On the universal embedding of the near hexagon related to the extended ternary Golay code

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Abstract

Let $\mathbb{E}_1$ be the near hexagon on 729 points related to the extended ternary Golay code. We prove in an entirely geometric way that the generating and embedding ranks of $\mathbb{E}_1$ are equal to 24. We also study the structure of the universal embedding $\widetilde{e}$ of $\mathbb{E}_1$. More precisely, we consider several nice subgeometries $\mathcal{A}$ of $\mathbb{E}_1$ and determine which kind of embedding $\widetilde{e}_{\mathcal{A}}$ is, where $\widetilde{e}_{\mathcal{A}}$ is the embedding of $\mathcal{A}$ induced by $\widetilde{e}$.

Keywords: generating rank, embedding rank, universal embedding, near hexagon, extended ternary Golay code
MSC2000: 05B25, 51A45

1 Introduction

In the literature, the embedding and generating ranks have been determined for several classes of point-line geometries, see Cooperstein [7] for a survey of the most important results obtained on this topic before the year 2003. In the present paper, we determine in an entirely geometric way the embedding and generating ranks of the near hexagon $\mathbb{E}_1$ on 729 points which is related to the extended ternary Golay code. Previously, only the embedding rank of that geometry was known. This embedding rank was determined in Brouwer et al. [4, p. 350] with the aid of a computer and in Yoshiara [16] with some group theoretical argument involving the Leech lattice.

Theorem 1.1 *The embedding and generating ranks of* $\mathbb{E}_1$ *are equal to* 24.

The near hexagon $\mathbb{E}_1$ has many subgeometries which are isomorphic to the point-line geometry $\mathcal{A}^*$ which we are going to define now. Embed $\mathrm{PG}(4,3)$ as a hyperplane in $\mathrm{PG}(5,3)$ and let X be a set of 6 points of $\mathrm{PG}(4,3)$ no five of which are contained in a hyperplane of $\mathrm{PG}(4,3)$. Then $\mathcal{A}^*$ is defined as follows:

- the points of $\mathcal{A}^*$ are the points of $\mathrm{PG}(5,3)$ not contained in $\mathrm{PG}(4,3)$;

• the lines of $\mathcal{A}^*$ are the lines of PG$(5,3)$ not contained in PG$(4,3)$ which contain a unique point of X;

• incidence is the one derived from PG$(5,3)$.

Any subgeometry of $\mathbb{E}_1$ which is isomorphic to $\mathcal{A}^*$ is called a *special subgeometry* of $\mathbb{E}_1$. For every $i \in \{1, 2, \ldots, 6\}$, the near hexagon $\mathbb{E}_1$ also has subgeometries which are isomorphic to the Hamming near $2i$-gon $H(i, 3)$. In Section 3.1, we will determine all special subgeometries of $\mathbb{E}_1$ as well as all subgeometries isomorphic to $H(i, 3)$, $i \in \{1, 2, \ldots, 6\}$.

An essential step in the proof that $\mathbb{E}_1$ can be generated by 24 points (see Theorem 1.1) is the proof that any special subgeometry of $\mathbb{E}_1$ can be generated by 22 points. This latter fact will also allow us to determine the embedding and generating ranks of $\mathcal{A}^*$.

Theorem 1.2 *The embedding and generating ranks of $\mathcal{A}^*$ are equal to* 22.

We are also interested in the structure of the universal embedding $\widetilde{e}$ of $\mathbb{E}_1$. More precisely, we are interested in the following kind of problem.

> Suppose $\mathcal{A}$ is a subgeometry of $\mathbb{E}_1$ and $\widetilde{e}_{\mathcal{A}}$ is the embedding of $\mathcal{A}$ induced by $\widetilde{e}$. What kind of embedding is $\widetilde{e}_{\mathcal{A}}$?

We will give an answer to the above question in the case that $\mathcal{A}$ is a special subgeometry or a subgeometry isomorphic to $H(i, 3)$, $i \in \{1, 2, 3, 4\}$.

Theorem 1.3 *Let $\widetilde{e}$ denote the universal embedding of $\mathbb{E}_1$, let $\mathcal{A}$ be a subgeometry of $\mathbb{E}_1$ which is either a special subgeometry or a subgeometry isomorphic to $H(i, 3)$ for some $i \in \{1, 2, 3, 4\}$. Let X denote the point set of $\mathcal{A}$. Then the following holds.*

(1) *The projective embedding of $\mathcal{A}$ induced by $\widetilde{e}$ is isomorphic to the universal embedding of $\mathcal{A}$.*

(2) *A point x of $\mathbb{E}_1$ belongs to X if and only if $\widetilde{e}(x) \in \langle \widetilde{e}(X) \rangle$.*

A description of the universal embedding of $\mathbb{E}_1$ was given in Yoshiara [16, Section 3.2]. This universal embedding was realized in the Leech lattice modulo 2. In Section 3.2, we give an explicit description of the universal embeddings of the Hamming near polygons $H(n, 3)$, $n \geq 1$.

2 Basic notions

2.1 Near polygons

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$ there exists a unique point on L nearest to x. Here, distances are measured in the collinearity graph Γ of $\mathcal{S}$. If $d \in \mathbb{N}$ is the diameter of Γ, then the near polygon is called a *near $2d$-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles.

A finite near $2d$-gon $\mathcal{S}$ with $d \geq 2$ is called *regular* if its collinearity graph is a so-called distance-regular graph (Brouwer et al. [5]). This implies that there exist constants s, t, t_i ($i \in \{2, \ldots, d-1\}$) such that every line is incident with precisely $s+1$ points, every point is incident with precisely $t+1$ lines and for every two points x and y at distance $i \in \{2, \ldots, d-1\}$ from each other, there are precisely $t_i + 1$ lines through y containing a (necessarily unique) point at distance $i-1$ from x. We call $(s, t, t_2, t_3, \ldots, t_{d-1})$ *the parameters of* $\mathcal{S}$.

Let $n, k \in \mathbb{N} \setminus \{0\}$ with $k \geq 2$ and put $A := \{1, 2, \ldots, k\}$. Let $H(n,k)$ denote the point-line geometry whose points are the elements of the cartesian power A^n and whose lines are all the sets of the form $\{a_1\} \times \cdots \times \{a_{i-1}\} \times A \times \{a_{i+1}\} \times \cdots \times \{a_n\}$, where i is some element of $\{1, 2, \ldots, n\}$ and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ are some elements of A (natural incidence). The point-line geometry $H(n,k)$ is a near $2n$-gon. It is called a *Hamming near polygon*.

Let $\mathbb{F}_3^{12}$ denote the 12-dimensional vector space over the field $\mathbb{F}_3$ of order 3 whose vectors are row matrices of length 12 with entries in $\mathbb{F}_3$. The 6 rows of the matrix

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}$$

generate a 6-dimensional subspace G_{12} of $\mathbb{F}_3^{12}$ which is called the *extended ternary Golay code*. By deleting one coordinate position, one gets a code (a subspace of $\mathbb{F}_3^{11}$) which was discovered by Golay [12]. Let $\mathbb{E}_1$ be the point-line geometry whose points are all the cosets of G_{12} and whose lines are all the triples of the form $\{\bar{v} + G_{12}, \bar{v} + \bar{e}_i + G_{12}, \bar{v} - \bar{e}_i + G_{12}\}$, with incidence being containment. Here, $\bar{v}$ is some vector of $\mathbb{F}_3^{12}$ and $\bar{e}_i$, $i \in \{1, 2, \ldots, 12\}$, denotes the row matrix all of whose entries are 0 except for the i-th one which is equal to 1. Shult and Yanushka [15, pp. 30–33] proved that $\mathbb{E}_1$ is a regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$. Brouwer [3] proved that $\mathbb{E}_1$ is the unique regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$. Every two points of $\mathbb{E}_1$ at distance 2 from each other are contained in a unique (3×3)-subgrid, called a *quad*, see Shult and Yanushka [15, Proposition 2.5]. If L_1 and L_2 are two lines meeting in a unique point, then also L_1 and L_2 are contained in a unique quad.

Another model for the near hexagon $\mathbb{E}_1$ was described in De Bruyn and De Clerck [11]. Let Π_∞ be a hyperplane of the projective space $\mathrm{PG}(6,3)$. For every set $\mathcal{K}$ of points of Π_∞, let $T_5^*(\mathcal{K})$ denote the point-line geometry whose points are the points of $\mathrm{PG}(6,3) \setminus \Pi_\infty$ and whose lines are those lines of $\mathrm{PG}(6,3)$ not contained in Π_∞ which intersect Π_∞ in a point of $\mathcal{K}$ (natural incidence). After fixing some reference system in Π_∞, the 12 columns of the matrix M define a set $\mathcal{K}^*$ of 12 points of Π_∞. By De Bruyn [9, Theorem 6.62(b)], the point-line geometry $T_5^*(\mathcal{K}^*)$ is a regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$. Hence, $T_5^*(\mathcal{K}^*) \cong \mathbb{E}_1$.

The set $\mathcal{K}^*$ satisfies several nice properties, see e.g. Coxeter [8]. Among them, we have the following ones which are of interest for the present paper.

• If Y is a set of $i \in \{1,2,3,4\}$ points of $\mathcal{K}^*$, then $\dim(\langle Y \rangle) = i-1$ and $\langle Y \rangle \cap \mathcal{K}^* = Y$.

• If Y is a set of 5 points of $\mathcal{K}^*$, then $\langle Y \rangle$ is a hyperplane of Π_∞ and $\langle Y \rangle$ contains precisely 6 points of $\mathcal{K}^*$. Hence, if $\mathcal{B}$ denotes the set of all sets B of size 6 which can be obtained by intersecting $\mathcal{K}^*$ with a hyperplane of Π_∞, then the point-line geometry with point set $\mathcal{K}^*$, line set $\mathcal{B}$ and natural incidence relation is a Steiner system $S(5,6,12)$. There is only one Steiner system with these parameters, see Beth, Jungnickel and Lenz [1, p. 240, Corollary 2.6]. In fact, this Steiner system is one of the small Witt designs.

• It is known (see [1, p. 238]) that the complement of a block of $S(5,6,12)$ is again a block of $S(5,6,12)$. Hence, if $\mathcal{K}^* = \{x_1, x_2, \ldots, x_{12}\}$ such that $\langle x_1, x_2, \ldots, x_6 \rangle$ is a hyperplane of Π_∞, then also $\langle x_7, x_8, \ldots, x_{12} \rangle$ is a hyperplane of Π_∞.

2.2 Generating rank, embedding rank and universal embedding

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. A *subspace* of $\mathcal{S}$ is a set of points which contains all the points of a line as soon as it contains at least two points of it. A *hyperplane* of $\mathcal{S}$ is a subspace distinct from $\mathcal{P}$ which meets each line. If X is a subspace of $\mathcal{S}$, then $\widetilde{X}$ denotes the subgeometry of $\mathcal{S}$ defined on the point set X by those lines of $\mathcal{S}$ which have all their points in X. If X is a set of points of $\mathcal{S}$, then $[X]$ denotes the smallest subspace of $\mathcal{S}$ containing the set X. We call $[X]$ the *subspace of* $\mathcal{S}$ *generated by* X. If $[X] = \mathcal{P}$, then we will also say that X *generates* $\mathcal{S}$ or that X is a *generating set* of $\mathcal{S}$. The minimal size of a generating set of $\mathcal{S}$ is called the *generating rank* of $\mathcal{S}$ and is denoted by $gr(\mathcal{S})$.

An *embedding* e of $\mathcal{S}$ into a projective space Σ is an injective mapping e from $\mathcal{P}$ to the point set of Σ satisfying: (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is contained in a line of Σ for every line L of $\mathcal{S}$. The embedding e is called *full* if $e(L)$ is a line of Σ for every line L of $\mathcal{S}$. If n is the maximal dimension of a projective space in which $\mathcal{S}$ has a full embedding, then the number $er(\mathcal{S}) := n+1$ is called the *embedding rank* of $\mathcal{S}$. Certainly, $er(\mathcal{S})$ is only defined when $\mathcal{S}$ admits a full embedding, in which case it holds that $er(\mathcal{S}) \leq gr(\mathcal{S})$. If e is a full embedding of $\mathcal{S}$ into a projective space Σ and if Π is a hyperplane of Σ, then $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. Any hyperplane of $\mathcal{S}$ which can be obtained in this way is said to *arise from* e.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a fully embeddable point-line geometry with three points on each line. Then by Ronan [13], $\mathcal{S}$ admits the so-called universal embedding and every hyperplane of $\mathcal{S}$ arises from this embedding. We now give a description of this universal embedding. Let V be a vector space over the field $\mathbb{F}_2$ of order 2 with a basis B whose vectors are indexed by the elements of $\mathcal{P}$, say $B = \{\overline{v}_x \,|\, x \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors $\overline{v}_{x_1} + \overline{v}_{x_2} + \overline{v}_{x_3}$ where $\{x_1, x_2, x_3\}$ is a line of $\mathcal{S}$. Then the map $x \in \mathcal{P} \mapsto \{\overline{v}_x + W, W\}$ defines a full embedding of $\mathcal{S}$ into the projective space $\mathrm{PG}(V/W)$ which is isomorphic to the universal embedding of $\mathcal{S}$. We have $er(\mathcal{S}) = \dim(V/W) = \dim(V) - \dim(W)$.

It is known that every dense near polygon with three points per line admits a full projective embedding (Brouwer and Shpectorov [6, Proposition 3.1(ii)]; De Bruyn [10, Proposition 3.11]). In particular, this holds for the near hexagon $\mathbb{E}_1$.

3 Some useful results

The aim of this section is to study some subgeometries of $\mathbb{E}_1$, to determine the generating rank, embedding rank and universal embedding of every Hamming near polygon $H(n, 3)$, $n \geq 2$, and to derive a lower bound for $er(\mathbb{E}_1)$.

3.1 Subgeometries of $\mathbb{E}_1$

As in Section 2.1, let Π_∞ be a hyperplane of $\mathrm{PG}(6, 3)$ and let $\mathcal{K}^*$ be a set of 12 points of Π_∞ defined by the columns of the matrix M. Then $T_5^*(\mathcal{K}^*) \cong \mathbb{E}_1$. If L is a line of $\mathrm{PG}(6, 3)$ not contained in Π_∞, then the unique point of $L \cap \Pi_\infty$ is called the *point at infinity* of L. Let V be a 7-dimensional vector space over $\mathbb{F}_3$ such that $\mathrm{PG}(6, 3) = \mathrm{PG}(V)$.

Let Y be a set of $i \in \{1, 2, \ldots, 6\}$ linearly independent points of $\mathcal{K}^*$ and let α be an i-dimensional subspace of $\mathrm{PG}(6, 3)$ which intersects Π_∞ in the subspace $\langle Y \rangle$. Then $S_\alpha := \alpha \setminus \Pi_\infty$ is a subspace of the near hexagon $T_5^*(\mathcal{K}^*)$. Let $\mathcal{A}$ be the subgeometry of $T_5^*(\mathcal{K}^*)$ defined on the set S_α by those lines of $T_5^*(\mathcal{K}^*)$ whose point at infinity belongs to Y. Notice that if $i \in \{1, 2, 3, 4\}$, then $\mathcal{A} = \widetilde{S_\alpha}$. By coordinatizing with respect to a basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_7\}$ of V for which $Y = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \ldots, \langle \bar{e}_i \rangle\}$ and $\langle \bar{e}_{i+1} \rangle \in \alpha \setminus \Pi_\infty$, we readily see that the geometry $\mathcal{A}$ must be isomorphic to $H(i, 3)$.

Theorem 3.1 *Every subgeometry $\mathcal{A}$ of $T_5^*(\mathcal{K}^*)$ isomorphic to $H(i, 3)$, $i \in \{1, 2, \ldots, 6\}$, is obtained in the above-described way.*

Proof. Let x be an arbitrary point of $\mathcal{A}$, let $L_1, L_2, \ldots, L_i$ denote the i lines of $\mathcal{A}$ through x and let y_j, $j \in \{1, 2, \ldots, i\}$, denote the point at infinity of L_j. Put $\alpha := \langle y_1, y_2, \ldots, y_i, x \rangle$. If $i \leq 5$, then the points $y_1, y_2, \ldots, y_i$ are linearly independent and hence α has dimension i. If $i = 6$, then α has dimension 5 or 6. The lines of $\mathcal{A}$ can be partitioned in a natural way into parallel classes[1]. Let $\mathcal{C}_j$, $j \in \{1, 2, \ldots, i\}$, denote the unique parallel class which contains the line L_j. If L and L' are two lines of $\mathcal{A}$, then we write $L \sim L'$ if L and L' are two disjoint lines of $\mathcal{A}$ which are contained in a (3×3)-subgrid.

Claim. *Every line of $\mathcal{C}_j$ has y_j as point at infinity.*
PROOF. Since y_j is the point at infinity of the line $L_j \in \mathcal{C}_j$, it suffices to prove that any two distinct lines K_1 and K_2 of $\mathcal{C}_j$ have the same point at infinity. Now, for two lines $K_1, K_2 \in \mathcal{C}_j$, there exist lines $M_0, M_1, \ldots, M_k \in \mathcal{C}_j$ (for some $k \in \mathbb{N}$) such that $K_1 = M_0 \sim M_1 \sim \cdots \sim M_k = K_2$. So, it suffices to consider lines $K_1, K_2 \in \mathcal{C}_j$ for which $K_1 \sim K_2$. Let $\mathcal{G}$ denote the unique (3×3)-subgrid containing K_1 and K_2. Let K_3 denote a line of $\mathcal{G}$ meeting K_1 and K_2, let z_j, $j \in \{1, 2, 3\}$, denote the point at infinity of the line K_j and let u denote the unique point in $K_1 \cap K_3$. Put $\beta := \langle z_1, z_3, u \rangle$. Then $S_\beta := \beta \setminus \Pi_\infty$

[1] Two lines L_1 and L_2 of a Hamming near polygon are called *parallel* if each point of L_1 has the same distance to L_2.

is a subspace of $T_5^*(\mathcal{K}^*)$ and $\widetilde{S_\beta}$ is a (3×3)-subgrid and hence a quad. Since there is only one quad through K_1 and K_3, we necessarily have $\mathcal{G}=\widetilde{S_\beta}$. Hence K_2 is contained in $\widetilde{S_\beta}$ and has either z_1 or z_3 as point at infinity. Since K_2 and K_3 meet, K_2 must have z_1 as point at infinity. So, K_1 and K_2 have the same point at infinity. (qed)

By the previous claim and the connectedness of $\mathcal{A}$, it now follows that every point of $\mathcal{A}$ belongs to $\alpha\setminus\Pi_\infty$. If $i\le 5$, then since $\mathcal{A}$ and $\alpha\setminus\Pi_\infty$ have the same number of points, namely 3^i, we see that $\alpha\setminus\Pi_\infty$ equals the point set of $\mathcal{A}$. If $i=6$, then since $\mathcal{A}$ contains 3^6 points, the subspace α must have dimension 6. So, also in this case $\alpha\setminus\Pi_\infty$ equals the point set of $\mathcal{A}$. Moreover, the points $y_1,y_2,\ldots,y_6$ are linearly independent. Taking the above Claim into account, we now see that $\mathcal{A}$ can be obtained as described before this theorem. ■

Let Z be a set of 6 points of $\mathcal{K}^*$ such that $\langle Z\rangle$ is a hyperplane of Π_∞. Let β be a 5-dimensional subspace of $\mathrm{PG}(6,3)$ which intersects Π_∞ in the subspace $\langle Z\rangle$. Then $S_\beta:=\beta\setminus\Pi_\infty$ is a subspace of the near hexagon $T_5^*(\mathcal{K}^*)$ and $\widetilde{S_\beta}$ is a special subgeometry of $T_5^*(\mathcal{K}^*)$.

Theorem 3.2 *Every special subgeometry of $T_5^*(\mathcal{K}^*)$ is obtained in the above-described way.*

Proof. Let $\mathcal{A}$ be a special subgeometry of $T_5^*(\mathcal{K}^*)$. Then there exists a subgeometry $\mathcal{A}'\cong H(5,3)$ of $\mathcal{A}$ whose point set $\mathcal{P}$ equals the point set of $\mathcal{A}$. Now, $\mathcal{A}'$ is also a subgeometry of $T_5^*(\mathcal{K}^*)$. So, by Theorem 3.1 there exists a set Z of 5 points of $\mathcal{K}^*$ and a 5-dimensional subspace β of $\mathrm{PG}(6,3)$ such that $\langle Z\rangle=\Pi_\infty\cap\beta$ and $\mathcal{P}=\beta\setminus\Pi_\infty$. Since $\langle Z\rangle$ contains precisely 6 points of $\mathcal{K}^*$, we necessarily have $\mathcal{A}\cong\widetilde{S_\beta}$. ■

3.2 Generating and embedding Hamming near polygons

Let $\mathcal{S}_1=(\mathcal{P}_1,\mathcal{L}_1,\mathrm{I}_1)$ and $\mathcal{S}_2=(\mathcal{P}_2,\mathcal{L}_2,\mathrm{I}_2)$ be two partial linear spaces. Without loss of generality, we may suppose that the sets $\mathcal{P}_1\times\mathcal{L}_2$ and $\mathcal{L}_1\times\mathcal{P}_2$ are disjoint. From $\mathcal{S}_1$ and $\mathcal{S}_2$, a new partial linear space $\mathcal{S}_1\times\mathcal{S}_2=(\mathcal{P},\mathcal{L},\mathrm{I})$ can be derived which is called the *direct product of* $\mathcal{S}_1$ and $\mathcal{S}_2$. The point set $\mathcal{P}$ of $\mathcal{S}_1\times\mathcal{S}_2$ is equal to the cartesian product $\mathcal{P}_1\times\mathcal{P}_2$ and the line set $\mathcal{L}$ of $\mathcal{S}_1\times\mathcal{S}_2$ is equal to $(\mathcal{P}_1\times\mathcal{L}_2)\cup(\mathcal{L}_1\times\mathcal{P}_2)$. A point (x,y) of $\mathcal{S}_1\times\mathcal{S}_2$ is incident with the line $(z,L)\in\mathcal{P}_1\times\mathcal{L}_2$ if and only if $x=z$ and $(y,L)\in\mathrm{I}_2$. The point (x,y) of $\mathcal{S}_1\times\mathcal{S}_2$ is incident with the line $(M,u)\in\mathcal{L}_1\times\mathcal{P}_2$ if and only if $(x,M)\in\mathrm{I}_1$ and $y=u$. If $\mathcal{S}_i$, $i\in\{1,2\}$, is a near $2n_i$-gon, then $\mathcal{S}_1\times\mathcal{S}_2$ is a near $2(n_1+n_2)$-gon. If $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are three partial linear spaces, then $\mathcal{S}_1\times\mathcal{S}_2\cong\mathcal{S}_2\times\mathcal{S}_1$ and $(\mathcal{S}_1\times\mathcal{S}_2)\times\mathcal{S}_3\cong\mathcal{S}_1\times(\mathcal{S}_2\times\mathcal{S}_3)$. So, the direct product $\mathcal{S}_1\times\mathcal{S}_2\times\cdots\times\mathcal{S}_k$ of $k\ge 2$ partial linear spaces $\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_k$ is well-defined. If we denote by $\mathbb{L}_k$ the line of size $k\ge 2$, then the direct product $\mathbb{L}_k\times\mathbb{L}_k\times\cdots\times\mathbb{L}_k$ of $n\ge 2$ isomorphic copies of $\mathbb{L}_k$ is isomorphic to the Hamming near $2n$-gon $H(n,k)$.

Lemma 3.3 *Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two partial linear spaces. If X_i, $i \in \{1, 2\}$, is a generating set of points of $\mathcal{S}_i$, then $X_1 \times X_2$ is a generating set of points of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. The partial linear space $\mathcal{S}_1 \times \mathcal{S}_2$ has subgeometries isomorphic to $\mathcal{S}_2$, namely for every $u \in \mathcal{P}_1$, $\{u\} \times \mathcal{P}_2$ is a subspace of $\mathcal{S}_1 \times \mathcal{S}_2$ and $\{u\} \times \mathcal{P}_2 \cong \mathcal{S}_2$. We observe the following:

(a) Since X_2 is a generating set of points of $\mathcal{S}_2$, $[\{x_1\} \times X_2] = \{x_1\} \times \mathcal{P}_2$ for every $x_1 \in X_1$. Hence, $\{x_1\} \times \mathcal{P}_2 \subseteq [X_1 \cup X_2]$ for every $x_1 \in X_1$.

(b) Suppose u_1, u_2 and u_3 are three mutually distinct points contained on some line of $\mathcal{S}_1$. Then $(\{u_1\} \times \mathcal{P}_2) \cup (\{u_2\} \times \mathcal{P}_2) \subseteq [X_1 \cup X_2]$ implies that $\{u_3\} \times \mathcal{P}_2 \subseteq [X_1 \cup X_2]$. For, every point $(u_3, v) \in \{u_3\} \times \mathcal{P}_2$ is contained on the line joining the point $(u_1, v) \in \{u_1\} \times \mathcal{P}_2$ with the point $(u_2, v) \in \{u_2\} \times \mathcal{P}_2$.

The lemma follows from (a) and (b) above. ■

Corollary 3.4 *Let $n, k \in \mathbb{N} \setminus \{0\}$ with $k \geq 2$. Then the Hamming near polygon $H(n, k)$ has a generating set of size 2^n.*

Proof. This follows from Lemma 3.3 taking into account that $H(n, k)$ is isomorphic to the direct product of n copies of $\mathbb{L}_k$ and that $\mathbb{L}_k$ is generated by two points. ■

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two partial linear spaces. Let e_i, $i \in \{1, 2\}$, be a (not necessarily full) projective embedding of $\mathcal{S}_i$ into $\mathrm{PG}(V_i)$, where V_i is some vector space over a field $\mathbb{K}$. For every point p of $\mathcal{S}_i$, $i \in \{1, 2\}$, let $\theta_i(p)$ denote a vector of V_i such that $e_i(p) = \langle \theta_i(p) \rangle$. Then the map $e_1 \otimes e_2 : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathrm{PG}(V_1 \otimes V_2) : (p_1, p_2) \mapsto \langle \theta_1(p_1) \otimes \theta_2(p_2) \rangle$ is a projective embedding of $\mathcal{S}_1 \times \mathcal{S}_2$ into $\mathrm{PG}(V_1 \otimes V_2)$. If e_1 and e_2 are full, then also $e_1 \otimes e_2$ is full.

Proposition 3.5 (1) *Let $n, k \in \mathbb{N} \setminus \{0\}$ with $k \geq 2$. Then the Hamming near polygon $H(n, k)$ has generating rank 2^n.*

(2) *Let V be a 2-dimensional vector space over $\mathbb{F}_2$ and let e be a full projective embedding of $\mathbb{L}_3$ into $\mathrm{PG}(V)$. Then $e \otimes e \otimes \cdots \otimes e$ (n times) is isomorphic to the universal embedding of $H(n, 3) \cong \mathbb{L}_3 \times \mathbb{L}_3 \times \cdots \times \mathbb{L}_3$.*

Proof. Let $\mathbb{K}$ be a field for which $k \leq |\mathbb{K}| + 1$, with $\mathbb{K} = \mathbb{F}_2$ if $k = 3$. Let V be a 2-dimensional vector space over $\mathbb{K}$ and let e be a projective embedding of $\mathbb{L}_k$ into $\mathrm{PG}(V)$. Since $e \otimes e \otimes \cdots \otimes e$ is a projective embedding of $H(n, k) \cong \mathbb{L}_k \times \mathbb{L}_k \times \cdots \times \mathbb{L}_k$ into $\mathrm{PG}(V \otimes V \otimes \cdots \otimes V)$, the generating rank of $H(n, k)$ is at least $\dim(V \otimes V \otimes \cdots \otimes V) = 2^n$. Corollary 3.4 now implies that the generating rank of $H(n, k)$ is equal to 2^n. In the case $k = 3$ (and $\mathbb{K} = \mathbb{F}_2$), we can say more, namely, that $e \otimes e \otimes \cdots \otimes e$ must be isomorphic to the universal embedding of $H(n, 3) \cong \mathbb{L}_3 \times \mathbb{L}_3 \times \cdots \times \mathbb{L}_3$. ■

3.3 A lower bound for $er(\mathbb{E}_1)$

The embedding rank $er(\mathbb{E}_1)$ of the near hexagon $\mathbb{E}_1$ is known to be equal to 24, see Brouwer et al. [4, p. 350] or Yoshiara [16, Theorem 1]. The fact that $er(\mathbb{E}_1) = 24$ was established in [4] with the aid of a computer and in [16] with some group theoretical argument involving the Leech lattice. In the present paper we determine $er(\mathbb{E}_1)$ in an entirely geometric way. The aim of this subsection is already to show that $er(\mathbb{E}_1) \geq 24$. The technique we will use to prove this is more or less standard (see e.g. [6, Section 5]). For the calculation of $er(\mathbb{E}_1)$ we need to calculate the $\mathbb{F}_2$-rank of a certain matrix N, which is very hard without a computer. The $\mathbb{R}$-rank of N is however easy to compute. This provides an upper bound for the $\mathbb{F}_2$-rank of N and a lower bound for $er(\mathbb{E}_1)$. We explain this method in detail.

There are standard techniques for calculating the eigenvalues (and corresponding multiplicities) of the collinearity graph Γ of $\mathbb{E}_1$, see [5] or [9, Section 3.3]. By [5, p. 427], the eigenvalues of Γ are 24 (with multiplicity 1), 6 (with multiplicity 264), -3 (with multiplicity 440) and -12 (with multiplicity 24). Let A be the collinearity matrix of $\mathbb{E}_1$, i.e. the adjacency matrix of Γ. The rows and columns of A are indexed by the points of $\mathbb{E}_1$, where we use the same ordering $p_1, p_2, \ldots, p_{729}$ of the points. The (i, j)-th entry of A is equal to 1 if $\mathrm{d}(p_i, p_j) = 1$ and equal 0 otherwise. Let N denote the incidence matrix of $\mathbb{E}_1$. The rows of N are indexed by the points of $\mathbb{E}_1$ (same ordering as before) and the columns of N are indexed by the lines of $\mathbb{E}_1$, with respect to a certain ordering $L_1, L_2, \ldots, L_{2916}$ of the lines. The (i, j)-th entry of N is equal to 1 if $p_i \in L_j$ and equal to 0 otherwise. We have

$$N \cdot N^T = 12 \cdot I_{729} + A,$$

where I_{729} is the (729×729)-identity matrix. By the explicit construction of the universal embedding given in Section 2.2, we have

$$er(\mathbb{E}_1) = 729 - rank_{\mathbb{F}_2}(N).$$

Since the multiplicity of the eigenvalue -12 of A is equal to 24, $rank_{\mathbb{F}_2}(N) \leq rank_{\mathbb{R}}(N) = rank_{\mathbb{R}}(NN^T) = rank(12 \cdot I_{729} + A) = 729 - 24 = 705$. It follows that

$$er(\mathbb{E}_1) = 729 - rank_{\mathbb{F}_2}(N) \geq 24.$$

4 A generating set for the geometry $\mathcal{A}^*$

As in Section 1, let X be a set of 6 points of $\mathrm{PG}(4, 3)$ no five of which are contained in a hyperplane of $\mathrm{PG}(4, 3)$ and suppose $\mathrm{PG}(4, 3)$ is embedded as a hyperplane in $\mathrm{PG}(5, 3)$. Suppose the point-line geometry $\mathcal{A}^*$ is derived from $(X, \mathrm{PG}(4, 3), \mathrm{PG}(5, 3))$ as explained in Section 1.

Lemma 4.1 *The generating rank of the geometry $\mathcal{A}^*$ is at most* 22.

Proof. Put $\mathrm{PG}(5,3) = \mathrm{PG}(V)$, where V is some 6-dimensional vector space over $\mathbb{F}_3$. We can choose a basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}$ of V such that $\mathrm{PG}(4,3) = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$ and $X = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_4 \rangle, \langle \bar{e}_5 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 + \bar{e}_5 \rangle\}$. We will denote the point $\langle X_1 \bar{e}_1 + X_2 \bar{e}_2 + \cdots + X_6 \bar{e}_6 \rangle$ of $\mathrm{PG}(5,3)$ also by $(X_1, X_2, \ldots, X_6)$. Let α be the subspace of $\mathrm{PG}(5,3)$ generated by the points $\langle \bar{e}_1 \rangle$, $\langle \bar{e}_2 \rangle$, $\langle \bar{e}_3 \rangle$, $\langle \bar{e}_4 \rangle$ and $\langle \bar{e}_6 \rangle$. Then $\alpha \setminus \mathrm{PG}(4,3)$ is a subspace of $\mathcal{A}^*$. The geometry induced on the set $\alpha \setminus \mathrm{PG}(4,3)$ is isomorphic to the Hamming near octagon $H(4,3)$. By Corollary 3.4, there exists a set Y_1 of 16 points of $\alpha \setminus \mathrm{PG}(4,3)$ such that $[Y_1] = \alpha \setminus \mathrm{PG}(4,3)$. So, $[Y_1]$ consists of all points of the form $(*,*,*,*,0,1)$. Now, put $Y_2 = \{(0,0,0,0,1,1), (1,0,0,0,1,1), (0,0,1,0,1,1), (1,0,1,0,1,1), (0,1,0,0,1,1), (1,1,0,0,1,1)\}$. We prove that $Y := Y_1 \cup Y_2$ generates $\mathcal{A}^*$.

Since the collinear points $(0,0,0,0,1,1)$ and $(1,0,0,0,1,1)$ belong to $[Y]$, every point of the form $(*,0,0,0,1,1)$ also belongs to $[Y]$. Since the collinear points $(0,0,1,0,1,1)$ and $(1,0,1,0,1,1)$ belong to $[Y]$, every point of the form $(*,0,1,0,1,1)$ also belongs to $[Y]$. Since the collinear points $(0,1,0,0,1,1)$ and $(1,1,0,0,1,1)$ belong to $[Y]$, every point of the form $(*,1,0,0,1,1)$ also belongs to $[Y]$. Since all points of the form $(*,0,0,0,1,1)$ and $(*,0,1,0,1,1)$ belong to $[Y]$, also all points of the form $(*,0,*,0,1,1)$ belong to $[Y]$. Since all points of the form $(*,0,0,0,1,1)$ and $(*,1,0,0,1,1)$ belong to $[Y]$, also all points of the form $(*,*,0,0,1,1)$ belong to $[Y]$. Summarizing, we have:

(I) all points of the form $(*,*,0,0,1,1)$ belong to $[Y]$;

(II) all points of the form $(*,0,*,0,1,1)$ belong to $[Y]$.

Let $x_1, x_2 \in \mathbb{F}_3$. Since the collinear points $(x_1, x_2, 0, 0, 1, 1)$ and $(x_1-1, x_2-1, -1, -1, 0, 1)$ belong to $[Y]$, also $(x_1+1, x_2+1, 1, 1, -1, 1)$ belongs to $[Y]$. Since the collinear points $(x_1, 0, x_2, 0, 1, 1)$ and $(x_1-1, -1, x_2-1, -1, 0, 1)$ belong to $[Y]$, also $(x_1+1, 1, x_2+1, 1, -1, 1)$ belongs to $[Y]$. Summarizing, we have:

(III) all points of the form $(*,*,1,1,-1,1)$ belong to $[Y]$;

(IV) all points of the form $(*,1,*,1,-1,1)$ belong to $[Y]$.

We will now make an observation. Suppose a certain point $(a,b,c,d,e,1)$ belongs to $[Y]$, where $e \neq 0$. Since $(a,b,c,d,e,1)$ is collinear with $(a,b,c,d,0,1) \in [Y]$, every point of the form $(a,b,c,d,*,1)$ belongs to $[Y]$. Applying this observation to (I), (II), (III) and (IV) above, we find:

(I') all points of the form $(*,*,0,0,*,1)$ belong to $[Y]$;

(II') all points of the form $(*,0,*,0,*,1)$ belong to $[Y]$;

(III') all points of the form $(*,*,1,1,*,1)$ belong to $[Y]$;

(IV') all points of the form $(*,1,*,1,*,1)$ belong to $[Y]$.

Let $x_1, x_2 \in \mathbb{F}_3$. Since the collinear points $(x_1, x_2, 0, 0, -1, 1)$ and $(x_1+1, x_2+1, 1, 1, 0, 1)$ belong to $[Y]$, also $(x_1-1, x_2-1, -1, -1, 1, 1)$ belongs to $[Y]$. By the above observation, also $(x_1-1, x_2-1, -1, -1, -1, 1) \in [Y]$. By (I') and (III'), we can now conclude that:

(I") all points of the form $(*,*,k,k,*,1)$ belong to $[Y]$.

Let $x_1, x_2 \in \mathbb{F}_3$. Since the collinear points $(x_1, 0, x_2, 0, -1, 1)$ and $(x_1+1, 1, x_2+1, 1, 0, 1)$ belong to $[Y]$, also $(x_1-1, -1, x_2-1, -1, 1, 1)$ belongs to $[Y]$. By the above observation, also $(x_1-1, -1, x_2-1, -1, -1, 1) \in [Y]$. By (II') and (IV'), we can now conclude that:

(II") all points of the form $(*,k,*,k,*,1)$ belong to $[Y]$.

We now prove that every point of the form $(a,b,c,d,e,1)$, $b \neq c$, belongs to $[Y]$. By (I"), this is true if $c = d$. By (II"), this is true if $b = d$. So, in the sequel, we may suppose that b, c and d are mutually distinct. Since the collinear points $(a,b,c,c,e,1)$ and $(a,b,c,b,e,1)$ belong to $[Y]$, also $(a,b,c,d,e,1)$ must belong to $[Y]$.

We now also prove that every point of the form $(a,b,b,d,e,1)$ belongs to $[Y]$. But this follows from the fact that the collinear points $(a,b,b+1,d,e,1)$ and $(a,b,b-1,d,e,1)$ belong to $[Y]$. ■

Lemma 4.2 *Let Y be a subset of size $i \in \{1,2,3,4\}$ of X, let α be an i-dimensional subspace of* $\mathrm{PG}(5,3)$ *which intersects* $\mathrm{PG}(4,3)$ *in the subspace $\langle Y \rangle$ and put $S := \alpha \setminus$* $\mathrm{PG}(4,3)$. *Then every generating set of size 2^i of $\widetilde{S} \cong H(i,3)$ can be extended to a generating set of size* 22 *of $\mathcal{A}^*$.*

Proof. As in the proof of Lemma 4.1, put $\mathrm{PG}(5,3) = \mathrm{PG}(V)$ where V is a 6-dimensional vector space over $\mathbb{F}_3$ and choose a basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}$ of V such that $\mathrm{PG}(4,3) = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$, $X = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_4 \rangle, \langle \bar{e}_5 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 + \bar{e}_5 \rangle\}$, $\langle \bar{e}_6 \rangle \in \alpha$ and $Y = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \ldots, \langle \bar{e}_i \rangle\}$. Let β be the subspace of $\mathrm{PG}(5,3)$ generated by the points $\langle \bar{e}_1 \rangle$, $\langle \bar{e}_2 \rangle$, $\langle \bar{e}_3 \rangle$, $\langle \bar{e}_4 \rangle$ and $\langle \bar{e}_6 \rangle$. Then $S' := \beta \setminus \mathrm{PG}(4,3)$ is a subspace of $\mathcal{A}^*$ and $\widetilde{S'} \cong H(4,3)$. Notice that $\widetilde{S}$ is a subgeometry of $\widetilde{S'}$. By Lemma 3.3, every generating set of size 2^i of $\widetilde{S}$ can be extended to a generating set Y_1 of size 16 of $\widetilde{S'}$. Now, by the proof of Lemma 4.1, Y_1 can be extended to a generating set of size 22 of $\mathcal{A}^*$. ■

5 Proofs of the main theorems

In this section, we prove Theorems 1.1, 1.2 and 1.3. Let Π_∞ be a hyperplane of the projective space $\mathrm{PG}(6,3)$. Let $\mathcal{K}^*$ be the set of 12 points of Π_∞ as defined in Section 2.1. Then the point-line geometry $T_5^*(\mathcal{K}^*)$ is isomorphic to the near hexagon $\mathbb{E}_1$. Recall that by Theorems 3.1 and 3.2 we know all subgeometries of $T_5^*(\mathcal{K}^*)$ which are either special or isomorphic to $H(i,3)$ for some $i \in \{1,2,3,4\}$.

Lemma 5.1 *Let $\mathcal{A}$ be a special subgeometry of $T_5^*(\mathcal{K}^*)$. Then any generating set Y of size* 22 *of $\mathcal{A}$ can be extended to a generating set of size* 24 *of $T_5^*(\mathcal{K}^*)$.*

Proof. Let α be the subspace of $\mathrm{PG}(6,3)$ generated by the points of $\mathcal{A}$. Let $x_1, x_2, \ldots, x_6$ denote the points of $\mathcal{K}^*$ contained in α and let $x_7, x_8, \ldots, x_{12}$ denote the remaining 6 points of $\mathcal{K}^*$. Then $\langle x_1, x_2, \ldots, x_6 \rangle$ and $\langle x_7, x_8, \ldots, x_{12} \rangle$ are two hyperplanes of Π_∞. Let y_1 and y_2 be two points of $T_5^*(\mathcal{K}^*)$ outside α such that $\beta_1 := \langle x_7, x_8, \ldots, x_{12}, y_1 \rangle$ and $\beta_2 :=$

$\langle x_7, x_8, \ldots, x_{12}, y_2\rangle$ are two distinct hyperplanes of $\mathrm{PG}(6,3)$ through $\langle x_7, x_8, \ldots, x_{12}\rangle$. We prove that $Y \cup \{y_1, y_2\}$ is a generating set of $T_5^*(\mathcal{K}^*)$.

Let $i \in \{1,2\}$. We first prove that every point of $\beta_i \setminus \Pi_\infty$ is contained in $[Y \cup \{y_i\}]$. The set $\beta_i \setminus \Pi_\infty$ is the point-set of a subgeometry $\mathcal{A}_i \cong H(5,3)$ of $T_5^*(\mathcal{K}^*)$ whose lines are those lines of $T_5^*(\mathcal{K}^*)$ contained in β_i whose points at infinity belong to the set $\{x_7, x_8, x_9, x_{10}, x_{11}\}$. Notice that $[Y] = \alpha \setminus \Pi_\infty$. Since $\alpha \cap \beta_i$ is a hyperplane of β_i, $O_i := [Y] \cap (\beta_i \setminus \Pi_\infty)$ is an ovoid of $\mathcal{A}_i$. Now, the complement of any ovoid of $H(5,3)$ is connected by Blok and Brouwer [2, Theorem 7.3] or Shult [14, Lemma 6.1]. Since $y_i \notin O_i$, we have $\beta_i \setminus \Pi_\infty \subseteq [Y \cup \{y_i\}]$.

Summarizing, we have $(\alpha \cup \beta_1 \cup \beta_2) \setminus \Pi_\infty \subseteq [Y \cup \{y_1, y_2\}]$. Now, let z be an arbitrary point of $T_5^*(\mathcal{K}^*)$. Since the line $x_1 z$ of $T_5^*(\mathcal{K}^*)$ contains two distinct points of $(\alpha \cup \beta_1 \cup \beta_2) \setminus \Pi_\infty$, namely the unique points in $x_1 z \cap \beta_1$ and $x_1 z \cap \beta_2$, we have $z \in [Y \cup \{y_1, y_2\}]$. Since z was an arbitrary point of $T_5^*(\mathcal{K}^*)$, $Y \cup \{y_1, y_2\}$ is a generating set of $T_5^*(\mathcal{K}^*)$. ■

Proposition 5.2 *The embedding and generating ranks of $T_5^*(\mathcal{K}^*)$ are equal to* 24.

Proof. By Lemmas 4.1 and 5.1 we have $gr(T_5^*(\mathcal{K}^*)) \leq 24$ and by Section 3.3 we know that $er(T_5^*(\mathcal{K}^*)) \geq 24$. Since $er(T_5^*(\mathcal{K}^*)) \leq gr(T_5^*(\mathcal{K}^*))$, we have $er(T_5^*(\mathcal{K}^*)) = gr(T_5^*(\mathcal{K}^*)) = 24$. ■

Proposition 5.3 *Let $\mathcal{A}$ be a special subgeometry of $T_5^*(\mathcal{K}^*)$. Then the embedding and generating ranks of $\mathcal{A}$ are equal to* 22. *Moreover, the projective embedding of $\mathcal{A}$ induced by the universal embedding of $T_5^*(\mathcal{K}^*)$ is isomorphic to the universal embedding of $\mathcal{A}$.*

Proof. Let $\widetilde{\Sigma}$ denote the projective space which affords the universal embedding $\widetilde{e}$ of $T_5^*(\mathcal{K}^*)$. By Lemmas 4.1 and 5.1, there exists a generating set $\{x_1, x_2, \ldots, x_{24}\}$ of size 24 of $T_5^*(\mathcal{K}^*)$ such that $\{x_1, x_2, \ldots, x_{22}\}$ is a generating set of $\mathcal{A}$. Since $er(T_5^*(\mathcal{K}^*)) = 24$, the points $\widetilde{e}(x_1), \widetilde{e}(x_2), \ldots, \widetilde{e}(x_{24})$ of $\widetilde{\Sigma}$ are linearly independent. So, the embedding $\widetilde{e}$ induces an embedding e of $\mathcal{A}$ into the 21-dimensional subspace $\langle \widetilde{e}(x_1), \widetilde{e}(x_2), \ldots, \widetilde{e}(x_{22})\rangle$ of $\widetilde{\Sigma}$. It follows that $er(\mathcal{A}) \geq 22$. By Lemma 4.1, $gr(\mathcal{A}) \leq 22$. Since $er(\mathcal{A}) \leq gr(\mathcal{A})$, we must have $er(\mathcal{A}) = gr(\mathcal{A}) = 22$. Hence, the embedding e must be universal. ■

Lemma 5.4 *Let $i \in \{1,2,3,4\}$ and let $\mathcal{A}$ be a subgeometry of $T_5^*(\mathcal{K}^*)$ isomorphic to $H(i,3)$. Then any generating set of size 2^i of $\mathcal{A}$ can be extended to a generating set of size* 24 *of $T_5^*(\mathcal{K}^*)$.*

Proof. This is a corollary of Lemmas 4.2 and 5.1. ■

Proposition 5.5 *Let $i \in \{1,2,3,4\}$ and let $\mathcal{A}$ be a subgeometry of $T_5^*(\mathcal{K}^*)$ isomorphic to $H(i,3)$. Then the projective embedding of $\mathcal{A}$ induced by the universal embedding of $T_5^*(\mathcal{K}^*)$ is isomorphic to the universal embedding of $\mathcal{A}$.*

Proof. Let $\widetilde{\Sigma}$ denote the projective space which affords the universal embedding $\widetilde{e}$ of $T_5^*(\mathcal{K}^*)$. By Corollary 3.4 and Lemma 5.4, there exists a generating set $\{x_1, x_2, \ldots, x_{24}\}$ of size 24 of $T_5^*(\mathcal{K}^*)$ such that $\{x_1, x_2, \ldots, x_{2^i}\}$ is a generating set of $\mathcal{A}$. Since $er(T_5^*(\mathcal{K}^*)) =$

24, the points $\widetilde{e}(x_1), \widetilde{e}(x_2), \ldots, \widetilde{e}(x_{24})$ of $\widetilde{\Sigma}$ are linearly independent. It follows that $\widetilde{e}$ induces an embedding e of $\mathcal{A}$ into the (2^i-1)-dimensional subspace $\langle \widetilde{e}(x_1), \widetilde{e}(x_2), \ldots, \widetilde{e}(x_{2^i}) \rangle$ of $\widetilde{\Sigma}$. By Proposition 3.5(2), e must be isomorphic to the universal embedding of $\mathcal{A}$. ■

Lemma 5.6 *Let S be a subspace of $T_5^*(\mathcal{K}^*)$ such that $\widetilde{S}$ is either a special subgeometry or a subgeometry isomorphic to $H(i,3)$ for some $i \in \{1,2,3,4\}$. Then S is the intersection of a number of hyperplanes of $T_5^*(\mathcal{K}^*)$.*

Proof. (1) First, suppose that S is a subspace of $T_5^*(\mathcal{K}^*)$ such that $\widetilde{S}$ is a special subgeometry of $T_5^*(\mathcal{K}^*)$. Let α be the subspace of PG$(6,3)$ generated by all points of S. Then $\alpha \cap \mathcal{K}^*$ is a set of 6 points, say $\alpha \cap \mathcal{K}^* = \{x_1, x_2, \ldots, x_6\}$. Put $\{x_7, x_8, \ldots, x_{12}\} = \mathcal{K}^* \setminus \{x_1, x_2, \ldots, x_6\}$. Now, $\langle x_1, x_2, \ldots, x_6 \rangle$ and $\langle x_7, x_8, \ldots, x_{12} \rangle$ are two hyperplanes of Π_∞. Let β_1 and β_2 be two distinct hyperplanes of PG$(6,3)$ through $\langle x_7, x_8, \ldots, x_{12} \rangle$ distinct from Π_∞. Put $S_i := \beta_i \setminus \Pi_\infty$, $i \in \{1,2\}$. It is easily seen that every line of $T_5^*(\mathcal{K}^*)$ is either contained in $S_i \cup S$ or intersects $S_i \cup S$ in a unique point. Hence, $S_i \cup S$ is a hyperplane of $T_5^*(\mathcal{K}^*)$. Clearly, $S = (S_1 \cup S) \cap (S_2 \cup S)$.

(2) Let $i \in \{1,2,3,4\}$ and let S be a subspace of $T_5^*(\mathcal{K}^*)$ such that $\widetilde{S}$ is a subgeometry of $T_5^*(\mathcal{K}^*)$ isomorphic to $H(i,3)$. If $\mathcal{F}$ denotes the set of all special subgeometries of $T_5^*(\mathcal{K}^*)$ containing all points of S, then using Theorems 3.1 and 3.2 it can readily be verified that $S = \bigcap_{F \in \mathcal{F}} \mathcal{P}_F$, where $\mathcal{P}_F$ denotes the point set of $F \in \mathcal{F}$. Since each $\mathcal{P}_F$, $F \in \mathcal{F}$, is the intersection of two hyperplanes, S is the intersection of a number of hyperplanes of $T_5^*(\mathcal{K}^*)$. ■

Proposition 5.7 *Let S be a subspace of $T_5^*(\mathcal{K}^*)$ such that $\widetilde{S}$ is either a special subgeometry or a subgeometry isomorphic to $H(i,3)$ for some $i \in \{1,2,3,4\}$. Let $\widetilde{e}$ denote the universal embedding of $T_5^*(\mathcal{K}^*)$. If x is a point of $T_5^*(\mathcal{K}^*)$, then $x \in S$ if and only if $\widetilde{e}(x) \in \langle \widetilde{e}(S) \rangle$.*

Proof. Let $H_1, H_2, \ldots, H_k$ be $k \geq 2$ hyperplanes of $T_5^*(\mathcal{K}^*)$ such that $S = H_1 \cap H_2 \cap \cdots \cap H_k$, let $\mathcal{P}$ denote the point set of $T_5^*(\mathcal{K}^*)$ and let $\widetilde{\Sigma} = $ PG$(23,2)$ be the projective space which affords the universal embedding of $T_5^*(\mathcal{K}^*)$. Recall that by Ronan [13], the hyperplane H_i, $i \in \{1,2,\ldots,k\}$, arises from $\widetilde{e}$, i.e. there is a hyperplane Π_i of $\widetilde{\Sigma}$ such that $\widetilde{e}(H_i) = \widetilde{e}(\mathcal{P}) \cap \Pi_i$. So, $\widetilde{e}(S) = \widetilde{e}(H_1 \cap H_2 \cap \cdots \cap H_k) = \widetilde{e}(H_1) \cap \widetilde{e}(H_2) \cap \cdots \cap \widetilde{e}(H_k) = \widetilde{e}(\mathcal{P}) \cap (\Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_k)$. Hence, $\langle \widetilde{e}(S) \rangle \subseteq \Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_k$. Since $\widetilde{e}(S) \subseteq \langle \widetilde{e}(S) \rangle \cap \widetilde{e}(\mathcal{P}) \subseteq \Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_k \cap \widetilde{e}(\mathcal{P}) = \widetilde{e}(S)$, we have $\widetilde{e}(S) = \langle \widetilde{e}(S) \rangle \cap \widetilde{e}(\mathcal{P})$. This is precisely what we needed to prove. ■

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