A spectrum result on maximal partial ovoids of the generalized quadrangle $Q(4, q)$, q even

C. Rößing L. Storme

January 11, 2008

Address of the authors:

C. Rößing: School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland (roessing@maths.ucd.ie)

L. Storme: Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281-S22, 9000 Ghent, Belgium (ls@cage.ugent.be, http://cage.ugent.be/ ∼ ls)

Proposed running head: A spectrum result on maximal partial ovoids

Corresponding author: L. Storme: ls@cage.ugent.be

Abstract

This article presents a spectrum result on maximal partial ovoids of the generalized quadrangle $Q(4,q)$, q even. We prove that for every integer k in an interval of, roughly, size $[q^2/10, 9q^2/10]$, there exists a maximal partial ovoid of size k on $Q(4,q)$, q even. Since the generalized quadrangle $W(q)$, q even, defined by a symplectic polarity of $PG(3,q)$, is isomorphic to the generalized quadrangle $Q(4,q)$, q even, the same result is obtained for maximal partial ovoids of $W(q)$, q even. Since a maximal partial ovoid of $W(q)$, q even, is also a minimal blocking set with respect to the planes of $PG(3,q)$, the same spectrum result is obtained for minimal blocking sets with respect to planes of $PG(3,q)$, q even. Finally, since minimal blocking sets with respect to planes in $PG(3,q)$ are tangency sets, they define partial 1-systems on the Klein quadric $Q^+(5,q)$, so we get the same spectrum result for maximal partial 1-systems of lines on the Klein quadric $Q^+(5,q)$, q even.

Key Words: generalized quadrangles, maximal partial ovoids, minimal blocking sets, maximal partial 1-systems.

1 Introduction

An incidence structure consisting of points and lines is called a *finite generalized quadrangle* $GQ(s,t)$ if the following axioms hold:

- every line is incident with $s+1$ points, and every point is incident with $t+1$ lines,
- two different lines can intersect in at most one point, and two different points can share at most one line, and
- for any non-incident point-line pair (P,l), there exists a unique line m and unique point Q such that P is incident with m, m is incident with Q, and Q is incident with l.

The parameters s and t are called the *order* of the generalized quadrangle. The points and lines of a non-singular 4-dimensional parabolic quadric $Q(4,q)$ are a classical example of a finite generalized quadrangle of order

$(s,t) = (q,q)$. The parabolic quadric $Q(4,q)$ of $PG(4,q)$ is the quadric having $X_0^2 + X_1X_2 + X_3X_4 = 0$ as canonical equation.

The other examples of classical generalized quadrangles are: (1) the non-singular 5-dimensional elliptic quadrics $Q^-(5,q)$, (2) the Hermitian varieties $H(3,q^2)$ and $H(4,q^2)$ in three and four dimensions, and (3) the points of $PG(3,q)$ and the totally isotropic lines under the symplectic *polarity* φ.

We refer to the standard reference [12] for more information on generalized quadrangles.

An *ovoid* $\mathbb{O}$ of a generalized quadrangle is a set of points such that every line of the generalized quadrangle is incident with exactly one point of $\mathbb{O}$. A *partial ovoid* is a set of points that shares at most one point with every line of the generalized quadrangle, and the partial ovoid is called *maximal* when it is not contained in a larger partial ovoid.

Particular interest has been paid to the existence and non-existence of ovoids in generalized quadrangles [15, 16]. The results of Ebert and Hirschfeld [7] translate into results on the smallest maximal partial ovoids of $Q^-(5,q)$. The result of Aguglia, Ebert, and Luyckx [1] presents the minimal size of a maximal partial ovoid of $H(3,q^2)$. Recently, research has been done to find spectra of sizes of maximal partial ovoids [3, 4], by using computer resources. We contribute to this study with a spectrum result on maximal partial ovoids of $Q(4,q)$, for q even.

As applications, we obtain similar spectrum results on: (1) maximal partial ovoids of $W(q)$, q even, (2) maximal partial spreads of $Q(4,q)$, q even, and $W(q)$, q even, (3) minimal blocking sets w.r.t. the planes of $PG(3,q)$, q even, and (4) maximal partial 1-systems on the Klein quadric $Q^+(5,q)$, q even.

2 The idea

We will use the ideas for the construction of minimal blocking sets in $PG(2,q^2)$ in the article of Szőnyi *et al* [14] for finding a spectrum for maximal partial ovoids of $Q(4,q)$, q even. In particular, we will need the statement introduced by Füredi on page 190 in the article [8]:

Corollary 2.1 *For a bipartite graph with bipartition $L \cup U$ where the degree of the elements in U is at least d, there is a set $L' \subseteq L$, for which*

$|L'| \leq |L| \frac{1+\log(|U|)}{d}$, *such that any element* $u \in U$ *is adjacent to at least one element of* L' .

The following setting is useful for our purposes. In the next section, we will discuss it in detail. Now, we want to focus on the application of the above corollary in our context. We refer to Figure 1.

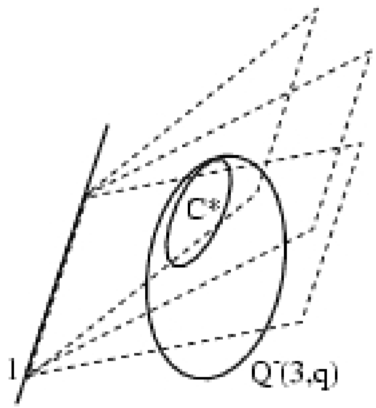

Figure 1: Conics of $Q^{-}(3,q)$ in planes through ℓ

Consider an elliptic quadric $Q^{-}(3,q)$ in $Q(4,q)$. Then $Q^{-}(3,q)$ is an ovoid of the generalized quadrangle $Q(4,q)$. Let ℓ be an external line to $Q^{-}(3,q)$, lying in the solid of $Q^{-}(3,q)$, and let C^{*} be one or possibly several conics on $Q^{-}(3,q)$, not lying in a plane through ℓ , intersected by the same planes through ℓ . Among the planes containing the line ℓ lying in the solid of $Q^{-}(3,q)$, there are two tangent planes to points R_1 and R_2 of $Q^{-}(3,q)$, and $q-1$ planes intersecting $Q^{-}(3,q)$ in a conic. Some of these planes intersect the conics C^{*} .

We are interested in the planes through ℓ intersecting the quadric $Q^{-}(3,q)$ in a conic. Among those planes, we choose $s-2$ planes out of which $r-1$ intersect the conics C^{*} . We now choose for U all conics of the quadric $Q^{-}(3,q)$; except for a small number of conics, in particular, those conics that lie in a plane containing ℓ . We isolate a particular group of $q+1$ conics passing through R_1 , but not through R_2 , intersected by the same $q/2+1$ conics in planes through ℓ . The $q/2$ conics of $Q^{-}(3,q)$ in planes through ℓ skew to this particular group of $q+1$ conics are the elements of L . An element of U is adjacent to an element of L when the two conics intersect in at least one point. Applying Corollary 2.1, we can reduce L to L' and still know that every conic in U intersects a conic of L'.

Then, in a first step, we can decrease the ovoid $Q^-(3,q)$ to a partial ovoid by omitting conics in planes through ℓ, but certainly not the conics in L', replacing those omitted conics by their polar points in $Q(4,q)$. Recall that q is even, thus the plane containing a conic also contains a point incident with all tangent lines to the conic, which is the nucleus of the conic.

The conics in C^* have to be intersected by the same planes containing ℓ. The following section will give the construction of these conics and show that we can replace them by their polar points without violating the properties of the partial ovoid constructed in the first step above.

3 Construction of maximal partial ovoids

Remark 3.1 A conic C of $Q(4,q)$, q even, has either one or $q+1$ *polar points* on $Q(4,q)$, i.e., there are either one or $q+1$ points of $Q(4,q)$ collinear with all $q+1$ points of C. A conic C of $Q(4,q)$ lying in a plane through the nucleus N of $Q(4,q)$ has $q+1$ polar points, while a conic C of $Q(4,q)$ lying in a plane, not passing through the nucleus N, has exactly one polar point.

A conic C contained in an elliptic quadric $Q^-(3,q)$ of $Q(4,q)$ only has one polar point. We want to replace a number of conics of the elliptic quadric $Q^-(3,q)$ by their polar point in order to get partial ovoids of different sizes. The aim is to do this in such a way that we get many different cardinalities for the maximal partial ovoids. Thus we want to be able to replace different numbers of conics, so we have to choose these conics in a way that their polar points are not collinear on $Q(4,q)$.

Let ℓ be an external line to $Q^-(3,q)$, then the polar line of ℓ w.r.t. $Q^-(3,q)$ is a bisecant intersecting $Q^-(3,q)$ in two points R_1 and R_2. The planes through R_1, R_2 intersect $Q^-(3,q)$ in a conic. The nuclei of these conics are the $q+1$ points on ℓ. The planes through ℓ consist of the tangent planes to $Q^-(3,q)$ in R_1 and R_2, and of $q-1$ planes which intersect $Q^-(3,q)$ in a conic $K^i, i=1,\ldots,q-1$. There is one polar point of $Q(4,q)$ collinear with the points of such a conic K^i, $i=1,\ldots,q-1$.

These $q-1$ polar points of the conics of $Q^-(3,q)$ in the planes through ℓ belong to the conic C which is the intersection of $Q(4,q)$ with the plane incident with the nucleus N of $Q(4,q)$ and the points R_1, R_2.

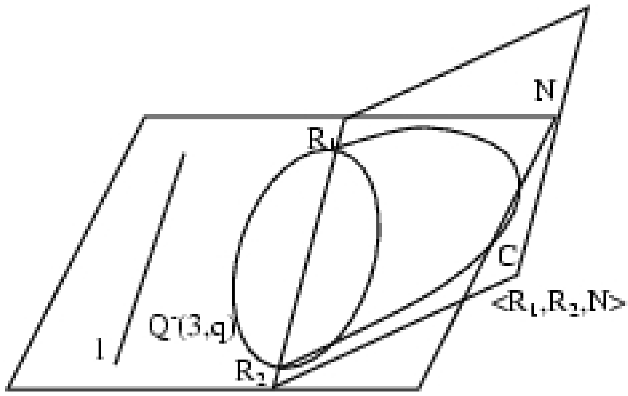

Figure 2: Conic C of the polar points of the conics in planes through ℓ

We look at the planes containing the external line ℓ. We can now replace some of these conics K^i by their polar point on C. If we keep $s-2$ conics $K^{q+2-s}, \dots, K^{q-1}$, and replace $q+1-s$ conics $K^1, \dots, K^{q+1-s}$ by their polar point, we get a partial ovoid $\mathbb{O}$ containing R_1, R_2, $s-2$ conics in planes through ℓ, and $q+1-s$ points being the polar points replacing the conics. So $\mathbb{O}$ is of size $2+(s-2)(q+1)+q+1-s$.

Now we need to introduce another set of conics; the conics denoted by C^* in the above section. Out of these, we will replace some by their polar point. Let us investigate conics of $Q^-(3,q)$ incident with R_1, but not R_2. There are $q+1$ pencils with carrier R_1, each containing q conics out of which one is incident with R_2. Thus we have $(q+1)(q-1)$ conics incident with R_1, but not R_2. These conics intersect $q/2+1$ planes through ℓ, one plane $\langle \ell, R_1 \rangle$ tangent to the elliptic quadric in R_1 and $q/2$ planes intersecting each conic in two points. We will first show that these conics form groups of $q+1$ conics which are intersected by the same $q/2+1$ planes containing ℓ. In these groups, there is exactly one conic of each pencil.

Lemma 3.2 *The $(q+1)(q-1)$ conics of the elliptic quadric $Q^-(3,q)$, incident with R_1 but not R_2, form groups of $q+1$ conics which are intersected by the same $q/2+1$ planes through the external line ℓ. Conics* *of the same group intersect in R_1, and every other point of such a conic is the intersection point with precisely one other conic of the group.*

Proof: The elliptic quadric $Q^-(3,q)$ is fixed by a 3-transitive group. The subgroup fixing R_1 and R_2 has size $q-1$. This group also fixes the polar line of R_1R_2, which is the line ℓ.

The elliptic quadric can be represented by the following equation: $X_0X_1 + f(X_2,X_3) = 0$, with $f(X_2,X_3) = aX_2^2 + bX_2X_3 + cX_3^2$ irreducible over $\mathbb{F}_q$. Then there is a cyclic group C_{q+1} of size $q+1$ fixing the quadratic form f. This group also operates cyclicly on the points of ℓ. Let R_1, R_2 have coordinates $R_1 = (1,0,0,0), R_2 = (0,1,0,0)$, then $\ell : X_0 = X_1 = 0$.

If we now fix a point on ℓ, for instance $P = (0,0,0,1)$, we get the mapping $\eta : (x_0,x_1,x_2,x_3) \mapsto (a^2x_0, x_1, ax_2, ax_3)$ fixing $Q^-(3,q)$.

If a is a generator of $\mathbb{F}_q^*$, then η defines a cyclic group C_{q-1} of order $q-1$.

Then η fixes the elliptic quadric and also the planes $\langle \ell, R_1\rangle : X_1 = 0$ and $\langle l, R_2\rangle : X_0 = 0$, where $X_1 = \alpha X_0$, for some $\alpha \neq 0$, are the secant planes to $Q^-(3,q)$ through ℓ.

If we consider the planes incident with the point P on ℓ and R_1, different from the plane through R_2 and the tangent plane in R_1, their intersection with $Q^-(3,q)$ is a conic and there are $q/2+1$ planes through ℓ intersecting this conic. In the quotient geometry $PG(1,q) = \mathbb{F}_q \cup \{\infty\}$ of ℓ, these planes correspond to a set of $q/2+1$ points where $\langle \ell, R_1\rangle$ corresponds to ∞ and the other $q/2$ planes define an additive subgroup of index 2 in $(\mathbb{F}_q,+)$, or a coset of an additive subgroup of index 2 in $(\mathbb{F}_q,+)$. The cyclic group C_{q-1} maps this subgroup of index 2 onto all subgroups of index 2 in $(\mathbb{F}_q,+)$, or this coset onto all cosets of these subgroups of index 2 in $(\mathbb{F}_q,+)$. Furthermore, C_{q-1} maps all conics in planes through the line $\langle P, R_1\rangle$, different from $\langle \ell, R_1, R_2\rangle$ and $T_{R_1}(Q^-(3,q))$, onto each other in a way that we see every subgroup exactly once.

The cyclic group C_{q+1} acts transitively on ℓ, so transitively on the possible lines PR_1, with $P \in \ell$. If C_{q+1} maps $P \in \ell$ onto $P' \in \ell$, the intersection conic between a plane through the line PR_1 and $Q^-(3,q)$ is mapped onto a conic in a plane through $P'R_1$ which is intersected by the same $q/2+1$ planes through ℓ, since C_{q+1} fixes the line R_1R_2 point by point. So for every point on ℓ, there is a unique conic intersected by the same $q/2+1$ planes through ℓ.

Definition 3.3 A *group* of conics of $Q^-(3,q)$ is a set C^* of $q+1$ conics $C^0,\dots,C^q$, through R_1, but not through R_2, intersected by the same $q/2+1$ planes through ℓ.

All the conics of a group must intersect in R_1 and in another point as their planes intersect in a line incident with R_1 which cannot be in the tangent plane $\langle \ell, R_1\rangle$ to $Q^-(3,q)$ in R_1.

We now show that these other intersection points are all different, thus that every point different from R_1 of every conic of a group is an intersection point with exactly one other conic of the group. The cyclic group C_{q+1} maps a conic C^0 onto conics $C^1,\dots,C^q$ which are intersected by the same $q/2+1$ planes through ℓ. One of these planes is the tangent plane $\langle \ell, R_1\rangle$; the other $q/2$ of those planes through ℓ are secant planes to $Q^-(3,q)$. We consider one such plane through ℓ and the intersection conic K^i with $Q^-(3,q)$. Let $R'_0,\dots,R'_q$ be the points of K^i and let γ be the generator of the group C_{q+1}. Thus $\gamma(R'_i) = R'_{i+1 \pmod q}$. This conic K^i shares two points with C^0, let us say R'_0, R'_j, then $\gamma^j(C^0)$ contains R'_j and $\gamma^{q+1-j}(C^0)$ contains R'_0.

We discuss in this way all points of $C^0 \setminus \{R_1\}$, as there are $q/2$ planes through ℓ intersecting such a conic C^0 of a given group in two points, and $q/2\cdot 2=q$ is the number of points of the conic of the given group, besides R_1. □

The idea is to replace some of these conics of a given group C^* by their polar point. As this new configuration is supposed to be a maximal partial ovoid, we have to know the incidences of these polar points. The following lemma shows that these polar points of the $q+1$ conics of a group C^* form a conic C' contained in the tangent cone $T_{R_1}(Q(4,q))$.

Lemma 3.4 *Consider a set C^* of $q+1$ conics $C^0,\dots,C^q$ incident with the point R_1 of the elliptic quadric $Q^-(3,q)$, but not incident with R_2, intersected by the same $q/2+1$ planes through the external line ℓ. The polar points of these conics form themselves a conic C' which lies in the* *tangent hyperplane of* $Q(4,q)$ *in* R_1 .

Proof: All these polar points lie in the tangent hyperplane $T_{R_1}(Q(4,q))$, since they are all incident with a line of $Q(4,q)$ through R_1.

We found the conics $C^0,\dots,C^q$ of $Q^-(3,q)$ in the foregoing corollary using the irreducible quadratic form $f(X_2,X_3)=aX_2^2+bX_2X_3+cX_3^2$. Embedding the elliptic quadric in $Q(4,q)$, we get $X_0X_1+aX_2^2+bX_2X_3+cX_3^2+X_4^2=0$. The cyclic group C_{q+1} from the proof of Corollary 3.2 can be rescaled and extended to a mapping η'

$$\eta' : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a' & b' & 0 \\ 0 & 0 & c' & d' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

fixing $Q(4,q)$, where the matrix

$$A=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

fixes the quadratic form $aX_2^2+bX_2X_3+cX_3^2$. The hyperplane $T_{R_1}(Q(4,q)) : X_1=0$ is fixed by η', thus by C_{q+1}. Furthermore, the hyperplanes $X_0=0$ and $X_4=0$ are fixed as well.

If $U=(u_0,0,0,0,u_4)$ belongs to $T_{R_1}(Q(4,q)) : X_1=0$, then $U=R_1$, so we can assume that $(u_2,u_3)\neq(0,0)$. If $U=(u_0,\dots,u_4)$, $(u_2,u_3)\neq(0,0)$, $U\in T_{R_1}(Q(4,q))$, then the images of U under C_{q+1} have coordinates

$$(u_0,u_1,A^j\begin{pmatrix} u_2 \\ u_3 \end{pmatrix},u_4).$$

An easy check shows that the images of U form a conic C' contained in $T_{R_1}(Q(4,q))\cap Q(4,q)$. The cyclic group C_{q+1} acts in one orbit on the $q+1$ conics of a group; so we have proven that the polar points of the conics of a group form a conic C' in $T_{R_1}(Q(4,q))\cap Q(4,q)$. $\square$

The conic C' in $T_{R_1}(Q(4,q))\cap Q(4,q)$ of the preceding corollary is skew to the conic $\langle \ell, N\rangle\cap Q(4,q)$, since this conic consists of the polar points of the conics in the planes through R_1R_2. Thus we can replace conics of $Q^-(3,q)$

in planes through ℓ and conics in a given group by their polar points, under certain restrictions. Assume again that we replace $q+1-s$ conics $K^1, \ldots, K^{q+1-s}$ being the intersection of planes containing the external line ℓ with the quadric $Q^-(3,q)$. Now we replace also t conics $C^1, \ldots, C^t$ out of $\{C^0, \ldots, C^q\}$ by their polar points to get more sizes for the maximal partial ovoids of $Q(4,q)$. Some of the points on the conics $C^1, \ldots, C^t$ were already cancelled when we replaced the conics $K^1, \ldots, K^{q+1-s}$ by their polar points, so we have to know how many conics of $\{K^1, \ldots, K^{q+1-s}\}$ intersect the t conics $C^1, \ldots, C^t$ in order to determine exactly the cardinality of the newly constructed maximal partial ovoids. Assume that we kept r of the conics in the planes through ℓ that intersect the t conics $C^1, \ldots, C^t$, including the tangent plane incident with R_1. The cardinality M of the partial ovoid $\mathbb{O}$ is then depending on how the $r-1$ conics out of $K^{q+2-s}, \ldots, K^{q-1}$ intersect $C^1, \ldots, C^t$. We have $2t(r-1)-u$ points of intersection between $C^1, \ldots, C^t$ and $K^{q+2-s}, \ldots, K^{q-1}$, where u is the number of intersection points of $C^1, \ldots, C^t$ and $K^{q+2-s}, \ldots, K^{q-1}$ lying in two of the conics $C^1, \ldots, C^t$.

In the next section, we will investigate the incidences between the intersection points among the conics $C^1, \ldots, C^t$ and the conics $K^{q+2-s}, \ldots, K^{q-1}$; now we say that there are u points of intersection. Then we get partial ovoids of size

$$
\begin{aligned}
M &= 2+(s-2)(q+1)+q+1-s-1-2t(r-1)+t+u, \\
&= (s-1)q-2tr+3t+u,
\end{aligned}
$$

where certain constraints apply for s and r, and where the term -1 comes from the fact that R_1 is also cancelled from $C^1, \ldots, C^t$, and the term $+t$ comes from the fact that $C^1, \ldots, C^t$ are replaced by their polar points.

Furthermore, we have to determine the bound on the cardinality of L' from Corollary 2.1, because $s-r \geq |L'|$. Now $|L'| \leq |L|\frac{1+\log(|U|)}{d}$ where the elements of U are the conics of $Q^-(3,q)$ besides

1. the $q-1$ conics lying in a plane containing the line ℓ,
2. the $q+1$ conics in a plane through R_1R_2, and
3. the conics of the selected group $C^* = \{C^0, \ldots, C^q\}$ of conics through

R_1, but not through R_2, intersected by the same $q/2+1$ planes through ℓ.

Note that $|U| \leq q^3+q^2 < (q+1)^3$. These $q/2$ conics in planes through ℓ skew to the conics of the group $\{C^0,\dots,C^q\}$ form the set L. A lower bound on the degree is given in [13, Lemma 2.12]; $d \geq \frac{1}{4}(q-1-6\sqrt{q})$. Together we get:

$$\begin{aligned} |L'| &\leq \frac{q}{2}\cdot\frac{1+\log((q+1)^3)}{\frac{1}{4}(q-1-6\sqrt{q})} \\ &\leq 2\cdot(1+3\log(q+1))\cdot\frac{q}{q-1-6\sqrt{q}}. \end{aligned}$$

For $q \geq 50$, $q/(q-1-6\sqrt{q}) \leq 8$ and we get $|L'| \leq 16(1+3\log(q+1))$.

Hence, the preceding results show that there exists, within the set of $q/2$ planes of L, a set L' of at most $16(1+3\log(q+1))$ planes such that every conic of $Q^-(3,q)$ in U, intersects at least one of the planes of L'. One of these planes could be the tangent plane $\langle \ell, R_2\rangle$. The symbol s in Step 4 of the summary of the construction stands for the planes $\langle \ell, R_1\rangle$, $\langle \ell, R_2\rangle$, and for the $s-2$ non-replaced conics in planes through ℓ. To make sure that also the plane $\langle \ell, R_2\rangle$ is counted within the symbol s, and since R_2 does not belong to the conics $C^0,\dots,C^q$, we increase the upper bound on the size of L' to $17+48\log(q+1)$.

To be sure that every conic in U intersects at least one conic of L', we do not replace the conics in L' by their polar points, and we impose the constraint $s-r \geq 17+48\log(q+1)$.

The following check also needs to be made. We are replacing conics in planes through ℓ by their polar points, which belong to the conic C, and we are also replacing conics in a selected group C^* of conics through R_1 by their polar points, which belong to the conic C'. We must verify whether a selected polar point on C' can be collinear on $Q(4,q)$ with a selected polar point on C.

It is impossible that a point on C' is collinear on $Q(4,q)$ with all the points of C. For the points of $Q(4,q)$ collinear with all the points of C are the polar points of the conics through R_1R_2, and they do not belong to C'.

The points of C' form an orbit under the cyclic group C_{q+1}, which fixes the conic C point by point. Hence, if the points of C' are collinear on

$Q(4,q)$ with points of C, then they are collinear with the same points of C. They certainly are collinear with R_1 since all the conics of a group C^* pass through R_1. Assume that the points of C' still are collinear with a second point R of C. Then R is the polar point of a conic D through ℓ. We prove that this conic D is skew to all the conics of the selected group C^* of conics through R_1.

Lemma 3.5 *Assume that the points of the conic C' are collinear on $Q(4,q)$ with a point R, different from R_1, of the conic C. Assume that R is the polar point of the conic D through ℓ, then D is skew to all the conics of the selected group C^* of conics through R_1.*

Proof: Suppose that D has an intersection point with such a conic. Then there are two intersection points T_1 and T_2 since the only plane through ℓ that intersects a conic of a group in one point, is the plane $\langle \ell, R_1 \rangle$.
Assume that T_1 and T_2 belong to the conic of the selected group through R_1 with polar point T on C'. We are assuming that this point T of C' is collinear with the point R of C; at most one of the points T_1 or T_2 can belong to the line TR. Assume that $T_2 \notin TR$, then T_2 is collinear with R since it belongs to the conic D which has R as its polar point, and T_2 is also collinear with T, but then there is a triangle of lines contained in $Q(4,q)$. This is impossible. $\square$

Since we will be selecting points of C' and of C to belong to the newly constructed partial ovoid $\mathbb{O}$, we need to avoid that these points are collinear. They can be collinear with only one point R of C, different from R_1, which is the polar point of a conic D skew to the selected group of conics through R_1. For this reason, we increase the upper bound on the size of L' by a unit to also include the conic D in L'. This gives the constraint $s - r \geq 18 + 48\log(q+1)$.

We now summarize the construction and prove that the newly constructed partial ovoids $\mathbb{O}$ are complete, under certain constraints.

Summary of the construction

1. Select an elliptic quadric $Q^-(3,q)$ contained in $Q(4,q)$, select an

external line ℓ to $Q^-(3,q)$ in the solid of $Q^-(3,q)$, and let R_1R_2 be the polar line of ℓ with respect to $Q^-(3,q)$, where $R_1, R_2 \in Q^-(3,q)$.

Let C be the conic of $Q(4,q)$ in the plane $\langle R_1, R_2, N\rangle$ containing the $q-1$ polar points of the $q-1$ conics $K^1, \ldots, K^{q-1}$ to $Q^-(3,q)$ in planes through ℓ.

2. Select a group C^* of $q+1$ conics $C^0, \ldots, C^q$ through R_1, intersected by the same $q/2+1$ planes through ℓ. Let C' be the conic in $T_{R_1}(Q(4,q))$ consisting of the polar points of the conics $C^0, \ldots, C^q$.

3. Let L' be the set of conics in planes through ℓ, skew to the given set of conics $C^0, \ldots, C^q$, whose existence is guaranteed by Corollary 2.1. Note that we increased the upper bound on the size of L' to $18+48\log(q+1)$ to guarantee that L' also includes the plane $\langle \ell, R_2\rangle$ and the conic D.

   The crucial property of the conics in the set L' is that every conic of $Q^-(3,q)$, not lying in a plane through ℓ or R_1R_2, and different from $C^0, \ldots, C^q$, intersects at least one of these conics in L' in at least one point.

4. We construct a new partial ovoid by selecting $q+1-s$ conics of $Q^-(3,q)$ in planes through ℓ, and by replacing them by their polar points on C. This gives a new partial ovoid of size $2+(s-2)(q+1)+q+1-s$. Note that we do not replace the conics in L', including the conic D, by their polar points.

5. We now select t conics $C^1, \ldots, C^t$ out of $C^0, \ldots, C^q$, and replace them by their polar points on C'.

   Assume that exactly $r-1$ out of the $s-2$ non-replaced conics $K^{q+2-s}, \ldots, K^{q-1}$ through ℓ intersect the conics $C^1, \ldots, C^t$; assume that they intersect in total in $2t(r-1)-u$ points.

   Then the newly constructed partial ovoid $\mathbb{O}$ has size

$$M \quad = \quad (s-1)q - 2tr + 3t + u.$$

It remains to be shown that such a partial ovoid $\mathbb{O}$ is complete. We know that a point in $Q^-(3,q) \setminus \mathbb{O}$ must lie on a conic which was replaced by its polar point. Thus this point is collinear with this polar point. So let us consider a point $P \in Q(4,q)$, but $P \notin Q^-(3,q)$ and $P \notin \mathbb{O}$, where we assume that P extends $\mathbb{O}$ to a larger partial ovoid. The tangent cone to $Q(4,q)$ in P intersects $Q^-(3,q)$ in a conic $\varphi(P)$. The plane of $\varphi(P)$ cannot contain the external line ℓ, for since $P \notin \mathbb{O}$, this conic in this plane of $\varphi(P)$ through ℓ would not have been cancelled from $Q^-(3,q)$; so this conic contains points of $\mathbb{O}$; hence P does not extend $\mathbb{O}$. So $\varphi(P)$ can either pass through R_1 and R_2, or be a conic of the selected group C^* of conics $C^0, \ldots, C^q$ which are intersected by the same $q/2+1$ planes containing ℓ, or be a conic not intersected by the same $q/2+1$ planes through ℓ as $C^0, \ldots, C^q$. If $R_1, R_2 \in \varphi(P)$, the nucleus of $\varphi(P)$ lies on ℓ. Thus every plane containing ℓ intersects $\varphi(P)$ in one point. But there are $s-2$ conics in planes through ℓ in $\mathbb{O}$, thus $\varphi(P)$ contains points of the partial ovoid $\mathbb{O}$, so P cannot extend $\mathbb{O}$ to a larger partial ovoid. If $\varphi(P)$ is intersected by $q/2+1$ planes through ℓ different from those intersecting $C^1, \ldots, C^t$, then $\varphi(P)$ intersects one of the conics in L' and P cannot extend $\mathbb{O}$. Otherwise, $\varphi(P)$ belongs to the group of conics $\{C^0, \ldots, C^q\}$, thus it intersects each of these conics $C^1, \ldots, C^t$ in R_1 and in one other point. Now $\varphi(P)$ has $2(r-1)$ points, different from R_1, in common with the $r-1$ non-cancelled conics in planes through ℓ which intersect the conics $C^0, \ldots, C^q$. So if $2(r-1) > t$, then $\varphi(P)$ contains at least one point of $\mathbb{O}$, thus P cannot extend $\mathbb{O}$ to a larger partial ovoid. Thus $\mathbb{O}$ is a maximal partial ovoid, if we impose the condition $r > (t+2)/2$.

We summarize the preceding results for future references.

Corollary 3.6 *The maximal partial ovoid $\mathbb{O}$ of $Q(4,q)$ has cardinality $M = (s-1)q - 2tr + 3t + u$, where the following constraints apply:*

1. $2 \le s \le q+1$,
2. $\frac{t+2}{2} < r \le q/2+1$,
3. *if* $s \ge q/2$, *then* $r \ge s - q/2$,
4. $s - r \ge 18 + 48\log(q+1)$.

The restrictions follow from the construction above and the application of Corollary 2.1 in the construction.

4 Selecting five conics

We wish to obtain maximal partial ovoids of different sizes, preferably the sizes should fill up an interval. The cardinality of the maximal partial ovoids we just constructed, is $M = (s-1)q - 2tr + 3t + u$, where we can let vary the parameters s, r, and u. From the previous section, we know that there are $\binom{t}{2}$ points, where the conics $C^1, \ldots, C^t$ intersect in a point different from R_1, out of which u are incident with a conic of $\{K^{q+2-s}, \ldots, K^{q-1}\}$. We will now show that if we choose for $C^1, \ldots, C^t$ five conics, thus $t = 5$, we get 10 points of intersection which can be made to belong/not belong to $K^{q+2-s}, \ldots, K^{q-1}$ in such a way that we can construct maximal partial ovoids of sizes $M = (s-1)q - 10r + 15, \ldots, M = (s-1)q - 10r + 25$. This way we can let vary s and r, and still get a continuous interval for the cardinalities.

Consider the $q+1$ conics $C^0, \ldots, C^q$ of the selected group. It follows from the proof of Corollary 3.2 that there is a cyclic group C_{q+1}, with generator α, acting transitively on these $q+1$ conics, and fixing all conics K^i in the planes through ℓ. Assume that $\alpha(C^i) = C^{i+1 \pmod{q+1}}$. We select the five conics $C^1, \ldots, C^5$ from the group of conics. Note that $t = 5$ implies $r \geq 4$ (Corollary 3.6 (2)).

Then 4 points of intersection are in one plane: $C^1 \cap C^2, C^2 \cap C^3, C^3 \cap C^4, C^4 \cap C^5 \in K^1$.

Since the two points of C^2 in K^1 lie already in a second conic, the intersection point $C^2 \cap C^4$ lies in another conic K^2. Then, by using α and α^{-1}, the intersection points $C^1 \cap C^3, C^2 \cap C^4, C^3 \cap C^5$ are in fact incident with K^2.

We still need to determine in which conics K^i the intersection points $C^1 \cap C^4, C^2 \cap C^5$, and $C^1 \cap C^5$ lie. Again, by using α, the first two of those three intersection points lie in the same conic K^i.

To simplify notations, we denote the intersection point of the conics C^i and C^j, different from R_1, by ij.

Lemma 4.1 *The points* 14 *and* 25 *lie in a conic* K^3*, different from* K^1 *and* K^2*.*

Proof: The point 25 does not lie in K^1, since the two points of C^2 in K^1

already lie on C^1 and C^3.

Suppose that 14 and 25 lie in K^2. Then the intersection points $24, 13, 35, 14, 25$ all lie in K^2. The conic K^2 is also stabilized by the cyclic group C_{q+1} generated by α. So these intersection points can be mapped onto each other by an appropriate power α^m of α. For instance, $\alpha^m(14) = 24$, then

$$\begin{cases} 1+m & \equiv & 4 \pmod{q+1}, \\ 4+m & \equiv & 2 \pmod{q+1}. \end{cases}$$

This implies that $m \equiv 3 \pmod{q+1}$ and that $m = -2 \pmod{q+1}$. So $5 \equiv 0 \pmod{q+1}$. This is impossible, if $q \geq 64$. □

Lemma 4.2 *The point* 15 *lies in a conic* K^4*, different from* K^1, K^2, K^3.

Proof: The point 15 does not lie in K^1 since the two points of C^1 in K^1 already lie on the conics C^0 and C^2.

Suppose that $15 \in K^2$, then K^2 contains the points $24, 13, 35, 15$. Again, there must be a power α^m of α mapping one of these intersection points on another intersection point lying in K^2. Assume that $\alpha^m(13) = 15$. Then

$$\begin{cases} 1+m & \equiv & 5 \pmod{q+1}, \\ 3+m & \equiv & 1 \pmod{q+1}. \end{cases}$$

This implies that $6 \equiv 0 \pmod{q+1}$. This is impossible if $q \geq 64$.

Suppose that 15 lies in K^3. Then K^3 contains the intersection points $14, 25, 15$. Assume that $\alpha^m(25) = 15$. Then

$$\begin{cases} 2+m & \equiv & 5 \pmod{q+1}, \\ 5+m & \equiv & 1 \pmod{q+1}. \end{cases}$$

This implies that $7 \equiv 0 \pmod{q+1}$. This is impossible if $q \geq 64$. □

We conclude that the 10 intersection points of the conics $C^1, \ldots, C^5$ lie in four conics K^1, K^2, K^3, K^4, containing respectively $4, 3, 2, 1$ intersection points.

With sums of the numbers $1, 2, 3, 4$, it is possible to form all numbers from 0 to 10, so we can get all possibilities $\pmod{10}$. We now show how we will apply this to get a sequence for the cardinalities for the maximal partial ovoids.

- $u = 0 : M = (s-1)q - 10r + 15.$

  We select none of the planes through ℓ with points of intersection. Then $1 \leq r \leq q/2 - 3$. The lower bound follows from the fact that the number r also includes the plane $\langle \ell, R_1 \rangle$ intersecting $C^1, \ldots, C^5$, and the upper bound from the fact that we need to avoid the four planes containing the intersection points.

- $u = 1 : M = (s-1)q - 10r + 16$.

  We select the plane K^4 with one point of intersection, but none of the other planes with intersection points, thus $2 \leq r \leq q/2 - 2$. The lower bound follows from the fact that the number r also includes the plane $\langle \ell, R_1 \rangle$ intersecting $C^1, \ldots, C^5$, while the upper bound $q/2 - 2$ comes from the fact that we need to avoid the three other planes containing intersection points.

- $u = 2 : M = (s-1)q - 10r + 17$.

  We select the plane K^3 with two points of intersection, but none of the other planes with intersection points, thus $2 \leq r \leq q/2 - 2$.

- $u = 3 : M = (s-1)q - 10r + 18$.

  We select the plane K^2 with three points of intersection, but none of the other planes with intersection points, thus $2 \leq r \leq q/2 - 2$.

- $u = 4 : M = (s-1)q - 10r + 19$.

  We select the plane K^1 with four points of intersection, but none of the other planes with intersection points, thus $2 \leq r \leq q/2 - 2$.

- $u = 5 : M = (s-1)q - 10r + 20$.

  We select the planes K^1 and K^4 with respectively four and one points of intersection, but none of the other planes with intersection points, thus $3 \leq r \leq q/2 - 1$.

- $u = 6 : M = (s-1)q - 10r + 21$.

  We select the planes K^1 and K^3 with respectively four and two points of intersection, but none of the other planes with intersection points, thus $3 \leq r \leq q/2 - 1$.

- $u = 7 : M = (s-1)q - 10r + 22$.

  We select the planes K^1 and K^2 with respectively four and three points of intersection, but none of the other planes with intersection points, thus $3 \leq r \leq q/2 - 1$.

- $u = 8 : M = (s-1)q - 10r + 23$.

  We select the planes K^1, K^2, and K^4 with respectively four, three, and one points of intersection, but not the plane K^3 with two intersection points, thus $4 \leq r \leq q/2$.

- $u = 9 : M = (s-1)q - 10r + 24$.

  We select the planes K^1, K^2, and K^3 with respectively four, three, and two points of intersection, but not the plane K^4 with one intersection point, thus $4 \leq r \leq q/2$.

- $u = 10 : M = (s-1)q - 10r + 25$.

  We select the planes K^1, K^2, K^3, and K^4 with respectively four, three, two, and one points of intersection, thus $5 \leq r \leq q/2 + 1$.

5 Calculation of the interval

For the spectrum, we do not wish to distinguish between the different cases for r from the above section. We impose $5 \leq r \leq q/2 - 3$ and get the interval $M = (s-1)q - 10r + 15, \ldots, M = (s-1)q - 10r + 25$, for a given pair (s, r). Together with the prior conditions from Corollary 3.6, we derive the following relevant constraints for s, r:

1. $r + 18 + \lfloor 48 \log(q+1) \rfloor \leq s$,
2. $5 \leq r \leq q/2 - 3$,
3. if $s \geq q/2$, then $r \geq s - q/2$.

We proceed as follows to find a non-interrupted interval of values of M for which a maximal partial ovoid of size M exists in $Q(4, q)$, q even. We explain the construction for $q = 2^{4h+1}$, so $q \equiv 2 \pmod 5$.

We know that $5 \leq r \leq q/2 - 3$. We first discuss the case $s \leq q/2 + 5$. For $s \leq q/2 + 5$, we can let start r with 5.

For a selected pair $(s,r) = (s,5)$, we find the sizes

$$\begin{aligned} M &= (s-1)q - 25, \\ &\vdots \\ M &= (s-1)q - 35. \end{aligned}$$

Now consider the value $s' = s+1$. We let vary r from 5 to $(q+48)/10$. This gives all values for M from

$$\begin{aligned} m &= sq - 25 \text{ for } r = 5, \\ &\vdots \\ M &= (s-1)q - 23 \text{ for } r = (q+48)/10, \\ &\vdots \\ M &= (s-1)q - 33 \text{ for } r = (q+48)/10. \end{aligned}$$

So all these values for s' fixed and $r \in [5, (q+48)/10]$ give a non-interrupted sequence of values which ends with the lower bound $M = (s-1)q-33$. But then $(s,r) = (s,5)$ gives the values $M = (s-1)q-34$ and $M = (s-1)q-35$. So we see that the fixed value s and the value $r = 5$ give the next smaller values. This enables us to get a large non-interrupted interval of integer values M for the size of maximal partial ovoids of $Q(4,q)$, q even.

We now discuss the case $s = q/2 + u$, with $u \geq 6$, so from the imposed conditions, $r \geq u$.

For $s = q/2 + u$ and $r = u$, we get the sizes

$$\begin{aligned} M &= q^2/2 + (u-1)q - 10u + 25, \\ &\vdots \\ M &= q^2/2 + (u-1)q - 10u + 15. \end{aligned}$$

For $s = q/2 + u + 1$ and $r = (q-2)/10 + u$, we get the sizes

$$\begin{aligned} M &= q^2/2 + (u-1)q - 10u + 27, \\ &\vdots \\ M &= q^2/2 + (u-1)q - 10u + 17. \end{aligned}$$

So for $s = q/2 + u + 1$ and $r \in [u+1, (q-2)/10 + u]$, the smallest size that is obtained, is equal to $M = q^2/2 + (u-1)q - 10u + 17$. Then, the values $(s, r) = (q/2 + u, u)$ give the next smaller values $M = q^2/2 + (u - 1)q - 10u + 16$ and $M = q^2/2 + (u-1)q - 10u + 15$.

So, also here, it is possible to find a large non-interrupted sequence of integer values M for the size of maximal partial ovoids of $Q(4, q)$, q even.

We now determine the smallest and the largest value of this non-interrupted sequence.

To determine the largest value, we note that we have to impose the upper bound $r = (q-2)/10 + u \leq q/2 - 3$, since we need to use the value $r = (q-2)/10 + u$ for $s = q/2 + u + 1$. So $u \leq (4q - 28)/10$, and so $s = q/2 + u + 1 \leq (9q - 18)/10$.

For $(s, r) = ((9q - 18)/10, (4q - 18)/10)$, the largest size is $M = (s-1)q - 10r + 25 = (9q^2 - 68q + 430)/10$.

To determine the smallest value, we note that for $s = 18 + 48\lfloor\log(q + 1)\rfloor + (q + 48)/10$, it is possible to let vary r in $r \in [5, (q + 48)/10]$. For $(s, r) = (18 + \lfloor 48\log(q+1)\rfloor + (q+48)/10, (q+48)/10)$, the smallest value for $M = (10q\lfloor 48\log(q+1)\rfloor + q^2 + 208q)/10 - 33$.

For $(s, r) = (17 + \lfloor 48\log(q+1)\rfloor + (q+48)/10, 5)$, we get the sizes

$$\begin{aligned} M &= (10q\lfloor 48\log(q+1)\rfloor + q^2 + 208q)/10 - 25, \\ &\vdots \\ M &= (10q\lfloor 48\log(q+1)\rfloor + q^2 + 208q)/10 - 35. \end{aligned}$$

So this interval gives values smaller than $M = (10q\lfloor 48\log(q+1)\rfloor + q^2 + 208q)/10 - 33$; we still have a non-interrupted sequence of values for M.

For $s = 17 + \lfloor 48\log(q+1)\rfloor + (q+48)/10$, necessarily, $r \leq (q+38)/10$. For $(s, r) = (17 + \lfloor 48\log(q+1)\rfloor + (q+48)/10, (q+38)/10)$, this gives the values

$$\begin{aligned} M &= (10q\lfloor 48\log(q+1)\rfloor + q^2 + 198q)/10 - 13, \\ &\vdots \\ M &= (10q\lfloor 48\log(q+1)\rfloor + q^2 + 198q)/10 - 23. \end{aligned}$$

For $(s,r) = (16 + \lfloor 48\log(q+1)\rfloor + (q+48)/10, 5)$, we get the sizes

$$\begin{aligned} M &= (10q\lfloor 48\log(q+1)\rfloor + q^2 + 198q)/10 - 25, \\ &\vdots \\ M &= (10q\lfloor 48\log(q+1)\rfloor + q^2 + 198q)/10 - 35. \end{aligned}$$

When comparing the last two sequences, we see that the value $M = (10q\lfloor 48\log(q+1)\rfloor + q^2 + 198q)/10 - 24$ is missing.
So the value where the non-interrupted sequence ends, is equal to $(10q\lfloor 48\log(q+1)\rfloor + q^2 + 198q)/10 - 23$.

We have determined the smallest and the largest value of the interval; we now state this in the following theorem, where we also give the intervals for the other values of q.

Theorem 5.1 *For $q = 2^t$, $t \geq 6$, the parabolic quadric $Q(4,q)$, q even, and the symplectic space $W(q)$, q even, have maximal partial ovoids for every value M in the interval*

- $q = 2^{4h}$:

$$M \in [\frac{q^2 + 194q + 10q\lfloor 48\log(q+1)\rfloor - 190}{10}, \frac{9q^2 - 69q + 440}{10}],$$

- $q = 2^{4h+1}$:

$$M \in [\frac{q^2 + 198q + 10q\lfloor 48\log(q+1)\rfloor - 230}{10}, \frac{9q^2 - 68q + 430}{10}],$$

- $M = 2^{4h+2}$:

$$M \in [\frac{q^2 + 196q + 10q\lfloor 48\log(q+1)\rfloor - 210}{10}, \frac{9q^2 - 66q + 410}{10}],$$

- $M = 2^{4h+3}$:

$$M \in [\frac{q^2 + 192q + 10q\lfloor 48\log(q+1)\rfloor - 170}{10}, \frac{9q^2 - 67q + 420}{10}].$$

Moreover, for every integer M in such an interval, there exists a minimal blocking set of size M, w.r.t. the planes, of $PG(3,q)$.

Proof: It is proven in [4] that a maximal partial ovoid of $W(q)$, q even, defines a minimal blocking set w.r.t. the planes of $PG(3,q)$. $\square$

Another application of our spectrum result is a spectrum result on maximal partial 1-systems of the Klein quadric $Q^+(5,q)$ [9, Section 15.4].

Definition 5.2 A *1-system* $\mathcal{M}$ on $Q^+(5,q)$ is a set of q^2+1 lines $\ell_1,\ldots,\ell_{q^2+1}$ on $Q^+(5,q)$ such that $\ell_i^\perp \cap \ell_j = \emptyset$, for all $i,j \in \{1,\ldots,q^2+1\}$, $i \neq j$.
A *partial 1-system* $\mathcal{M}$ on $Q^+(5,q)$ is a set of $s \leq q^2+1$ lines $\ell_1,\ldots,\ell_s$ on $Q^+(5,q)$ such that $\ell_i^\perp \cap \ell_j = \emptyset$, for all $i,j \in \{1,\ldots,s\}$, $i \neq j$.

A line of the Klein quadric lies in two planes of the Klein quadric. The above definition of 1-system is equivalent to the definition that a 1-system $\mathcal{M}$ on $Q^+(5,q)$ is a set of q^2+1 lines $\ell_1,\ldots,\ell_{q^2+1}$ on $Q^+(5,q)$ such that every line ℓ_j is skew to the two planes of the Klein quadric through any line ℓ_i, for all $i,j \in \{1,\ldots,q^2+1\}$, $i \neq j$. A similar observation can be made regarding the definition of a partial 1-system.

Via the Klein correspondence, points of the Klein quadric correspond to lines of $PG(3,q)$, and lines of the Klein quadric correspond to planar pencils of $PG(3,q)$, i.e., they correspond to the lines of $PG(3,q)$ through a point R in a plane Π passing through R.

A *tangency set* $\mathcal{T}$ of $PG(3,q)$ is a set of points of $PG(3,q)$, such that for every point $R \in \mathcal{T}$, there is a plane Π_R intersecting $\mathcal{T}$ only in R. It is proven in [11] that a tangency set in $PG(3,q)$ is equivalent to a partial 1-system on the Klein quadric.

A minimal blocking set B w.r.t. the planes of $PG(3,q)$ is an example of a tangency set; thus we can apply the results of Theorem 5.1.

Corollary 5.3 *For every value M belonging to one of the intervals of Theorem 5.1, there exists a maximal partial 1-system of size M on the Klein quadric $Q^+(5,q)$.*

We now present the other known results on the size of maximal partial ovoids of $Q(4,q)$, q even. The theoretical results of [2, 4], together with the computer-aided results of [4], indicate that for the smallest possible sizes (approximately $q+1$) and the largest possible sizes (approximately q^2+1) of maximal partial ovoids on $Q(4,q)$, q even, there exist integer values for M for which there do not exist maximal partial ovoids of $Q(4,q)$, q even. We refer to [4] for the computer-aided data; here we present the main theoretical results.

We first present the results on large maximal partial ovoids.

Theorem 5.4 (Brown, De Beule, and Storme [2]) *(1) The maximal size of a partial ovoid of* $Q(4,q)$ *,* q *even, is* q^2+1 *, which is the size of an ovoid of* $Q(4,q)$ *.*

(2) The size of the largest maximal partial ovoid of $Q(4,q)$ *,* q *even, different from an ovoid, is* q^2-q+1 *, so there do not exist maximal partial ovoids of* $Q(4,q)$ *,* q *even, with size in* $[q^2-q+2,q^2]$ *.*

Theorem 5.5 (Cimráková, De Winter, Fack, and Storme [4]) *The generalized quadrangle* $Q(4,q)$ *,* q *even, has maximal partial ovoids of size* q^2-2q+3 *.*

We now present the results on small maximal partial ovoids.

Theorem 5.6 (Cimrákova, De Winter, Fack, and Storme [4]) *(1) The smallest maximal partial ovoids of* $Q(4,q)$ *,* q *even, have size* $q+1$ *, and are equal to conics, lying in a plane not containing the nucleus* N *of* $Q(4,q)$ *.*

(2) The generalized quadrangle $Q(4,q)$ *,* q *even, has maximal partial ovoids of size* $2q+1$ *, and of size* $3q-1$ *if* $q\geq 4$ *.*

For results regarding the exclusion of some values k , with $k\in[q+2,2q]$, for the size of maximal partial ovoids of $Q(4,q)$, q even, we refer to [4].

References

[1] A. Aguglia, G.L. Ebert, and D. Luyckx. On partial ovoids of Hermitian surfaces. *Bull. Belg. Math. Soc. Simon Stevin* **12** (2005), 641–650.

[2] M.R. Brown, J. De Beule, and L. Storme. Maximal partial spreads of $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$. *European J. Combin.* **24** (2003), no. 1, 73–84.

[3] M. Cimráková and V. Fack. Clique algorithms for finding substructures in generalized quadrangles. *J. Combin. Math. and Combin. Computing*, to appear.

[4] M. Cimráková, S. De Winter, V. Fack, and L. Storme. On the smallest maximal partial ovoids and spreads of the generalized quadrangles $W(q)$ and $Q(4,q)$. *European J. Combin.* **28** (2007), 1934–1942.

[5] M. Cimráková. *Search algorithms for substructures in generalized quadrangles.* PhD Thesis, Ghent University, Belgium.

[6] P. Dembowski. *Finite Geometries.* Ergebnisse der Mathematik und ihrer Grenzgebiete **44**, Springer Berlin, (1968).

[7] G.L. Ebert and J.W.P. Hirschfeld. Complete systems of lines on a Hermitian surface over a finite field. *Des. Codes Cryptogr.* **17** (1999), 253–268.

[8] Z. Füredi. Matchings and covers in hypergraphs. *Graphs and Combin.* **4** (1988), 115–206.

[9] J.W.P. Hirschfeld. *Finite Projective Spaces of Three Dimensions,* Oxford University Press, Oxford, 1985.

[10] J.W.P. Hirschfeld. *Projective Geometry over Finite Fields.* Oxford Mathematical Monographs, Oxford University Press, New York, 1998.

[11] K. Metsch and L. Storme. Tangency sets in $PG(3,q)$. *J. Combin. Des.*, to appear.

[12] S.E. Payne and J.A. Thas. *Finite Generalized Quadrangles.* Research Notes in Mathematics, 110, Pitman (Advanced Publishing Program), Boston, MA, (1984), vi+312 pp.

[13] L. Storme and T. Szőnyi. Intersection of arcs and normal rational curves in spaces of even characteristic. *J. Geom.* **51** (1994), 150–166.

[14] T. Szőnyi, A. Cossidente, A. Gács, C. Mengyán, A. Siciliano, and Zs. Weiner. On Large Minimal Blocking Sets in $PG(2,q)$. *J. Combin. Des.* **13** (2005), 25–41.

[15] J.A. Thas. Old and new results on spreads and ovoids of finite classical polar spaces. Combinatorics '90 (Gaeta, 1990), *Ann. Discrete Math.* **52** (1992).

[16] J.A. Thas. Ovoids, spreads and m-systems of finite classical polar spaces. Survey in combinatorics, 2001 (Sussex), 241–267, London Math. Soc. Lecture Note Ser., 288, Cambridge Univ. Press, Cambridge (2001).