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CYCLE-FREE CUTS OF MUTUAL RANK PROBABILITY RELATIONS

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It is well known that the linear extension majority (LEM) relation of a poset of size $n \geq 9$ can contain cycles. In this paper we are interested in obtaining minimum cutting levels α_m such that the crisp relation obtained from the mutual rank probability relation by setting to 0 its elements smaller than or equal to α_m, and to 1 its other elements, is free from cycles of length m. In a first part, theoretical upper bounds for α_m are derived using known transitivity properties of the mutual rank probability relation. Next, we experimentally obtain minimum cutting levels for posets of size $n \leq 13$. We study the posets requiring these cutting levels in order to have a cycle-free strict cut of their mutual rank probability relation. Finally, a lower bound for the minimum cutting level α_4 is computed. To accomplish this, a family of posets is used that is inspired by the experimentally obtained 12-element poset requiring the highest cutting level to avoid cycles of length 4.

Keywords: partially ordered set, linear extension majority cycle, mutual rank probability relation, minimum cutting level, cycle-free cut

Classification: 06A06, 06A07

1. INTRODUCTION

When considering the probability space consisting of the set of linear extensions (also called topological orderings) of a given poset P equipped with the uniform probability measure, the mutual rank probability relation appears naturally. It expresses the probability that an element $x \in P$ has a higher position than an element $y \in P$ in a linear extension of P sampled uniformly at random. This relation plays an important role, both from an application [6] as well as from a theoretical [5, 12, 17] point of view. Although the study of the type of transitivity exhibited by mutual rank probability relations has received considerable attention [5, 12, 19, 23], this transitivity remains far from characterized. It is, however, well known that mutual rank probability relations are in general not weakly stochastic transitive [5], allowing for the occurrence of cycles in the linear extension majority (LEM) relation of posets of size $n \geq 9$.

Quite some attention has been given to such LEM cycles in the literature. Examples of posets with LEM cycles are given in [1, 11, 12, 13, 14, 16, 18], frequency estimates for LEM cycles have been reported in [15, 17], and the occurrence of LEM cycles in

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certain subclasses of posets has been studied in [2, 9]. Moreover, in previous work [7], the present authors have succeeded in counting the posets of size $n \leq 13$ with LEM cycles. Besides the fact that the existence of LEM cycles is an intriguing phenomenon in its own right, a better understanding of LEM cycles might help in the ongoing quest to characterize the transitivity of mutual rank probability relations.

In this paper we are interested in obtaining minimum cutting levels α_m such that the strict α_m-cut of the mutual rank probability relation M_P, obtained by setting to 0 its elements smaller than or equal to α_m and to 1 its other elements, is free from cycles of size m. In other words, we want to obtain the smallest number α_m such that at least one mutual rank probability in any LEM cycle of size m is smaller than or equal to α_m.

The outline of this paper is as follows. In Section 3, we invoke known transitivity properties to establish theoretical upper bounds for the minimum cutting level α_m. In Section 4, minimum cutting levels for posets of size $n \leq 13$ are obtained experimentally. Moreover, posets requiring these minimum cutting levels in order to have a cycle-free strict cut of their mutual rank probability relation are studied in a modest attempt to deepen the understanding of the occurrence of LEM cycles. Finally, in Section 5 a lower bound for the minimum cutting level α_4 to avoid LEM cycles of length 4 is computed. This lower bound for α_4 implies that the theoretical upper bound for α_4 is quite tight.

2. PRELIMINARIES

A binary relation $\leq_P$ on a set P is called an *order relation* if it is reflexive ($x \leq_P x$), antisymmetric ($x \leq_P y$ and $y \leq_P x$ imply $x =_P y$) and transitive ($x \leq_P y$ and $y \leq_P z$ imply $x \leq_P z$). A *linear order relation* $\leq_P$ is an order relation in which every two elements are comparable ($x \leq_P y$ or $y \leq_P x$). If $x \leq_P y$ and $x \neq y$, we write $x <_P y$. If neither $x \leq_P y$ nor $x \geq_P y$, we say that x and y are *incomparable* and write $x \,||_P\, y$. A couple $(P, \leq_P)$, where P is a set of objects and $\leq_P$ is an order relation on P, is called a partially ordered set or *poset* for short. The *size* of a poset $(P, \leq_P)$ is defined as the cardinality of P. A poset of size n will be called an n-element poset for short. The dual poset of $(P, \leq_P)$, denoted as $(P, \leq_P^\top)$, is the poset consisting of the same set P and the converse relation $\leq_P^\top$ of $\leq_P$, i. e. $x \leq_P^\top y$ if and only if $y \leq_P x$, for all $x, y \in P$.

The binary relation $\prec_P$, for which it holds that $(x, y) \in \prec_P$ if and only if $x <_P y$ and there exists no $z \in P$ such that $x <_P z <_P y$, is called the *covering relation* of $(P, \leq_P)$. The covering relation $\prec_P$ of a poset $(P, \leq_P)$ can be conveniently represented by a so-called *Hasse diagram* where a sequence of connected lines upwards from x to y is present if and only if $x <_P y$. Examples of representations of posets by such Hasse diagrams can be found in the appendix of this paper.

Let Q be a set and R and S two binary relations on Q. If $R \subset S$, then (Q, S) is called an extension of (Q, R). A *linear extension* of a poset $(P, \leq_P)$ is an extension $(P, \leq_L)$ for which $\leq_L$ is a linear order relation. The *mutual rank probability* $\mathrm{p}(x > y)$ of two elements x and y of a poset $(P, \leq_P)$ is defined as the probability that $x >_L y$ in a linear extension $(P, \leq_L)$ that has been sampled uniformly at random from the set of linear extensions of $(P, \leq_P)$. Stated differently, it is the number of linear extensions of $(P, \leq_P)$ in which $x >_L y$, divided by the number of linear extensions of $(P, \leq_P)$. The mutual rank probability relation M_P is the $[0, 1]$-valued binary relation on P defined by $M_P(x, y) = \mathrm{p}(x > y)$ for all $x, y \in P$ where $x \neq y$ and $M_P(x, x) = 1/2$ for all $x \in P$.

Note that M_P is a so-called reciprocal relation since $M_P(x,y) + M_P(y,x) = 1$.

The *linear extension majority (LEM) relation* [20] of a poset P is the antisymmetric binary relation $\succ_{\mathrm{LEM}}$ on P such that $x \succ_{\mathrm{LEM}} y$ if $\mathrm{p}(x > y) > \mathrm{p}(y > x)$. Due to the reciprocity of the mutual rank probability relation, it is equivalent to state that $x \succ_{\mathrm{LEM}} y$ if $\mathrm{p}(x > y) > 1/2$. It is well known [10] that the linear extension majority relation $\succ_{\mathrm{LEM}}$ can contain cycles, i. e. subsets $\{x_1, x_2, \ldots, x_m\}$ of elements of P such that $x_1 \succ_{\mathrm{LEM}} x_2 \succ_{\mathrm{LEM}} \cdots \succ_{\mathrm{LEM}} x_m \succ_{\mathrm{LEM}} x_1$, and thus is not transitive. These cycles are referred to as *LEM cycles* on m elements, or *m-cycles* for short.

The *strict α-cut*, with $\alpha \in [1/2, 1[$, of a reciprocal relation Q on a set A is the crisp relation Q^α on A defined by

$$Q^\alpha(x,y) = \begin{cases} 1, & \text{if } Q(x,y) > \alpha\,, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the LEM relation is nothing else but the strict 1/2-cut of M_P.

We define the *minimum cutting level α_m* as the smallest number such that for any finite poset the strict α_m-cut of the corresponding mutual rank probability relation is free of cycles of length m.

3. THEORETICAL UPPER BOUNDS FOR α_m

The problem of characterizing the transitivity exhibited by the mutual rank probability relation M_P was originally raised by Fishburn [12]. For any $u, v \in [0,1]$, define $\delta(u,v)$ as

$$\delta(u,v) = \inf\{M_P(a,c) \mid M_P(a,b) \geq u \,\wedge\, M_P(b,c) \geq v\}\,, \tag{1}$$

where the infimum is taken over all finite posets $(P, \leq_P)$ and all $(a,b,c) \in P^3$. His problem can be elegantly reformulated as to identify the conjunctor $\delta : [0,1]^2 \to [0,1]$ such that the inequality

$$\delta(M_P(a,b), M_P(b,c)) \leq M_P(a,c)$$

holds for all finite posets $(P, \leq_P)$ and all $(a,b,c) \in P^3$. A non-trivial lower bound for $\delta(u,v)$ was obtained by Kahn and Yu [19] via geometric arguments. For any $u, v \in [0,1]$, define $\delta^*(u,v)$ as

$$\delta^*(u,v) = \inf\{\mathrm{Prob}(Y_i > Y_k) \mid \mathrm{Prob}(Y_i > Y_j) \geq u \,\wedge\, \mathrm{Prob}(Y_j > Y_k) \geq v\}\,,$$

where the infimum is taken over all random $Y = (Y_1, Y_2, \ldots, Y_n)$ chosen uniformly from some n-dimensional compact convex subset of $\mathbb{R}^n$. Kahn and Yu have shown that

$$\delta^*(u,v) = \begin{cases} 0, & \text{if } u+v < 1\,, \\ \min(u,v), & \text{if } u+v \geq 1 + \min(u^2, v^2)\,, \\ \dfrac{(1-u)(1-v)}{u+v-2\sqrt{u+v-1}}, & \text{otherwise}\,. \end{cases} \tag{2}$$

A reciprocal relation Q on a set A is called δ^*-transitive if it holds for all $a, b, c \in A$ that

$$\delta^*(Q(a,b), Q(b,c)) \leq Q(a,c)\,. \tag{3}$$

As Kahn and Yu have shown that Fishburn's problem can be embedded in this more general setting, the function δ^* provides a lower bound for δ. Thus, for any poset P the mutual rank probability relation M_P is δ^*-transitive. Clearly, the set of mutual rank probability relations M_P of all finite posets P is a subset of the set of all δ^*-transitive reciprocal relations. Therefore, an upper bound $\bar{\alpha}_m$ for the minimum cutting level for the latter type of relations is necessarily an upper bound for α_m.

Let $\Delta = \{(u,v) \in [0,1]^2 \mid u+v \geq 1\}$ and consider the mapping $\gamma : \Delta \to [0,1]$ defined by

$$\gamma(u,v) = \frac{(1-u)(1-v)}{u+v-2\sqrt{u+v-1}}\,.$$

We first prove the following lemmas.

Lemma 3.1. For every $u,v,x,y \in [0,1]$ such that $x > u$, $y > v$ and $u+v \geq 1$, it holds that $\delta^*(x,y) > \delta^*(u,v)$.

Proof.

(i) Assume that $1 \leq u+v < 1+\min(u^2,v^2)$. It follows that $\delta^*(u,v) = \gamma(u,v)$. It can be shown easily that the partial derivative of γ w.r.t. u is strictly positive if and only if $1 \leq u+v < 1+v^2$, which is satisfied by assumption. Analogously, it can be shown that the partial derivative of γ w.r.t. v is strictly positive if and only if $1 \leq u+v < 1+u^2$. It follows that γ is strictly increasing in each variable, and a fortiori that $\delta^*(x,y) > \delta^*(u,v)$.

(ii) Assume that $u+v \geq 1+\min(u^2,v^2)$. It follows that $\delta^*(u,v) = \min(u,v)$ and thus immediately that $\delta^*(x,y) > \delta^*(u,v)$.

Finally, we prove that δ^* is continuous on the set $\{(u,v) \in [0,1]^2 \mid u+v = 1+\min(u^2,v^2)\}$, i.e. that on this set it holds that

$$\frac{(1-u)(1-v)}{u+v-2\sqrt{u+v-1}} = \min(u,v)\,.$$

(i) Assume that $u < v$. As $u+v = 1+u^2$, it holds that $v = 1+u^2-u$. This implies

$$\gamma(u,v) = \frac{u(1-u)^2}{(1-u)^2} = u = \min(u,v)\,.$$

(ii) Assume that $u \geq v$. As $u+v = 1+v^2$, it holds that $u = 1+v^2-v$. This implies

$$\gamma(u,v) = \frac{v(1-v)^2}{(1-v)^2} = v = \min(u,v)\,.$$

□

Lemma 3.2. It holds that $\gamma \leq \delta^*$ on Δ.

Proof. We first prove that $\gamma(u,v) \le u$. Expressing that $\gamma(u,v) \le u$ is equivalent to the condition

$$u(u+v) - 2u\sqrt{u+v-1} \ge (1-u)(1-v)\,,$$

which can be rewritten as

$$u^2 + (u+v-1) - 2u\sqrt{u+v-1} \ge 0\,,$$

or, equivalently,

$$(u - \sqrt{u+v-1})^2 \ge 0\,,$$

which is trivially satisfied on Δ. The case $\gamma(u,v) \le v$ is completely analogous. □

Lemma 3.3. An upper bound on the minimum cutting level for a δ^*-transitive reciprocal relation Q_4 on a set $\{x_0, x_1, x_2, x_3\}$ is given by $\bar{\alpha}_4 = 2 - \sqrt{2}$.

Proof. We will use shorthands $t_i = Q_4(x_i, x_{(i+1) \bmod 4})$ and $s_i = Q_4(x_i, x_{(i+2) \bmod 4})$ for $i \in \{0,1,2,3\}$.

We first assume that $t_0 = t_1 = t_2 = t_3 = t > 0.5$. In order for Q_4 to be δ^*-transitive, i.e. to fulfill condition (3), it is necessary that $s_0 \ge \delta^*(t,t)$ and $s_2 \ge \delta^*(t,t)$, or, due to reciprocity, that $s_0 \le 1 - \delta^*(t,t)$. Therefore, it has to hold that $\delta^*(t,t) \le 1 - \delta^*(t,t)$, or, equivalently, that $\delta^*(t,t) \le 0.5$. Lemma 3.2 implies that $\gamma(t,t) \le 0.5$, yielding an upper bound $\bar{\alpha}_4 = 2 - \sqrt{2}$ on t. It is easily shown that $\delta^*(t,t) = 0.5$ and that Q_4 is δ^*-transitive when $t = \bar{\alpha}_4$.

We will now show that Q_4 is not δ^*-transitive in the case where $t_i > t = \bar{\alpha}_4$ for every $i \in \{0,1,2,3\}$. Assume to the contrary that Q_4 is transitive. This implies that $s_0 \ge \delta^*(t_0, t_1)$. From $t_i > t$ for every $i \in \{0,1,2,3\}$, it follows using Lemma 3.1 that

$$s_0 \ge \delta^*(t_0, t_1) > \delta^*(t,t) = 0.5\,,$$

and, due to reciprocity of Q_4, that

$$s_2 < 0.5\,.$$

However, δ^*-transitivity of Q_4 implies that

$$s_2 \ge \delta^*(t_2, t_3) > \delta^*(t,t) = 0.5\,,$$

which is a contradiction.

As the minimum cutting level can only be determined by the minimum of the t_i's, no additional cases need to be considered. Therefore, $\bar{\alpha}_4 = 2 - \sqrt{2}$ is an upper bound on the minimum cutting level for Q_4. □

Lemma 3.4. An upper bound on the minimum cutting level for a δ^*-transitive reciprocal relation Q_5 on a set $\{x_0, x_1, \ldots, x_4\}$ is given by $\bar{\alpha}_5 \approx 0.6057$.

Proof. We will use the shorthands $t_i = Q_5(x_i, x_{(i+1) \bmod 5})$ and $s_i = Q_5(x_i, x_{(i+2) \bmod 5})$ for $i \in \{0,1,\ldots,4\}$.

We first assume that $t_0 = t_1 = \ldots = t_4 = t > 0.5$. In order for Q_5 to be δ^*-transitive, it is necessary that $s_0 \geq \delta^*(t,t)$ and that $1 - s_3 \geq \delta^*(s_0, t)$, implying $1 - s_3 \geq \delta^*(\delta^*(t,t),t)$ as δ^* is increasing. Moreover, it should hold that $s_3 \geq \delta^*(t,t)$, implying the condition $\delta^*(\delta^*(t,t),t) \leq 1 - \delta^*(t,t)$. Due to Lemma 3.2, it thus should hold that $\gamma(\gamma(t,t),t) \leq 1 - \gamma(t,t)$, yielding an upper bound $\bar{\alpha}_5$ on t, with $\bar{\alpha}_5 \approx 0.6057$ the root of $\gamma(\gamma(t,t),t) = 1 - \gamma(t,t)$ in the interval $[0.5, 1]$. Furthermore, it is easily shown that Q_5 is δ^*-transitive when $t = \bar{\alpha}_5$.

We will now show that Q_5 is not δ^*-transitive in the case where $t_i > t = \bar{\alpha}_5$ for every $i \in \{0, 1, \ldots, 4\}$. Assume to the contrary that Q_5 is δ^*-transitive. This implies that $s_0 \geq \delta^*(t_0, t_1)$. From $t_i > t$ for every $i \in \{0, 1, \ldots, 4\}$, it follows using Lemma 3.1 that

$$s_0 \geq \delta^*(t_0, t_1) > \delta^*(t,t) = \delta^*(\bar{\alpha}_5, \bar{\alpha}_5)\,,$$

and, again using Lemma 3.1, that

$$1 - s_3 \geq \delta^*(s_0, t_2) > \delta^*(\delta^*(\bar{\alpha}_5, \bar{\alpha}_5), \bar{\alpha}_5)\,,$$

and, due to the reciprocity of Q_5, that

$$s_3 < 1 - \delta^*(\delta^*(\bar{\alpha}_5, \bar{\alpha}_5), \bar{\alpha}_5)\,.$$

However, δ^*-transitivity of Q_5 implies that

$$s_3 \geq \delta^*(t_3, t_4) > \delta^*(t,t) = \delta^*(\bar{\alpha}_5, \bar{\alpha}_5)\,,$$

a contradiction as

$$\delta^*(\delta^*(\bar{\alpha}_5, \bar{\alpha}_5), \bar{\alpha}_5) = 1 - \delta^*(\bar{\alpha}_5, \bar{\alpha}_5)\,.$$

As the minimum cutting level can only be determined by the minimum of the t_i's, no additional cases need to be considered. Therefore, $\bar{\alpha}_5 \approx 0.6057$ is an upper bound on the minimum cutting level for Q_5. □

The upper bounds $\bar{\alpha}_m$ for the minimum cutting levels α_m with $m \geq 6$ can be obtained in a similar way. In Table 1 the upper bounds $\bar{\alpha}_m$ are shown for $m \in \{1, 2, \ldots, 13\}$.

Finally, it should be mentioned that it is known from the work of Yu [23] that the strict ρ-cut of any mutual rank probability relation at the value

$$\rho = \frac{1 + (\sqrt{2} - 1)\sqrt{2\sqrt{2} - 1}}{2} \approx 0.7800$$

yields a crisp relation that is transitive, and thus is obviously free of m-cycles for any $m > 0$. Therefore, it must hold that

$$\lim_{m \to \infty} \alpha_m \leq \rho\,.$$

m	$\bar{\alpha}_m$
3	0.5556
4	0.5858
5	0.6057
6	0.6201
7	0.6312
8	0.6400
9	0.6473
10	0.6535
11	0.6587
12	0.6632
13	0.6672

Tab. 1. The upper bounds $\bar{\alpha}_m$ on the minimum cutting levels α_m for $m \in \{1, 2, \ldots, 13\}$.

4. MINIMUM CUTTING LEVELS FOR POSETS OF SIZE $n \leq 13$

Note that the theoretical considerations on the minimum cutting levels from the previous section concern all δ^*-transitive reciprocal relations, a superset of the set of mutual rank probability relations M_P of all finite posets. It can therefore be expected that the given bounds are not tight when we restrict to posets of some given size. In this section we will experimentally compute the exact minimum cutting levels for posets of size $n \leq 13$.

The present authors have shown in [8] that the mutual rank probability relation can be computed using the lattice of ideals representation of a poset, without necessitating the enumeration of all linear extensions. Although this approach requires additional memory for storing the lattice of ideals, due to the fact that the size of this lattice for small posets remains limited, it is ideally suited for obtaining the mutual rank probability relation of posets of size $n \leq 13$ quickly. A combination of the poset generation algorithm of Brinkmann and McKay [3] and the algorithm to compute the mutual rank probability relation for a given poset enabled us to obtain exact counts for posets of size $n \leq 13$ [7].

We adapted this algorithm to keep track of the minimum cutting levels avoiding m-cycles in any poset of size n, which we will denote as α_m^n. In Table 2 these minimum cutting levels α_m^n are shown. Note that since no posets of size $n \leq 13$ exist with 8-cycles, the minimum cutting level for $m = 8$ is 0.5.

In order to verify the correctness of the implementation of the algorithms used, for posets of size $n \leq 9$ all mutual rank probability relations were compared with the results obtained by an independent approach based on enumerating all linear extensions for each poset by means of the Varol–Rotem algorithm [22]. As an additional verification of the implementation of the algorithms, the mutual rank probabilities of all posets requiring the highest cutting levels shown in Table 2 have been verified using an implementation of an algorithm of Pruesse et al. [21] based on the generation of all linear extensions of a poset.

Since one can trivially construct a poset of size $n + 1$ from a poset of size n with an

$n\backslash m$	3	4	5	6	7
9	**0.5031**	0.5	0.5	0.5	0.5
10	**0.5040**	**0.5028**	0.5	0.5	0.5
11	**0.5062**	0.5028	0.5	0.5	0.5
12	**0.5074**	**0.5087**	**0.5004**	**0.5024**	0.5
13	**0.5089**	0.5087	**0.5029**	**0.5025**	**0.5002**

Tab. 2. Minimum cutting level α_m^n to avoid m-cycles in posets of size $n = 9, \ldots, 13$ for $m = 3, \ldots, 7$.

equal minimum cutting level by adding an element that is either smaller than, larger than or incomparable to the given n elements, the minimum cutting levels α_m^n are increasing in n. In Table 2 one can observe that for $n = 11$ no higher cutting level for avoiding 4-cycles is found than for $n = 10$ since $\alpha_4^{11} = \alpha_4^{10}$, and similarly it is found that $\alpha_4^{13} = \alpha_4^{12}$. Further, note that a cutting level $\alpha_m^n = 0.5$ indicates that no LEM cycles of length m are possible in n-element posets.

In Figures 6 – 19 the posets requiring the non-trivial minimum cutting levels indicated in boldface in Table 2 are depicted by their Hasse diagrams. Note that the dual of a poset has an equal minimum cutting level, and is therefore not shown. However, four depicted posets are identical to their dual posets (Figures 6, 10, 11 and 19). We also mention that some posets have multiple LEM cycles with an identical cutting level, while others have LEM cycles of different lengths. The 9-element poset in Figure 6, for example, has three 3-cycles with identical probabilities, while for the 12-element poset in Figure 10, aside from the cycle with length 3, a 4-cycle is present, since it holds that

$$p(5 > 7) = p(7 > 8) = \frac{6184}{12244}\,.$$

The 12-element poset in Figure 12 has cycles of length 3, 4, 5 and 6. The poset in Figure 19 even has cycles of length 3, 4, 5, 6 and 7. Furthermore, the poset in Figure 13 also has a 3-cycle, the poset in Figure 16 has cycles of length 3 and 4, and the posets in Figures 17 and 18 both have 3-cycles. For some minimum cutting levels multiple posets, aside from their dual versions, are found. This is the case for the posets of size 13 in Figures 14 and 15 which attain the minimum cutting level for 3-cycles. The same is true for 6-cycles in Figures 17 and 18.

One of the aims of this experiment was to find common properties for posets with LEM cycles or to see a common structure emerging in the posets requiring the minimum cutting level. Indeed, if some common properties are found it might be possible to confine the search space to one or more subclasses of posets, or at least to rule out several hopefully large enough subclasses. By doing so, one could hope to take a step further and to find all posets with LEM cycles for $n = 14$ or maybe even $n = 15$. However, to our surprise, the posets have little in common. Because there are no common (sub)structures, even for posets of small size, it is not clear whether a small number of large subclasses to be ruled out can be identified. However, the symmetric and relatively simple structure of the 12-element poset in Figure 11 requiring the minimum cutting level

α_4 will inspire us in the next section to generalize it and to find a lower bound for α_4 for increasing poset size. Note that it might be possible to derive even better lower bounds for α_4 from other posets yet to be identified.

5. A LOWER BOUND FOR α_4

Consider in Figure 1 a generalization of the poset in Figure 11 requiring the minimum cutting level α_4^{12} to avoid cycles of length 4 in posets with 12 elements. Note that the parameters p and q represent numbers of elements and that setting $p = 1$ and $q = 0$ yields the original poset in Figure 11.

We observed that by increasing q, the minimum cutting level to avoid cycles of length 4 increases as well. The same observation was made for increasing p. Moreover, although we will not prove it here, there is exactly one cycle of length 4, consisting of the elements $\omega_1, \omega_2, \omega_3$ and ω_4, requiring this cutting level. For the purpose of this paper, a proof is not needed. A minimum cutting level that avoids this cycle is a lower bound for the minimum cutting level. Due to symmetry it holds that $\mathrm{p}(\omega_2 > \omega_1) = \mathrm{p}(\omega_4 > \omega_3)$ and $\mathrm{p}(\omega_1 > \omega_4) = \mathrm{p}(\omega_3 > \omega_2)$. If we denote the strict order relation of the dual poset as $>^\top$, it holds that $\mathrm{p}(\omega_2 > \omega_1) = \mathrm{p}(\omega_4 >^\top \omega_1)$ and therefore that $\mathrm{p}(\omega_2 > \omega_1) = \mathrm{p}(\omega_1 > \omega_4)$. The probabilities in the cycle are thus identical, i. e. $\mathrm{p}(\omega_2 > \omega_1) = \mathrm{p}(\omega_1 > \omega_4) = \mathrm{p}(\omega_4 > \omega_3) = \mathrm{p}(\omega_3 > \omega_2)$. It would be interesting to derive an analytical expression for e. g. $\mathrm{p}(\omega_2 > \omega_1)$ as a function of p and q, since it would yield minimum cutting levels for avoiding cycles of length 4 in a family of posets which seems promising for obtaining a lower bound.

As a first step, we count the number of linear extensions of the poset, as this number will be the denominator of the rational value of $\mathrm{p}(\omega_2 > \omega_1)$. Note that the four indicated elements a, b, c and d in Figure 1 can appear in four different orders in a linear extension:

$$\begin{aligned} &a < b < c < d\,, \\ &a < b < d < c\,, \\ &b < a < c < d\,, \\ &b < a < d < c\,. \end{aligned}$$

Again due to symmetry, the number of linear extensions where $a < b$ is identical to that where $b < a$. We can therefore restrict to the orders $a < b < c < d$ and $a < b < d < c$ and multiply the expression found by 2 in order to obtain the total number of linear extensions.

Since we will often need the number of linear extensions of the poset consisting of two disjoint chains with lengths i and j with no comparabilities between them, in what follows we will denote this number as $\kappa(i, j)$, i. e. we define

$$\kappa(i, j) = \binom{i+j}{j}.$$

The total number of linear extensions $N(p, q)$ can now be written as the following summation

$$\sum_{i_1=0}^{q} \kappa(i_1, q) \sum_{i_2=0}^{2p+1} \kappa(q - i_1, i_2) \left[N^{cd}(p, q, i_1, i_2) + N^{dc}(p, q, i_1, i_2)\right]. \tag{4}$$

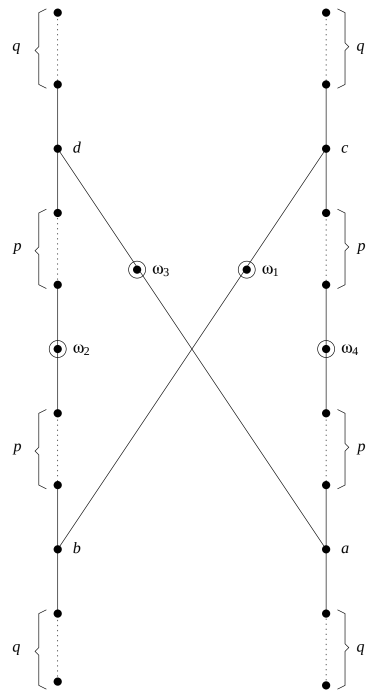


Fig. 1. Generalization of the poset in Figure 11 requiring the minimum cutting level α_4^{12} to avoid cycles of length $l \leq 4$ in posets with 12 elements.

Note that the index i_1 represents the number of elements below b that are also below a in the linear extension, and that the index i_2 represents the elements of the chain of length $2p+1$ between a and c that are below b in the linear extension. The functions N^{cd} and N^{dc} cover the case that $c < d$ and $c > d$, respectively, and are defined as follows:

$$N^{cd}(p,q,i_1,i_2) = \sum_{i_3=0}^{2p+1} \kappa(i_3, 2p+1-i_2) \sum_{i_4=0}^{q} n^{cd}\kappa(2p+1-i_3, i_4)\kappa(q, q-i_4)\,,$$

$$N^{dc}(p,q,i_1,i_2) = \sum_{i_3=0}^{2p+1-i_2} \kappa(2p+1, i_3) \sum_{i_4=0}^{q} n^{dc}\kappa(i_4, 2p+1-i_2-i_3)\kappa(q-i_4, q)\,.$$

The factors n^{cd} and n^{dc} will be defined below and count the number of ways ω_1 and ω_3 can be placed in the linear extensions. In the case $c < d$, i. e. in the function N^{cd}, the index i_3 represents the number of elements from the chain between b and d of length $2p+1$ that are below c in the linear extension. The index i_4 represents the number of elements above c that are below d in the linear extension. In the case $d < c$, i. e. in the function N^{dc}, the index i_3 represents the number of elements in the chain of length $2p+1$ between a and c that are above b and below d in the linear extension. The index i_4 represents the number of elements above d that are below c in the linear extension.

We denote the minimal number of elements in the poset that are smaller than ω_3 in all linear extensions under consideration as $\underline{\omega}_3$. In other words, $\underline{\omega}_3$ is a lower bound for the position of ω_3. Similarly, we denote the upper bound on the position of ω_3 when $c < d$ as $\overline{\omega}_3^{cd}$ and the upper bound on the position of ω_3 when $d < c$ as $\overline{\omega}_3^{dc}$. These bounds are given as follows:

$$\begin{aligned}\underline{\omega}_3 &= q + i_1 + 1\,,\\ \overline{\omega}_3^{cd} &= 4p + 2q + i_4 + 5\,,\\ \overline{\omega}_3^{dc} &= 2p + 2q + i_2 + i_3 + 3\,.\end{aligned}$$

Analogously, we obtain a lower bound $\underline{\omega}_1$ and two upper bounds $\overline{\omega}_1^{cd}$ and $\overline{\omega}_1^{dc}$ on the position of ω_1,

$$\begin{aligned}\underline{\omega}_1 &= 2q + i_2 + 2\,,\\ \overline{\omega}_1^{cd} &= 2p + 2q + i_3 + 3\,,\\ \overline{\omega}_1^{dc} &= 4p + 2q + i_4 + 5\,.\end{aligned}$$

Consider the case $c < d$. The element ω_1 can be freely inserted between positions $\underline{\omega}_1$ and $\overline{\omega}_1^{cd}$, and similarly, the element ω_3 can be inserted between $\underline{\omega}_3$ and $\overline{\omega}_3^{cd}$. However, as can be seen in Figure 2, between $\underline{\omega}_1$ and $\overline{\omega}_1^{cd}$ an additional position for ω_3 appears due to the insertion of ω_1. In order to account for all possible positions of the two elements ω_1 and ω_3 in all linear extensions, we therefore have to add the term $\overline{\omega}_1^{cd} - \underline{\omega}_1 + 1$ as to obtain

$$n^{cd} = (\overline{\omega}_1^{cd} - \underline{\omega}_1 + 1)(\overline{\omega}_3^{cd} - \underline{\omega}_3 + 1) + \overline{\omega}_1^{cd} - \underline{\omega}_1 + 1\,. \tag{5}$$

When $d < c$ a similar argument holds as shown in Figure 3:

$$n^{dc} = (\overline{\omega}_1^{dc} - \underline{\omega}_1 + 1)(\overline{\omega}_3^{dc} - \underline{\omega}_3 + 1) + \overline{\omega}_3^{dc} - \underline{\omega}_1 + 1\,. \tag{6}$$

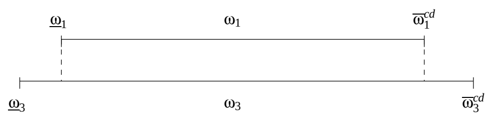


Fig. 2. Lower and upper bounds on the positions of ω_1 and ω_3 when $c < d$.

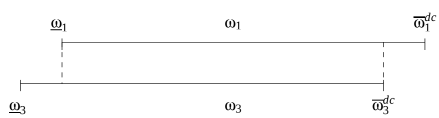


Fig. 3. Lower and upper bounds on the positions of ω_1 and ω_3 when $d < c$.

After simplifying expressions (5) and (6), we obtain

$$\begin{aligned} n^{cd} &= (2p - i_2 + i_3 + 2)(4p + q - i_1 + i_4 + 6)\,, \\ n^{dc} &= (4p - i_2 + i_4 + 4)(2p + q - i_1 + i_2 + i_3 + 3) + (2p + i_3 + 2)\,. \end{aligned}$$

By noting that

$$\binom{j+i}{i} = (-1)^i \binom{-j-1}{i}\,, \tag{7}$$

it can be easily proven that the equality

$$\sum_{i_1=0}^{q} \kappa(q, i_1)\kappa(i_2, q - i_1) = \kappa(q + i_2 + 1, q) \tag{8}$$

holds, where we have used the well-known Chu–Vandermonde identity (see e. g. page 44 of [4]), valid for $r, s \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\sum_{m=0}^{n} \binom{r}{m}\binom{s}{n-m} = \binom{r+s}{n}\,.$$

Analogously, it can be proven that the following equality holds

$$\sum_{i_1=0}^{q} (q - i_1 + i_2 + 1)\kappa(q, i_1)\kappa(i_2, q - i_1) = (i_2 + 1)\kappa(q + i_2 + 2, q)\,, \tag{9}$$

such that if we define $t = 2p + 1$, it is easily verified that $N(p, q)$ in expression (4) can be rewritten as

$$\begin{aligned} &\sum_{i_2=0}^{t}\sum_{i_3=0}^{t} \kappa(t - i_2, i_3) \sum_{i_4=0}^{q} \kappa(i_4, t - i_3)\kappa(q - i_4, q) \\ &\qquad \cdot \big[(t - i_2 + i_3 + 1)(i_2 + 1)\kappa(q + i_2 + 2, q) \\ &\qquad + (t - i_2 + i_3 + 1)(2t - i_2 + i_4 + 3)\kappa(q + i_2 + 1, q)\big] \\ &+ \sum_{i_2=0}^{t}\sum_{i_3=0}^{t-i_2} \kappa(t, i_3) \sum_{i_4=0}^{q} \kappa(i_4, t - i_2 - i_3)\kappa(q - i_4, q) \\ &\qquad \cdot \big[(2t - i_2 + i_4 + 2)(i_2 + 1)\kappa(q + i_2 + 2, q) \\ &\qquad + (2t - i_2 + i_4 + 3)(t + i_3 + 1)\kappa(q + i_2 + 1, q)\big]\,. \end{aligned} \tag{10}$$

In analogy to equalities (8) and (9), as q in the first argument of the first function κ in (8) could have been any arbitrary number, it is found that

$$\begin{aligned} \sum_{i_4=0}^{q} \kappa(i_4, t - i_3)\kappa(q - i_4, q) &= \sum_{i_4=0}^{q} \kappa(q - i_4, t - i_3)\kappa(i_4, q) \\ &= \kappa(t + q - i_3 + 1, q)\,, \end{aligned}$$

and, by replacing i_3 with i_2+i_3,

$$\sum_{i_4=0}^{q}\kappa(i_4,t-i_2-i_3)\kappa(q-i_4,q)=\sum_{i_4=0}^{q}\kappa(q-i_4,t-i_2-i_3)\kappa(i_4,q)$$
$$=\kappa(t+q-i_2-i_3+1,q)\,.$$

Moreover, by taking $i_2=t-i_3$ in (9) and by reindexing $i_4=q-i_1$,

$$\sum_{i_4=0}^{q}(t-i_3+i_4+1)\kappa(i_4,t-i_3)\kappa(q-i_4,q)$$
$$=\sum_{i_4=0}^{q}(t+q-i_3-i_4+1)\kappa(q-i_4,t-i_3)\kappa(i_4,q)$$
$$=(t-i_3+1)\kappa(t+q-i_3+2,q)$$

and, by replacing i_3 with i_2+i_3,

$$\sum_{i_4=0}^{q}(t-i_2-i_3+i_4+1)\kappa(i_4,t-i_2-i_3)\kappa(q-i_4,q)$$
$$=\sum_{i_4=0}^{q}(t+q-i_2-i_3-i_4+1)\kappa(q-i_4,t-i_2-i_3)\kappa(i_4,q)$$
$$=(t-i_2-i_3+1)\kappa(t+q-i_2-i_3+2,q)\,,$$

such that expression (10) simplifies to

$$\sum_{i_2=0}^{t}\sum_{i_3=0}^{t}\kappa(t-i_2,i_3)$$
$$\cdot\,[(t-i_2+i_3+1)(i_2+1)\kappa(t+q-i_3+1,q)\kappa(q+i_2+2,q)$$
$$+\;(t-i_2+i_3+1)(t-i_3+1)\kappa(t+q-i_3+2,q)\kappa(q+i_2+1,q)$$
$$+\;(t-i_2+i_3+1)(t-i_2+i_3+2)\kappa(t+q-i_3+1,q)\kappa(q+i_2+1,q)]$$

$$+\sum_{i_2=0}^{t}\sum_{i_3=0}^{t-i_2}\kappa(t,i_3)$$
$$\cdot\,[(t-i_2-i_3+1)(i_2+1)\kappa(t+q-i_2-i_3+2,q)\kappa(q+i_2+2,q)$$
$$+\;(t+i_3+1)(i_2+1)\kappa(t+q-i_2-i_3+1,q)\kappa(q+i_2+2,q)$$
$$+\;(t+i_3+1)(t-i_2-i_3+1)\kappa(t+q-i_2-i_3+2,q)\kappa(q+i_2+1,q)$$
$$+\;(t+i_3+1)(t+i_3+2)\kappa(t+q-i_2-i_3+1,q)\kappa(q+i_2+1,q)]\,. \qquad (11)$$

As a next step we calculate the number of linear extensions where $\omega_2 < \omega_1$, which is identical to the number of linear extensions of the poset in Figure 4. We will use an analogous technique, but since symmetry is lost in this case, it is necessary to consider both cases $a < \omega_2$ and $a > \omega_2$.

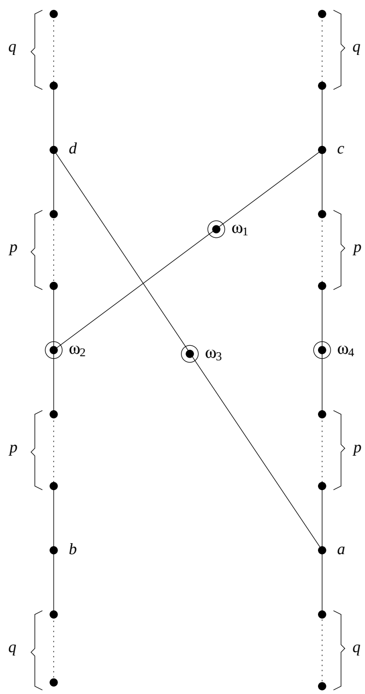


Fig. 4. Addition of the ordered pair $\omega_2 < \omega_1$ to the poset in Figure 1.

The number of linear extensions where $a < \omega_2$, denoted as $M_{a\omega_2}(p,q)$, is given by

$$\sum_{i_1=0}^{q+p+1} \kappa(i_1,q) \sum_{i_2=0}^{2p+1} \kappa(p+q+1-i_1,i_2) \left[M^{cd}_{a\omega_2}(p,q,i_1,i_2)+M^{dc}_{a\omega_2}(p,q,i_1,i_2)\right] . \qquad (12)$$

Note that the index i_1 represents the number of elements below ω_2 that are below a in the linear extension. The index i_2 represents the number of elements in the chain of length $2p+1$ between a and c that are below ω_2 in the linear extension. The two functions $M^{cd}_{a\omega_2}$ and $M^{dc}_{a\omega_2}$ cover the case that $c < d$ and $c > d$, respectively:

$$M^{cd}_{a\omega_2}(p,q,i_1,i_2) = \sum_{i_3=0}^{p} \kappa(i_3, 2p+1-i_2) \sum_{i_4=0}^{q} m^{cd}_{a\omega_2} \kappa(p-i_3,i_4)\kappa(q,q-i_4)\,,$$

$$M^{dc}_{a\omega_2}(p,q,i_1,i_2) = \sum_{i_3=0}^{2p+1-i_2} \kappa(p,i_3) \sum_{i_4=0}^{q} m^{dc}_{a\omega_2} \kappa(i_4, 2p+1-i_2-i_3)\kappa(q-i_4,q)\,.$$

The factors $m^{cd}_{a\omega_2}$ and $m^{dc}_{a\omega_2}$ will be defined below and count the number of ways ω_1 and ω_3 can be placed in the linear extension. In the case $c < d$, i. e. in the function $M^{cd}_{a\omega_2}$, the index i_3 represents the number of elements in the chain of length p between ω_2 and d that are below c in the linear extension. The index i_4 represents the number of elements above c that are below d in the linear extension. In the case $c > d$, i. e. in the function $M^{dc}_{a\omega_2}$, the index i_3 represents the number of elements in the chain of length p between ω_4 and c that are below d in a linear extension. The index i_4 represents the number of elements above d that are below c in the linear extension.

The bounds on ω_1 and ω_3 are as follows

$$\begin{aligned}
\underline{\omega}_3 &= q + i_1 + 1\,,\\
\overline{\omega}_3^{cd} &= 4p + 2q + i_4 + 5\,,\\
\overline{\omega}_3^{dc} &= 2p + 2q + i_2 + i_3 + 3\,,\\
\underline{\omega}_1 &= p + 2q + i_2 + 3\,,\\
\overline{\omega}_1^{cd} &= 3p + 2q + i_3 + 4\,,\\
\overline{\omega}_1^{dc} &= 4p + 2q + i_4 + 5\,,
\end{aligned}$$

while

$$\begin{aligned}
m^{cd}_{a\omega_2} &= (\overline{\omega}_1^{cd} - \underline{\omega}_1 + 1)(\overline{\omega}_3^{cd} - \underline{\omega}_3 + 1) + \overline{\omega}_1^{cd} - \underline{\omega}_1 + 1\,,\\
m^{dc}_{a\omega_2} &= (\overline{\omega}_1^{dc} - \underline{\omega}_1 + 1)(\overline{\omega}_3^{dc} - \underline{\omega}_3 + 1) + \overline{\omega}_3^{dc} - \underline{\omega}_1 + 1\,.
\end{aligned}$$

After simplification, we obtain $M_{a\omega_2}(p,q) =$

$$\begin{aligned}
&\sum_{i_2=0}^{2p+1}\sum_{i_3=0}^{p}\kappa(i_3, 2p+1-i_2)\,[(2p+i_3-i_2+2)\\
&\quad\cdot\{(i_2+1)\kappa(p+q-i_3+1,q)\kappa(p+q+1,q+i_2+2)\\
&\quad+\ (2p-i_2+i_3+3)\kappa(p+q+1,q+i_2+1)\kappa(p+q-i_3+1,q)\\
&\quad+\ (p-i_3+1)\kappa(q+i_2+1,p+q+1)\kappa(p+q-i_3+2,q)\}]
\end{aligned}$$

$$\begin{aligned}
&+\sum_{i_2=0}^{2p+1}\sum_{i_3=0}^{2p+1-i_2}\kappa(i_3,p)\,[(p+i_3+1)\\
&\quad\cdot\{(2p-i_2-i_3+2)\kappa(q+i_2+1,p+q+1)\kappa(2p+q-i_2-i_3+3,q)\\
&\quad+\ (p+i_3+2)\kappa(q+i_2+1,p+q+1)\kappa(2p+q-i_2-i_3+2,q)\}\\
&\quad+(i_2+1)\\
&\quad\cdot\{(p+i_3+1)\kappa(q+i_2+2,p+q+1)\kappa(2p+q-i_2-i_3+2,q)\\
&\quad+\ (2p-i_2-i_3+2)\kappa(q+i_2+2,p+q+1)\kappa(2p+q-i_2-i_3+3,q)\}]\,. \qquad (13)
\end{aligned}$$

The number of linear extensions where $a > \omega_2$, denoted as $M_{\omega_2 a}(p,q)$, is given by

$$\sum_{i_1=0}^{q}\kappa(p+q+1,i_1)\sum_{i_2=0}^{p}\kappa(i_2,q-i_1)\left[M^{cd}_{\omega_2 a}(p,q,i_1,i_2)+M^{dc}_{\omega_2 a}(p,q,i_1,i_2)\right]\,. \qquad (14)$$

Note that the index i_1 represents the number of elements below a that are below ω_2 in the linear extension. The index i_2 represents the number of elements in the chain of length p between ω_2 and d that are below a in the linear extension. The two functions $M^{cd}_{\omega_2 a}$ and $M^{dc}_{\omega_2 a}$ cover the case that $c < d$ and $c > d$, respectively:

$$M^{cd}_{\omega_2 a}(p,q,i_1,i_2) = \sum_{i_3=0}^{p-i_2} \kappa(i_3, 2p+1) \sum_{i_4=0}^{q} m^{cd}_{\omega_2 a} \kappa(p-i_2-i_3, i_4)\kappa(q, q-i_4)\,,$$
$$M^{dc}_{\omega_2 a}(p,q,i_1,i_2) = \sum_{i_3=0}^{2p+1} \kappa(p-i_2, i_3) \sum_{i_4=0}^{q} m^{dc}_{\omega_2 a} \kappa(i_4, 2p-i_3+1)\kappa(q-i_4, q)\,.$$

The factors $m^{cd}_{\omega_2 a}$ and $m^{dc}_{\omega_2 a}$ will be defined below and count the number of ways ω_1 and ω_3 can be placed in the linear extension. In the case $c < d$, i. e. in the function $M^{cd}_{\omega_2 a}$, the index i_3 represents the number of elements in the chain between ω_2 and d that are below c in the linear extension. The index i_4 represents the number of elements above c that are below d in the linear extension. In the case $c > d$, i. e. in the function $M^{dc}_{\omega_2 a}$, the index i_3 represents the number of elements in the chain of length $2p+1$ between a and c that are below d in a linear extension. The index i_4 represents the number of elements above d that are below c in the linear extension.
The bounds on ω_1 and ω_3 are as follows

$$\begin{aligned}
\underline{\omega}_1 &= p+q+i_1+2\,,\\
\overline{\omega}_1^{cd} &= 3p+2q+i_2+i_3+4\,,\\
\overline{\omega}_1^{dc} &= 4p+2q+i_4+5\,,\\
\underline{\omega}_3 &= p+2q+i_2+3\,,\\
\overline{\omega}_3^{cd} &= 4p+2q+i_4+5\,,\\
\overline{\omega}_3^{dc} &= 2p+2q+i_3+3\,,
\end{aligned}$$

while
$$\begin{aligned}
m^{cd}_{\omega_2 a} &= (\overline{\omega}_1^{cd} - \underline{\omega}_1 + 1)(\overline{\omega}_3^{cd} - \underline{\omega}_3 + 1) + \overline{\omega}_1^{cd} - \underline{\omega}_1 + 1\,,\\
m^{dc}_{\omega_2 a} &= (\overline{\omega}_1^{dc} - \underline{\omega}_1 + 1)(\overline{\omega}_3^{dc} - \underline{\omega}_3 + 1) + \overline{\omega}_3^{dc} - \underline{\omega}_1 + 1\,.
\end{aligned}$$

After simplification, we obtain $M_{\omega_2 a}(p,q) =$

$$\begin{aligned}
&\sum_{i_2=0}^{p}\sum_{i_3=0}^{2p+1} \kappa(p-i_2, i_3)\,[(p-i_2+i_3+1)\\
&\quad \cdot \{(2p-i_3+2)\kappa(p+q+i_2+2,q)\kappa(2p+q-i_3+3,q)\\
&\quad + (p-i_2+i_3+2)\kappa(p+q+i_2+2,q)\kappa(2p+q-i_3+2,q)\\
&\quad + (i_2+1)\kappa(p+q+i_2+3,q)\kappa(2p+q-i_3+2,q)\}]\\
&+\sum_{i_2=0}^{p}\sum_{i_3=0}^{p-i_2} \kappa(2p+1, i_3)\,[(2p+i_3+2)\\
&\quad \cdot \{(p-i_2-i_3+1)\kappa(p+q-i_2-i_3+2,q)\kappa(p+q+i_2+2,q)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \ (2p+i_3+3)\kappa(p+q-i_2-i_3+1,q)\kappa(p+q+i_2+2,q)\} \\
&+(i_2+1)\cdot \\
&\quad \{(p-i_2-i_3+1)\kappa(p+q-i_2-i_3+2,q)\kappa(p+q+i_2+3,q) \\
&\quad + \ (2p+i_3+2)\kappa(p+q-i_2-i_3+1,q)\kappa(p+q+i_2+3,q)\}] \ . \qquad (15)
\end{aligned}
$$

It is clear that for arbitrary p and q, the mutual rank probability $\mathrm{p}(\omega_2 > \omega_1)$ is given by the expression

$$\mathrm{p}(\omega_2 > \omega_1) = 1 - \frac{M_{a\omega_2}(p,q) + M_{\omega_2 a}(p,q)}{2N(p,q)} \, . \qquad (16)$$

We will now consider the case where $q \to \infty$. We remark that for the functions in expressions (11), (13) and (15) having the form

$$\kappa(q+i,q+j) = \binom{2q+i+j}{q+j} = \frac{(2q+i+j)!}{(q+i)! \cdot (q+j)!}$$

Stirling's approximation can be used, i. e.

$$f(q)! \approx \sqrt{2\pi \cdot f(q)} \cdot f(q)^{f(q)} \cdot e^{-f(q)} \qquad \text{when } q \to \infty \, ,$$

leading to

$$\kappa(q+i,q+j) \approx \frac{2^{2q+i+j}}{\sqrt{\pi \cdot n}} \qquad \text{when } q \to \infty \, .$$

Due to the nature of the fraction in expression (16), it is equivalent to substitute

$$\kappa(q+i,q+j) \quad \text{by} \quad 2^{i+j} \, . \qquad (17)$$

It is now feasible to compute $\mathrm{p}(\omega_2 > \omega_1)$ for given p when $q \to \infty$. Some values are given in Table 3 and a plot is shown in Figure 5.

p	$\mathrm{p}(\omega_2 > \omega_1)$	
1	8/15	≈ 0.5333
20	419/731	≈ 0.5732
40	1051/1826	≈ 0.5756
60	1158/2009	≈ 0.5764
80	12223/21190	≈ 0.5768
100	3163/5481	≈ 0.5771
120	9071/15714	≈ 0.5773
140	18464/31979	≈ 0.5774
160	16041/27778	≈ 0.5775
180	10133/17545	≈ 0.5775
200	74953/129766	≈ 0.5776

Tab. 3. The mutual rank probabilities $\mathrm{p}(\omega_2 > \omega_1)$ for given p when $q \to \infty$.

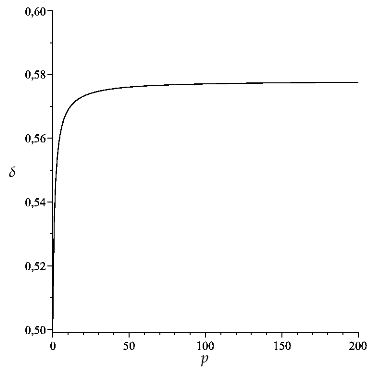
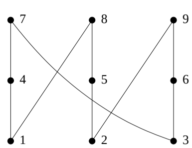


Fig. 5. A plot of the mutual rank probabilities $p(\omega_2 > \omega_1)$ for $q \to \infty$.

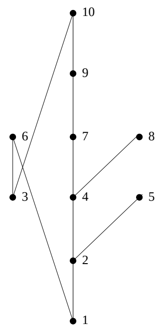
As can be seen, the minimum cutting level quickly increases for increasing values of p, but soon the rate at which the function increases diminishes to attain values slightly below 0.58. Recall that the upper bound $\bar{\alpha}_4$ on α_4 is approximately 0.5858, such that we obtain a quite narrow interval for the minimum cutting level α_4. It comes as no surprise that we do not attain $\bar{\alpha}_4$ since, as already mentioned, this upper bound is obtained by using δ^*-transitivity exhibited by a more general setting in which mutual rank probability relations can be embedded. Moreover, it can be expected that posets with more than 12 elements that do not fall into this family of posets provide tighter lower bounds. Nevertheless, the narrow interval between lower and upper bound is an indication that δ^*-transitivity is situated quite closely to the transitivity exhibited by mutual rank probability relations.

6. APPENDIX: POSETS REQUIRING MINIMUM CUTTING LEVELS α_M^N



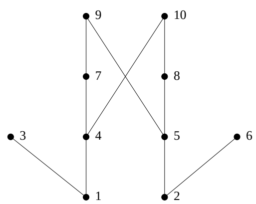
$$p(7 > 8) = p(8 > 9) = p(9 > 7) = \frac{720}{1431} \approx 0,5031$$
$$p(4 > 5) = p(5 > 6) = p(6 > 4) = \frac{720}{1431} \approx 0,5031$$
$$p(1 > 2) = p(2 > 3) = p(3 > 1) = \frac{720}{1431} \approx 0,5031$$

Fig. 6. 9-element poset with a LEM cycle requiring the minimum cutting level α_3^9.



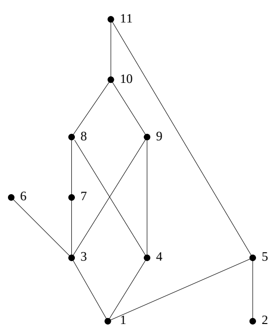
$$p(8 > 6) = p(6 > 9) = \frac{508}{1008} \approx 0,5040$$
$$p(9 > 8) = \frac{512}{1008}$$

Fig. 7. 10-element poset with a LEM cycle requiring the minimum cutting level α_3^{10}.



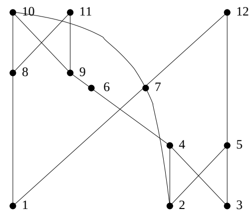
$$p(7 > 3) = p(3 > 8) = p(8 > 6) = p(6 > 7) = \frac{1765}{3510} \approx 0,5028$$

Fig. 8. 10-element poset with a LEM cycle requiring the minimum cutting level α_4^{10}.



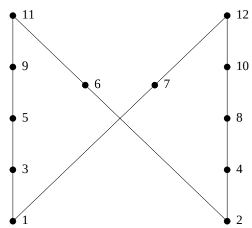
$$p(5 > 8) = \frac{1146}{2260}$$
$$p(8 > 6) = \frac{1144}{2260} \approx 0,5062$$
$$p(6 > 5) = \frac{1145}{2260}$$

Fig. 9. 11-element poset with a LEM cycle requiring the minimum cutting level α_3^{11}.



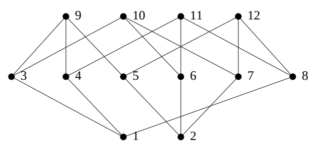
$$p(8 > 6) = p(6 > 5) = \frac{6214}{12244}$$
$$p(5 > 8) = \frac{6212}{12244} \approx 0,5074$$

Fig. 10. 12-element poset with a LEM cycle requiring the minimum cutting level α_3^{12}.



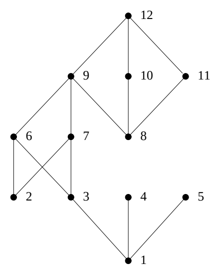
$$p(5 > 7) = p(7 > 8) = p(8 > 6) = p(6 > 5) = \frac{7396}{14540} \approx 0,5087$$

Fig. 11. 12-element poset with a LEM cycle requiring the minimum cutting level α_4^{12}.



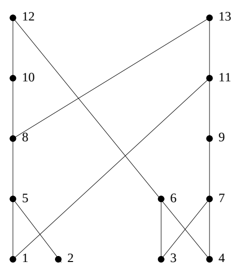
$$p(5 > 4) = p(4 > 3) = \frac{60400}{120640}$$
$$p(3 > 6) = p(6 > 8) = p(8 > 5) = \frac{60368}{120640} \approx 0,5004$$

Fig. 12. 12-element poset with a LEM cycle requiring the minimum cutting level α_5^{12}.



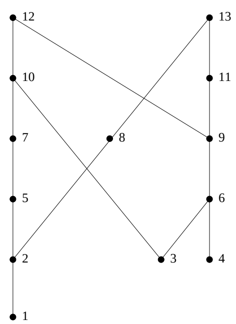
$$p(7 > 4) = p(6 > 5) = \frac{46392}{92336} \approx 0,5024$$
$$p(4 > 10) = p(5 > 11) = \frac{46560}{92336}$$
$$p(10 > 6) = p(11 > 7) = \frac{46850}{92336}$$

Fig. 13. 12-element poset with a LEM cycle requiring the minimum cutting level α_6^{12}.



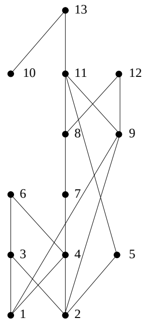
$$p(6 > 8) = \frac{12240}{24022}$$
$$p(8 > 9) = \frac{12262}{24022}$$
$$p(9 > 6) = \frac{12224}{24022} \approx 0,5089$$

Fig. 14. First 13-element poset with a LEM cycle requiring the minimum cutting level α_3^{13}.



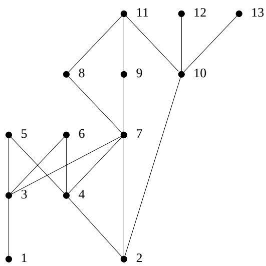
$$p(7 > 8) = \frac{6112}{12011} \approx 0,5089$$
$$p(8 > 9) = \frac{6120}{12011}$$
$$p(9 > 7) = \frac{6131}{12011}$$

Fig. 15. Second 13-element poset with a LEM cycle requiring the minimum cutting level α_3^{13}.



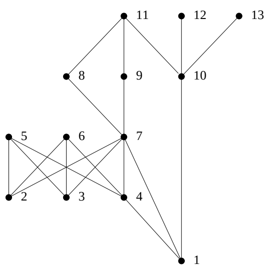
$$p(10 > 9) = \frac{33871}{67242}$$
$$p(9 > 5) = \frac{33916}{67242}$$
$$p(5 > 7) = \frac{33816}{67242} \approx 0,5029$$
$$p(7 > 3) = \frac{33834}{67242}$$
$$p(3 > 10) = \frac{34151}{67242}$$

Fig. 16. 13-element poset with a LEM cycle requiring the minimum cutting level α_5^{13}.



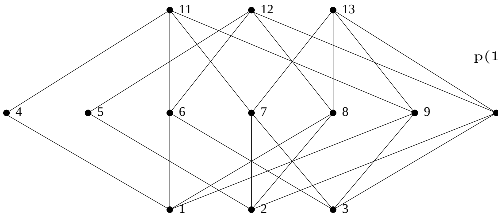
$$p(12 > 9) = p(13 > 8) = \frac{66354}{131472}$$
$$p(9 > 6) = p(8 > 5) = \frac{66060}{131472} \approx 0,5025$$
$$p(6 > 13) = p(5 > 12) = \frac{66306}{131472}$$

Fig. 17. First 13-element poset with a LEM cycle requiring the minimum cutting level α_6^{13}.



$$p(12 > 9) = p(13 > 8) = \frac{132708}{262944}$$
$$p(9 > 6) = p(8 > 5) = \frac{132120}{262944} \approx 0,5025$$
$$p(6 > 13) = p(5 > 12) = \frac{132612}{262944}$$

Fig. 18. Second 13-element poset with a LEM cycle requiring the minimum cutting level α_6^{13}.



$$p(11 > 6) = p(6 > 8) = p(4 > 11) = \frac{268352}{536510} \approx 0,5002$$
$$p(8 > 10) = p(5 > 4) = \frac{268384}{536510}$$
$$p(10 > 12) = p(12 > 5) = \frac{268465}{536510}$$

Fig. 19. 13-element poset with a LEM cycle requiring the minimum cutting level α_7^{13}.

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