Hyperplanes of $DW(5, \mathbb{K})$ with $\mathbb{K}$ a perfect field of characteristic 2

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Abstract

Let $\mathbb{K}$ be a perfect field of characteristic 2. In this paper, we classify all hyperplanes of the symplectic dual polar space $DW(5, \mathbb{K})$ that arise from its Grassmann embedding. We show that the number of isomorphism classes of such hyperplanes is equal to $5 + N$, where N is the number of equivalence classes of the following equivalence relation R on the set $\{\lambda \in \mathbb{K} \,|\, X^2 + \lambda X + 1 \text{ is irreducible in } \mathbb{K}[X]\}$: $(\lambda_1, \lambda_2) \in R$ whenever there exists an automorphism σ of $\mathbb{K}$ and an $a \in \mathbb{K}$ such that $(\lambda_2^\sigma)^{-1} = \lambda_1^{-1} + a^2 + a$.

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1 Introduction

Let $n \geq 2$, let $\mathbb{K}$ be a perfect field of characteristic 2 and let V be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a nondegenerate alternating bilinear form. With this bilinear form there corresponds a symplectic polarity ζ of the projective space $\mathrm{PG}(V) = \mathrm{PG}(2n-1, \mathbb{K})$.

Associated with the polarity ζ there is a symplectic polar space $W(2n-1, \mathbb{K})$ (see Tits [29]) and a symplectic dual polar space $DW(2n-1, \mathbb{K})$ (see Cameron [5]). The singular subspaces of $W(2n-1, \mathbb{K})$ are the subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$ which are absolute with respect to ζ. We denote by $\mathcal{P}$ the set of all maximal singular subspaces of $W(2n-1, \mathbb{K})$. For every next-to-maximal singular subspace β of $W(2n-1, \mathbb{K})$, let L_β denote the set of all maximal singular subspaces of $W(2n-1, \mathbb{K})$ containing β, and let $\mathcal{L}$ denote the set of all sets L_β which can be obtained in this way. Then $DW(2n-1, \mathbb{K})$ is the point-line geometry with point-set $\mathcal{P}$ and line-set $\mathcal{L}$.

Let $\bigwedge^n V$ denote the n-th exterior power of V. For every point $\alpha = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ of $DW(2n-1, \mathbb{K})$, let $e(\alpha)$ be the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ of $\mathrm{PG}(\bigwedge^n V)$. The subspace Σ of $\mathrm{PG}(\bigwedge^n V)$ generated by all points $e(\alpha)$, $\alpha \in \mathcal{P}$, is $\left(\binom{2n}{n} - \binom{2n}{n-2} - 1\right)$-dimensional (see e.g. Brouwer [3] or De Bruyn [16]). By Cooperstein [10], the map $\alpha \mapsto e(\alpha)$ defines a full projective embedding of $DW(2n-1, \mathbb{K})$ into Σ. In other words, e is an injective mapping from the point-set of $DW(2n-1, \mathbb{K})$ to the point-set of Σ mapping lines of $DW(2n-1, \mathbb{K})$ to (full) lines of Σ such that the image of e generates the whole projective space Σ. The embedding e is called the *Grassmann embedding* of $DW(2n-1, \mathbb{K})$.

A set $S \neq \mathcal{P}$ of points of $DW(2n-1, \mathbb{K})$ is called a *hyperplane* of $DW(2n-1, \mathbb{K})$ if every line of $DW(2n-1, \mathbb{K})$ intersects S in either the whole line or a singleton. If Π is a hyperplane of the projective space Σ, then $e^{-1}(\Pi \cap e(\mathcal{P}))$ is a hyperplane of $DW(2n-1, \mathbb{K})$. We say that the hyperplane $e^{-1}(\Pi \cap e(\mathcal{P}))$ *arises from* (the Grassmann embedding) e. The aim of this paper is to determine the isomorphism classes of hyperplanes of $DW(5, \mathbb{K})$ that arise from its Grassmann embedding. Except for the case $\mathbb{K} \cong \mathbb{F}_2$ the hyperplanes of $DW(5, \mathbb{K})$ that arise from some projective embedding are precisely the hyperplanes of $DW(5, \mathbb{K})$ that arise from the Grassmann embedding (see the remark at the end of this section).

If x and y are two points of $DW(2n-1, \mathbb{K})$, then we denote by $\mathrm{d}(x, y)$ the distance between x and y in the collinearity graph Δ of $DW(2n-1, \mathbb{K})$ (which has diameter n). The dual polar space $DW(2n-1, \mathbb{K})$ is a *near polygon* ([28], [11]) which means that for every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x. A set X of points of $DW(2n-1, \mathbb{K})$ is called *connected* if the subgraph of Δ induced on X is connected. For every point x of $DW(2n-1, \mathbb{K})$ and every $i \in \mathbb{N}$, $\Delta_i(x)$ denotes the set of points of $DW(2n-1, \mathbb{K})$ at distance i from x. We also define $x^\perp := \Delta_0(x) \cup \Delta_1(x)$. For every nonempty set X of points and every $i \in \mathbb{N}$, $\Delta_i(X)$ is the set of all points y for which $\mathrm{d}(y, X) := \min\{\mathrm{d}(y, x) \,|\, x \in X\} = i$. If x is a point of $DW(2n-1, \mathbb{K})$, then the set H_x of points of $DW(2n-1, \mathbb{K})$ at distance at most $n-1$ from x is a hyperplane of $DW(2n-1, \mathbb{K})$, called the *singular hyperplane* of $DW(2n-1, \mathbb{K})$ *with deepest point* x. The singular hyperplanes of $DW(2n-1, \mathbb{K})$ arise from the Grassmann embedding of $DW(2n-1, \mathbb{K})$, see e.g. Cardinali, De Bruyn and Pasini [7, Section 4.3] or De Bruyn [15, Proposition 2.15].

By Shult [26, Lemma 6.1], every hyperplane of $DW(2n-1, \mathbb{K})$ is a maximal subspace of $DW(2n-1, \mathbb{K})$ and hence its complement is connected. This fact also implies that if H is a hyperplane of $DW(2n-1, \mathbb{K})$ arising from the Grassmann embedding e of $DW(2n-1, \mathbb{K})$, then $\langle e(H) \rangle_\Sigma$ is a hyperplane of Σ and $\langle e(H)\rangle_\Sigma \cap e(\mathcal{P}) = e(H)$. If H_1 and H_2 are two distinct hyperplanes of $DW(2n-1,\mathbb{K})$ arising from e, then we denote by $[H_1,H_2]^*$ the set of all hyperplanes of $DW(2n-1,\mathbb{K})$ of the form $e^{-1}(e(\mathcal{P})\cap\Pi)$ where Π is some hyperplane of Σ containing $\langle e(H_1)\rangle_\Sigma \cap \langle e(H_1)\rangle_\Sigma$. We also define $(H_1,H_2)^* := [H_1,H_2]^* \setminus \{H_1,H_2\}$.

A *quad* of $DW(2n-1,\mathbb{K})$ is the set of all maximal singular subspaces of $W(2n-1,\mathbb{K})$ containing a given $(n-3)$-dimensional singular subspace of $W(2n-1,\mathbb{K})$. The lines and quads through a given point x of $DW(2n-1,\mathbb{K})$ define a point-line geometry $Res(x)$ (natural incidence) which is a projective space isomorphic to $\mathrm{PG}(n-1,\mathbb{K})$. The points and lines of $DW(2n-1,\mathbb{K})$ contained in a quad Q define a point-line geometry $\widetilde{Q}$ which is a generalized quadrangle isomorphic to $DW(3,\mathbb{K}) \cong Q(4,\mathbb{K})$. The Grassmann embedding $e: DW(2n-1,\mathbb{K}) \to \Sigma$ of $DW(2n-1,\mathbb{K})$ induces a full embedding e_Q of $\widetilde{Q}$ into the subspace $\langle e(Q)\rangle_\Sigma$ of Σ. This embedding is isomorphic to the Grassmann embedding of $DW(3,\mathbb{K})$, see e.g. Cardinali, De Bruyn and Pasini [7, Proposition 4.10]. (Although the discussion there was limited to the finite case, the arguments work as well for the infinite case.) The Grassmann embedding of $DW(3,\mathbb{K})$ is isomorphic to the natural embedding of $Q(4,\mathbb{K})$ into $\mathrm{PG}(4,\mathbb{K})$. It is easy to verify that every hyperplane of $Q(4,\mathbb{K})$ is either a singular hyperplane, a full subgrid or an ovoid, an ovoid being a set of points intersecting each line in a singleton. Every singular hyperplane or full subgrid of $Q(4,\mathbb{K})$ arises from the natural embedding of $Q(4,\mathbb{K})$ into $\mathrm{PG}(4,\mathbb{K})$. This is not necessarily true for the ovoids. If an ovoid of $Q(4,\mathbb{K})$ arises from the natural embedding of $Q(4,\mathbb{K})$ into $\mathrm{PG}(4,\mathbb{K})$, then it is called *classical*. So, a classical ovoid is a nonsingular quadric of Witt index 1 in a hyperplane of $\mathrm{PG}(4,\mathbb{K})$.

A *max* of $DW(2n-1,\mathbb{K})$ is the set of all maximal singular subspaces of $W(2n-1,\mathbb{K})$ through a given point x of $W(2n-1,\mathbb{K})$. The points and lines contained in a max M define a point-line geometry $\widetilde{M}$ which is isomorphic to $DW(2n-3,\mathbb{K})$ if $n \geq 3$. If A is a hyperplane of $\widetilde{M}$, then $H_A := \Delta_0(A) \cup \Delta_1(A) = M \cup \Delta_1(A)$ is a hyperplane of $DW(2n-1,\mathbb{K})$, called the *extension of A* ([19, Proposition 1]). The extension of a singular hyperplane of $\widetilde{M}$ is a singular hyperplane of $DW(2n-1,\mathbb{K})$. The extension of a full subgrid of a quad of $DW(5,\mathbb{K})$ arises from the Grassmann embedding of $DW(5,\mathbb{K})$, see [15, Section 2.3]. In Section 3 (more precisely Lemma 3.7), we will show that also the extension of a classical ovoid of a quad of $DW(5,\mathbb{K})$ arises from the Grassmann embedding. If M is a max of $DW(2n-1,\mathbb{K})$ and x is a point not contained in M, then x is collinear with a unique point $\pi_M(x)$ of M, called the *projection* of x onto M. Moreover, $\mathrm{d}(x,y) = 1 + \mathrm{d}(\pi_M(x),y)$ for every point $y \in M$. If M_1 and M_2 are two disjoint maxes, then the map $x \mapsto \pi_{M_2}(x)$ defines an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$, see e.g. [11, Theorem 1.10].

Consider the polar space $Q(2n, \mathbb{K})$ related to a nonsingular quadric of Witt-index n of $\mathrm{PG}(2n, \mathbb{K})$ and let $DQ(2n, \mathbb{K})$ denote the associated dual polar space. Since $\mathbb{K}$ is a perfect field of characteristic 2, the dual polar spaces $DW(2n-1, \mathbb{K})$ and $DQ(2n, \mathbb{K})$ are isomorphic (see e.g. De Bruyn and Pasini [18]). The dual polar space $DQ(2n, \mathbb{K})$ has a full embedding into the projective space $\mathrm{PG}(2^n - 1, \mathbb{K})$ which is called the spin embedding of $DQ(2n, \mathbb{K})$, see Chevalley [9] or Buekenhout and Cameron [4]. If $e : DW(2n-1, \mathbb{K}) \to \Sigma$ denotes the Grassmann embedding of $DW(2n-1, \mathbb{K})$, then the intersection $\mathcal{N}$ of all subspaces $\langle e(H_x) \rangle_\Sigma$, $x \in \mathcal{P}$, is called the *nucleus* of e. By Cardinali, De Bruyn and Pasini [7, Section 4.1], $\dim(\Sigma) - \dim(\mathcal{N}) = 2^n$; hence, $\dim(\mathcal{N}) = \binom{2n}{n} - \binom{2n}{n-2} - 2^n - 1$. The hyperplanes of $DW(2n-1, \mathbb{K})$ that arise from the spin embedding are precisely the hyperplanes H of $DW(2n-1, \mathbb{K})$ that arise from e and that satisfy $\mathcal{N} \subseteq \langle e(H) \rangle_\Sigma$. Hence, if H_1 and H_2 are two distinct hyperplanes of $DW(2n-1, \mathbb{K})$ that arise from the spin embedding, then also every hyperplane of $[H_1, H_2]^*$ arises from the spin embedding.

The isomorphism between the dual polar spaces $DW(5, \mathbb{K})$ and $DQ(6, \mathbb{K})$ plays a crucial role in this paper. The reason why we have imposed the restriction that $\mathbb{K}$ is a perfect field of characteristic 2 is that this isomorphism fails to hold for other fields. We will now discuss some properties of the hyperplanes of $DW(5, \mathbb{K}) \cong DQ(6, \mathbb{K})$ that arise from its spin embedding. Proofs of these facts can be found in the papers De Bruyn [13], Pralle [24], Shult [25] and Shult & Thas [27]. There are two types of hyperplanes of $DW(6, \mathbb{K}) \cong DQ(6, \mathbb{K})$ that arise from its spin embedding: the singular hyperplanes and the so-called hexagonal hyperplanes. The points and lines contained in a hexagonal hyperplane define a split-Cayley hexagon $H(\mathbb{K})$. If H is a hexagonal hyperplane of $DQ(6, \mathbb{K})$, then for every quad Q of $DQ(6, \mathbb{K})$, $Q \cap H$ is a singular hyperplane of Q. Moreover, for every point $x \in H$, there exists a unique quad Q through x for which $x^\perp \cap H = x^\perp \cap Q = Q \cap H$.

In this paper, we prove the following theorem.

Theorem 1.1 *Let $\mathbb{K}$ be a perfect field of characteristic 2 and let H be a hyperplane of $DW(5, \mathbb{K})$ arising from the Grassmann embedding. Then H is one of the following:*

(1) *a singular hyperplane of $DW(5, \mathbb{K})$;*
(2) *a hexagonal hyperplane of $DW(5, \mathbb{K})$;*
(3) *the extension of a full subgrid of a quad of $DW(5, \mathbb{K})$;*
(4) *the extension of a classical ovoid of a quad of $DW(5, \mathbb{K})$;*

(5) *a hyperplane belonging to some set* $(H_G, H_x)^*$ *where* G *is a full subgrid of a quad* Q *of* $DW(5, \mathbb{K})$ *and* x *is a point of* $DW(5, \mathbb{K})$ *not contained in* Q *for which* $\pi_Q(x) \in G$*;*

(6) *a hyperplane belonging to some set* $(H_G, H_x)^*$ *where* G *is a full subgrid of a quad* Q *of* $DW(5, \mathbb{K})$ *and* x *is a point of* $DW(5, \mathbb{K})$ *not contained in* Q *for which* $\pi_Q(x) \notin G$*.*

The 6 hyperplane classes mentioned in Theorem 1.1 can be distinguished as follows. For a hyperplane H of $DW(5, \mathbb{K})$, let D_H denote the set of quads of $DW(5, \mathbb{K})$ that are contained in H. In case (1), D_H consists of all quads of $DW(5, \mathbb{K})$ which contain the deepest point of H. In case (2), $D_H = \emptyset$ since every quad Q intersects H in a singular hyperplane of $\widetilde{Q}$. In case (3), D_H consists of all quads which contain a line of the grid which defines H. In case (4), D_H consists of the unique quad which carries the ovoid which defines H. In case (5), D_H defines a nonempty and nondegenerate conic in the dual projective plane of $Res(\pi_Q(x))$. In case (6), $D_H = \emptyset$ and there exists a quad Q for which $Q \cap H$ is not a singular hyperplane of $\widetilde{Q}$.

Regarding the uniqueness of the hyperplanes in each of the 6 classes mentioned in Theorem 1.1, we can say the following:

Theorem 1.2 *For each of the classes corresponding to* (1)*,* (2)*,* (3)*,* (5) *or* (6) *of Theorem 1.1, there exists up to isomorphism a unique hyperplane. Two extensions of classical ovoids are isomorphic if and only if the ovoids of* $Q(4, \mathbb{K})$ *from which they arise are isomorphic.*

It remains to determine how many isomorphism classes of classical ovoids of $Q(4, \mathbb{K})$ there are. Take a reference system in the projective space $\mathrm{PG}(4, \mathbb{K})$ and suppose $Q(4, \mathbb{K})$ is associated with the quadric $Q \leftrightarrow X_0^2 + X_1X_2 + X_3X_4 = 0$ of $\mathrm{PG}(4, \mathbb{K})$. For every $\lambda \in \mathbb{K}$, let π_λ be the hyperplane $X_4 = X_3 + \lambda X_0$ of $\mathrm{PG}(4, \mathbb{K})$ and put $O_\lambda := Q \cap \pi_\lambda$. The equation of O_λ induced on the hyperplane π_λ is $X_1X_2 + (X_0^2 + \lambda X_0X_3 + X_3^2)$. So, O_λ is a (classical) ovoid of $Q(4, \mathbb{K})$ if and only if $\lambda \in \Omega := \{\lambda \in \mathbb{K} \,|\, X^2 + \lambda X + 1 \text{ is irreducible in } \mathbb{K}[X]\}$. Define the following equivalence relation R on the set Ω: $(\lambda_1, \lambda_2) \in R$ whenever there exists an automorphism σ of $\mathbb{K}$ and an $a \in \mathbb{K}$ such that $(\lambda_2^\sigma)^{-1} = \lambda_1^{-1} + a^2 + a$. Then we show the following:

Theorem 1.3 *Let* $\mathbb{K}$ *be a perfect field of characteristic* 2*. Then:*

(i) *Every classical ovoid* O *of* $Q(4, \mathbb{K})$ *is isomorphic to an ovoid* O_λ *for some* $\lambda \in \Omega$*.*

(*ii*) *If* $\lambda_1, \lambda_2 \in \Omega$*, then the classical ovoids* O_{λ_1} *and* O_{λ_2} *of* $Q(4, \mathbb{K})$ *are isomorphic if and only if* $(\lambda_1, \lambda_2) \in R$*.*

Hence, we can say the following:

Corollary 1.4 *Let* $\mathbb{K}$ *be a perfect field of characteristic* 2*. Then:*
(*i*) *The number of nonisomorphic classical ovoids of* $Q(4, \mathbb{K})$ *is equal to the number* N *of classes of the equivalence relation* R*.*
(*ii*) *The number of nonisomorphic hyperplanes of* $DW(5, \mathbb{K})$ *is equal to* $5 + N$*.*

The results mentioned in Theorems 1.1 and 1.2 were already known if $\mathbb{K}$ is a finite field of characteristic 2, see [14]. The proofs given in [14] however make use of several counting arguments. The key result which allows us to avoid all counting arguments is Lemma 4.1 whose proof relies very much on a recent result of Blok, Cardinali and De Bruyn [1] (see also [8]) on the nucleus of the Grassmann embedding of $DW(5, \mathbb{K})$. Some of the lemmas mentioned in that paper are essentially contained in [14] since their proofs do not essentially make use of the finiteness of the field. Some other lemmas require an adaptation of the arguments so that their proofs would also work in the infinite case. We have decided to include also complete proofs of these lemmas in order to be able to offer the reader complete, self-contained and streamlined proofs for Theorems 1.1 and 1.2.

Remark. If $|\mathbb{K}| \neq 2$, then the Grassmann embedding of $DW(5, \mathbb{K})$ is the so-called absolutely universal embedding of $DW(5, \mathbb{K})$, see [10], [17] and [20]. In that case, the hyperplanes of $DW(5, \mathbb{K})$ that arise from some projective embedding are precisely the hyperplanes of $DW(5, \mathbb{K})$ arising from the Grassmann embedding. If $|\mathbb{K}| = 2$, then the Grassmann embedding is not the absolutely universal embedding of $DW(5, \mathbb{K}) = DW(5, 2)$, see e.g. Blokhuis and Brouwer [2] or Li [21]. The dual polar space $DW(5, 2)$ has 6 isomorphism classes of hyperplanes which do not arise from the Grassmann embedding, see [24] or [14].

2 Some properties of the automorphism group of $DW(2n-1, \mathbb{K})$

Let $W(2n-1, \mathbb{K})$, $n \geq 2$, be the symplectic polar space associated with a nondegenerate alternating bilinear form $(\cdot, \cdot)$ of a $2n$-dimensional vector space

V over a field $\mathbb{K}$. Suppose g is an element of $\Gamma L(V)$ for which there exists an $a_g \in \mathbb{K}\setminus\{0\}$ and an automorphism σ_g of $\mathbb{K}$ such that $(g(\bar{x}), g(\bar{y})) = a_g \cdot (\bar{x}, \bar{y})^{\sigma_g}$ for all $\bar{x}, \bar{y} \in V$. Then the map $\langle \bar{x} \rangle \mapsto \langle g(\bar{x}) \rangle$ defines an automorphism of $W(2n-1, \mathbb{K})$. Conversely, every automorphism of $W(2n-1, \mathbb{K})$ is obtained in this way.

Let $\mathcal{A}$ denote the full automorphism group of $DW(2n-1, \mathbb{K})$. Then every element of $\mathcal{A}$ is induced by an automorphism of $W(2n-1, \mathbb{K})$, and conversely. The following properties are easily verified taking into account the above description of the automorphisms of $\mathcal{A}$ (some of them also follow from Witt's theorem):

(P1) $\mathcal{A}$ acts transitively on the set of points of $DW(2n-1, \mathbb{K})$.

(P2) $\mathcal{A}$ acts transitively on the set of ordered pairs (x_1, x_2) where x_1 and x_2 are two opposite points of $DW(2n-1, \mathbb{K})$.

(P3) $\mathcal{A}$ acts transitively on the set of maxes of $DW(2n-1, \mathbb{K})$.

(P4) If $\theta \in \mathcal{A}$ fixes the point x of $DW(2n-1, \mathbb{K})$, then θ trivially induces an automorphism of $Res(x) \cong \mathrm{PG}(n-1, \mathbb{K})$. Conversely, if $n \geq 3$ then every automorphism of $Res(x)$ is induced by an automorphism of $DW(2n-1, \mathbb{K})$ fixing x.

(P5) If $n \geq 3$, if M is a max of $DW(2n-1, \mathbb{K})$ and if θ is an automorphism of the point-line geometry $\widetilde{M}$, then there exists an automorphism θ' of $DW(2n-1, \mathbb{K})$ such that $\theta'(x) = \theta(x)$ for every $x \in M$.

(P6) The automorphism group of $W(2n-1, \mathbb{K})$ acts transitively on the set of hyperbolic lines of $W(2n-1, \mathbb{K})$. [With a hyperbolic line we mean a line of $\mathrm{PG}(2n-1, \mathbb{K})$ which is not a totally isotropic line of $W(2n-1, \mathbb{K})$.]

(P7) If e denotes the Grassmann embedding of $DW(2n-1, \mathbb{K})$ into $\Sigma = \mathrm{PG}(\bigwedge^n V)$, then for every automorphism θ of $DW(2n-1, \mathbb{K})$, there exists an automorphism $\widetilde{\theta}$ of Σ such that $e(\theta(x)) = \widetilde{\theta}(e(x))$ for every point x of $DW(2n-1, \mathbb{K})$. If θ is associated with a projectivity of $\mathrm{PG}(2n-1, \mathbb{K})$, then $\widetilde{\theta}$ is a projectivity of Σ. (Every $g \in \Gamma L(V)$ naturally induces an element $\widetilde{g} \in \Gamma L(\bigwedge^n V)$, and the automorphisms of $\mathbb{K}$ corresponding to g and $\widetilde{g}$ coincide.) Property (P7) implies that if a hyperplane H of $DW(2n-1, \mathbb{K})$ arises from e, then also every hyperplane $\theta(H)$, $\theta \in \mathcal{A}$, arises from e.

Lemma 2.1 *Let $n \geq 2$. For every max M of $DW(2n-1, \mathbb{K})$, there exists a group T_M of automorphisms of $DW(2n-1, \mathbb{K})$ satisfying:*

(i) *every element of* T_M *fixes* M *pointwise;*

(ii) *if* L *is a line meeting* M *in a unique point* z*, then* T_M *acts regularly on* $L \setminus \{z\}$*.*

Proof. Let $\langle \bar{x}^* \rangle$ denote the point of $W(2n-1, \mathbb{K})$ corresponding to the max M of $DW(2n-1, \mathbb{K})$. For every $k \in \mathbb{K}$, the symplectic transvection $\bar{y} \mapsto \bar{y} - k(\bar{x}^*, \bar{y})\bar{x}^*$ of $GL(V)$ defines an automorphism of $W(2n-1, \mathbb{K})$ and hence also an automorphism τ_k of $DW(2n-1, \mathbb{K})$. Put $T_M := \{\tau_k \,|\, k \in \mathbb{K}\}$. Then T_M is a group of automorphisms of $DW(2n-1, \mathbb{K})$ fixing M pointwise.

Now, let L be a line meeting M in a unique point z. Then L corresponds to an $(n-2)$-dimensional singular subspace $\beta = \langle \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1} \rangle$ of $W(2n-1, \mathbb{K})$ and $\alpha := \langle \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, \bar{x}^* \rangle$ is the $(n-1)$-dimensional singular subspace of $W(2n-1, \mathbb{K})$ corresponding to z. Let $\langle \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n \rangle$ be an $(n-1)$-dimensional singular subspace through β distinct from α. Then the points of $L \setminus \{z\}$ correspond to the $(n-1)$-dimensional singular subspaces $\langle \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, \bar{x}_n + \lambda \bar{x}^* \rangle$, $\lambda \in \mathbb{K} \setminus \{0\}$. It is now straightforward to verify that T_M acts regularly on the set $L \setminus \{z\}$ (notice that $(\bar{x}_1, \bar{x}^*) = \ldots = (\bar{x}_{n-1}, \bar{x}^*) = 0$ and $(\bar{x}_n, \bar{x}^*) \neq 0$). ■

Lemma 2.2 *The automorphism group of* $Q(4, \mathbb{K})$ *acts transitively on the set of full subgrids of* $Q(4, \mathbb{K})$*.*

Proof. Let ζ be a symplectic polarity of $\mathrm{PG}(3, \mathbb{K})$ giving rise to $DW(3, \mathbb{K}) \cong Q(4, \mathbb{K})$. For every full subgrid G of $Q(4, \mathbb{K})$ there exists a hyperbolic line L of $W(3, \mathbb{K})$ such that the points of G correspond to the totally isotropic lines of $W(3, \mathbb{K})$ meeting L and L^ζ. The lemma now follows from Property (P6). ■

Lemma 2.3 *The automorphism group of* $Q(4, \mathbb{K})$ *acts transitively on the set of all pairs* (G, x) *where* G *is a full subgrid of* $Q(4, \mathbb{K})$ *and* x *is a point of* $Q(4, \mathbb{K})$ *not contained in* G*.*

Proof. By Lemma 2.2, the automorphism group of $Q(4, \mathbb{K})$ acts transitively on the set of full subgrids of $Q(4, \mathbb{K})$. If G is a full subgrid of $Q(4, \mathbb{K})$, then G is a hyperplane and hence its complement is connected. So, it suffices to prove that for any full subgrid G of $Q(4, \mathbb{K})$ and any two distinct collinear points x_1 and x_2 of $Q(4, \mathbb{K})$ not contained in G, there exists an automorphism of $Q(4, \mathbb{K})$ stabilizing G and mapping x_1 to x_2. For such a choice of G, x_1 and x_2, let x denote the unique point in $x_1 x_2 \cap G$ and let L denote a line of G containing x. Then there exists a unique automorphism in T_L mapping x_1 to x_2. This automorphism of T_L stabilizes G. ■

Lemma 2.4 *The automorphism group of $DW(5,\mathbb{K})$ acts transitively on the pairs (G,x) where G is a full subgrid of a quad and x is a point of $\Delta_2(G)$.*

Proof. The automorphism group of $DW(5,\mathbb{K})$ acts transitively on the set of full subgrids by Properties (P3)+(P5) and Lemma 2.2. Now, fix a certain full subgrid G and let Q denote the unique quad containing G. Then $\Delta_2(G)$ is connected since it is the complement of a hyperplane. So, it suffices to prove that for any two distinct collinear points $x_1, x_2 \in \Delta_2(G)$, there exists an automorphism of $DW(5,\mathbb{K})$ stabilizing G and mapping x_1 to x_2. Let x denote the unique point of the line x_1x_2 contained in $G \cup \Delta_1(G)$. If $x \in Q$, put $M := Q$; otherwise, let M denote one of the two quads of $DW(5,\mathbb{K})$ through x intersecting G in a line. By Lemma 2.1, there exists an automorphism of T_M mapping x_1 to x_2. This automorphism stabilizes G. ∎

Lemma 2.5 *Let $\mathbb{K}$ be a perfect field of characteristic 2. Let x_1 and x_2 be two points of $DW(5,\mathbb{K})$ at distance 3 from each other. Then there exists a line L in $DW(5,\mathbb{K})$ satisfying the following: (i) $d(x_1,L) = d(x_2,L) = 2$; (ii) $\pi_L(x_1) \neq \pi_L(x_2)$; (iii) for any two points $y_1, y_2 \in L \setminus \{\pi_L(x_1), \pi_L(x_2)\}$, there exists an automorphism θ of $DW(5,\mathbb{K})$ fixing x_1 and x_2, stabilizing L and mapping y_1 to y_2.*

Proof. Choose a reference system such that the polar space $W(5,\mathbb{K})$ is described by the following alternating form:

$$(X_0Y_3 - X_3Y_0) + (X_1Y_4 - X_4Y_1) + (X_2Y_5 - X_5Y_2).$$

Without loss of generality (see Property (P2)), we may suppose that $x_1 \leftrightarrow X_3 = X_4 = X_5 = 0$ and $x_2 \leftrightarrow X_0 = X_1 = X_2 = 0$. Let L be the following line of $DW(5,\mathbb{K})$: $L \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_2 = X_5 = 0$. The points $p_1 \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_5 = 0$ and $p_2 \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_2 = 0$ belong to L. Moreover, $\mathrm{d}(x_1,p_1) = \mathrm{d}(x_2,p_2) = 2$ and $\mathrm{d}(x_1,p_2) = \mathrm{d}(x_2,p_1) = 3$. The other points of L are given by the equations $X_0 - X_3 = X_1 - X_4 = X_2 - \mu X_5 = 0$, $\mu \in \mathbb{K} \setminus \{0\}$, and lie at distance 3 from x_1 and x_2. Now, choose two arbitrary points y_1 and y_2 in $L \setminus \{p_1,p_2\}$. So, there exist $\mu_1, \mu_2 \in \mathbb{K} \setminus \{0\}$ such that $y_i \leftrightarrow X_0 - X_3 = X_1 - X_4 = X_2 - \mu_i X_5 = 0$, $i \in \{1,2\}$. Since $\mathbb{K}$ is perfect, there exists a $k \in \mathbb{K} \setminus \{0\}$ such that $k^2 = \frac{\mu_2}{\mu_1}$. The map $(X_0,X_1,X_2,X_3,X_4,X_5) \mapsto (X_0,X_1,kX_2,X_3,X_4,\frac{X_5}{k})$ induces an automorphism θ of $DW(5,\mathbb{K})$ fixing x_1 and x_2, stabilizing L and mapping y_1 to y_2. ∎

Lemma 2.6 *Let $\mathbb{K}$ be a perfect field of characteristic 2. Let G be a full subgrid of a quad Q of $DW(5,\mathbb{K})$ and let p_1 be an arbitrary point of $\Delta_2(G)$.*

Then there exists a line L satisfying the following properties: (i) L intersects Q in a point p_2 of $\Delta_3(p_1)\setminus G$; (ii) for every two points $y_1, y_2 \in L\setminus\{p_2, \pi_L(p_1)\}$, there exists an automorphism θ of $DW(5,\mathbb{K})$ fixing p_1, stabilizing G and L, and mapping y_1 to y_2.

Proof. Suppose first that $\mathbb{K} \cong \mathbb{F}_2$. Let p_2 be a point of $Q \setminus (G \cup \pi_Q(p_1)^\perp)$ and let L denote an arbitrary line through p_2 not contained in Q. Then $|L \setminus \{p_2, \pi_L(p_1)\}| = 1$ and so condition (ii) holds: since $y_1 = y_2$, we can take for θ the trivial automorphism.

Suppose $\mathbb{K}$ is not isomorphic to $\mathbb{F}_2$. The point p_1 corresponds to a totally isotropic plane α_1 of $W(5,\mathbb{K})$. There exists a nonisotropic plane α_2 such that the singular point x_{α_2} of α_2 corresponds to the quad Q and the points of G correspond to the totally isotropic planes of $W(5,\mathbb{K})$ which intersect α_2 in a line through x_{α_2}. (Recall that with every full subgrid of $Q(4,\mathbb{K})$ there corresponds a pair of orthogonal hyperbolic lines of $W(3,\mathbb{K})$, see the proof of Lemma 2.2.) Since $p_1 \notin Q$ and $\pi_Q(p_1) \notin G$, α_1 and α_2 are disjoint.

Now, choose a reference system such that the polar space $W(5,\mathbb{K})$ is described by the following alternating form:

$$(X_0Y_3 - X_3Y_0) + (X_1Y_4 - X_4Y_1) + (X_2Y_5 - X_5Y_2).$$

Without loss of generality (see Lemma 2.4), we may suppose that $\alpha_1 \leftrightarrow X_0 = X_1 = X_2 = 0$ and $\alpha_2 \leftrightarrow X_3 = X_4 = X_0 - X_5 = 0$. One readily verifies that α_2 is a nonisotropic plane and that the point $(0,1,0,0,0,0)$ is its singular point. Now, choose a $\delta \in \mathbb{K} \setminus \{0,1\}$ and let L be the following line of $DW(5,\mathbb{K})$: $X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 = X_4 = 0$. Put $L \cap Q = \{p_2\}$. Then p_2 is the following point of Q: $X_0 - \delta X_5 = \delta X_3 - X_2 = X_4 = 0$. Obviously, $\mathrm{d}(p_1, p_2) = 3$. Since the system $X_3 = X_4 = X_0 - X_5 = 0$, $X_0 - \delta X_5 = \delta X_3 - X_2 = X_4 = 0$ has only the point $(0,1,0,0,0,0)$ as solution, $p_2 \notin G$. The point $\pi_L(p_1)$ has the following equation: $X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 = 0$. A point y of $L \setminus \{p_2, \pi_L(p_1)\}$ has the following equation for a certain $\mu \in \mathbb{K} \setminus \{0\}$: $X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 - \mu X_4 = 0$. Now, let y_1, y_2 be arbitrary points of $L \setminus \{p_2, \pi_L(p_1)\}$ and let $\mu_1, \mu_2 \in \mathbb{K} \setminus \{0\}$ such that $y_i \leftrightarrow X_0 - \delta X_5 = X_2 - \delta X_3 = X_1 - \mu_i X_4 = 0$ for every $i \in \{1,2\}$. Let $k \in \mathbb{K} \setminus \{0\}$ such that $k^2 = \frac{\mu_2}{\mu_1}$, then the map $(X_0, X_1, X_2, X_3, X_4, X_5) \mapsto (X_0, kX_1, X_2, X_3, \frac{X_4}{k}, X_5)$ induces an automorphism of $DW(5,\mathbb{K})$ satisfying all required properties. ∎

3 Regarding the sets $[H_1, H_2]^*$

Throughout this section, $\mathbb{K}$ denotes a perfect field of characteristic 2.

Lemma 3.1 *If G is a full subgrid of $Q(4,\mathbb{K})$, then for every point $x \in G$, $(G, x^{\perp})^*$ only contains full subgrids.*

Proof. If L is one of the two lines through x which are contained in G, then since $L \subseteq G$ and $L \subseteq x^{\perp}$, L is also contained in any hyperplane of $(G, x^{\perp})^*$. If L is a line through x not contained in G, then since $L \subseteq x^{\perp}$ and $L \not\subseteq G$, L cannot be contained in any of the hyperplanes of $(G, x^{\perp})^*$. So, for any hyperplane H of $(G, x^{\perp})^*$, precisely two lines through x are contained in H; hence, H is a full subgrid. ∎

Lemma 3.2 *If x_1 and x_2 are two distinct points of $Q(4,\mathbb{K})$, then any hyperplane of $[x_1^{\perp}, x_2^{\perp}]^*$ is singular.*

Proof. The spin embedding of $Q(4,\mathbb{K})$ is isomorphic to the natural embedding of $W(3,\mathbb{K})$ into $\mathrm{PG}(3,\mathbb{K})$ and hence the hyperplanes arising from it are precisely the singular hyperplanes of $Q(4,\mathbb{K})$. Now, since $x_1^{\perp}$ and $x_2^{\perp}$ arise from the spin embedding of $Q(4,\mathbb{K})$ also any hyperplane of $[x_1^{\perp}, x_2^{\perp}]^*$ arises from the spin embedding and hence is singular. ∎

Lemma 3.3 *Let M be a max of $DW(2n-1,\mathbb{K})$ and let A_1, A_2 be two distinct hyperplanes of $\widetilde{M}$. If H is a hyperplane of $DW(2n-1,\mathbb{K})$ satisfying $H \cap H_{A_1} = H_{A_1} \cap H_{A_2} = H \cap H_{A_2}$, then $H = H_{A_3}$ for some hyperplane A_3 of $\widetilde{M}$ satisfying $A_1 \cap A_3 = A_1 \cap A_2 = A_2 \cap A_3$.*

Proof. Notice first that for every hyperplane A of $\widetilde{M}$, $H_A = \bigcup_{x \in A} x^{\perp}$.

We have $M \subseteq H_{A_1} \cap H_{A_2} \subseteq H$. We show that for any $x \in M$, either $x^{\perp} \subseteq H$ or $x^{\perp} \cap H = x^{\perp} \cap M$. If this would not be the case, then there exist two lines L_1 and L_2 through x not contained in M such that $L_1 \subseteq H$ and $L_2 \not\subseteq H$. Let Q denote the unique quad through L_1 and L_2 and let L_3 be the line $Q \cap M$. Now, $Q \cap H$ is a hyperplane of $\widetilde{Q}$ which is necessary a full subgrid since $L_1, L_3 \subseteq H$ and $L_2 \not\subseteq H$. Let y denote a point of $L_3 \cap A_1$ and let L_4 denote the unique line of $Q \cap H$ through y distinct from L_3. Since $H \cap H_{A_1} = H_{A_1} \cap H_{A_2} = H \cap H_{A_2}$, we would have the following: (i) $L_4 \subseteq H_{A_2}$; (ii) any line through y not contained in $M \cup L_4$ is not contained in H_{A_2}. This is clearly not possible. Hence, either $x^{\perp} \subseteq H$ or $x^{\perp} \cap H = x^{\perp} \cap M$.

Now, let A_3 denote the set of points of M satisfying $x^{\perp} \subseteq H$. Let M' denote a max disjoint from M and put $A_i' := \pi_{M'}(A_i)$, $i \in \{1,2,3\}$. Since $A_3' = H \cap M'$, A_3' is a hyperplane of $\widetilde{M'}$. So, since the projection from M' onto M is an isomorphism, A_3 is a hyperplane of $\widetilde{M}$ and $H = H_{A_3}$. Since $H \cap H_{A_1} = H_{A_1} \cap H_{A_2} = H \cap H_{A_2}$, we have $A_1' \cap A_3' = A_1' \cap A_2' = A_2' \cap A_3'$. Hence, also $A_1 \cap A_3 = A_1 \cap A_2 = A_2 \cap A_3$. ∎

Lemma 3.4 *Let Q be a quad of $DW(5,\mathbb{K})$ and let A and B be two distinct hyperplanes of $\widetilde{Q}$ which are not ovoids. Then $[H_A, H_B]^* = \{H_C \,|\, C \in [A,B]^*\}$.*

Proof. Let Q' be a quad disjoint from Q and put $A' := \pi_{Q'}(A)$ and $B' := \pi_{Q'}(B)$. Then $A' \neq B'$. Let $e : DW(5,\mathbb{K}) \to \Sigma$ denote the Grassmann embedding of $DW(5,\mathbb{K})$ and let $e_{Q'} : \widetilde{Q'} \to \Sigma'$ be the embedding of $\widetilde{Q'}$ induced by e. Recall that $e_{Q'}$ is isomorphic to the Grassmann embedding of $Q(4,\mathbb{K})$. Let Σ_A and Σ_B denote the hyperplanes of Σ giving rise to H_A and H_B, respectively. [Recall that the extension of any singular hyperplane or any full subgrid of $\widetilde{Q}$ arises from the Grassmann embedding of $DW(5,\mathbb{K})$.] Then since $A' = H_A \cap Q'$, the hyperplane A' of $\widetilde{Q'}$ arises from $e_{Q'}$, more precisely from the hyperplane $\Sigma_A \cap \Sigma'$ of Σ'. Similarly, the hyperplane B' arises from the hyperplane $\Sigma_B \cap \Sigma'$ of Σ'. Now, the hyperplanes of Σ' through $(\Sigma_A \cap \Sigma') \cap (\Sigma_B \cap \Sigma')$ are precisely the hyperplanes of the form $\Pi \cap \Sigma'$ where Π is some hyperplane of Σ through $\Sigma_A \cap \Sigma_B$. This implies that $\{H \cap Q' \,|\, H \in [H_A, H_B]^*\} = [A', B']^*$. By Lemma 3.3, every hyperplane of $[H_A, H_B]^*$ is the extension of a hyperplane of $\widetilde{Q}$. Hence, $[H_A, H_B]^* = \{H_C \,|\, C \subseteq Q \text{ and } \pi_{Q'}(C) \in [A', B']^*\} = \{H_C \,|\, C \in [A,B]^*\}$. ∎

Lemma 3.5 *If x_1 and x_2 are two distinct points of $DW(5,\mathbb{K})$ at distance at most 2 from each other, then any hyperplane of $[H_{x_1}, H_{x_2}]^*$ is singular.*

Proof. Let Q denote an arbitrary quad containing x_1 and x_2. Then H_{x_i}, $i \in \{1,2\}$, is the extension of the singular hyperplane $x_i^\perp \cap Q$ of $\widetilde{Q}$. The lemma now immediately follows from Lemmas 3.2 and 3.4. ∎

Lemma 3.6 *If O is a classical ovoid of $Q(4,\mathbb{K})$, then there exists a full subgrid G of $Q(4,\mathbb{K})$ and a point $x \notin G$ such that $O \in [G, x^\perp]^*$.*

Proof. Let x be a point of $Q(4,\mathbb{K})$ not contained in O, let y be a point of O collinear with x and let z be a point collinear with y at distance 2 from x. Since $y \in O \cap x^\perp$, y is contained in any hyperplane of $(x^\perp, O)^*$. Since $z \notin x^\perp \cup O$, there exists a unique hyperplane $H^* \in (x^\perp, O)^*$ containing z. The hyperplane H^* contains the line yz and hence has to be either a singular hyperplane (necessarily distinct from $x^\perp$) or a full subgrid. If H^* would we a singular hyperplane, then by Lemma 3.2, also $O \in [H^*, x^\perp]^*$ would be singular, a contradiction. So, H^* is a full subgrid and $O \in (H^*, x^\perp)^*$. Since $x \notin O$ and $x \in x^\perp$, x cannot belong to H^*. ∎

Lemma 3.7 *The extension of a classical ovoid O of a quad Q of $DW(5,\mathbb{K})$ arises from the Grassmann embedding of $DW(5,\mathbb{K})$.*

Proof. By Lemma 3.6, there exists a full subgrid G of $\widetilde{Q}$ and a point $x \in Q \setminus G$ such that $O \in (G, x^{\perp})^*$. By Lemma 3.4, $H_O \in [H_x, H_G]^*$; hence, H_O arises from the Grassmann embedding of $DW(5, \mathbb{K})$. ∎

Lemma 3.8 *If x_1 and x_2 are two points of $DW(5, \mathbb{K})$ at distance 3 from each other, then every hyperplane of $(H_{x_1}, H_{x_2})^*$ is hexagonal.*

Proof. Since H_{x_1} and H_{x_2} arise from the spin embedding of $DW(5, \mathbb{K})$, also every hyperplane of $(H_{x_1}, H_{x_2})^*$ arises from the spin embedding of $DW(5, \mathbb{K})$. So, any hyperplane H of $(H_{x_1}, H_{x_2})^*$ is either singular or hexagonal. It suffices to show that every quad Q intersects H in a singular hyperplane of $\widetilde{Q}$. If $x_i \in Q$ for some $i \in \{1, 2\}$, then since $Q \subseteq H_{x_i}$ and $Q \cap H_{x_{3-i}}$ is the singular hyperplane of $\widetilde{Q}$ with deepest point $\pi_Q(x_{3-i})$, also $H \cap Q$ is the singular hyperplane of $\widetilde{Q}$ with deepest point $\pi_Q(x_{3-i})$. If $x_1, x_2 \notin Q$, then $\pi_Q(x_1) \neq \pi_Q(x_2)$ (since $\mathrm{d}(x_1, x_2) = 3$) and $H \cap Q \in [\pi_Q(x_1)^{\perp} \cap Q, \pi_Q(x_2)^{\perp} \cap Q]^*$ (look at the embedding space); by Lemma 3.2, $H \cap Q$ is a singular hyperplane of Q. ∎

Lemma 3.9 *If H is a hexagonal hyperplane of $DW(5, \mathbb{K})$, then for every point x_1 of $DW(5, \mathbb{K})$ not contained in H, there exists a unique point $x_2 \neq x_1$ such that $H \in (H_{x_1}, H_{x_2})^*$. The point x_2 lies at distance 3 from x_1.*

Proof. Let y be a point of H collinear with x_1, let Q denote the unique quad through y such that $Q \cap H = y^{\perp} \cap Q$ and let $z \in \Delta_2(y) \cap Q$. Since $y^{\perp} \cap Q \subseteq H \cap H_{x_1}$, $y^{\perp} \cap Q$ is contained in any hyperplane of $(H, H_{x_1})^*$. Since $z \notin H \cup H_{x_1}$, there exists a unique hyperplane $H^* \in (H, H_{x_1})^*$ containing z. Since H and H_{x_1} arise from the spin embedding of $DW(5, \mathbb{K})$, also H^* arises from the spin embedding and hence is either singular or hexagonal. Since $y^{\perp} \cap Q \subseteq H^*$ and $z \in H^*$, $Q \subseteq H^*$ and hence H^* is singular with deepest point belonging to Q. Since $H \in [H_{x_1}, H^*]^*$, the deepest point x_2 of H^* lies at distance 3 from x_1 by Lemma 3.5. If there would exist a point $x_2' \notin \{x_1, x_2\}$ such that $H \in (H_{x_1}, H_{x_2'})^*$, then $H_{x_2'} \in [H_{x_1}, H]^* = [H_{x_1}, H_{x_2}]^*$, contradicting Lemma 3.8. ∎

4 Proof of Theorem 1.1

Throughout this section, $\mathbb{K}$ denotes a perfect field of characteristic 2.

Lemma 4.1 *Let H be a hyperplane of $DW(5, \mathbb{K})$ arising from the Grassmann embedding and let $\mathcal{Q}_H$ denote the set of quads of $DW(5, \mathbb{K})$ which either are contained in H or intersect H in a singular hyperplane of $\widetilde{Q}$. Then the following holds:*

(1) *If H arises from the spin embedding of $DW(5,\mathbb{K})$, then $\mathcal{Q}_H$ coincides with the set of all quads of $DW(5,\mathbb{K})$.*

(2) *If H does not arise from the spin embedding of $DW(5,\mathbb{K})$, then there exists a quad Q^* of $DW(5,\mathbb{K})$ such that $\mathcal{Q}_H$ consists of all the quads of $DW(5,\mathbb{K})$ which meet Q^*.*

Moreover, if H_1 and H_2 are two distinct hyperplanes of $DW(5,\mathbb{K})$ arising from the Grassmann embedding of $DW(5,\mathbb{K})$ for which $\mathcal{Q}_{H_1} = \mathcal{Q}_{H_2}$, then $[H_1, H_2]^*$ contains a hyperplane that arises from the spin embedding of $DW(5,\mathbb{K})$.

Proof. Let e denote the Grassmann embedding of $DW(5,\mathbb{K})$ into $\Sigma \cong \mathrm{PG}(13,\mathbb{K})$ and let $\mathcal{N}$ denote the nucleus of e. Then $\dim(\mathcal{N}) = 5$. For every quad Q of $DW(5,\mathbb{K})$, e induces a full embedding e_Q of $\widetilde{Q}$ into the subspace $\langle e(Q)\rangle$ of Σ which is isomorphic to the Grassmann embedding of $Q(4,\mathbb{K})$. Let $f(Q)$ denote the nucleus of the embedding e_Q and let $g(Q)$ denote the point of $W(5,\mathbb{K})$ corresponding to the quad Q. By Blok, Cardinali and De Bruyn [1] (see also Cardinali and Lunardon [8] for the finite case), $f \circ g^{-1}$ defines a full projective embedding of $W(5,\mathbb{K})$ into $\mathcal{N}$ which is (necessarily) isomorphic to the natural embedding of $W(5,\mathbb{K})$ into $\mathrm{PG}(5,\mathbb{K})$. Now, let U denote the set of points contained in $\mathcal{N} \cap \langle e(H)\rangle$. Let x be an arbitrary point of $\mathcal{N}$ and put $Q = f^{-1}(x)$. If $x \in U$, then the space $\langle e(H)\rangle \cap \langle e(Q)\rangle$ contains the nucleus of e_Q and hence intersects $e(Q)$ in $e(A)$ where A is either Q or a singular hyperplane of $\widetilde{Q}$. If $x \notin U$, then $\langle e(H)\rangle \cap \langle e(Q)\rangle$ does not contain the nucleus of e_Q and hence intersects $e(Q)$ in $e(A)$ where A is either a full subgrid or a classical ovoid of $\widetilde{Q}$. If follows that $\mathcal{Q}_H = f^{-1}(U)$. If H arises from the spin embedding of $DW(5,\mathbb{K})$, then $U = \mathcal{N}$ and $\mathcal{Q}_H$ coincides with the whole set of quads of $DW(5,\mathbb{K})$. If H does not arise from the spin embedding of $DW(5,\mathbb{K})$, then U is a hyperplane of $\mathcal{N}$ and $g \circ f^{-1}(U)$ is a hyperplane of $W(5,\mathbb{K})$ which consists of all the points of $W(5,\mathbb{K})$ which are equal to or collinear with a given point x^* of $W(5,\mathbb{K})$. Hence, $\mathcal{Q}_H = f^{-1}(U)$ consists of all quads of $DW(5,\mathbb{K})$ which meet $Q^* := g^{-1}(x^*)$. This proves the first part of the lemma.

Suppose now that H_1 and H_2 are two distinct hyperplanes of $DW(5,\mathbb{K})$ arising from e for which $\mathcal{Q}_{H_1} = \mathcal{Q}_{H_2}$. If $\mathcal{Q}_{H_1} = \mathcal{Q}_{H_2}$ consists of all the quads of $DW(5,\mathbb{K})$, then H_1 and H_2 arise from the spin embedding of $DW(5,\mathbb{K})$ and hence also all hyperplanes of $[H_1, H_2]^*$. So, suppose $\mathcal{Q}_{H_1} = \mathcal{Q}_{H_2}$ does not coincide with the whole set of quads of $DW(5,\mathbb{K})$. Then by the above discussion, $\langle e(H_1)\rangle \cap \mathcal{N} = \langle e(H_2)\rangle \cap \mathcal{N}$ is a hyperplane of $\mathcal{N}$. Now, let α denote the hyperplane of Σ generated by the subspaces $\mathcal{N}$ and $\langle e(H_1)\rangle \cap$

$\langle e(H_2)\rangle$. Then the hyperplane $e^{-1}(\alpha \cap e(\mathcal{P}))$ arises from the spin embedding of $DW(5,\mathbb{K})$ and belongs to $[H_1, H_2]^*$. ■

We are now ready to give a proof of Theorem 1.1. If H arises from the spin embedding of $DW(5,\mathbb{K})$, then H is either a singular hyperplane or a hexagonal hyperplane of $DW(5,\mathbb{K})$.

Suppose H does not arise from the spin embedding of $DW(5,\mathbb{K})$. Then by Lemma 4.1, there exists a quad Q such that $\mathcal{Q}_H$ consists of all the quads of $DW(5,\mathbb{K})$ which meet Q. Now, let G be an arbitrary full subgrid of Q such that $H_G \neq H$. Then $\mathcal{Q}_{H_G} = \mathcal{Q}_H$. Hence, by Lemma 4.1, there exists a hyperplane $H' \in (H_G, H)^*$ that arises from the spin embedding of $DW(5,\mathbb{K})$. We have $H \in (H', H_G)^*$.

We now prove that there exists a point x in $DW(5,\mathbb{K})$ and a full subgrid G' of Q such that $H \in [H_x, H_{G'}]^*$. Obviously, this is the case if H' is singular (take for x the deepest point of H' and $G' = G$). So, suppose H' is hexagonal. Let y be an arbitrary point of $G \setminus H'$. Then by Lemma 3.9, there exists a unique point x at distance 3 from y such that $H' \in (H_x, H_y)^*$. Since $H \in (H', H_G)^*$ and $H' \in (H_x, H_y)^*$, there exists a hyperplane $H'' \in (H_y, H_G)^*$ such that $H \in (H_x, H'')^*$. By Lemmas 3.1 and 3.4, H'' is the extension of a certain full subgrid G' of Q. So, $H \in (H_x, H_{G'})^*$.

Notice that if $x \in Q$, then $H \in (H_x, H_G)^*$ is the extension of a classical ovoid or a full subgrid of $\widetilde{Q}$ by Lemma 3.4 and the fact that H is not singular. Theorem 1.1 now readily follows.

5 Proof of Theorem 1.2

By Property (P1), there exists up to isomorphism a unique singular hyperplane of $DW(5,\mathbb{K})$.

By Properties (P3)+(P5) and Lemma 2.2, there exists up to isomorphism a unique hyperplane of $DW(5,\mathbb{K})$ that arises by extending a full subgrid of a quad.

By Properties (P2)+(P7) and Lemmas 2.5 + 3.9, there exists up to isomorphism a unique hexagonal hyperplane in $DW(5,\mathbb{K})$.

By Property (P7) and Lemmas 2.4 + 2.6, there exists up to isomorphism a unique hyperplane of $DW(5,\mathbb{K})$ which belongs to some set of the form $(H_G, H_x)^*$ where G is a full subgrid of a quad of $DW(5,\mathbb{K})$ and $x \in \Delta_2(G)$.

Lemma 5.1 *Let O_i, $i \in \{1,2\}$, be an ovoid of a quad Q_i. Then $H_{O_1} \cong H_{O_2}$ if and only if there exists an isomorphism θ from $\widetilde{Q}_1$ to $\widetilde{Q}_2$ mapping O_1 to O_2.*

Proof. Suppose there exists an isomorphism θ from $\widetilde{Q}_1$ to $\widetilde{Q}_2$ mapping O_1 to O_2. Let θ_1' be an arbitrary automorphism of $DW(5,\mathbb{K})$ mapping Q_2 to Q_1 (recall Property (P3)) and let θ_1 be the isomorphism from $\widetilde{Q}_2$ to $\widetilde{Q}_1$ induced by θ_1'. Then $\theta_3 := \theta_1 \circ \theta$ is an automorphism of $\widetilde{Q}_1$ which extends to an automorphism θ_3' of $DW(5,\mathbb{K})$ (recall Property (P5)). Clearly, the automorphism ${\theta_1'}^{-1} \circ \theta_3'$ of $DW(5,\mathbb{K})$ maps H_{O_1} to H_{O_2}.

Conversely, if θ' is an automorphism of $DW(5,\mathbb{K})$ mapping H_{O_1} to H_{O_2}, then since O_i, $i \in \{1,2\}$, is the set of all points $x \in H_{O_i}$ for which $x^\perp \subseteq H_i$, θ' induces an isomorphism θ from $\widetilde{Q}_1$ to $\widetilde{Q}_2$ mapping O_1 to O_2. ■

The following lemma finishes the proof of Theorem 1.2.

Lemma 5.2 *For every $i \in \{1,2\}$, let G_i be a full subgrid of a quad Q_i of $DW(5,\mathbb{K})$, let x_i be a point of $\Delta_1(G_i) \cap \Delta_1(Q_i)$ and let H_i be a hyperplane of the set $(H_{G_i}, H_{x_i})^*$. Then the hyperplanes H_1 and H_2 are isomorphic.*

Proof. Let $i \in \{1,2\}$. Put $y_i := \pi_{Q_i}(x_i)$. Then $y_i^\perp \subseteq H_i$ since $y_i^\perp \subseteq H_{x_i}$ and $y_i^\perp \subseteq H_{G_i}$. So, if Q is a quad through y_i, then either $Q \subseteq H_i$ or $Q \cap H_i = y_i^\perp \cap Q$. Let U_i denote the set of quads through y_i contained in H_i. Since H_i is a maximal subspace of $DW(5,\mathbb{K})$ and $H_i \neq H_{y_i}$, there exists a point $z_i \in H_i \cap \Delta_3(y_i)$. The map which associates with every line L through z_i the unique quad through y_i meeting L defines an isomorphism between $Res(z_i)$ and the dual of $Res(y_i)$. Let $\mathcal{L}_i$ denote the set of lines through z_i meeting a quad of U_i. Then $\mathcal{L}_i$ coincides with the set of lines through z_i contained in H_i. By Cardinali and De Bruyn [6, Corollary 1.5] (see also Pasini [22, Theorem 9.3]), $\mathcal{L}_i$ is a possibly degenerate conic of $Res(z_i)$. Hence, U_i defines a possibly degenerate conic in the dual of $Res(y_i)$. These conics are nonempty and nondegenerate by (1) and (2) below. [Notice that a conic of $\mathrm{PG}(2,\mathbb{K})$ is nonempty and nondegenerate if and only if it contains at least 2 points and no lines.]

(1) It holds that $|U_i| \geq 2$. For, if R_1 and R_2 denote the two quads through $x_i y_i$ meeting G in a line, then $R_1, R_2 \subseteq H_i$ since $R_1, R_2 \subseteq H_{x_i} \cap H_{G_i}$.

(2) We claim that there exists no line L through y_i with the property that every quad through L is contained in H_i. If R is a quad through $x_i y_i$ intersecting Q_i in a line which is not contained in G_i, then since $R \subseteq H_{x_i}$ and $R \not\subseteq H_{G_i}$, R is not contained in H_i. Hence, the claim holds if $L = x_i y_i$ or if L is not contained in $R_1 \cup R_2$. Suppose now that $L \neq x_i y_i$ and that L is contained in R_j for a certain $j \in \{1,2\}$. Then the unique quad through L and $R_{3-j} \cap Q$ is not contained in H_i since it is contained in H_{G_i} but not in H_{x_i}.

Notice that there exists up to isomorphism only 1 nonempty and nondegenerate conic in $\mathrm{PG}(2,\mathbb{K})$, namely the one which is described by the equation $X_0^2 + X_1X_2 = 0$ with respect to some reference system.

CLAIM: *H_i is the unique hyperplane of $DW(5,\mathbb{K})$ arising from the Grassmann embedding of $DW(5,\mathbb{K})$ which contains $y_i^\perp$ and every line of $\mathcal{L}_i$.*
PROOF. Put $\alpha_1 = \langle e(y_i^\perp)\rangle_\Sigma$ and $\alpha_2 = \langle e(z_i^\perp)\rangle_\Sigma$ where $e : DW(5,\mathbb{K}) \to \Sigma \cong \mathrm{PG}(13,\mathbb{K})$ denotes the Grassmann embedding of $DW(5,\mathbb{K})$. Since y_i and z_i are opposite points, $\Sigma = \langle \alpha_1, \alpha_2\rangle$ and $\dim(\alpha_1) = \dim(\alpha_2) = 6$, see e.g. [12]. By [6, Theorem 1.3], for every hyperplane α_2' of α_2 through $e(z_i)$, the set of lines L through z_i for which $e(L) \subseteq \alpha_2'$ is a conic $C(\alpha_2')$ of $Res(z_i)$. Moreover, there exist reference systems in $Res(z_i)$ and the quotient space $\alpha_2/e(z_i)$ such that if $\alpha_2'/e(z_i)$ is given by the equation $a_{00}Y_0 + a_{01}Y_1 + a_{02}Y_2 + a_{11}Y_3 + a_{12}Y_4 + a_{22}Y_5 = 0$, then $C(\alpha_2')$ is given by the equation $a_{00}X_0^2 + a_{11}X_1^2 + a_{22}X_2^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{12}X_1X_2 = 0$. The map $\alpha_2' \mapsto C(\alpha_2')$ is not necessarily injective. However, since the equation of a nonempty nondegenerate conic of $\mathrm{PG}(2,\mathbb{K})$ is uniquely determined up to a nonzero factor, there exists a unique hyperplane α_2^* in α_2 through $e(z_i)$ for which $C(\alpha_2^*) = \mathcal{L}_i$.

It is now clear that the unique hyperplane of $DW(5,\mathbb{K})$ arising from e and containing $y_i^\perp$ and $\bigcup_{L\in\mathcal{L}_i} L$ coincides with the hyperplane of $DW(5,\mathbb{K})$ arising from the hyperplane $\langle \alpha_2, \alpha_2^*\rangle$ of Σ. (qed)

By Properties (P1) and (P4), there now exists an automorphism θ of $DW(5,\mathbb{K})$ mapping y_1 to y_2 and U_1 to U_2. Now, let L^* be a line through z_1 not contained in $\mathcal{L}_1$, i.e. not meeting U_1. Then $\theta(L^*)$ does not meet any quad of $\theta(U_1) = U_2$. So, $\theta(L^*)$ contains a unique point of $H_2 \cap \Delta_3(y_2)$. Without loss of generality, we may suppose that this point is equal to z_2. (Recall that the only restriction on the choice of z_2 was that it is a point of $H_2 \cap \Delta_3(y_2)$.) Now, let M denote the unique quad through y_2 meeting $\theta(L^*)$. Then there exists a unique element $\theta' \in T_M$ mapping $\theta(z_1)$ to z_2. The automorphism θ' fixes y_2 and every quad through y_2 since every such quad intersects M in a line. Now, the automorphism $\theta' \circ \theta$ maps y_1 to y_2, U_1 to U_2 and z_1 to z_2. Hence, it also maps $\mathcal{L}_1$ to $\mathcal{L}_2$. So, $\theta' \circ \theta(H_1)$ is a hyperplane of $DW(5,\mathbb{K})$ containing $y_2^\perp$ and $\bigcup_{L\in\mathcal{L}_2} L$. Moreover, $\theta' \circ \theta(H_1)$ arises from the Grassmann embedding of $DW(5,\mathbb{K})$ by Property (P7). By the previous claim, we necessarily have $\theta' \circ \theta(H_1) = H_2$. ■

Remark. The claims mentioned after Theorem 1.1 should now be all clear for the first 5 classes of hyperplanes. Suppose now that H is a hyperplane belonging to the 6th class. Suppose $H \in (H_G, H_x)^*$, where G is a full subgrid of a quad Q of $DW(5,\mathbb{K})$ and x is a point of $DW(5,\mathbb{K})$ contained in $\Delta_2(G)$. If R is a quad through x, then since $R \subseteq H_x$ and $R \not\subseteq H_G$, R is not contained in H. If R is a quad not containing x, then $R \cap H_x$ is the singular hyperplane of $\widetilde{R}$ with deepest point $\pi_R(x)$. If R would be contained in H, then also $R \cap H_G$ would be the singular hyperplane of $\widetilde{R}$ with deepest point $\pi_R(x)$. This would imply that $R \cap Q$ is a line of Q which intersects G in the unique point $\pi_R(x)$. But this is impossible since $x \in \Delta_2(G)$.

Hence, no quad of $DW(5,\mathbb{K})$ is contained in H. Observe also that if R is a quad through x disjoint from Q, then $H \cap R = \pi_R(G)$ since $R \subseteq H_x$ and $R \cap H_G = \pi_R(G)$.

6 Proof of Theorem 1.3

Throughout this section, $\mathbb{K}$ denotes a perfect field of characteristic 2.

Lemma 6.1 *Let $\lambda_1, \lambda_2 \in \mathbb{K} \setminus \{0\}$, $a \in \mathbb{K}$ and σ an automorphism of $\mathbb{K}$ such that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2^\sigma} + a^2 + a = 0$. If the polynomial $X^2 + \lambda_1 X + 1$ is irreducible in $\mathbb{K}[X]$, then also the polynomial $X^2 + \lambda_2 X + 1$ is irreducible in $\mathbb{K}[X]$.*

Proof. If $\lambda \in \mathbb{K} \setminus \{0\}$, then $\frac{1}{\lambda^2}(X^2 + \lambda X + 1) = (\frac{X}{\lambda})^2 + \frac{X}{\lambda} + \frac{1}{\lambda^2}$ and hence $X^2 + \lambda X + 1$ is irreducible (in $\mathbb{K}[X]$) if and only if $X^2 + X + \frac{1}{\lambda^2}$ is irreducible. So, $X^2 + X + \frac{1}{\lambda_1^2}$ is irreducible. Now, since $X^2 + X + \frac{1}{\lambda_1^2} = (X + a^2)^2 + (X + a^2) + \frac{1}{\lambda_1^2} + a^4 + a^2 = (X + a^2)^2 + (X + a^2) + (\frac{1}{\lambda_2^\sigma})^2$, also the polynomials $X^2 + X + (\frac{1}{\lambda_2^\sigma})^2$ and $X^2 + \lambda_2^\sigma X + 1$ are irreducible. Hence, also the polynomial $X^2 + \lambda_2 X + 1$ is irreducible. ∎

Now, let Ω denote the set of all elements $\lambda \in \mathbb{K}$ for which the polynomial $X^2 + \lambda X + 1$ is irreducible in $\mathbb{K}[X]$. We define the following relation R on the set Ω. We say that $(\lambda_1, \lambda_2) \in R$ if and only if there exists an $a \in \mathbb{K}$ and an automorphism σ of $\mathbb{K}$ such that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2^\sigma} + a^2 + a = 0$. It is straightforward to verify that R is an equivalence relation.

Now, choose a reference system in $\mathrm{PG}(4,\mathbb{K})$ and suppose $Q(4,\mathbb{K})$ is the generalized quadrangle associated with the quadric $Q \leftrightarrow X_0^2 + X_1X_2 + X_3X_4 = 0$ of $\mathrm{PG}(4,\mathbb{K})$. For every automorphism σ of $\mathbb{K}$, let θ_σ denote the following automorphism of $\mathrm{PG}(4,\mathbb{K})$: $(X_0, X_1, X_2, X_3, X_4) \mapsto (X_0^\sigma, X_1^\sigma, X_2^\sigma, X_3^\sigma, X_4^\sigma)$. Then θ_σ stabilizes Q. For every $\lambda \in \mathbb{K}$, let π_λ be the hyperplane $X_4 = X_3 + \lambda X_0$ of $\mathrm{PG}(4,\mathbb{K})$ and put $O_\lambda := Q \cap \pi_\lambda$. Then O_λ is a (classical) ovoid of $Q(4,\mathbb{K})$ if and only if $\lambda \in \Omega$. The following lemma is precisely Theorem 1.3(i).

Lemma 6.2 *Every classical ovoid O of $Q(4,\mathbb{K})$ is isomorphic to an ovoid O_λ for some $\lambda \in \Omega$.*

Proof. By Lemma 3.6, there exists a full subgrid G of $Q(4,\mathbb{K})$ and a point x of $Q(4,\mathbb{K})$ not contained in G such that $O \in (G, x^\perp)^*$. By Lemma 2.3, we may without loss of generality suppose that $x = (1,0,0,1,1)$ and that G is described by the equations $X_0 = 0$, $X_0^2 + X_1X_2 + X_3X_4 = 0$. The set $x^\perp$ is described by the equations $X_3 + X_4 = 0$, $X_0^2 + X_1X_2 + X_3X_4 = 0$. So, there exists a $\lambda \in \mathbb{K}\setminus\{0\}$ such that O is described by the equations $X_4 = X_3 + \lambda X_0$, $X_0^2 + X_1X_2 + X_3X_4 = 0$, i.e. $O = O_\lambda$. Since O is a classical ovoid, $\lambda \in \Omega$. ■

Lemma 6.3 *Let $\lambda_1, \lambda_2 \in \Omega$. If there exists a projectivity μ of* $\mathrm{PG}(4,\mathbb{K})$ *stabilizing the quadric $X_0^2 + X_1X_2 + X_3X_4 = 0$ and mapping the hyperplane $X_4 + X_3 + \lambda_1 X_0 = 0$ to the hyperplane $X_4 + X_3 + \lambda_2 X_0 = 0$, then there exists an $a \in \mathbb{K}$ such that $\frac{1}{\lambda_2} + \frac{1}{\lambda_1} + a^2 + a = 0$.*

Proof. Let $\overline{\mathbb{K}}$ denote a given algebraic closure of $\mathbb{K}$ and let $\mathbb{K}_i$, $i \in \{1,2\}$, denote the splitting field in $\overline{\mathbb{K}}$ of the quadratic polynomial $X^2 + \lambda_i X + 1 \in \mathbb{K}[X]$. Then $\mathbb{K}_i$ is also the splitting field of the polynomial $X^2 + X + \frac{1}{\lambda_i^2}$. For every $i \in \{1,2\}$, let $\mathcal{P}_i$ (respectively $\mathcal{P}_i'$), denote the set of points of $\mathrm{PG}(4,\mathbb{K})$ (respectively $\mathrm{PG}(4,\mathbb{K}_1)$) defined by the equations

$$\begin{cases} X_4 + X_3 + \lambda_i X_0 &= 0, \\ X_0^2 + X_1X_2 + X_3(X_3 + \lambda_i X_0) &= 0. \end{cases}$$

Then $\mu(\mathcal{P}_1) = \mathcal{P}_2$. Regarding μ as a projectivity of $\mathrm{PG}(4,\mathbb{K}_1)$, we have $\mu(\mathcal{P}_1') = \mathcal{P}_2'$. So, $\mathcal{P}_2'$ is a hyperbolic quadric in the hyperplane $X_4 + X_3 + \lambda_2 X_0 = 0$ of $\mathrm{PG}(4,\mathbb{K}_1)$. This is only possible when $\mathbb{K}_2 \subseteq \mathbb{K}_1$. Applying the same reasoning to the projectivity μ^{-1}, we find $\mathbb{K}_1 \subseteq \mathbb{K}_2$. Hence, $\mathbb{K}_1 = \mathbb{K}_2$.

Let $\delta \in \mathbb{K}_1$ be a root of the polynomial $X^2 + X + \frac{1}{\lambda_1^2}$. Since $\mathbb{K}_1 = \mathbb{K}_2$ can be regarded as a two-dimensional vector space over $\mathbb{K}$, there exist $b, c \in \mathbb{K}$ such that $b\delta + c$ is a root of the polynomial $X^2 + X + \frac{1}{\lambda_2^2}$. Since $X^2 + X + \frac{1}{\lambda_2^2}$ is irreducible, $b\delta + c \notin \mathbb{K}$ and hence $b \neq 0$. We have $\delta^2 = \delta + \frac{1}{\lambda_1^2}$ and $(b\delta + c)^2 + (b\delta + c) + \frac{1}{\lambda_2^2} = (b^2 + b)\delta + \frac{b^2}{\lambda_1^2} + \frac{1}{\lambda_2^2} + c^2 + c = 0$. Since $\delta \in \mathbb{K}_1 \setminus \mathbb{K}$, $b^2 + b = 0$, i.e. $b = 1$. Hence, $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + c^2 + c = 0$. If $a \in \mathbb{K}$ denotes the square root of c, then $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + a^2 + a = 0$. ■

Lemma 6.4 *Let $\lambda_1, \lambda_2 \in \Omega$. If there exists an automorphism θ of* $\mathrm{PG}(4,\mathbb{K})$ *stabilizing the quadric $X_0^2 + X_1X_2 + X_3X_4 = 0$ and mapping the hyperplane $X_4 + X_3 + \lambda_1 X_0 = 0$ to the hyperplane $X_4 + X_3 + \lambda_2 X_0 = 0$, then there exists an automorphism σ of $\mathbb{K}$ and an $a \in \mathbb{K}$ such that $\frac{1}{\lambda_2^\sigma} + \frac{1}{\lambda_1} + a^2 + a = 0$.*

Proof. Let θ be an automorphism of $\mathrm{PG}(4,\mathbb{K})$ satisfying the conditions of the lemma. Then $\theta = \theta_{\sigma^{-1}} \circ \mu$ for some automorphism σ of $\mathbb{K}$ and some projectivity μ of $\mathrm{PG}(4,\mathbb{K})$. The automorphism $\theta_{\sigma^{-1}}$ stabilizes the quadric $X_0^2 + X_1X_2 + X_3X_4 = 0$ and maps the hyperplane $X_4 + X_3 + \lambda_2^\sigma X_0 = 0$ to the hyperplane $X_4 + X_3 + \lambda_2 X_0 = 0$. So, the projectivity μ also stabilizes the quadric and maps the hyperplane $X_4 + X_3 + \lambda_1 X_0 = 0$ to the hyperplane $X_4 + X_3 + \lambda_2^\sigma X_0 = 0$. By Lemma 6.3, there exists an $a \in \mathbb{K}$ such that $\frac{1}{\lambda_2^\sigma} + \frac{1}{\lambda_1} + a^2 + a = 0$. ■

The proofs of the following two lemmas are straightforward.

Lemma 6.5 *Let $\lambda_1, \lambda_2 \in \mathbb{K} \setminus \{0\}$ and $a \in \mathbb{K}$ such that $\frac{1}{\lambda_2} + \frac{1}{\lambda_1} + a^2 + a = 0$. Then the projectivity*

$$\mu : \begin{cases} X_0 \mapsto X_0 + (a^2\lambda_1)X_3, \quad X_1 \mapsto X_1, \quad X_2 \mapsto X_2, \\ X_3 \mapsto \frac{\lambda_1}{\lambda_2}X_3, \quad X_4 \mapsto \frac{\lambda_2}{\lambda_1}X_4 + (a^4\lambda_1\lambda_2)X_3, \end{cases}$$

of $\mathrm{PG}(4,\mathbb{K})$ *stabilizes the quadric $X_0^2 + X_1X_2 + X_3X_4 = 0$ and maps the hyperplane $X_4 + X_3 + \lambda_1 X_0 = 0$ to the hyperplane $X_4 + X_3 + \lambda_2 X_0 = 0$.*

Lemma 6.6 *Let $\lambda \in \mathbb{K}$ and let σ be an automorphism of $\mathbb{K}$. Then the automorphism θ_σ of* $\mathrm{PG}(4,\mathbb{K})$ *stabilizes the quadric $X_0^2 + X_1X_2 + X_3X_4 = 0$ and maps the hyperplane $X_4 + X_3 + \lambda X_0 = 0$ to the hyperplane $X_4 + X_3 + \lambda^\sigma X_0 = 0$.*

Theorem 1.3(ii) is now an immediate corollary of Lemmas 6.4, 6.5 and 6.6.

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