Equivalence properties of 3D conformal transformations and their application to reverse transformations

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ABSTRACT

Seven-parameter conformal coordinate transformations, also known as Helmert transformations, can be constructed in more than one way. Two possible orderings of the rotations are in common use, giving rise to Helmert versions 1 and 2. It is demonstrated how the rotation parameters of either version can be converted into the rotation parameters of the other. This is useful when software is designed for the other version. It also enables computation of the same-formula inverse transformation by changing the sign of the equivalent 'other version' parameters. These results were primarily intended for conformal transformations between geodetic datums. They can, however, be extended to coordinate transformations in disciplines such as photogrammetry where rotations sometimes exceed 90 degrees.
Author Keywords: conformal transformations, Helmert transformations, reverse transformations, inverse transformations

Introduction

In this paper, a transformation in three dimensions converts coordinates from one coordinate reference system (CRS) to another. The CRSs for which a transformation is designed are commonly described as *source* and *target*, with associated subscripts *s* and *t*. That source-to-target transformation is sometimes referred to as the *forward* transformation. The *reverse* transformation can be regarded as the corresponding operation that converts target coordinates back to source coordinates.

The transformation is *conformal* if it totally preserves shape, in which case it is often referred to as a similarity transformation. A conformal transformation applies rotations exactly, as opposed to a near-conformal transformation which simplifies the rotation matrix by linearisation. It is widely referred to as a Helmert transformation and is designated as such in this paper. See for example Sjöberg (2013) and Watson (2005).

It should be noted that a few sources, notably NATO (2001), use the name 'Helmert transformation' for the simplified form attributed to Bursa (1962) and Wolf (1963). Ordnance Survey (2018, pp. 35-37) treats 'Helmert transformation' as a generic term covering both the conformal version and the simplified form (which is given without reference to Bursa or Wolf).

The Bursa-Wolf transformation is often used instead of Helmert, because the difference is negligible if the rotation parameters are small. Rapp (1993, p. 61) summarises the effect for rotations of the order of 1″, 3″ and 9″, based on the research of Malys (1988). For geodetic transformations involving rotations of more than a few arc-seconds, the Helmert form is needed to ensure conformality. Going beyond geodetic datum transformations, into photogrammetry for example, transformations can involve rotations of many degrees, so simplifications of Helmert are not an option.

The Helmert transformation in its general form involves 7 parameters. These consist of 3 shift (or translation) parameters, a scaling factor and 3 rotations. There are 6 possible orderings of the rotations and these affect the Helmert formula. However, only 2 are used in practice, with the Y-rotation applied between the other two. This paper will show how the rotation parameters which produce one version of Helmert can be converted into rotation parameters which produce the other version.

The Helmert transformation formula can be rearranged to give an exact reverse formula. There appears, however, to be a preference for using the original formula with different parameters. Often the reverse parameters are taken to be the forward parameters with signs reversed, and the results are only approximate. Aktuğ (2009) has attempted to derive exact inverse parameters, but it will be shown that here again the results are only approximate.

This paper will show how to obtain the same-formula inverse parameters that give an exact reverse Helmert transformation. ('Exact' in this context means consistency between the reverse transformation and the forward transformation, since the latter is only a model.) The proof uses the relationship between the two versions of Helmert.

This paper only considers the effect of Helmert transformations on Cartesian coordinates. If the area of application is geodesy, where latitude, longitude and height are frequently used, coordinate conversion to and from Cartesians is often required. Traditional methods for doing this can be found in Heiskanen and Moritz (1967, pp. 182-183), although the method of obtaining latitude from Cartesians is iterative. Alternative methods of computing latitude from Cartesians are summarised in Featherstone and Claessens (2008).

Helmert parameters

The Helmert transformation has 7 parameters which determine what is done to the position coordinates in the source CRS to obtain position coordinates in the target CRS. They consist of the following:

Translation parameters

These are additions to the Cartesian coordinates in the transformation from source to target. They are denoted $\Delta X, \Delta Y, \Delta Z$.

Scaling parameter

These are distance-conversion factors in the directions of the axes. A scaling parameter can be expressed as a scaling multiple S or as a scale-change ΔS (where S=1+ΔS). ΔS is often given in parts per million (ppm).

Rotation parameters

These describe rotation of position vectors about Cartesian axes, or rotation of Cartesian axes when points are considered as fixed. Position vector (PV) rotations are illustrated in Fig. 1. The sign convention adopted here is that positive rotations are counter-clockwise about Cartesian axis when viewed from the positive side of the origin, as in Fig. 1. One characteristic is that a positive rotation about the Z-axis has the effect of increasing longitude. Ordnance Survey (2018, p. 37) says of PV rotations, 'It is the form in most common use in Europe (particularly in the oil and gas industry), is used by the International Association of Geodesy (IAG) and recommended by ISO (2007) and is EPSG dataset coordinate operation method code 1033'. This is the convention adopted for this paper.

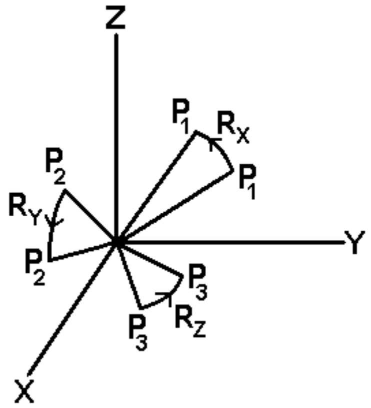

Fig. 1. Position vector (PV) rotations.

It should be noted that some authors and researchers prefer to use coordinate frame (CF) rotations. According to this, the sign convention is that positive rotations of the axis-planes are counter-clockwise when viewed from the positive side of the origin, as in Fig. 2. CF rotation parameters are opposite in sign to PV rotation parameters. Ordnance Survey (2018, 37) says the CF convention 'is common in the USA oil and gas industry and is EPSG dataset coordinate operation method code 1032'.

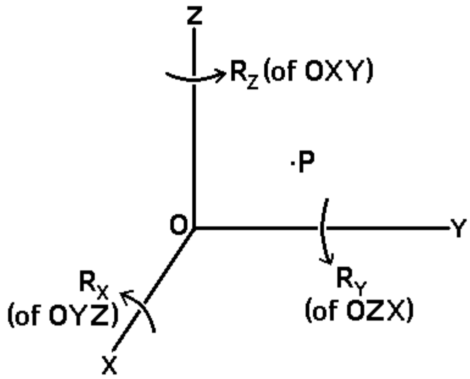

Fig. 2. Coordinate frame (CF) rotations.

Where cited references have used the CF convention shown in Fig. 2, this has not been remarked upon. The terms affected can easily be converted to comply with the PV convention, so the use of CF rotations can be regarded as a presentational detail. The results in this paper can be adapted for use with CF rotations.

The two versions of Helmert

Given the parameters ΔX, ΔY, ΔZ, S (or ΔS), R_X, R_Y and R_Z, it is convenient to denote $\cos R_X$ by c_X, $\sin R_X$ by s_X, etc.; this convention is used in Deakin (2006). Using **R** to denote the rotation matrix, the transformation equation is

$$\begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} + (1+\Delta S)\mathbf{R} \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} \tag{1}$$

The precise form of **R** depends on the order in which the rotations are applied. For the purposes of this paper:

- Version 1 is where R_X is applied first and R_Z last. Sources using this permutation include Deakin (2006), Fan (2005), Harvey (1986), Reit (1998) and Watson (2005).
- Version 2 is where R_Z is applied first and R_X last. Sources using this permutation include Awange and Grafarend (2002), Sjöberg (2013) and Wang *et al* (2018).

There are four other possible permutations of the rotations R_X, R_Y and R_Z. However, the author has seen no evidence that any of them are used.

In the case of Version 1, **R** is given by

$$\mathbf{R}_{ZYX} = \begin{bmatrix} c_Z & -s_Z & 0 \\ s_Z & c_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_Y & 0 & s_Y \\ 0 & 1 & 0 \\ -s_Y & 0 & c_Y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_X & -s_X \\ 0 & s_X & c_X \end{bmatrix} \tag{2}$$

It is easily verified that

$$\mathbf{R}_{ZYX} = \begin{bmatrix} c_Y c_Z & s_X s_Y c_Z - c_X s_Z & s_X s_Z + c_X s_Y c_Z \\ c_Y s_Z & c_X c_Z + s_X s_Y s_Z & c_X s_Y s_Z - s_X c_Z \\ -s_Y & s_X c_Y & c_X c_Y \end{bmatrix} \tag{3}$$

If the rotations are within the range -180° to 180° rather than -90° to 90°, the set of rotations leading to a given rotation matrix is not unique. If the parameters R_X, R_Y and R_Z which give rise to the rotation matrix in equation (3) are replaced by $R_X \pm 180°$, $\pm 180° - R_Y$ and $R_Z \pm 180°$, the rotation matrix is unchanged. This is because all the sine and cosine terms in equation (3) will change sign except for s_Y. For example, the case where $R_X = -50°$, $R_Y = 94°$ and $R_Z = 10°$ will give the same Version-1 rotation matrix as $R_X = 130°$, $R_Y = 86°$ and $R_Z = -170°$.

The non-uniqueness of the rotation-parameter set is known in photogrammetric circles; see for example Aimaiti (2015, p. 13). The author recommends the proof given above for its simplicity.

In the case of Version 2, **R** is given by

$$\mathbf{R}_{XYZ}=\begin{bmatrix}1&0&0\\0&c_X&-s_X\\0&s_X&c_X\end{bmatrix}\begin{bmatrix}c_Y&0&s_Y\\0&1&0\\-s_Y&0&c_Y\end{bmatrix}\begin{bmatrix}c_Z&-s_Z&0\\s_Z&c_Z&0\\0&0&1\end{bmatrix} \quad (4)$$

It is easily verified that

$$\mathbf{R}_{XYZ}=\begin{bmatrix}c_Yc_Z&-c_Ys_Z&s_Y\\s_Xs_Yc_Z+c_Xs_Z&c_Xc_Z-s_Xs_Ys_Z&-s_Xc_Y\\s_Xs_Z-c_Xs_Yc_Z&c_Xs_Ys_Z+s_Xc_Z&c_Xc_Y\end{bmatrix} \quad (5)$$

If the rotations are within the range -180° to 180° rather than -90° to 90°, the set of rotations leading to a given rotation matrix is not unique. If the parameters R_X, R_Y and R_Z which give rise to the rotation matrix in equation (5) are replaced by $R_X \pm 180°$, $\pm 180° - R_Y$ and $R_Z \pm 180°$, the rotation matrix is unchanged. The reason is exactly the same as for the Version-1 rotation matrix. For example, the case where $R_X = 87°$, $R_Y = 150°$ and $R_Z = -85°$ will give the same Version-1 rotation matrix as $R_X = -93°$, $R_Y = 30°$ and $R_Z = 95°$.

One common feature of the two versions of the rotation matrix is that they have the same linearised version. Noting that in this instance the rotations are in radians, that matrix is $\begin{bmatrix}1&-R_Z&R_Y\\R_Z&1&-R_X\\-R_Y&R_X&1\end{bmatrix}$. This is the simplified rotation matrix used in the Bursa-Wolf method which, as stated earlier, is only valid for very small rotations.

Among authors and researchers who use the rigorous rotation matrix, there has been a tendency to adopt one of the matrices in equations (3) and (5) to the exclusion of the other. As a result, the possible equivalence of rotation-parameter sets between Versions 1 and 2 appears to have been unexplored.

The following algorithms demonstrate how to convert the rotations in one version of Helmert to the equivalent rotations in the other.

Rotation-parameter conversion from Version 1 to Version 2

For the moment, it is assumed that rotations are numerically smaller than 90°. Generalisations, including the special case where arctangent values are undefined, will be considered later.

The initial Version-1 rotations R_X, R_Y and R_Z are applied as per (2). By (3),

$$\begin{bmatrix}r_{1,1}&r_{1,2}&r_{1,3}\\r_{2,1}&r_{2,2}&r_{2,3}\\r_{3,1}&r_{3,2}&r_{3,3}\end{bmatrix}=\begin{bmatrix}c_Yc_Z&s_Xs_Yc_Z-c_Xs_Z&s_Xs_Z+c_Xs_Yc_Z\\c_Ys_Z&c_Xc_Z+s_Xs_Ys_Z&c_Xs_Ys_Z-s_Xc_Z\\-s_Y&s_Xc_Y&c_Xc_Y\end{bmatrix} \quad (6)$$

For this exercise, R'_X, R'_Y and R'_Z denote the equivalent Version-2 rotations. Bearing in mind (5), the first objective is to find $c'_X, s'_X, c'_Y, s'_Y, c'_Z, s'_Z$ such that

$$\begin{bmatrix}c'_Yc'_Z&-c'_Ys'_Z&s'_Y\\s'_Xs'_Yc'_Z+c'_Xs'_Z&c'_Xc'_Z-s'_Xs'_Ys'_Z&-s'_Xc'_Y\\s'_Xs'_Z-c'_Xs'_Yc'_Z&c'_Xs'_Ys'_Z+s'_Xc'_Z&c'_Xc'_Y\end{bmatrix}=\begin{bmatrix}r_{1,1}&r_{1,2}&r_{1,3}\\r_{2,1}&r_{2,2}&r_{2,3}\\r_{3,1}&r_{3,2}&r_{3,3}\end{bmatrix} \quad (7)$$

Clearly,

$$s'_Y = r_{1,3} = s_Xs_Z + c_Xs_Yc_Z. \quad (8)$$

On the basis that s'_Y is the sine of R'_Y,

$$(c'_Y)^2 = 1-(s'_Y)^2 = 1-(s_Xs_Z+c_Xs_Yc_Z)^2. \quad (9)$$

Given the nature of the cosine function, the numerically-smallest possible value of R'_Y satisfies

$$c'_Y = \sqrt{1-(s'_Y)^2}. \quad (10)$$

Equating the expressions for $r_{3,3}, r_{2,3}, r_{1,1}, r_{1,2}$ respectively,

$$c'_Xc'_Y = c_Xc_Y, \quad (11)$$

$$s'_Xc'_Y = s_Xc_Z - c_Xs_Ys_Z, \quad (12)$$

$$c'_Yc'_Z = c_Yc_Z, \quad (13)$$

$$c'_Y s'_Z = c_X s_Z - s_X s_Y c_Z. \quad (14)$$

Since c'_Y is positive, the Version-2 rotations can be computed from

$$R'_X = \arctan2(c'_X c'_Y, s'_X c'_Y), \quad R'_Y = \arctan2(c'_Y, s'_Y), \quad R'_Z = \arctan2(c'_Y c'_Z, c'_Y s'_Z) \quad (15)$$

The arctan2(x,y) function is arctan(y/x) in the range -180° to 180° such that it always has the same sign as x. Programming languages usually have a function corresponding to arctan2. In VBA, a user-defined version is needed; suitable code can be found, for example, in Ruffhead (2016). The use of arctan2 rather than arctan is not strictly necessary when rotations are known to be small, but its presence here is to enable generalisation.

The solution for the Version-2 rotations only uses the five identities from the first row and first column of the rotation matrix. Verification of the other four identities is given in Appendix A.

Example 1

If the Version-1 rotation parameters are -33.88457022″, 70.66260075″ and -9.39541463″, equation (3) will produce the following rotation matrix:

$$\mathbf{R} = \begin{bmatrix} 0.9999999403 & 0.0000454940 & 0.0003425894 \\ -0.0000455503 & 0.9999999855 & 0.0001642614 \\ -0.0003425819 & -0.0001642770 & 0.9999999278 \end{bmatrix} \quad (16)$$

Application of equations (8) to (15) will show that -33.88135347″, 70.66414317″ and -9.38380681″ are the equivalent Version-2 rotation parameters.

Rotation-parameter conversion from Version 2 to Version 1

For the moment, it is assumed that rotations are numerically smaller than 90°. Generalisations, including the special case where arctangent values are undefined, will be considered later.

The initial Version-2 rotations R_X, R_Y and R_Z are applied as per (4). By (5),

$$\begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} = \begin{bmatrix} c_Y c_Z & -c_Y s_Z & s_Y \\ s_X s_Y c_Z + c_X s_Z & c_X c_Z - s_X s_Y s_Z & -s_X c_Y \\ s_X s_Z - c_X s_Y c_Z & c_X s_Y s_Z + s_X c_Z & c_X c_Y \end{bmatrix}. \quad (17)$$

For this exercise, R'_X, R'_Y and R'_Z denote the equivalent Version-1 rotations. Bearing in mind (3), the first objective is to find $c'_X, s'_X, c'_Y, s'_Y, c'_Z, s'_Z$ such that

$$\begin{bmatrix} c'_Y c'_Z & s'_X s'_Y c'_Z - c'_X s'_Z & s'_X s'_Z + c'_X s'_Y c'_Z \\ c'_Y s'_Z & c'_X c'_Z + s'_X s'_Y s'_Z & c'_X s'_Y s'_Z - s'_X c'_Z \\ -s'_Y & s'_X c'_Y & c'_X c'_Y \end{bmatrix} = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}. \quad (18)$$

Clearly,

$$s'_Y = -r_{3,1} = c_X s_Y c_Z - s_X s_Z. \quad (19)$$

On the basis that s'_Y is the sine of R'_Y,

$$(c'_Y)^2 = 1 - (s'_Y)^2 = 1 - (c_X s_Y c_Z - s_X s_Z)^2. \quad (20)$$

Given the nature of the cosine function, the numerically-smallest possible value of R'_Y satisfies

$$c'_Y = \sqrt{1 - (s'_Y)^2}. \quad (21)$$

Equating the expressions for $r_{3,3}, r_{3,2}, r_{1,1}, r_{2,1}$ respectively,

$$c'_X c'_Y = c_X c_Y, \quad (22)$$

$$c'_Y s'_X = c_X s_Y s_Z + s_X c_Z, \quad (23)$$

$$c'_Y c'_Z = c_Y c_Z, \quad (24)$$

$$c'_Y s'_Z = s_X s_Y c_Z + c_X s_Z. \quad (25)$$

Since c'_Y is positive, the Version-1 rotations can be computed from

$$R'_X = \arctan2(c'_X c'_Y, s'_X c'_Y), \quad R'_Y = \arctan2(c'_Y, s'_Y), \quad R'_Z = \arctan2(c'_Y c'_Z, c'_Y s'_Z). \quad (26)$$

The arctan2(x,y) function is arctan(y/x) in the range -180° to 180° such that it always has the same sign as x.

The solution for the Version-1 rotations only uses the five identities from the third row and first column of the rotation matrix. Verification of the other four identities is given in Appendix B.

Equations (19) to (26) can be applied to reproduce example 1 in reverse.

Application to reverse (inverse) transformations

For either version of Helmert, the reverse transformation to Helmert can be obtained exactly by rearranging (1):

$$\begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} = \frac{1}{(1+\Delta S)} \mathbf{R}^{-1} \begin{bmatrix} X_t - \Delta X \\ Y_t - \Delta Y \\ Z_t - \Delta Z \end{bmatrix}. \tag{27}$$

Equation (27) provides a simple and exact method of computing the reverse transformation, particularly as **R** is orthogonal ($\mathbf{R}^{-1} = \mathbf{R}^T$, which is easily verified by computing $\mathbf{R}^T\mathbf{R}$). The method is noted in Reit (2009). However, there appears to be a preference among users for obtaining the reverse transformation by using the original formula with different parameters. Unfortunately, there is a tendency to simply reverse the signs of the shifts, scale-change and rotations. This only gives approximate results, and the larger the non-shift parameters the larger the errors. In the extreme cases of WGS 84 back to Fatu Iva 1972 and RGR 1992 back to Réunion 1947, the distance errors are 0.970 m and 0.404 m respectively.

Aktuğ (2009) describes a derivation of the inverse datum transformation, but claims that 'the scale and the rotation parameters will be the same as the direct transformation parameters with opposite signs'. This is not correct, since 1-ΔS differs (if only slightly) from 1/(1+ΔS). Also, his consideration of rotations makes no allowance for the order in which they are applied.

Nevertheless, it is possible to obtain reverse parameters that enable the Helmert formula to be applied in the other direction (target to source coordinates). In other words, given forward parameters ΔX, ΔY, ΔZ, S, R_X, R_Y, R_Z, there are corresponding 'same-formula inverse' parameters ΔX^{SFI}, ΔY^{SFI}, ΔZ^{SFI}, S^{SFI}, R_X^{SFI}, R_Y^{SFI}, R_Z^{SFI} which make the Helmert formula exact, at least in terms of consistency with the forward transformation.

This is shown below for both versions of Helmert.

Same-formula inverse parameters for Version 1 of Helmert

From (27) and the fact that **R** is orthogonal,

$$\begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} = \frac{1}{(1+\Delta S)} \mathbf{R}^{T} \begin{bmatrix} -\Delta X \\ -\Delta Y \\ -\Delta Z \end{bmatrix} + \frac{1}{(1+\Delta S)} \mathbf{R}^{T} \begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} \tag{28}$$

The algorithm for rotation-parameter conversion (Version 1 to Version 2) is applied to obtain the equivalent Version-2 rotations R'_X, R'_Y and R'_Z. The corresponding trigonometric quantities c'_X, s'_X, etc. are also computed.

$$\begin{aligned}
\mathbf{R}^T &= \left\{ \begin{bmatrix} c_Z & -s_Z & 0 \\ s_Z & c_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_Y & 0 & s_Y \\ 0 & 1 & 0 \\ -s_Y & 0 & c_Y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_X & -s_X \\ 0 & s_X & c_X \end{bmatrix} \right\}^T \\
&= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & c'_X & -s'_X \\ 0 & s'_X & c'_X \end{bmatrix} \begin{bmatrix} c'_Y & 0 & s'_Y \\ 0 & 1 & 0 \\ -s'_Y & 0 & c'_Y \end{bmatrix} \begin{bmatrix} c'_Z & -s'_Z & 0 \\ s'_Z & c'_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}^T \\
&= \begin{bmatrix} c'_Z & s'_Z & 0 \\ -s'_Z & c'_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c'_Y & 0 & -s'_Y \\ 0 & 1 & 0 \\ s'_Y & 0 & c'_Y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c'_X & s'_X \\ 0 & -s'_X & c'_X \end{bmatrix}
\end{aligned} \tag{29}$$

Comparing this with equation (2) and noting that $s'_X = -\sin(-R'_X)$, etc, it follows that

$$R_X^{SFI} = -R'_X, \quad R_Y^{SFI} = -R'_Y, \quad R_Z^{SFI} = -R'_Z. \tag{30}$$

From (28), the same-formula inverse parameters for scale and shift are given by the following formulae:

$$S^{SFI} = \frac{1}{1+\Delta S} = 1 + \frac{(-\Delta S)}{1+\Delta S} \tag{31}$$

$$\begin{bmatrix} \Delta X^{SFI} \\ \Delta Y^{SFI} \\ \Delta Z^{SFI} \end{bmatrix} = \frac{1}{(1+\Delta S)} \mathbf{R}^T \begin{bmatrix} -\Delta X \\ -\Delta Y \\ -\Delta Z \end{bmatrix} \tag{32}$$

In equation (32) for Version 1 of Helmert, $\mathbf{R}^T$ can be deduced from equation (3).

Example 2

Consider the Version-1 Helmert transformation with parameters ΔX = 346.90967 m, ΔY = 1078.23235 m, ΔZ = 2623.87087 m, ΔS = 186.1299981 ppm, R_X = -33.88457022″, R_Y = 70.66260075″ and R_Z = -9.395414631″. Some of these parameters are unusually large for a geodetic datum transformation, but it is suitable for converting Fatu Iva 1972 coordinates to WGS 84. It is the conformal transformation that comes closest to fitting the Bursa-Wolf transformation given in ESRI (2012).

R is the matrix given in Example 1. It was shown in that example that the equivalent Version-2 rotation parameters are -33.88135347″, 70.66414317″ and -9.38380681″. Equations (30) to (32) are applied. The same-formula inverse parameters (for use in Helmert Version 1) are -345.8972629 m, -1077.61650 m, -2623.67829 m, -186.0953602 ppm, 33.88135347″, -70.66414317″ and 9.38380681″.

Same-formula inverse parameters for Version 2 of Helmert

Equation (28) applies in this case also.

The algorithm for rotation-parameter conversion (Version 2 to Version 1) is applied to obtain the equivalent Version-1 rotations R'_X, R'_Y and R'_Z. The corresponding trigonometric quantities c'_X, s'_X, etc. are also computed.

The same-formula inverse parameters can be obtained from equations (30), (31) and (32), the proofs being entirely analogous to those given for the Version-1 case. In equation (32) for Version 2 of Helmert, $\mathbf{R}^T$ can be deduced from equation (5).

Equations (30) to (32) can be applied to reproduce example 2 in reverse.

Applications involving large rotations

Besides geodesy, conformal 3D transformations occur in a number of research areas. These include photogrammetry, geographical information science (GIS) and computer vision. In some areas, particularly in photogrammetry, conformal transformations involve rotations that are sometimes larger than 90° in magnitude. Deakin (1998) discusses conformal transformations of practical use in the construction industry, where the source coordinates are in the XYZ survey system and the target coordinates are in an ENU (East, North, Up) design system.

The algorithms given are still valid, but need two modifications:

- allowance for equivalent sets of rotation parameters that produce the same rotation matrices;
- an alternative process in the case where the cosine of R'_Y is zero, so as to avoid R'_X and R'_Z being undefined.

Rotation-parameter conversion from Version 1 to Version 2 is carried out as before. Given Version-1 rotations R_X, R_Y and R_Z, the method produces Version-2 rotations R'_X, R'_Y and R'_Z which give the same rotation matrix. It has already been noted that R'_Y will be in the range -90° to 90°. However, there is an alternative solution for which R'_Y is outside this range. That solution is $R'_X - \mathrm{sgn}(R'_X)180°$, $\mathrm{sgn}(R'_Y)180° - R'_Y$ and $R'_Z - \mathrm{sgn}(R'_Z)180°$. The solutions are equivalent because they give rise to the same rotation matrix.

The only problem occurs when $c'_Y = 0$, because (15) will leave R'_X and R'_Z undefined. From (11) and (13), this scenario only occurs if $c_Y = 0$ or $c_X = c_Z = 0$. This in turn only occurs if R_Y is an odd multiple of 90º or R_X and R_Z are both odd multiples of 90º. Table 1 (derived from trigonometrical identities) covers all special cases, although the equivalent Version-2 rotations are not necessarily unique.

Same-formula inverse parameters for Version 1 of Helmert can be computed by the algorithm already given, although it is only one set of SFI parameters. These can be denoted ΔX^{SFI}, ΔY^{SFI}, ΔZ^{SFI}, S^{SFI}, R_X^{SFI}, R_Y^{SFI} and R_Z^{SFI}. Of these, R_Y^{SFI} will be in the range -90° to 90°. The alternative – but equivalent – set of inverse parameters consists of ΔX^{SFI}, ΔY^{SFI}, ΔZ^{SFI}, S^{SFI}, $R_X^{SFI} - \mathrm{sgn}(R_X^{SFI})180°$, $\mathrm{sgn}(R_Y^{SFI})180° - R_Y^{SFI}$ and $R_Z^{SFI} - \mathrm{sgn}(R_Z^{SFI})180°$.

Table 1 Special cases of rotation conversions from Helmert Version 1 to Helmert Version 2

Helmert Version-1 rotations R_X, R_Y, R_Z	Equivalent Helmert Version-2 rotations R'_X, R'_Y, R'_Z
$90°, R_Y, 90°$	$90°, 90°, -R_Y$
$90°, R_Y, -90°$	$R_Y, -90°, -90°$
$-90°, R_Y, 90°$	$-90°, -90°, R_Y$
$-90°, R_Y, -90°$	$R_Y, 90°, -90°$
$R_X, 90°, R_Z$	$-90°, R_X - R_Z + 90°, 90°$
$R_X, -90°, R_Z$	$90°, R_X + R_Z - 90°, 90°$

Rotation-parameter conversion from Version 2 to Version 1 is carried out as before. Given Version-1 rotations R_X, R_Y and R_Z, the method produces Version-1 rotations R'_X, R'_Y and R'_Z which give the same rotation matrix. It has already been noted that R'_Y will be in the range -90° to 90°. However, there is an alternative solution for which R'_Y is outside this range. That solution is $R'_X - \text{sgn}(R'_X)180°$, $\text{sgn}(R'_Y)180° - R'_Y$ and $R'_Z - \text{sgn}(R'_Z)180°$. The solutions are equivalent because they give rise to the same rotation matrix.

The only problem occurs when $c'_Y = 0$ because (26) will leave R'_X and R'_Z undefined. From (22) and (24), this scenario only occurs if $c_Y = 0$ or $c_X = c_Z = 0$. This in turn only occurs if R_Y is an odd multiple of 90º or R_X and R_Z are both odd multiples of 90º. Table 2 (derived from trigonometrical identities) covers all special cases, although the equivalent Version-1 rotations are not necessarily unique.

Same-formula inverse parameters for Version 2 of Helmert can be computed by the algorithm already given, although it is only one set of SFI parameters. These can be denoted ΔX^{SFI}, ΔY^{SFI}, ΔZ^{SFI}, S^{SFI}, R_X^{SFI}, R_Y^{SFI} and R_Z^{SFI}. Of these, R_Y^{SFI} will be in the range -90° to 90°. The alternative – but equivalent – set of inverse parameters consists of ΔX^{SFI}, ΔY^{SFI}, ΔZ^{SFI}, S^{SFI}, $R_X^{SFI} - \text{sgn}(R_X^{SFI})180°$, $\text{sgn}(R_Y^{SFI})180° - R_Y^{SFI}$ and $R_Z^{SFI} - \text{sgn}(R_Z^{SFI})180°$.

Table 2: Special cases of rotation conversions from Helmert Version 2 to Helmert Version 1

Helmert Version-2 rotations R_X, R_Y, R_Z	Equivalent Helmert Version-1 rotations R'_X, R'_Y, R'_Z
$90°, R_Y, 90°$	$R_Y, -90°, 90°$
$90°, R_Y, -90°$	$R_Y, 90°, -90°$
$-90°, R_Y, 90°$	$R_Y, 90°, 90°,$
$-90°, R_Y, -90°$	$-90°, -90°, -R_Y$
$R_X, 90°, R_Z$	$-90°, R_X + R_Z + 90°, -90°$
$R_X, -90°, R_Z$	$90°, R_X - R_Z - 90°, -90°$

Example 3

Consider the Version-1 Helmert transformation with parameters ΔX = 197.306 m, ΔY = 157.968 m, ΔZ = 562.462 m, ΔS = 36.78040521 ppm, R_X = -50.05177814º, R_Y = 94.03082206º and R_Z = 10.12609423º. This is fictitious, but the rotations are close to those of the example in Deakin (1998).

Equation (3) will produce the following rotation matrix:

$$
\mathbf{R} = \begin{bmatrix} -0.0691981588 & -0.8657066577 & 0.4957454968 \\ -0.0123585870 & 0.4976424599 & 0.8672942104 \\ -0.9975263807 & 0.0538884487 & -0.0451348530 \end{bmatrix} \tag{33}
$$

Application of equations (8) to (15) will show that -92.97904193º, 29.71892126º and 94.57008241º are equivalent Version-2 rotation parameters. Equations (30) to (32) are applied. The first set of same-formula inverse parameters (for use in Helmert Version 1) consists of 576.65495 m, 61.88505 m, -209.42395 m, -36.77905246 ppm, 92.97904193º, -29.71892126º and -94.57008241º.

An alternative but equivalent set of 'same-parameter inverse' parameters consists of 576.65495 m, 61.88505 m, -209.42395 m, -36.77905246 ppm, -87.02095807º, -150.28107874º and 85.42991759º.

Equations (30) to (32) can be applied to reproduce example 3 in reverse (for Version 2 to Version 1).

Conclusions

The two commonly-used versions of the Helmert transformation involve different sets of rotation parameters, and the larger the non-shift parameters, the larger the differences. However, there are simple conversion formulae to convert Version-1 rotations into Version-2 rotations, and vice-versa. One application is that parameters obtained for one version of Helmert can be converted for use in software designed for the other version.

Where mathematical models of 3D transformations are conformal, it is desirable that the inverse transformations should be exact relative to the forward transformations. This is partly to avoid additional errors which can add up at individual points if transformations occur regularly between CRSs. It also has the merit of giving users of positioning instruments confidence in the built-in transformation software. This is achievable through the use of a 'same-formula inverse' method, provided the inverse parameters are computed by the algorithm recommended for the appropriate version of Helmert.

Both these results can be extended to Helmert transformations with large rotations, by making allowance for sets of rotation parameters that are equivalent in terms of the overall matrix.

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References

Aimaiti, N., 2015. A Comparison of Rotation Parameterisations for Bundle Adjustment. Master's Thesis in Computational Science and Engineering, Umeå University, Department of Computer Science, SE-901 87, Umeå, Sweden.

Aktuğ, B., 2009. Inverse and Compound Datum/Frame Transformations. *Journal of Surveying Engineering*, 135 (2) 46-55.

Awange, L. J., and Grafarend, W. E., 2002. Linearized Least Squares and nonlinear Gauss-Jacobi combinatorial algorithm applied to the 7-parameter datum transformation $\mathbb{C}_7(3)$ problem. *ZFV* 127 (2), 109-116.

Bursa, M., 1962. The Theory of the Determination of the Non-parallelism of the Minor Axis of the Reference Ellipsoid and the Inertial Polar Axis of the Earth, and the planes of the Initial Astronomic and Geodetic Meridians from Observation of Artificial Earth Satellites. *Studia Geophysica et Geodetica*, 6 (3), 209-214.

Deakin, R. E., 1998. 3D Coordinate Transformations. *Surveying and Land Information Systems*, 58 (4), 223-234.

Deakin, R. E., 2006. A Note on the Bursa-Wolf and Molodensky-Badekas Transformations. Department of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia, 21 pages.

ESRI , 2012. ArcGIS 10.1 Geographic and Vertical Transformation Tables. Resources.arcgis.com/en/help/main/10.1/003r/pdf/geographic_transformations.pdf

Fan, H., 2005. Three-Dimensional Coordinate Transformations with large rotations and scale change. International Workshop on Education in Geospatial Information Technology, 27-28 October, Technical University of Moldova, Chisinau, Moldova.

Featherstone, W. E., and Claessens, S. J., 2008. Closed-Form Transformation Between Geodetic and Ellipsoidal Coordinates. *Studia Geophysica et Geodaetica*, 52, 1-18.

Harvey, B. R., 1986. Transformation of 3D Coordinates. *The Australian Surveyor*, 33 (2), 105-125.

Heiskanen, W.A., and Moritz, H., 1967. Physical Geodesy. W. H. Freeman San Francisco.

ISO, 2007. *ISO 19111, Geographical information – Spatial referencing by coordinates* (2nd ed.). Geneva: International Organization for Standardization.

Malys, S., 1988. Dispersion and Correlations among Transformation Parameters, Relating Two Satellite Reference Frames. Report No. 329, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, Ohio.

NATO, 2001. Standardization Agreement (STANAG) 2211 IGEO, Geodetic Datums, Projections, Grids and Grid References, Edition 6, NATO Military Agency for Standardization, Brussels.

Ordnance Survey, 2018. A guide to coordinate systems in Great Britain. Version 3.3. https://www.ordnancesurvey.co.uk/docs/support/guide-coordinate-systems-great-britain.pdf?awc=2495_1472758581_2be7907c343c32b09a8d5171103197d7

Rapp, R. H., 1993. *Geometric Geodesy Part II*. Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, Ohio.

Reit, B.-G., 1998. The 7-parameter transformation to a horizontal geodetic datum. *Survey Review*, 34 (268), 400-404.

Reit, B.-G., 2009. *On geodetic transformations*. (LMV-Report 2010:1). Lantmäteriet — the Swedish mapping, cadastral and land registration authority. 62 pages.

Ruffhead, A.C., 2016. The SMITSWAM method of datum transformations consisting of Standard Molodensky in two stages with applied misclosures. *Survey Review*, 48 (350), 376-384.

Sjöberg, L. E., 2013. Closed-form and iterative weighted least squares solutions of Helmert transformation parameters. *Journal of Geodetic Science* 3 (1), 7-11.

Wang, Q., et al., 2018. Representation of the rotation parameter estimation errors in the Helmert transformation model. *Survey Review*, 50 (358), 69-81.

Watson, G. A., 2005. Computing Helmert transformations. *Journal of Computational and Applied Mathematics*, 197 (2), 387-394.

Wolf, H., 1963. Geometric connection and re-orientation of three-dimensional triangulation nets. *Bulletin Géodésique*, 68, 165-169.

Appendix A: Helmert equivalence verification (Version 1 to Version 2)

Equations (6) to (15) describe how to convert Version-1 parameters R_X, R_Y and R_Z into Version-2 parameters R'_X, R'_Y and R'_Z which lead to the same rotation matrix. However, the following identities were left unverified.

$$s'_X s'_Y c'_Z + c'_X s'_Z = r_{2,1}. \tag{34}$$

$$c'_X c'_Z - s'_X s'_Y s'_Z = r_{2,2}. \tag{35}$$

$$s'_X s'_Z - c'_X s'_Y c'_Z = r_{3,1}. \tag{36}$$

$$c'_X s'_Y s'_Z + s'_X c'_Z = r_{3,2}. \tag{37}$$

The proofs make use of equations (8) to (9) and (11) to (14). They also use the trigonometrical identity $\sin^2\theta + \cos^2\theta = 1$ and its variations, notably $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$ and $\sin^2\theta - \cos^2\theta = 1 - 2\cos^2\theta$.

The special case when $c'_Y = 0$ is covered by Table 1 and is not considered in this appendix. The proofs all involve $(c'_Y)^2$ as a common factor in both parts of a quotient. From (9),

$$(c'_Y)^2 = 1 - s_X^2 s_Z^2 - c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z. \tag{38}$$

Since $1 - s_X^2 s_Z^2 = (s_Z^2 + c_Z^2) - (1 - c_X^2) s_Z^2$, (38) can be rewritten as

$$(c'_Y)^2 = c_Z^2 + c_X^2 s_Z^2 - c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z. \tag{39}$$

Proof of (34):

$$s'_X s'_Y c'_Z + c'_X s'_Z = (r_{2,3}/c'_Y) r_{1,3} (r_{1,1}/c'_Y) + (r_{3,3}/c'_Y)(-r_{1,2}/c'_Y)$$

$$= \left[(s_X c_Z - c_X s_Y s_Z)(s_X s_Z + c_X s_Y c_Z) c_Y c_Z + c_X c_Y (c_X s_Z - s_X s_Y c_Z)\right] / (c'_Y)^2$$

$$= \left[\left(s_X^2 s_Z c_Z^2 + s_X c_X s_Y c_Z^3 - s_X c_X s_Y s_Z^2 c_Z - c_X^2 s_Y^2 s_Z c_Z^2 + c_X^2 s_Z - s_X c_X s_Y c_Z\right) c_Y\right] / (c'_Y)^2$$

$$= \left[\left(c_X^2 s_Z + s_X^2 s_Z c_Z^2 - c_X^2 s_Y^2 s_Z c_Z^2 + (s_X c_X s_Y c_Z^3 - s_X c_X s_Y c_Z) - s_X c_X s_Y s_Z^2 c_Z\right) c_Y\right] / (c'_Y)^2$$

$$= \left[\left(c_X^2 s_Z + s_X^2 s_Z c_Z^2 - c_X^2 s_Y^2 s_Z c_Z^2 - s_X c_X s_Y s_Z^2 c_Z - s_X c_X s_Y s_Z^2 c_Z\right) c_Y\right] / (c'_Y)^2 \quad \text{as } c_Z^2 - 1 = -s_Z^2$$

$$= \left[\left(c_X^2 + s_X^2 c_Z^2 - c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) s_Z c_Y\right] / (c'_Y)^2$$

$$= c_Y s_Z \quad \text{by (39)}$$

$$= r_{2,1}.$$

Proof of (35):

$$c'_X c'_Z - s'_X s'_Y s'_Z = (r_{3,3}/c'_Y)(r_{1,1}/c'_Y) - (-r_{2,3}/c'_Y) r_{1,3} (-r_{1,2}/c'_Y)$$

$$= \left[(c_X c_Y)(c_Y c_Z) - (s_X c_Z - c_X s_Y s_Z)(s_X s_Z + c_X s_Y c_Z)(c_X s_Z - s_X s_Y c_Z)\right] / (c'_Y)^2$$

$$= \left[c_X c_Y^2 c_Z + (c_X s_Y s_Z - s_X c_Z)\left(s_X c_X s_Z^2 - s_X^2 s_Y s_Z c_Z + c_X^2 s_Y s_Z c_Z - s_X c_X s_Y^2 c_Z^2\right)\right] / (c'_Y)^2$$

$$= \left[c_X c_Y^2 c_Z + s_X c_X^2 s_Y s_Z^3 - s_X^2 c_X s_Y^2 s_Z^2 c_Z + c_X^3 s_Y^2 s_Z^2 c_Z - s_X c_X^2 s_Y^3 s_Z c_Z^2 - s_X^2 c_X s_Z^2 c_Z + s_X^3 s_Y s_Z c_Z^2 - s_X c_X^2 s_Y s_Z c_Z^2 + s_X^2 c_X s_Y^2 c_Z^3\right] / (c'_Y)^2$$

$$= \left[c_X c_Y^2 c_Z + c_X^3 s_Y^2 s_Z^2 c_Z - s_X^2 c_X s_Z^2 c_Z + s_X^2 c_X s_Y^2 c_Z^3 - s_X c_X^2 s_Y s_Z c_Z^2 + s_X c_X^2 s_Y s_Z^3 + s_X^3 s_Y s_Z c_Z^2 - s_X c_X^2 s_Y^3 s_Z c_Z^2 - s_X^2 c_X s_Y^2 s_Z^2 c_Z\right] / (c'_Y)^2$$

by rearrangement of terms

$$= \left[c_X c_Y^2 c_Z + s_X^2 c_X s_Y^2 s_Z^2 c_Z + c_X^3 s_Y^2 s_Z^2 c_Z + s_X^2 c_X s_Y^2 c_Z^3 - s_X^2 c_X s_Z^2 c_Z - 2 s_X c_X^2 s_Y s_Z c_Z^2 \right.$$
$$\left. + s_X c_X^2 s_Y s_Z^3 + s_X c_X^2 s_Y s_Z c_Z^2 + s_X^3 s_Y s_Z c_Z^2 - s_X c_X^2 s_Y^3 s_Z c_Z^2 - 2 s_X^2 c_X s_Y^2 s_Z^2 c_Z\right] / (c'_Y)^2$$

by introducing terms that cancel

$$= \left(c_Y^2 + s_X^2 s_Y^2 s_Z^2 + c_X^2 s_Y^2 s_Z^2 - s_X^2 s_Z^2 + s_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) c_X c_Z / (c'_Y)^2 + \left(c_X^2 s_Z^2 + c_X^2 c_Z^2 + s_X^2 c_Z^2 - c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) s_X s_Y s_Z / (c'_Y)^2$$

$$= \left(c_Y^2 + s_Y^2 s_Z^2 - s_X^2 s_Z^2 + s_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) c_X c_Z / (c'_Y)^2$$

$$+ \left(s_X^2 c_Z^2 + c_Z^2 - c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) s_X s_Y s_Z / (c'_Y)^2 \text{ as } c_X^2 + s_X^2 = 1$$

$$= \left(1 - s_Y^2 + s_Y^2 (1 - c_Z^2) - s_X^2 s_Z^2 + (1 - c_X^2) s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) c_X c_Z / (c'_Y)^2$$

$$+ \left(s_X^2 c_Z^2 + c_Z^2 - c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z\right) s_X s_Y s_Z / (c'_Y)^2$$

$$= (c'_Y)^2 c_X c_Z / (c'_Y)^2 + (c'_Y)^2 s_X s_Y s_Z / (c'_Y)^2 \qquad \text{by (38) and (39)}$$

$$= c_X c_Z + s_X s_Y s_Z$$

$$= r_{2,2}.$$

Proof of (36):

$$s'_X s'_Z - c'_X s'_Y c'_Z = (-r_{2,3} / c'_Y)(-r_{1,2} / c'_Y) - (r_{3,3} / c'_Y) r_{1,3} (r_{1,1} / c'_Y)$$

$$= \left[(s_X c_Z - c_X s_Y s_Z)(c_X s_Z - s_X s_Y c_Z) - c_X c_Y (s_X s_Z + c_X s_Y c_Z) c_Y c_Z\right] / (c'_Y)^2$$

$$= \left[s_X c_X s_Z c_Z - s_X^2 s_Y c_Z^2 - c_X^2 s_Y s_Z^2 + s_X c_X s_Y^2 s_Z c_Z - s_X c_X c_Y^2 s_Z c_Z - c_X^2 s_Y c_Y^2 c_Z^2\right] / (c'_Y)^2$$

$$= \left[s_X c_X s_Z c_Z - s_X^2 s_Y c_Z^2 - c_X^2 s_Y s_Z^2 + 2 s_X c_X s_Y^2 s_Z c_Z - s_X c_X s_Z c_Z - c_X^2 s_Y c_Y^2 c_Z^2\right] / (c'_Y)^2$$

$$\text{as } s_Y^2 - c_Y^2 = 2 s_Y^2 - 1$$

$$= \left[- s_X^2 s_Y c_Z^2 - c_X^2 s_Y s_Z^2 + 2 s_X c_X s_Y^2 s_Z c_Z - c_X^2 s_Y (1 - s_Y^2) c_Z^2\right] / (c'_Y)^2$$

$$= \left[- s_X^2 s_Y c_Z^2 - c_X^2 s_Y s_Z^2 + 2 s_X c_X s_Y^2 s_Z c_Z - c_X^2 s_Y c_Z^2 + c_X^2 s_Y^3 c_Z^2\right] / (c'_Y)^2$$

$$= \left[- s_Y c_Z^2 - c_X^2 s_Y s_Z^2 + 2 s_X c_X s_Y^2 s_Z c_Z + c_X^2 s_Y^3 c_Z^2\right] / (c'_Y)^2 \quad \text{as} - s_X^2 - c_X^2 = -1$$

$$= -s_Y \qquad \text{by (39)}$$

$$= r_{3,1}.$$

Proof of (37):

$$c'_X s'_Y s'_Z + s'_X c'_Z = (r_{3,3} / c'_Y) r_{1,3} (-r_{1,2} / c'_Y) + (-r_{2,3} / c'_Y)(r_{1,1} / c'_Y)$$

$$= \left[c_X c_Y (s_X s_Z + c_X s_Y c_Z)(c_X s_Z - s_X s_Y c_Z) + (s_X c_Z - c_X s_Y s_Z) c_Y c_Z\right] / (c'_Y)^2$$

$$= \left[\left(s_X c_X^2 s_Z^2 - s_X^2 c_X s_Y s_Z c_Z + c_X^3 s_Y s_Z c_Z - s_X c_X^2 s_Y^2 c_Z^2 + s_X c_Z^2 - c_X s_Y s_Z c_Z\right) c_Y\right] / (c'_Y)^2$$

$$= \left[\left(s_X c_X^2 s_Z^2 - s_X c_X^2 s_Y^2 c_Z^2 + c_X s_Y s_Z c_Z - 2 s_X^2 c_X s_Y s_Z c_Z + s_X c_Z^2 - c_X s_Y s_Z c_Z\right) c_Y\right] / (c'_Y)^2$$

$$\text{since } - s_X^2 + c_X^2 = 1 - 2 s_X^2$$

$$= \left(s_X c_X^2 s_Z^2 - s_X c_X^2 s_Y^2 c_Z^2 - 2 s_X^2 c_X s_Y s_Z c_Z + s_X c_Z^2\right) c_Y / (c'_Y)^2$$

$$= s_X c_Y \qquad \text{by (39)}$$

$$= r_{3,2}.$$

Appendix B: Helmert equivalence verification (Version 2 to Version 1)

Equations (17) to (26) describe how to convert Version-2 parameters R_X, R_Y and R_Z into Version-1 parameters R'_X, R'_Y and R'_Z which lead to the same rotation matrix. However, the following identities were left unverified.

$$s'_X s'_Y c'_Z - c'_X s'_Z = r_{1,2}. \tag{40}$$

$$s'_X s'_Z + c'_X s'_Y c'_Z = r_{1,3}. \tag{41}$$

$$c'_X c'_Z + s'_X s'_Y s'_Z = r_{2,2}. \tag{42}$$

$$c'_X s'_Y s'_Z - s'_X c'_Z = r_{2,3}. \tag{43}$$

The proofs make use of equations (19) to (20) and (22) to (25). They also use the trigonometrical identity $\sin^2\theta+\cos^2\theta=1$ and its variations, notably $\cos^2\theta-\sin^2\theta=1-2\sin^2\theta$ and $\sin^2\theta-\cos^2\theta=1-2\cos^2\theta$.

The special case when $c'_Y=0$ is covered by Table 2 and is not considered in this appendix. The proofs all involve $(c'_Y)^2$ as a common factor in both parts of a quotient. From (20),

$$(c'_Y)^2=1-s_X^2s_Z^2-c_X^2s_Y^2c_Z^2+2s_Xc_Xs_Ys_Zc_Z. \tag{44}$$

Since $1-s_X^2s_Z^2=(s_X^2+c_Z^2)-s_Z^2(1-c_X^2)$, (44) can be rewritten as

$$(c'_Y)^2=c_X^2+s_X^2c_Z^2-c_X^2s_Y^2c_Z^2+2s_Xc_Xs_Ys_Zc_Z. \tag{45}$$

Since $1-s_X^2s_Z^2=(s_Z^2+c_Z^2)-(1-c_X^2)s_Z^2$, (44) can also be rewritten as

$$(c'_Y)^2=c_Z^2+c_X^2s_Z^2-c_X^2s_Y^2c_Z^2+2s_Xc_Xs_Ys_Zc_Z. \tag{46}$$

Proof of (40):

$$s'_Xs'_Yc'_Z-c'_Xs'_Z=(r_{3,2}/c'_Y)(-r_{3,1})(r_{1,1}/c'_Y)-(r_{3,3}/c'_Y)(r_{2,1}/c'_Y)$$
$$=\left[(c_Xs_Ys_Z+s_Xc_Z)(c_Xs_Yc_Z-s_Xs_Z)c_Yc_Z-c_Xc_Y(s_Xs_Yc_Z+c_Xs_Z)\right]/(c'_Y)^2$$
$$=\left[\left(c_X^2s_Y^2s_Zc_Z^2-s_Xc_Xs_Ys_Z^2c_Z+s_Xc_Xs_Yc_Z^3-s_X^2s_Zc_Z^2-s_Xc_Xs_Yc_Z-c_X^2s_Z\right)c_Y\right]/(c'_Y)^2$$
$$=\left[\left(c_X^2s_Y^2s_Zc_Z^2-c_X^2s_Z-s_X^2s_Zc_Z^2+s_Xc_Xs_Yc_Z^3-s_Xc_Xs_Yc_Z-s_Xc_Xs_Ys_Z^2c_Z\right)c_Y\right]/(c'_Y)^2$$
by rearrangement of terms
$$=\left[\left(c_X^2s_Y^2s_Zc_Z^2-c_X^2s_Z-s_X^2s_Zc_Z^2+s_Xc_Xs_Yc_Z^3-s_Xc_Xs_Yc_Z^3-2s_Xc_Xs_Ys_Z^2c_Z\right)c_Y\right]/(c'_Y)^2$$
as $-1-s_Z^2=-c_Z^2-2s_Z^2$
$$=\left[\left(c_X^2s_Y^2s_Zc_Z^2-c_X^2s_Z-s_X^2s_Zc_Z^2-2s_Xc_Xs_Ys_Z^2c_Z\right)c_Y\right]/(c'_Y)^2$$
$$=-(c'_Y)^2c_Ys_Z/(c'_Y)^2 \quad \text{by (45)}$$
$$=-c_Ys_Z$$
$$=r_{1,2}.$$

Proof of (41):

$$s'_Xs'_Z+c'_Xs'_Yc'_Z=(r_{3,2}/c'_Y)(r_{2,1}/c'_Y)+(r_{3,3}/c'_Y)(-r_{3,1})(r_{1,1}/c'_Y)$$
$$=\left[(c_Xs_Ys_Z+s_Xc_Z)(s_Xs_Yc_Z+c_Xs_Z)+c_Xc_Y(c_Xs_Yc_Z-s_Xs_Z)c_Yc_Z\right]/(c'_Y)^2$$
$$=\left[s_Xc_Xs_Y^2s_Zc_Z+c_X^2s_Ys_Z^2+s_X^2s_Yc_Z^2+s_Xc_Xs_Zc_Z+c_X^2s_Yc_Y^2c_Z^2-s_Xc_Xc_Y^2s_Zc_Z\right]/(c'_Y)^2$$
$$=\left[s_Xc_Xs_Y^2s_Zc_Z-s_Xc_Xc_Y^2s_Zc_Z+s_Xc_Xs_Zc_Z+c_X^2s_Ys_Z^2+s_X^2s_Yc_Z^2+c_X^2s_Yc_Y^2c_Z^2\right]/(c'_Y)^2$$
by rearrangement of terms
$$=\left[2s_Xc_Xs_Y^2s_Zc_Z-s_Xc_Xs_Zc_Z+s_Xc_Xs_Zc_Z+c_X^2s_Ys_Z^2+s_X^2s_Yc_Z^2+c_X^2s_Yc_Y^2c_Z^2\right]/(c'_Y)^2$$
since $s_Y^2-c_Y^2=2s_Y^2-1$
$$=\left[2s_Xc_Xs_Y^2s_Zc_Z+c_X^2s_Ys_Z^2+(1-c_X^2)s_Yc_Z^2+c_X^2s_Y(1-s_Y^2)c_Z^2\right]/(c'_Y)^2$$
$$=\left[2s_Xc_Xs_Y^2s_Zc_Z-c_X^2s_Yc_Z^2+s_X^2s_Yc_Z^2+s_Yc_Z^2\right]/(c'_Y)^2$$
$$=s_Y(c'_Y)^2/(c'_Y)^2 \quad \text{by (46)}$$
$$=s_Y$$
$$=r_{1,3}.$$

Proof of (42):

$$c'_X c'_Z + s'_X s'_Y s'_Z = (r_{3,3}/c'_Y)(r_{1,1}/c'_Y) + (r_{3,2}/c'_Y)(-r_{1,3})(r_{2,1}/c'_Y)$$

$$= \left[(c_X c_Y)(c_Y c_Z) + (c_X s_Y s_Z + s_X c_Z)(c_X s_Y c_Z - s_X s_Z)(s_X s_Y c_Z + c_X s_Z)\right] / (c'_Y)^2$$

$$= \left[c_X c_Y^2 c_Z + (c_X s_Y s_Z + s_X c_Z)\left(s_X c_X s_Y^2 c_Z^2 + c_X^2 s_Y s_Z c_Z - s_X^2 s_Y s_Z c_Z - s_X c_X s_Z^2\right)\right] / (c'_Y)^2$$

$$= \left[c_X c_Y^2 c_Z + s_X c_X^2 s_Y^3 s_Z c_Z^2 + c_X^3 s_Y^2 s_Z^2 c_Z - s_X^2 c_X s_Y^2 s_Z^2 c_Z - s_X c_X^2 s_Y s_Z^3\right.$$
$$\left. + s_X^2 c_X s_y^2 c_Z^3 + s_X c_X^2 s_Y s_Z c_Z^2 - s_X^3 s_Y s_Z c_Z^2 - s_X^2 c_X s_Z^2 c_Z\right] / (c'_Y)^2$$

$$= \left[c_X^3 s_Y^2 s_Z^2 c_Z + c_X c_Y^2 c_Z + s_X c_X^2 s_Y s_Z c_Z^2 - s_X^2 c_X s_Z^2 c_Z\right.$$
$$\left. + s_X c_X^2 s_Y^3 s_Z c_Z^2 - s_X^3 s_Y s_Z c_Z^2 + (s_X^2 c_X s_Y^2 c_Z^3 - s_X^2 c_X s_Y^2 s_Z^2 c_Z) - s_X c_X^2 s_Y s_Z^3\right] / (c'_Y)^2$$

by rearrangement of terms

$$= \left[c_X^3 s_Y^2 s_Z^2 c_Z + c_X c_Y^2 c_Z + s_X c_X^2 s_Y s_Z c_Z^2 - s_X^2 c_X s_Z^2 c_Z\right.$$
$$\left. + s_X c_X^2 s_Y^3 s_Z c_Z^2 - s_X^3 s_Y s_Z c_Z^2 + (s_X^2 c_X s_Y^2 c_Z - 2 s_X^2 c_X s_Y^2 s_Z^2 c_Z) + (s_X c_X^2 s_Y s_Z c_Z^2 - s_X c_X^2 s_Y s_Z)\right] / (c'_Y)^2$$

since $c_Z^2 - s_Z^2 = 1 - 2s_Z^2$ and $-s_Z^2 = c_Z^2 - 1$

$$= \left[c_X^3 s_Y^2 s_Z^2 c_Z + c_X c_Y^2 c_Z + 2 s_X c_X^2 s_Y s_Z c_Z^2 + s_X^2 c_X s_Y^2 c_Z - s_X^2 c_X s_Z^2 c_Z + s_X c_X^2 s_Y^3 s_Z c_Z^2 - s_X^3 s_Y s_Z c_Z^2 - 2 s_X^2 c_X s_Y^2 s_Z^2 c_Z - s_X c_X^2 s_Y s_Z^3\right] / (c'_Y)^2$$

$$= c_X c_Z \left(c_X^2 s_Y^2 s_Z^2 + c_Y^2 + 2 s_X c_X s_Y s_Z c_Z + s_X^2 s_Y^2 - s_X^2 s_Z^2\right) / (c'_Y)^2 + \left(c_X^2 s_Y^2 c_Z^2 - s_X^2 c_Z^2 + 2 s_X c_X s_Y s_Z c_Z - c_X^2\right) s_X s_Y s_Z / (c'_Y)^2$$

$$= c_X c_Z \left(c_X^2 s_Y^2 (1 - c_Z^2) + c_Y^2 + s_X^2 s_Y^2 - s_X^2 s_Z^2 + 2 s_X c_X s_Y s_Z c_Z\right) / (c'_Y)^2 + \left(c_X^2 s_Y^2 c_Z^2 - s_X^2 c_Z^2 + 2 s_X c_X s_Y s_Z c_Z - c_X^2\right) s_X s_Y s_Z / (c'_Y)^2$$

$$= c_X c_Z \left(c_X^2 s_Y^2 - c_X^2 s_Y^2 c_Z^2 + c_Y^2 + s_X^2 s_Y^2 - s_X^2 s_Z^2 + 2 s_X c_X s_Y s_Z c_Z\right) / (c'_Y)^2$$
$$+ \left(c_X^2 s_Y^2 c_Z^2 - s_X^2 c_Z^2 + 2 s_X c_X s_Y s_Z c_Z - c_X^2\right) s_X s_Y s_Z / (c'_Y)^2$$

$$\left.\begin{aligned} &= c_X c_Z \left(s_Y^2 - c_X^2 s_Y^2 c_Z^2 + c_Y^2 - s_X^2 s_Z^2 + 2 s_X c_X s_Y s_Z c_Z\right) / (c'_Y)^2 \\ &\quad + \left(c_X^2 s_Y^2 c_Z^2 - s_X^2 c_Z^2 + 2 s_X c_X s_Y s_Z c_Z - c_X^2\right) s_X s_Y s_Z / (c'_Y)^2 \end{aligned}\right\} \text{ as } c_X^2 + s_X^2 = 1$$

$$\left.\begin{aligned} &= c_X c_Z \left(1 - c_X^2 s_Y^2 c_Z^2 - s_X^2 s_Z^2 + 2 s_X c_X s_Y s_Z c_Z\right) / (c'_Y)^2 \\ &\quad + \left(c_X^2 s_Y^2 c_Z^2 - s_X^2 c_Z^2 + 2 s_X c_X s_Y s_Z c_Z - c_X^2\right) s_X s_Y s_Z / (c'_Y)^2 \end{aligned}\right\} \text{ as } s_Y^2 + c_Y^2 = 1$$

$$= c_X c_Z (c'_Y)^2 / (c'_Y)^2 + (-c'_Y)^2 s_X s_Y s_Z / (c'_Y)^2 \} \text{ by (44) and (45)}$$

$$= c_X c_Z - s_X s_Y s_Z$$

$$= r_{2,2}.$$

Proof of (43):

$$c'_X s'_Y s'_Z - s'_X c'_Z = (r_{3,3}/c'_Y) r_{1,3} (r_{2,1}/c'_Y) - (r_{3,2}/c'_Y)(r_{1,1}/c'_Y)$$

$$= \left[c_X c_Y (c_X s_Y c_Z - s_X s_Z)(s_X s_Y c_Z + c_X s_Z) - (c_X s_Y s_Z + s_X c_Z) c_Y c_Z\right] / (c'_Y)^2$$

$$= \left[\left(s_X c_X^2 s_Y^2 c_Z^2 + c_X^3 s_Y s_Z c_Z - s_X^2 c_X s_Y s_Z c_Z - s_X c_X^2 s_Z^2 - c_X s_Y s_Z c_Z - s_X c_Z^2\right) c_Y\right] / (c'_Y)^2$$

$$= \left[\left(s_X c_X^2 s_Y^2 c_Z^2 - c_X s_Y s_Z c_Z + (c_X^3 s_Y s_Z c_Z - s_X^2 c_X s_Y s_Z c_Z) - s_X c_X^2 s_Z^2 - s_X c_Z^2\right) c_Y\right] / (c'_Y)^2$$

by rearrangement of terms

$$= \left[\left(s_X c_X^2 s_Y^2 c_Z^2 - c_X s_Y s_Z c_Z + (c_X s_Y s_Z c_Z - 2 s_X^2 c_X s_Y s_Z c_Z) - s_X c_X^2 s_Z^2 - s_X c_Z^2\right) c_Y\right] / (c'_Y)^2$$

as $c_X^2 - s_X^2 = 1 - 2s_X^2$

$$= \left[\left(s_X c_X^2 s_Y^2 c_Z^2 - 2 s_X^2 c_X s_Y s_Z c_Z - s_X c_X^2 s_Z^2 - s_X c_Z^2\right) c_Y\right] / (c'_Y)^2$$

$$= \left(c_X^2 s_Y^2 c_Z^2 - 2 s_X c_X s_Y s_Z c_Z - c_X^2 s_Z^2 - c_Z^2\right) s_X c_Y / (c'_Y)^2$$

$$= (-c'_Y)^2 s_X c_Y / (c'_Y)^2 \quad \text{by (46)}$$

$$= -s_X c_Y$$

$$= r_{2,3}.$$