

Dual jet bundles, Hamiltonian systems and connections

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Abstract

We define Hamilton–De Donder systems on a dual jet bundle, and show that they are variational in a general sense. We explore the relationship between these systems and Ehresmann connections. We also consider regularity conditions for such systems and show that, when regular, they arise from Lagrangian systems on the original jet bundle.

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1 Introduction

In this paper we consider those second-order partial differential equations arising as the variational equations of a regular Lagrangian. We explain the Hamiltonian formalism for these equations: more precisely, we examine the geometry of the Hamilton–De Donder equations for a regular Lagrangian, both on a jet bundle and also on the corresponding dual jet bundle, and we compare the two approaches. To each regular Lagrangian system we associate a Hamiltonian system, and we establish conditions under which the reverse correspondence holds.

The main approach of the paper involves the use of connections related to the Hamiltonian systems and the corresponding second-order partial differential equations. Our main results concern the construction of connections satisfying the condition that every integral section is an extremal of the variational problem, and conversely we show that every extremal can be embedded in a connection of this kind. We give an explicit description of all such connections, and in this way we can describe all the local solutions of the Euler–Lagrange equations in terms of the related connections.

The point of view we have adopted is a generalization of the Hamiltonian formalism of symplectic geometry: in that case, with a single independent variable, the extremals of a regular variational problem can be expressed as the integral curves of a vector field [13]. It is well known that there are several ways of formulating a generalization to the case of several independent variables, as needed for the study of PDEs and field theory, and our approach uses a multisymplectic formulation on dual jet bundles [9]. We concentrate on the most familiar case of the Hamilton–De Donder equations of a regular Lagrangian, and we also investigate the same equations using the geometric framework originated by

Goldschmidt and Sternberg [8] (following De Donder [4]). We see that there is a strong parallel between these two approaches and the corresponding geometric structures. Some selected references are [1, 2, 3, 5, 6, 7, 10, 11, 14].

The structure of this paper is as follows. In Section 2 we review the affine duality structure of jet bundles in n independent variables and give an invariant definition of those systems of first-order partial differential equations that may be described as being of Hamilton–De Donder type and are therefore generated by a Hamiltonian. We show that such equations are variational, and give conditions for a general system of first-order PDEs in the correct format to be Hamilton–De Donder equations. This approach, which we subsequently refine in Section 5, also gives us the solution of the inverse variational problem in a Hamiltonian formulation, so that we can see when a given system of first order PDEs is a Hamilton–De Donder system related to a regular Lagrangian, and it also allows us to reconstruct the corresponding Lagrangian system of second-order variational PDEs.

In Section 3 we move on to consider the compatibility relationship between Hamilton–De Donder equations and the equations arising from an Ehresmann connection, showing that a global compatible connection always exists, and that every solution of the Hamilton–De Donder equations is also, locally, an integral section of some compatible connection; this allows us to give a definition of a completely integrable Hamilton–De Donder system. In Section 4 we compare this to the theory of Goldschmidt and Sternberg, where Hamilton–De Donder equations are defined on a jet bundle rather than its dual, and the corresponding connections are semispray connections. We are able to give an explicit formula for the family of connections satisfying these conditions. Finally in Section 5 we consider both Lagrangian and Hamiltonian structures together, showing that if a Lagrangian is regular in the usual sense then there is a corresponding Hamiltonian structure, and that conversely if the Hamiltonian is ‘regular’ then it arises from a Lagrangian; we also revisit the inverse problem in this context.

2 Hamiltonian systems on jet bundles

2.1 Dual jet bundles

Let $\pi : Y \rightarrow X$ be a fibred manifold with $\dim X = n$, $\dim Y = n + m$ and let $\pi_1 : J^1Y \rightarrow X$ be its first jet bundle over X . The manifold J^1Y is also the total space of an affine bundle $\pi_{1,0} : J^1Y \rightarrow Y$ modelled on the vector bundle $V_X Y \otimes \pi^* T^* X \rightarrow Y$, where $V_X Y \rightarrow Y$ is the bundle of vectors tangent to Y and vertical over X . We shall use the notation $J_y^1 Y$ for a fibre $\pi_{1,0}^{-1}(y) \subset J^1Y$ where $y \in Y$, and put $x = \pi(y) \in X$.

To see how this affine structure arises, let γ be a local section of π satisfying $\gamma(x) = y$. The equivalence class of local sections γ defining the jet $j_x^1 \gamma$ is the same as the equivalence class defining the tangent map $(T\gamma)_x : T_x X \rightarrow T_y Y$, so that we may identify $j_x^1 \gamma \in J_y^1 Y$ with

$$(T\gamma)_x \in \text{Hom}(T_x X, T_y Y) \cong T_y Y \otimes T_x^* X,$$

where we consider the latter space as a subspace of the tensor space $T_y Y \otimes T_y^* Y$. Taking

(x^i, y^σ) , $1 \leq i \leq n$, $1 \leq \sigma \leq m$ as local fibred coordinates on Y , with $(x^i, y^\sigma, y_j^\sigma)$ as the associated coordinates on J^1Y , the jet $j_x^1\gamma$ corresponds to the tensor

$$\left(\frac{\partial}{\partial x^i} + y_i^\sigma(j_x^1\gamma) \frac{\partial}{\partial y^\sigma} \right) \otimes dx^i$$

at $y \in Y$, where the coefficient of $\partial/\partial x^j$ is δ_i^j because γ is a local section.

Every affine space of dimension m is associated with a vector space of dimension $m + 1$, its *extended dual*, whose elements are the real-valued affine maps on the space. Applying this construction to the fibres J_y^1Y of the affine bundle $\pi_{1,0}$ gives the manifold $J^\dagger Y$, the extended dual of J^1Y . If $\phi \in J^\dagger Y$ and $\phi : J_y^1Y \rightarrow \mathbb{R}$ we put $\pi_{1,0}^\dagger(\phi) = y$, and then $\pi_{1,0}^\dagger : J^\dagger Y \rightarrow Y$ becomes a vector bundle.

Each fibre $J_y^\dagger Y$ of the extended dual is a vector space containing a distinguished subspace of constant maps. The quotient space will be denoted J_y^*Y , and the union of all the quotient spaces forms another vector bundle, the *reduced dual* J^*Y . We shall let $\rho : J^\dagger Y \rightarrow J^*Y$ denote the quotient map, and $\pi_{1,0}^* : J^*Y \rightarrow Y$ denote the induced map. The composition $\pi \circ \pi_{1,0}^* : J^*Y \rightarrow X$ will be denoted by τ . We can see how all these maps fit together in the following diagram.

$$\begin{array}{ccccc}
 & & J^1Y & & \\
 & & \searrow^{\pi_1} & & \\
 & & & \searrow^{\pi_{1,0}} & \\
 & & & & Y & \xrightarrow{\pi} & X \\
 & \nearrow^{\pi_{1,0}^\dagger} & & \nearrow^{\pi_{1,0}^*} & & & \\
 J^\dagger Y & \xrightarrow{\rho} & J^*Y & \xrightarrow{\tau} & X & & \\
 & & \nearrow^{\pi_{1,0}^*} & & & &
 \end{array}$$

If y is in the domain of the chart (x^i, y^σ) , let $y_{i(y)}^\sigma$ be the restrictions of the induced coordinate functions y_i^σ to the fibre J_y^1Y , so that $y_{i(y)}^\sigma$ are affine functions and are elements of the fibre $J_y^\dagger Y$. The constant function $1_{(y)} : J_y^1Y \rightarrow \mathbb{R}$ is also an affine function, and together the functions $(1_{(y)}, y_{i(y)}^\sigma)$ form a basis for the vector space $J_y^\dagger Y$. Denote the dual basis of $(J_y^\dagger Y)^*$ by $(P_{(y)}, P_{\sigma(y)}^i)$; then the functions P, P_σ^i defined locally on J^1Y by these fibre functions are smooth, and the functions $(x^i, y^\sigma, P, P_\sigma^i)$ form an adapted coordinate system. Similarly the functions $(x^i, y^\sigma, P_\sigma^i)$ are coordinates on the reduced dual J^*Y .

Now suppose that there is a given volume n -form ω_0 on X such that the coordinates x^i satisfy $\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, and consider the bundle of cotangent n -vectors

$$\bigwedge_1^n T^*Y = \{ \theta \in \bigwedge^n T^*Y : \xi, \eta \in V_X Y \Rightarrow i_\xi i_\eta \theta = 0 \},$$

so that an element θ of this bundle may be written in the coordinates (x^i, y^σ) as

$$\theta = \theta_0 \omega_0 + \theta_\sigma^i dy^\sigma \wedge \omega_i$$

where $\omega_i = i_{\partial/\partial x^i} \omega_0$. We may identify this bundle with the extended dual in the following way. Take $\theta \in \bigwedge_1^n T^*Y$, and let γ be any local section satisfying $\gamma(x) = y$. The pull-back

cotangent n -vector $\gamma^*(\theta) \in \bigwedge^n T_x^* X$ depends only on the value and first derivatives of γ at x , and so the function

$$\tilde{\theta} : J_y^1 Y \rightarrow \mathbb{R}, \quad \tilde{\theta}(j_x^1 \gamma) \omega_0 = \gamma^*(\theta)$$

is well-defined. In coordinates

$$\tilde{\theta}(j_x^1 \gamma) = \theta_0 + \theta_\sigma^i \left. \frac{d\gamma^\sigma}{dx^i} \right|_x = \theta_0 + \theta_\sigma^i y_i^\sigma(j_x^1 \gamma),$$

showing that $\tilde{\theta}$ is an affine function. This correspondence is a linear isomorphism of the fibres, under which the basis $(1_{(y)}, y_{i(y)}^\sigma)$ of $J_y^1 Y$ corresponds to the basis $(\omega_0|_y, dy^\sigma \wedge \omega_i|_y)$ of $\bigwedge_1^n T_y^* Y$, and gives rise to a diffeomorphism $J^\dagger Y \cong \bigwedge_1^n T^* Y$. The canonical n -form

$$\Theta = P \omega_0 + P_\sigma^i dy^\sigma \wedge \omega_i$$

on $\bigwedge_1^n T^* Y$ may therefore be regarded as an n -form on the extended dual $J^\dagger Y$, and similarly the $(n+1)$ -form $\Omega = d\Theta$ may be regarded as an $(n+1)$ -form on $J^\dagger Y$. As Ω is a multisymplectic form, $(J^\dagger Y, \Omega)$ becomes a multisymplectic manifold. Note that, owing to the affine structure, the multisymplectic form Ω is *canonical for the manifold $J^\dagger Y$* , although it is not the only multisymplectic form on $J^\dagger Y$. For reasons which will become clear later on, we shall also call Ω the *Hamilton–De Donder* multisymplectic form.

2.2 Hamilton–De Donder systems

Any local section \mathfrak{h} of the quotient projection $\rho : J^\dagger Y \rightarrow J^* Y$, with domain $U \subset J^* Y$, gives rise to a local ‘Hamilton–De Donder system’ in the following way. Consider the pullbacks $\Theta_{\mathfrak{h}} = \mathfrak{h}^* \Theta$, $\Omega_{\mathfrak{h}} = \mathfrak{h}^* \Omega$ of the canonical forms to U , and let $\mathcal{H} = -(P \circ \mathfrak{h})$ be (minus) the nontrivial component of the section \mathfrak{h} in the given coordinates, so that \mathcal{H} is a function defined on U ; then

$$\Theta_{\mathfrak{h}} = -\mathcal{H} \omega_0 + P_\sigma^i dy^\sigma \wedge \omega_i, \quad \Omega_{\mathfrak{h}} = -d\mathcal{H} \wedge \omega_0 + dP_\sigma^i \wedge dy^\sigma \wedge \omega_i.$$

We now define the Hamilton–De Donder system $\mathcal{D}_{\mathfrak{h}}$ to be the exterior differential system on U generated by the family of n -forms $i_\xi \Omega_{\mathfrak{h}}$, where ξ runs over all vector fields on U that are vertical over X . Taking for ξ each of the basis vector fields $(\partial/\partial y^\sigma, \partial/\partial P_\sigma^i)$, we see that locally

$$(2.1) \quad \mathcal{D}_{\mathfrak{h}} = \text{span}\{\eta_\sigma, \eta_i^\sigma\}, \quad \eta_\sigma = -\frac{\partial \mathcal{H}}{\partial y^\sigma} \omega_0 - dP_\sigma^i \wedge \omega_i, \quad \eta_i^\sigma = -\frac{\partial \mathcal{H}}{\partial P_\sigma^i} \omega_0 + dy^\sigma \wedge \omega_i.$$

If ψ is a local section of $\tau : J^* Y \rightarrow X$ then ψ annihilates $\mathcal{D}_{\mathfrak{h}}$ (that is, ψ is a solution of the Hamilton–De Donder exterior differential system) if and only if ψ satisfies the equation

$$(2.2) \quad \psi^*(i_\xi \Omega_{\mathfrak{h}}) = 0 \quad \text{for every vector field } \xi \text{ on } J^* Y, \text{ vertical over } X.$$

In coordinates,

$$(2.3) \quad -\frac{\partial \mathcal{H}}{\partial y^\sigma} - \frac{\partial \psi_\sigma^i}{\partial x^i} = 0, \quad -\frac{\partial \mathcal{H}}{\partial P_\sigma^i} + \frac{\partial \psi^\sigma}{\partial x^i} = 0$$

for the components $\psi^\sigma = y^\sigma \circ \psi$ and $\psi^i = P_\sigma^i \circ \psi$ of ψ . (The apparently perverse choice of sign for these equations is needed to ensure that, as we shall see in the next section, the equations are strictly variational without needing a multiplier.) We shall refer to these equations as *Hamilton equations of De Donder type*, or as *Hamilton–De Donder equations*. The images of solutions ψ are immersed n -dimensional submanifolds of the $(n + m + nm)$ -dimensional manifold J^*Y .

Remark 2.1. It is easy to check that condition (2.2) is equivalent to

$$(2.4) \quad \psi^*(i_\xi \Omega_h) = 0 \quad \text{for every vector field } \xi \text{ on } J^*Y.$$

So considering the exterior differential system $\tilde{\mathcal{D}}_h$ generated by n -forms $i_\xi \Omega_h$ where ξ runs over *all* vector fields on U , we can see that \mathcal{D}_h and $\tilde{\mathcal{D}}_h$ have the same set of solutions, even though $\mathcal{D}_h \subsetneq \tilde{\mathcal{D}}_h$.

Theorem 2.2. *The ideal generated by \mathcal{D}_h is a differential ideal, and its Cauchy distribution is trivial.*

Proof. As

$$dx^j \wedge \eta_\sigma = dP_\sigma^j \wedge \omega_0, \quad dx^i \wedge \eta_i^\sigma = -n dy^\sigma \wedge \omega_0,$$

we obtain

$$\begin{aligned} d\eta_\sigma &= -\frac{\partial^2 \mathcal{H}}{\partial y^\sigma \partial y^\nu} dy^\nu \wedge \omega_0 - \frac{\partial^2 \mathcal{H}}{\partial y^\sigma \partial P_\nu^j} dP_\nu^j \wedge \omega_0 \\ &= \frac{1}{n} \frac{\partial^2 \mathcal{H}}{\partial y^\sigma \partial y^\nu} dx^i \wedge \eta_i^\nu - \frac{\partial^2 \mathcal{H}}{\partial y^\sigma \partial P_\nu^j} dx^j \wedge \eta_\nu \\ d\eta_i^\sigma &= -\frac{\partial^2 \mathcal{H}}{\partial P_\sigma^i \partial y^\nu} dy^\nu \wedge \omega_0 - \frac{\partial^2 \mathcal{H}}{\partial P_\sigma^i \partial P_\nu^j} dP_\nu^j \wedge \omega_0 \\ &= \frac{1}{n} \frac{\partial^2 \mathcal{H}}{\partial P_\sigma^i \partial y^\nu} dx^i \wedge \eta_i^\nu - \frac{\partial^2 \mathcal{H}}{\partial P_\sigma^i \partial P_\nu^j} dx^j \wedge \eta_\nu \end{aligned}$$

proving the first of the assertions. We also observe that ζ belongs to the Cauchy distribution only if $i_\zeta \eta_\sigma = 0$ and $i_\zeta \eta_i^\sigma = 0$, showing that $\zeta = 0$. \square

2.3 Variational principles for Hamiltonian systems

Our aim now is to look for a variational principle whose extremals are local sections ψ of the fibred manifold $\tau : J^*Y \rightarrow X$ that are precisely the solutions of the Hamilton–De Donder equations (2.3). These equations are system of first order PDEs over the fibred manifold $\tau : J^*Y \rightarrow X$ and define a submanifold of $J^1(J^*Y)$ projecting to $U \subset J^*Y$, so that we can tackle the problem directly by applying the Helmholtz conditions to the equations.

Given fibred coordinates $(x^i, z^I) = (x^i, y^\sigma, P_\sigma^i)$ on J^*Y , denote the associated coordinates on $J^1(J^*Y)$ by $(x^i, z^I, z_{|j}^I) = (x^i, y^\sigma, P_\sigma^i, y_{|j}^\sigma, P_{\sigma|j}^i)$ so that equations (2.3) take the form

$$\left(-\frac{\partial \mathcal{H}}{\partial y^\sigma} - P_{\sigma|i}^i\right) \circ j^1\psi = 0, \quad \left(-\frac{\partial \mathcal{H}}{\partial P_\sigma^i} + y_{|i}^\sigma\right) \circ j^1\psi = 0.$$

Writing E for the $(n+1)$ -form

$$E = E_J dz^J \wedge \omega_0 = \left(-\frac{\partial \mathcal{H}}{\partial y^\sigma} - P_{\sigma|i}^i\right) dy^\sigma \wedge \omega_0 + \left(-\frac{\partial \mathcal{H}}{\partial P_\sigma^i} + y_{|i}^\sigma\right) dP_\sigma^i \wedge \omega_0$$

and substituting into the Helmholtz conditions

$$\frac{\partial E_I}{\partial z_{|i}^J} + \frac{\partial E_J}{\partial z_{|i}^I} = 0, \quad \frac{\partial E_I}{\partial z^J} - \frac{\partial E_J}{\partial z^I} + d_i \frac{\partial E_J}{\partial z_{|i}^I} = 0$$

we see that the conditions are satisfied, showing that the Hamilton–De Donder equations are variational. A corresponding local Lagrangian can then be computed using the Tonti formula

$$\mathcal{L} = z^I \int_0^1 (E_I \circ \chi) du \quad \text{where} \quad \chi(u, (x^i, z^J, z_{|i}^J)) = (x^i, uz^J, uz_{|i}^J)$$

defined on a proper star-shaped neighborhood of each point in $\tau_{1,0}^{-1}(U)$. Using the explicit formula for the functions E_I then gives

$$\begin{aligned} \mathcal{L}_h &= y^\sigma \int_0^1 \left(-\frac{\partial \mathcal{H}}{\partial y^\sigma} - P_{\sigma|i}^i\right) \circ \chi du + P_\sigma^i \int_0^1 \left(-\frac{\partial \mathcal{H}}{\partial P_\sigma^i} + y_{|i}^\sigma\right) \circ \chi du \\ (2.5) \quad &= -y^\sigma \int_0^1 \left(\frac{\partial \mathcal{H}}{\partial y^\sigma} \circ \chi\right) du - P_\sigma^i \int_0^1 \left(\frac{\partial \mathcal{H}}{\partial P_\sigma^i} \circ \chi\right) du + \frac{1}{2}(P_\sigma^i y_{|i}^\sigma - y^\sigma P_{\sigma|i}^i) \\ &= -\mathcal{H} + P_\sigma^i y_{|i}^\sigma \\ &+ \text{a term of the form } d_i f^i, \text{ providing zero Euler–Lagrange expressions.} \end{aligned}$$

If $D \subset X$ is a connected compact n -dimensional submanifold with boundary, the action yielding the Hamilton–De Donder equations is therefore the function

$$\psi \mapsto \int_D J^1\psi^* (-\mathcal{H} + P_\sigma^i y_{|i}^\sigma) dx^1 \wedge \cdots \wedge dx^n$$

on the set of sections of the fibred manifold $\tau : J^*Y \rightarrow X$ satisfying $D \subset \text{Dom } \psi$ and $\text{Im } \psi \subset \text{Dom } \mathcal{L}_h$; variations are then vector fields on $U \subset J^*Y$ projectable to X . Note that the Lagrangian \mathcal{L}_h is a first order Lagrangian that is affine in the first derivatives, so that its Euler–Lagrange equations are indeed first order equations.

There is an alternative, more geometric, approach to this problem, obtained from a direct use of the fibred manifold $\tau : J^*Y \rightarrow X$ and its jet prolongation $\tau_1 : J^1(J^*Y) \rightarrow X$.

Let h and p_1 denote the horizontalization and 1-contactization operators with respect to τ_1 , and let

$$\vartheta^\sigma = dy^\sigma - y_{|j}^\sigma dx^j, \quad \pi_\sigma^i = dP_\sigma^i - P_{\sigma|j}^i dx^j$$

denote the corresponding local contact 1-forms. We use the pullback $\Theta_h = -\mathcal{H}\omega_0 + P_\sigma^i dy^\sigma \wedge \omega_i$ of the canonical n -form on J^*Y , defined on the domain U of the hamiltonian section h .

Theorem 2.3. *The Hamilton–De Donder equations of the $(n + 1)$ -form $\Omega_h = d\Theta_h$ are the Euler–Lagrange equations of the Lagrangian*

$$(2.6) \quad \Lambda_h = h(\Theta_h) = (-\mathcal{H} + P_\sigma^i y_{|i}^\sigma) \omega_0.$$

Proof. Although this follows immediately from the definition of the Tonti Lagrangian

$$\mathcal{L}_h = -\mathcal{H} + P_\sigma^i y_{|i}^\sigma$$

given above, we can also provide a different argument: it is easy to see that the Cartan form Θ_{Λ_h} of the Lagrangian Λ_h is the pullback of Θ_h to $J^1(J^*Y)$, so the Euler–Lagrange equations for Λ_h are exactly equations (2.2) and are given in coordinates by (2.3). \square

For this reason we shall call the Euler–Lagrange form E_h of the Lagrangian $\Lambda_h = h(\Theta_h)$ the *Hamilton–De Donder form*; its coordinate expression is of course

$$(2.7) \quad \begin{aligned} E_h &= p_1 d\Theta_{\Lambda_h} = p_1 \Omega_h = p_1 (-d\mathcal{H} \wedge \omega_0 + (\pi_\sigma^i + P_{\sigma|j}^i dx^j) \wedge (\vartheta^\sigma + y_{|j}^\sigma dx^j) \wedge \omega_i) \\ &= -\left(\frac{\partial \mathcal{H}}{\partial y^\sigma} + P_{\sigma|i}^i\right) \vartheta^\sigma \wedge \omega_0 - \left(\frac{\partial \mathcal{H}}{\partial P_\sigma^i} - y_{|i}^\sigma\right) \pi_\sigma^i \wedge \omega_0. \end{aligned}$$

As the Hamilton–De Donder equations are variational, we shall use the term *Hamilton–De Donder extremal* to denote any section ψ of $\tau : J^*Y \rightarrow X$ that is a solution of the Hamilton–De Donder equations.

2.4 The inverse problem

The Hamilton–De Donder equations are a system of first order PDEs which, as we have seen, come from a variational principle on the dual jet bundle. We may therefore consider the *inverse problem* which can be stated in general terms as follows.

Given a system of first-order PDEs

$$E_J(x^i, z^K, z_i^K) = 0,$$

is there a Lagrangian such that these equations are equations for the extremals of that Lagrangian?

This is a familiar problem which, when expressed more precisely as a question about the corresponding *dynamical form*, may be considered on two levels, local and global. In

both cases the solution is well known. In our case we let $\tau : J^*Y \rightarrow X$ be the reduced dual bundle, and we say that a system of $m + mn$ differential equations

$$(2.8) \quad \frac{\partial y^\sigma}{\partial x^i} - F_i^\sigma = 0, \quad \frac{\partial P_\sigma^j}{\partial x^j} + G_\sigma = 0, \quad 1 \leq \sigma \leq m, \quad 1 \leq i \leq n$$

for local sections $\psi(x) = (x^i(x), y^\sigma(x), P_\sigma^j(x))$ of τ is locally variational if, for each point of J^*Y , there is a Lagrangian function L defined on some neighbourhood of that point such that the Euler–Lagrange expressions of L coincide with the given equations “as they stand”. Applying the known results, we obtain the following.

Corollary 2.4. *The system of equations (2.8) is locally variational if and only if the functions F_i^σ and G_σ obey the integrability conditions*

$$(2.9) \quad \frac{\partial G_\sigma}{\partial y^\nu} - \frac{\partial G_\nu}{\partial y^\sigma} = 0, \quad \frac{\partial F_i^\sigma}{\partial P_\nu^k} - \frac{\partial F_k^\nu}{\partial P_\sigma^i} = 0, \quad \frac{\partial G_\sigma}{\partial P_\nu^k} - \frac{\partial F_k^\nu}{\partial y^\sigma} = 0.$$

If this is the case then, locally,

$$F_i^\sigma = \frac{\partial \mathcal{H}}{\partial P_\sigma^i}, \quad G_\sigma = \frac{\partial \mathcal{H}}{\partial y^\sigma},$$

where

$$\mathcal{H} = y^\sigma \int_0^1 (G_\sigma \circ \chi) du + P_\sigma^j \int_0^1 (F_j^\sigma \circ \chi) du.$$

A corresponding local Lagrangian function on an open subset of $J^1(J^*Y)$ is

$$\mathcal{L} = -\mathcal{H} + P_\sigma^i y_{|i}^\sigma.$$

Furthermore, the function \mathcal{H} defines a local Hamiltonian section \mathfrak{h} of the projection $\rho : J^1 J^*Y \rightarrow J^*Y$ such that equations (2.8) represent a local Hamiltonian system.

In the language of PDEs this result concerns a system of first order PDEs for a vector function and a tensor function of n independent variables, such that the equations for the vector function are in the explicit form, and those for the tensor function are in divergence form.

3 Hamiltonian connections

3.1 Fields of Hamilton–De Donder extremals

We now investigate the geometric meaning of the Hamilton–De Donder equations by considering them in the context of Ehresmann connections. We can see immediately, from their structure, that the Hamilton–De Donder equations are not equations for the paths of a single Ehresmann connection when $n \geq 2$, in contrast to the case of Hamiltonian mechanics. We can, however, construct various local Ehresmann connections whose solutions are always Hamilton–De Donder extremals.

To see this, consider for example the coordinate equations

$$\frac{\partial \psi^\sigma}{\partial x^i} = \frac{\partial \mathcal{H}}{\partial P_\sigma^i}, \quad \frac{\partial \psi_\sigma^1}{\partial x^1} = -\frac{\partial \mathcal{H}}{\partial y^\sigma}, \quad \frac{\partial \psi_\sigma^1}{\partial x^a} = 0, \quad \frac{\partial \psi_\sigma^b}{\partial x^i} = 0$$

where the indices a, b run from 2 to n , so that these are the equations of a connection in the given chart. Any local integral section of this connection will clearly be a Hamilton–De Donder extremal, although the converse need not hold: it need not be the case that every Hamilton–De Donder extremal is also a solution of the connection equations. We can obviously write down many other sets of connection equations satisfying a similar property.

To approach this problem more systematically, let $\hat{\Gamma}$ denote a local Ehresmann connection on $\tau : J^*Y \rightarrow X$, so that $\hat{\Gamma}$ is a local section of the fibred manifold $\tau_{1,0} : J^1(J^*Y) \rightarrow J^*Y$. Suppose that the domain of $\hat{\Gamma}$ contains U , the domain of Ω_h , and that $\hat{\Gamma}$ is represented by its horizontal projector

$$\Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \Gamma_j^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{\sigma j}^i \frac{\partial}{\partial P_\sigma^i} \right),$$

where $\Gamma_j^\sigma = y_{|j}^\sigma \circ \hat{\Gamma}$ and $\Gamma_{\sigma j}^i = P_{\sigma|j}^i \circ \hat{\Gamma}$ are the components of the section $\hat{\Gamma}$ in the local coordinates; we then see that

$$i_\Gamma \Omega_h = (n-1)\Omega_h - \left(\Gamma_{\sigma i}^i + \frac{\partial \mathcal{H}}{\partial y^\sigma} \right) dy^\sigma \wedge \omega_0 + \left(\Gamma_i^\sigma - \frac{\partial \mathcal{H}}{\partial P_\sigma^i} \right) dP_\sigma^i \wedge \omega_0.$$

If a local section ψ of $\tau : J^*Y \rightarrow X$ is an integral section of $\hat{\Gamma}$ then it will satisfy the condition $\hat{\Gamma} \circ \psi = j^1\psi$, so that in coordinates it will satisfy

$$\frac{\partial \psi^\sigma}{\partial x^j} = \Gamma_j^\sigma, \quad \frac{\partial \psi_\sigma^i}{\partial x^j} = \Gamma_{\sigma j}^i.$$

We therefore have the following result (cf. [5]).

Theorem 3.1. *If an Ehresmann connection $\hat{\Gamma}$ on $\tau : J^*Y \rightarrow X$ satisfies the compatibility condition*

$$(3.1) \quad i_\Gamma \Omega_h = (n-1)\Omega_h$$

then any integral section of $\hat{\Gamma}$ is a solution of the Hamilton–De Donder equations (2.3), and so is a Hamilton–De Donder extremal.

We can express this compatibility condition in terms of the Hamilton–De Donder form $E_h = p_1 \Omega_h$.

Theorem 3.2. *For any Ehresmann connection $\hat{\Gamma}$ on the fibred manifold $\tau_{1,0} : J^1(J^*Y) \rightarrow J^*Y$ the compatibility condition (3.1) is equivalent to the condition*

$$(3.2) \quad \hat{\Gamma}^* E_h = 0.$$

Proof. By (2.7),

$$\hat{\Gamma}^* E_{\mathfrak{h}} = -\left(\frac{\partial \mathcal{H}}{\partial y^\sigma} + \Gamma_{\sigma i}^i\right) \vartheta^\sigma \wedge \omega_0 - \left(\frac{\partial \mathcal{H}}{\partial P_\sigma^i} - \Gamma_i^\sigma\right) \pi_\sigma^i \wedge \omega_0,$$

so that $\hat{\Gamma}^* E_{\mathfrak{h}} = 0$ if and only if

$$\Gamma_i^\sigma = \frac{\partial \mathcal{H}}{\partial P_\sigma^i}, \quad \Gamma_{\sigma i}^i = -\frac{\partial \mathcal{H}}{\partial y^\sigma},$$

and this holds if and only if equation (3.1) holds. \square

We shall therefore call connections compatible with a Hamilton–De Donder system *fields of Hamilton–De Donder extremals*. Note, however, that in general there is no guarantee that such a connection will have any integral sections at all, even if the Hamilton–De Donder equations have solutions.

Although connections compatible with a Hamilton–De Donder system are not unique, we may nevertheless describe them all locally.

Theorem 3.3. *Let $\Omega_{\mathfrak{h}}$ be a Hamilton–De Donder system on $U \subset J^*Y$. If $\hat{\Gamma}$ is an Ehresmann connection on $\tau : J^*Y \rightarrow X$ that is compatible with $\Omega_{\mathfrak{h}}$ then $\hat{\Gamma}$ is described locally by the horizontal projector*

$$(3.3) \quad \Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \frac{\partial \mathcal{H}}{\partial P_\sigma^j} \frac{\partial}{\partial y^\sigma} - \left(\frac{1}{n} \delta_j^i \frac{\partial \mathcal{H}}{\partial y^\sigma} + F_{\sigma j}^i \right) \frac{\partial}{\partial P_\sigma^i} \right),$$

where for each index σ , the matrix of functions $(F_{\sigma j}^i)$ is an arbitrary $(n \times n)$ -matrix on U , traceless at each point of U .

Proof. It is sufficient to check the formula for $\Gamma_{\sigma j}^i$. Fixing σ , considering $(\Gamma_{\sigma j}^i)$ as a $(n \times n)$ -matrix on U and applying the trace decomposition formula, we see that $\Gamma_{\sigma j}^i$ may be expressed as

$$(3.4) \quad \Gamma_{\sigma j}^i = (\Gamma_{\sigma j}^i - \frac{1}{n} \delta_j^i \operatorname{Tr} \Gamma_\sigma) + \frac{1}{n} \delta_j^i \operatorname{Tr} \Gamma_\sigma.$$

Putting

$$-F_{\sigma j}^i = \Gamma_{\sigma j}^i - \frac{1}{n} \delta_j^i \operatorname{Tr} \Gamma_\sigma$$

and using the fact that $\operatorname{Tr} \Gamma_\sigma = -\partial H / \partial y^\sigma$, we obtain the result. \square

So we have found a family of Ehresmann connections such that every integral section of each connection is a Hamilton–De Donder extremal. In particular, for every *integrable* connection $\hat{\Gamma}$ in this family (in the sense of Frobenius complete integrability) this means that the corresponding maximal Hamilton–De Donder extremals form a n -dimensional foliation of the open submanifold $\operatorname{Dom} \hat{\Gamma} \subset J^*Y$.

It is now natural to ask whether the family of Ehresmann connections characterized by Theorem 3.3 includes (at least locally) *all* Hamilton–De Donder extremals. The answer is positive: every Hamilton–De Donder extremal is locally an integral section of some compatible connection, which is maximal in the sense that it is defined on the domain U of \mathfrak{h} .

Lemma 3.4. *If h is a section of $\rho : J^\dagger Y \rightarrow J^*Y$ defined on $U \subset J^*Y$ then there is a connection $\hat{\Gamma}_0$ on U satisfying $i_{\Gamma_0}\Omega_h = (n-1)\Omega_h$. If, in particular, h is a global section of $\rho : J^\dagger Y \rightarrow J^*Y$ then $\hat{\Gamma}_0$ is a global connection.*

Proof. We consider the global case first, and use a partition of unity argument. Let (V_α) be an open cover of Y by domains of adapted charts and put $V_\alpha^* = (\pi_{1,0}^*)^{-1}(V_\alpha)$, so that (V_α^*) is an open cover of J^*Y by domains of adapted charts. Let (f_α) be a partition of unity subordinate to (V_α^*) .

In the chart corresponding to V_α^* define the functions Γ_i^σ and $\Gamma_{\sigma j}^i$ by

$$\Gamma_i^\sigma = \frac{\partial \mathcal{H}}{\partial P_\sigma^i}, \quad \Gamma_{\sigma j}^i = -\frac{1}{n} \delta_j^i \frac{\partial \mathcal{H}}{\partial y^\sigma};$$

if we put

$$\Gamma_\alpha = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \Gamma_j^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{\sigma j}^i \frac{\partial}{\partial P_\sigma^i} \right)$$

then Γ_α is the horizontal projector of a connection defined on V_α^* satisfying the condition that $i_{\Gamma_\alpha}\Omega_h = (n-1)\Omega_h$.

Now put $\Gamma_0 = \sum_\alpha f_\alpha \Gamma_\alpha$, so that

$$i_{\Gamma_0}\Omega_h = \sum_\alpha i_{f_\alpha \Gamma_\alpha}\Omega_h = \sum_\alpha f_\alpha (i_{\Gamma_\alpha}\Omega_h) = \left(\sum_\alpha f_\alpha \right) (n-1)\Omega_h = (n-1)\Omega_h.$$

The argument when h is defined on a nonempty open subset of J^*Y is identical. \square

Theorem 3.5. *Let h be a section of $\rho : J^\dagger Y \rightarrow J^*Y$ defined on $U \subset J^*Y$, and suppose that W is a nonempty open subset of $\tau(U) \subset X$. Let ψ be a local section of $\tau : J^*Y \rightarrow X$ defined on W and satisfying $\psi^*(i_\xi\Omega_h) = 0$ for every vertical vector field ξ on $\tau^{-1}(W)$. For each $x \in W$ there is then a connection $\hat{\Gamma}$, defined on $U \subset J^*Y$ and satisfying the compatibility condition $i_{\hat{\Gamma}}\Omega_h = (n-1)\Omega_h$, such that for some neighbourhood N of $\psi(x)$ the restriction $\psi|_{\tau(N)}$ is an integral section of $\hat{\Gamma}$.*

Proof. Let V be the domain of a chart on Y with $\pi_{1,0}^*(\psi(x)) \in V$, and put $\bar{V} = V \cap \pi^{-1}(W)$ and $\bar{W} = \pi(\bar{V})$. Put $\bar{V}^* = (\pi_{1,0}^*)^{-1}(\bar{V})$, and let $\nu : \bar{N} \rightarrow \psi(\bar{W})$ be an open tubular neighbourhood of the image submanifold $\psi(\bar{W}) \subset \bar{V}^*$. Define the functions Γ_i^σ and $\Gamma_{\sigma j}^i$ on \bar{N} by

$$\Gamma_i^\sigma = \frac{\partial \mathcal{H}}{\partial P_\sigma^i}, \quad \Gamma_{\sigma j}^i = \nu^* \left(\frac{\partial \psi^i}{\partial x^j} \right) - \frac{1}{n} \delta_j^i \left(\nu^* \left(\frac{\partial \psi^k}{\partial x^k} \right) + \frac{\partial \mathcal{H}}{\partial y^\sigma} \right),$$

so putting

$$\bar{\Gamma} = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \Gamma_j^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{\sigma j}^i \frac{\partial}{\partial P_\sigma^i} \right),$$

we see that $\bar{\Gamma}$ is the horizontal projector of a connection defined on \bar{N} satisfying the conditions that $i_{\bar{\Gamma}}\Omega_h = (n-1)\Omega_h$, and that $\psi|_{\tau(\bar{N})}$ is an integral section of $\bar{\Gamma}$.

Now let $\hat{\Gamma}_0$ be a global compatible connection defined on $U \subset J^*Y$ as in Lemma 3.4. Let (f, g) be a partition of unity subordinate to the open cover $(\bar{N}, U - \{\psi(x)\})$ of U , and let $N \subset \bar{N}$ be a neighbourhood of $\psi(x)$ such that $f|_N = 1$ and $g|_N = 0$. Then $\Gamma = f\bar{\Gamma} + g\Gamma_0$ is the horizontal projector of a connection on $U \subset J^*Y$ satisfying the compatibility condition $i_\Gamma\Omega_h = (n-1)\Omega_h$, and such that the restriction $\psi|_{\tau(N)}$ is an integral section of $\hat{\Gamma}$. \square

Combining Theorems 3.1 and 3.5 we obtain the following ‘‘Main Theorem’’.

Theorem 3.6. *Every solution of the Hamilton-De Donder equations can be locally embedded in a field of Hamilton-De Donder extremals.*

3.2 Integrable Hamiltonian connections

Let $\hat{\Gamma}$ be an Ehresmann connection on $U \subset J^*Y$, given by its horizontal projector

$$\Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \Gamma_j^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{\sigma j}^i \frac{\partial}{\partial P_\sigma^i} \right),$$

and let

$$(3.5) \quad \Gamma_{(j)} = \frac{\partial}{\partial x^j} + \Gamma_j^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{\sigma j}^i \frac{\partial}{\partial P_\sigma^i}$$

be local generators of the corresponding horizontal distribution. The connection is flat, in other words its curvature vanishes, if and only if the horizontal distribution is Frobenius integrable. Notice that the Lie bracket of any two horizontal vector fields (3.5) is a vector field vertical over X , so that the integrability conditions are

$$[\Gamma_{(i)}, \Gamma_{(j)}] = 0,$$

or, more explicitly,

$$\Gamma_{(i)}(\Gamma_{(j)}^\sigma) - \Gamma_{(j)}(\Gamma_{(i)}^\sigma) = 0, \quad \Gamma_{(i)}(\Gamma_{\sigma j}^k) - \Gamma_{(j)}(\Gamma_{\sigma i}^k) = 0.$$

Assuming that $\hat{\Gamma}$ is compatible with Ω_h , these integrability conditions take the form

$$\Gamma_{(i)}\left(\frac{\partial \mathcal{H}}{\partial P_\sigma^j}\right) - \Gamma_{(j)}\left(\frac{\partial \mathcal{H}}{\partial P_\sigma^i}\right) = 0,$$

$$\Gamma_{(i)}\left(\frac{1}{n}\delta_j^k \frac{\partial \mathcal{H}}{\partial y^\sigma} + F_{\sigma j}^k\right) - \Gamma_{(j)}\left(\frac{1}{n}\delta_i^k \frac{\partial \mathcal{H}}{\partial y^\sigma} + F_{\sigma i}^k\right) = 0.$$

Rewriting them as conditions for the unknown functions $F_{\sigma j}^i$ we can see that the first set represents *algebraic conditions* on the functions $F_{\sigma j}^i$, while the second represents *differential conditions* on these functions. We therefore obtain the following result.

Proposition 3.7. *Let $\hat{\Gamma}$ be an Ehresmann connection on $U \subset J^*Y$ compatible with the Hamilton-De Donder form Ω_h ; then $\hat{\Gamma}$ is flat if and only if the horizontal distribution $\text{span}\{\Gamma_{(i)}, 1 \leq i \leq n\}$ is completely integrable, and this holds if and only if the functions $F_{\sigma i}^j$ in the coordinate expression (3.3) satisfy*

(1) for each index σ , $\text{Tr}(F_\sigma) = 0$;

(2) algebraic conditions

$$i \neq j: \quad F_{\sigma i}^k \frac{\partial^2 \mathcal{H}}{\partial P_\nu^j \partial P_\sigma^k} - F_{\sigma j}^k \frac{\partial^2 \mathcal{H}}{\partial P_\nu^i \partial P_\sigma^k} = \frac{1}{n} \frac{\partial \mathcal{H}}{\partial y^\sigma} \left(\frac{\partial^2 \mathcal{H}}{\partial P_\nu^j \partial P_\sigma^i} - \frac{\partial^2 \mathcal{H}}{\partial P_\nu^i \partial P_\sigma^j} \right) \\ + \frac{\partial^2 \mathcal{H}}{\partial x^j \partial P_\nu^i} - \frac{\partial^2 \mathcal{H}}{\partial x^i \partial P_\nu^j} + \frac{\partial \mathcal{H}}{\partial P_\sigma^j} \frac{\partial^2 \mathcal{H}}{\partial y^\sigma \partial P_\nu^i} - \frac{\partial \mathcal{H}}{\partial P_\sigma^i} \frac{\partial^2 \mathcal{H}}{\partial y^\sigma \partial P_\nu^j};$$

(3) differential conditions

$$i \neq j = k: \quad \Gamma_{(i)}(F_{\sigma j}^j) - \Gamma_{(j)}(F_{\sigma i}^j) = -\frac{1}{n} \Gamma_{(i)} \left(\frac{\partial \mathcal{H}}{\partial y^\sigma} \right) \quad \text{no summation} \\ i \neq j \neq k: \quad \Gamma_{(i)}(F_{\sigma j}^k) - \Gamma_{(j)}(F_{\sigma i}^k) = 0.$$

Corollary 3.8. *If $\hat{\Gamma}$ is a flat connection on U compatible with $\Omega_{\mathfrak{h}}$ then the integral sections of $\hat{\Gamma}$ form an n -dimensional foliation of U , and the leaves of this foliation are solutions of the Hamilton–De Donder equations. It follows that the Cauchy problem for the given Hamilton–De Donder equations has at least one maximal solution for any given initial condition on U , corresponding to the unique maximal integral manifold passing through that point of U .*

Definition 3.9. Let U be an open subset of J^*Y . We say that a Hamilton–De Donder system is *Cauchy integrable* on U if the Cauchy problem for the given Hamilton–De Donder equations has, for every initial condition in U , at least one maximal solution.

We therefore see that a sufficient condition for the Cauchy integrability of $\Omega_{\mathfrak{h}}$ on U is the existence on U of a flat Ehresmann connection compatible with $\Omega_{\mathfrak{h}}$. Indeed, there is a distinguished family of Hamiltonian systems with the following desirable integrability properties.

Definition 3.10. We say that a Hamilton–De Donder system $\Omega_{\mathfrak{h}}$ is *completely integrable* if every solution (that is, every local section ψ of J^*Y satisfying the Hamilton–De Donder equations) can be locally embedded in a flat compatible Ehresmann connection.

4 Hamilton–De Donder equations in J^1Y

4.1 The Goldschmidt–Sternberg setting

So far we have considered Hamiltonian systems in dual jet bundles, and the corresponding Hamilton–De Donder equations have arisen from the canonical multisymplectic structure as equations for sections of the bundle $\tau : J^*Y \rightarrow X$, dual to a jet bundle $\pi_1 : J^1Y \rightarrow X$. This model generalizes Hamiltonian systems on symplectic manifolds to the case of several independent variables.

We shall now take a different approach, looking at Hamilton–De Donder theory in the setting of Goldschmidt and Sternberg [8], who considered Hamilton’s equations as

equations for sections of the bundle $\pi_1 : J^1Y \rightarrow X$ that arise from a variational problem on $\pi : Y \rightarrow X$.

Let λ be a Lagrangian on J^1Y , and consider its Poincaré–Cartan form

$$(4.1) \quad \theta_\lambda = L\omega_0 + \frac{\partial L}{\partial y_i^\sigma} \omega^\sigma \wedge \omega_i.$$

If the Poincaré–Cartan form is expressed in the canonical basis given by differentials of the coordinate functions, rather than in the basis adapted to the contact structure as above, the components of θ_λ are called the *momenta* and the *Hamiltonian* of λ . We then obtain

$$(4.2) \quad \theta_\lambda = -H\omega_0 + p_\sigma^i dy^\sigma \wedge \omega_i,$$

where the momenta and Hamiltonian are local functions on J^1Y given by

$$(4.3) \quad p_\sigma^i = \frac{\partial L}{\partial y_i^\sigma}, \quad H = -L + \frac{\partial L}{\partial y_i^\sigma} y_i^\sigma = -L + p_\sigma^i y_i^\sigma.$$

Using the first variation formula it can be shown that a local section γ of π is an extremal of the action of λ if and only if it satisfies the Euler–Lagrange equations

$$(4.4) \quad j^1\gamma^* i_\xi d\theta_\lambda = 0 \quad \text{for every vector field } \xi \text{ on } J^1Y \text{ vertical over } X.$$

Denote by \mathcal{D}_λ the exterior differential system generated by n -forms $i_\xi d\theta_\lambda$ where ξ runs over all vector fields on J^1Y that are vertical over X . Then, in terms of exterior differential systems theory, equations (4.4) can be viewed as equations for *special solutions* of \mathcal{D}_λ , namely those which are local sections of π_1 and have the form of *prolongations* of local sections of π . The equations for *all integral sections* ϕ of \mathcal{D}_λ (that is, sections that need not be prolongations) take the form

$$(4.5) \quad \phi^* i_\xi d\theta_\lambda = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1Y,$$

and are called *Hamilton–De Donder equations*: the terminology arises from the local expression of these equations in the Legendre coordinates of a regular Lagrangian. Recall that a Lagrangian is called *regular* if

$$(4.6) \quad \det\left(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}\right) \neq 0;$$

the momenta p_σ^i of a regular Lagrangian are independent, and can be used to define new local coordinates $(x^i, y^\sigma, p_\sigma^i)$ on J^1Y called *Legendre coordinates*. In these Legendre coordinates, and using the Hamiltonian H , we see that local generators of \mathcal{D}_λ are

$$(4.7) \quad -\frac{\partial H}{\partial y^\sigma} \omega_0 - dp_\sigma^i \wedge \omega_i, \quad -\frac{\partial H}{\partial p_\sigma^i} \omega_0 + dy^\sigma \wedge \omega_i.$$

We may then use equations (4.5) to obtain

$$(4.8) \quad \frac{\partial y^\sigma}{\partial x^i} = \frac{\partial H}{\partial p_\sigma^i}, \quad \frac{\partial p_\sigma^i}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma},$$

the Hamilton equations as introduced by De Donder [4].

We can see that equations (4.5) are more general than (4.8), as they are coordinate-independent and do not require regularity of the Lagrangian. Compared with the Euler–Lagrange equations, equations (4.5) generally have more solutions. It can, however, be shown that if λ satisfies the above regularity condition then the Hamilton–De Donder equations (4.5) are *equivalent* to the Euler–Lagrange equations, in the sense that solutions of the former equations must now necessarily be prolongations, and so are in bijective correspondence with solutions of the latter equations.

Following Krupka and Stepankova [12], we may now consider the first jet prolongation of the fibred manifold π_1 . Given a Lagrangian λ on J^1Y , put

$$\tilde{\lambda} = \tilde{h}\theta_\lambda$$

where \tilde{h} represents horizontalization with respect to the projection π_1 , so that $\tilde{\lambda}$ is a horizontal form (a Lagrangian) on $J^1(J^1Y)$. Using $\theta_{\tilde{\lambda}} = (\pi_1)_{1,0}^*\theta_\lambda$, we see that the Euler–Lagrange form of $\tilde{\lambda}$ is $E_{\tilde{\lambda}} = \tilde{p}_1 d\theta_\lambda$, and that $(\pi_1)_{1,0}^*d\theta_\lambda$ is projectable to J^1Y . We immediately obtain the following result [12].

Theorem 4.1. *The Hamilton–De Donder equations (4.5) are variational, and they are Euler–Lagrange equations of the Lagrangian $\tilde{\lambda}$.*

4.2 Semispray connections in Hamilton–De Donder theory

Now consider the bundle of semiholonomic jets $\hat{J}^2Y \rightarrow X$, and let $\iota : J^2Y \rightarrow \hat{J}^2Y$ be the natural inclusion. Let $\hat{\Gamma}$ be a local semiholonomic semispray connection on J^1Y , so that $\hat{\Gamma}$ is a local section $J^1Y \rightarrow \hat{J}^2Y$. With such connection we can associate a vector-valued one-form

$$\Gamma = dx^i \otimes \left(\frac{\partial}{\partial x^i} + y_i^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{ij}^\sigma \frac{\partial}{\partial y_j^\sigma} \right)$$

called its horizontal form, and also a horizontal distribution spanned locally by the semisprays

$$\Gamma_{(i)} = \frac{\partial}{\partial x^i} + y_i^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{ij}^\sigma \frac{\partial}{\partial y_j^\sigma},$$

where Γ_{ij}^σ are the components of $\hat{\Gamma}$. Using the fibre product decomposition

$$\hat{J}^2Y \cong J^2Y \times_{J^1Y} \left(\wedge^2 T^*X \otimes VY \right)$$

([15], Theorem 5.3.4) we see that the symmetric part of a semiholonomic connection $\hat{\Gamma}$ is a holonomic connection $\hat{\Gamma}_0$, with $y_{ij}^\sigma \circ \hat{\Gamma}_0 = \Gamma_{(ij)}^\sigma$, and the skew-symmetric part is a vector-valued 2-form on J^1Y , the *torsion*

$$\mathbb{T} = \Gamma_{[ij]}^\sigma (dx^i \wedge dx^j) \otimes \frac{\partial}{\partial y^\sigma}$$

where for each index σ we let $\Gamma_{(ij)}^\sigma$ and $\Gamma_{[ij]}^\sigma$ denote the symmetric and skew-symmetric parts of the matrix $\Gamma^\sigma = (\Gamma_{ij}^\sigma)$. Using the vertical lift correspondence

$$S : \left(dx^i \otimes \frac{\partial}{\partial y^\sigma} \right) \mapsto \frac{\partial}{\partial y_i^\sigma}, \quad S \left((dx^i \wedge dx^j) \otimes \frac{\partial}{\partial y^\sigma} \right) = \frac{1}{2} \left(dx^i \otimes \frac{\partial}{\partial y_j^\sigma} - dx^j \otimes \frac{\partial}{\partial y_i^\sigma} \right)$$

we may therefore write the horizontal projector of $\hat{\Gamma}$ as a sum

$$\Gamma = dx^i \otimes \left(\frac{\partial}{\partial x^i} + y_i^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{(ij)}^\sigma \frac{\partial}{\partial y_j^\sigma} \right) + \Gamma_{[ij]}^\sigma S \left((dx^i \wedge dx^j) \otimes \frac{\partial}{\partial y^\sigma} \right).$$

A local section γ of π such that $\text{Im } j^1\gamma \subset \text{Dom } \hat{\Gamma}$ will be called an *integral section* of the semiholonomic connection $\hat{\Gamma}$ if

$$\hat{\Gamma}_0 \circ j^1\gamma = j^2\gamma,$$

and we see that such integral sections are solutions of a system of second order PDEs given by

$$\frac{\partial^2 \gamma^\sigma}{\partial x^i \partial x^j} = \Gamma_{(ij)}^\sigma.$$

Indeed, if ϕ is a local section of $\pi_1 : J^1Y \rightarrow X$ such that $\text{Im } \phi \subset \text{Dom } \hat{\Gamma}$, and if $\hat{\Gamma} \circ \phi = j^1\phi \in \hat{J}^2Y \subset J^1J^1Y$ (so that ϕ is an integral section in the usual sense of $\hat{\Gamma}$, where the latter is regarded as an ordinary Ehresmann connection on $J^1Y \rightarrow X$) then

$$\phi_i^\sigma = \frac{\partial \phi^\sigma}{\partial x^i}$$

so that $\phi = j^1\gamma$ for some local section γ of π , and then γ is an integral section of $\hat{\Gamma}$ as defined above.

Now given a Lagrangian λ on J^1Y , we say that a (holonomic or semiholonomic) semispray connection $\hat{\Gamma}$ is λ -compatible if

$$i_\Gamma d\theta_\lambda = (n-1)d\theta_\lambda.$$

Theorem 4.2.

- (1) A holonomic semispray connection $\hat{\Gamma}$ is λ -compatible if and only if $\hat{\Gamma}^*E_\lambda = 0$.

(2) If the matrix

$$B_{\sigma\nu}^{ij} = \frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}$$

is symmetric in i, j , namely

$$B_{\sigma\nu}^{ij} = \frac{\partial E_\sigma}{\partial y_{ij}^\nu},$$

then a semiholonomic semispray connection $\hat{\Gamma}$ is λ -compatible if and only if $\hat{\Gamma}_0^* E_\lambda = 0$.

Proof. The result follows from a direct computation. In fibred coordinates we have

$$d\theta_\lambda = A_\sigma \omega^\sigma \wedge \omega_0 + \frac{\partial^2 L}{\partial y^\nu \partial y_i^\sigma} \omega^\nu \wedge \omega^\sigma \wedge \omega_i + B_{\sigma\nu}^{ij} dy_j^\nu \wedge \omega^\sigma \wedge \omega_i,$$

where we have put

$$A_\sigma = \frac{\partial L}{\partial y^\sigma} - \frac{\partial^2 L}{\partial x^j \partial y_j^\sigma} - \frac{\partial^2 L}{\partial y^\nu \partial y_j^\sigma} y_j^\nu, \quad B_{\sigma\nu}^{ij} = \frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu}$$

so that $E_\sigma = A_\sigma - B_{\sigma\nu}^{ij} y_{ij}^\nu = A_\sigma - B_{\sigma\nu}^{(ij)} y_{ij}^\nu$. We then see that

$$i_\Gamma d\theta_\lambda = (n-1)d\theta_\lambda + (A_\sigma - B_{\sigma\nu}^{ij} \Gamma_{ij}^\nu) \omega^\sigma \wedge \omega_0 = (n-1)d\theta_\lambda + \hat{\Gamma}_0^* E_\lambda - B_{\sigma\nu}^{[ij]} \Gamma_{[ij]}^\nu \omega^\sigma \wedge \omega_0.$$

□

From now on we shall assume that λ is a *regular Lagrangian*. Recall that in this case

- (i) around every point in $J^1 Y$ there are Legendre coordinates $(x^i, y^\sigma, p_\sigma^j)$;
- (ii) the Hamilton–De Donder equations are equivalent to the Euler–Lagrange equations of λ , so that *the only solutions of the Hamilton–De Donder equations are prolongations of extremals*.

In Legendre coordinates the horizontal form Γ of a semispray connection $\hat{\Gamma}$ takes the form

$$\Gamma = dx^i \otimes \left(\frac{\partial}{\partial x^i} + y_i^\sigma \frac{\partial}{\partial y^\sigma} + \Gamma_{\sigma i}^j \frac{\partial}{\partial p_\sigma^j} \right),$$

where now the y_i^σ are considered as functions of the Legendre coordinates; the transformation rule between the remaining components in fibred and Legendre coordinates becomes

$$\Gamma_{\sigma i}^j = \frac{\partial p_\sigma^j}{\partial y_k^\nu} \Gamma_{ik}^\nu.$$

In Legendre coordinates we have

$$d\theta_\lambda = -dH \wedge \omega_0 + dp_\sigma^i \wedge dy^\sigma \wedge \omega_i \quad \text{and} \quad y_i^\sigma = \frac{\partial H}{\partial p_\sigma^i},$$

and so the compatibility condition applied to $d\theta_\lambda$ gives

$$\Gamma_{\sigma i}^i = -\frac{\partial H}{\partial y^\sigma} = \frac{\partial L}{\partial y^\sigma}.$$

Using the results of the previous section, we can now obtain similar results for *semispray connections compatible with regular Lagrangians*. First we discuss the *existence* of λ -compatible connections. From the formulas above it is clear that the local existence is guaranteed: indeed we can obtain such a local connection simply by putting, in any Legendre chart,

$$\Gamma_{\sigma j}^i = -\frac{1}{n} \delta_j^i \frac{\partial H}{\partial y^\sigma}.$$

But the same partition of unity arguments we used in the proof of Lemma 3.4 can also be applied in the present situation, to give the following result.

Theorem 4.3. *Given a regular Lagrangian λ on J^1Y , there exists a λ -compatible semispray connection $\hat{\Gamma} : J^1Y \rightarrow \hat{J}^2Y$.*

The existence of a family of λ -compatible connections is again obtained with help of the trace decomposition.

Theorem 4.4. *If λ is a regular Lagrangian then any local λ -compatible semispray connection is, in Legendre coordinates, described by the horizontal projector*

$$(4.9) \quad \Gamma = dx^j \otimes \left(\frac{\partial}{\partial x^j} + \frac{\partial H}{\partial p_\sigma^j} \frac{\partial}{\partial y^\sigma} - \left(\frac{1}{n} \delta_j^i \frac{\partial H}{\partial y^\sigma} + F_{\sigma j}^i \right) \frac{\partial}{\partial p_\sigma^i} \right),$$

where for each index σ the matrix of functions $(F_{\sigma j}^i)$ is a traceless $(n \times n)$ -matrix.

Our final task is to establish a relationship between solutions of the compatible connections and solutions of the Euler–Lagrange equations. However, guided by the corresponding theorems in the previous section, and under our regularity assumption on the Lagrangian, the results follow immediately.

Theorem 4.5. *Let λ be a regular Lagrangian. If a local semispray connection $\hat{\Gamma} : J^1Y \rightarrow \hat{J}^2Y$ is λ -compatible then the prolongation of any integral section of $\hat{\Gamma}$ is a solution of the Hamilton–De Donder equations (4.8), so that any integral section is an extremal.*

Theorem 4.6. *Let λ be a regular Lagrangian on J^1Y , and let γ be a local section of $\pi : Y \rightarrow X$ defined on a nonempty open subset $W \subset X$ and satisfying the Euler–Lagrange equations $j^1\gamma^* i_\xi d\theta_\lambda = 0$ for every vertical vector field ξ on $\pi_1^{-1}(W)$. For each $x \in W$ there is a λ -compatible semispray connection $\hat{\Gamma}$ such that, for some neighbourhood N of $j^1\gamma(x)$, the restriction $\gamma|_{\pi_1(N)}$ is an integral section of $\hat{\Gamma}$.*

Once again the proof is completely analogous to the proof of Theorem 3.5, and relies on the construction of a suitable compatible semispray connection defined on an open tubular neighborhood ν around the image of the given prolonged extremal. If we denote by $(x^i, u^\sigma, v_\sigma^j)$ the components of $j^1\gamma$ in Legendre coordinates, we can take the “momentum components” of $\hat{\Gamma}$ in the form

$$\Gamma_{\sigma j}^i = \nu^* \left(\frac{\partial v_\sigma^i}{\partial x^j} \right) - \frac{1}{n} \delta_j^i \left(\nu^* \left(\frac{\partial v_\sigma^k}{\partial x^k} \right) + \frac{\partial H}{\partial y^\sigma} \right).$$

A global compatible semispray connection such that γ is its (local) integral section is then obtained by the partition of unity arguments from the local connection above and a global connection, the existence of which we have proved in Theorem 4.3.

In summary, therefore, for every regular Lagrangian we have found a family of semispray connections, parametrized by traceless matrices, and such that every integral section of each connection is an extremal, a solution of the Euler–Lagrange equations. Moreover, the integral sections of this family of semispray connections include, at least locally, *all* extremals.

Remark 4.7. If we consider the integrability of λ -compatible *holonomic* semispray connections, we have symmetry of the connection components in fibered coordinates, and we know that $y_i^\sigma = \partial H / \partial p_i^\sigma$. Consequently, the integrability conditions for the connection are simpler. The “algebraic conditions” for the unknown functions F

$$\Gamma_{(i)} \left(\frac{\partial H}{\partial p_\sigma^j} \right) = \Gamma_{(j)} \left(\frac{\partial H}{\partial p_\sigma^i} \right)$$

become just

$$\Gamma_{(i)}(y_j^\sigma) = \Gamma_{(j)}(y_i^\sigma),$$

which is equivalent to $\Gamma_{ij}^\sigma = \Gamma_{ji}^\sigma$. But this is an identity due to the symmetry, and is hence redundant. It follows that the integrability conditions in fibered coordinates reduce to just the “differential conditions” for the functions F , and these are the conditions for the curvature of the connection to vanish. It should, however, be noted that, as equations for the unknown functions F , these conditions are more complicated than the “differential” integrability conditions in Legendre coordinates.

5 Lagrangian-Hamiltonian duality

Our aim now is to establish a duality relationship between the “Lagrangian side” and “Hamiltonian side” for Hamilton–De Donder systems.

5.1 From Lagrangian to Hamiltonian

Consider the affine dual $J^1Y \xrightarrow{\rho} J^*Y \xrightarrow{\tau} X$ for a fibred manifold $\pi : Y \rightarrow X$ as above, with the canonical n -form $\Theta = P\omega_0 + P_\sigma^i dy^\sigma \wedge \omega_i$ and the canonical multisymplectic form $\Omega = d\Theta$ on J^1Y . On $\pi_1 : J^1Y \rightarrow X$ let a Hamilton–De Donder system be given by a Lagrangian λ and its Poincaré–Cartan form θ_λ . Both the sides of the picture can be related by means of a map $\text{Leg} : J^1Y \rightarrow J^*Y$ fibred over the identity of Y , defined by

$$(5.1) \quad \text{Leg}^* \Theta = \theta_\lambda.$$

We call Leg an *extended Legendre map* and equation (5.1) a *duality equation*. We also construct a composite map $\text{leg} : J^1Y \rightarrow J^*Y$ by setting $\text{leg} = \rho \circ \text{Leg}$. This is called the *reduced Legendre map*.

In local fibred coordinates on J^1Y such that $\omega_0 = dx^1 \wedge \cdots \wedge dx^n$ and related canonical coordinates $(x^i, y^\sigma, P, P_\sigma^j)$ and $(x^i, y^\sigma, P_\sigma^j)$ on J^1Y and J^*Y respectively, the maps Leg and leg are given in coordinates by

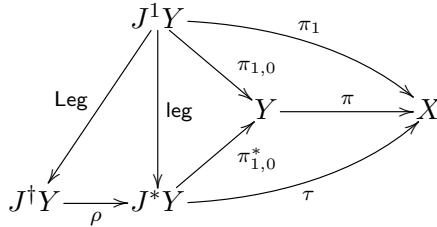
$$P \circ \text{Leg} = L - \frac{\partial L}{\partial y_j^\sigma} y_j^\sigma, \quad P_\sigma^j \circ \text{Leg} = \frac{\partial L}{\partial y_j^\sigma}$$

and

$$P_\sigma^j \circ \text{leg} = \frac{\partial L}{\partial y_j^\sigma},$$

respectively.

If the Lagrangian is regular we can choose local Legendre coordinates on J^1Y , and then leg is represented by the identity mapping.



We immediately obtain the following result.

Theorem 5.1. *If λ is regular then every extended Legendre map is an immersion and every corresponding reduced Legendre map is a local diffeomorphism.*

We say that λ is *hyper-regular* if there is an extended Legendre map Leg which is defined globally and which has the property that the corresponding reduced Legendre map leg is a diffeomorphism. Any such map gives rise to a global Hamiltonian section \mathfrak{h} by setting $\mathfrak{h} = \text{Leg} \circ \text{leg}^{-1}$. If λ is regular rather than hyper-regular then we may construct local sections \mathfrak{h} in this way. The negative coordinate representative of such a section \mathfrak{h} will be the function $\mathcal{H} = -(P \circ \mathfrak{h})$, a *Hamiltonian function* corresponding to λ .

With help of the definition of the Legendre maps, and using Legendre coordinates, we easily obtain the following results.

Theorem 5.2. *For a hyper-regular Lagrangian λ on J^1Y the following duality assertions hold:*

- (1) $\text{leg} \circ \mathcal{H} = H$;
- (2) $(j^1 \text{leg})^* \Lambda = \tilde{\lambda}$;
- (3) $\text{leg}^* \Omega_{\mathfrak{h}} = \text{leg}^* h^* \Omega = d\theta_{\lambda}$;
- (4) $\text{rank } h^* \Omega = \text{rank } \mathcal{D}_{\mathfrak{h}} = \text{rank } d\theta_{\lambda} = \text{rank } \mathcal{D}_{\lambda} = m + nm$;
- (5) $\text{leg}^* \mathcal{D}_{\mathfrak{h}} = \mathcal{D}_{\lambda}$;
- (6) *if $\psi : X \rightarrow J^*Y$ is an integral section of $\mathcal{D}_{\mathfrak{h}}$ then $\text{leg}^{-1} \circ \psi = j^1 \gamma$ where γ is a section of $\pi : Y \rightarrow X$, and furthermore $j^1 \gamma$ is an integral section of \mathcal{D}_{λ} ;*
- (7) *every integral section of \mathcal{D}_{λ} is of the form $j^1 \gamma$, and $\psi = \text{leg} \circ j^1 \gamma$ is an integral section of $\mathcal{D}_{\mathfrak{h}}$.*

5.2 From Hamiltonian to Lagrangian

Now let $\Omega_{\mathfrak{h}}$ be a Hamiltonian system on the dual jet bundle $\tau : J^*Y \rightarrow X$. We have seen that such a system is variational, in that there is a Lagrangian form $\Lambda_{\mathfrak{h}} = \mathcal{L}_{\mathfrak{h}} \omega_0$ whose extremals ψ are precisely the solutions of the Hamilton–De Donder system $\mathcal{D}_{\mathfrak{h}}$. But $\Lambda_{\mathfrak{h}}$ is a form on $J^1 J^*Y$; we now wish to see if there is, at least locally, a Lagrangian form on the original jet manifold J^1Y whose extremals are related to the solutions ψ .

Let $U \subset J^*Y$ be the domain of a chart $(x^i, y^{\sigma}, P_{\sigma}^i)$ contained within the domain of \mathfrak{h} , and define a map $\text{heg} : U \rightarrow J^1Y$ fibred over the identity on Y by

$$(5.2) \quad y_i^{\sigma} \circ \text{heg} = \frac{\partial \mathcal{H}}{\partial P_{\sigma}^i}.$$

If heg is a local diffeomorphism, so that \mathcal{H} satisfies the regularity condition

$$(5.3) \quad \det \left(\frac{\partial^2 \mathcal{H}}{\partial P_{\sigma}^i \partial P_{\tau}^j} \right) \neq 0,$$

we may construct a local Lagrangian $\lambda = L\omega_0$ on J^1Y by setting

$$(5.4) \quad \lambda = h(\theta_{\lambda}) \quad \text{where} \quad \theta_{\lambda} = (\text{heg}^{-1})^* \Theta_{\mathfrak{h}},$$

so that

$$(5.5) \quad L \circ \text{heg} = -\mathcal{H} + \frac{\partial \mathcal{H}}{\partial P_{\sigma}^i} P_{\sigma}^i.$$

Theorem 5.3. *Let $\Omega_{\mathfrak{h}}$ be a Hamilton–De Donder system on J^*Y satisfying the regularity condition (5.3) on U . The local Lagrangian $\lambda = L\omega_0$ (5.5) is then regular and satisfies $\text{leg} = \text{heg}^{-1}$; furthermore, the solutions of the Euler–Lagrange equations for λ are in bijective correspondence with the Hamilton–De Donder extremals of $\Omega_{\mathfrak{h}}$.*

Proof. From (5.5) we see that

$$\frac{\partial}{\partial P_\nu^j}(L \circ \text{heg}) = \frac{\partial^2 \mathcal{H}}{\partial P_\sigma^i \partial P_\nu^j} P_\sigma^i,$$

whereas a direct calculation using the chain rule gives

$$\frac{\partial}{\partial P_\nu^j}(L \circ \text{heg}) = \left(\frac{\partial L}{\partial y_i^\sigma} \circ \text{heg} \right) \frac{\partial}{\partial P_\nu^j}(y_i^\sigma \circ \text{heg}) = \left(\frac{\partial L}{\partial y_i^\sigma} \circ \text{heg} \right) \frac{\partial^2 \mathcal{H}}{\partial P_\sigma^i \partial P_\nu^j}.$$

It then follows by regularity of Ω_h that

$$P_\sigma^i \circ \text{heg}^{-1} = \frac{\partial L}{\partial y_i^\sigma} = P_\sigma^i \circ \text{leg},$$

so that $\text{leg} = \text{heg}^{-1}$ is a local diffeomorphism.

We now observe from (5.5) that

$$\mathcal{H} \circ \text{leg} = -L + \left(\frac{\partial \mathcal{H}}{\partial P_\sigma^i} \circ \text{leg} \right) (P_\sigma^i \circ \text{leg}) = -L + y_i^\sigma \frac{\partial L}{\partial y_i^\sigma}$$

so that \mathcal{H} is the Hamiltonian function associated classically with the Lagrangian function L . Any extremal ψ of the Hamilton-De Donder system Ω_h thus satisfies the condition that $\text{heg} \circ \psi = j^1 \gamma$ for some extremal γ of L , and conversely if γ is an extremal of L then $\text{leg} \circ j^1 \gamma$ is an extremal of Ω_h . \square

5.3 The inverse problem

We remark finally that this procedure may be applied also to a family of first-order equations

$$(5.6) \quad \frac{\partial y^\sigma}{\partial x^i} - F_i^\sigma = 0, \quad \frac{\partial P_\sigma^i}{\partial x^i} + G_\sigma = 0$$

where F_i^σ, G_σ are functions defined locally on J^*Y ; such functions may, for example, be obtained from an Ehresmann connection Γ on $\tau : J^*Y \rightarrow X$ where $F_i^\sigma = y_{|i}^\sigma \circ \Gamma$ and $G_\sigma = -P_{\sigma|i}^i \circ \Gamma$. Writing E for the $(n+1)$ -form

$$E = (G_\sigma + P_{\sigma|i}^i) dy^\sigma \wedge \omega_0 + (-F_i^\sigma + y_{|i}^\sigma) dP_\sigma^i \wedge \omega_0$$

we see that Corollary 2.4 can be refined as follows.

Theorem 5.4. *A system of first order PDEs (5.6) is a Hamilton-De Donder system, equivalent to the Euler-Lagrange equations of a (local) regular Lagrangian L , if and only if*

$$\frac{\partial G_\sigma}{\partial y^\nu} = \frac{\partial G_\nu}{\partial y^\sigma}, \quad \frac{\partial F_i^\sigma}{\partial P_\nu^k} = \frac{\partial F_k^\nu}{\partial P_\sigma^i}, \quad \frac{\partial G_\sigma}{\partial P_\nu^i} = \frac{\partial F_i^\nu}{\partial y^\sigma}$$

and

$$\det\left(\frac{\partial F_i^\sigma}{\partial P_\nu^i}\right) \neq 0.$$

If these conditions hold then

$$L = F_i^\sigma P_\sigma^i + y^\sigma \int_0^1 (G_\sigma \circ \chi) du + P_\sigma^j \int_0^1 (F_j^\sigma \circ \chi) du,$$

and the inverse Legendre transformation (giving “velocities” y_j^σ) is defined by the equations $y_j^\sigma = F_j^\sigma$.

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