end{pmatrix} \tag{6}$$

Theorem 6 will show that Σ_2-transfer as stated in (1) is provable in ERNA + Π_1-TRANS+TRANS$^+$. Therefore, the latter theory will be abbreviated to ERNA+ Σ_2-TRANS. Since we want to use Herbrand's theorem to prove the consistency of ERNA + Σ_2-TRANS, we have to skolemize the axiom schema TRANS$^+$, as it is not quantifier-free.

3. **Theorem.** *The axiom schema* TRANS^+ *can be skolemized by introducing two Skolem constants* n_S *and* k_S *per axiom. The quantifier-free skolemization is*

$$(7)\quad \neg\varphi(n_1, \min_{\neg\varphi}(n_1)) \vee \left(\begin{array}{c} n_2 \textit{ is finite } \to (\exists m \leq k_S)\neg\varphi(n_2, m) \wedge k_S \textit{ is finite} \\ \vee \\ n_3 \textit{ is finite} \to \varphi(n_S, n_3) \wedge n_S \textit{ is finite} \end{array} \right)$$

where n_1*,* n_2 *and* n_3 *are free variables.*

Proof. In (6) we can replace the bounded formula $(\forall m \leq k)\varphi(n, m)$ by a quantifier-free equivalent, see [5, Theorem 52]. Resolving the implications, renaming some variables and introducing the minimum operator turns (6) into

$$(\forall n_1)\neg\varphi(n_1, \min_{\neg\varphi}(n_1)) \vee \left(\begin{array}{c} (\exists^{st} k)(\forall^{st} n_2)(\exists m \leq k)\neg\varphi(n_2, m) \\ \vee \\ (\exists^{st} n)(\forall^{st} n_3)\varphi(n, n_3) \end{array} \right).$$

Introducing two Skolem constants n_S and k_S, we obtain

$$(\forall n_1)\neg\varphi(n_1, \min_{\neg\varphi}(n_1)) \vee \left(\begin{array}{c} (\forall^{st} n_2)(\exists m \leq k_S)\neg\varphi(n_2, m) \wedge k_S \text{ is finite} \\ \vee \\ (\forall^{st} n_3)\varphi(n_S, n_3) \wedge n_S \text{ is finite} \end{array} \right),$$

hence (7), after bringing the three universal quantifiers $(\forall n_1)$, $(\forall^{st} n_2)$ and $(\forall^{st} n_3)$ to the front. □

As is usual, we will denote the skolemization of TRANS^+ by $(\text{TRANS}^+)^S$.

In spite of the abundance of quantifiers in (6), its skolemization turns out to be easy: no new Skolem functions are required besides the minimum operator, which is actually a Skolem function for (a version of) the induction axioms. Before going into the consistency of $\text{ERNA} + \Sigma_2$-TRANS, we briefly recall the consistency proof of $\text{ERNA} + \Pi_1$-TRANS.

4. **Definition.** If τ is an individual constant, the depth $d(\tau)$ is zero. For a term $f(x_1, \ldots, x_k)$ we put $d(f(x_1, \ldots, x_l)) = \max(d(x_1), \ldots d(x_k)) + 1$.

By Herbrand's theorem, we have to prove that any finite set T of instantiated axioms of $\text{ERNA} + \Pi_1$-TRANS is consistent. Consider such a set T. Assume that $\varphi_1, \ldots, \varphi_N$ are all the formulas whose corresponding Π_1-TRANS axiom is in T. Let D be an upper bound for the depths of the functions in T. Then we define numbers a_D, b_D and c_D in at most ND steps, and subsequently a mapping 'val' on the terms τ of T such that $|\text{val}(\tau)| \in \{0\} \cup [1/c_D, 1/b_D] \cup [1/a_D, a_D] \cup [b_D, c_D]$. In particular, all functions $f(\vec{x})$, except for min and the constants ω and ε, and all relations $R(\vec{x})$, except for $\approx$, are interpreted as their 'homeomorphic' image:

$$(8)\qquad \text{val}(f(\vec{x})) = f(\text{val}(\vec{x})) \text{ and } \text{val}(R(\vec{x}) \text{ is true}) \leftrightarrow R(\text{val}(\vec{x})) \text{ is true}.$$

Moreover, all finite terms τ of T have $|\text{val}(\tau)| \in [0, a_D]$, all infinitesimal terms $|\text{val}(\tau)| \in [1/c_D, 1/b_D]$ and all infinite terms $|\text{val}(\tau)| \in [b_D, c_D]$. If τ appears in T, '$\tau \approx 0$' is interpreted as $|\text{val}(\tau)| \leq 1/b_D$ and 'τ is finite' as $|\text{val}(\tau)| \leq a_D$. This makes sense, given that infinitesimals are mapped into $[-1/b_D, -1/c_D] \cup [1/c_D, 1/b_D]$ and finite terms into $[-a_D, a_D]$. All axioms, including those in the schema Π_1-TRANS, are then verified to be true under this interpretation.

5. **Theorem.** $\text{ERNA} + \Sigma_2$-TRANS *is consistent and its consistency can be proved in* PRA.

Proof. Using (8) and induction on formula length, one readily verifies that $\text{val}(\varphi) = \varphi$ for every standard quantifier-free formula φ.

Now, consider an arbitrary finite set T of instantiated axioms of ERNA + Π_1-TRANS + $(\text{TRANS}^+)^S$. Assume that all terms appearing in T have depth at most D. Let $\varphi_1(n,m),\dots,\varphi_N(n,m)$ be the formulas of $(\text{TRANS}^+)^S$ whose corresponding axiom is in T and let T' be T without the axioms of $(\text{TRANS}^+)^S$. In T', replace each of the constants n_S and k_S by the standard term in T with the largest value and which is not a Skolem constant. If $T\setminus T'$ contains terms not occurring in T', which are not Skolem constants, add field axioms instantiated with those terms to T'. After that, T' is a finite set of instantiated axioms of ERNA+Π_1-TRANS and has a valid interpretation 'val' into the rationals. Therefore, if τ appears in T and is not a Skolem constant, $|\text{val}(\tau)|\in\{0\}\cup[1/c_D,1/b_D]\cup[1/a_D,a_D]\cup[b_D,c_D]$. Note that the replacement of the Skolem constants in T' allows us to obtain an interpretation of T' and that the addition of field axioms to T' is required to guarantee that all terms in T (not just T'), except for Skolem constants, have an interpretation. Next, we show how to interpret the Skolem constants in $[0,a_D]$.

Set $O=[0,a_D]\cup[b_D,c_D]$, and let $\varphi(n,m)$ be any of the $\varphi_i(n,m)$ with $1\le i\le N$. Since $\text{val}(\varphi)=\varphi$ for standard quantifier-free φ, the formula (7) will be true under val if

(9)
$$(\forall n\in O)\neg\varphi\big(n,\text{val}(\min_{\neg\varphi}(n))\big)\vee\left(\begin{array}{c}(\forall n\le a_D)(\exists m\le \text{val}(k_S))\neg\varphi(n,m)\wedge\text{val}(k_S)\le a_D\\ \vee\\ (\forall m\le a_D)\varphi(n_S,m)\wedge \text{val}(n_S)\le a_D\end{array}\right)$$

for a suitable interpretation of k_S and n_S. To define $\text{val}(n_S)$ and $\text{val}(k_S)$, consider the following sentence

$$(\forall n\in O)(\exists m\in O)\neg\varphi(n,m)\vee\left(\begin{array}{c}(\exists k\le a_D)(\forall n\le a_D)(\exists m\le k)\neg\varphi(n,m)\\ \vee\\ (\exists n\le a_D)(\forall m\le a_D)\varphi(n,m)\end{array}\right).\tag{10}$$

It is a tautology, because the the second part of the disjunction implies the negation of the third. Hence, one of the three members of the disjunction holds. If it is the first, set $n_S=k_S=0$. If it is the subformula starting with $(\exists k\le a_D)$, define k_S as any of these k's, and set $n_S=0$; likewise for the third member. With this interpretation, the Skolem constants satisfy all axioms in T' and hence the replacement in the previous paragraph does not affect the proof.

Repeating this procedure for the other formulas of $\varphi_1,\dots,\varphi_N$, we give valid interpretations to all sentences in T. Herbrand's theorem implies the consistency of ERNA + Π_1-TRANS + $(\text{TRANS}^+)^S$ and, a fortiori, of ERNA + Σ_2-TRANS. □

Now we prove the main result of this section, viz. that ERNA + Σ_2-TRANS has Σ_2-transfer.

6. **Theorem.** *In* ERNA + Σ_2-TRANS*, the* Σ_2*-transfer principle, stated in* (1)*, holds.*

Proof. By theorem 5 we know that we can consistently add the axiom schema 2 to ERNA + Π_1-TRANS. In the extended theory, theorem 1 yields that (6) implies (3). For the inverse implication, assume $(\exists^{st}n)(\forall^{st}m)\varphi(n,m)$ and fix $n_0\in\mathbb{N}$ such that $(\forall^{st}m)\varphi(n_0,m)$. By Π_1-transfer, this implies $(\forall m)\varphi(n_0,m)$ and hence $(\exists n)(\forall m)\varphi(n,m)$. □

Using pairing functions, we immediately obtain

7. **Corollary.** *In* ERNA + Σ_2-TRANS *we have, for every quantifier-free formula* φ *of* L^{st} *not involving* min*, that*

$$(\exists x_1)\dots(\exists x_k)(\forall y_1)\dots(\forall y_l)\varphi(\vec{x},\vec{y})\leftrightarrow(\exists^{st}x_1)\dots(\exists^{st}x_k)(\forall^{st}y_1)\dots(\forall^{st}y_l)\varphi(\vec{x},\vec{y})$$

and

$$(\forall^{st}x_1)\dots(\forall^{st}x_k)(\exists^{st}y_1)\dots(\exists^{st}y_l)\varphi(\vec{x},\vec{y}) \leftrightarrow (\forall x_1)\dots(\forall x_k)(\exists y_1)\dots(\exists y_l)\varphi(\vec{x},\vec{y}).$$

The direct approach to Σ_2 and Π_2-transfer would have been to add to ERNA the schema consisting of every axiom of the form (3). But a consistency proof for the resulting theory does not seem easy to find. Moreover, our indirect approach offers some advantages in terms of computational complexity. Indeed, if M is the number of Σ_2-transfer formulas in T, N the number of Π_1-transfer formulas in T and D the maximum term depth of T, the number of steps, i.e. applications of the iterated exponential, required in our consistency proof is at most $ND + M$. If the technique used in the consistency of ERNA + Π_1-TRANS were to prove applicable here, it could require up to MND steps.

To conclude this section, let us observe that our axiom schema TRANS$^+$ has not been formulated in the most economical way; for some ‘reducible’ formulas, the transfer property (3) is provovable in ERNA + Π_1-TRANS.

8. **Definition.** A formula φ is said to be ‘irreducible (w.r.t. ERNA + Π_1-TRANS)’ if and only if we have

$$(\forall^{st}k)(\exists^{st}n)(\forall m \le k)\varphi(n,m).$$

9. **Theorem.** *For every reducible quantifier-free formula $\varphi(n,m)$ of L^{st} not involving* min*, the downward Σ_2-transfer property* (3) *is provable in* ERNA + Π_1-TRANS.

Proof. Assume $(\forall^{st}n)(\exists^{st}m)\neg\varphi(n,m)$; since φ is reducible, there is a $k_0 \in \mathbb{N}$ such that $(\forall^{st}n)(\exists m \le k_0)\neg\varphi(n,m)$. By Π_1-transfer, $(\forall n)(\exists m \le k_0)\neg\varphi(n,m)$, hence $(\forall n)(\exists m)\neg\varphi(n,m)$. □

Using reducibility, we could replace TRANS$^+$ with the following alternative axiom schema.

10. **Axiom schema** (alternative Σ_2-Transfer)**.** *For every irreducible quantifier-free formula $\varphi(n,m)$ of L^{st} not involving* min*, we have*

$$(\exists n)(\forall m)\varphi(n,m) \to (\exists^{st}n)(\forall^{st}m)\varphi(n,m) \tag{11}$$

This schema makes the ‘transfer’ character more apparent, but obscures the deeper reason for its effectiveness. The consistency of ERNA + schema 10 is implied by that of ERNA + Σ_2-TRANS, but could be established independently in exactly the same way.

2. ERNA + Saturation

Besides transfer, saturation is an equally important nonstandard tool (see [6] and [11]). If κ is an infinite cardinal, κ-saturation means that a family of less than κ internal sets has a nonempty intersection as soon as it has the finite intersection property. The most applicable case is that of $\kappa = \aleph_1$. By the so-called Internal Definition Principle [11, 3.4.6], an internal set has the form $\{x \in S|\varphi(x)\}$ for some internal set S and internal formula φ. Hence, $\aleph_1$-saturation amounts to the following: a sequence of internal sets $\{x \in S_i|\varphi(i,x)\}$ $(i \in \mathbb{N})$ with the finite intersection property has a nonempty intersection. For sets of hyperreal numbers, this property can be written as:

$$(\forall n \in \mathbb{N})(\exists x \in {}^*\mathbb{R})(\forall i \le n)\varphi(i,x) \to (\exists x \in {}^*\mathbb{R})(\forall n \in \mathbb{N})\varphi(n,x)$$

This formulation can easily be translated into the set-free language of ERNA and gives rise to our Saturation axiom schema below. While Transfer was obtained for Σ_2 and Π_2-formulas, there are no conditions on quantifier complexity for Saturation.

11. **Axiom schema** (Saturation)**.** *For every internal formula* $\varphi(n,x)$ *not involving* min*, we have*

$$(\forall^{st} n)(\exists x)(\forall i \leq n)\varphi(i,x) \to (\exists x)(\forall^{st} n)\varphi(n,x). \tag{12}$$

This schema will be referred to as 'SAT'; if φ is restricted to Σ_k or Π_k-formulas, we call it SAT_k.

Skolemizing removes the existential quantifiers from a formula, replacing them with Skolem functions. Thanks to the special form of SAT, in which both φ and $\neg\varphi$ appear if the implication is resolved, we can remove *all* quantifiers from φ. The resulting 'alternative skolemization' of SAT is denoted by 'SAT^S'. It proves the same theorems as SAT, just like the regular skolemization would.

Note that Saturation cannot be obtained using overflow, as in ERNA the latter principle is limited to quantifier-free formulas ([5, Theorem 54]).

12. **Theorem.** *Every axiom of* SAT *can be written as*

$$\big((\forall x)(\exists i \leq n_S)(n_S \textit{ is finite } \wedge \phi(i,x))\big) \vee (\forall^{st} n)\phi(n,x_S), \tag{13}$$

where ϕ *is some quantifier-free internal formula and* x_S *and* n_S *are Skolem constants varying with the original* φ*.*

Proof. First assume that φ is quantifier-free. Resolving the implication in (12), we obtain

$$(\exists^{st} n)(\forall y)(\exists i \leq n)\neg\varphi(i,y) \vee (\exists x)(\forall^{st} n)\varphi(n,x), \tag{14}$$

which implies

$$(\exists^{st} n)(\exists x)\big((\forall y)(\exists i \leq n)\neg\varphi(i,y) \vee (\forall^{st} n)\varphi(n,x)\big).$$

Introducing Skolem constants n_S and x_S, and renaming variables, we get

$$(\forall x)(\exists i \leq n_S)(n_S \text{ is finite } \wedge \neg\varphi(i,x)) \vee (\forall^{st} n)\varphi(n,x_S),$$

which is just (13).

Now assume $\varphi \in \Sigma_k \cup \Pi_k$ for $k > 0$. Using pairing functions, we reduce all blocks of universal/existential quantifiers to a single quantifier. Let $\psi(n,x,m_1,\dots,m_k)$ be $\varphi(n,x)$ with its k alternating quantifiers stripped off. Bring (12) in the form (14). Then either $\neg\varphi$ or φ contains $(\exists m_1)\psi(n,x,m_1,\dots,m_k)$. Both cases being analogous, we treat the latter. Replace $(\exists m_1)\psi(n,x,m_1,m_2,\dots,m_k)$ in φ with $\psi(n,x,f(n,x,m_1,m_2,\dots,m_k),m_2,\dots,m_k)$, where f is an appropriate Skolem function. Likewise, replace $(\forall m_1)\psi(n,x,m_1,\dots,m_k)$ in $\neg\varphi$ with the equivalent formula $\neg\psi(n,x,f(n,x,m_1,m_2,\dots,m_k),m_2,\dots,m_k)$. Replace the original φ and $\neg\varphi$ in (12) with these modified versions. After that, $\neg\varphi$ contains the existential formula

$$(\exists m_2)\neg\psi(n,x,f(n,m_1,m_2,\dots,m_k),m_2,\dots,m_k)$$

and we can repeat the above process another $k-1$ times to obtain quantifier-free versions of φ and $\neg\varphi$, denoted by ϕ and $\neg\phi$, respectively. □

Collection axioms could be used to reduce quantifier complexity, but these are not available in ERNA. A proof with such a lower quantifier complexity would not be essentially different from ours.

We are now ready to prove the consistency of ERNA+SAT; in fact, we prove the consistency of ERNA + SAT^S. A consistency proof for a 'base theory' with extra axioms has already occurred in theorem 5, namely for ERNA + Σ_2-TRANS. Here, we use the same technique again.

13. **Theorem.** ERNA + SAT^S*, a fortiori* ERNA + SAT*, is consistent and the consistency of* SAT_0 *can be proved in* PRA*.*

Proof. Consider an arbitrary finite set T of instantiated axioms of $\mathrm{ERNA} + \mathrm{SAT}^S$. The set T' is defined in the same way as in the proof of theorem 5, with the only exception that x_S is replaced by that term in T which has largest value and is not a Skolem constant.

Set $O = [0, a_D] \cup [b_D, c_D]$, and let $\varphi(n, m)$ be any of the $\varphi_i(n, m)$ with $1 \leq i \leq N$. Then (13) will be true under val if

$$\begin{array}{c}(\forall x \in O)\big(\exists i \leq \mathrm{val}(n_S)\big)\big(\mathrm{val}(n_S) \leq a_D \wedge \mathrm{val}(\neg\varphi(i, x))\big) \\ \vee \\ (\forall n \leq a_D)\mathrm{val}\big(\varphi(n, \mathrm{val}(x_S))\big)\end{array} \tag{15}$$

for a suitable x_S and n_S. To define $\mathrm{val}(n_S)$ and $\mathrm{val}(x_S)$, consider the sentence

$$\begin{array}{c}(\exists n \leq a_D)(\forall x \in O)(\exists i \leq n)\mathrm{val}(\neg\varphi(i, x)) \\ \vee \\ (\exists x \in O)(\forall n \leq a_D)\mathrm{val}(\varphi(n, x))\end{array}. \tag{16}$$

It is a tautology, because the first part of the disjunction implies the negation of the second. Hence, one of the two members of the disjunction holds. If it is the first, define n_S as any of these $n \leq a_D$, and set $x_S = 0$; if it is the second, define x_S as any of these $x \in O$, and set $n_S = 0$.

Repeating this procedure for the other formulas of $\varphi_1, \ldots, \varphi_N$, we give valid interpretations to all sentences in T.

Since for a quantifier-free formula φ, there holds $\varphi^S \equiv \varphi$, it is clear from above that the consistency proof of $\mathrm{ERNA} + \mathrm{SAT}_0$ takes place in PRA. □

Next, we use SAT to prove a fundamental principle of nonstandard arithmetic.

14. **Theorem** (Overflow in the rationals)**.** *If the internal formula $\varphi(x)$ holds for all $x \approx 0$, it holds for all $|x| \leq x_0$, where x_0 is some positive rational.*

Proof. The assumption implies that $(\forall x)(\exists^{st} n)\left(|x| < \frac{1}{n} \to \varphi(x)\right)$ is true, and the contraposition of the SAT-axiom for $|x| < \frac{1}{n} \wedge \neg\varphi(x)$ shows that $(\exists^{st} n)(\forall x)(\exists i \leq n)\left(|x| < \frac{1}{i} \to \varphi(x)\right)$. Hence, $(\exists^{st} n)(\forall x)\left(|x| < \frac{1}{n} \to \varphi(x)\right)$. □

Incidentally, this implies that there is no internal formula equivalent to $x \approx 0$. Indeed, if $\varphi(x)$ is internal, we cannot have that $\varphi(x) \leftrightarrow x \approx 0$, because overflow in the rationals would imply $|x| \leq x_0 \to \varphi(x)$ for some rational $x_0 \neq 0$. This yields $x_0 \approx 0$, a contradiction as $x_0 \neq 0$ is standard.

We now consider an equivalent, intuitively more appealing, formulation of the saturation schema SAT_0. A similar result for the more general schema SAT could be proved along the same lines, but its statement would lack transparency due to the presence of Skolem functions in the formula φ appearing in SAT.

15. **Theorem.** *In* ERNA*, the schema* SAT_0 *is equivalent to the statement:*

(17) *a finite internal function not involving* min *is finitely bounded.*

Proof. As each instance of the condition (17) is equivalent to a sentence of the form

$$(\forall x)(\exists^{st} n)(|f(x)| \leq n) \to (\exists^{st} m)(\forall x)(\exists i \leq m)(|f(x)| \leq i), \tag{18}$$

it is clearly included in SAT_0. For the other direction, we prove that (17) implies the counterposition of every axiom of SAT_0. To this end, assume that $(\forall x)(\exists^{st} n)\neg\varphi(x, n)$ for some quantifier-free formula φ not involving min. By [5, Theorem 58], there is an internal function $f(x)$ not involving min, such that for all x

$$f(x) \text{ is finite } \wedge \neg\varphi(x, f(x)).$$

By (17), $(\forall x)(|f(x)| \leq n_0)$ for some $n_0 \in \mathbb{N}$. This implies $(\forall x)\big(|f(x)| \leq n_0 \wedge \neg\varphi(x, f(x))\big)$ and finally $(\exists^{st} n)(\forall x)(\exists i \leq n)\neg\varphi(x, i)$. Hence, SAT_0 can be derived from (17). □

Many external functions are finite everywhere, but not finitely bounded, e.g. the function $g(x)$ which is equal to x for finite x and zero otherwise. This shows that external functions can make (17) false. Given theorem 15, this example illustrates in an elementary way that SAT is false for external objects.

The very proof of theorem 13 reveals a deeper reason to restrict SAT to internal formulas. The formula $(\forall i \leq n)\varphi(i, x)$ cannot be written in a quantifier-free form, as required by Herbrand's theorem, if φ is external. The formula has to be replaced by $\prod_{i=0}^{n} T_\varphi(x) > 0$, but it is intrisically impossible to prove anything about sums and products of characteristic functions of *external* formulas, as the proof of [5, Theorem 58] would result in a function calculating the smallest infinite number.

3. Generalizing the scope of Transfer

Both Π_1 and Σ_2-transfer are limited to formulas of L^{st}. Hence, a formula cannot be transferred if it contains, for instance, ERNA's cosine $\sum_{n=0}^{\omega}(-1)^n \frac{x^{2n}}{(2n)!}$ or similar objects not definable in L^{st}. In this section, we widen the scope of transfer so as to be applicable to such objects.

First we label some terms which, though not part of L^{st}, are 'nearly as good' as standard for the purpose of transfer. As in [5, Notation 57], the variable ω' in $(\forall\omega')$ runs over the infinite hypernaturals.

16. **Definition.** Let the term $\tau(n, \vec{x})$ be standard, i.e. not involve ω or $\approx$. We say that $\tau(\omega, \vec{x})$ is *near-standard* if

$$(\forall \vec{x})(\forall \omega')(\tau(\omega, \vec{x}) \approx \tau(\omega', \vec{x})). \tag{19}$$

An atomic inequalitity $\tau(\omega, \vec{x}) \leq \sigma(\omega, \vec{x})$ is called near-standard if both members are. Since $x = y$ is equivalent to $x \leq y \wedge x \geq y$, and $\mathcal{N}(x)$ to $\lceil x \rceil = |x|$, any formula $\varphi(\omega, \vec{x})$ can be assumed to consist entirely of atomic inequalities; it is called near-standard if it is made up of near-standard atomic inequalities.

Full transfer for near-standard formulas is impossible. Thus, the implication $|x| < 1 \to \frac{1}{|x|} > 1 + \frac{1}{\omega}$ is true for all *standard* x, but false for $x = \frac{2\omega}{2\omega+1}$. But the weaker implication $|x| < 1 \to \frac{1}{|x|} \gtrapprox 1 + \frac{1}{\omega}$ does hold for all x, and this is the idea behind generalized transfer, to be considered next. We need a few definitions, first 'positive' and 'negative' occurrence of subformulas (see [1], [2], [8]). Intuitively speaking, an occurrence of a subformula B in A is positive (negative) if, after resolving the implications outside B and pushing all negations inward, but not inside B, there is no (one) negation in front of B. Thus, in

$$(\neg(B \to C) \wedge (D \to B)) \to \neg D,$$

all occurrences of B are negative, C has one positive occurrence and D occurs both positively and negatively. The formal definition is as follows.

17. **Definition.** Given a formula A, an occurrence of a subformula B, and an occurrence of a logical connective α in A, we say that B is *negatively bound* by α if either α is a negation $\neg$ and B is in its scope, or α is an implication $\to$ and B is a subformula of the antecedent. The subformula B is said to *occur negatively* (positively) in A if B is negatively bound by an odd (even) number of connectives of A.

18. **Definition.** We write $a \ll b$ for $a < b \wedge a \not\approx b$ and $a \lessapprox b$ for $a < b \vee a \approx b$. Given a near-standard formula $\varphi(\omega, \vec{x})$, let $\overline{\varphi}(\omega, \vec{x})$ be the formula obtained by replacing every positive (negative) occurrence of a near-standard inequality $\leq$ with $\lessapprox$ ($\ll$).

For atomic formulas, it is immediate that $\overline{\varphi}$ is weaker than φ; to verify the same for general φ, proceed by induction on complexity.

19. **Theorem** (Generalized Transfer). *Let $\varphi(x, y)$ and $\psi(x)$ be near-standard quantifier-free internal formulas not involving* min. *In* ERNA + Σ_2-TRANS *we have that*

$$(\forall^{st} x)\psi(x) \rightarrow (\forall x)\overline{\psi}(x) \tag{20}$$

and

$$(\forall^{st} x)(\exists^{st} y)\varphi(x, y) \rightarrow (\forall x)(\exists y)\overline{\varphi}(x, y). \tag{21}$$

Proof.
We will prove the Π_2-case (21); for a proof of the Π_1-case (20), omit one quantifier.

Using induction on the number of connectives in φ, we see that the only near-standard atomic subformulas in $\neg\overline{\varphi}$, if the negation has been pushed inwards, are formulas with $\ll$ or $\gg$. Therefore, it suffices to treat the atomic case where φ is $\tau_1(\omega, x, y) \leq \tau_2(\omega, x, y)$. So assume $(\forall^{st} x)(\exists^{st} y)\varphi(x, y)$. We have to prove that $(\forall x)(\exists y)\tau_1(\omega, x, y) \lessapprox \tau_2(\omega, x, y)$. If not, $(\exists x)(\forall y)(\tau_1(\omega, x, y) \gg \tau_2(\omega, x, y))$. Since both τ_1 and τ_2 are near-standard, they vary infinitesimally if ω is replaced with another infinite hypernatural. This property implies $(\exists x)(\forall y)(\forall m \geq \omega)(\tau_1(m, x, y) \gg \tau_2(m, x, y))$, hence the *standard* sentence

$$(\exists x)(\exists n)(\forall y)(\forall m \geq n)(\tau_1(m, x, y) > \tau_2(m, x, y)).$$

Using Σ_2-transfer, we obtain

$$(\exists^{st} x)(\exists^{st} n)(\forall^{st} y)(\forall^{st} m \geq n)(\tau_1(m, x, y) > \tau_2(m, x, y)).$$

Let x_0 and n_0 be standard numbers such that $(\forall^{st} y)(\forall^{st} m \geq n_0)(\tau_1(m, x_0, y) > \tau_2(m, x_0, y))$. By Π_1-transfer, $(\forall y)(\forall m \geq n_0)(\tau_1(m, x_0, y) > \tau_2(m, x_0, y))$ and since n_0 is finite, we have $(\forall y)(\tau_1(\omega, x_0, y) > \tau_2(\omega, x_0, y))$. As x_0 is standard, $(\exists^{st} x)(\forall^{st} y)(\tau_1(\omega, x, y) > \tau_2(\omega, x, y))$, contradicting the assumption. □

The near-standard condition (19) can be omitted in the special case we consider next.

20. **Theorem** (Generalized Transfer, special case). *Let $\psi(x)$ be a quantifier-free formula not involving* min*, whose only nonstandard terms are finite and of the form $\tau(\omega)$, with τ internal. In* ERNA + Π_1-TRANS *we have that*

$$(\forall^{st} x)\psi(x) \rightarrow (\forall x)\overline{\psi}(x).$$

Proof. Again, it suffices to consider the atomic case. Assume that $\tau_1(x)$ is standard and that $(\forall^{st} x)(\tau_1(x) \leq \tau(\omega))$, where $\tau(\omega)$ is finite. If $(\exists x)(\tau_1(x) \gg \tau(\omega))$, choose such an $x = x_0$. Then [5, Theorem 56] guarantees the existence of a rational number q such that $\tau_1(x_0) \geq q > \tau(\omega)$. From $(\exists x)(\tau_1(x) \geq q)$ we obtain by Σ_1-transfer that $(\exists^{st} x)(\tau_1(x) \geq q)$, hence $(\exists^{st} x)(\tau_1(x) > \tau(\omega))$. This contradicts the assumption. □

4. ERNA, NQA$^+$ and Transfer

In this last section, we prove several results concerning the (relative) strength of fragments of ERNA and NQA$^+$ plus transfer. For NQA$^+$, see [5] and [7]. We also refer to NQA$^\emptyset$ (ERNA$^\emptyset$), which is NQA$^+$ (ERNA) without minimization axioms.

Our first theorem does away with the external and internal minimum in the consistency proof of ERNA$^\emptyset+\Sigma_2$-TRANS. The gain is considerable, because treating minimization (especially the external min) takes up a large portion of ERNA's consistency proof (see [9]). Note also that Σ_2-transfer is powerful enough to have NQA$^\emptyset+\Sigma_2$-TRANS interpret a strong induction theory in spite of lacking induction axioms.

21. **Theorem** (min$_\varphi$-redundancy)**.** ERNA$^\emptyset+\Sigma_2$-TRANS *and* ERNA$+\Sigma_2$-TRANS *prove the same theorems.*

Proof. First we treat the external minimum schema. Assume $\varphi(n,\vec{x})$ as in [5, Axiom Schema 37], i.e. quantifier-free and not involving ω or min. Fix a natural n. Let φ' be φ with all positive occurrences of $\tau_i(n,\vec{x})\approx 0$ replaced with $(\forall^{st} n_i)(|\tau_i(n,\vec{x})|<1/n_i)$, where n_i is a new variable not appearing in φ. Do the same for the negative occurrences, using new variables m_i. Bringing all quantifiers in $\varphi'(n,\vec{x})$ to the front, we obtain

$$(\exists^{st} m_1)\dots(\exists^{st} m_l)(\forall^{st} n_1)\dots(\forall^{st} n_k)\psi(n,\vec{x},\vec{n},\vec{m})$$

where ψ is quantifier-free and standard. By Σ_2-transfer, this is equivalent to

$$(\exists m_1)\dots(\exists m_l)(\forall n_1)\dots(\forall n_k)\psi(n,\vec{x},\vec{n},\vec{m}).$$

If we return the quantifiers to their original places, all external atomic formulas $\tau_i(n,\vec{x})\approx 0$ have become $(\forall n_i)(|\tau_i(n,\vec{x})|<1/n_i)$ or, equivalently, $\tau_i(n,\vec{x})=0$. If $\varphi''(n,\vec{x})$ is φ with all symbols $\approx$ replaced with $=$, we have proved that $\varphi''(n,\vec{x})$ is equivalent to $\varphi(n,\vec{x})$. By [5, Theorem 58], ERNA$^\emptyset$ has a function which calculates the least n such that $\varphi''(n,\vec{x})$, if such there are. This function replaces the external minimum operator min$_\varphi$.

Now for the internal minimum schema. Assume $\varphi(n,\vec{x})$ as in [5, Axiom Schema 31], i.e. quantifier-free and not involving $\approx$ or min. Let $\varphi(n,\vec{x},m)$ be $\varphi(n,\vec{x})$ with all occurrences of ω replaced with the new variable m. By [5, Theorem 58], ERNA$^\emptyset$ has a function which, for every finite m, calculates the least $n\le\omega$ such that $\varphi(n,\vec{x},m)$, if such there are. Then the sentence

$$(\forall l\le\omega)\neg\varphi(l,\vec{x},m)\vee(\exists n\le\omega)\big(\varphi(n,\vec{x},m)\wedge(\forall k<n)\neg\varphi(k,\vec{x},m)\big)$$

is true for all natural m. Using Σ_1-transfer, we obtain

$$(\forall^{st} m)\Big((\forall^{st} l)\neg\varphi(l,\vec{x},m)\vee(\exists^{st} n)\big(\varphi(n,\vec{x},m)\wedge(\forall k<n)\neg\varphi(k,\vec{x},m)\big)\Big)$$

and Π_2-transfer implies

$$(\forall m)\Big((\forall l)\neg\varphi(l,\vec{x},m)\vee(\exists n)\big(\varphi(n,\vec{x},m)\wedge(\forall k<n)\neg\varphi(k,\vec{x},m)\big)\Big).$$

If we fix $m=\omega$, the skolemization of the resulting sentence is exactly the axiom of the internal minimum schema for $\varphi(n,\vec{x})$. Since a theory and its skolemization prove the same theorems, we are done. □

Note that, in order to prove that standard terms are finite for finite input, one needs external induction, which is equivalent to external minimization. Hence it is not possible to prove external minimization without transfer by arguing that, as φ does not involve ω, all terms appearing in φ are standard and hence not-infinitesimal, unless zero.

22. **Corollary.** *The theories* $I\Sigma_1$ *and* WKL_0 *can be interpreted in* $\mathrm{NQA}^{\emptyset}+\Sigma_2$-TRANS *and in* $\mathrm{ERNA}^{\emptyset}+\Sigma_2$-TRANS.

Proof. The proof of theorem 21 also implies that $\mathrm{NQA}^{\emptyset}+\Sigma_2$-TRANS proves internal minimization. By the above theorem, the same holds for $\mathrm{ERNA}^{\emptyset}+\Sigma_2$-TRANS. To obtain the interpretation required, apply [5, Theorem 45], replacing $\mathrm{NQA}^{\emptyset}$ by $\mathrm{ERNA}^{\emptyset}$. □

In [7], Rössler and Jeřábek provide their theory NQA^+ with an interpretation of $I\Sigma_1$ and WKL_0 by regarding Σ_1-induction as general quantifier-free external induction. In [5], we use Π_1-transfer to obtain an interpretation of those theories in a fragment of NQA^+ without external induction. As a generalization to these results, the following theorem shows that even stronger theories like $I\Sigma_2$ and $B\Sigma_2$ can be interpreted in ERNA and NQA^+ plus transfer.

23. **Theorem.**
(1) *The theory* $I\Sigma_2$ *can be interpreted in* $\mathrm{NQA}^+ + \Pi_1$-TRANS.
(2) *The theory* $B\Sigma_2$ *can be interpreted in* $\mathrm{ERNA} + \Pi_1$-TRANS.

Proof. For the notations '$\mathbb{FN}(n)$', '$\mu m \leq \nu_0$' and 'O-MIN$^{\mathrm{st}}$', we refer to [7]. Additionally we assume $n, m, k, l, \ldots$ to be *hypernatural* variables, i.e. satisfying the predicate $\mathbb{N}$ of $\mathrm{NQA}^{\emptyset}$. In [7], the interpretation of $I\Sigma_1$ in NQA^+ is based on replacing all arithmetical Σ_1-formulas with quantifications relativized to $\mathbb{FN}(n)$, which are in turn replaced by external open formulas, provided by [7, Lemma 4.2]. This being done, the Σ_1-induction axioms of $I\Sigma_1$ are interpreted as instances of external open induction, which are implied by the schema O-MIN$^{\mathrm{st}}$ of NQA^+.

To interpret $I\Sigma_2$ in $\mathrm{NQA}^+ + \Pi_1$-TRANS, we start from the interpretation of arithmetical Σ_2-formulas as quantifications relativized to $\mathbb{FN}(n)$. From [7, Lemma 2.4], it follows that the $\mathrm{NQA}^{\emptyset}$-term

$$m_{\varphi,\nu_0}(\vec{n}) := (\mu m \leq \nu_0(t_{(\forall k\leq \nu_0)\varphi(m,k,\vec{n})}(m,\vec{n}) = 1))$$

is definable in NQA^+. Now $\mathbb{FN}(m_{\varphi,\nu_0}(\vec{n}))$ implies $(\exists m)(\mathbb{FN}(m)\wedge(\forall k \leq \nu_o)\varphi(m,k,\vec{n}))$, hence $(\exists m)\big(\mathbb{FN}(m) \wedge (\forall k)(\mathbb{FN}(k) \to \varphi(m,k,\vec{n})\big)$. On the other hand, if m_0 is such that $\mathbb{FN}(m_0) \wedge (\forall k)(\mathbb{FN}(k) \to \varphi(m_0,k,\vec{n})$, Π_1-transfer applied to $(\forall k)(\mathbb{FN}(k) \to \varphi(m_0,k,\vec{n}))$ implies $(\forall k)(\mathbb{N}(k) \to \varphi(m_0,k,\vec{n}))$, hence certainly $(\forall k \leq \nu_0)\varphi(m_0,k,\vec{n})$. Now $m_{\varphi,\nu_0}(\vec{n})$ is standard; in fact it is at most m_0, because it is the least of the m satisfying $(\forall k \leq \nu_0)\varphi(m,k,\vec{n})$. Thus, $\mathrm{NQA}^+ + \Pi_1$-TRANS proves the equivalence

$$(\exists m)\big(\mathbb{FN}(m) \wedge (\forall k)(\mathbb{FN}(k) \to \varphi(m,k,\vec{n})\big) \leftrightarrow \mathbb{FN}(m_{\varphi,\nu_0}(\vec{n})). \tag{22}$$

It follows that, all arithmetical Σ_2-formulas being replaced with quantifications relativized to $\mathbb{FN}(x)$, the interpreted Σ_2-induction axioms of $I\Sigma_2$ are equivalent to instances of external open induction. Hence, they follow from O-MIN$^{\mathrm{st}}$.

For the second, we also interpret the quantifiers $(\exists n)$ and $(\forall m)$, occurring in formulas of $B\Sigma_2$, as $(\exists^{st} n)$ and $(\forall^{st} m)$, respectively, in $\mathrm{ERNA} + \Pi_1$-TRANS. Fix $k_0 \in \mathbb{N}$ and let $\varphi(k,l)$ be the Σ_2 sentence $(\exists n)(\forall m)\varphi_0(n,m,k,l)$ with φ_0 quantifier-free. Then the interpretation of the antecedent of the REPL-axiom of $B\Sigma_2$ for φ is

$$(\forall k \leq k_0)(\exists^{st} l)(\exists^{st} n)(\forall^{st} m)\varphi_0(n,m,k,l).$$

Using Π_1-transfer for suitable $k, l, n \in \mathbb{N}$, we obtain

$$(\forall k \leq k_0)(\exists^{st} l)(\exists^{st} n)(\forall m)\varphi_0(n,m,k,l),$$

hence certainly

$$(\forall k \leq k_0)(\exists^{st} l)(\exists^{st} n)(\forall m \leq \omega)\varphi_0(n,m,k,l).$$

Using a binary pairing function, we reduce $(\exists^{st} l)$ and $(\exists^{st} n)$ to a single quantifier $(\exists^{st} N)$. By [5, Theorem 58], ERNA$^{\emptyset}$ has an internal function $f(k)$ which calculates the least of these. Defining $l_0 = \sum_{k=0}^{k_0} f(k)$, we find

$$(\forall k \leq k_0)(\exists l \leq l_0)(\exists n \leq l_0)(\forall m \leq \omega)\varphi_0(n, m, k, l),$$

which yields

$$(\exists^{st} l_0)(\forall k \leq k_0)(\exists l \leq l_0)(\exists^{st} n)(\forall^{st} m)\varphi_0(n, m, k, l),$$

i.e. the consequent of the interpretation of the REPL-axiom of φ. □

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