

Homogeneity and projective equivalence of differential equation fields

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Abstract

We propose definitions of homogeneity and projective equivalence for systems of ordinary differential equations of order greater than two, which allow us to generalize the concept of a spray (for systems of order two). We show that the Euler-Lagrange fields of parametric Lagrangians of order greater than one which are regular (in a natural sense that we define) form a projective equivalence class of homogeneous systems. We show further that the geodesics, or base integral curves, of projectively equivalent homogeneous differential equation fields are the same apart from orientation-preserving reparametrization; that is, homogeneous differential equation fields determine systems of paths.

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1 Introduction

This paper is concerned with systems of ordinary differential equations

$$y_{n+1}^i = \Gamma^i(y^i, y_1^i, \dots, y_n^i); \quad y_r^i = \frac{d^r y^i}{dx^r}$$

of order $n + 1$, $n \geq 1$, in m dependent variables y^i , $i = 1, 2, \dots, m$ and one independent variable x . Second-order systems of this kind ($n = 1$) have been studied intensively using methods of differential geometry, by taking advantage of the fact that the equations are those satisfied by the integral curves of a vector field of a certain type, a so-called second-order differential equation field (often abbreviated to SODE) on the tangent bundle TM of an m -dimensional differential manifold M , or some open submanifold of it. Higher-order systems may be studied by analogous methods, where now the differential equation vector field in question lives (in the case of equations of order $n + 1$) on the bundle $T^n M$ of n -velocities, in other words n -jets at zero of curves in M (whose domains contain zero), or some open submanifold of it. This is the approach adopted here. (For a recent account of the geometric theory of differential equations, covering both second- and higher-order systems, and containing many references, see [1]. However, this reference is mainly concerned with aspects of the theory not covered here, namely generalized connections and related matters.)

In the theory of second-order differential equation fields an important role is played by a class of fields which are homogeneous in a certain sense. Let $T_\circ M$ be the slit tangent bundle of M (TM with the zero section deleted). Let Δ be the Liouville vector field, that is, the infinitesimal generator of dilations of the fibres of $T_\circ M$. A second-order differential equation field Γ on $T_\circ M$ is called a spray if it satisfies $[\Delta, \Gamma] = \Gamma$. The canonical spray of a Finsler space, whose integral curves project onto geodesics with constant speed parametrization, is a helpful example. If Γ is a spray then the distribution \mathfrak{D}^+ spanned by Γ and Δ is involutive. Any second-order differential equation field Γ' contained in \mathfrak{D}^+ is said to be projectively equivalent to Γ , and its base integral curves are obtained from those of Γ by reparametrization. Slightly more generally, for the purposes of this paper we shall say that a second-order differential equation field Γ is homogeneous if the distribution \mathfrak{D}^+ spanned by Γ and Δ is involutive. It is not difficult to show that if Γ is homogeneous in this sense then there is a projectively equivalent second-order differential equation field which is a spray. In Finsler geometry the canonical spray is the Euler-Lagrange field of the energy, which is a regular Lagrangian. But instead of the energy one may consider the

Finsler function itself as a Lagrangian. Because of its assumed homogeneity it is parametric: that is to say, the corresponding action integral is invariant under reparametrizations. Such a parametric Lagrangian determines, not a single Euler-Lagrange field, but a projective equivalence class of second-order differential equation fields, which in the Finslerian case is the projective class of the canonical spray.

Our purpose in the present paper is to generalize the concept of homogeneity so that it applies to differential equation fields of higher order. For this purpose we introduce the jet group of order n , L^n , which is defined as follows. Consider the local diffeomorphisms ϕ of \mathbb{R} , defined in a neighbourhood of zero and satisfying $\phi(0) = 0$. Then L^n is the set of n -jets at zero of such local diffeomorphisms; it is a group under composition, and even a Lie group. Moreover, it acts to the right on $T^n M$. Let \mathfrak{D} be the distribution spanned by the generators of the action. We say that a differential equation field Γ is homogeneous if the distribution \mathfrak{D}^+ spanned by Γ and \mathfrak{D} is involutive. This generalizes the concept described above for second-order differential equation fields because when $n = 1$, \mathfrak{D} is just the one-dimensional distribution spanned by Δ . (A preliminary attempt to examine homogeneity of higher-order differential equation fields is to be found in [2]; however, this paper is again concerned mainly with connection theory.)

Section 3 below is devoted to the definition of homogeneity and its immediate consequences.

With this definition we show that the other properties of homogeneous second-order differential equation fields discussed above also generalize. The extremals of a Lagrangian of order n satisfy differential equations of order $2n$. If the Lagrangian is parametric, so that it satisfies the so-called Zermelo conditions, then its extremals are determined only up to reparametrization. We show in Section 4 that, provided it is sufficiently regular, a parametric Lagrangian determines a projective equivalence class of homogeneous differential equation fields. This result gives us the reassurance that our definition is not vacuous.

In Section 5 we show that the base integral curves of projectively equivalent homogeneous differential equation fields differ only in parametrization.

Section 6 of the paper contains some examples, while Section 2 is devoted to the requisite geometry of jet manifolds and jet groups.

We shall use the Einstein summation convention for coordinate indices such as i ; other sums will be indicated explicitly.

2 Geometrical background

We consider differentiable manifolds of class C^∞ which are Hausdorff, second-countable and (unless otherwise specified) connected.

Let M be such a manifold with $\dim M = m$ and with local coordinates (y^i) . Consider curves $\gamma : (a, b) \rightarrow M$, where $0 \in (a, b)$, and let $T^n M$ be the set of n -jets at zero $\{j_0^n \gamma\}$, with coordinates (y_r^i) for $0 \leq r \leq n$ given by

$$y_r^i(j_0^n \gamma) = \left. \frac{d^r \gamma^i}{dx^r} \right|_0.$$

It is a standard result that $T^n M$ is a manifold with $\dim T^n M = (n+1)m$, and that the maps $\tau_n : T^n M \rightarrow M$ and $\tau_{n,n'} : T^n M \rightarrow T^{n'} M$ ($n' < n$) given by

$$\tau_n(j_0^n \gamma) = \gamma(0), \quad \tau_{n,n'}(j_0^n \gamma) = j_0^{n'} \gamma$$

are fibre bundles.

The manifold $T^{n+1} M$ has a natural identification with a submanifold of $TT^n M$, obtained because the map $t \mapsto j_t^n \gamma$ defines a curve

$$\tilde{j}^n \gamma : t \mapsto j_0^n(\gamma \circ \mathbb{T}_t)$$

in $T^n M$, where $\mathbb{T}_t : \mathbb{R} \rightarrow \mathbb{R}$ is the translation $x \mapsto x + t$. We therefore obtain the inclusion

$$T^{n+1} M \rightarrow TT^n M, \quad j_0^{n+1} \gamma \mapsto j_0 \tilde{j}^n \gamma$$

given in coordinates by $y_{r+1}^i \mapsto \dot{y}_r^i$. We may think of this inclusion as a section of the pull-back vector bundle $\tau_{n+1,n}^* TT^n M \rightarrow T^{n+1} M$, in other words as a vector field along the projection $\tau_{n+1,n}$; with such an interpretation it is the total derivative

$$d_{\mathbb{T}} = \sum_{r=0}^n y_{r+1}^i \frac{\partial}{\partial y_r^i}.$$

The projection $\tau_{n+1,n} : T^{n+1} M \rightarrow T^n M$ is in fact an affine bundle, modelled on the vector bundle $V\tau_{n,n-1} \rightarrow T^n M$ of very vertical tangent vectors on $T^n M$. The affine action is just addition in the fibres of $TT^n M \rightarrow T^n M$:

$$(y_r^i; 0, \dots, 0, z^i) + (y_r^i; \dot{y}_0^i, \dots, \dot{y}_{n-1}^i, \dot{y}_n^i) \mapsto (y_r^i; \dot{y}_0^i, \dots, \dot{y}_{n-1}^i, z^i + \dot{y}_n^i)$$

gives rise, using the identification $\dot{y}_r^i = y_{r+1}^i$ on $T^{n+1} M$, to

$$(y_r^i; z^i) + (y_r^i; y_{n+1}^i) \mapsto (y_r^i; z^i + y_{n+1}^i).$$

We shall make use later of the vertical endomorphism on $T^n M$. This is the type $(1, 1)$ tensor field S given in coordinates by

$$S = \sum_{r=1}^n r \frac{\partial}{\partial y_r^i} \otimes dy_{r-1}^i;$$

it is canonically defined, and generalizes the well-known vertical endomorphism or tangent structure on TM . It is not difficult to show that for any 1-form α on $T^n M$,

$$S(d_T \alpha) - d_T(S\alpha) = \tau_{n+1,n}^* \alpha$$

(where in the first instance S is the vertical endomorphism on $T^{n+1}M$, in the second the one on $T^n M$). This may conveniently be written $Sd_T - d_T S = 1$, with the pull-back map understood (note that we assume action on 1-forms; there is a more complicated formula for action on forms of higher degree).

A section $\tilde{\Gamma} : T^n M \rightarrow T^{n+1}M$ of the affine bundle $\tau_{n+1,n}$ is called a *differential equation field*; in coordinates it is indeed a differential equation

$$y_{n+1}^i = \Gamma^i(y_0^i, \dots, y_n^i).$$

A curve $\gamma : (a, b) \rightarrow M$ is a geodesic of the equation if $\tilde{j}^{n+1}\gamma = \tilde{\Gamma} \circ \tilde{j}^n \gamma$. Composing $\tilde{\Gamma}$ with the inclusion $T^{n+1}M \rightarrow TT^n M$ gives a vector field Γ on $T^n M$ of the particular form

$$\Gamma = \sum_{r=0}^{n-1} y_{r+1}^i \frac{\partial}{\partial y_r^i} + \Gamma^i \frac{\partial}{\partial y_n^i};$$

every integral curve $\tilde{\gamma} : (a, b) \rightarrow T^n M$ of Γ is of the form $\tilde{\gamma} = \tilde{j}^n \gamma$ for some geodesic γ of $\tilde{\Gamma}$; that is to say, the geodesics of $\tilde{\Gamma}$ are the base integral curves of Γ . We use the term ‘differential equation field’ to refer to Γ as well as $\tilde{\Gamma}$.

Now consider local diffeomorphisms ϕ of \mathbb{R} defined in a neighbourhood of zero and satisfying $\phi(0) = 0$. Let L^n be the set of n -jets at zero $\{j_0^n \phi\}$ of these local diffeomorphisms; this is a group under composition,

$$j_0^n \phi_1 \cdot j_0^n \phi_2 = j_0^n (\phi_1 \circ \phi_2)$$

and is a Lie group. As a manifold it has two connected components. The component of the identity is a subgroup L^{n+} of index 2 containing jets of local diffeomorphisms satisfying $\phi'(0) > 0$. The map

$$j^n \phi_0 \mapsto (\phi'(0), \phi''(0), \dots, \phi^{(n)}(0)) \in \mathbb{R}^n$$

is a global coordinate system on L^n (and L^{n+}). We can obtain an explicit formula for the product in these coordinates by using an expression for the n th derivative of a composition of functions. This is Faà di Bruno's formula [4], which we take in the form involving the Bell polynomials:

$$(\xi \circ \eta)^{(n)}(x) = \sum_{r=1}^n \xi^{(r)}(\eta(x)) B_n^r \left(\eta'(x), \eta''(x), \dots, \eta^{(n+1-r)}(x) \right);$$

here ξ and η are functions of some variable t with (in our case) $\xi(0) = \eta(0) = 0$, and $B_n^r(\eta_1, \eta_2, \dots, \eta_{n+1-r})$ is a polynomial in the $n+1-r$ variables η_p , for which the following explicit formula is known:

$$\begin{aligned} & B_n^r(\eta_1, \eta_2, \dots, \eta_{n+1-r}) \\ &= \sum \frac{n!}{q_1! q_2! \dots q_{n+1-r}!} \left(\frac{\eta_1}{1!} \right)^{q_1} \left(\frac{\eta_2}{2!} \right)^{q_2} \dots \left(\frac{\eta_{n+1-r}}{(n+1-r)!} \right)^{q_{n+1-r}} \end{aligned}$$

where the sum is over all non-negative integers $q_1, q_2, \dots, q_{n+1-r}$ such that $q_1 + q_2 + \dots + q_{n+1-r} = r$ and $q_1 + 2q_2 + \dots + (n+1-r)q_{n+1-r} = n$. (This formula may easily be derived by taking $\xi(x) = x^r$ and expressing $\eta(x)$ as a formal power series in x :

$$\eta(x) = \frac{\eta_1}{1!}x + \frac{\eta_2}{2!}x^2 + \frac{\eta_3}{3!}x^3 + \dots;$$

then $(r!/n!)B_n^r(\eta_1, \eta_2, \dots, \eta_{n+1-r})$ is the coefficient of x^n in the formal power series for $(\eta(x))^r$. In principle B_n^r could depend on η_p with $p > n+1-r$, but in practice, since $\eta(x)$ contains no constant term there can be no contribution to x^n in $(\eta(x))^r$ coming from the term $\eta_p x^p/p!$ if $p > n+1-r$.) To express multiplication in L^n in terms of the global coordinate system introduced above, a vectorial notation is convenient. We denote by $\boldsymbol{\xi}, \boldsymbol{\eta}$ elements of \mathbb{R}^n , considered as row vectors, and by $\mathbf{B}(\boldsymbol{\eta})$ the (upper triangular) matrix whose p, q entry is $B_q^p(\eta_1, \eta_2, \dots)$. Then the product in L^n is given by

$$\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \boldsymbol{\xi} \mathbf{B}(\boldsymbol{\eta});$$

we must of course assume that ξ_1 and η_1 are non-zero. Note that the matrix which determines multiplication in $L^{n'}$ with $n' < n$ is just the $n' \times n'$ submatrix of $\mathbf{B}(\boldsymbol{\eta})$ in the upper left corner. For instance, taking $n = 4$ gives

$$\mathbf{B}(\boldsymbol{\eta}) = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ 0 & \eta_1^2 & 3\eta_1\eta_2 & 4\eta_1\eta_3 + 3\eta_2^2 \\ 0 & 0 & \eta_1^3 & 6\eta_1^2\eta_2 \\ 0 & 0 & 0 & \eta_1^4 \end{pmatrix}.$$

There are obvious projections $\lambda_{n,1} : L^n \rightarrow L^1$ and $\lambda_{n,1}^+ : L^{n+} \rightarrow L^{1+}$ given by $j_0^n \phi \mapsto j_0^1 \phi$; they are homomorphisms because

$$\lambda_{n,1}(j_0^n \phi_1 \cdot j_0^n \phi_2) = \lambda_{n,1}(j_0^n(\phi_1 \circ \phi_2)) = j_0^1(\phi_1 \circ \phi_2) = j_0^1 \phi_1 \cdot j_0^1 \phi_2.$$

The kernel of $\lambda_{n,1}$ is therefore a normal subgroup $K^n \triangleleft L^n$. By inspection of the matrix \mathbf{B} above in the case $n = 4$ we see that K^2 and K^3 are abelian but K^4 is not; indeed K^n is non-abelian whenever $n \geq 4$.

We may, in addition, identify L^1 with a subgroup of L^n by mapping $j_0^1 \phi$ to $j_0^n \mu_{\phi'(0)}$, where $\mu_s : \mathbb{R} \rightarrow \mathbb{R}$ is the multiplication diffeomorphism $\mu_s(t) = st$ for $s \neq 0$. Thus L^n may be regarded as a semidirect product $L^1 \rtimes K^n$, using the action of L^1 on K^n by conjugation.

Furthermore, we may write $L^n = L^2 \rtimes J^n$, where J^n is the kernel of the homomorphism $\lambda_{n,2} : L^n \rightarrow L^2$ given by $j_0^n \phi \mapsto j_0^2 \phi$. For the inclusion of L^2 in L^n we take, as a representative of $j_0^2 \phi$, not the quadratic polynomial $\phi(x) = ax + \frac{1}{2}bx^2$ but the Möbius transformation

$$\phi(x) = \frac{ax}{1 - a^{-1}bx}$$

because the composition of two of these transformations is another map of the same form. By calculating the derivatives of ϕ one finds a polynomial representative of $j_0^n \phi$,

$$\sum_{r=1}^n \frac{b^{r-1}x^r}{a^{r-2}}$$

(see [5]). Note that the kernels of the homomorphisms $\lambda_{n,r}$ do not give semidirect product decompositions when $r > 2$.

We next obtain a basis for the Lie algebra \mathfrak{l}^n of the jet group L^n in terms of the coordinates defined above. For this purpose we need the following lemma.

Lemma 1. *For given $n, p = 1, 2, \dots, n$, and $r = 1, 2, \dots, n + 1 - p$,*

$$\left. \frac{\partial B_n^p}{\partial \eta_r} \right|_{(1,0,\dots,0)} = 0$$

unless $p = n + 1 - r$, when

$$\left. \frac{\partial B_n^p}{\partial \eta_r} \right|_{(1,0,\dots,0)} = \frac{n!}{(n-r)!r!}.$$

Proof. We consider the cases $r = 1$, $r \geq 2$ separately. We have

$$\begin{aligned} \frac{\partial B_n^p}{\partial \eta_1} &= \sum \frac{n!}{(q_1 - 1)! q_2! \cdots q_{n+1-p}! 1!} \\ &\quad \times \left(\frac{\eta_1}{1!}\right)^{q_1-1} \left(\frac{\eta_2}{2!}\right)^{q_2} \cdots \left(\frac{\eta_{n+1-r}}{(n+1-r)!}\right)^{q_{n+1-r}} \end{aligned}$$

where the sum is over all non-negative integers $q_1, q_2, \dots, q_{n+1-r}$ with $q_1 > 0$ such that $q_1 + q_2 + \dots + q_{n+1-r} = p$ and $q_1 + 2q_2 + \dots + (n+1-r)q_{n+1-r} = n$. With $\eta_1 = 1$, $\eta_r = 0$ for $r \geq 2$ we get a non-zero contribution to the sum only if $q_r = 0$ for $r \geq 2$, when we must have $q_1 = p$ and $q_1 = n$. That is, there is no non-zero term in the sum unless $p = n$, and so

$$\left. \frac{\partial B_n^p}{\partial \eta_1} \right|_{(1,0,\dots,0)} = \begin{cases} 0 & p \neq n \\ \frac{n!}{(n-1)! 1!} & p = n \end{cases} .$$

For $r \geq 2$ on the other hand

$$\begin{aligned} \frac{\partial B_n^p}{\partial \eta_r} &= \sum \frac{n!}{q_1! \cdots (q_r - 1)! \cdots q_{n+1-p}! r!} \\ &\quad \times \left(\frac{\eta_1}{1!}\right)^{q_1} \cdots \left(\frac{\eta_r}{r!}\right)^{q_r-1} \cdots \left(\frac{\eta_{n+1-r}}{(n+1-r)!}\right)^{q_{n+1-r}} \end{aligned}$$

where the sum is now over all non-negative integers $q_1, q_2, \dots, q_{n+1-r}$ with $q_r > 0$ such that $q_1 + q_2 + \dots + q_{n+1-r} = p$ and $q_1 + 2q_2 + \dots + (n+1-r)q_{n+1-r} = n$. With $\eta_1 = 1$, $\eta_s = 0$ for $s \geq 2$ we get a non-zero contribution to the sum only if $q_r = 1$, $q_s = 0$ for $s \geq 2$, $s \neq r$. We must therefore have $q_1 + 1 = p$ and $q_1 + r = n$, that is, $p = n + 1 - r$ and $q_1 = n - r$. Then

$$\left. \frac{\partial B_n^p}{\partial \xi_r} \right|_{(1,0,\dots,0)} = \begin{cases} 0 & p \neq n + 1 - r \\ \frac{n!}{(n-r)! r!} & p = n + 1 - r \end{cases} . \quad \square$$

We denote by δ^r , $r = 1, 2, \dots, n$, the left-invariant vector field on L^n which takes the value

$$r! \frac{\partial}{\partial y_r}$$

at the identity. Here (y_1, y_2, \dots, y_n) are the coordinates on L^n . The factor $r!$ is included for later convenience. We now obtain an explicit expression for δ^r in terms of coordinates.

Proposition 1.

$$\delta^r = \sum_{s=r}^n \frac{s!}{(s-r)!} y_{s+1-r} \frac{\partial}{\partial y_s}.$$

Proof. Let $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be a generic point of L^n . The identity \mathbf{e} has coordinates $(1, 0, \dots, 0)$. We have

$$\delta_{\mathbf{y}}^r = L_{\mathbf{y}}|_{*\mathbf{e}} \delta_{\mathbf{e}}^r$$

where $L_{\mathbf{y}}$ is left multiplication by \mathbf{y} : $L_{\mathbf{y}}\boldsymbol{\eta} = \mathbf{y}\mathbf{B}(\boldsymbol{\eta})$. Then

$$\begin{aligned} L_{\mathbf{y}}|_{*\mathbf{e}} \frac{\partial}{\partial y^r} &= \sum_{s=1}^n \sum_{p=1}^s y_p \frac{\partial B_s^p}{\partial \eta_r} \Big|_{(1,0,\dots,0)} \frac{\partial}{\partial y_s} \\ &= \sum_{s=r}^n \frac{s!}{(s-r)! r!} y_{s+1-r} \frac{\partial}{\partial y_s}. \end{aligned} \quad \square$$

Corollary 1.

$$[\delta^r, \delta^s] = \begin{cases} (r-s)\delta^{r+s-1} & \text{if } r+s \leq n+1 \\ 0 & \text{otherwise} \end{cases}.$$

□

One reason for introducing the factor $r!$ in the definition of δ^r is to simplify the expression for the bracket.

We can now make some observations about the Lie algebra \mathfrak{l}^n which correspond to properties of the Lie group L^n mentioned earlier. In the first place, $\{\delta^1, \delta^2\}$ span a subalgebra. Furthermore, for any $p = 1, 2, \dots, n$, $\langle \delta^r : r \geq p \rangle$ is an ideal in \mathfrak{l}^n (since for any $r \geq p$ and $s \geq 1$, $r+s-1 \geq p$). Let $\mathfrak{l}_p^n \subset \mathfrak{l}^n$ be the ideal just defined (so that in particular $\mathfrak{l}^n = \mathfrak{l}_1^n$). Then for $p > 1$, $\mathfrak{l}^n / \mathfrak{l}_p^n \sim \mathfrak{l}^{p-1}$.

The ideals \mathfrak{l}_p^n for $p > 1$ are nilpotent. It is known that for a nilpotent Lie algebra, exponentiation is a surjective map onto the simply connected Lie group of which it is the algebra; it follows in particular that exponentiation maps \mathfrak{l}_2^n onto K^n (see [5]). But it is possible to prove this directly, as we now show. The Lie algebra \mathfrak{l}^n consists of linear vector fields on \mathbb{R}^n , and so exponentiation in the group L^n coincides with matrix exponentiation. In the case of an element $\kappa = \sum_{r=2}^n k_r \delta^r$ of \mathfrak{l}_2^n the matrix \mathbf{K}_n to be exponentiated is strictly lower triangular: its elements are given by

$$(\mathbf{K}_n)_{rs} = \begin{cases} 0 & r < s \\ \frac{r!}{(s-1)!} k_{r+1-s} & r \geq s \end{cases}.$$

Notice that the $(n-1) \times (n-1)$ submatrix in the upper left corner is just the matrix \mathbf{K}_{n-1} corresponding to the element $\sum_{r=2}^{n-1} k_r \delta^r$ of \mathfrak{l}_2^{n-1} . As an example, with $n=5$ we have

$$\mathbf{K}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{2!}{0!}k_2 & 0 & 0 & 0 & 0 \\ \frac{3!}{0!}k_3 & \frac{3!}{1!}k_2 & 0 & 0 & 0 \\ \frac{4!}{0!}k_4 & \frac{4!}{1!}k_3 & \frac{4!}{2!}k_2 & 0 & 0 \\ \frac{5!}{0!}k_5 & \frac{5!}{1!}k_4 & \frac{5!}{2!}k_3 & \frac{5!}{3!}k_2 & 0 \end{pmatrix}.$$

Proposition 2. *For any $\mathbf{y} = (1, y_2, y_3, \dots, y_n)^T \in K^n$ there is $\kappa \in \mathfrak{l}_2^n$ such that $\mathbf{y} = \exp \kappa$.*

Proof. As a column vector in \mathbb{R}^n , $\exp \kappa \in L^n$ is just $(\exp \mathbf{K}_n)\mathbf{e}$, where $\mathbf{e} = (1, 0, \dots, 0)^T$; that is to say, $\exp \kappa$ is the left column of $\exp \mathbf{K}_n$. It is easy to see that the first $n-1$ entries in the left column of $\exp \mathbf{K}_n$ comprise the left column of the $(n-1) \times (n-1)$ matrix $\exp \mathbf{K}_{n-1}$, while the lower left corner of $\exp \mathbf{K}_n$ is of the form $n!k_n + p(k_2, k_3, \dots, k_{n-1})$ where p is a polynomial in the indicated variables. Since \mathbf{K}_n is strictly lower triangular, $\exp \mathbf{K}_n$ has 1s down the diagonal. We have to show that for any y_2, y_3, \dots, y_n we can choose k_2, k_3, \dots, k_n such that the left column of $\exp \mathbf{K}_n$ is $(1, y_2, y_3, \dots, y_n)^T$. We proceed by induction. Thus we assume that we can find k_2, k_3, \dots, k_{n-1} such that the left column of $\exp \mathbf{K}_{n-1}$ is $(1, y_2, y_3, \dots, y_{n-1})^T$. With these values of k_2, k_3, \dots, k_{n-1} we set $k_n = (y_n - p(k_2, k_3, \dots, k_{n-1}))/n!$; then the left column of $\exp \mathbf{K}_n$ is $(1, y_2, y_3, \dots, y_n)^T$ as required. We have

$$\exp \mathbf{K}_2 = \begin{pmatrix} 1 & 0 \\ 2!k_2 & 1 \end{pmatrix},$$

so the first step certainly works. \square

We next turn to the relationship between \mathfrak{l}^n and certain vector fields on \mathbb{R} . An element δ of \mathfrak{l}^n determines a one-parameter subgroup of L^n , namely $t \mapsto \exp(t\delta)$: observe first that the integral curve of δ through $\mathbf{y} \in L^n$ is $t \mapsto \mathbf{y} \cdot \exp(t\delta)$. Now let ϕ_t be a one-parameter group of diffeomorphisms of \mathbb{R} such that $\phi_t(0) = 0$. Then $\tilde{\phi}_t = j_0^n \phi_t$ is evidently a one-parameter subgroup of L^n whose infinitesimal generator, a vector field χ on L^n , is left invariant and therefore an element of \mathfrak{l}^n . For any $\mathbf{y} \in L^n$, $\chi_{\mathbf{y}}$ is the tangent

vector at $t = 0$ to the curve $t \mapsto \mathbf{y} \cdot \tilde{\phi}_t$. But the s th component of $\mathbf{y} \cdot \tilde{\phi}_t$ is given by

$$(\mathbf{y} \cdot \tilde{\phi}_t)_s = \sum_{p=1}^s y_p B_s^p \left(\frac{\partial \phi_t}{\partial x} \Big|_0, \frac{\partial^2 \phi_t}{\partial x^2} \Big|_0, \dots, \frac{\partial^{s+1-p} \phi_t}{\partial x^{s+p-1}} \Big|_0 \right) y_p.$$

Note that since $\phi_0(x) = x$, the argument of B_s^p at $t = 0$ is $(1, 0, \dots, 0)$, while

$$\frac{\partial}{\partial t} \left(\frac{\partial^r \phi_t}{\partial x^r} \right)_{(0,0)} = \frac{\partial^r}{\partial x^r} \left(\frac{\partial \phi_t}{\partial t} \right)_{(0,0)} = \frac{d^r X}{dx^r} \Big|_0$$

where $X(x)\partial/\partial x$ is the infinitesimal generator of ϕ_t (a vector field on \mathbb{R}). On differentiating the formula for $(\mathbf{y} \cdot \tilde{\phi}_t)_s$ with respect to t at $t = 0$ we obtain

$$\chi = \sum_{s=1}^n \left(\sum_{p=1}^s \frac{s!}{(s-p)!p!} \frac{d^p X}{dx^p} \Big|_0 y_{s+1-p} \right) \frac{\partial}{\partial y_s} = \sum_{p=1}^n \frac{1}{p!} \frac{d^p X}{dx^p} \Big|_0 \delta^p.$$

In fact

$$\Phi : X \mapsto \sum_{p=1}^n \frac{1}{p!} \frac{d^p X}{dx^p} \Big|_0 \delta^p$$

is a linear map from the space of vector fields on \mathbb{R} vanishing at the origin onto \mathfrak{l}^n , such that $\Phi(X\partial/\partial x)$ depends only on $j_0^n X$. In particular, $\Phi(x^r \partial/\partial x) = \delta^r$ for $r = 1, 2, \dots, n$ while $\Phi(x^r \partial/\partial x) = 0$ for $r > n$. Now

$$\left[x^r \frac{\partial}{\partial x}, x^s \frac{\partial}{\partial x} \right] = (s-r)x^{r+s-1} \frac{\partial}{\partial x},$$

so Φ is an anti-homomorphism.

Let \mathfrak{p} be the Lie algebra of vector fields on \mathbb{R} whose coefficients are formal power series in x , and let \mathfrak{p}^n be the subalgebra of those vector fields which vanish to order n at 0, that is, whose coefficient begins with x^{n+1} . Then \mathfrak{p}^0 is the subalgebra of formal power series vector fields which vanish at 0; and for $n > 0$, \mathfrak{p}^n is an ideal in \mathfrak{p}^0 . If we think of Φ as a map $\mathfrak{p}^0 \rightarrow \mathfrak{l}^n$ it is a surjective anti-homomorphism with kernel \mathfrak{p}^n , and therefore defines an anti-isomorphism $\mathfrak{p}^0/\mathfrak{p}^n \rightarrow \mathfrak{l}^n$. To put things another way, we can realise \mathfrak{l}^n as the space of vector fields on \mathbb{R} whose coefficients are polynomials of order n which vanish at the origin, with bracket the negative of the ordinary bracket of vector fields followed by truncation at order n . This alternative realisation is found elsewhere in the literature. The fact that the bracket is

related to the negative of the ordinary bracket is not too surprising when one recalls that if the diffeomorphism group of \mathbb{R} is regarded as an infinite dimensional Lie group, its Lie algebra is the space of vector fields on \mathbb{R} with compact support, but with bracket the negative of the ordinary vector field bracket.

The group L^n has a right action α_n on $T^n M$ given by composition of jets,

$$\alpha_n : L^n \times T^n M \rightarrow T^n M, \quad (j_0^n \phi, j_0^n \gamma) \mapsto j_0^n (\gamma \circ \phi);$$

the action is fibred over the identity on M . It restricts to the ‘regular’ submanifold $T_\circ^n M$ (that is, of n -jets $j_0^n \gamma$ where γ is a curve with $\gamma'(0) \neq 0$, so that γ is an immersion near zero) because composing a diffeomorphism with an immersion gives another immersion. We may use Faà di Bruno’s formula recursively to see that the restricted action is free, using the coordinates (y_r^i) , $1 \leq i \leq n$, defined on complete fibres over M : by regularity at least one coordinate y_1^i must be non-zero at any given point, so if $\alpha_n(j_0^n \phi, j_0^n \gamma) = j_0^n \gamma$ we see successively that $\phi'(0) = 1$ and then that $\phi''(0) = 0$, $\phi'''(0) = 0$, \dots , so that $j_0^n \phi = 1_{L^n}$. The orbit space of $T_\circ^n M$ under the action α_n has a manifold structure (and is, indeed, a Hausdorff manifold); we shall denote it by $PT^n M$. The subgroup L^{n+} acts in the same way, and its orbit space will be denoted by $P^+T^n M$; this is a double cover of $PT^n M$. Let $\rho_n : T_\circ^n M \rightarrow PT^n M$ and $\rho_n^+ : T_\circ^n M \rightarrow P^+T^n M$ be the projections.

In the present work we shall in effect be interested in the circumstances when a differential equation field on $T^n M$ (more accurately, on $T_\circ^n M$) ‘passes to the quotient’ to determine a line-element field on $PT^n M$ or an oriented line-element field on $P^+T^n M$. In view of the identification of $T^{n+1} M$ with an affine sub-bundle of $TT^n M$ over $T^n M$, it is of some interest to consider the identification of PT^{n+1} with a submanifold of $PT(PT^n M)$. One may do this, for instance, by using the action of the tangent group TL^n on $TT_\circ^n M$ to give $T(PT^n M)$, and then the action of L^1 on the open submanifold $T_\circ(PT^n M)$ to give $PT(PT^n M)$. In fact it is known that PT^{n+1} is an affine bundle over $PT^n M$, even though $PT(PT^n M) \rightarrow PT^n M$ certainly does not have an affine structure, as its fibres are compact (they are projective spaces): see, for instance, the discussion in [6], which uses the fact that the kernel of the homomorphism $L^{n+1} \rightarrow L^n$ is abelian. Similar arguments hold in the oriented case.

We shall need to know the fundamental vector fields on $T^n M$ of the action of L^n . We denote by Δ^r the fundamental vector field corresponding to $\delta^r \in \mathfrak{l}^n$. Then Δ^r is the infinitesimal generator of the one-parameter group $R_{\exp(t\delta^r)}$ (where R denotes the right action). In terms of the coordinates (y_r^i)

on $T^n M$, the action of $\eta \in L^n$ is given by

$$(R_\eta(y_s^j))_r^i = \sum_{p=1}^r y_p^i B_r^p(\eta_1, \eta_2, \dots, \eta_{r+1-p}).$$

By a by now familiar type of argument invoking Lemma 1 we obtain

Proposition 3.

$$\Delta^r = \sum_{s=r}^n \frac{s!}{(s-r)!} y_{s+1-r}^i \frac{\partial}{\partial y_s^i}.$$

In particular Δ^1 , which corresponds to the infinitesimal generator of dilations of \mathbb{R} , is given by

$$\Delta^1 = y_1^i \frac{\partial}{\partial y_1^i} + 2y_2^i \frac{\partial}{\partial y_2^i} + \dots + ny_n^i \frac{\partial}{\partial y_n^i}.$$

Corollary 2.

$$[\Delta^r, \Delta^s] = \begin{cases} (r-s)\Delta^{r+s-1} & \text{if } r+s \leq n+1 \\ 0 & \text{otherwise} \end{cases}.$$

Corollary 3. *In terms of the vertical endomorphism S*

1. a vector field Γ on $T_\circ^n M$ is a differential equation field if and only if $S(\Gamma) = \Delta^1$;
2. $\Delta^{r+1} = S(\Delta^r) = S^r(\Delta^1) = S^{r+1}(\Gamma)$.

We shall not use the results of Corollary 3 directly, but we mention them because they provide an alternative starting point for the geometrical analysis of differential equation fields which may be found elsewhere in the literature.

We shall however use in Section 4 a result which is related to item 1 of Corollary 3, which we now explain. We denote by i the operator of interior product of a vector field with a form, so that for example for a 1-form α , $i_X \alpha = \alpha(X)$. We extend this notation to apply to the total derivative: if α is a 1-form on $T_\circ^n M$ then $i_T \alpha$ is a function on $T_\circ^{n+1} M$. Then $i_T(S\alpha) = i_{\Delta^1}(\tau_{n+1,n}^* \alpha)$, which may conveniently be written $i_T S = i_{\Delta^1}$, with the pull-back map understood. Note that we assert only that this formula holds for action on 1-forms (there is a more complicated formula for action on forms of higher degree). Similarly, in relation to item 2, we have (again for action on 1-forms) $i_T S^r = i_{\Delta^r}$.

3 Homogeneous higher-order systems

We now consider differential equation (vector) fields on $T_{\circ}^n M$.

Definition 1. A differential equation field Γ on $T_{\circ}^n M$ is homogeneous if the distribution \mathfrak{D}^+ spanned by Γ and the Δ^r is involutive.

Proposition 4. The differential equation field Γ is homogeneous if and only if there are functions λ^r , $r = 1, 2, \dots, n$ such that

$$[\Delta^1, \Gamma] = \Gamma + \lambda^1 \Delta^n, \quad [\Delta^r, \Gamma] = r\Delta^{r-1} + \lambda^r \Delta^n, \quad r = 2, 3, \dots, n.$$

When they exist, such functions must satisfy the following consistency conditions, where $1 < r, s \leq n$, $r \neq s$:

$$\begin{aligned} \Delta^1(\lambda^r) - \Delta^r(\lambda^1) &= (n+1-r)\lambda^r; \\ \Delta^r(\lambda^s) - \Delta^s(\lambda^r) &= (r-s)\lambda^{r+s-1}, \quad r+s \leq n+1; \\ \Delta^r(\lambda^s) - \Delta^s(\lambda^r) &= -(n+1)(r-s), \quad r+s = n+2; \\ \Delta^r(\lambda^s) - \Delta^s(\lambda^r) &= 0, \quad r+s > n+2. \end{aligned}$$

Proof. A straightforward coordinate calculation shows that $[\Delta^1, \Gamma] - \Gamma$ and $[\Delta^r, \Gamma] - r\Delta^{r-1}$ are very vertical. Thus in order for \mathfrak{D}^+ to be involutive there must be functions λ^r such that

$$[\Delta^1, \Gamma] = \Gamma + \lambda^1 \Delta^n, \quad [\Delta^r, \Gamma] = r\Delta^{r-1} + \lambda^r \Delta^n, \quad r = 2, 3, \dots, n.$$

The consistency conditions follow from the Jacobi identities. \square

In terms of coordinates the conditions for

$$\Gamma = \sum_{r=0}^{n-1} y_{r+1}^i \frac{\partial}{\partial y_r^i} + \Gamma^i \frac{\partial}{\partial y_n^i}$$

to be homogeneous are

$$\Delta^1(\Gamma^i) = (n+1)\Gamma^i + n!\lambda^1 y_1^i, \quad \Delta^r(\Gamma^i) = \frac{(n+1)!}{(n+1-r)!} y_{n+2-r}^i + n!\lambda^r y_1^i.$$

If Γ is a differential equation field such that \mathfrak{D}^+ is involutive, so is $\Gamma + \mu\Delta^n$ for any function μ on $T_{\circ}^n M$, and of course it belongs to the same involutive distribution.

One cannot, in general, demand that for $n \geq 3$ a differential equation field Γ satisfies the strong conditions

$$[\Delta^1, \Gamma] = \Gamma, \quad [\Delta^r, \Gamma] = r\Delta^{r-1}, \quad r = 2, 3, \dots, n;$$

the Jacobi identities would be inconsistent with the bracket relations for the Δ^r . (This point is discussed more fully in [2], where examples of equations which do satisfy the conditions for $n = 2$ are given.) Recall that $\{\Delta^1, \Delta^2\}$ span a subalgebra of \mathfrak{D} ; it is easily verified that the conditions

$$[\Delta^1, \Gamma] = \Gamma, \quad [\Delta^2, \Gamma] = 2\Delta^1$$

are consistent. We shall now show that for any homogeneous $\hat{\Gamma}$ it is possible to find μ such that if $\Gamma = \hat{\Gamma} + \mu\Delta^n$ then Γ satisfies the conditions above (as well as $[\Delta^r, \Gamma] = r\Delta^{r-1} + \lambda^r\Delta^n$ for $r > 2$); in other words, if Γ is homogeneous, without loss of generality we may assume that $\lambda^1 = \lambda^2 = 0$. We need a couple of lemmas.

Lemma 2. *There is a submanifold \mathcal{S} of $T^n M$ of codimension 1 such that Δ^1 is transverse to \mathcal{S} and Δ^2 is tangent to it. There is a submanifold \mathcal{S}' of \mathcal{S} of codimension 1 such that $\Delta^2|_{\mathcal{S}'}$ is transverse to \mathcal{S}' .*

Proof. Let g be any Riemannian metric on M . Then $g_{ij}y_1^i y_1^j$ is a well-defined function on $T^n M$ (because the y_1^i transform as the components of tangent vectors), and $g_{ij}y_1^i y_1^j > 0$ on $T^n M$. Let $\varphi(y) = \sqrt{g_{ij}y_1^i y_1^j}$, and $\mathcal{S} = \{y \in T^n M : \varphi(y) = 1\}$. Then $\Delta^1(\varphi) = \varphi$ and $\Delta^2(\varphi) = 0$, so Δ^1 and Δ^2 are respectively transverse and tangent to \mathcal{S} . Next, let $\varphi' = \frac{1}{2}\Gamma(\varphi)$ for any differential equation field Γ on $T^n M$. We have

$$\Delta^2(\varphi') = \frac{1}{2}\Delta^2(\Gamma(\varphi)) = \frac{1}{2}\Gamma(\Delta^2(\varphi)) + \Delta^1(\varphi) = \varphi,$$

since $[\Delta^2, \Gamma] - 2\Delta^1$ is very vertical. Now

$$2\varphi\varphi' = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}y_1^i y_1^j y_1^k + g_{ij}y_1^i y_2^j = g_{ij}y_1^i (y_2^j + \Gamma_{kl}^j y_1^k y_1^l),$$

where the Γ_{ij}^k are the connection coefficients of the Levi-Civita connection of the metric g , from which it is clear that $\mathcal{S}' = \{y \in \mathcal{S} : \varphi'(y) = 0\}$ is a codimension 1 submanifold of \mathcal{S} , and $\Delta^2|_{\mathcal{S}'}$ is transverse to it. \square

Lemma 3. *Let X be a complete vector field on a manifold M , with 1-parameter group ϕ_t , and \mathcal{S} a codimension 1 submanifold of M transverse*

to X such that $\{\phi_t(\mathcal{S}) : t \in \mathbb{R}\} = M$. Let f be any smooth function on M , k any constant, and z_0 any smooth function on \mathcal{S} . Then there is a unique smooth function z on M such that $X(z) + kz = f$ and $z|_{\mathcal{S}} = z_0$. In particular, if $f = 0$ and $z_0 = 0$ then $z = 0$.

Proof. One can integrate the differential equation

$$\frac{dz}{dt} + kz = f(\phi_t(x)),$$

where x is any fixed point of \mathcal{S} , by the integrating factor method. \square

(The restriction that k should be constant covers the situation encountered below, but clearly more general equations could be considered.)

Theorem 1. *Let $\hat{\Gamma}$ be a homogeneous higher-order differential equation field on $T_{\circ}^n M$, so that \mathfrak{D}^+ is involutive. Then there is a higher-order differential equation field $\Gamma = \hat{\Gamma} + \mu\Delta^n \in \mathfrak{D}^+$ such that*

$$[\Delta^1, \Gamma] = \Gamma, \quad [\Delta^2, \Gamma] = 2\Delta^1.$$

Proof. Let \mathcal{S} be a codimension 1 submanifold of $T_{\circ}^n M$ such that Δ^1 is transverse to \mathcal{S} and Δ^2 is tangent to it, and \mathcal{S}' a codimension 1 submanifold of \mathcal{S} such that $\Delta^2|_{\mathcal{S}}$ is transverse to \mathcal{S}' .

We may write

$$[\Delta^1, \hat{\Gamma}] = \hat{\Gamma} + \hat{\lambda}^1 \Delta^n, \quad [\Delta^2, \hat{\Gamma}] = 2\Delta^1 + \hat{\lambda}^2 \Delta^n.$$

Then with $\Gamma = \hat{\Gamma} + \mu\Delta^n$,

$$[\Delta^1, \Gamma] = \Gamma + (\Delta^1(\mu) + (2 - n)\mu + \hat{\lambda}^1)\Delta^n.$$

By Lemma 3 we can solve the equation $\Delta^1(\mu) + (2 - n)\mu = -\hat{\lambda}^1$, assigning the value of μ on \mathcal{S} arbitrarily. Also,

$$[\Delta^2, \Gamma] = 2\Delta^1 + (\Delta^2(\mu) + \hat{\lambda}^2)\Delta^n.$$

Now Δ^2 is tangent to \mathcal{S} , and μ is undetermined on that submanifold, so we may solve the equation $\Delta^2(\mu) = -\hat{\lambda}^2$ there, assigning the value of μ on \mathcal{S}' arbitrarily. With μ satisfying these two equations, for Γ we have $\lambda^1 = 0$ everywhere on $T_{\circ}^n M$ and $\lambda^2 = 0$ on \mathcal{S} . But from the consistency condition for λ^1 and λ^2 applied to Γ we have $\Delta^1(\lambda^2) = (n - 1)\lambda^2$ on $T_{\circ}^n M$. It follows from Lemma 3 and the fact that $\lambda^2 = 0$ on \mathcal{S}_2 that $\lambda^2 = 0$ everywhere. \square

We suggest, somewhat tentatively, that homogeneous higher-order differential equation fields such that $[\Delta^1, \Gamma] = \Gamma$ and $[\Delta^2, \Gamma] = 2\Delta^1$ should be considered as generalized sprays.

Finally in this section we observe that a homogeneous differential equation field on $T_\circ^n M$ does indeed pass to the quotient under the action of L^{n+} .

Theorem 2. *Let Γ be a homogeneous differential equation field on $T_\circ^n M$. Then the involutive distribution \mathfrak{D}^+ defines an oriented line-element field on $P^+T^n M$.*

Proof. The distribution \mathfrak{D}^+ evidently projects to a line-element field on $P^+T^n M$. To orient it we take, at any point $\rho_n^+(j_0^n \gamma)$, the positive multiples of $\rho_{n*}^+ \Gamma_{j_0^n \gamma}$ for any representative point $j_0^n \gamma \in T_\circ^n M$. \square

4 Parametric Lagrangians

To show that the theory described above is not vacuous, we devote this section to an important source of examples, the higher-order differential equation fields obtained from parametric variational problems. (Related results, but for second-order variational problems only, may be found in [8]. Multiple-integral parametric variational problems are discussed at length in [3], and indeed the methods used here are related to those used in that paper, but of course specialized to the case of a single integral.)

Let L be a Lagrangian function, defined on $T_\circ^n M$. The *Hilbert form* of L is the 1-form

$$\vartheta = \sum_{p=0}^{n-1} \frac{(-1)^p}{(p+1)!} d_{\mathbb{T}}^p S^{p+1} dL$$

defined on $T_\circ^{2n-1} M$. This may be used to construct the Euler-Lagrange form of the Lagrangian,

$$\varepsilon = dL - d_{\mathbb{T}} \vartheta$$

defined on $T_\circ^{2n} M$. By construction, therefore,

$$\varepsilon = dL - d_{\mathbb{T}} \sum_{p=0}^{n-1} \frac{(-1)^p}{(p+1)!} d_{\mathbb{T}}^p S^{p+1} dL = \sum_{p=0}^n \frac{(-1)^p}{p!} d_{\mathbb{T}}^p S^p dL.$$

We may see that ε is horizontal over M , because $Sd_{\mathbb{T}} = d_{\mathbb{T}} S + 1$ (modulo pullback), so that

$$Sd_{\mathbb{T}}^p = d_{\mathbb{T}}^p S + pd_{\mathbb{T}}^{p-1}$$

and therefore

$$Sd_{\mathbb{T}}^p S^p = d_{\mathbb{T}}^p S^{p+1} + pd_{\mathbb{T}}^{p-1} S^p.$$

Thus $S\varepsilon$ is a collapsing sum, and indeed

$$S\varepsilon = \sum_{p=0}^n \frac{(-1)^p}{p!} d_{\mathbb{T}}^p S^{p+1} dL + \sum_{p=1}^n \frac{p(-1)^p}{p!} d_{\mathbb{T}}^{p-1} S^p dL = \frac{(-1)^n}{n!} d_{\mathbb{T}}^n S^{n+1} dL = 0$$

as L has order n .

Lemma 4. *The Euler-Lagrange form ε vanishes along the extremals of any fixed-endpoint variational problem defined by L .*

Proof. An extremal of such a problem is a curve γ in M such that

$$\int_a^b (\tilde{j}^n \gamma)^* \mathcal{L}_{X^n} L dt = 0$$

for any vector field X on M vanishing at the (fixed) image of the endpoints of $[a, b]$, where X^n denotes the prolongation of X to $T_{\circ}^n M$. Then

$$\int_a^b (\tilde{j}^n \gamma)^* \mathcal{L}_{X^n} L dt = \int_a^b (\tilde{j}^n \gamma)^* (i_{X^n} dL) dt + \int_a^b (\tilde{j}^n \gamma)^* (di_{X^n} L) dt;$$

but

$$\int_a^b (\tilde{j}^n \gamma)^* (di_{X^n} L) dt = \int_a^b d((\tilde{j}^n \gamma)^* i_{X^n} L) dt = [(\tilde{j}^n \gamma)^* i_{X^n} L]_a^b = 0$$

as X vanishes at the endpoints, so that

$$\int_a^b (\tilde{j}^n \gamma)^* \mathcal{L}_{X^n} L dt = \int_a^b (\tilde{j}^n \gamma)^* (i_{X^n} dL) dt = \int_a^b (\tilde{j}^{2n} \gamma)^* (i_{X^{2n}} (\varepsilon + d_{\mathbb{T}} \vartheta)) dt.$$

But now,

$$\begin{aligned} \int_a^b (\tilde{j}^{2n} \gamma)^* (i_{X^{2n}} d_{\mathbb{T}} \vartheta) dt &= \int_a^b (\tilde{j}^{2n} \gamma)^* (d_{\mathbb{T}} i_{X^{2n-1}} \vartheta) dt \\ &= \int_a^b d((\tilde{j}^{2n-1} \gamma)^* (i_{X^{2n-1}} \vartheta)) dt = [(\tilde{j}^{2n-1} \gamma)^* (i_{X^{2n-1}} \vartheta)]_a^b = 0 \end{aligned}$$

because vector field prolongations commute with total derivatives, and the pull-back of $d_{\mathbb{T}}$ to \mathbb{R} is d . Thus

$$\int_a^b (\tilde{j}^n \gamma)^* \mathcal{L}_{X^n} L dt = \int_a^b (\tilde{j}^{2n} \gamma)^* (i_{X^{2n}} \varepsilon) dt = \int_a^b (\tilde{j}^{2n} \gamma)^* (i_X \varepsilon) dt$$

because ε is horizontal over M . As X is arbitrary, it follows that $(\tilde{j}^{2n} \gamma)^* \varepsilon = 0$. \square

Now suppose that L is positively homogeneous; that is, that it satisfies the Zermelo conditions

$$\Delta^1(L) = L, \quad \Delta^r(L) = 0 \quad (r \geq 2).$$

Such a Lagrangian is called a *parametric Lagrangian*, and the geodesics of the corresponding variational problem are invariant under orientation-preserving reparametrizations [3]; in other words, they are *paths*. We shall show that (subject to suitable regularity conditions) they are also the geodesics of homogeneous differential equation fields. In order to investigate this, we first establish a technical lemma.

Lemma 5. *The Hilbert form ϑ and the Euler-Lagrange form ε satisfy*

$$i_{\mathbb{T}}\vartheta = L, \quad i_{\mathbb{T}}d\vartheta = -\varepsilon$$

(omitting the pull-back maps).

Proof. From $i_{\mathbb{T}}d_{\mathbb{T}} = d_{\mathbb{T}}i_{\mathbb{T}}$ and $i_{\mathbb{T}}S^p = i_{\Delta^p}$ we obtain

$$\begin{aligned} i_{\mathbb{T}}d_{\mathbb{T}}^p S^{p+1}dL &= d_{\mathbb{T}}^p i_{\mathbb{T}}S^{p+1}dL = d_{\mathbb{T}}^p i_{\Delta^{p+1}}dL \\ &= \begin{cases} i_{\Delta^1}dL = L & (p = 0) \\ 0 & (p > 0) \end{cases} \end{aligned}$$

so that

$$i_{\mathbb{T}}\vartheta = \sum_{p=0}^{n-1} \frac{(-1)^p}{(p+1)!} i_{\mathbb{T}}d_{\mathbb{T}}^p S^{p+1}dL = L.$$

In addition,

$$d_{\mathbb{T}}\vartheta = di_{\mathbb{T}}\vartheta + i_{\mathbb{T}}d\vartheta = dL + i_{\mathbb{T}}d\vartheta$$

so that, from the definition of ε ,

$$\varepsilon = dL - d_{\mathbb{T}}\vartheta = -i_{\mathbb{T}}d\vartheta. \quad \square$$

Proposition 5. *The characteristic distribution of the 2-form $d\vartheta$ satisfies*

$$\{\Gamma, \Delta^1, \Delta^2, \dots, \Delta^{2n-1}\} \subset \ker d\vartheta$$

whenever Γ is a differential equation field on $T_{\circ}^n M$ whose geodesics are extremals of the variational problem with fixed endpoints defined by L .

Proof. We remark first that ϑ satisfies $i_{\Delta^r}\vartheta = 0$ and $\mathcal{L}_{\Delta^r}\vartheta = 0$ for $1 \leq r \leq 2n - 1$ ([3], Proposition 6.1 and Theorem 6.4). It follows that

$$i_{\Delta^r}d\vartheta = \mathcal{L}_{\Delta^r}\vartheta - di_{\Delta^r}\vartheta = 0,$$

and so $\{\Delta^1, \Delta^2, \dots, \Delta^{2n-1}\} \subset \ker d\vartheta$.

Now let $\tilde{\Gamma}$ be a differential equation field on $T_{\circ}^{2n-1}M$, so that

$$\tilde{\Gamma} : T_{\circ}^{2n-1}M \rightarrow T_{\circ}^{2n}M,$$

and let $\Gamma : T_{\circ}^{2n-1}M \rightarrow i(T_{\circ}^{2n}M) \subset TT_{\circ}^{2n-1}M$ be the corresponding vector field.

Suppose in particular that the integral curves of Γ (which must, necessarily, be prolongations $\tilde{j}^{2n-1}\gamma$ of curves γ in M) are such that γ is always an extremal of the variational problem. Each point of $T_{\circ}^{2n-1}M$ must lie on such an integral curve, and we may suppose (by translation in the domain) that the point in question is $\tilde{j}^{2n-1}\gamma(0)$, in other words that the point may be written as $j_0^{2n-1}\gamma$.

The integral curve property means that $\tilde{\Gamma}(j_0^{2n-1}\gamma) = j_0^{2n}\gamma$, so that

$$\Gamma_{j_0^{2n-1}\gamma} = (i \circ \tilde{\Gamma})(j_0^{2n-1}\gamma) = i(j_0^{2n}\gamma);$$

thus for any vector $\xi \in T_{j_0^{2n-1}\gamma}T_{\circ}^{2n-1}M$ we have

$$\begin{aligned} d\vartheta_{j_0^{2n-1}\gamma}(\Gamma_{j_0^{2n-1}\gamma}, \xi) &= d\vartheta_{j_0^{2n-1}\gamma}(i(j_0^{2n}\gamma), \xi) \\ &= \langle (i_{\Gamma}d\vartheta)_{j_0^{2n}\gamma}, \xi \rangle \\ &= \langle -\varepsilon_{j_0^{2n}\gamma}, \xi \rangle \\ &= 0 \end{aligned}$$

because γ is an extremal of L so that $\varepsilon_{j_0^{2n}\gamma} = 0$. This calculation also uses the fact that the contraction i_{Γ} is just the inclusion map $i : T_{\circ}^{2n}M \rightarrow TT_{\circ}^{2n-1}M$ in disguise. \square

We say that the Lagrangian is *regular* if the characteristic distribution of $d\vartheta$ has dimension $2n$. Any differential equation field in this distribution is an Euler-Lagrange field of the variational problem. Since the characteristic distribution of a closed 2-form is involutive we have the following theorem.

Theorem 3. *Any Euler-Lagrange field of a regular parametric Lagrangian is homogeneous.*

5 Projective equivalence and reparametrization

We now return to the study of general higher-order differential equation fields.

Definition 2. *Let Γ, Γ' be homogeneous. Then if $\Gamma' - \Gamma = \mu\Delta^n$ for some function μ we say that Γ and Γ' are projectively equivalent.*

Projective equivalence is an equivalence relation. The projective equivalence class of a homogeneous differential equation field consists of all the differential equation fields in the involutive distribution \mathfrak{D}^+ . We may summarize the result of Theorem 1 by saying that every projective equivalence class of a homogeneous differential equation field contains a generalized spray.

We have seen that the Euler-Lagrange fields of a regular parametric Lagrangian consist of a projective equivalence class of homogeneous differential equation fields. So we should expect in this case that the geodesics of projectively equivalent homogeneous differential equation fields should define the same paths, that is, differ only in parametrization. In this section we shall show that this is true for any projective equivalence class of homogeneous differential equation fields, whether or not they come from a parametric Lagrangian.

In fact we shall prove two results about projective equivalence in homogeneous systems: firstly, if Γ' is projectively equivalent to Γ then a geodesic of Γ' is a reparametrization of a geodesic of Γ ; secondly, the jet group acts on differential equation fields in such a way as to map any homogeneous field to one which is projectively equivalent to it.

Theorem 4. *Let Γ be a homogeneous differential equation field, and Γ' another which is projectively equivalent to it. Let γ' be a geodesic of Γ' . Then there is a geodesic γ of Γ such that γ' is obtained from γ by an orientation-preserving reparametrization.*

Proof. By assumption the distribution \mathfrak{D}^+ is involutive, and thus integrable. Let \mathfrak{L} be a leaf of \mathfrak{D}^+ . Then Γ' is tangent to \mathfrak{L} , and any integral curve of Γ' which meets \mathfrak{L} lies in it. If γ' is a geodesic of Γ' then $\tilde{j}^n\gamma'$ is an integral curve of Γ' : let \mathfrak{L} be in fact the leaf of \mathfrak{D}^+ containing (the image of) $\tilde{j}^n\gamma'$. Since $\mathfrak{D} \subset \mathfrak{D}^+$, \mathfrak{L} is invariant under the action of L^{n+} , the identity component of the jet group L^n , and consists of the orbits of points on $\tilde{j}^n\gamma'$ under the action of L^{n+} . Under projection $\tau_n : T^n M \rightarrow M$, we have of course $\tau_n \circ \tilde{j}^n\gamma' = \gamma'$, and for any $g \in L^{n+}$, $\tau_n \circ g = \tau_n$. Thus $\tau_n(\mathfrak{L})$ is the 1-dimensional submanifold of M which is the oriented path of γ' . Now Γ also belongs to \mathfrak{D}^+ , and so likewise any integral curve of Γ which meets \mathfrak{L}

lies in it. Take any point of \mathfrak{L} , and the integral curve of Γ through that point: it is of the form $\tilde{j}^n \gamma$ where γ is a geodesic of Γ . Then $\gamma = \tau_n \circ \tilde{j}^n \gamma$ lies in $\tau_n(\mathfrak{L})$, that is to say, γ' and γ have the same oriented path. Since we are restricted to $(y_1^i) \neq 0$, we must assume that γ and γ' have nowhere vanishing tangent vectors. It follows that one is a reparametrization of the other, by a reparametrization that preserves orientation. \square

Corollary 4. *Let Γ be a homogeneous differential equation field of order $n + 1$. Let γ be the geodesic of Γ with initial conditions $(y^i, y_1^i, \dots, y_n^i)$. Let (z_1^i, \dots, z_n^i) be the image of (y_1^i, \dots, y_n^i) under some element of L^{n+} (so that (z_1^i) is a positive scalar multiple of (y_1^i)). Then the geodesic of Γ with initial conditions $(y^i, z_1^i, \dots, z_n^i)$ is obtained from γ by an orientation-preserving reparametrization.*

The second result is rather more complicated to prove, and requires some preliminaries.

We shall need to distinguish notationally between the fundamental vector fields of the actions of L^n on $T^n M$ and L^{n+1} on $T^{n+1} M$. Let us denote by Δ_n^r the vector field operating on $T^n M$: that is to say

$$\Delta_n^r = \sum_{s=r}^n \frac{s!}{(s-r)!} y_{s+1-r}^i \frac{\partial}{\partial y_s^i}.$$

With a bit of licence we can write, for $r = 1, 2, \dots, n$,

$$\Delta_{n+1}^r = \Delta_n^r + \frac{(n+1)!}{(n+1-r)!} y_{n+2-r}^i \frac{\partial}{\partial y_{n+1}^i},$$

while

$$\Delta_{n+1}^{n+1} = (n+1)! y_1^i \frac{\partial}{\partial y_{n+1}^i}.$$

In particular,

$$\Delta_{n+1}^r(y_{n+1}^i) = \frac{(n+1)!}{(n+1-r)!} y_{n+2-r}^i, \quad \Delta_{n+1}^{n+1}(y_{n+1}^i) = (n+1)! y_1^i.$$

Now the homogeneity conditions for an $(n+1)$ th order differential equation field Γ , when expressed in terms of the Γ^i , are

$$\Delta_n^1(\Gamma^i) = (n+1)\Gamma^i + n!\lambda^1 y_1^i, \quad \Delta_n^r(\Gamma^i) = \frac{(n+1)!}{(n+1-r)!} y_{n+2-r}^i + n!\lambda^r y_1^i.$$

Comparison with the formulae for $\Delta_{n+1}^r(y_{n+1}^i)$ is suggestive.

Consider the section $\tilde{\Gamma}$ of $T_{\circ}^{n+1}M \rightarrow T_{\circ}^n M$ corresponding to Γ , given by $y_{n+1}^i = \Gamma^i(y_0^j, y_1^j, \dots, y_n^j)$. We have, on $\text{im } \tilde{\Gamma}$, with $r = 2, \dots, n$,

$$\begin{aligned}\Delta_{n+1}^1(y_{n+1}^i - \Gamma^i) &= (n+1)(y_{n+1}^i - \Gamma^i) - n!\lambda^1 y_1^i = -n!\lambda^1 y_1^i \\ \Delta_{n+1}^r(y_{n+1}^i - \Gamma^i) &= -n!\lambda^r y_1^i \\ \Delta_{n+1}^{n+1}(y_{n+1}^i - \Gamma^i) &= (n+1)!y_1^i.\end{aligned}$$

One way of restating this is that for $r = 1, 2, \dots, n$ the vector fields

$$\Delta_{n+1}^r + \frac{\lambda^r}{(n+1)} \Delta_{n+1}^{n+1}$$

(strictly speaking one should write $\tau_{n+1,n}^* \lambda^r$ rather than just λ^r) are tangent to $\text{im } \tilde{\Gamma}$. Set

$$\Delta_{n+1}^r + \frac{\lambda^r}{(n+1)} \Delta_{n+1}^{n+1} = \tilde{\Delta}^r.$$

The $\tilde{\Delta}^r$ are well-defined vector fields on $T^{n+1}M$.

Proposition 6. For $r, s = 1, 2, \dots, n$

$$[\tilde{\Delta}^r, \tilde{\Delta}^s] = \begin{cases} (r-s)\tilde{\Delta}^{r+s-1} & \text{if } r+s \leq n+1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We have

$$[\tilde{\Delta}^1, \tilde{\Delta}^r] = [\Delta_{n+1}^1, \Delta_{n+1}^r] + \frac{(\Delta_n^1(\lambda^r) - \Delta_n^r(\lambda^1) - n\lambda^r)}{(n+1)} \Delta_{n+1}^{n+1}$$

while

$$[\tilde{\Delta}^r, \tilde{\Delta}^s] = [\Delta_{n+1}^r, \Delta_{n+1}^s] + \frac{(\Delta_n^r(\lambda^s) - \Delta_n^s(\lambda^r))}{(n+1)} \Delta_{n+1}^{n+1}$$

(the λ^r are functions on $T^n M$). The stated results follow easily, using the consistency conditions on the λ^r . The one case which is not entirely straightforward is $r+s = n+2$, when we have

$$\begin{aligned}[\tilde{\Delta}^r, \tilde{\Delta}^s] &= [\Delta_{n+1}^r, \Delta_{n+1}^s] + \frac{(\Delta_n^r(\lambda^s) - \Delta_n^s(\lambda^r))}{(n+1)} \Delta_{n+1}^{n+1} \\ &= (r-s)\Delta_{n+1}^{n+1} - \frac{(n+1)(r-s)}{(n+1)} \Delta_{n+1}^{n+1} = 0,\end{aligned}$$

since in this case $\Delta_n^r(\lambda^s) - \Delta_n^s(\lambda^r) = -(n+1)(r-s)$. \square

That is to say, when Γ is homogeneous there is a representation of the algebra \mathfrak{l}^n on $\text{im } \tilde{\Gamma}$.

Notice also that for $r \geq 2$, $[\Delta_{n+1}^{n+1}, \tilde{\Delta}^r] = 0$, since $[\Delta_{n+1}^{n+1}, \Delta_{n+1}^r] = 0$ and $\Delta_{n+1}^{n+1}(\lambda^r) = 0$.

Let ϕ be an orientation-preserving local diffeomorphism of \mathbb{R} leaving 0 fixed. Let ϕ_n be the diffeomorphism of $T^n M$ given by the action of $j_0^n \phi \in L^n$. We have $\tau_{n+1,n} \circ \phi_{n+1} = \phi_n \circ \tau_{n+1,n}$. Let Γ be an $(n+1)$ st-order differential equation field, $\tilde{\Gamma} : T_\circ^{n+1} M \rightarrow T_\circ^{n+1} M$ the corresponding section. Since $\tau_{n+1,n} \circ \tilde{\Gamma} = \text{id}_{T_\circ^n M}$, $\phi_{n+1}^{-1} \circ \tilde{\Gamma} \circ \phi_n$ is also a section of $\tau_{n+1,n}$, and so defines a new $(n+1)$ st-order differential equation field, say Γ_ϕ . Suppose that Γ is any homogeneous $(n+1)$ st-order differential equation field. We shall show that for any ϕ , there is a function ν_ϕ on $T_\circ^n M$ such that

$$\Gamma_\phi^i(y_r^j) = \Gamma^i(y_r^j) + \nu_\phi(y_r^j)y_1^i.$$

Note that when this holds $\Gamma_\phi = \Gamma + \nu_\phi \Delta_n^n$, so Γ_ϕ is projectively equivalent to Γ . From the point of view of representation by sections one might say that $\tilde{\Gamma}_\phi$ is obtained by sliding $\tilde{\Gamma}$ along the integral curves of Δ_{n+1}^{n+1} . To be more precise, recalling that $T_\circ^{n+1} M$ is an affine bundle over $T_\circ^n M$ modelled on the bundle of very vertical tangent vectors on $T_\circ^n M$, $\tilde{\Gamma}_\phi$ is the translate of $\tilde{\Gamma}$ by the very vertical vector field $\nu_\phi \Delta_{n+1}^{n+1}$.

Theorem 5. *Let Γ be a homogeneous differential equation field and let ϕ be an orientation-preserving local diffeomorphism of \mathbb{R} leaving 0 fixed. Then Γ_ϕ is projectively equivalent to Γ .*

Proof. Note first that $\Gamma_{\phi_1 \circ \phi_2} = (\Gamma_{\phi_1})_{\phi_2}$. So if for some ϕ_1, ϕ_2 and every homogeneous Γ , it is the case that Γ_{ϕ_1} is projectively equivalent to Γ and Γ_{ϕ_2} is projectively equivalent to Γ then $\Gamma_{\phi_1 \circ \phi_2}$ is projectively equivalent to Γ . We pointed out in Section 2 that the finite order jet group L^m is the semi-direct product of L^1 and K^m , the subgroup consisting of the jets of diffeomorphisms with derivative at 0 equal to the identity, and further that every element of the latter is the exponential of some element of its Lie algebra. So the result about Γ_ϕ , or equivalently $\tilde{\Gamma}_\phi$, will be proved if it can be proved when $\phi(x) = kx$ (k a positive constant) and when ϕ_{n+1} is the exponential of an element of $\langle \Delta_{n+1}^r \rangle$ for $r = 2, \dots, n$.

Consider first the case where $\phi(x) = kx$. Then $\phi_n(y_r^i) = (k^r y_r^i)$. We have $\Delta_n^1(\Gamma^i) = (n+1)\Gamma^i + n!\lambda^1 y_1^i$, which integrates to give

$$e^{-(n+1)t} \Gamma^i(e^{rt} y_r^j) - \Gamma^i(y_r^j) = n! \int_0^t e^{-ns} \lambda^1(e^{rs} y_r^j) y_1^i ds.$$

Then

$$\begin{aligned}\tilde{\Gamma}_\phi(y_r^j) &= \phi_{n+1}^{-1} \tilde{\Gamma}(\phi_n(y_r^j)) = \phi_{n+1}^{-1}(k^r y_r^i, \Gamma^i(k^r y_r^j)) \\ &= (y_r^i, k^{-(n+1)} \Gamma^i(k^r y_r^j)) \\ &= (y_r^i, \Gamma^i(y_r^j) + \nu_\phi(y_r^j) y_1^i)\end{aligned}$$

where

$$\nu_\phi(y_r^j) = n! \int_0^{\log k} e^{-ns} \lambda^1(e^{rs} y_r^j) ds.$$

Next, let κ_t be the 1-parameter group generated by $\sum_{p=2}^{n+1} k_p \Delta_{n+1}^p$ for some constants k_p . Now

$$\sum_{p=2}^{n+1} k_p \Delta_{n+1}^p = \sum_{p=2}^n (k_p \tilde{\Delta}^p) + \left(k_{n+1} - \frac{1}{n+1} \sum_{p=2}^n k_p \lambda^p \right) \Delta_{n+1}^{n+1}.$$

Let $\tilde{\kappa}_t$ be the flow generated by $\sum_{p=2}^n (k_p \tilde{\Delta}^p)$, and $\bar{\kappa}_t$ the flow generated by $\sum_{p=2}^n (k_p \Delta_n^p)$, so that

$$\bar{\kappa}_t \circ \tau_{n+1,n} = \tau_{n+1,n} \circ \tilde{\kappa}_t = \tau_{n+1,n} \circ \kappa_t.$$

For any $(y_r^i) \in T^n M$ let $\tau(t, y_r^i)$ be the solution of the first-order ordinary differential equation

$$\frac{d\tau}{dt} = k_{n+1} - \frac{1}{n+1} \sum_{p=2}^n k_p \lambda^p(\bar{\kappa}_t(y_r^j))$$

such that $\tau(0, y_r^i) = 0$. We claim that

$$\kappa_t(\tilde{\Gamma}(y_r^i)) = \phi_{\tau(t, y_r^i)}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i))),$$

where ϕ_t^{n+1} is the 1-parameter group generated by Δ_{n+1}^{n+1} . Consider the $(\dim T^n M + 1)$ -dimensional submanifold S of $T_{\circ}^{n+1} M$ consisting of points $\phi_t^{n+1}(\tilde{\Gamma}(y_r^i))$, $t \in \mathbb{R}$, $(y_r^i) \in T_{\circ}^n M$. Since $\sum_{p=2}^n (k_p \tilde{\Delta}^p)$ is tangent to $\text{im } \tilde{\Gamma}$, $\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i)) \in \text{im } \tilde{\Gamma}$ for all t , and so $\phi_{\tau(t, y_r^i)}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i))) \in S$ for all t . We can use coordinates (t, y_r^i) on S ; with respect to these coordinates ϕ_t^{n+1} is just translation of the first coordinate, and

$$\phi_{\tau(t, y_r^i)}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i))) = (\tau(t, y_r^i), \bar{\kappa}_t(y_r^i)).$$

Moreover, since $[\Delta_{n+1}^{n+1}, \tilde{\Delta}^r] = 0$ the coordinate representation of $\tilde{\Delta}^r$ coincides with its coordinate representation on $\text{im } \tilde{\Gamma}$, which is the same as the coordinate representation of Δ_n^r . Thus the tangent vector to the curve $t \mapsto \phi_{\tau(t, y_r^i)}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i)))$ at t , in coordinate form, is

$$\dot{\tau}(t, y_r^i) \frac{\partial}{\partial t} + \sum_{p=2}^n k_p \Delta_n^p(\bar{\kappa}_t(y_r^i)).$$

But $\partial/\partial t$ is the coordinate representation of Δ_{n+1}^{n+1} on S , and we conclude that $t \mapsto \phi_{\tau(t, y_r^i)}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i)))$ is an integral curve of the vector field

$$\left(k_{n+1} - \frac{1}{n+1} \sum_{p=2}^n k_p \lambda^p \right) \Delta_{n+1}^{n+1} + \sum_{p=2}^n (k_p \tilde{\Delta}^p) = \sum_{p=2}^{n+1} k_p \Delta_{n+1}^p.$$

Since $\phi_{\tau(0, y_r^i)}^{n+1}(\tilde{\kappa}_0(\tilde{\Gamma}(y_r^i))) = \tilde{\Gamma}(y_r^i)$, we see that $t \mapsto \phi_{\tau(t, y_r^i)}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(y_r^i)))$ is the integral curve of $\sum_{p=2}^{n+1} k_p \Delta_{n+1}^p$ starting at $\tilde{\Gamma}(y_r^i)$, that is, it coincides with $\kappa_t(\tilde{\Gamma}(y_r^i))$ as claimed.

It follows that

$$\kappa_t(\tilde{\Gamma}(\bar{\kappa}_{-t}(y_r^i))) = \phi_{\tau(t, \bar{\kappa}_{-t}(y_r^i))}^{n+1}(\tilde{\kappa}_t(\tilde{\Gamma}(\bar{\kappa}_{-t}(y_r^i)))).$$

But $\bar{\kappa}_t$ maps $\text{im } \tilde{\Gamma}$ to itself, and $\tau_{n+1, n} \circ \bar{\kappa}_t = \bar{\kappa}_t \circ \tau_{n+1, n}$, so $\bar{\kappa}_t(\tilde{\Gamma}(\bar{\kappa}_{-t}(y_r^i))) = \tilde{\Gamma}(y_r^i)$. Thus

$$\kappa_t(\tilde{\Gamma}(\bar{\kappa}_{-t}(y_r^i))) = \phi_{\tau(t, \bar{\kappa}_{-t}(y_r^i))}^{n+1}(\tilde{\Gamma}(y_r^i)) = (y_r^i, \Gamma^i(y_r^j) + \tau(t, \bar{\kappa}_{-t}(y_r^j))y_1^i).$$

This establishes the required result with $\kappa_t = \phi_{n+1}^{-1}$ and $\nu_\phi(y_r^i) = \tau(t, \phi_n(y_r^i))$. \square

This result shows that a projective equivalence class is invariant under the action of the group L^{n+} on $T_\circ^n M$.

6 Examples

Take M to be Euclidean space of dimension m and the y^i to be Euclidean coordinates. With the scalar product etc. denoted in the usual way, consider the third-order differential equation field defined by

$$\Gamma^i(y_1, y_2) = - \left(\frac{2|y_1|^2|y_2|^2 + (y_1 \cdot y_2)^2}{2|y_1|^4} \right) y_1^i + 3 \frac{(y_1 \cdot y_2)}{|y_1|^2} y_2^i.$$

This is easily seen to be homogeneous, with $\lambda_1 = \lambda_2 = 0$.

We now explain the geometrical significance of this third-order system. Denote the differential equation field by Γ . It is easy to see that

$$\begin{aligned}\Gamma(|y_1|^2) &= 2(y_1 \cdot y_2) \\ \Gamma(y_1 \cdot y_2) &= \frac{2(y_1 \cdot y_2)}{|y_1|^2}.\end{aligned}$$

It follows that Γ is tangent to the submanifold on which $|y_1| = 1$, $y_1 \cdot y_2 = 0$. This consists of the 2-jets of curves parametrized by Euclidean arc-length. The restriction of Γ^i to this submanifold is just $-|y_2|^2 y_1^i$. The corresponding vector differential equation, considering the y^i as the components of a vector \mathbf{r} with respect to an orthonormal frame and using overdots to indicate differentiation with respect to arc-length s , may be written

$$\ddot{\mathbf{r}} = -|\dot{\mathbf{r}}|^2 \dot{\mathbf{r}}.$$

Along any solution curve of this differential equation the 2-plane spanned by $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ is constant, and so the curve is a plane curve. Moreover, $|\dot{\mathbf{r}}|$ is the curvature of the curve, and

$$\frac{d}{ds} |\dot{\mathbf{r}}|^2 = 2\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -2|\dot{\mathbf{r}}|^2 \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = 0.$$

So the curve is a plane curve of constant curvature, that is, a circle (or if the curvature is zero, a straight line).

Now the submanifold $|y_1| = 1$, $y_1 \cdot y_2 = 0$ in this case is just the submanifold \mathcal{S}' of Lemma 2. It follows that every point of $T_o^2 M$ can be written in the form $\phi_{t_1}^1(\phi_{t_2}^2(z))$ for some $z \in \mathcal{S}'$, where ϕ_t^1 and ϕ_t^2 are the 1-parameter groups generated by Δ^1 and Δ^2 . So in particular every point of $T_o^2 M$ is the image by some jet map of a point of \mathcal{S}' . It follows from Corollary 4 that the homogeneous system we started with also has the property that its geodesics are circles.

There will of course be other homogeneous differential equation fields with the same property, namely those projectively equivalent to Γ . For example we could take the simpler system with

$$\Gamma^i(y_1, y_2) = 3 \frac{(y_1 \cdot y_2)}{|y_1|^2} y_2^i.$$

The corresponding field is homogeneous, now with $\lambda^2 = (y_1 \cdot y_2)/|y_1|^2$ (but $\lambda^1 = 0$ still). However, this field is not tangent to \mathcal{S}' , so while it is true (by

Theorem 4) that its geodesics are circles, it is not so straightforward to see this.

We next exhibit a fourth-order system in Euclidean space derived from a parametric second-order Lagrangian. Consider the Lagrangian function L on T^2_0M given (in terms of Euclidean coordinates (y^i) as before) by

$$L(y, y_1, y_2) = \frac{|y_1|^2|y_2|^2 - (y_1 \cdot y_2)^2}{|y_1|^5}.$$

This satisfies $\Delta^1(L) = L$ and $\Delta^2(L) = 0$, so is parametric. In fact L is closely related to the (first) curvature of a curve, which may be considered as a function κ on T^2_0M : we have $L = \kappa^2|y_1|$ (see for example [9]). By deriving the Euler-Lagrange equations for L one obtains the projective equivalence class of fourth-order equations, necessarily homogeneous, containing the one given by

$$\Gamma^i = -3\frac{(y_2 \cdot y_3)}{|y_1|^2}y_1^i + \left(\frac{5|y_1|^2|y_2|^2 - 35(y_1 \cdot y_2)^2 + 8|y_1|^2(y_1 \cdot y_3)}{2|y_1|^4} \right) y_2^i + 6\frac{(y_1 \cdot y_2)}{|y_1|^2}y_3^i.$$

For this particular representative Γ we have $\lambda^1 = 0$, but λ^2 is nonzero, and so of course is λ^3 . We find that $\Gamma(|y_1|, \Gamma(y_1 \cdot y_2)$ and $\Gamma(y_1 \cdot y_3 + |y_2|^2)$ are contained in the ring generated by these functions, which means that Γ is tangent to the submanifold \mathcal{S}'' of T^2_0M where $|y_1| = 1$, $y_1 \cdot y_2 = y_1 \cdot y_3 + |y_2|^2 = 0$, whose points consist of jets of curves parametrized by arc-length. On \mathcal{S}'' we obtain the differential equation

$$\ddot{\mathbf{r}} = -3(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}} - \frac{3}{2}|\ddot{\mathbf{r}}|^2\dot{\mathbf{r}}$$

(in terms of arc-length parameter). This may be more succinctly written as

$$\frac{d}{ds} \left(\ddot{\mathbf{r}} + \frac{3}{2}|\ddot{\mathbf{r}}|^2\dot{\mathbf{r}} \right) = 0.$$

This is the Euclidean version of an equation discussed recently by Matsyuk in the context of ‘Zitterbewegung’ [7] (eq (38) with $A = 0$ and $R = 0$).

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