The hyperplanes of $DQ^-(7, \mathbb{K})$ arising from embedding

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Abstract

We determine all hyperplanes of the dual polar space $DQ^-(7, \mathbb{K})$ which arise from embedding. This extends one of the results of [5] to the infinite case.

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1 Introduction

Let Π be a nondegenerate polar space of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The geometry Δ is called a *dual polar space of rank* n ([1]). The dual polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles ([9]). Every dual polar space Δ is a so-called *near polygon*. This means that for every point x and every line L, there exists a unique point on L nearest to x. Here, distances are measured in the collinearity graph of Δ. If $i \in \mathbb{N}$ and x is a point of Δ, then $\Delta_i(x)$ denotes the set of points at distance i from x. We also define $x^\perp := \Delta_0(x) \cup \Delta_1(x)$ for every point x of Δ. A set S of points of Δ is called a *subspace* if every line which has at least two points in S has all its points in S. A subspace S is called *convex* if every point on a shortest path between two points of S is also contained in S.

There exists a bijective correspondence between the non-empty convex subspaces of Δ and the possibly empty singular subspaces of Π. If α is a possibly empty singular subspace of Π, then the set of all maximal singular subspaces containing α is a convex subspace of Δ. Conversely, every convex subspace of Δ is obtained in this way. The maximal distance between two points of a convex subspace F of Δ is called the *diameter* of F. The convex subspaces of diameter 2, 3, respectively $n-1$, are called the *quads*, *hexes*, respectively *maxes*, of Δ. If F is a convex subspace of diameter $\delta \in \{2, \ldots, n\}$, then the point-line geometry Δ_F induced on F is a dual polar space of rank δ. In particular, if Q is a quad, then Δ_Q is a generalized quadrangle.

If F is a convex subspace of Δ, then for every point x of Δ, there exists a unique point $\pi_F(x) \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. The point $\pi_F(x)$ is called the *projection* of x onto F. If F is a max and $x \notin F$, then $\pi_F(x)$ is collinear with x. If F_1 and F_2 are two disjoint maxes, then the restriction of π_{F_2} to F_1 is an isomorphism from Δ_{F_1} to Δ_{F_2}. If F_1 and F_2 are two distinct maxes, then either $F_1 \cap F_2 = \emptyset$ or $F_1 \cap F_2$ is a convex subspace of diameter $n-2$. The set of convex subspaces through a point x of Δ define a projective space of dimension $n-1$ which we will denote by $Res_\Delta(x)$.

A *hyperplane* of a dual polar space is a proper subspace which meets each line. Suppose H is a hyperplane of a thick dual polar space Δ and Q is a quad of Δ. Then either $Q \subset H$ or $Q \cap H$ is a hyperplane of Q. So, by Theorem 2.3.1 of Payne and Thas [9], one of the following cases occurs: (i) $Q \subseteq H$; (ii) $Q \cap H = x^\perp \cap Q$ for a certain point $x \in Q$; (iii) $Q \cap H$ is an ovoid of Q; (iv) $Q \cap H$ is a subquadrangle of Q. (We recall here that an *ovoid* is a set of points meeting each line in a unique point.) If case (i), case (ii), case (iii), respectively case (iv) occurs, then Q is called *deep*, *singular*, *ovoidal*, respectively *subquadrangular* (*with respect to* H). If case (ii) occurs, then x is called the deep point of Q.

A *full embedding* of a dual polar space Δ into a projective space Σ is an injective mapping e from the point-set P of Δ to the point-set of Σ satisfying (i) $\langle e(P) \rangle = \Sigma$ and (ii) $e(L)$ is a line of Σ for every line L of Δ. If $e : \Delta \to \Sigma$ is a full embedding of Δ into Σ, then for every hyperplane α of Σ, the set $e^{-1}(e(P) \cap \alpha)$ is a hyperplane of Δ. We say that the hyperplane $e^{-1}(e(P) \cap \alpha)$ *arises from the embedding* e.

We now describe the class of dual polar spaces under consideration in this paper. Let $n \geq 2$ and let $\mathbb{K}, \mathbb{K}'$ be fields such that $\mathbb{K}'$ is quadratic Galois extension of $\mathbb{K}$. Let θ denote the unique nontrivial element in the Galois group $Gal(\mathbb{K}'/\mathbb{K})$. For all $i, j \in \{0, \ldots, 2n+1\}$ with $i \leq j$, let $a_{ij} \in \mathbb{K}$ such that $q(\overline{X}) = \sum_{0 \leq i \leq j \leq 2n+1} a_{ij} X_i X_j$ is a quadratic form defining a quadric $Q^-(2n+1, \mathbb{K})$ of Witt index n in $\mathrm{PG}(2n+1, \mathbb{K})$ and a quadric $Q^+(2n+1, \mathbb{K}')$ of Witt index $n+1$ in $\mathrm{PG}(2n+1, \mathbb{K}')$. Let $DQ^-(2n+1, \mathbb{K})$ denote the dual polar space associated with $Q^-(2n+1, \mathbb{K})$. So, the points and lines of $DQ^-(2n+1, \mathbb{K})$ are the $(n-1)$-dimensional, respectively $(n-2)$-dimensional, subspaces of $Q^-(2n+1, \mathbb{K})$ and incidence is reverse containment. If F is a convex subspace of $\Delta = DQ^-(2n+1, \mathbb{K})$ of diameter $\delta \in \{2, \ldots, n\}$, then $\Delta_F \cong DQ^-(2\delta+1, \mathbb{K})$. If $n = 2$, then $DQ^-(2n+1, \mathbb{K}) = DQ^-(5, \mathbb{K})$ is a generalized quadrangle which is isomorphic to the generalized quadrangle $H(3, \mathbb{K}', \theta)$ of the points and lines of a nonsingular θ-Hermitian variety of Witt index 2 in $\mathrm{PG}(3, \mathbb{K}')$.

The dual polar space $DQ^-(2n+1, \mathbb{K})$ admits up to isomorphism a unique full embedding e into the projective space $\mathrm{PG}(2^n-1, \mathbb{K}')$, see Cooperstein and Shult [4] (for the finite case) and De Bruyn [8] (general case). This embedding is called the *spin embedding* of $DQ^-(2n+1, \mathbb{K})$. If F is a convex subspace of diameter $\delta \in \{2, \ldots, n\}$ of $DQ^-(2n+1, \mathbb{K})$, then e induces an embedding e_F of $\Delta_F \cong DQ^-(2\delta+1, \mathbb{K})$ into a subspace of $\mathrm{PG}(2^n-1, \mathbb{K}')$. This embedding is isomorphic to the spin embedding of $DQ^-(2\delta+1, \mathbb{K})$, see e.g. Theorem 1.6 of Cardinali, De Bruyn and Pasini [2]. If $n = 2$, then $DQ^-(2n+1, \mathbb{K}) = DQ^-(5, \mathbb{K})$ and the image of e is just a nonsingular θ-Hermitian variety of Witt index 2 in $\mathrm{PG}(3, \mathbb{K}')$. (This explains the isomorphism $DQ^-(5, \mathbb{K}) \cong H(3, \mathbb{K}', \theta)$.) An ovoid of $DQ^-(5, \mathbb{K})$ is called *classical* if it arises from the spin embedding of $DQ^-(5, \mathbb{K})$.

Suppose now $n = 3$ and consider the dual polar space $DQ^-(7, \mathbb{K})$.

If x is a point of $DQ^-(7, \mathbb{K})$, then the set H_x of points of $DQ^-(7, \mathbb{K})$ at distance at most 2 from x is a hyperplane of $DQ^-(7, \mathbb{K})$. This hyperplane is called the *singular hyperplane* of $DQ^-(7, \mathbb{K})$ with *deepest point* x.

Let Q be a quad of $DQ^-(7, \mathbb{K})$ and O an ovoid of Q. Then the set of points of $DQ^-(7, \mathbb{K})$ at distance at most 1 from O is a hyperplane of $DQ^-(7, \mathbb{K})$. We call this hyperplane the *extension of* O.

Now, let α_1 and α_2 be two disjoint planes of $Q^-(7, \mathbb{K})$ and let α be a hyperplane of $\mathrm{PG}(7, \mathbb{K})$ containing α_1 and α_2. Then $\alpha \cap Q^-(7, \mathbb{K})$ is a nonsingular quadric of Witt index 3 of α. Denote this quadric by $Q(6, \mathbb{K})$ and its associated dual polar space by $DQ(6, \mathbb{K})$. The following proposition was proved by Shult [11] (for the finite case) and Pralle [10] (for the general case).

Proposition 1.1 ([10], [11]) *The dual polar space $DQ(6,\mathbb{K})$ has hyperplanes with respect to which every quad is singular. The points and lines contained in any such hyperplane define a split-Cayley generalized hexagon $H(\mathbb{K})$.*

Conversely, if H is a hyperplane of a thick dual polar space Δ of rank 3 such that every quad of Δ is singular with respect to H, then Δ is the dual polar space associated with a nonsingular quadric of Witt index 3 in a 6-dimensional projective space over a field.

If H is a hyperplane of $DQ(6,\mathbb{K})$ such that every quad of $DQ(6,\mathbb{K})$ is singular with respect to H, then H is called a *hexagonal hyperplane of* $DQ(6,\mathbb{K})$. Now, for a hexagonal hyperplane H of $DQ(6,\mathbb{K})$, put $\overline{H} := H \cup U$, where U is the set of generators of $Q^-(7,\mathbb{K})$ intersecting α in a line L, which regarded as line of $DQ(6,\mathbb{K})$ is contained in H. Then by Pralle [10], $\overline{H}$ is a hyperplane of $DQ^-(7,\mathbb{K})$. We call any such hyperplane of $DQ^-(7,\mathbb{K})$ a *hexagonal hyperplane of* $DQ^-(7,\mathbb{K})$.

The following is the main result of this paper.

Theorem 1.2 *The hyperplanes of the dual polar space $DQ^-(7,\mathbb{K})$ which arise from its spin embedding are precisely the following hyperplanes:*

(1) *the singular hyperplanes of $DQ^-(7,\mathbb{K})$;*
(2) *the extensions of the classical ovoids of the quads of $DQ^-(7,\mathbb{K})$;*
(3) *the hexagonal hyperplanes of $DQ^-(7,\mathbb{K})$.*

Since the spin embedding of $DQ^-(7,\mathbb{K})$ is the so-called absolutely universal embedding of $DQ^-(7,\mathbb{K})$ (see De Bruyn [8, Corollary 1.4]), we have

Corollary 1.3 *The hyperplanes of the dual polar space $DQ^-(7,\mathbb{K})$ which arise from some projective embedding are precisely the singular hyperplanes, the extensions of the classical ovoids of the quads and the hexagonal hyperplanes.*

Along our way, we will also prove the following result regarding the structure of hyperplanes of $DQ^-(2n+1,\mathbb{K})$, $n \geq 2$, which arise from its spin embedding.

Theorem 1.4 *Let H be a hyperplane of the dual polar space $\Delta = DQ^-(2n+1, \mathbb{K})$, $n \geq 2$, arising from its spin embedding, and let x be a point of H. Then the set of lines through x contained in H is a subspace of co-dimension at most 2 of the projective space $Res_\Delta(x) \cong \mathrm{PG}(n-1, \mathbb{K})$.*

Remark. For finite fields $\mathbb{K}$, Theorem 1.2 was already proved in De Bruyn [5, Theorem 1.5]. Several arguments in the proof of [5] however only work in the finite case (counting arguments; a line and a Hermitian variety of $\mathrm{PG}(3, q^2)$ always meet).

2 Proof of Theorem 1.2

Lemma 2.1 *Let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic extension of $\mathbb{K}$. Let $\mathrm{PG}(2, \mathbb{K}')$ be the Desarguesian projective plane coordinatized in the natural way by the field $\mathbb{K}'$. Let $\mathrm{PG}(2, \mathbb{K})$ denote the subplane of $\mathrm{PG}(2, \mathbb{K}')$ consisting of those points of $\mathrm{PG}(2, \mathbb{K}')$ whose coordinates can be chosen in the subfield $\mathbb{K}$. Then any line of $\mathrm{PG}(2, \mathbb{K}')$ intersects $\mathrm{PG}(2, \mathbb{K})$ in either a point or a line of $\mathrm{PG}(2, \mathbb{K})$.*

Proof. Let ϵ be an arbitrary element of $\mathbb{K}' \setminus \mathbb{K}$. Then $\{1, \epsilon\}$ is a basis of $\mathbb{K}'$ regarded as two-dimensional vector space over $\mathbb{K}$. Let L be an arbitrary line of $\mathrm{PG}(2, \mathbb{K}')$. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{K}$, not all zero, such that L consists of all points (X_0, X_1, X_2) satisfying

$$(\alpha_1 + \alpha_2\epsilon)X_0 + (\beta_1 + \beta_2\epsilon)X_1 + (\gamma_1 + \gamma_2\epsilon)X_2 = 0.$$

If $(k_0, k_1, k_2) \in \mathrm{PG}(2, \mathbb{K}) \cap L$, then

$$\begin{cases} \alpha_1 k_0 + \beta_1 k_1 + \gamma_1 k_2 &= 0, \\ \alpha_2 k_0 + \beta_2 k_1 + \gamma_2 k_2 &= 0. \end{cases}$$

Let r be the rank of the matrix:

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}.$$

If $r = 1$, then L intersects $\mathrm{PG}(2, \mathbb{K})$ in a line of $\mathrm{PG}(2, \mathbb{K})$. If $r = 2$, then L intersects $\mathrm{PG}(2, \mathbb{K})$ in a point. ■

Lemma 2.2 *Let Δ be a dual polar space of type $DQ^-(2n+1, \mathbb{K})$, $n \geq 2$, and let H be a hyperplane of Δ. Then there are no quads of Δ which are subquadrangular with respect to H.*

Proof. In De Bruyn [8, Lemma 3.1] it was proved that a generalized quadrangle of type $DQ^-(5, \mathbb{K})$ does not have proper subquadrangles. Notice also that every quad of Δ is isomorphic to a generalized quadrangle of type $DQ^-(5, \mathbb{K})$. ∎

Lemma 2.3 *Let Δ be a dual polar space of type $DQ^-(2n+1, \mathbb{K})$, $n \geq 2$, and let H be a hyperplane of Δ. For every point x of H, the set $\Lambda_H(x)$ of lines of Δ through x contained in H is a set of points of the $(n-1)$-dimensional projective space $Res_\Delta(x)$. This set $\Lambda_H(x)$ is a possibly empty subspace of $Res_\Delta(x)$.*

Proof. Suppose L_1 and L_2 are two distinct lines through x contained in H. Let Q denote the unique quad through L_1 and L_2. By Lemma 2.2, there are two possibilities.

(1) Q is singular with respect to H. Then $Q \cap H = x^\perp \cap Q$.

(2) Q is deep with respect to H.

In either case, any line of Q through x is contained in H. It follows that $\Lambda_H(x)$ is a subspace of $Res_\Delta(x)$. ∎

Lemma 2.4 *Let Δ be a dual polar space of type $DQ^-(7, \mathbb{K})$, let Q be a quad of Δ and let H be a hyperplane of $DQ^-(7, \mathbb{K})$ containing Q. Then H is either a singular hyperplane or the extension of an ovoid of Q.*

Proof. By Lemma 2.3, there are two possibilities for a point x of Q: either $x^\perp \cap H = x^\perp$ or $x^\perp \cap H = x^\perp \cap Q$. Let A denote the set of all points $x \in Q$ for which $x^\perp \subseteq H$. Since every point outside Q is collinear with a unique point of Q, we have

$$H = Q \cup \Big(\bigcup_{x \in A} x^\perp \Big).$$

Since H does not coincide with the whole point-set of Δ, $A \neq Q$. Now, let Q' denote an arbitrary quad of Δ disjoint from Q. Then $\pi_{Q'}(A) = H \cap Q'$ is a hyperplane of Q' which is either a singular hyperplane or an ovoid of Q' by Lemma 2.2.

Suppose $H \cap Q' = y^\perp \cap Q'$ for a certain point y of Q'. Then $A = \pi_Q(H \cap Q') = z^\perp \cap Q$ where $z := \pi_Q(y)$. It readily follows that $H = Q \cup \left(\bigcup_{x \in A} x^\perp \right)$ is the singular hyperplane of Δ with deepest point z.

Suppose $H \cap Q' = O'$ for some ovoid O' of Q'. Then $A = \pi_Q(O')$ is an ovoid of Q isomorphic to the ovoid O' of Q'. Clearly, $H = Q \cup \left(\bigcup_{x \in A} x^\perp \right)$ is the extension of the ovoid A of Q. ∎

Proposition 2.5 *Let Δ be a dual polar space of type $DQ^-(7, \mathbb{K})$ and let H be one of the following hyperplanes of $DQ^-(7, \mathbb{K})$: (i) a singular hyperplane; (ii) the extension of a classical ovoid in a quad; (iii) a hexagonal hyperplane. Then H arises from the spin embedding of Δ.*

Proof. Let $e : \Delta \rightarrow \Sigma$ denote the spin embedding of Δ into the projective space $\Sigma \cong \mathrm{PG}(7, \mathbb{K}')$, where $\mathbb{K}'$ is the quadratic extension of $\mathbb{K}$ associated with $DQ^-(7, \mathbb{K})$. In De Bruyn [8, Theorem 1.2], it was proved that every singular hyperplane of Δ arises from e. By Theorem 1.2 of De Bruyn [7], also every hexagonal hyperplane of $DQ^-(7, \mathbb{K})$ arises from e. Suppose now that H is the extension of a classical ovoid O of a quad Q. Let Q' denote an arbitrary quad of Δ disjoint from Q. Then $O' := \pi_{Q'}(O)$ is a classical ovoid of Q'. By De Bruyn [6, Theorem 1.1 (5)], $\Sigma_1 := \langle e(Q) \rangle$ and $\Sigma_2 := \langle e(Q') \rangle$ are two disjoint 3-spaces of Σ. Moreover, the embedding e induces a full embedding e_1 of Q into Σ_1 and a full embedding e_2 of Q' into Σ_2. These embeddings are isomorphic to the spin embedding of $DQ^-(5, \mathbb{K})$. So, there exists a plane α in Σ_2 such that $e_2^{-1}(e_2(Q') \cap \alpha) = O'$. Now, let H' denote the hyperplane of Δ arising from the hyperplane $\langle \Sigma_1, \alpha \rangle$ of Σ. Then H' contains Q and O'. By (the proof of) Lemma 2.4, H' coincides with the extension of O, i.e. $H' = H$. This proves the proposition. ∎

Lemma 2.6 *Let Δ be a dual polar space of type $DQ^-(7, \mathbb{K})$ and let H be a hyperplane of Δ arising from its spin embedding. Then for every point x of H, $\Lambda_H(x)$ is a nonempty subspace of the projective plane $Res_\Delta(x) \cong \mathrm{PG}(2, \mathbb{K})$.*

Proof. Let $e : \Delta \rightarrow \Sigma$ denote the spin embedding of Δ and let x be a point of Δ. Then $\dim \langle e(x^\perp) \rangle = 3$ by De Bruyn [6, Theorem 1.6]. So, the quotient space $\Sigma_x = \langle e(x^\perp) \rangle / e(x)$ has dimension 2. For every line L of Δ through x, let $f(L)$ denote the point $e(L)/e(x)$ of Σ_x. Clearly, f is an injection from the set of points of $Res_\Delta(x)$ to the set of points of Σ_x. Let Q be a quad through x and let $\mathcal{L}_Q$ be the set of lines of Q through x. The embedding e

induces an embedding of Q into a 3-space of Σ which is isomorphic to the spin embedding of Δ_Q. Hence, $\dim(\langle e(x^\perp \cap Q)\rangle) = 2$. It follows that f maps the lines of $\mathcal{L}_Q$ into a line $f(Q)$ of Σ_x. If L is a line through x not contained in Q, then $f(L)$ is not contained in $f(Q)$ by Theorem 1.1 (3) of De Bruyn [6]. Hence, f defines an embedding of $Res_\Delta(x) \cong \mathrm{PG}(2, \mathbb{K})$ into a subgeometry of $\Sigma_x \cong \mathrm{PG}(2, \mathbb{K}')$.

Let V be an 8-dimensional vector space over $\mathbb{K}'$ such that $\Sigma = \mathrm{PG}(V)$. Choose a nonzero vector $\bar{e}_1 \in V$ and a 3-dimensional subspace W of V such that $e(x) = \langle \bar{e}_1 \rangle$ and $\langle e(x^\perp)\rangle = \mathrm{PG}(\langle \bar{e}_1, W\rangle)$. Let Q^* be a given quad through x. The embedding of Q^* into $\langle e(Q^*)\rangle$ induced by e is isomorphic to the spin embedding of $DQ^-(5, \mathbb{K})$. So, $e(Q^*)$ is a nondegenerate θ-Hermitian variety of Witt index 2 of $\langle e(Q^*)\rangle \cong \mathrm{PG}(3, \mathbb{K}')$. Notice that there exists a natural bijective correspondence between the points of Σ_x and those of $\mathrm{PG}(W)$. Since $e(Q^*)$ is a nondegenerate θ-Hermitian variety of Witt index 2 of $\langle e(Q^*)\rangle \cong \mathrm{PG}(3, \mathbb{K}')$, we may suppose that we have chosen V, $\bar{e}_1$ and W in such a way that $f(\mathcal{L}_{Q^*})$ corresponds to a Baer-$\mathbb{K}$-subline of $\mathrm{PG}(W)$, i.e. we may suppose that there exist vectors $\bar{f}_1', \bar{f}_2' \in W$ such that

(1) $f(\mathcal{L}_{Q^*})$ corresponds to the set of all points of $\mathrm{PG}(W)$ of the form $\langle k_1 \bar{f}_1' + k_2 \bar{f}_2'\rangle$ where $k_1, k_2 \in \mathbb{K}$ with $(k_1, k_2) \neq (0, 0)$.

On the other hand, since f defines an embedding of $Res_\Delta(x) \cong \mathrm{PG}(2, \mathbb{K})$ into a subgeometry of $\Sigma_x \cong \mathrm{PG}(2, \mathbb{K}')$, then there exists a basis $\{\bar{e}_2, \bar{e}_3, \bar{e}_4\}$ of W and a subfield $\mathbb{K}_0$ of $\mathbb{K}'$ isomorphic to $\mathbb{K}$ such that the image of f corresponds to the set of all points of $\mathrm{PG}(W)$ of the form $\langle k_2\bar{e}_2 + k_3\bar{e}_3 + k_4\bar{e}_4\rangle$, where k_2, k_3 and k_4 are elements of $\mathbb{K}_0$ not all equal to 0. Hence, there exist vectors $\bar{f}_1, \bar{f}_2 \in W$ such that

(2) $f(\mathcal{L}_{Q^*})$ corresponds to the set of all points of $\mathrm{PG}(W)$ of the form $\langle k_1 \bar{f}_1 + k_2 \bar{f}_2\rangle$ where $k_1, k_2 \in \mathbb{K}_0$ with $(k_1, k_2) \neq (0, 0)$.

The statements (1) and (2) force the subfields $\mathbb{K}_0$ and $\mathbb{K}$ of $\mathbb{K}'$ to coincide. It follows that the image of f corresponds to the set of all points of $\mathrm{PG}(W)$ of the form $\langle k_2\bar{e}_2 + k_3\bar{e}_3 + k_4\bar{e}_4\rangle$ where k_2, k_3 and k_4 are elements of $\mathbb{K}$ not all equal to 0.

Now, suppose that H is a hyperplane of Δ through x arising from the embedding e. So, $H = e^{-1}(e(\Delta) \cap \alpha)$ for a certain hyperplane α of Σ through $e(x)$. If α contains $\langle e(x^\perp)\rangle$, then $x^\perp \subseteq H$. If α does not contain $\langle e(x^\perp)\rangle$, then

α defines a hyperplane of Σ_x. By Lemma 2.1 and the previous paragraph, it then readily follows that one of the following two cases occurs: (i) there exists a unique line through x contained in H; (ii) there exists a quad R through x such that the set of lines through x contained in H coincides with the set of lines through x contained in R. This proves the lemma. ∎

The following proposition is precisely Theorem 1.4.

Proposition 2.7 *Let Δ be a dual polar space of type $DQ^-(2n+1, \mathbb{K})$, $n \geq 2$, and let H be a hyperplane of Δ arising from its spin embedding. Then for every $x \in H$, $\Lambda_H(x)$ is a subspace of co-dimension at most 2 of $Res_\Delta(x) \cong$ PG$(n-1, \mathbb{K})$.*

Proof. By Lemma 2.3, $\Lambda_H(x)$ is a subspace of $Res_\Delta(x)$. Suppose the co-dimension of $\Lambda_H(x)$ is at least 3. Then there exists a hex F through x such that $x^\perp \cap (H \cap F) = \{x\}$. The hyperplane $H \cap F$ of F arises from the embedding e_F of $\Delta_F \cong DQ^-(7, \mathbb{K})$ induced by the spin embedding of Δ. Recall that e_F is isomorphic to the spin embedding of $DQ^-(7, \mathbb{K})$. A contradiction is now obtained by applying Lemma 2.6 to the hyperplane $H \cap F$ of F and the point $x \in H \cap F$. So, the co-dimension of $\Lambda_H(x)$ is at most 2. ∎

The following proposition in combination with Proposition 2.5 finishes the proof of Theorem 1.2.

Proposition 2.8 *Let Δ be a dual polar space of type $DQ^-(7, \mathbb{K})$. If H is a hyperplane of Δ arising from its spin embedding e, then H is either a singular hyperplane, the extension of a classical ovoid in a quad or a hexagonal hyperplane.*

Proof. Suppose first that H contains a quad Q. Then by Lemma 2.4, H is either a singular hyperplane or the extension of an ovoid in Q. Suppose H is the extension of the ovoid O in Q. Let Q' be a quad of Δ disjoint from Q and put $O' := \pi_{Q'}(O)$. Then $H \cap Q' = O'$. Since H arises from the embedding e, the ovoid O' of Q' arises from the embedding of Q' induced by e and hence is classical. As a consequence, also the ovoid $O = \pi_Q(O')$ of Q is classical.

Suppose now that H does not contain quads. We show that H does not contain points x for which $x^\perp \subseteq H$. Suppose to the contrary that $x^\perp \subseteq H$. If $y \in \Delta_2(x) \cap H$, then the unique quad Q through x and y contains $x^\perp \cap Q$

and y and hence is completely contained in H, a contradiction. Hence, $\Delta_2(x) \cap H = \emptyset$. Obviously, $\Delta_3(x) \cap H \neq \emptyset$. (A line containing a point of $\Delta_3(x)$ meets H necessarily in a point of $\Delta_3(x)$.) If $y \in \Delta_3(x) \cap H$, then by Lemma 2.6, there exists a line through y contained in H. This line contains a point at distance 2 from x, a contradiction. Hence, H does not contain points x for which $x^\perp \subseteq H$.

Since H does not contain quads, every quad of Δ is singular or ovoidal with respect to H (recall Lemma 2.2).

Claim I. *If L is a line of Δ, then either all quads through L are singular or precisely one quad through L is singular. So, there exist singular quads.*
PROOF. If $L \subseteq H$, then all quads through L are singular. Suppose therefore that $L \cap H$ is a singleton $\{x\}$. By Lemma 2.6, there are two possibilities:

(1) there exists a unique line M through x which is contained in H. Then the unique quad through L and M is the unique singular quad through L.

(2) There exists a quad Q through x such that the lines through x contained in H are precisely the lines through x contained in Q. Every quad through L intersects Q in a line through x and hence is singular. (eop)

Claim II. *For every quad Q which is singular with respect to H, there exists a quad disjoint from Q which is singular with respect to H.*
PROOF. Let $x \in Q$ such that $Q \cap H = x^\perp \cap Q$. Let Q_1 denote an arbitrary quad through x distinct from Q. Since $Q_1 \cap Q$ is a line contained in H, Q_1 is singular. Hence, $Q_1 \cap H = x_1^\perp \cap Q_1$ for a certain point $x_1 \in Q_1$. Obviously, $x_1 \in Q_1 \cap Q$. Since $x^\perp \cap H = x^\perp \cap Q$, $x_1 \neq x$. Now, let Q_2 denote a quad through x_1 not containing $Q_1 \cap Q$. Since $Q_1 \cap Q_2$ is a line contained in H, Q_2 is singular with respect to H. So, $Q_2 \cap H = x_2^\perp \cap Q_2$ for a certain point $x_2 \in Q_2$. Since $Q_2 \cap Q \not\subseteq H$, $x_2 \notin Q$. Now, there exists a line in Q_2 through x_2 not meeting Q. Any quad through that line disjoint from Q is singular, proving the claim. (eop)

Claim III. *If x is a point of Δ such that there exists a quad Q through x such that $Q \cap H = x^\perp \cap H$, then every quad through x is singular.*
PROOF. This follows from the fact that any quad through x contains a line (of Q) which is completely contained in H. (eop)

Let S denote the set of quads of Δ which are singular with respect to H. Let $\widetilde{S}$ denote the set of points of $Q^-(7, \mathbb{K})$ corresponding to the elements of S. By Claim I, $\widetilde{S}$ is a subspace of $Q^-(7, \mathbb{K})$ and hence carries the structure of a polar space Π_0. It is important to notice that this polar space Π_0 is nondegenerate by Claim II. By Claim III, we know that the rank of Π_0 is equal to 3. By Claim I, Π_0 is either $Q^-(7,\mathbb{K})$ or a hyperplane of $Q^-(7,\mathbb{K})$. In the latter case, it follows from Cohen and Shult [3, Theorem 5.12] that Π_0 is obtained by intersecting $Q^-(7,\mathbb{K})$ with a hyperplane of the ambient projective space of $Q^-(7,\mathbb{K})$; hence $\Pi_0 \cong Q(6,\mathbb{K})$. (Also in [10], it was shown that Π_0 is a polar space (see Proposition 15), but the possible complication that Π_0 might be degenerate - which is impossible by Claim II - seems not to be discussed there.)

If $\Pi_0 = Q^-(7,\mathbb{K})$, then every quad of $DQ^-(7,\mathbb{K})$ is singular with respect to H. This is impossible by Proposition 1.1. So, $\Pi_0 = Q(6,\mathbb{K})$. Then H is a hexagonal hyperplane of $DQ^-(7,\mathbb{K})$ by Pralle [10, Theorem 2 + Corollary 1], Lemma 2.6 and the fact that H does not contain quads. For reasons of completeness, we give here a complete proof of this fact.

Claim IV. *For every line L of $Q(6,\mathbb{K})$, there exists a generator of $Q(6,\mathbb{K})$ through L belonging to H.*
PROOF. We regard L as a line of $DQ(6,\mathbb{K})$. We must show that there exists a point x on L every quad through which is singular, or equivalently (see Lemma 2.6), a point x on L for which $\Lambda_H(x)$ is a line of $Res_\Delta(x)$. If L is not contained in H, then the unique point of $L\cap H$ satisfies this property (Recall the proof of Claim I and the fact that every quad through L is singular). If L is contained in H, then we can take for x the deepest point of any singular quad through L. (eop)

Now, let $\widetilde{H}$ denote the set of points of $DQ(6,\mathbb{K})$ which are also points of H. The following claim follows from Claim IV and the fact that H is a subspace.

Claim V. *A generator α of $Q^-(7,\mathbb{K})$ not contained in $Q(6,\mathbb{K})$ belongs to H if and only if every generator of $Q(6,\mathbb{K})$ through $\alpha\cap Q(6,\mathbb{K})$ belongs to $\widetilde{H}$.*

Since H is a proper subspace of $DQ^-(7,\mathbb{K})$, $\widetilde{H}$ is a proper subset of the point set of $DQ(6,\mathbb{K})$. By Claim IV and the fact that H is a subspace, $\widetilde{H}$ is a hyperplane of $DQ(6,\mathbb{K})$. Every quad Q of $DQ(6,\mathbb{K})$ is properly contained in a singular quad $\widetilde{Q}$ of $DQ^-(7,\mathbb{K})$. Every quad of $DQ^-(7,\mathbb{K})$ through the deep point of $\widetilde{Q}$ is singular and hence the deep point of $\widetilde{Q}$ is a point of $DQ(6,\mathbb{K})$, i.e. a point of Q. It follows that Q is singular with respect to $\widetilde{H}$. So, $\widetilde{H}$ is a hexagonal hyperplane of $DQ(6,\mathbb{K})$. By Claim V, it now follows that H is a hexagonal hyperplane of $DQ^-(7,\mathbb{K})$. ■

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