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Phase-space consistency of stellar dynamical models determined by separable augmented densities

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ABSTRACT

Assuming the separable augmented density, it is always possible to construct a distribution function of a spherical population with any given density and anisotropy. We consider under what conditions the distribution constructed as such is in fact non-negative everywhere in the accessible phase space. We first generalize the known necessary conditions on the augmented density using fractional calculus. The condition on the radius part $R(r^2)$ (whose logarithmic derivative is the anisotropy parameter) is equivalent to the complete monotonicity of $w^{-1}R(w^{-1})$. The condition on the potential part on the other hand is given by its derivative up to any order not greater than $\frac{3}{2}-\beta_0$ being non-negative where β_0 is the central anisotropy parameter. We also derive a specialized inversion formula for the distribution from the separable augmented density, which leads to sufficient conditions on separable augmented densities for the non-negativity of the distribution. These last conditions are generalizations of the similar condition derived earlier for the generalized Cuddeford system to arbitrary separable systems.

Key words: methods: analytical – galaxies: kinematics and dynamics – dark matter.

1 INTRODUCTION

Except maybe in our imagination, nothing is exactly spherically symmetric in our Universe. Yet spherical models by virtue of simplicity have widely been adopted as the default route when we embark on something new to investigate. What is surprising is that insights obtained from these 'spherical cows' appear to be helpful at all for our understanding of the 'real' Universe. This is particularly true for dynamical models of stellar systems. Models of spherical stellar systems are not only useful to approximate putative dark haloes or any actual roundish aggregate system found in the sky but also important to provide the simplest test ground for the physical principles and understanding of structures governed by them.

It was Dejonghe (1986) who had first used *augmented densities* (i.e. extensions of the density profile into bivariate functions of the potential and radius) of a spherical system to build a dynamical model of spherical stellar systems. Whilst the information contained in the distribution function and the corresponding augmented density is mathematically equivalent, the approach through the augmented density, in particular for such systems with anisotropic velocity distributions, is advantageous since its relations to directly observable quantities are simpler than those of the distribution function. That is to say, it is in principle trivial to find an augmented density with desired behaviours of observables unlike distribution functions, observables resulting from which are only available through moment integrals. For example, *an* augmented density $\tilde{\nu}(\Psi, r^2)$ (and subsequently a distribution function via algorithmic inversions) can be found from arbitrarily specified profiles of the density $\nu(r)$ and the anisotropy parameter such that $\tilde{\nu}(\Psi, r^2) = P(\Psi)R(r^2)$ where $P[\Psi(r)] = \nu(r)/R(r^2)$ and $R(r^2)$ is given by equation (14) from the prescribed anisotropy (Qian & Hunter 1995; Baes & Van Hese 2007).

A drawback of this approach is that one does not know a priori whether the spherical system described by the given augmented density is consistent with being built by a physical distribution, that is, non-negative everywhere in the accessible phase space (*the phase-space consistency*). For some systems however where the inversion algorithm reduces to a single integral quadrature such as the constant anisotropy system (see e.g. Evans & An 2006), the criteria on the augmented density for the phase-space consistency have been derived. For instance, Ciotti & Pellegrini (1992) had discovered necessary and sufficient conditions for the non-negativity of the Osipkov (1979)–Merritt (1985) distribution expressed in terms of the corresponding augmented density, and Ciotti & Morganti (2010a) extended these to be applicable to the multicomponent generalized Cuddeford system. Ciotti & Morganti (2010b) have essentially hypothesized that the necessary conditions of Ciotti & Morganti (2010a), which concerns the behaviour of the potential-dependent parts of augmented densities, may be applicable to any system for which the potential and radial dependences of the augmented density are multiplicatively separable. This has been subsequently proven by Van Hese, Baes & Dejonghe (2011) and An (2011a)

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whereas An (2011b) was able to find necessary conditions on the radius-dependent parts of separable augmented densities, which results in the constraints on the behaviour of the anisotropy parameter that can be consistent with separable augmented densities.

This paper continues the study of the phase-space consistency criteria for separable augmented densities. As its logical conclusion, we attempt to provide an answer to the question under what conditions the distribution constructed from a separable augmented density is non-negative everywhere in the entire accessible subvolume of the phase space. This paper is organized as follows. We start by reviewing the concepts of the distribution function and augmented density in Section 2, in which we also present a result (equation 5) that leads to many of the main arguments. Using this, in Section 3 we elucidate the relation amongst the distribution function, the augmented density and the observables. The main findings of this paper are provided in Section 4 where necessary conditions on separable augmented densities for the phase-space consistency are presented, and in Section 5 where corresponding sufficient conditions are given. In Section 6 we present an application on the parametrization of the anisotropy suitable for practical modelling. This paper concludes with the summary of findings in Section 7. Mathematical ideas used in this paper are reviewed in the appendices.

2 MODELS FOR SPHERICAL DYNAMICAL SYSTEMS

2.1 Distribution function

Let $F(\boldsymbol{r};\boldsymbol{v})$ be a steady-state phase-space distribution such that $\int_S F\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}$ is the number of tracers in any measurable phase-space volume S. Here $\boldsymbol{r}$ is the position vector in the configuration space and $\boldsymbol{v}=\dot{\boldsymbol{r}}$ is the velocity. Assuming spherical symmetry, the distribution is invariant under any orthogonal transformation, which implies that $F(\boldsymbol{r};\boldsymbol{v})=F(r;v_\mathrm{r},v_\mathrm{t})$ where $r=\|\boldsymbol{r}\|$ is the radial distance, $v_\mathrm{r}=\boldsymbol{v}\boldsymbol{\cdot}\hat{\boldsymbol{r}}$ and $v_\mathrm{t}=\|\boldsymbol{v}-v_\mathrm{r}\hat{\boldsymbol{r}}\|$ are the radial and tangential velocities with $\hat{\boldsymbol{r}}=\boldsymbol{r}/r$ being the radial unit vector. If we adopt the spherical polar coordinate (r,θ,ϕ), these are also given by $\|\boldsymbol{v}\|^2=v^2=v_\mathrm{r}^2+v_\mathrm{t}^2$ and $v_\mathrm{t}^2=v_\theta^2+v_\phi^2$ where $(v_\mathrm{r},v_\theta,v_\phi)=(\dot r,r\dot\theta,r\dot\phi\sin\theta)$ are the velocity components projected on to the associated orthonormal basis. Moreover, the Jeans theorem indicates that if the given distribution function (df) is a solution to the collisionless Boltzmann equation with a *generic* static spherical potential $\Phi(r)$, it must be in the form of $F(E,L^2)$ where $E=\Psi(r)-v^2/2$ and $L=rv_\mathrm{t}$ are the two isotropic isolating integrals admitted by all generic static spherical potentials, namely the specific binding energy and the magnitude of the specific angular momentum. Here,

$$\Psi(r)\equiv\begin{cases}\Phi(r_\mathrm{out})-\Phi(r) & \text{if } r_\mathrm{out}\text{ is finite}\\ \Phi(\infty)-\Phi(r) & \text{if } r_\mathrm{out}=\infty\text{ and }|\Phi(\infty)|<\infty\\ -\Phi(r) & \text{if } r_\mathrm{out}=\infty\text{ and }\Phi(\infty)\to\infty\end{cases}\tag{1}$$

is the relative potential with respect to the boundary r_out. The system that is not confined within a finite boundary radius is represented by $r_\mathrm{out}=\infty$ with $\Phi(\infty)=\lim_{r\to\infty}\Phi(r)$. If r_out or $\Phi(\infty)$ is finite, then $F(E<0,L^2)=0$ because by definition $E\geq0$ for all tracers bound to the system (and bounded by $r\leq r_\mathrm{out}$).

2.2 Augmented density

Integrating $F(E,L^2)$ over the velocity space results in

$$\tilde\nu(\Psi,r^2)\equiv\iiint\mathrm{d}^3\boldsymbol{v}\,F\left(E=\Psi-\frac{v^2}{2},L^2=r^2v_\mathrm{t}^2\right),\tag{2}$$

a bivariate function of Ψ and r^2, that is, the augmented density (AD). The integral is over the whole velocity subspace, but if r_out or $\Phi(\infty)$ is finite, it is essentially within the sphere $v^2\leq2\Psi$ since $F(E<0,L^2)=0$ for these cases. With $\Psi(r)$ specified, the AD yields the local density via $\nu(r)=\tilde\nu[\Psi(r),r^2]$. Similarly, the augmented moment functions (n.b., $\tilde\nu=m_{0,0}$) are given by

$$\begin{aligned}m_{k,n}(\Psi,r^2)&\equiv\iiint\mathrm{d}^3\boldsymbol{v}\,v_\mathrm{r}^{2k}v_\mathrm{t}^{2n}F\left(E=\Psi-\frac{v^2}{2},L^2=r^2v_\mathrm{t}^2\right)\\&=4\pi\iint_{v_\mathrm{r}\geq0,v_\mathrm{t}\geq0\,(v^2\leq2\Psi)}\mathrm{d}v_\mathrm{r}\,\mathrm{d}v_\mathrm{t}\,v_\mathrm{r}^{2k}v_\mathrm{t}^{2n+1}\\&\quad\times F\left(\Psi-\frac{v_\mathrm{r}^2+v_\mathrm{t}^2}{2},r^2v_\mathrm{t}^2\right).\end{aligned}\tag{3a}$$

Changing the integration variables to (E,L^2), these are represented to be a set of integral transformations of the df,

$$\begin{aligned}m_{k,n}&=\frac{2\pi}{r^{2n+2}}\iint_T\mathrm{d}E\,\mathrm{d}L^2K^{k-\frac12}L^{2n}F(E,L^2)\\&=\frac{2\pi}{r^{2n+2}}\iint_{E\geq E_0,L^2\geq0}\mathrm{d}E\,\mathrm{d}L^2\Theta(K)\,|K|^{k-\frac12}L^{2n}F(E,L^2).\end{aligned}\tag{3b}$$

Here $\Theta(x)$ is the Heaviside unit-step function and

$$E_0\equiv\begin{cases}0 & \text{if } r_\mathrm{out}\text{ or }\Phi(\infty)\text{ is finite}\\ -\infty & \text{if }\lim_{r\to\infty}\Psi(r)=-\Phi(\infty)\to-\infty\end{cases}\tag{4}$$

is the lower bound of the binding energy. The transform kernel is $K(E,L^2;\Psi,r^2)\equiv2(\Psi-E)-L^2r^{-2}$, which is v_r^2 expressed as a function of 4-tuple $(E,L^2;\Psi,r^2)$. Finally, the domain of (E,L^2) space in which the integral is performed is $T\equiv\{(E,L^2)\mid E\geq E_0,L^2\geq0,K\geq0\}$.

An (2011a) has shown that the Abel transformation of the augmented moment function results in an integral transformation of the df similar to equation (3b) but with different powers on K and L^2. This is generalized by means of *fractional calculus* (Appendix A1), that is, for any pair of non-negative reals $\xi\geq\mu\geq0$,

$$\begin{aligned}&{}_{E_0}D_\Psi{}^{\mu}\left[{}_0I_{r^2}{}^{\xi-\frac12}\left(\frac{\tilde\nu}{r^{2\xi-1}}\right)\right]\\&=\begin{cases}\dfrac{2^{\mu+1}\pi^{\frac32}r^{2\xi-3}}{\Gamma(\xi-\mu)}\displaystyle\iint_T\mathrm{d}E\,\mathrm{d}L^2\frac{K^{\xi-\mu-1}}{L^{2\xi-1}}F(E,L^2) & (\xi>\mu)\\ 2^{\xi}\pi^{\frac32}r^{2\xi-3}\displaystyle\int_0^{L_\mathrm{m}^2}\frac{\mathrm{d}L^2}{L^{2\xi-1}}F\left(\Psi-\frac{L^2}{2r^2},L^2\right) & (\xi=\mu),\end{cases}\end{aligned}\tag{5a}$$

$$\begin{aligned}&{}_0D_{r^2}{}^{\mu}\left(r^{2\mu}{}_{E_0}I_\Psi{}^{\xi-\frac12}\tilde\nu\right)\\&=\begin{cases}\dfrac{2^{\frac32-\xi}\pi^{\frac32}}{r^{2\mu+2}\Gamma(\xi-\mu)}\displaystyle\iint_T\mathrm{d}E\,\mathrm{d}L^2K^{\xi-\mu-1}L^{2\mu}F(E,L^2) & (\xi>\mu)\\ \dfrac{\pi^{\frac32}}{2^{\mu-\frac12}r^{2\mu+2}}\displaystyle\int_0^{L_\mathrm{m}^2}\mathrm{d}L^2L^{2\mu}F\left(\Psi-\frac{L^2}{2r^2},L^2\right) & (\xi=\mu),\end{cases}\end{aligned}\tag{5b}$$

where $\Gamma(x)$ is the gamma function and the operators ${}_aI_x{}^\lambda$ and ${}_aD_x{}^\lambda$ are as defined in Appendix A1. In addition,

$$L_\mathrm{m}^2\equiv\begin{cases}2r^2\Psi & \text{if } E_0=0\\ \infty & \text{if } E_0=-\infty.\end{cases}\tag{6}$$

Derivations are provided in Appendix B.

3 MOMENT SEQUENCES AND AUGMENTED DENSITIES

The knowledge of $\tilde\nu(\Psi,r^2)$ is mathematically equivalent to knowing $F(E,L^2)$. In particular, once the potential $\Psi=\Psi(r)$ is specified, the specification of the AD completely determines a unique spherical dynamic system in equilibrium. In light of equation (5), here we seek a possible 'physical interpretation' of the AD in relation to the df for describing dynamic systems.

Consider the moment sequence of the df restricted along $K = 0$,

$$\mathscr{M}_\mu(\Psi, r^2) \equiv \frac{(2\pi)^{\frac{3}{2}}}{(2r^2)^{\mu+1}} \int_0^{L_\mathrm{m}^2} \mathrm{d}L^2 L^{2\mu} F\left(\Psi - \frac{L^2}{2r^2},\ L^2\right)$$
$$= \begin{cases} \Psi^{\mu+1} \int_0^1 \mathrm{d}y\, y^\mu \mathscr{F}(y\Psi; \Psi, r^2) & (E_0 = 0,\ L_\mathrm{m}^2 = 2r^2\Psi) \\ \int_0^\infty \mathrm{d}Y\, Y^\mu \mathscr{F}(Y; \Psi, r^2) & (E_0 = -\infty,\ L_\mathrm{m}^2 = \infty), \end{cases} \quad (7a)$$

where

$$\mathscr{F}(Y; \Psi, r^2) \equiv (2\pi)^{\frac{3}{2}} F(\Psi - Y, 2r^2 Y). \quad (7b)$$

Then equations (5) indicate that

$$\mathscr{M}_\mu = \begin{cases} {}_{E_0}I_\Psi{}^{\mu-1/2}\,{}_0D_{r^2}{}^\mu (r^{2\mu}\tilde{\nu}) & (\mu \geq 2^{-1}) \\ {}_{E_0}D_\Psi{}^{1/2-\mu}\,{}_0D_{r^2}{}^\mu (r^{2\mu}\tilde{\nu}) & (0 \leq \mu \leq 2^{-1}) \\ {}_{E_0}D_\Psi{}^{\xi+1/2}\,{}_0I_{r^2}{}^\xi \left(r^{-2\xi}\tilde{\nu}\right) & (\xi = -\mu \geq 0)\,. \end{cases} \quad (8a)$$

In particular, if μ is a non-negative integer, this results in

$$\mathscr{M}_0 = \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial \Psi} \int_{E_0}^\Psi \frac{\tilde{\nu}(Q, r^2)\,\mathrm{d}Q}{\sqrt{\Psi - Q}}$$
$$\mathscr{M}_n = \frac{1}{\left(\frac{1}{2}\right)_{n-1}^+ \sqrt{\pi}} \int_{E_0}^\Psi \mathrm{d}Q (\Psi - Q)^{n-\frac{3}{2}} \left(\frac{\partial}{\partial r^2}\right)^n \left[r^{2n}\tilde{\nu}(Q, r^2)\right], \quad (8b)$$

where $n = 1, 2, \dots$ and $(a)_n^+ = \prod_{j=1}^n (a - 1 + j)$ is the *rising* sequential product. In other words, $\tilde{\nu}(\Psi, r^2)$ directly determines the entire moment sequences along a fixed sectional line in (E, L^2) space. The AD in this sense is similar to the *moment generating function* or the *characteristic function* for the df as a probability density. With varying (Ψ, r^2), the $K = 0$ lines sweep the whole accessible (E, L^2) space, and thus $\tilde{\nu}(\Psi, r^2)$ in principle uniquely determines $F(E, L^2)$. Explicit inversion algorithms from $\tilde{\nu}(\Psi, r^2)$ to $F(E, L^2)$ are available in literature utilizing either the known inverse of named integral transforms (e.g. Lynden-Bell 1962; Dejonghe 1986) or complex contour integrals (e.g. Hunter & Qian 1993).

Next, we consider what information on the physical properties of the system is sufficient to specify a unique AD. For this, equation (5b) indicates that the even-order (augmented) velocity moments are related to the AD (cf. Dejonghe & Merritt 1992, equation 13) as in

$$m_{k,n}(\Psi, r^2) = \frac{2^{k+n}\Gamma\left(k + \frac{1}{2}\right)}{\sqrt{\pi} r^{2n+2}} \left(r^4 \frac{\partial}{\partial r^2}\right)^n \left(r^2\,{}_{E_0}I_\Psi{}^{n+k}\tilde{\nu}\right)$$
$$= 2^{k+n} \left(\frac{1}{2}\right)_k^+ {}_{E_0}I_\Psi{}^{k+n} \left[{}_0D_{r^2}{}^n (r^{2n}\tilde{\nu})\right]. \quad (9a)$$

Here note $\sqrt{\pi}\,(\frac{1}{2})_k^+ = \Gamma(k+\frac{1}{2})$. Given the potential $\Psi(r)$, specifying the AD completely fixes every (in principle observable) velocity moment with equation (9a) such that

$$\overline{v_\mathrm{r}^{2k} v_\mathrm{t}^{2n}} = \frac{m_{k,n}[\Psi(r), r^2]}{\tilde{\nu}[\Psi(r), r^2]}. \quad (9b)$$

Conversely, equation (9a) for $(k, n) = (\mu + 1, 0)$, that is, $m_{\mu+1,0} = 2^{\mu+1}(\frac{1}{2})_{\mu+1}^+\,{}_{E_0}I_\Psi{}^{\mu+1}\tilde{\nu}$ *at a fixed* r reduces to

$$\mathscr{V}_\mu(r) \equiv \frac{\mu!\overline{v_\mathrm{r}^{2(\mu+1)}}}{2^{\mu+1}\left(\frac{1}{2}\right)_{\mu+1}^+}$$
$$= \begin{cases} [\Psi(r)]^{\mu+1} \int_0^1 \mathrm{d}q\, q^\mu \mathscr{P}[q\Psi(r); r] & (E_0 = 0) \\ \int_0^\infty \mathrm{d}Q\, Q^\mu \mathscr{P}(Q; r) & (E_0 = -\infty), \end{cases} \quad (10a)$$

where

$$\mathscr{P}(Q; r) \equiv \frac{\tilde{\nu}[\Psi(r) - Q, r^2]}{\nu(r)}. \quad (10b)$$

That is, given the local density $\nu(r)$ and the potential $\Psi(r)$, the infinite set of the radial velocity moments in every order consists in the moment sequence of the AD considered as a distribution of Ψ at fixed r. The problem is reducible to the *Hausdorff* (for $E_0 = 0$) or the *Stieltjes* (for $E_0 = -\infty$) *moment problems*. With the infinite sequence of the radial velocity moments as functions of r, the AD can then be uniquely determined at least formally by such means as e.g. the Hilbert basis or the Laplace and/or Fourier transform (cf. the moment generating function and the characteristic function).

The final information required for the full specification of the system is the determination of the potential. The self-consistent potential is determined through the Poisson equation. In fact, if the mass-to-light ratio is constant, $\Psi(r)$ may be fixed by solving the ordinary differential equation on $\Psi(r)$ that results from the spherical Poisson equation with the source term given by $\nu = \tilde{\nu}(\Psi, r^2)$. Alternatively, from equation (9a), we deduce for $k \geq 1$ that

$$\frac{\partial m_{k,n}}{\partial \Psi} = (2k - 1)\, m_{k-1,n},$$
$$\frac{\partial (r^{2n+2} m_{k,n})}{\partial r^2} = \left(k - \frac{1}{2}\right) r^{2n} m_{k-1,n+1}. \quad (11a)$$

Consequently the total radial derivative of $m_{k,n}$ for $k \geq 1$ results in

$$\frac{\mathrm{d}m_{k,n}}{\mathrm{d}r} = \frac{2m_{k,n}}{r}\left[\frac{\partial \log(r^{2n+2} m_{k,n})}{\partial \log r^2} - (n+1)\right] + \frac{\mathrm{d}\Psi}{\mathrm{d}r}\frac{\partial m_{k,n}}{\partial \Psi}$$
$$= -\frac{2(n+1)m_{k,n} - (2k-1)m_{k-1,n+1}}{r} + (2k-1)m_{k-1,n}\frac{\mathrm{d}\Psi}{\mathrm{d}r}. \quad (11b)$$

With $\Psi = \Psi(r)$ and $m_{k,n}[\Psi(r), r^2] = \nu\overline{v_\mathrm{r}^{2k} v_\mathrm{t}^{2n}}$, this may be solved for $\mathrm{d}\Psi/\mathrm{d}r$ if the required velocity moments as a function of r are known. For the simplest case $(k, n) = (1, 0)$, this reduces to the spherical (second-order steady-state) Jeans equation.

4 NECESSARY CONDITIONS FOR SEPARABLE AUGMENTED DENSITIES

In the following, we limit our concern to the cases for which the potential and the radius dependences of the AD are multiplicatively separable such that

$$\tilde{\nu}(\Psi, r^2) = P(\Psi)R(r^2). \quad (12)$$

In addition to mathematical expediency, this assumption is also notable because under the separability assumption in equation (12), the radius part $R(r^2)$ of the AD alone uniquely specifies the so-called Binney anisotropy parameter,

$$\beta(r) \equiv 1 - \frac{\overline{v_\mathrm{t}^2}}{2\overline{v_\mathrm{r}^2}} = 1 - \frac{m_{0,1}[\Psi(r), r^2]}{2m_{1,0}[\Psi(r), r^2]}$$
$$= 1 - \frac{1}{m_{1,0}}\frac{\partial(r^2 m_{1,0})}{\partial r^2} = -\left.\frac{\partial \log m_{1,0}}{\partial \log r^2}\right|_{\Psi(r), r^2} \quad (13)$$

such that (Dejonghe 1986; Qian & Hunter 1995)

$$\beta(r) = -\frac{\mathrm{d}\log R(r^2)}{\mathrm{d}\log r^2}, \qquad \frac{R(r^2)}{R(r_0^2)} = \exp\left(\int_r^{r_0} \frac{2\beta(s)}{s}\mathrm{d}s\right). \quad (14)$$

Some applications are found in Baes & Van Hese (2007) whilst An (2011b) discusses implications of the separability assumption.

4.1 Conditions on the radius part

An (2011b) has argued that (hereafter $x \equiv r^2$)

$$R_{(n)}(x) \equiv \frac{\mathrm{d}^n[x^n R(x)]}{\mathrm{d}x^n} \geq 0 \qquad (x > 0,\ n = 0, 1, 2, \ldots) \tag{15}$$

for the radius part $R(x)$ of equation (12) is necessary for the non-negativity of the corresponding df. Here we derive several equivalent statements of this condition.

First of these is

$${}_0D_x{}^{\mu}(x^{\mu} R) \geq 0 \qquad (x > 0,\ \mu \geq 0). \tag{16}$$

This follows equation (5b), which indicates that for $0 \leq \mu \leq \xi$

$${}_0D_x{}^{\mu}\left(x^{\mu}\,{}_{E_0}I_{\Psi}{}^{\xi-\frac{1}{2}}\tilde{\nu}\right) = {}_{E_0}I_{\Psi}{}^{\xi-\frac{1}{2}}P(\Psi) \times {}_0D_x{}^{\mu}[x^{\mu}R(x)] \geq 0 \tag{17}$$

given equation (12). Since $P \geq 0$ is obviously necessary, equation (16) follows this and Lemma A7, which implies that ${}_{E_0}I_{\Psi}{}^{\xi-\frac{1}{2}}P > 0$ for $\xi \geq \frac{1}{2}$. It is trivial that equation (16) implies equation (15) as the latter is the restriction of the former for an integer $\mu = n$. The opposite implication follows Corollary A35. That is to say, equation (15) for a particular positive integer n implies equation (16) for $\mu \in [n-1, n]$, and thus equation (16) for $\mu \geq 0$ follows equation (15) for all positive integers n.

Next, equation (A31) indicates that

$$R_{(n)}(x) = \frac{1}{x^{n+1}}\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^n [xR(x)] = (-1)^n w^{n+1}\left.\frac{\mathrm{d}^n\mathcal{R}(w)}{\mathrm{d}w^n}\right|_{w=x^{-1}}, \tag{18}$$

where

$$\mathcal{R}(w) \equiv \frac{R(w^{-1})}{w}. \tag{19}$$

Hence equation (15) is also equivalent to

$$\begin{aligned}&\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^n [xR(x)] \geq 0 \qquad (x > 0,\ n = 0, 1, 2, \ldots),\\ &(-1)^n\frac{\mathrm{d}^n\mathcal{R}(w)}{\mathrm{d}w^n} \geq 0 \qquad (w > 0,\ n = 0, 1, 2, \ldots).\end{aligned} \tag{20}$$

The last is equivalent to saying that the function $\mathcal{R}(w)$ defined in equation (19) is a *completely monotonic* (Definition A12) function of w. The *Bernstein theorem* (Theorem A17) then implies that $\mathcal{R}(w)$ is representable as the Laplace transform of a non-negative function. In other words, there exists a non-negative function $\phi(t) \geq 0$ of $t > 0$ such that $\mathcal{R}(w) = \underset{t\to w}{\mathcal{L}}[\phi(t)]$. The inverse Laplace transformation may be found using the *Post–Widder formula* (equation A11), which, thanks to equation (18), reduces to

$$\phi(t) \equiv \underset{w\to t}{\mathcal{L}^{-1}}[\mathcal{R}(w)] = \lim_{n\to\infty}\frac{1}{n!}R_{(n)}\left(\frac{t}{n}\right). \tag{21}$$

Thus we find another equivalent necessary condition,

$$\lim_{n\to\infty}\frac{1}{n!}\left.\frac{\mathrm{d}^n[x^nR(x)]}{\mathrm{d}x^n}\right|_{x=t/n} \geq 0 \qquad (t > 0). \tag{22}$$

It is obvious that equation (15) implies equation (22), provided that it converges. The converse on the other hand follows the Bernstein theorem and the Post–Widder formula. However, the conditional equivalence given its convergence may also be inferred from Corollary A33. By definition, equation (22) indicates that there exists a sufficiently large integer $m > 0$ such that $R_{(n)}(x) \geq 0$ for $^{\forall}n \geq m$ and $x > 0$. Corollary A33 then suggests that $R_{(m-1)}(x) \geq 0$ for $x > 0$, and equation (15) follows subsequent successive arguments with descending subscripts of $R_{(n)}(x)$.

4.2 Conditions on the potential part

Van Hese et al. (2011) have proven that given equation (12), $P^{(k)}(\Psi) \geq 0$ for all accessible Ψ and any non-negative integer k not greater than $\frac{3}{2} - \beta_0$, where β_0 is the limit of the anisotropy parameter at the centre, is necessary for the df to be non-negative. We shall show that this generalizes incorporating fractional derivatives.

If the AD is given as in equation (12), equation (5a) results in

$${}_{E_0}D_{\Psi}{}^{\mu}\,{}_0I_x{}^{\xi-\frac{1}{2}}\left(\frac{\tilde{\nu}}{x^{\xi-1/2}}\right) = {}_{E_0}D_{\Psi}{}^{\mu}P \times {}_0I_x{}^{\xi-\frac{1}{2}}\left(\frac{R}{x^{\xi-1/2}}\right) \geq 0, \tag{23}$$

for $0 \leq \mu \leq \xi$. Since $R(x) \geq 0$ is again trivially necessary, ${}_0I_x{}^{\lambda}(x^{-\lambda}R) > 0$ for $x > 0$ and any $\lambda \geq 0$ unless $R(x) = 0$ *almost everywhere* in $x \equiv r^2 \in [0, \infty)$ (Lemma A7). Ignoring pathological cases, we conclude that equation (23) implies that

$$0 < {}_0I_x{}^{\lambda}(x^{-\lambda}R) < \infty \implies {}_{E_0}D_{\Psi}{}^{\mu}P \geq 0 \qquad (\mu \leq \lambda + \tfrac{1}{2}). \tag{24}$$

With $\lambda = 0$, this indicates that ${}_{E_0}D_{\Psi}{}^{\mu}P \geq 0$ for $\mu \leq \frac{1}{2}$. For $\lambda > 0$ on the other hand, equation (24) implies that if $x^{-\lambda}R(x)\,\mathrm{d}x$ is integrable over $x = 0$, then ${}_{E_0}D_{\Psi}{}^{\mu}P \geq 0$ for $\mu \leq \lambda + \frac{1}{2}$ and all accessible Ψ is necessary for a non-negative df. Alternatively, ${}_{E_0}D_{\Psi}{}^{\mu}P \geq 0$ with a fixed $\mu > \frac{1}{2}$ is necessary for the df to be non-negative if there exists $\lambda \geq \mu - \frac{1}{2}$ such that ${}_0I_x{}^{\lambda}(x^{-\lambda}R)$ is well defined.

Equation (24) is yet inconclusive regarding whether ${}_{E_0}D_{\Psi}{}^{\frac{3}{2}-\beta}P \geq 0$ is necessary for the phase-space consistency given $R(x) \sim x^{-\beta}$ with $\beta < 1$ as $x \to 0$, which is in fact necessary as shown below. For this, we first note that if $h(t)$ is right continuous at $t = a$,

$$\lim_{\epsilon\to 0^+}\epsilon\int_a^{\bar{t}}\frac{h(t)\,\mathrm{d}t}{(t-a)^{1-\epsilon}} = \lim_{t\to a^+}h(t) = h(a) \qquad (a < \bar{t}). \tag{25}$$

This applied to the left-hand side of equation (5a) results in

$$\lim_{\xi\to(\frac{3}{2}-\eta)^-}\left(\tfrac{3}{2}-\eta-\xi\right){}_0I_x{}^{\xi-\frac{1}{2}}\left(\frac{\tilde{\nu}}{x^{\xi-1/2}}\right) = \frac{\hat{P}_{\eta}(\Psi)}{x^{\eta}\Gamma(1-\eta)}, \tag{26a}$$

where $\eta < 1$ and

$$\hat{P}_{\eta}(\Psi) = \lim\nolimits_{x\to 0^+}x^{\eta}\tilde{\nu}(\Psi, x). \tag{26b}$$

Equation (5a) then results in the formula

$${}_{E_0}D_{\Psi}{}^{\mu}\hat{P}_{\eta}(\Psi) = 2^{\frac{3}{2}-\eta}\pi^{\frac{3}{2}}\Gamma(1-\eta)\,{}_{E_0}I_{\Psi}{}^{\frac{3}{2}-\eta-\mu}\tilde{g}_{\eta}(\Psi) \geq 0, \tag{26c}$$

where

$$\tilde{g}_{\eta}(E) = \lim\nolimits_{L^2\to 0^+}L^{2\eta}F(E, L^2). \tag{26d}$$

For $\mu < \frac{3}{2} - \eta$, this is derived with the limit $\xi \to (\frac{3}{2}-\eta)^-$ while maintaining $\mu < \xi < \frac{3}{2} - \eta$. For $\mu = \frac{3}{2} - \eta$ on the other hand, the same limit is taken with $\mu = \xi$. Hence, equation (26c) is valid for $\mu \leq \frac{3}{2} - \eta$ and $\eta < 1$, provided that ${}_0I_x{}^{\xi-\frac{1}{2}}(x^{\frac{1}{2}-\xi}\tilde{\nu})$ is well defined for $\xi < \frac{3}{2} - \eta$ (n.b., the integrability of the same for $\xi = \frac{3}{2} - \eta$ is actually *not* required for its validity). The non-negativity of equation (26c) follows the non-negativity of $F(E, L^2)$. Of particular interests is equation (26c) for $\mu = 0$ and $\frac{3}{2} - \eta$,

$$\begin{aligned}\hat{P}_{\eta}(\Psi) &= 2^{\frac{3}{2}-\eta}\pi^{\frac{3}{2}}\Gamma(1-\eta)\,{}_{E_0}I_{\Psi}{}^{\frac{3}{2}-\eta}\tilde{g}_{\eta}(\Psi),\\ \tilde{g}_{\eta}(\Psi) &= \frac{{}_{E_0}D_{\Psi}{}^{\frac{3}{2}-\eta}\hat{P}_{\eta}(\Psi)}{2^{3/2-\eta}\pi^{3/2}\Gamma(1-\eta)},\end{aligned} \tag{27}$$

which give explicit formulae for $\hat{P}_{\eta}(\Psi)$ and $\tilde{g}_{\eta}(\Psi)$ from each other.

For a separable AD given as in equation (12), we have

$$\hat{P}_{\eta}(\Psi) = \hat{R}_{\eta}P(\Psi), \qquad \hat{R}_{\eta} = \lim\nolimits_{x\to 0^+}x^{\eta}R(x). \tag{28}$$

Therefore, equation (26c) indicates that

$$0 < \hat{R}_\eta < \infty \quad \Longrightarrow \ {}_{E_0}D_\Psi{}^\mu P \geq 0 \qquad (\mu \leq \tfrac{3}{2} - \eta). \tag{29}$$

That is, if there exists $\eta < 1$ such that $\hat{R}_\eta$ is a positive finite constant, then ${}_{E_0}D_\Psi{}^\mu P \geq 0$ for $^\forall\mu \leq \frac{3}{2} - \eta$. This encompasses equation (24), which is seen as follows: if $\hat{R}_\eta$ is non-zero finite for $\eta < 1$, then $R \sim x^{-\eta}$ as $x \to 0$. Hence ${}_0I_x{}^\lambda(x^{-\lambda}R)$ converges for $\lambda < 1 - \eta$, and so if $\mu \leq \lambda + \frac{1}{2}$ and ${}_0I_x{}^\lambda(x^{-\lambda}R)$ is well defined, then $\mu < \frac{3}{2} - \eta$.

For example, with a constant anisotropy system of $R(x) = x^{-\beta}$, we find that $\hat{R}_\beta = 1$ whilst the convergence condition reduces to

$${}_0I_x{}^\lambda(x^{-\lambda}R) = \frac{1}{\Gamma(\lambda)}\int_0^x \frac{(x-s)^{\lambda-1}\,\mathrm{d}s}{s^{\lambda+\beta}} = \frac{\Gamma(1-\beta-\lambda)}{x^\beta\Gamma(1-\beta)} < \infty, \tag{30}$$

which converges for $0 \leq \lambda < 1 - \beta$. It follows that equation (24) indicates that ${}_{E_0}D_\Psi{}^\mu P \geq 0$ for $\mu \leq \lambda + \frac{1}{2} < \frac{3}{2} - \beta$ is necessary for the df to be non-negative whereas equation (26c) suggests the same for $\mu \leq \frac{3}{2} - \beta$ (and $\beta < 1$).

5 SUFFICIENT CONDITIONS FOR PHASE-SPACE CONSISTENCY

In the companion paper (Van Hese, An & Baes, in preparation), we derive the necessary *and* sufficient condition for the df with $E_0 = 0$ to be non-negative, expressed in terms of the integro-differential constraints of the AD. This is achieved by reducing the problem to the Hausdorff moment problem, according to which the df is non-negative if and only if the moment sequence of equation (7) is a *completely monotone sequence*.[1] Since the moment sequence is generated by the AD using equation (8), this condition is expressible in terms of finite differences of integro-differential operations on the AD.

With a separable AD, Van Hese et al. (in preparation) also derive a simple sufficient (but not necessary) condition composed of two pieces, each of which only involves the potential or the radius part separately but not together. In this paper we derive an alternative sufficient condition for a separable AD to be resulted from a non-negative df, which turns out to be equivalent to that of Van Hese et al. (in preparation). The derivation here is based on the properties of completely monotonic functions and the Laplace transform. In the following, we consider only the case that $E_0 = 0$ and $L_\mathrm{m}^2 = 2r^2\Psi$, that is, the df has a compact support and $F(E < 0, L^2) = 0$.

5.1 Sufficient conditions on a separable augmented density

Inverting equation (3b) for $F(E, L^2)$ is formally equivalent to recovering the two-integral even df, $F^+(E, J_z^2)$, from the axisymmetric density $\nu[\Psi(R^2, z^2), R^2]$ (Hunter & Qian 1993). One notable inversion formula of this kind is that of Lynden-Bell (1962) who had utilized the Laplace transform. This suggests that $\phi(t)$ in equation (21) should be related to $F(E, L^2)$. In Appendix C we do in fact find that the df that builds the separable AD of equation (12) with $E_0 = 0$ is recovered via the inverse Laplace transform given by

$$F(E, L^2) = \mathop{\mathcal{L}^{-1}}_{s\to E}\left[\frac{s^{\frac{3}{2}}\mathcal{P}(s)}{(2\pi)^{3/2}}\phi\left(\frac{sL^2}{2}\right)\right], \tag{31}$$

where $\mathcal{P}(s) \equiv \mathop{\mathcal{L}}_{\Psi\to s}[P(\Psi)]$ is the Laplace transformation of $P(\Psi)$ and $\phi(t)$ is as defined in equation (21).

By the Bernstein theorem, equation (31) is non-negative if and only if its Laplace transform is a completely monotonic function of $s > 0$ for all accessible L^2. However $\mathcal{P}(s)$ is already completely monotonic since $P(\Psi) \geq 0$. Thus, that $s^{\frac{3}{2}}\phi(sL^2/2)$ is a completely monotonic function of $s > 0$ for any $L^2 \geq 0$ is in fact sufficient for the df to be non-negative (Lemma A14-5). Equivalently, since

$$\left.\frac{\mathrm{d}^n[t^{\frac{3}{2}}\phi(t)]}{\mathrm{d}t^n}\right|_{t=sL^2/2} = \left(\frac{L^2}{2}\right)^{\frac{3}{2}-n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[s^{\frac{3}{2}}\phi\left(\frac{sL^2}{2}\right)\right], \tag{32}$$

the condition is equivalent to the complete monotonicity of $t^{\frac{3}{2}}\phi(t)$. Unfortunately, this is too severe to be physical,[2] which is inferred in reference to the constant anisotropy model given by $R(x) = x^{-\beta}$ and $\phi(t) = t^{-\beta}/\Gamma(1-\beta)$. The condition for this system reduces to

$$\frac{\left(\beta - \frac{3}{2}\right)_n^+}{\Gamma(1-\beta)}\frac{1}{t^{\beta+n-3/2}} \geq 0 \qquad (t > 0,\ n = 0, 1, 2, \ldots), \tag{33}$$

which cannot be satisfied for any constant $\beta < 1$.

Nevertheless, the preceding discussion extends to yield useful sufficient conditions: that is, for any fixed λ, the conditions that

$$(-1)^n\frac{\mathrm{d}^n[s^\lambda\mathcal{P}(s)]}{\mathrm{d}s^n} \geq 0 \qquad (s > 0,\ n = 0, 1, 2, \ldots) \tag{34}$$

and

$$(-1)^n\frac{\mathrm{d}^n[t^{\frac{3}{2}-\lambda}\phi(t)]}{\mathrm{d}t^n} \geq 0 \qquad (t > 0,\ n = 0, 1, 2, \ldots) \tag{35}$$

are jointly sufficient to imply equation (C7) being completely monotonic and consequently the df in equation (31) being non-negative. With increasing λ, the constraint in equation (34) tightens whereas the condition in equation (35) becomes strictly weaker. In other words, with a larger λ, the smaller subset of functions $P(\Psi)$ will lead to $s^\lambda\mathcal{P}(s)$ being completely monotonic. At the same time if $\phi(t)$ satisfies equation (35) for a fixed $\lambda = \lambda_0$, the same condition for any larger $\lambda \geq \lambda_0$ automatically holds. Both of these are easily inferred using Corollary A15.

5.1.1 The condition on R(x) equivalent to equation (35)

To translate equation (35) into a direct constraint on $R(x)$, we first assume the existence of $\phi(t)$, the validity of equation (21) and its non-negativity, that is, $\phi(t) \geq 0$ for $t > 0$, which are all necessary. Substituting equation (21) into equation (35) then results in

$$\begin{aligned}(-1)^n\frac{\mathrm{d}^n[t^{\frac{3}{2}-\lambda}\phi(t)]}{\mathrm{d}t^n} &= \lim_{k\to\infty}\frac{(-1)^n}{k!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[t^{\frac{3}{2}-\lambda}R_{(k)}\left(\frac{t}{k}\right)\right] \\ &= \lim_{k\to\infty}\frac{(-1)^n}{k!k^{n+\lambda-3/2}}\left.\frac{\mathrm{d}^n[x^{\frac{3}{2}-\lambda}R_{(k)}(x)]}{\mathrm{d}x^n}\right|_{x=t/k}.\end{aligned} \tag{36}$$

Provided that this converges, equation (35) is equivalent to insisting that there exists an integer $m > 0$ such that, for all integers $^\forall k \geq m$,

$$(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{x^{\frac{3}{2}-\lambda}\frac{\mathrm{d}^k[x^kR(x)]}{\mathrm{d}x^k}\right\} \geq 0 \qquad (x > 0,\ n = 0, 1, 2, \ldots). \tag{37}$$

[1] A sequence $(a_0, a_1, a_2, \ldots)$ is completely monotone if and only if $(-1)^k\Delta^ka_j \geq 0$ for all non-negative integer pairs k and j. Here Δ is the finite difference operator such that $\Delta^{k+1}a_j = \Delta^ka_{j+1} - \Delta^ka_j$ and $\Delta^0a_j = a_j$.

[2] If the Laplace transform of $\phi(t)$ exists, then $\phi(t)$ cannot diverge faster than t^{-1} as $t \to 0$. Consequently, $\lim_{t\to0}t^{3/2}\phi(t) \to 0$ and thus $t^{3/2}\phi(t)$ cannot be completely monotonic because the limit suggests that $t^{3/2}\phi(t)$ should be negative or increasing in some interval $t \in (0, t_0)$ where $^\exists t_0 > 0$.

In other words, the complete monotonicity of $x^{\frac{3}{2}-\lambda}R_{(k)}(x)$ for *all sufficiently large* integers k is equivalent to equation (35), that is, the complete monotonicity of $t^{\frac{3}{2}-\lambda}\phi(t)$. In fact, equation (35) is equivalent to equation (37) for not only all sufficiently large integers but also all non-negative integers k, which follows successive applications of Theorem A36 with descending subscripts k (the opposite implication is trivial). Note that the condition as stated in this last form, that is, equation (37) for all non-negative integers k, is the same as noted by Van Hese et al. (in preparation).

5.1.2 *The condition on $P(\Psi)$ equivalent to equation (34)*

Explicit constraints on $P(\Psi)$ resulting from equation (34) are expressible by means of fractional calculus. First, equations (A9) and (A10) indicate that (n.b., ${}_0I_\Psi{}^{1-\delta}P(0)=0$ from Corollary A9)

$$s^\lambda\mathcal{P}(s)=s^{\mu+1-(1-\delta)}\underset{\Psi\to s}{\mathcal{L}}[P(\Psi)]=s^{\mu+1}\underset{\Psi\to s}{\mathcal{L}}\left[{}_0I_\Psi{}^{1-\delta}P(\Psi)\right]$$
$$=\underset{\Psi\to s}{\mathcal{L}}\left[{}_0D_\Psi{}^{\lambda}P(\Psi)\right]+\textstyle\sum_{j=1}^{\mu}s^{j-1}{}_0D_\Psi{}^{\lambda-j}P(0), \qquad (38)$$

where $\mu=\lfloor\lambda\rfloor$ and $\delta=\lambda-\mu$ $(0\le\delta<1)$ are the integer floor and the fractional part of λ. This suggests that for $\lambda\ge0$, together

$${}_0D_\Psi{}^{\lambda}P(\Psi)\ge0 \qquad (\Psi>0) \qquad (39)$$

and

$${}_0I_\Psi{}^{1-\delta}P(0)={}_0D_\Psi{}^{\delta}P(0)=\cdots={}_0D_\Psi{}^{\lambda-1}P(0)=0 \qquad (40)$$

are sufficient for $s^\lambda\mathcal{P}(s)$ to be completely monotonic. Note, provided that $P(\Psi)$ is right continuous at $\Psi=0$, that ${}_0I_\Psi{}^{1-\delta}P(0)=0$ (Corollary A9), which is taken as granted henceforth. If $\lambda=p+1$ is a positive integer, equations (39) and (40) reduce to

$$P^{(p+1)}(\Psi)\ge0 \quad\text{and}\quad P(0)=\cdots=P^{(p)}(0)=0. \qquad (41)$$

For $0\le\delta<1$ on the other hand, equation (40) may also be replaced with the same boundary condition as in equation (41). That is to say, $P^{(0)}(0)=\cdots=P^{(n)}(0)=0$ actually implies ${}_0D_\Psi{}^{n+\delta}P(0)=0$ for $0<\delta<1$ (Lemma A37), and thus it follows that for $\lambda\ge1$,

$$P^{(0)}(0)=\cdots=P^{(\lfloor\lambda\rfloor-1)}(0) \qquad (42)$$

also implies equation (40) (they are identical if $\delta=0$). Therefore, together equations (39) and (42) also consist in a sufficient condition for $s^\lambda\mathcal{P}(s)$ to be completely monotonic at a fixed λ. The condition as expressed with equation (42) is also useful because equation (A7) indicates that equation (39) is then equivalent to

$${}_0D_\Psi{}^{\lambda}P=\frac{1}{\Gamma(1-\delta)}\frac{\mathrm{d}^{1+\mu-n}}{\mathrm{d}\Psi^{1+\mu-n}}\int_0^\Psi\frac{P^{(n)}(Q)\,\mathrm{d}Q}{(\Psi-Q)^\delta}\ge0, \qquad (43)$$

where n is any non-negative integer not greater than λ.

Again, the joint condition of equations (39) and (42) becomes strictly stronger as λ increases in accordance with the restriction on the complete monotonicity of $s^\lambda\mathcal{P}(s)$. This is seen with equation (A6) for $0\le\epsilon\le\lambda$ given equation (40) or (42), that is, ${}_0I_\Psi{}^{\epsilon}\left({}_0D_\Psi{}^{\lambda}P\right)={}_0D_\Psi{}^{\lambda-\epsilon}P$. Therefore, ${}_0D_\Psi{}^{\lambda}P(\Psi)\ge0$ implies ${}_0D_\Psi{}^{\xi}P(\Psi)\ge0$ for $0\le\xi\le\lambda$. The similar implications of equation (42) with descending λ are trivial.

5.2 Constant anisotropy models

Let us consider the constant anisotropy model given with

$$R(x)=x^{-\beta}, \qquad \mathcal{R}(w)=w^{\beta-1}, \qquad R_{(n)}(x)=(1-\beta)_n^+x^{-\beta}, \qquad (44a)$$

which satisfies the necessary condition in Section 4.1 if and only if $\beta\le1$ (cf. Lemma A13). The function $\phi(t)$ as defined in equation (21) for $\beta<1$ is found using either $\underset{s\to t}{\mathcal{L}}[s^{a-1}]=t^{-a}\Gamma(a)$ with $a>0$ or $\lim_{n\to\infty}(n!n^z)/(1+z)_n^+=\Gamma(1+z)$ so that

$$\phi(t)=\frac{1}{t^\beta\Gamma(1-\beta)} \qquad (\beta<1). \qquad (44b)$$

For $\beta=1$, formally $\phi(t)$ results in the Dirac delta. Although this case will not be discussed explicitly here (see Appendix D instead), the following result actually extends for $\beta\le1$.

Equations (35) and (37) now reduce to

$$(-1)^n\,\frac{\mathrm{d}^n[t^{\frac{3}{2}-\lambda}\phi(t)]}{\mathrm{d}t^n}=\frac{1}{\Gamma(1-\beta)}\frac{\left(\beta+\lambda-\frac{3}{2}\right)_n^+}{t^{\beta+n+\lambda-3/2}}\ge0,$$
$$(-1)^n\,\frac{\mathrm{d}^n[x^{\frac{3}{2}-\lambda}R_{(k)}(x)]}{\mathrm{d}x^n}=(1-\beta)_k^+\frac{\left(\beta+\lambda-\frac{3}{2}\right)_n^+}{x^{\beta+n+\lambda-3/2}}\ge0. \qquad (45)$$

For $\beta<1$, this is equivalent to $\beta+\lambda\ge\frac{3}{2}$. It follows that if $R(x)=x^{-\beta}$ with $\frac{1}{2}-p\le\beta<1$ where p is a non-negative integer, then $P(\Psi)$ satisfying equation (41) is sufficient for the existence of a non-negative df (cf. Ciotti & Morganti 2010a). In general for any *real* $\lambda>\frac{1}{2}$, if $R(x)=x^{-\beta}$ with $\frac{3}{2}-\lambda\le\beta<1$, equations (39) and (42) constitute a sufficient condition for the phase-space consistency.

For a fixed $\beta<1$, this indicates that, if there exists $\lambda\ge\frac{3}{2}-\beta$ such that equations (39) and (42) hold for $P(\Psi)$, then $\tilde{\nu}=r^{-2\beta}P$ guarantees the non-negativity of the corresponding df. Here the existence of such λ further implies ${}_0D_\Psi{}^{\xi}P\ge0$ for $0\le{}^\forall\xi\le{}^\exists\lambda$ whilst Section 4.2 suggests that ${}_0D_\Psi{}^{\mu}P\ge0$ for ${}^\forall\mu\le\frac{3}{2}-\beta$ is necessary for the df inverted from $\tilde{\nu}=r^{-2\beta}P$ to be non-negative. It follows that if $\tilde{\nu}(\Psi,r^2)=r^{-2\beta}P(\Psi)$, then ${}_0D_\Psi{}^{\frac{3}{2}-\beta}P\ge0$ is the necessary *and* sufficient condition for the phase-space consistency. In fact, here $P(\Psi)=\hat{P}_\beta(\Psi)$ and $F(E,L^2)=\tilde{g}_\beta(E)L^{-2\beta}$ where $\hat{P}_\beta(\Psi)$ and $\tilde{g}_\beta(E)$ are as defined in equations (26b) and (26d) with $\eta=\beta$. Hence equation (27) results in the inversion formula $(\beta<1)$,

$$F(E,L^2)=\frac{{}_0D_E{}^{\frac{3}{2}-\beta}P(E)}{2^{3/2-\beta}\pi^{3/2}\Gamma(1-\beta)L^{2\beta}} \quad\Longleftarrow\quad \tilde{\nu}(\Psi,r^2)=\frac{P(\Psi)}{r^{2\beta}}. \qquad (46)$$

This is just the generalized Eddington inversion formula (e.g. Evans & An 2006) for constant anisotropy systems. That ${}_0D_\Psi{}^{\frac{3}{2}-\beta}P(\Psi)\ge0$ is necessary and sufficient for the existence of a non-negative df is its trivial consequence.

6 FAMILY OF MONOTONIC ANISOTROPY PARAMETERS

Consider the anisotropy parameter (Baes & Van Hese 2007),

$$\beta(r)=\frac{\beta_1r_\mathrm{a}^{2s}+\beta_2r^{2s}}{r_\mathrm{a}^{2s}+r^{2s}} \qquad (s>0,\ r_\mathrm{a}>0). \qquad (47a)$$

If the spherical system is characterized by a separable AD as in equation (12), this follows the radial function (cf. equation 14)

$$R(x)=x^{-\beta_1}(1+x^s)^{-\zeta} \qquad \text{where } s\zeta=\beta_2-\beta_1,$$
$$\mathcal{R}(w)=w^{-1}R(w^{-1})=w^{\beta_1-1}(1+w^{-s})^{-\zeta}=w^{\beta_2-1}(1+w^s)^{-\zeta}. \qquad (47b)$$

Hereafter we set $r_\mathrm{a}=1$ (i.e. $x=r^2/r_\mathrm{a}^2$), but this has no effect on the following discussion whatsoever.

Note that $R_{(1)}(x)\ge0$ for $x>0$ restricts $\beta_1,\beta_2\le1$. In fact,

Theorem 6.1 (An 2011b). *$R(x)$ given by equation* (47b) *with* $0 < s \leq 1$ *and* $\beta_1, \beta_2 \leq 1$ *satisfies the necessary condition in Section* 4.1.

This is easily deduced from Corollary A16. However, the situation for $s > 1$ is inconclusive. On one hand, if $\beta_2 = 1 > \beta_1$, then $\mathcal{R}''(w) < 0$ for $w^s < (s-1)/(2-\beta_1)$ and so the condition fails for $s > 1$. An (2011b) on the other hand has found that the condition is met for all $s > 0$ if ζ is zero or a negative integer. It appears that for $s > 1$, there may exist a *proper* subset of parameter combinations $\beta_1, \beta_2 \leq 1$ that satisfies the necessary condition of equation (15), but we have not been able to establish the concrete criteria.

The necessary condition on the potential part in Section 4.2 on the other hand is straightforward since $R(x) \sim x^{-\beta_1}$ as $x \to 0$. That is,

Theorem 6.2. *If the AD is given by equation* (12) *with $R(x)$ of equation* (47b), *the potential part $P(\Psi)$ must satisfy*

$$_{E_0}D_\Psi{}^\lambda P(\Psi) \geq 0 \qquad \text{for } {}^\forall\lambda \leq \tfrac{3}{2} - \beta_1 \tag{48}$$

in order for the df to be non-negative.

Here also note $\beta_1 \leq 1$ and thus $_{E_0}D_\Psi{}^\lambda P \geq 0$ for any $\lambda \leq \frac{1}{2}$.

6.1 Sufficient conditions for a non-negative df with $0 < s \leq 1$

By Theorem A25, equation (21) results in

$$\phi(t) = t^{-\beta_1} E^\zeta_{s,1-\beta_1}\left(-t^s\right) \tag{49}$$

for $R(x)$ in equation (47b) with $s > 0$ and $\beta_1 < 1$ (for $\beta_1 = 1$ see Appendix D). Here $E^\lambda_{p,b}(z)$ is as defined in equation (A21).

We consider sufficient conditions to guarantee the phase-space consistency for a separable AD with $R(x)$ in equation (47b) with $0 < s \leq 1$ (and $E_0 = 0$). In Section 5.2, we have argued that for $\beta_1 = \beta_2 < 1$, if there exists $\lambda \geq \frac{3}{2} - \beta_1$ such that $_0D_\Psi{}^\lambda P \geq 0$ and $P(0) = \cdots = P^{(\lfloor\lambda\rfloor - 1)}(0) = 0$, then the df with $E_0 = 0$ inverted from $\tilde{\nu} = r^{-2\beta_1} P(\Psi)$ is non-negative everywhere. This follows from the fact that $t^{\frac{3}{2}-\lambda}\phi(t) = t^{\frac{3}{2}-\lambda-\beta_1}/\,\Gamma(1-\beta)$ is completely monotonic for $\lambda \geq \frac{3}{2} - \beta_1$. As with $\phi(t)$ in equation (49), if $\zeta > 0$, then $t^{\frac{3}{2}-\lambda}\phi(t)$ is completely monotonic for $\lambda \geq \frac{3}{2} - \beta_1$ (Theorem A27), and thus

Theorem 6.3. *For $E_0 = 0$ and $R(x)$ given by equation* (47b) *with $0 < s \leq 1$ and $\beta_1 < \beta_2 \leq 1$, if there exists $\lambda \geq \frac{3}{2} - \beta_1$ such that $_0D_\Psi{}^\lambda P \geq 0$ and $P(0) = \cdots = P^{(\lfloor\lambda\rfloor-1)}(0) = 0$, then the df inverted from $\tilde{\nu} = P(\Psi)R(r^2)$ is non-negative*.

This actually extends to $\beta_1 \leq \beta_2 \leq 1$ (Section 5.2 and Appendix D). Also the $(s, \beta_2) = (1, 1)$ case results in the Cuddeford (1991) system and thus this with an integer $\lambda \geq \frac{3}{2} - \beta_1$ reproduces the sufficient condition of Ciotti & Morganti (2010a, equation 27 or 28 with $m = \lfloor \frac{3}{2} - \beta_1 \rfloor$). Finally if $P(0) = \cdots = P^{(\lfloor \frac{1}{2} - \beta \rfloor)}(0) = 0$, then $_0D_\Psi{}^{\frac{3}{2}-\beta_1} P \geq 0$ is the necessary *and* sufficient condition for the phase-space consistency given $E_0 = 0$ and $R(x)$ with $0 < s \leq 1$ and $\beta_1 \leq \beta_2 \leq 1$.

For $\zeta \leq 0$ on the other hand, thanks to Theorems A28 and A29 (see again Appendix D for $\beta_1 = 1$), we find

Theorem 6.4. *For $E_0 = 0$ and $R(x)$ given by equation* (47b) *with $0 < s \leq 1$ and $\beta_2 \leq \beta_1 \leq 1$, if there exists $\lambda \geq \frac{3}{2} - \beta_1 + sn$, where $n = \lceil(\beta_1 - \beta_2)/s\rceil$ is the smallest integer that is not less than $(\beta_1 - \beta_2)/s$, such that $_0D_\Psi{}^\lambda P \geq 0$* and $P(0) = \ldots = P^{(\lfloor\lambda\rfloor-1)}(0) = 0$, *then the df inverted from $\tilde{\nu} = P(\Psi)R(r^2)$ is non-negative*.

Theorem 6.5. *For $E_0 = 0$ and $R(x)$ given by equation* (47b) *with $0 < s \leq 1$, $\beta_2 \leq \beta_1 \leq 1$ and $\beta_2 \leq 1 - s$, if there exists $\lambda \geq \frac{3}{2} - \beta_2$ such that $_0D_\Psi{}^\lambda P \geq 0$ and $P(0) = \cdots = P^{(\lfloor\lambda\rfloor-1)}(0) = 0$, then the df inverted from $\tilde{\nu} = P(\Psi)R(r^2)$ is non-negative*.

7 SUMMARY

The main findings of this paper are summarized as follows.

(i) We have argued that a unique AD $\tilde{\nu}(\Psi, r^2)$ (and subsequently the df) is specified given the potential $\Psi(r)$ and the density profile $\nu(r)$ once the infinite set of the radial velocity moments in every order (equivalently the complete radial velocity distribution) as a function of the radius is available (cf. Dejonghe & Merritt 1992).

(ii) We have also shown that the set of fractional calculus operations on the AD listed in equation (8) provides with the complete moment sequence of the df along $K(E, L^2; \Psi, r^2) = 0$ as shown in equation (7). We infer from this that the AD that ensures the non-negativity of the df may be deduced by analogy to the classical moment problem in probability theory (Van Hese et al. in preparation).

(iii) This introduces the set of necessary conditions on the AD for the non-negativity of the df. If the AD is multiplicatively separable into functions of the potential and the radius dependences like equation (12), this results in the necessary condition stated by An (2011b), that is, equation (15) for the radius part of the AD. We have also discovered a few equivalent statements of this condition, notably the complete monotonicity of the function $\mathcal{R}(w)$ defined in equation (19) as well as equation (22).

(iv) The similar argument for the potential part of a separable AD on the other hand recovers the conditions derived by Van Hese et al. (2011) and An (2011a), which are further generalized with fractional calculus to indicate that $_{E_0}D_\Psi{}^\mu P \geq 0$ for all accessible Ψ is necessary if $\mu \leq \frac{1}{2}$ or there exists $\lambda \geq \mu - \frac{1}{2}$ such that $_0I_{r^2}{}^\lambda[r^{-2\lambda}R(r^2)]$ is well defined or ${}^\exists\beta \leq \frac{3}{2} - \mu$ such that $\lim_{r^2\to0^+} r^{2\beta}R(r^2)$ is non-zero and finite.

(v) The df of an escapable system with a separable AD may be inverted from the latter utilizing the inverse Laplace transform as in equation (31). The non-negativity of the resulting df is guaranteed if its Laplace transformation is completely monotonic. From this we have found that the joint condition at a fixed λ composed of equation (37) for $R(x)$ with all non-negative integer pairs n and k, and equations (39) and (42) for $P(\Psi)$ is sufficient to imply the phase-space consistency of the system corresponding to $\tilde{\nu}(\Psi, r^2) = P(\Psi)R(r^2)$.

(vi) With $R(x)$ given by equation (47b) with $0 < s \leq 1$ and β_1, $\beta_2 \leq 1$, the condition $_{E_0}D_\Psi{}^\lambda P \geq 0$ for ${}^\forall\lambda \leq \frac{3}{2} - \beta_1$ is necessary in order for the AD $P(\Psi)R(r^2)$ to correspond to a non-negative df. For an escapable system with the same $R(x)$, if there exists $\lambda \geq \frac{3}{2} - \min(\beta_1, \beta_2)$ such that equations (39) and (42) hold for $P(\Psi)$, then the AD $P(\Psi)R(r^2)$ guarantees the phase-space consistency, unless $1 - s < \beta_2 < \beta_1 < 1$. If $1 - s < \beta_2 < \beta_1 < 1$ on the other hand, we at this point only find a slightly restrictive sufficient condition with ${}^\exists\lambda \geq \frac{3}{2} - (\beta_1 - s) > \frac{3}{2} - \beta_2 > \frac{3}{2} - \beta_1 > \frac{1}{2}$ (n.b., $\beta_1 - s < 1 - s < \beta_2 < \beta_1 < 1$).

Finally, we briefly consider possible generalizations of our conditions to inseparable ADs. First we note that it is possible to write down the necessary *and* sufficient condition for the phase-space consistency of any (i.e. not necessarily separable) AD by means of completely monotone sequences as developed by Van Hese et al. (in preparation) although its actual algebraic expression appears to be rather cumbersome. Secondly, whilst the necessary conditions discussed in Section 4 are not directly applicable for inseparable ADs, the idea behind their derivations is none the less valid in general and straightforward to extend for arbitrary ADs. Lastly, if the AD were to be given by a sum of separable components, the joint sufficient conditions applied for each component would be sufficient for the phase-space consistency of the whole system thanks to the linearity of the transformation from the df to the AD (however, the similar argument for the necessary condition is invalid).

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APPENDIX A: MATHEMATICAL PRELIMINARY

A1 Fractional calculus

Although it is not usually a part of typical curricula of mathematical methods, the concept of fractional calculus, if not by its name, appears not infrequently in problems of dynamical systems (e.g. Lake 1981; Pedraza, Ramos-Caro & González 2008). For more backgrounds and details than those provided here, see e.g. Srivastava & Saxena (2001) and references therein.

Definition A1. *For any non-negative real* $\lambda \geq 0$, *the Riemann–Liouville integral operator is defined to be*

$$ {}_aI_x{}^{\lambda} f \equiv \begin{cases} \dfrac{1}{\Gamma(\lambda)}\displaystyle\int_a^x (x-y)^{\lambda-1} f(y)\,\mathrm{d}y & (\lambda > 0) \\ f(x) & (\lambda = 0), \end{cases} \tag{A1} $$

where $\Gamma(x)$ *is the gamma function*.

For $0 < \lambda < 1$, this is also recognized as the Abel transform with the classical case corresponding to $\lambda = \frac{1}{2}$. Next we define

Definition A2. *The fractional derivative for* $\lambda \geq 0$ *given by*

$$ \begin{aligned} {}_aD_x{}^{\lambda} f &\equiv \frac{\mathrm{d}^{\lceil\lambda\rceil}}{\mathrm{d}x^{\lceil\lambda\rceil}}\, {}_aI_x{}^{\lceil\lambda\rceil-\lambda} f \\ &= \begin{cases} \dfrac{1}{\Gamma(\lceil\lambda\rceil-\lambda)} \dfrac{\mathrm{d}^{\lceil\lambda\rceil}}{\mathrm{d}x^{\lceil\lambda\rceil}} \displaystyle\int_a^x \frac{f(y)\,\mathrm{d}y}{(x-y)^{\lambda-\lfloor\lambda\rfloor}} & (\lfloor\lambda\rfloor < \lambda < \lceil\lambda\rceil) \\ f^{(\lambda)}(x) & (\lambda = \lfloor\lambda\rfloor = \lceil\lambda\rceil), \end{cases} \end{aligned} \tag{A2} $$

where $\lceil\lambda\rceil$ *and* $\lfloor\lambda\rfloor$ *are the integer ceiling and floor of* λ, *respectively*.

The definitions are extended to include a negative index using

Definition A3. *For arbitrary real* λ,

$$ {}_aI_x{}^{-\lambda} f = {}_aD_x{}^{\lambda} f \quad \textit{and vice versa}. \tag{A3} $$

The basic result regarding these operators is the composite rules

$$ \begin{aligned} {}_aI_x{}^{\xi}\left({}_aI_x{}^{\lambda} f\right) &= {}_aI_x{}^{\xi+\lambda} f, \\ {}_aD_x{}^{\xi}\left({}_aI_x{}^{\lambda} f\right) &= \begin{cases} {}_aI_x{}^{\lambda-\xi} f & (\xi \leq \lambda) \\ {}_aD_x{}^{\xi-\lambda} f & (\xi \geq \lambda) \end{cases} \end{aligned} \tag{A4} $$

for $\lambda, \xi \geq 0$, provided that all the integrals in their definitions absolutely converge. These are shown by direct calculations utilizing the Fubini theorem and the Euler integral of the first kind for the beta function. Equations (A4) are however not valid for negative indices λ or ξ without modification involving the boundary terms.

For proper results, we first observe for $\xi \geq 0$ that

$$ {}_aI_x{}^{\xi+1} f'(x) = {}_aI_x{}^{\xi} f(x) - \frac{(x-a)^{\xi} f(a)}{\Gamma(1+\xi)}. \tag{A5} $$

For $\xi > 0$, this is shown via integration by part whilst the $\xi = 0$ case results from the fundamental theorem of calculus. Using equations (A4) and (A5) (and Corollary A9), we then find that for $\lambda, \xi \geq 0$,

$$ \begin{aligned} {}_aI_x{}^{\xi}\left({}_aD_x{}^{\lambda} f\right) &= {}_aD_x{}^{\lambda}\left({}_aI_x{}^{\xi} f\right) \\ &\quad - \sum_{k=1}^{\lfloor\lambda\rfloor} \frac{(\xi)_k^-\, {}_aD_x{}^{\lambda-k} f(a)}{\Gamma(1+\xi)} (x-a)^{\xi-k}, \\ {}_aD_x{}^{\xi}\left({}_aD_x{}^{\lambda} f\right) &= {}_aD_x{}^{\xi+\lambda} f \\ &\quad - \sum_{k=1}^{\lfloor\lambda\rfloor} \frac{(-1)^{n+k}(\delta)_{n+k}^+}{\Gamma(1-\delta)} \frac{{}_aD_x{}^{\lambda-k} f(a)}{(x-a)^{k+\xi}}, \end{aligned} \tag{A6} $$

where $n = \lfloor\xi\rfloor$ and $\delta = \xi - \lfloor\xi\rfloor$, assuming that all the integrals in their definitions absolutely converge. Here

$$ (a)_n^+ \equiv \textstyle\prod_{j=1}^n (a-1+j) \quad \text{and} \quad (a)_n^- \equiv \prod_{j=1}^n (a+1-j) $$

are the *rising* and *falling* sequential products, which are related to each other via $(-a)_n^- = (-1)^n (a)_n^+$ and $(a)_n^- = (a-n+1)_n^+$. Both are also referred to as the Pochhammer symbol: $(a)_n^+$ follows the analyst's convention whilst $(a)_n^-$ does the combinatorist's. Equation (A5) also implies that the fractional derivative of a positive non-integer order may alternatively be given by

$$ {}_aD_x{}^\lambda f = \frac{\mathrm{d}^{\lceil\lambda\rceil-n}}{\mathrm{d}x^{\lceil\lambda\rceil-n}}\,{}_aI_x{}^{\lceil\lambda\rceil-\lambda} f^{(n)} + \sum_{k=0}^{n-1} \frac{(-1)^{\lfloor\lambda\rfloor-k}(\delta)^+_{\lfloor\lambda\rfloor-k}\, f^{(k)}(a)}{\Gamma(1-\delta)\,(x-a)^{\lambda-k}}, \tag{A7}$$

where $\delta = \lambda - \lfloor\lambda\rfloor$ is the fractional part of λ and $n = 0, 1, \ldots, \lceil\lambda\rceil$.

We formalize a fact, which is important for our purpose, namely

Lemma A7. *For $\lambda > 0$ and $x > a$, if $f(y) \ge 0$ for $^\forall y \in [a, x]$, then ${}_aI_x{}^\lambda f(x) > 0$, unless $f = 0$ almost everywhere in $[a, x]$, that is, provided that the support of f in (a, x) has non-zero measure.*

This is trivial by the definition of ${}_aI_x{}^\lambda$. Next we note

Lemma A8. *For a finite a,*

$$ {}_aI_x{}^\lambda f(x) \sim \frac{f(a)}{\Gamma(\lambda+1)}(x-a)^\lambda \qquad \text{as } x \to a^+, \tag{A8}$$

which is valid for $\lambda \ge 0$ if $f(x)$ is right continuous at $x = a$ or for $\lambda \ge -1$ if $f(x)$ is right differentiable at $x = a$.

This immediately implies that

Corollary A9. *If $f(x)$ is right continuous at $x = a$ ($a \ne \pm\infty$) and $f(a)$ is finite, then ${}_aI_x{}^\lambda f(a) = 0$ for $\lambda > 0$.*

Next, we examine the behaviour of fractional calculus operators under the Laplace transform. The basic result is for $\lambda \ge 0$

$$ s^{-\lambda} \mathop{\mathcal{L}}_{x\to s}[f(x)] = \mathop{\mathcal{L}}_{x\to s}\left[{}_0I_x{}^\lambda f(x)\right]. \tag{A9}$$

This is shown through direct calculations utilizing the Fubini theorem and the Euler integral of the second kind for the gamma function. The Laplace transform of fractional derivatives is then found by combining equation (A9) with

$$ s^{n+1} \mathop{\mathcal{L}}_{x\to s}[f(x)] = \mathop{\mathcal{L}}_{x\to s}[f^{(n+1)}(x)] + \textstyle\sum_{j=0}^n s^j f^{(n-j)}(0), \tag{A10}$$

which is valid given that the Laplace transform converges. Note that equation (A10) is proven for $n = 0$ via integration by part and the induction completes its proof for any non-negative integer n.

A2 Post–Widder formula and completely monotonic functions

Theorem A11 (Post–Widder). *If $\phi(t)$ is continuous for $t \ge 0$ and there exist reals $^\exists A > 0$ and $^\exists b$ such that $\mathrm{e}^{-bt}|\phi(t)| \le A$ for $^\forall t > 0$, then the Laplace transform $\mathop{\mathcal{L}}_{t\to x}[\phi(t)] \equiv \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-xt}\phi(t)$ converges and is infinitely differentiable in $x > b$. Moreover, $\phi(t)$ may be inverted from its Laplace transformation $f(x) = \mathop{\mathcal{L}}_{t\to x}[\phi(t)]$ via the differential inversion formula* (Post 1930; Widder 1941)

$$ \phi(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} f^{(n)}\left(\frac{n}{t}\right) \qquad (t > 0). \tag{A11}$$

This formula is usually named after Emil Leon Post (1897–1954) or together with David Vernon Widder (1898–1990). The proof may be found in a standard text on the Laplace transform.

Definition A12. *A smooth function $f(t)$ of $t > 0$ is said to be completely monotonic (cm henceforth) if and only if*

$$ (-1)^n f^{(n)}(t) \ge 0 \qquad (t > 0,\ n = 0, 1, 2, \ldots). \tag{A12}$$

The archetypal example of cm functions is $f(t) = \mathrm{e}^{-t}$. Other elementary examples of cm functions include

Lemma A13. *$f(t) = \ln(1 + t^{-1})$ is a cm function of $t > 0$ whilst $f(t) = t^{-\delta}$ for $t > 0$ is cm if and only if $\delta \ge 0$.*

Proof. This is shown via direct calculations. That is, for $n \ge 0$

$$ \frac{\mathrm{d}^{n+1}\ln(1+t^{-1})}{\mathrm{d}t^{n+1}} = (-1)^{n+1} n! \left[\frac{1}{t^{n+1}} - \frac{1}{(1+t)^{n+1}}\right], \tag{A13}$$

$$ \frac{\mathrm{d}^n t^{-\delta}}{\mathrm{d}t^n} = (-\delta)_n^- t^{-\delta-n} = (-1)^n \frac{(\delta)_n^+}{t^{n+\delta}}. \tag{A14}$$

Some basic properties of cm functions are

Lemma A14. *Let $f(t)$ and $g(t)$ be cm functions of $t > 0$. Then*

1. *$(-1)^n f^{(n)}(t)$ for any non-negative integer n is cm.*
2. *If $F(t) \ge 0$ in $(0, \infty)$ and $f(t) = -F'(t)$, then $F(t)$ is cm.*
3. *$\int_t^\infty f(s)\,\mathrm{d}s$ is cm, provided that it converges.*
4. *$af(t) + bg(t)$ is cm where a and b are non-negative constants.*
5. *$f(t) \times g(t)$ is cm.*
6. *If $F(t) > 0$ in $(0, \infty)$ and $f(t) = F'(t)$, then $(g \circ F)(t)$ is cm.*
7. *$\exp[f(t)]$ is cm.*

Here *1–4* are trivial whilst *5* follows direct calculations using the Leibniz rule. The last two may be shown by means of the Faà di Bruno formula, that is,

$$ (g\circ F)^{(n)}(t) = \textstyle\sum_{k=0}^n g^{(k)}[F(t)]\, B_{n,k}\left[f(t), f'(t), \ldots, f^{(n-k)}(t)\right]. \tag{A15}$$

Here $F'(t) = f(t)$ and $B_{n,k}$ is the Bell polynomial,

$$ B_{n,k}(x_0, \ldots, x_{n-k}) \equiv \sum_{(j_0, j_1, \ldots)}{}' \frac{n!}{j_0!\,j_1!\cdots}\left(\frac{x_0}{1!}\right)^{j_0}\left(\frac{x_1}{2!}\right)^{j_1}\cdots, \tag{A16}$$

where the summation is over all sequences $(j_0, j_1, \ldots)$ of non-negative integers constrained such that

$$ \textstyle\sum_{m=0} j_m = k; \qquad \sum_{m=0}(m+1)j_m = n. \tag{A17}$$

Note then $\sum_{m=0} mj_m = n - k$ and thus $j_m = 0$ for $^\forall m > n - k$ (n.b., if otherwise, $j_m \ge 1$ for $^\exists m > n - k$ and so $\sum_{m=0} mj_m > n - k$, which is contradictory). Next, $n - k - \sum_{m=0} j_{2m+1} = 2\sum_{m=0} m(j_{2m} + j_{2m+1})$ is even. This implies that if f is cm, the parity of $B_{n,k}$ in equation (A15) is $(-1)^{n-k}$. Hence, given that g is also cm, the parity of every term of equation (A15) is $(-1)^n$, which proves *6*. Equation (A15) also indicates that

$$ \frac{\mathrm{d}^n \exp[f(t)]}{\mathrm{d}t^n} = \exp[f(t)]\, B_n\left[f'(t), f''(t), \ldots, f^{(n-k+1)}(t)\right], \tag{A18}$$

where B_n is the nth complete Bell polynomial,

$$ B_n(x_0, \ldots, x_{n-1}) \equiv \textstyle\sum_{k=1}^n B_{n,k}(x_0, \ldots, x_{n-k}). \tag{A19}$$

Note that $n - \sum_{m=0} j_{2m} = 2\sum_{m=0} m(j_{2m-1} + j_{2m})$ is even. Hence if f is cm, the parity of B_n in equation (A18) is $(-1)^n$ and so follows *7*.

Corollary A15. *Let $g(t)$ be cm; then both $t^{-\delta}g(t)$ with $\delta \ge 0$ and $g(t^p)$ with $0 < p \le 1$ are cm.*

Proof. The first is obvious thanks to Lemmas A13 and A14-*5*. The last follows Lemma A14-*6* with $F(t) = t^p$ since $F' = pt^{p-1}$ for $0 < p \le 1$ is cm. Q.E.D.

Corollary A16. *For* $0 < p \le 1$ *and* $a, b \ge 0$, *these are cm*:

$$f(t) = t^{-a}(1+t^{p})^{-b}, \qquad f(t) = t^{-a}(1+t^{-p})^{b}. \tag{A20}$$

Proof. Let $F(t) = c + t^{p}$. Then $F' = pt^{p-1}$ is cm for $0 < p \le 1$. Hence first $(g \circ F)(t) = (1 + t^{p})^{-b}$ with $c = 1$ and $g(w) = w^{-b}$ for $0 < p \le 1$ and $b \ge 0$ is cm. Next, with $c = 0$ and $g(w) = b \ln(1 + w^{-1})$, we find that $(g \circ F)(t) = b \ln(1 + t^{-p})$ is cm for $0 < p \le 1$ and $b \ge 0$, and so is $(1 + t^{-p})^{b} = \exp[b \ln(1 + t^{-p})]$. The final conclusion follows Corollary A15. Q.E.D.

The fundamental result characterizing cm functions (Bernstein 1928; Widder 1941) is due to Sergei Natanovich Bernstein (1880–1968).

Theorem A17 (Hausdorff–Bernstein–Widder). *A smooth function* $f(x)$ *of* $x > 0$ *is cm if and only if* $f(x) = \int_0^\infty e^{-xt}\,d\mu(t)$ *where* $\mu(t)$ *is the Borel measure on* $[0, \infty)$, *that is, there exists a non-negative distribution* $\phi(t) \ge 0$ *of* $t > 0$ *such that* $f(x) = \underset{t\to x}{\mathcal{L}}[\phi(t)]$.

The ‘if’ part is elementary. Although the complete proof of the ‘only if’ part is beyond our scope, the partial proof follows the Post–Widder formula. That is, if the inverse Laplace transform $\phi(t)$ of a cm function $f(x)$ is well defined, then equation (A11), provided that it converges, indicates that $\phi(t)$ must be non-negative.

A3 Generalized Mittag-Leffler function

Let us consider a particular generalized hypergeometric function

Definition A21.

$$E^{\lambda}_{p,b}(z) \equiv \sum_{k=0}^{\infty} \frac{(\lambda)^{+}_{k}}{\Gamma(pk+b)} \frac{z^{k}}{k!} \qquad (p > 0). \tag{A21}$$

This is absolutely convergent for $p > 0$ and all z, and thus is an entire function of z with $p > 0$. The function defined as such is the generalization of the Mittag-Leffler function introduced by Prabhakar (1971, see also Haubold, Mathai & Saxena 2011) with $E^{1}_{p,b}(z) = E_{p,b}(z)$ and $E^{1}_{p,1}(z) = E_{p}(z)$. If $p = 1$ on the other hand, the definition results in the Kummer confluent hypergeometric function of the first kind, that is, $E^{\lambda}_{1,b}(z) = {}_1\tilde{F}_1(\lambda; b; z) = {}_1F_1(\lambda; b; z)/\Gamma(b)$.

Some operational properties of the generalized Mittag-Leffler function may be derived directly through term-by-term calculations on its definition. Important for our purpose amongst them are

$$\frac{d^{n} E^{\lambda}_{p,b}(-z)}{dz^{n}} = (-1)^{n} (\lambda)^{+}_{n} E^{\lambda+n}_{p,b+pn}(-z), \tag{A22}$$

$$(1-\lambda)^{+}_{n}\, {}_0I_z{}^{n} E^{\lambda}_{p,b}(-z) = E^{\lambda-n}_{p,b-pn}(-z) - \sum_{k=0}^{n-1} \frac{(n-\lambda)^{-}_{k} z^{k}}{k!\Gamma(b-pn+pk)}, \tag{A23}$$

$$\frac{d[z^{\lambda} E^{\lambda}_{p,b}(-z)]}{dz} = \lambda z^{\lambda-1} E^{\lambda+1}_{p,b}(-z) \tag{A24}$$

for a non-negative integer n.

Our interest in the generalized Mittag-Leffler function mostly hinges on the particular Laplace transform, namely

Theorem A25. *For* $b, p > 0$,

$$\underset{t\to w}{\mathcal{L}}\left[t^{b-1} E^{\lambda}_{p,b}(-t^{p})\right] = \frac{1}{w^{b}} \left(1 + \frac{1}{w^{p}}\right)^{-\lambda} = \frac{1}{w^{b-p\lambda}(1+w^{p})^{\lambda}}. \tag{A25a}$$

This is shown by direct term-by-term integrations that result in

$$\int_0^\infty dt\, e^{-wt} t^{b-1} E^{\lambda}_{p,b}(-t^{p}) = \sum_{k=0}^{\infty} \frac{(-1)^{k} (\lambda)^{+}_{k}}{k!\, w^{pk+b}}, \tag{A25b}$$

and assembling back the binomial series of $(1 + w^{-p})^{-\lambda}$.

Lemma A26. *If* $0 < p \le 1$, $b > 0$, *and* $b \ge p\lambda$, *then* $E^{\lambda}_{p,b}(-z) \ge 0$ *is non-negative for all* $z > 0$.

Proof. By Corollary A16, the Laplace transformation in Theorem A25 is a cm function of $w > 0$ for $0 < p \le 1$ either if $b \ge 0$ and $\lambda \le 0$ or if $b - p\lambda \ge 0$ and $\lambda \ge 0$. The Bernstein theorem then indicates that if $0 < p \le 1$, $b > 0$ and $b \ge p\lambda$, then $t^{b-1} E^{\lambda}_{p,b}(-t^{p}) \ge 0$ for $t > 0$ and thus $E^{\lambda}_{p,b}(-z) \ge 0$ for $z > 0$.

Given equation (A22), this further indicates that

Theorem A27. *If* $0 < p \le 1$ *and* $0 < p\lambda \le b$, *then* $E^{\lambda}_{p,b}(-z)$ *and* $E^{\lambda}_{p,b}(-t^{p})$ *are cm functions of* $z > 0$ *and* $t > 0$.

For $\lambda = -\xi \le 0$ on the other hand, we find:

Theorem A28. *If* $0 < p \le 1$, $\xi \ge 0$ *and* $b > 0$, *then* $z^{-\lceil\xi\rceil} E^{-\xi}_{p,b}(-z)$ *and subsequently* $t^{-p\lceil\xi\rceil} E^{-\xi}_{p,b}(-t^{p})$ *are cm.*

Theorem A29. *If* $0 < p \le 1$, $\xi \ge 0$, $b > 0$ *and* $b \ge p(1-\xi)$, *then* $z^{-\xi} E^{-\xi}_{p,b}(-z)$ *and* $t^{-p\xi} E^{-\xi}_{p,b}(-t^{p})$ *are cm.*

For a non-negative integer $\xi = \lceil\xi\rceil = \mu$, these are trivial since $E^{-\mu}_{p,b}(-z)$ then reduces to a μth polynomial of z with all positive coefficients and subsequently

$$z^{-\mu} E^{-\mu}_{p,b}(-z) = \sum_{k=0}^{\mu} \binom{\mu}{k} \frac{z^{-(\mu-k)}}{\Gamma(b+pk)}. \tag{A26}$$

Next, equation (A22) for $\lambda = -\xi \le 0$ and $n = \lceil\xi\rceil$ results in

$$\frac{d^{\lceil\xi\rceil} E^{-\xi}_{p,b}(-z)}{dz^{\lceil\xi\rceil}} = (1-\epsilon)^{+}_{\lceil\xi\rceil} E^{\epsilon}_{p,b+p\lceil\xi\rceil}(-z), \tag{A27a}$$

where $0 \le \epsilon = \lceil\xi\rceil - \xi < 1$. Now it follows from equation (A23) that

$$(1-\epsilon)^{+}_{\lceil\xi\rceil}\, {}_0I_z{}^{\lceil\xi\rceil} E^{\epsilon}_{p,b+p\lceil\xi\rceil}(-z) = E^{-\xi}_{p,b}(-z) - \sum_{k=0}^{\lceil\xi\rceil-1} \binom{\xi}{k} \frac{z^{k}}{\Gamma(b+pk)}. \tag{A27b}$$

For $\xi > 0$ (n.b., then $\lceil\xi\rceil \ge 1$), this results in

$$z^{-\lceil\xi\rceil} E^{-\xi}_{p,b}(-z) = \sum_{k=0}^{\lceil\xi\rceil-1} \binom{\xi}{k} \frac{z^{-(\lceil\xi\rceil-k)}}{\Gamma(b+pk)} + \frac{(1-\epsilon)^{+}_{\lceil\xi\rceil}}{(\lceil\xi\rceil-1)!} \int_0^1 du\,(1-u)^{\lceil\xi\rceil-1} E^{\epsilon}_{p,b+p\lceil\xi\rceil}(-uz). \tag{A28}$$

Theorem A28 (for a non-integer $\xi > 0$) follows this since

$$\frac{d^{n}}{ds^{n}} \int_0^1 du\,(1-u)^{k} f(su) = \int_0^1 du\,(1-u)^{k} u^{n} f^{(n)}(su), \tag{A29}$$

and $E^{\epsilon}_{p,b+p\lceil\xi\rceil}(-z)$ is cm given $b + p\lceil\xi\rceil - p\epsilon = b + p\xi > 0$ (Theorem A27). Theorem A29 is proven by equation (A24), that is,

$$-\frac{d[z^{-\xi} E^{-\xi}_{p,b}(-z)]}{dz} = \frac{\xi E^{1-\xi}_{p,b}(-z)}{z^{\xi+1}} = \frac{\xi z^{-\lceil\xi-1\rceil} E^{-(\xi-1)}_{p,b}(-z)}{z^{2-\epsilon}}, \tag{A30}$$

which is cm either if $0 < p \le 1$, $b > 0$ and $\xi \ge 1$ (Theorem A28) or if $0 < p \le 1$, $0 \le \xi < 1$ and $b \ge p(1-\xi)$ (Theorem A27).

A4 Miscellaneous

Lemma A31 (An 2011b, theorem A3).

$$\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)^n(xf) = x^{n+1}\frac{\mathrm{d}^n(x^n f)}{\mathrm{d}x^n} \tag{A31}$$

for any non-negative integer n and arbitrary function f(x).

This may be proven by induction on n. It is also equivalent to

Lemma A32. (An 2011b, corollary A4).

$$x^n f_{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1}f_{(n)}(x)\right] \quad \text{where } f_{(n)}(x) \equiv \frac{\mathrm{d}^n[x^n f(x)]}{\mathrm{d}x^n}. \tag{A32}$$

Thanks to the fundamental theorem of calculus indicating

$$x^{n+1}f_{(n)}(x) = x^{n+1}f_{(n)}(x)\Big|_{x=0} + \int_0^x y^n f_{(n+1)}(y)\,\mathrm{d}y, \tag{A33}$$

We also find

Corollary A33. *For a non-negative integer n, if* $f_{(n+1)}(x) \geq 0$ *for* $x > 0$ *and* $f_{(n)}(0)$ *is finite, then* $f_{(n)}(x) \geq 0$ *for* $x > 0$.

Lemma A32 generalizes with fractional calculus. In particular,

Lemma A34. *For a non-negative integer n and* $0 \leq \delta < 1$,

$$\begin{aligned} x^{n+1}\,{}_0D_x^{\ n+\delta}(x^{n+\delta}f) &= {}_0I_x^{\ 1-\delta}\left[x^{n+\delta}f_{(n+1)}(x)\right], \\ x^{n+\delta}\ f_{(n+1)}(x) &= {}_0D_x^{\ 1-\delta}\left[x^{n+1}{}_0D_x^{\ n+\delta}(x^{n+\delta}f)\right]. \end{aligned} \tag{A34}$$

This follows

$$\begin{aligned} {}_0I_x^{\ 1-\delta}(x^{n+\delta}f) &= \frac{x^{n+1}}{\Gamma(1-\delta)}\int_0^1\frac{t^{n+\delta}f(xt)\,\mathrm{d}t}{(1-t)^\delta}, \\ {}_0D_x^{\ n+\delta}(x^{n+\delta}f) &= \frac{1}{\Gamma(1-\delta)}\int_0^1\frac{\mathrm{d}t\,t^{n+\delta}}{(1-t)^\delta}\frac{\mathrm{d}^{n+1}[x^{n+1}f(xt)]}{\mathrm{d}x^{n+1}} \\ &= \frac{1}{x^{n+1}\Gamma(1-\delta)}\int_0^x\frac{y^{n+\delta}f_{(n+1)}(y)\,\mathrm{d}y}{(x-y)^\delta}. \end{aligned} \tag{A35}$$

Note that equations (A34) for $\delta = 0$ reduce to equations (A32) and (A33). Together Lemmas A7 and A34 generalize Corollary A33.

Corollary A35. *For a non-negative integer n, if* $f_{(n+1)}(x) \geq 0$ *for* $x > 0$, *then* ${}_0D_x^{\ \mu}(x^\mu f) \geq 0$ *for* $x > 0$ *and* $n \leq \mu \leq n+1$.

Corollary A33 may in fact be generalized alternatively, namely,

Theorem A36. *For a non-negative integer n, if* $x^a f_{(n+1)}(x)$ *is cm, then* $x^a f_{(n)}(x)$ *is also cm.*

Proof. Suppose that $x^a f_{(n+1)}$ is cm. Then by the Bernstein theorem, there exists a non-negative function $h(u) \geq 0$ of $u > 0$ such that

$$x^a f_{(n+1)}(x) = \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-xu}h(u). \tag{A36a}$$

The complete monotonicity of $x^a f_{(n)}$ can then be shown directly using equation (A33), which indicates that

$$\begin{aligned} x^a f_{(n)} &= x^{a-n-1}\int_0^x \mathrm{d}y\,y^n f_{(n+1)}(y) = \int_0^1\mathrm{d}t\,t^{n-a}\int_0^\infty\mathrm{d}u\,\mathrm{e}^{-xtu}h(u), \\ \frac{\mathrm{d}^k[x^a f_{(n)}]}{\mathrm{d}x^k} &= (-1)^k\int_0^1\mathrm{d}t\,t^{n+k-a}\int_0^\infty\mathrm{d}u\,\mathrm{e}^{-xtu}u^k h(u). \end{aligned} \tag{A36b}$$

Finally, we also note

Lemma A37. *For a non-negative integer n, if* $f^{(n+1)}(a)$ *is finite and* $f^{(0)}(a) = \cdots = f^{(n)}(a) = 0$, *then* ${}_aD_x^{\ n+\delta}f(a) = 0$ *for* $0 \leq \delta < 1$.

Proof. Here we assume $a = 0$, but the similar argument holds for any finite a accompanied by a simple translation. First,

$${}_0I_x^{\ 1-\delta}f(x) = \frac{x^{1-\delta}}{\Gamma(1-\delta)}\int_0^1\frac{f(xt)\,\mathrm{d}t}{(1-t)^\delta}, \tag{A37a}$$

$${}_0D_x^{\ n+\delta}f(x) = \frac{1}{\Gamma(1-\delta)}\int_0^1\frac{\mathrm{d}^{n+1}[y^{1-\delta}f(y)]}{\mathrm{d}y^{n+1}}\Bigg|_{y=xt}\frac{t^{n+\delta}\,\mathrm{d}t}{(1-t)^\delta}. \tag{A37b}$$

Here the latter follows the former because

$$\frac{\mathrm{d}^{n+1}[x^{1-\delta}f(xt)]}{\mathrm{d}x^{n+1}} = t^{n+\delta}\frac{\mathrm{d}^{n+1}[y^{1-\delta}f(y)]}{\mathrm{d}y^{n+1}}\Bigg|_{y=xt}. \tag{A37c}$$

Finally, given the Leibniz rule,

$$\begin{aligned} \frac{\mathrm{d}^{n+1}[y^{1-\delta}f(y)]}{\mathrm{d}y^{n+1}} &= y^{1-\delta}f^{(n+1)}(y) \\ &\quad +(1-\delta)\sum_{k=0}^n(-1)^{n-k}\binom{n+1}{k}(\delta)^+_{n-k}\frac{f^{(k)}(y)}{y^{n+\delta-k}}, \end{aligned} \tag{A37d}$$

which identically vanishes for $y = 0$ if the condition part of Lemma A37 with $a = 0$ holds. Here the conclusion follows as the integrand of equation (A37b) with $x = 0$ is also zero.

APPENDIX B: DERIVATIONS OF Equations (5)

First we establish for any $s > -1$ and $\lambda \geq 0$ that

$${}_0I_{r^2}^{\ \lambda}\left(r^{2s}\iint_T\mathrm{d}E\,\mathrm{d}L^2K^sG\right) = \frac{r^{2(s+\lambda)}}{2^\lambda(s+1)^+_\lambda}\iint_T\frac{K^{s+\lambda}G\,\mathrm{d}E\,\mathrm{d}L^2}{(\Psi-E)^\lambda}, \tag{B1a}$$

$${}_0I_{r^2}^{\ \lambda}\left(\frac{1}{r^{2\lambda+2}}\iint_T\mathrm{d}E\,\mathrm{d}L^2K^sG\right) = \frac{r^{2\lambda-2}}{(s+1)^+_\lambda}\iint_T\frac{K^{s+\lambda}G\,\mathrm{d}E\,\mathrm{d}L^2}{L^{2\lambda}}, \tag{B1b}$$

$${}_{E_0}I_\Psi^{\ \lambda}\iint_T\mathrm{d}E\,\mathrm{d}L^2K^sG = \frac{1}{2^\lambda(s+1)^+_\lambda}\iint_T\mathrm{d}E\,\mathrm{d}L^2K^{s+\lambda}G, \tag{B1c}$$

provided that all integrals converge and the Ψ and r^2 dependences of an arbitrary integrable function $G = G(E, L^2)$ are only through E and L^2 – here and henceforth trivial arguments of $G(E, L^2)$ are suppressed for the sake of brevity. In addition,

$$\frac{1}{(s+1)^+_\lambda} = \frac{\Gamma(s+1)}{\Gamma(s+\lambda+1)} = (s)^-_{-\lambda}$$

is the generalized Pochhammer symbol. These are demonstrated by direct calculations utilizing the Fubini theorem that are identical to that of An (2011a) except for different arguments involved in the Euler integral for the beta function. We also find additional properties of the integral transform in the form of equation (3b), namely, for any $s > -1$ and a non-negative integer $n \geq 0$,

$$\begin{aligned} &\frac{\partial^n}{\partial\Psi^n}\iint_T\mathrm{d}E\,\mathrm{d}L^2K^sG \\ &\quad = \begin{cases} 2^n(s)^-_n\displaystyle\iint_T\mathrm{d}E\,\mathrm{d}L^2K^{s-n}G & (n < s+1) \\ 2^s s!\displaystyle\int_0^{L^2_\mathrm{m}}\mathrm{d}L^2G\left(\Psi-\frac{L^2}{2r^2}, L^2\right) & (n = s+1), \end{cases} \end{aligned} \tag{B2a}$$

$$\left(r^4\frac{\partial}{\partial r^2}\right)^n \iint_T \mathrm{d}E\,\mathrm{d}L^2 K^s G = \begin{cases} (s)_n^- \displaystyle\iint_T \mathrm{d}E\,\mathrm{d}L^2 K^{s-n} L^{2n} G & (n < s+1) \\ \dfrac{s!}{2}\displaystyle\int_0^{L_\mathrm{m}^2} \mathrm{d}L^2 L^{2s+2} G\left(\Psi - \frac{L^2}{2r^2}, L^2\right) & (n = s+1). \end{cases} \tag{B2b}$$

With $\tilde{\nu} = m_{0,0}(\Psi, r^2)$ in equation (3b), these then result in

$$\frac{\partial^n}{\partial\Psi^n}\left[{}_0I_{r^2}{}^{\xi-\frac{1}{2}}\left(\frac{\tilde{\nu}}{r^{2\xi-1}}\right)\right] = \begin{cases} \dfrac{2^{n+1}\pi^{\frac{3}{2}} r^{2\xi-3}}{\Gamma(\xi-n)} \displaystyle\iint_T \mathrm{d}E\,\mathrm{d}L^2 \frac{K^{\xi-n-1}}{L^{2\xi-1}} F(E,L^2) & (n<\xi) \\ 2^\xi \pi^{\frac{3}{2}} r^{2\xi-3} \displaystyle\int_0^{L_\mathrm{m}^2} \frac{\mathrm{d}L^2}{L^{2\xi-1}} F\left(\Psi - \frac{L^2}{2r^2}, L^2\right) & (n=\xi), \end{cases} \tag{B3a}$$

$$\left(r^4\frac{\partial}{\partial r^2}\right)^n \left(r^2\,{}_{E_0}I_\Psi{}^{\xi-\frac{1}{2}}\tilde{\nu}\right) = \begin{cases} \dfrac{2^{\frac{3}{2}-\xi}\pi^{\frac{3}{2}}}{\Gamma(\xi-n)} \displaystyle\iint_T \mathrm{d}E\,\mathrm{d}L^2 K^{\xi-n-1} L^{2n} F(E,L^2) & (n<\xi) \\ 2^{\frac{1}{2}-\xi}\pi^{\frac{3}{2}} \displaystyle\int_0^{L_\mathrm{m}^2} \mathrm{d}L^2 L^{2\xi} F\left(\Psi - \frac{L^2}{2r^2}, L^2\right) & (n=\xi), \end{cases} \tag{B3b}$$

where n is again a non-negative integer and $\xi \geq \frac{1}{2}$.

Equation (5a) for $\xi \geq \frac{1}{2}$ is a straightforward generalization of equation (B3a) from an integer n to a real $\mu \leq \xi$, which is similarly shown through direct calculations using equations (B1) and (B2) assuming that all the integrals converge. Next equation (5a) for $\xi = \frac{1}{2}$ is identical to equation (B3b) with $n = 0$ (and $\xi = \frac{1}{2} - \mu$), since ${}_{E_0}I_\Psi{}^{\xi-\frac{1}{2}}\tilde{\nu} = {}_{E_0}D_\Psi{}^{\frac{1}{2}-\xi}\tilde{\nu}$. Hence, it is inferred that equation (B3b) is in fact valid for not only $\xi \geq \frac{1}{2}$ but also $\xi \geq 0$ (n.b., $0 \leq n \leq \xi$ and so if $0 \leq \xi \leq \frac{1}{2}$, then $n = 0$).

A generalization of equation (B3b) from an integer n to a real μ (cf. equation A31) and the extension of equation (5a) to $\xi \geq 0$ are possible although demonstrating them through direct calculations is comparatively nontrivial. Instead, we follow an indirect route to derive the generalization of equation (B3b). First, equation (B3b) with $(n, \xi) = (0, \mu)$ and equation (B1a) with $G = F$ and $(s, \lambda) = (\mu - 1, 1 - \delta)$ where $\delta = \mu - \lfloor\mu\rfloor$ together indicate that

$${}_0I_{r^2}{}^{1-\delta}\left(r^{2\mu}\,{}_{E_0}I_\Psi{}^{\mu-\frac{1}{2}}\tilde{\nu}\right) = \frac{\pi^{\frac{3}{2}} r^{2\lfloor\mu\rfloor}}{2^{\lfloor\mu\rfloor-\frac{1}{2}}\lfloor\mu\rfloor!} \iint_T \mathrm{d}E\,\mathrm{d}L^2 \frac{K^{\lfloor\mu\rfloor}F(E,L^2)}{(\Psi-E)^{1-\delta}} \tag{B4}$$

for $\mu > 0$ and $0 < \delta < 1$. Applying $[r^4(\partial/\partial r^2)]^{\lfloor\mu\rfloor+1}$ on this after dividing by $r^{2\lfloor\mu\rfloor}$ (equation B2b) and using equation (A31), we find that

$$\begin{aligned} {}_0D_{r^2}{}^\mu\left(r^{2\mu}\,{}_{E_0}I_\Psi{}^{\mu-\frac{1}{2}}\tilde{\nu}\right) &= \frac{\pi^{\frac{3}{2}}}{2^{\mu-\frac{1}{2}} r^{2\mu+2}} \int_0^{L_\mathrm{m}^2} \mathrm{d}L^2 L^{2\mu} F\left(\Psi - \frac{L^2}{2r^2}, L^2\right) \\ &= (2\pi)^{\frac{3}{2}} \int_{E_0}^{\Psi} \mathrm{d}E\,(\Psi-E)^\mu F\left[E, 2r^2(\Psi-E)\right], \end{aligned} \tag{B5}$$

which is the $\xi = \mu$ case of equation (5b). Note that, thanks to equation (A31), this is consistent with the case $n = \xi$ of equation (B3b).

Thus, equation (B5) is actually valid for any $\mu \geq 0$ including integer values. Finally, let us apply ${}_{E_0}I_\Psi{}^{\xi-\mu}$ to equation (B5). It then follows the Fubini theorem that for $0 \leq \mu < \xi$

$${}_0D_{r^2}{}^\mu\left(r^{2\mu}\,{}_{E_0}I_\Psi{}^{\xi-\frac{1}{2}}\tilde{\nu}\right) = \frac{(2\pi)^{\frac{3}{2}}}{2^\xi r^{2\mu+2}\Gamma(\xi-\mu)} \times \iint_T \mathrm{d}E\,\mathrm{d}L^2 K^{\xi-\mu-1} L^{2\mu} F(E,L^2), \tag{B6}$$

which recovers the remaining part ($\xi > \mu$) of equation (5b). Equations (B5) and (B6) together (i.e. equation 5b) constitute the generalization of equation (B3b) from an integer n to a real μ, which is valid for any pair (μ, ξ) with $0 \leq \mu \leq \xi$.

Lastly, note that the index transform $(\mu, \xi) \to (\frac{1}{2} - \xi, \frac{1}{2} - \mu)$ sends equation (5a) to (5b) and vice versa. Therefore equation (5b) with $0 \leq \mu \leq \xi \leq \frac{1}{2}$ here implies that equation (5a) is also valid for any μ and ξ with $0 \leq \mu \leq \xi \leq \frac{1}{2}$, too.

APPENDIX C: DERIVATION OF Equation (31)

We first apply the Laplace transform on Ψ to equation (3b),

$$\begin{aligned} \mathop{\mathcal{L}}_{\Psi\to s}\left[\tilde{\nu}(\Psi, r^2)\right] &= \int_0^\infty \mathrm{d}\Psi\,\mathrm{e}^{-s\Psi}\tilde{\nu}(\Psi, r^2) \\ &= \frac{2\pi}{r^2}\iint_{E\geq 0, L^2\geq 0} \mathrm{d}E\,\mathrm{d}L^2 F(E,L^2) \int_0^\infty \mathrm{d}\Psi\,\mathrm{e}^{-s\Psi}\frac{\Theta(K)}{\sqrt{|K|}}. \end{aligned} \tag{C1}$$

The inner integral in the last line reduces to

$$\int_0^\infty \mathrm{d}\Psi\,\mathrm{e}^{-s\Psi}\frac{\Theta(K)}{\sqrt{|K|}} = \sqrt{\frac{\pi}{2s}}\,\mathrm{e}^{-sE}\exp\left(-\frac{sL^2}{2r^2}\right), \tag{C2}$$

and consequently we find that

$$\mathop{\mathcal{L}}_{\Psi\to s}[\tilde{\nu}] = \frac{\sqrt{2}\pi^{\frac{3}{2}}}{\sqrt{s}r^2}\int_0^\infty \mathrm{d}L^2 \exp\left(-\frac{sL^2}{2r^2}\right)\int_0^\infty \mathrm{d}E\,\mathrm{e}^{-sE}F(E,L^2). \tag{C3}$$

Substituting the variables $t = \frac{1}{2}sL^2$ and $w = r^{-2}$, this reduces to

$$\mathop{\mathcal{L}}_{\Psi\to s}\left[\tilde{\nu}(\Psi, w^{-1})\right] = \left(\frac{2\pi}{s}\right)^{\frac{3}{2}} w \mathop{\mathcal{L}}_{t\to w}\left[\int_0^\infty \mathrm{d}E\,\mathrm{e}^{-sE}F\left(E, \frac{2t}{s}\right)\right]. \tag{C4}$$

If the AD is separable as in equation (12), then

$$w^{-1}\mathop{\mathcal{L}}_{\Psi\to s}\left[\tilde{\nu}(\Psi, w^{-1})\right] = \mathcal{R}(w)\mathop{\mathcal{L}}_{\Psi\to s}[P(\Psi)] = \mathcal{P}(s)\mathop{\mathcal{L}}_{t\to w}[\phi(t)], \tag{C5}$$

where $\mathcal{P}(s) \equiv \mathop{\mathcal{L}}_{\Psi\to s}[P(\Psi)]$ and $\mathcal{R}(w) = \mathop{\mathcal{L}}_{t\to w}[\phi(t)]$. Given that the inverse Laplace transformation is unique, equations (C4) and (C5) together then imply

$$\mathcal{P}(s)\,\phi(t) = \left(\frac{2\pi}{s}\right)^{\frac{3}{2}}\int_0^\infty \mathrm{d}E\,\mathrm{e}^{-sE}F\left(E, \frac{2t}{s}\right), \tag{C6}$$

and reinstating $t = \frac{1}{2}sL^2$ then leads to

$$\frac{s^{\frac{3}{2}}\mathcal{P}(s)}{(2\pi)^{3/2}}\phi\left(\frac{sL^2}{2}\right) = \int_0^\infty \mathrm{d}E\,\mathrm{e}^{-sE}F(E,L^2) = \mathop{\mathcal{L}}_{E\to s}\left[F(E,L^2)\right]. \tag{C7}$$

Equation (31) is simply the inversion of this.

APPENDIX D: THE $\beta_1 = 1$ CASES

D1 The $\beta = 1$ constant anisotropy model

Let us consider the df given by

$$\sqrt{2}\pi^{\frac{3}{2}} F(E, L^2) = f(E)\delta(L^2), \tag{D1}$$

where $f(E)$ is an arbitrary function of E and $\delta(L^2)$ is the Dirac delta. This df corresponds to the spherical system entirely built by radial orbits, that is, the $\beta = 1$ constant anisotropy model. Given that $K(L^2 = 0) = 2(\Psi - E)$, the corresponding AD is found to be

$$\tilde{\nu}(\Psi, r^2) = \frac{1}{r^2}\sqrt{\frac{2}{\pi}}\int_{E_0}^{\Psi} \frac{f(E)\,\mathrm{d}E}{\sqrt{2(\Psi - E)}} = r^{-2}\,{}_{E_0}I_\Psi^{\frac{1}{2}} f(\Psi), \tag{D2}$$

which is separable as in equation (12) with $P(\Psi) = {}_{E_0}I_\Psi^{\frac{1}{2}} f(\Psi)$ and $R(x) = x^{-1}$. The AD is easily inverted to the df, $f(E) = {}_{E_0}D_E^{\frac{1}{2}} P(E)$, whose non-negativity is also the necessary *and* sufficient condition for the phase-space consistency. This is consistent with the discussion in Section 5.2 being applicable for $\beta \le 1$. Note that $R(x) = x^{-1}$ is the natural limit of the constant anisotropy model in equation (44a) to $\beta = 1$.

We find that ${}_0I_x^{\lambda} x^{-1-\lambda} \to \infty$, ${}_0I_x^{1-\delta} x^{\lambda-1} = x^n \Gamma(\lambda)/n!$ and ${}_0D_x^{\lambda} x^{\lambda-1} = 0$ for $\lambda = n + \delta > 0$, whilst ${}_0I_x^{0} x^{-1} = {}_0D_x^{0} x^{-1} = x^{-1}$. Hence, $R = x^{-1}$ satisfies the necessary condition in equation (15). Moreover, equations (5) still hold with non-trivial cases indicating ${}_{E_0}D_\Psi^{\mu} P = {}_{E_0}I_\Psi^{\frac{1}{2}-\mu} f(\Psi)$, whose non-negativity for $^\forall\mu \le \frac{1}{2}$ is the same necessary condition for $P(\Psi)$ discussed in Section 4.2.

From $R(x) = x^{-1}$, we also find $\mathcal{R}(w) = 1$ and $\phi(t) = \delta(t)$. Although equation (35) strictly is then trivial as $\delta(t) = 0$ for $t > 0$, this interpretation of equation (35) seems improper considering that the Dirac delta is not differentiable at $t = 0$. Equation (37) on the other hand reduces to $x^{\frac{1}{2}-\lambda}$ being cm since $R_{(0)}(x) = R(x) = x^{-1}$ and $R_{(n)}(x) = 0$ for any positive integer n. The sufficient condition following this, that is, equations (39) and (42) for $^\exists\lambda \ge \frac{1}{2}$, is in fact a proper one, as is the natural limiting case of the constant anisotropy model for $\beta = 1$. It appears that for $R \sim x^{-1}$ as $x \sim 0$ (and $\lim_{w\to\infty} \mathcal{R}$ being nonzero finite), we may consider $\phi(t) \sim t^{-1}$ as $t \sim 0$ for the purpose of applying equation (35).

D2 Equation (47b) with $\beta_1 = 1$

The discussion on necessary conditions (Section 4) is valid inclusively for $\beta_1 \le 1$. That is, equation (47b) with $\beta_1 = 1$ still requires to satisfy equation (15) – if $0 < p \le 1$, this is automatically met – in order for the df to be non-negative whereas the potential-dependent part is restricted to be ${}_{E_0}D_\Psi^{\frac{1}{2}} P \ge 0$ for the phase-space consistency.

The complication arises however for $\beta_1 = 1$ in regard to sufficient conditions discussed in Section 6.1. The main difficulty is due to the fact that $\lim_{x\to 0} x R(x) = \lim_{w\to\infty} \mathcal{R}(w) = 1$ is non-zero. Whilst this indicate $\phi \sim t^{-1}$ for $t \sim 0$, this behaviour is incompatible with the convergence of the Laplace transform. The formal solution follows adopting $\lim_{a\to 1^-} x^{-a}/\Gamma(1-a) = \delta(x)$. Then, the function $\phi(t)$ in equation (49) with $\beta_1 = 1$ is in fact the inverse Laplace transform of '$\mathcal{R}(w) - 1$' whilst the 'true' inverse transform of $\mathcal{R}(w)$ with $\beta_1 = 1$ is given by '$\phi(t) + \delta(t)$'. For example, since $1/\Gamma(0) = 0$, the $k = 0$ term in equation (A21) for $E^{\lambda}_{p,0}$ does not contribute. Hence, equation (A25) can in fact be well defined for the $b = 0$ case too. In particular, $\underset{t\to w}{\mathcal{L}}[t^{-1}E^{\lambda}_{p,0}(-t^p)] = (1 + w^{-p})^{-\lambda} - 1$. Since $(1 + w^{-p})^{-\lambda} \ge 1$ for $w > 0$ and $\lambda \le 0$, it follows that if $0 < p \le 1$ and $\lambda \le 0$, this is also cm and $E^{\lambda}_{p,0}(-z) \ge 0$ for $z > 0$. Given that $\underset{t\to w}{\mathcal{L}}[\delta(t)] = 1$, we also find from this that $\underset{t\to w}{\mathcal{L}}[\delta(t) + t^{-1}E^{-\xi}_{p,0}(-t^p)] = (1 + w^{-p})^{\xi}$.

For the specific discussion concerning sufficient conditions for the phase-space consistency, consider $P(\Psi)R(r^2) = P(\Psi)R_0(r^2) + r^{-2}P(\Psi)$ where $R_0(x) = R(x) - x^{-1}$. From the corresponding df with $E_0 = 0$, it is obvious that the corresponding sufficient condition is ${}_0D_\Psi^{\frac{1}{2}} P \ge 0$ together with those derived in Section 5 with $R_0(x)$. In addition, Theorems A27–A29 actually extend to $b = 0$ thanks to the non-negativity of $E^{\lambda}_{p,0}(-z) \ge 0$. It follows that the theorems in Section 6.1 also hold inclusively for $\beta_1 = 1$.

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