Inequalities for regular near polygons, with applications to m-ovoids

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Abstract

We derive two sets of inequalities for regular near polygons and study the case where one or more of these inequalities become equalities. This will allow us to obtain two characterization results for dual polar spaces. Our investigation will also have implications for triple intersection numbers and m-ovoids in regular near polygons. In particular, we obtain new results on triple intersection numbers in generalized hexagons of order (s, s^3), $s \geq 2$, and prove that no finite generalized hexagon of order (s, s^3), $s \geq 2$, can have 1-ovoids. We also show that in one case, the existence of a 1-ovoid would allow a construction of a strongly regular graph srg$(47125, 12012, 3575, 2886)$.

Key words: regular near polygons, generalized polygons, inequalities, m-ovoids, distance-2-ovoids

1. Introduction

A point-line incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *partial linear space* if every line is incident with at least two points and if every two distinct points are incident with at most one line. The *collinearity graph* of $\mathcal{S}$ has as vertices the points of $\mathcal{S}$, with two distinct points being adjacent whenever they are incident with a common line. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *near polygon* if for every point x and every line L, there exists a unique point on L nearest to x. Here, distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph Γ of $\mathcal{S}$. If $d \in \mathbb{N}$ is the diameter of Γ, then the near polygon is also

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called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles. If $x \in \mathcal{P}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x. If $\emptyset \neq X \subseteq \mathcal{P}$, then $\Gamma_i(X)$ denotes the set of points at distance i from X, i.e. the set of all $y \in \mathcal{P}$ for which $\mathrm{d}(y, X) := \min\{\mathrm{d}(y, x) \,|\, x \in X\} = i$.

A near polygon $\mathcal{S}$ is said to have *order* (s, t), where $s, t \geq 1$, if every line of $\mathcal{S}$ is incident with precisely $s+1$ points and if every point of $\mathcal{S}$ is incident with precisely $t+1$ lines. A near $2d$-gon $\mathcal{S}$ with $d \geq 2$ is called *regular* if there exist constants s, t and t_i, $i \in \{0, 1, \dots, d\}$, such that $\mathcal{S}$ has order (s, t) and for every two points x and y at distance i from each other, there are precisely $t_i + 1$ lines through y containing a (necessarily unique) point at distance $i-1$ from x. If this holds, then $(t_0, t_1, t_d) = (-1, 0, t)$ and we say that $\mathcal{S}$ is *regular with parameters* $(s, t_2, t_3, \dots, t_{d-1}, t)$. The finite regular near polygons are precisely those near polygons whose collinearity graph is distance-regular. The intersection numbers a_i, b_i, c_i ($i \in \{0, 1, \dots, d\}$) of the distance-regular graph associated with a regular near $2d$-gon $\mathcal{S}$ are easily derived from its parameters $(s, t_2, t_3, \dots, t_{d-1}, t)$. Indeed, we have that $a_i = (s-1)(t_i+1)$, $b_i = s(t - t_i)$ and $c_i = t_i + 1$ for every $i \in \{0, 1, \dots, d\}$. A regular near $2d$-gon with parameters $(s, t_2, t_3, \dots, t_{d-1}, t)$ is a generalized $2d$-gon if $t_i = 0$ for every $i \in \{2, 3, \dots, d-1\}$. We note that all finite regular near $2d$-gons with $d \geq 4$, $c_2 > 2$ and $s \geq 2$ have been classified. By results of Brouwer & Cohen [4, Corollary 2, p. 195], Brouwer & Wilbrink [8, Section (m)] and De Bruyn [15] we know that every such regular near $2d$-gon is a so-called dual polar space.

The main results of this paper will be discussed and proved in Section 3. In Theorem 3.2, we will prove that if $\mathcal{S}$ is a regular near $2d$-gon with parameters $(s, t_2, t_3, \dots, t_{d-1}, t)$, $s \geq 2$, and associated intersection numbers a_i, b_i, c_i ($i \in \{0, 1, \dots, d\}$), then $c_2 \leq s^2 + 1$ and

$$\frac{(s^i - 1)(c_{i-1} - s^{i-2})}{s^{i-2} - 1} \leq c_i \leq \frac{(s^i + 1)(c_{i-1} + s^{i-2})}{s^{i-2} + 1} \qquad (1)$$

for all $i \in \{3, 4, \dots, d\}$. The two inequalities given in (1) extend a result of Neumaier [31, Theorem 3.1] who already proved the validity of the upper bound in (1) in case i is odd and the validity of the lower bound in case i is even. Hence the two inequalities of Neumaier hold regardless of the parity of i. The upper bound with $i = d = 3$ is also known as the *Mathon bound* [30].

We will discuss the structure of those regular near polygons for which at least one of the bounds in (1) is attained. Our investigation leads to two characterization results for dual polar spaces (Theorems 3.4 and 3.5). In case of equality in the obtained bounds, we are also able to derive some relations between the intersection numbers (Theorem 3.7) and between triple intersection numbers (Theorem 3.8 and Corollary 3.9). We also obtain a number of results on m-ovoids of a regular near polygon, an *m-ovoid* being a set of points intersecting each line in precisely m points. Also for m-ovoids some results regarding intersection sizes are obtained (Theorem 3.12, Corollary 3.13 and Corollary 3.15). This will allow us to give a new proof for the non-existence of 1-ovoids in the dual polar spaces $DQ(2d, q)$ and $DW(2d-1, q)$ for every $d \geq 3$ (Corollary 3.14). We also prove that with every 1-ovoid of a regular near hexagon for which the Mathon bound is attained, there is associated a strongly regular graph (Theorem 3.17). The fact that the parameters and the multiplicities of the eigenvalues of this strongly regular graph are nonnegative integers puts restrictions on the parameters of the regular near hexagon. In particular, this allows us to prove that no generalized hexagon of order (s, s^3), $s \geq 2$, can have 1-ovoids (Corollary 3.19). This will have implications regarding the non-existence of certain semi-finite generalized hexagons (Corollary 3.20).

2. Preliminaries

In this section, we collect some known facts about distance-regular graphs that will be useful later. We refer to Brouwer, Cohen & Neumaier [5] for proofs and much more information on these graphs.

Let Γ be a finite undirected connected graph, without loops or multiple edges, with vertex set Ω and diameter $d \geq 2$. The *distance-i-relation* R_i in Γ consists of all pairs of vertices at distance i. The graph Γ is called *distance-regular* if there exist natural numbers a_i, b_i, c_i ($i \in \{0, 1, \dots, d\}$), known as the *intersection numbers*, such that for any two vertices x and y at distance i from each other, we have $|\Gamma_i(x) \cap \Gamma_1(y)| = a_i$, $|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i$ and $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i$. For the remainder of this section, we assume Γ is a distance-regular graph. The number $|\Gamma_i(x) \cap \Gamma_j(y)|$ with x and y points at distance k then only depends on i, j and k, and is denoted by $p^k_{i,j}$. We also define $k_i := p^0_{i,i}, \forall i \in \{0, 1, \dots, d\}$, the number of vertices at distance i from any given vertex. Note that $a_i = p^i_{1,i}$, $b_i = p^i_{1,i+1}$, $c_i = p^i_{1,i-1}$ and $k_1 = b_0 = a_i + b_i + c_i$ for every $i \in \{0, 1, \dots, d\}$. We also have $k_{i+1} = k_i b_i / c_{i+1}$ for every $i \in \{0, 1, \dots, d-1\}$ and $|\Omega| = \sum_{i=0}^{d} k_i$.

Put $\Omega = \{p_1, p_2, \ldots, p_{|\Omega|}\}$ and let A_i, $i \in \{0, 1, \ldots, d\}$, be the $(|\Omega| \times |\Omega|)$-matrix over $\mathbb{R}$ whose (j,k)-th entry is equal to 1 if $\mathrm{d}(p_j, p_k) = i$ and equal to 0 otherwise. Clearly, $A_0 = I$ and $A_0 + A_1 + \cdots + A_d = J$, where I denotes the identity matrix and J the all-one matrix.

The real vector space spanned by $\{A_0, A_1, \ldots, A_d\}$ is a commutative $(d+1)$-dimensional algebra of symmetric matrices, known as the *Bose-Mesner algebra*. It can be shown that the Bose-Mesner algebra has a unique basis $\{E_0, E_1, \ldots, E_d\}$ of minimal idempotents for which $E_i E_j = \delta_{ij} E_i$, $\forall i, j \in \{0, 1, \ldots, d\}$, $E_0 + E_1 + \cdots + E_d = I$ and $E_0 = J/|\Omega|$. These minimal idempotents are positive semidefinite.

The adjacency matrix A_1 of Γ has exactly $d+1$ distinct eigenvalues. There exists a bijective correspondence between these $d+1$ eigenvalues and the $d+1$ minimal idempotents. Indeed, for every minimal idempotent E there exists a unique eigenvalue λ such that $A_1 E = \lambda E$, and then the column span of E is precisely the (right) eigenspace of A_1 for λ.

The *dual eigenvalue sequence* of a minimal idempotent E is the unique sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ of real numbers such that $E = |\Omega|^{-1}(\theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d)$. We then say Γ is *Q-polynomial* with respect to E if there is a (necessarily unique) ordering $E_0 = J/|\Omega|, E_1 = E, E_2, \ldots, E_d$ of the minimal idempotents such that every E_j, $j \in \{0, 1, \ldots, d\}$, can be written in the form $|\Omega|^{-1} \sum_{i=0}^{d} q_j(\theta_i^*) A_i$ for some real polynomial q_j of degree j. If θ is the eigenvalue for A_1 corresponding to E, then we also say Γ is *Q-polynomial with respect to θ*.

The distance-regular graph Γ is said to have *classical parameters* (d, b, α, β) where $\alpha \in \mathbb{R}$ and $b, \beta \in \mathbb{R} \setminus \{0\}$ if

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right), \quad (2)$$

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b \right) \quad (3)$$

for every $i \in \{0, 1, \ldots, d\}$. Here, $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_b = i$ if $b = 1$ and $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_b = (b^i - 1)/(b - 1)$ if $b \neq 1$. Graphs with classical parameters are Q-polynomial with respect to $\left[\begin{smallmatrix} d-1 \\ 1 \end{smallmatrix}\right]_b (\beta - \alpha) - 1$.

We say that a graph with v vertices is *strongly regular with parameters* (v, k, λ, μ), and write $\mathrm{srg}(v, k, \lambda, \mu)$, if every vertex has exactly $k \geq 1$ neighbors, and every two distinct vertices have exactly λ or μ neighbors in common, depending on whether these two vertices are adjacent or not. The strongly regular graphs $\mathrm{srg}(v, k, \lambda, \mu)$ with $k < v - 1$ and $\mu > 0$ are precisely the distance-regular graphs of diameter 2, and are Q-polynomial.

The finite dual polar spaces of rank $d \geq 2$ constitute an important class of regular near $2d$-gons. In a finite projective space Σ of (projective) dimension at least three, the subspaces that are contained in a given nonsingular quadric or are totally isotropic with respect to a given symplectic or Hermitian polarity of Σ define a so-called polar space. Its rank d is the vector dimension of the maximal totally isotropic subspaces. With each such polar space P there is associated a dual polar space Δ, see Cameron [10]. This Δ is the point-line geometry whose points and lines are the maximal and next-to-maximal subspaces of P, respectively, with incidence being reverse containment. In the corresponding collinearity graph, two vertices are at distance i when their intersection has dimension $d - i$.

3. Main results

Throughout this section, $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ denotes a finite regular near $2d$-gon, $d \geq 2$, with parameters $(s, t_2, t_3, \ldots, t_{d-1}, t)$, $s \geq 2$, and associated intersection numbers a_i, b_i, c_i ($i \in \{0, 1, \ldots, d\}$). We denote by Γ the collinearity graph of $\mathcal{S}$. Similarly as in Section 2, if $i, j, k \in \{0, 1, \ldots, d\}$ and x, y are two points at distance k from each other, then $p^k_{i,j}$ denotes the number of points at distance i from x and distance j from y. We also define $k_i := p^0_{i,i}$.

Suppose $\mathcal{P} = \{p_1, p_2, \ldots, p_{|\mathcal{P}|}\}$ and let A_i, $i \in \{0, 1, \ldots, d\}$, be the $(|\mathcal{P}| \times |\mathcal{P}|)$-matrix over $\mathbb{R}$ whose (j, k)-th entry is equal to 1 if $\mathrm{d}(p_j, p_k) = i$ and equal to 0 otherwise. We also define

$$M := \sum_{i=0}^{d} \Big(-\frac{1}{s}\Big)^i A_i.$$

For every subset $X \subseteq \mathcal{P}$, let χ_X be the *characteristic vector* of X, i.e. the i-th entry of X is equal to 1 if $p_i \in X$ and equal to 0 otherwise. We regard χ_X as a column matrix.

For a proof of the following lemma, see e.g. Vanhove [41, Lemma 1].

Lemma 3.1. *The element M of the Bose-Mesner algebra of Γ is a minimal idempotent up to a positive scalar, and its column span is precisely the eigenspace of the eigenvalue $-(t + 1)$ of A_1.*

In this section, we obtain the main results of this paper. We start by proving the two inequalities already mentioned in Section 1.

Theorem 3.2. *Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, with $s \geq 2$ and $d \geq 2$. Then $c_2 \leq s^2 + 1$ and*

$$\frac{(s^i - 1)(c_{i-1} - s^{i-2})}{s^{i-2} - 1} \leq c_i \leq \frac{(s^i + 1)(c_{i-1} + s^{i-2})}{s^{i-2} + 1} \tag{4}$$

for all $i \in \{3, 4, \ldots, d\}$. Suppose x and y are two points of $\mathcal{S}$ at distance i from each other where $3 \leq i \leq d$, and put $Z := \Gamma_1(x) \cap \Gamma_{i-1}(y)$ and $Z' := \Gamma_{i-1}(x) \cap \Gamma_1(y)$. Then the following holds.

(a) *If $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$ then $Mv = 0$ where*

$$v = s(c_{i-1} - (-1)^i s^{i-2})(\chi_{\{x\}} - \chi_{\{y\}}) + (\chi_Z - \chi_{Z'}).$$

(b) *If $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$ then $Mv = 0$ where*

$$v = s(c_{i-1} + (-1)^i s^{i-2})(\chi_{\{x\}} + \chi_{\{y\}}) + (\chi_Z + \chi_{Z'}).$$

Proof. Let i be a fixed element of $\{2, 3, \ldots, d\}$ and put $v = \alpha\chi_{\{x\}} + \beta\chi_{\{y\}} + \gamma\chi_Z + \delta\chi_{Z'}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We will compute $s^i(v^T M v)$. If $j \in \{0, 1, \ldots, d\}$, then by definition of the sets Z and Z', we have

$$(\chi_{\{x\}})^T A_j \chi_{\{x\}} = (\chi_{\{y\}})^T A_j \chi_{\{y\}} = \delta_{0j}, (\chi_{\{x\}})^T A_j \chi_{\{y\}} = \delta_{ij},$$

$$(\chi_{\{x\}})^T A_j \chi_Z = (\chi_{\{y\}})^T A_j \chi_{Z'} = \delta_{1j} c_i, \quad (\chi_{\{x\}})^T A_j \chi_{Z'} = (\chi_{\{y\}})^T A_j \chi_Z = \delta_{i-1,j} c_i,$$

$$(\chi_Z)^T A_0 \chi_Z = (\chi_{Z'})^T A_0 \chi_{Z'} = c_i, \quad (\chi_Z)^T A_2 \chi_Z = (\chi_{Z'})^T A_2 \chi_{Z'} = c_i(c_i - 1),$$

$$(\chi_Z)^T A_j \chi_Z = (\chi_{Z'})^T A_j \chi_{Z'} = 0 \text{ if } j \notin \{0, 2\}.$$

Suppose now that $p_x \in Z$ and $p_y \in Z'$. Since $\mathrm{d}(p_x, y) = i-1$ and $\mathrm{d}(y, p_y) = 1$, we have $\mathrm{d}(p_x, p_y) \in \{i-2, i-1, i\}$ by the triangle inequality. We prove that the possibility $\mathrm{d}(p_x, p_y) = i-1$ cannot occur. If $\mathrm{d}(p_x, p_y) = i-1$, then the fact that also $\mathrm{d}(p_x, y) = i-1$ would imply that there exists a point p_y' on the line $p_y y$ at distance $i-2$ from p_x, but at distance i from x. This is impossible since x and p_x are collinear.

If $p_x \in Z$, then $\Gamma_{i-2}(p_x) \cap \Gamma_1(y) \subseteq \Gamma_{i-1}(x) \cap \Gamma_1(y) = Z'$. Hence, $|\Gamma_{i-2}(p_x) \cap Z'| = c_{i-1}$ and $|\Gamma_i(p_x) \cap Z'| = c_i - c_{i-1}$. So, we have

$$(\chi_Z)^T A_{i-2} \chi_{Z'} = c_i c_{i-1}, \quad (\chi_Z)^T A_i \chi_{Z'} = c_i(c_i - c_{i-1}), \quad (\chi_Z)^T A_j \chi_{Z'} = 0 \text{ if } j \notin \{i-2, i\}.$$

By the above equalities, we readily see that $s^i(v^T M v) = [\alpha,\beta,\gamma,\delta]\cdot F\cdot[\alpha,\beta,\gamma,\delta]^T$, where F is the following (4×4)-matrix:

$$F := \begin{bmatrix} s^i & (-1)^i & -s^{i-1}c_i & (-1)^{i-1}sc_i \\ (-1)^i & s^i & (-1)^{i-1}sc_i & -s^{i-1}c_i \\ -s^{i-1}c_i & (-1)^{i-1}sc_i & s^i\left(c_i+\frac{c_i(c_i-1)}{s^2}\right) & s^i\left(\frac{c_ic_{i-1}}{(-s)^{i-2}}+\frac{c_i(c_i-c_{i-1})}{(-s)^i}\right) \\ (-1)^{i-1}sc_i & -s^{i-1}c_i & s^i\left(\frac{c_ic_{i-1}}{(-s)^{i-2}}+\frac{c_i(c_i-c_{i-1})}{(-s)^i}\right) & s^i\left(c_i+\frac{c_i(c_i-1)}{s^2}\right) \end{bmatrix}.$$

By Lemma 3.1, the matrix M is positive semidefinite. So, also F is positive semidefinite, implying that the determinant of every principle submatrix of F is nonnegative. The fact that the determinant of the principle submatrix determined by the first three rows and columns is nonnegative implies that

$$c_i \le \frac{s^{2i}-1}{s^2-1}. \tag{5}$$

In particular, we have $c_2 \le s^2+1$. (The inequality (5) was already derived in Vanhove [41, Theorem 1]).

Suppose now that $i\ge 3$. Then $\det(F)$ is equal to

$$-c_i^2(s^2-1)^2(s^{i-2}+1)(s^{i-2}-1)\Big(c_i-\frac{(s^i-1)(c_{i-1}-s^{i-2})}{s^{i-2}-1}\Big)\Big(c_i-\frac{(s^i+1)(c_{i-1}+s^{i-2})}{s^{i-2}+1}\Big).$$

The fact that $c_{i-1}\le (s^{2(i-1)}-1)/(s^2-1)$ implies that

$$\frac{(s^i-1)(c_{i-1}-s^{i-2})}{s^{i-2}-1} \le \frac{(s^i+1)(c_{i-1}+s^{i-2})}{s^{i-2}+1}.$$

Now, $\det(F)\ge 0$ implies that

$$\frac{(s^i-1)(c_{i-1}-s^{i-2})}{s^{i-2}-1} \le c_i \le \frac{(s^i+1)(c_{i-1}+s^{i-2})}{s^{i-2}+1},$$

finishing the proof of the first part of the theorem.

Now, we have $Mv=0 \Longleftrightarrow v^TMv=0 \Longleftrightarrow [\alpha,\beta,\gamma,\delta]\cdot F\cdot[\alpha,\beta,\gamma,\delta]^T = 0 \Longleftrightarrow F\cdot[\alpha,\beta,\gamma,\delta]^T=0$. If

$$\begin{cases} c_i = \frac{(s^i-(-1)^i)(c_{i-1}-(-1)^i s^{i-2})}{s^{i-2}-(-1)^i}, \\ (\alpha,\beta,\gamma,\delta) = (s(c_{i-1}-(-1)^i s^{i-2}), -s(c_{i-1}-(-1)^i s^{i-2}), 1, -1), \end{cases}$$

or

$$\begin{cases} c_i = \frac{(s^i+(-1)^i)(c_{i-1}+(-1)^i s^{i-2})}{s^{i-2}+(-1)^i}, \\ (\alpha,\beta,\gamma,\delta) = (s(c_{i-1}+(-1)^i s^{i-2}), s(c_{i-1}+(-1)^i s^{i-2}), 1, 1), \end{cases}$$

then $F \cdot [\alpha,\beta,\gamma,\delta]^T = 0$ and hence $Mv = 0$, finishing the proof of the last part of the theorem. □

Remark 3.3. In the proof of Theorem 3.2, other principle submatrices of F have to be positive semidefinite as well of course, but this yields no stronger inequalities.

The upper bound $c_2 \leq s^2 + 1$ in Theorem 3.2 is also a consequence of Higman's inequality for generalized quadrangles [27, (6.4)] and the fact that regular near $2d$-gons with parameters $(s, t_2, t_3, \ldots, t_{d-1}, t)$ satisfying $s \geq 2$ and $c_2 \geq 2$ admit quads (Shult & Yanushka [34, Proposition 2.5]). If $i = d = 3$, then the upper bound in Theorem 3.2 is the so-called *Mathon bound* [30] for regular near hexagons with parameters (s, t_2, t), $s \geq 2$; $t + 1 \leq (t_2 + s + 1)(s^2 - s + 1)$. If $t_2 = 0$, then the Mathon bound reduces to the well-known *Haemers-Roos inequality* [26] for generalized hexagons of order (s, t), $s \geq 2$; $t \leq s^3$. We call a regular near hexagon with parameters (s, t_2, t), $s \geq 2$, for which $t + 1$ equals the Mathon bound $(t_2 + s + 1)(s^2 - s + 1)$ a *maximal regular near hexagon.*

An inequality of Brouwer & Wilbrink [8, p. 161] states that $c_d \leq (s^2 + 1)c_{d-1}$ if d is even. If we compare this upper bound for c_d with the one provided by (4), we see that the inequality of Brouwer and Wilbrink is stronger if $c_{d-1} < \frac{(s^d+1)s^{d-4}}{s^{d-4}+1}$ and weaker if $c_{d-1} > \frac{(s^d+1)s^{d-4}}{s^{d-4}+1}$. Hiraki & Koolen [28, Theorem 1(1)] also obtained an upper bound for c_d in terms of s and c_{d-1}. Also this bound is sometimes stronger and sometimes weaker than the one given by (4).

As already mentioned in Section 1, the two inequalities given by (4) extend a result of Neumaier [31, Theorem 3.1] who already proved the validity of the upper bound in (4) in the case i is odd and the validity of the lower bound in the case i is even. The upper bound for i even and the lower bound for i odd also follow from an inequality obtained in very recent independent work of Tonejc [40] on general distance-regular graphs.

We now give a list of all known regular near $2d$-gons, $d \geq 3$ and $s \geq 2$, for which equality holds in one of the inequalities in (4).

• Suppose $\mathcal{S}$ is the Hermitian dual polar space $DH(2d-1, q^2)$, $d \geq 3$, associated with a Hermitian polarity of the projective space $\mathrm{PG}(2d-1, q^2)$.

(The collinearity graph is also denoted by ${}^2A_{2d-1}(q)$.) Then $s = q$ and $c_i = (q^{2i}-1)/(q^2-1)$ for every $i \in \{0,1,\ldots,d\}$. The lower and upper bound in (4) is attained for any $i \in \{3,4,\ldots,d\}$.

• Suppose $\mathcal{S}$ is one of the following dual polar spaces of rank $d \geq 3$: (i) the orthogonal dual polar space $DQ(2d,q)$ associated with a nonsingular parabolic quadric of $\mathrm{PG}(2d,q)$; (ii) the symplectic dual polar space $DW(2d-1,q)$ associated with a symplectic polarity of $\mathrm{PG}(2d-1,q)$. (The collinearity graphs are also denoted by $B_d(q)$ and $C_d(q)$, respectively.) Observe that $DQ(2d,q) \cong DW(2d-1,q)$ if and only if q is even. Then $s = q$ and $c_i = (q^i-1)/(q-1)$ for every $i \in \{0,1,\ldots,d\}$. The lower bound in (4) is attained for any $i \in \{3,4,\ldots,d\}$.

• Suppose $\mathcal{S}$ is a (maximal) generalized hexagon of order (s,s^3), $s \geq 2$. Then the upper bound in (4) is attained if $i = 3$. The only known generalized hexagons of order (s,s^3), $s \geq 2$, are the dual twisted triality hexagons $\mathrm{T}(q,q^3)$, where q is some prime power. These generalized hexagons were constructed by Tits [39]. (The collinearity graphs of these generalized hexagons are also denoted by ${}^3D_{4,2}(q)$.)

• Suppose $\mathcal{S}$ is the unique regular near hexagon with parameters $(s,t_2,t) = (2,1,11)$ (Brouwer [2]; Shult & Yanushka [34, Section 2.5]), also denoted by $\mathbb{E}_1$. An explicit description of this near hexagon will be given in the appendix. The regular near hexagon $\mathbb{E}_1$ is maximal as the upper bound in (4) is attained if $i = 3$.

• Suppose $\mathcal{S}$ is the unique regular near hexagon with parameters $(s,t_2,t) = (2,2,14)$ (Brouwer [3]; Shult & Yanushka [34, Section 2.5]). Then $\mathcal{S}$ is isomorphic to the point-line geometry $\mathbb{E}_2$ whose points are the blocks of the unique Steiner system $S(5,8,24)$ and whose lines are the triples of mutually disjoint blocks (natural incidence). The regular near hexagon $\mathbb{E}_2$ is maximal as the upper bound in (4) is attained if $i = 3$.

In Vanhove [41, Theorem 1], it was proved that $c_i \leq (s^{2i}-1)/(s^2-1)$ for every $i \in \{2,3,\ldots,d\}$ (see also the proof of Theorem 3.2). In [41], the regular near polygons for which some c_i, $i \in \{2,3,\ldots,d\}$, attains the upper bound $(s^{2i}-1)/(s^2-1)$ were not yet classified. By relying on the inequalities given in (4), we are now able to achieve this goal if $d \geq 3$.

Theorem 3.4. *Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, with $s \geq 2$ and $d \geq 3$. If $c_i = (s^{2i}-1)/(s^2-1)$ for a certain $i \in \{2,3,\ldots,d\}$, then s is a prime power and $\mathcal{S}$ is isomorphic to the Hermitian dual polar space $DH(2d-1,s^2)$.*

Proof. First, suppose $c_j = (s^{2j}-1)/(s^2-1)$ for some $j \in \{2,3,\ldots,d-1\}$.

By Theorem 3.2, we then have

$$\frac{s^{2(j+1)}-1}{s^2-1} = \frac{(s^{j+1}-1)(c_j - s^{j-1})}{s^{j-1}-1} \leq c_{j+1}.$$

Together with $c_{j+1} \leq (s^{2(j+1)}-1)/(s^2-1)$ this implies that $c_{j+1} = (s^{2(j+1)} - 1)/(s^2-1)$.

Next, suppose $c_j = (s^{2j}-1)/(s^2-1)$ for some $j \in \{3, 4, \ldots, d\}$. By Theorem 3.2, we then have

$$\frac{s^{2j}-1}{s^2-1} = c_j \leq \frac{(s^j+1)(c_{j-1}+s^{j-2})}{s^{j-2}+1},$$

and hence $(s^{2(j-1)}-1)/(s^2-1) \leq c_{j-1}$. The fact that $c_{j-1} \leq (s^{2(j-1)} - 1)/(s^2-1)$ again implies that $c_{j-1} = (s^{2(j-1)}-1)/(s^2-1)$.

By a straightforward inductive argument, we now see that $c_j = (s^{2j} - 1)/(s^2-1)$ for every $j \in \{2, 3, \ldots, d\}$. Results of Cameron [10] and Brouwer & Wilbrink [8, Lemma 26] (see also Brouwer, Cohen & Neumaier [5, Theorem 9.4.4]) now imply that $\mathcal{S}$ is a dual polar space. The fact that $t_2 = s^2$ then implies that s is a prime power and that $\mathcal{S}$ is isomorphic to $DH(2d-1, s^2)$. □

In case one of the intersection numbers c_i, $i \in \{3, 4, \ldots, d\}$, attains the lower bound in (4), we only have a complete classification if $i = 3$.

Theorem 3.5. *Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, with $s \geq 2$ and $d \geq 3$. If c_3 attains the lower bound $(s^2+s+1)(c_2-s)$ in (4), then s must be a prime power and $\mathcal{S}$ is isomorphic to either $DQ(2d, s)$, $DW(2d-1, s)$ or $DH(2d-1, s^2)$.*

Proof. Note that since $c_3 = (s^2+s+1)(c_2-s) > 0$, we must have $c_2 > s$ and thus $t_2 \geq s$. We know from Brouwer & Wilbrink [8, Theorem 4] (mentioned as Theorem 2.3 in De Bruyn [14]) that $\mathcal{S}$ has subgeometries, the so-called *hexes*, that are regular near hexagons with parameters $(s, t_2, t_3) = (s, t_2, (s^2 + s+1)(t_2+1-s)-1)$. For each such regular near hexagon, an inequality by Brouwer & Wilbrink [8, p. 161] yields that

$$\begin{aligned} 0 &\leq t_3^2 - ((s^2+1)(t_2+1)-1)t_3 + s^4(t_2+1) \\ &= s(s^2+s+1)(t_2-s)(t_2-s^2). \end{aligned}$$

Since $t_2 \geq s$ and $t_2 = c_2 - 1 \leq s^2$ (see for instance Theorem 3.2), we see that $t_2 = s$ or $t_2 = s^2$. Hence $c_3 - 1 = t_3 = t_2(t_2+1)$. Results of Cameron

[10] and Brouwer & Wilbrink [8, Lemma 26] (see also Brouwer, Cohen & Neumaier [5, Theorem 9.4.4]) imply that each of the hexes is a dual polar space. Brouwer & Cohen [4, Corollary 2, p. 195] then implies that $\mathcal{S}$ itself must also be a dual polar space. The fact that $t_2 \in \{s, s^2\}$ then forces s to be a prime power and $\mathcal{S}$ to be isomorphic to either $DQ(2d, s)$, $DW(2d-1, s)$ or $DH(2d-1, s^2)$. □

In case c_3 attains the upper bound in (4), the following can be said by relying on Terwilliger's work.

Theorem 3.6. *Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, with $s \geq 2$ and $d \geq 3$. Then the following are equivalent:*

(a) $c_3 = (s^2 - s + 1)(c_2 + s)$;
(b) *for every even $i \in \{3, 4, \dots, d\}$, the lower bound in (4) is attained, and for every odd $i \in \{3, 4, \dots, d\}$, the upper bound in (4) is attained;*
(c) *Γ is Q-polynomial with respect to the eigenvalue $-t-1$;*
(d) *Γ has classical parameters $(d, -s, \alpha, \beta)$ for certain $\alpha, \beta \in \mathbb{R}$.*

If any of these conditions hold, then we have

$$\alpha = -\frac{s+t_2}{s-1}, \qquad \beta = s + \frac{(-s)^{d-1}-1}{s^2-1} s(s+t_2),$$

and hence

$$c_i = -\frac{(-s)^i - 1}{s+1}\Big(1 + \frac{s+t_2}{s^2-1}((-s)^{i-1}-1)\Big), \quad \forall i \in \{0, 1, \dots, d\}.$$

Moreover, if $1 \leq h \leq d$ and $0 \leq i, j \leq d$, then for any two points x and y with $\mathrm{d}(x, y) = h$, we then have $Mv = 0$, where $Z = \Gamma_i(x) \cap \Gamma_j(y)$, $Z' = \Gamma_j(x) \cap \Gamma_i(y)$ and

$$v = -p^h_{i,j} \frac{(-\frac{1}{s})^i - (-\frac{1}{s})^j}{1 - (-\frac{1}{s})^h} (\chi_{\{x\}} - \chi_{\{y\}}) + (\chi_Z - \chi_{Z'}).$$

Proof. We know from Lemma 3.1 that the idempotent corresponding to eigenvalue $-(t+1)$ has, up to a positive scalar, the following dual eigenvalue sequence: $1, -1/s, \dots, (-1/s)^d$. The equivalences now follow immediately from Terwilliger [35, Theorem 4.2]. The precise values of α and β follow from equations (2) and (3), taking into account that $c_2 = t_2 + 1$ and $b_0 = a_1 + b_1 + c_1 = sc_1 + b_1$. The equality $Mv = 0$ follows from the equivalence $(v) \leftrightarrow (vii)$ in Terwilliger [35, Theorem 3.3]. □

We note that Weng [43, Theorem C] has classified all regular near $2d$-gons with Q-polynomial collinearity graphs (with respect to any eigenvalue) with $d \geq 4$, $c_2 \geq 2$ and $s \geq 2$.

The fact that one of the bounds in (4) is attained implies some relations between the intersection numbers.

Theorem 3.7. *Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, with $s \geq 2$ and $d \geq 3$.*

(a) *If $c_i = (s^i - 1)(c_{i-1} - s^{i-2})/(s^{i-2} - 1)$ for a certain $i \in \{3, 4, \dots, d\}$, then for every $j \in \{0, 1, \dots, i\} \setminus \{\frac{i}{2}\}$, we have*

$$c_{i-1} - s^{i-2} = \frac{s^{i-2} - 1}{s^{2j-i} - 1} c_j + \frac{s^{i-2} - 1}{s^{i-2j} - 1} c_{i-j}. \tag{6}$$

(b) *If $c_i = (s^i + 1)(c_{i-1} + s^{i-2})/(s^{i-2} + 1)$ for a certain $i \in \{1, 2, \dots, d\}$, then for every $j \in \{0, 1, \dots, i\}$, we have*

$$c_{i-1} + s^{i-2} = \frac{s^{i-2} + 1}{s^{2j-i} + 1} c_j + \frac{s^{i-2} + 1}{s^{i-2j} + 1} c_{i-j}. \tag{7}$$

In particular, we have $c_{i/2} = (c_{i-1} + s^{i-2})/(s^{i-2} + 1) = c_i/(s^i + 1)$ if i is even.

Proof. Note that claim (b) is correct if $i \in \{1, 2\}$. Now consider two points x and y at distance $i \geq 3$, and let p be a point at distance j from x and at distance $i-j$ from y. Suppose $Z = \Gamma_1(x) \cap \Gamma_{i-1}(y)$ and $Z' = \Gamma_{i-1}(x) \cap \Gamma_1(y)$. The point p will be at distance $j-1$ from exactly c_j elements in Z. No point $p_x \in Z$ can be at distance j from p, since then the line xp_x would contain a point p'_x with $\mathrm{d}(p'_x, p) = j - 1$, $\mathrm{d}(p, y) = i - j$ and $\mathrm{d}(p'_x, y) = i$, which is impossible. Hence p is at distance $j+1$ from the remaining $c_i - c_j$ elements of Z. Similarly, the point p will be at distance $(i-j) - 1$ from c_{i-j} elements of Z' and at distance $(i-j) + 1$ from the remaining $c_i - c_{i-j}$ elements of Z'. Note that $(\chi_{\{p\}})^T A_i \chi_X = |\Gamma_i(p) \cap X|$ for any subset X of $\mathcal{P}$. If $c_i = (s^i - 1)(c_{i-1} - s^{i-2})/(s^{i-2} - 1)$, then take $v = s(c_{i-1} - s^{i-2})(\chi_{\{x\}} - (-1)^i \chi_{\{y\}}) + (\chi_Z - (-1)^i \chi_{Z'})$, and if $c_i = (s^i + 1)(c_{i-1} + s^{i-2})/(s^{i-2} + 1)$, then take $v = s(c_{i-1} + s^{i-2})(\chi_{\{x\}} + (-1)^i \chi_{\{y\}}) + (\chi_Z + (-1)^i \chi_{Z'})$. Working out $(\chi_{\{p\}})^T M v = 0$ (see Theorem 3.2) now yields an equation in c_{i-1}, c_{i-j} and c_j, which we can solve for c_{i-1}, unless $i = 2j$ and c_i attains the lower bound in (4). □

Observe that equality in (6) and (7) in Theorem 3.7 trivially holds if $j \in \{0, 1, i-1, i\}$. Also, (6) and (7) remain unchanged if we replace j by $i-j$.

In case c_d attains one of the two bounds in (4), the following can be said regarding triple intersection numbers. This will give some alternative explanation for some properties of well-known near $2d$-gons.

Theorem 3.8. *Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, with $s \geq 2$ and $d \geq 3$. The following holds for three points x, y and z of $\mathcal{S}$, pairwise at distance d.*

(a) *If $c_d = (s^d - (-1)^d)(c_{d-1} - (-1)^d s^{d-2})/(s^{d-2} - (-1)^d)$ then*

$$|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)|.$$

(b) *If $c_d = (s^d + (-1)^d)(c_{d-1} + (-1)^d s^{d-2})/(s^{d-2} + (-1)^d)$ then*

$$\begin{aligned} |\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| &= |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| \\ &= \frac{(s^{d-1} + (-1)^d)(c_{d-1} + (-1)^d s^{d-2})}{s^{d-2} + (-1)^d}. \end{aligned}$$

Proof. Suppose $Z = \Gamma_1(x) \cap \Gamma_{d-1}(y)$ and $Z' = \Gamma_{d-1}(x) \cap \Gamma_1(y)$. Note that $(\chi_{\{z\}})^T A_d \chi_{\{x\}} = (\chi_{\{z\}})^T A_d \chi_{\{y\}} = 1$ and that $(\chi_{\{z\}})^T A_i \chi_{\{x\}} = (\chi_{\{z\}})^T A_i \chi_{\{y\}} = 0$ for $i \neq d$. Note also that points in Z or Z' can only be at distance $d-1$ or d from z. We have $(\chi_{\{z\}})^T A_i \chi_Z = 0$ if $0 \leq i \leq d-2$, $(\chi_{\{z\}})^T A_{d-1} \chi_Z = |\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)|$, $(\chi_{\{z\}})^T A_d \chi_Z = c_d - |\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)|$, and similarly $(\chi_{\{z\}})^T A_i \chi_{Z'} = 0$ if $0 \leq i \leq d-2$, $(\chi_{\{z\}})^T A_{d-1} \chi_{Z'} = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)|$, $(\chi_{\{z\}})^T A_d \chi_{Z'} = c_d - |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)|$.

(a) Suppose $c_d = (s^d - (-1)^d)(c_{d-1} - (-1)^d s^{d-2})/(s^{d-2} - (-1)^d)$. We know from Theorem 3.2 that $Mv = 0$ if $v = s(c_{d-1} - (-1)^d s^{d-2})(\chi_{\{x\}} - \chi_{\{y\}}) + (\chi_Z - \chi_{Z'})$. If we denote $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)|$ by n_x and $|\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)|$ by n_y, then

$$\begin{aligned} 0 = (\chi_{\{z\}})^T M v &= (\chi_{\{z\}})^T M (\chi_Z - \chi_{Z'}) \\ &= n_x \Big(-\frac{1}{s}\Big)^{d-1} + (c_d - n_x)\Big(-\frac{1}{s}\Big)^d - n_y \Big(-\frac{1}{s}\Big)^{d-1} - (c_d - n_y)\Big(-\frac{1}{s}\Big)^d \\ &= (n_x - n_y)\Big(\Big(-\frac{1}{s}\Big)^{d-1} - \Big(-\frac{1}{s}\Big)^d\Big) \end{aligned}$$

and hence $n_x = n_y$.

(b) Suppose $c_d = (s^d + (-1)^d)(c_{d-1} + (-1)^d s^{d-2})/(s^{d-2} + (-1)^d)$. By Theorem 3.2, $Mv = 0$ if $v = s(c_{d-1} + (-1)^d s^{d-2})(\chi_{\{x\}} + \chi_{\{y\}}) + (\chi_Z + \chi_{Z'})$. Working out $(\chi_{\{z\}})^T Mv = 0$ now yields:

$$|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| + |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| =$$

$$2\frac{(s^{d-1} + (-1)^d)(c_{d-1} + (-1)^d s^{d-2})}{s^{d-2} + (-1)^d}.$$

Completely analogously, we also have

$$|\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| + |\Gamma_{d-1}(x) \cap \Gamma_{d-1}(y) \cap \Gamma_1(z)| =$$

$$|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| + |\Gamma_{d-1}(x) \cap \Gamma_{d-1}(y) \cap \Gamma_1(z)| =$$
$$2\frac{(s^{d-1} + (-1)^d)(c_{d-1} + (-1)^d s^{d-2})}{s^{d-2} + (-1)^d},$$

which allows us to compute $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)|$ and $|\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)|$. □

In the special case that $\mathcal{S}$ is isomorphic to the Hermitian dual polar space $DH(2d-1, q^2)$, $d \geq 3$, Theorem 3.8 implies that $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| = (q^{d-1} + (-1)^d)(q^d - (-1)^d)/(q^2 - 1)$. This fact also follows from Thas [37, Lemma, p. 538].

In the special case that $\mathcal{S}$ is isomorphic to the symplectic dual polar space $DW(2d-1, q)$, $d \geq 3$, we know from Theorem 3.8 that $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)|$ if d is even and $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| = (q^{d-1} - 1)/(q - 1)$ if d is odd. However, it follows from Klein, Metsch & Storme [29, Theorem 21] that $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| \in \{(q^{d/2} - 1)(q^{d/2-1} + 1)/(q - 1), (q^{d/2} + 1)(q^{d/2-1} - 1)/(q - 1)\}$ if d is even and q odd, $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| \in \{(q^{d-1} - 1)/(q - 1), (q^d - 1)/(q - 1)\}$ if d and q are even, and $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| = (q^{d-1} - 1)/(q - 1)$ if d is odd.

In the case that $\mathcal{S}$ is isomorphic to the dual polar space $DQ(2d, q)$, $d \geq 3$, an approach similar to the one of [37, Lemma, p. 538] or [29, Theorem 21] allows one to prove that $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| \in \{(q^{d-1} - 1)/(q - 1), (q^d - 1)/(q - 1)\}$ if d is even, and $|\Gamma_1(x) \cap \Gamma_{d-1}(y) \cap \Gamma_{d-1}(z)| = |\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_{d-1}(z)| = (q^{d-1} - 1)/(q - 1)$ if d is odd.

Haemers [25, Theorem 5.2.6] proved that any (maximal) generalized hexagon of order (s, s^3), $s \geq 2$, satisfies some very nice combinatorial properties. More precisely, he showed that the number $p_{ijk}(L, x, y)$ of points at distance i from a line L, at distance j from a point x and at distance k from a point y only depends on i, j, k and the "configuration" induced on L, x and y (and not on the particular choice of L, x and y). The following corollary of Theorem 3.8 gives a new combinatorial property, similar to the one obtained by Haemers, that needs to be satisfied for a maximal generalized hexagon.

Corollary 3.9. *If $\mathcal{S}$ is a generalized hexagon of order (s, s^3), $s \geq 2$, then $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ for any three points x, y and z of $\mathcal{S}$, pairwise at distance* 3.

Our next results regard m-ovoids of regular near polygons. Recall that an m-ovoid of $\mathcal{S}$ is a set of points such that each line contains exactly m of its elements. One easily proves that an m-ovoid must have size $(m/(s+1)) \cdot |\mathcal{P}|$. The notion "$m$-ovoid" was introduced for generalized quadrangles by Thas [36]. Investigations of such substructures using algebraic graph theory were done in Eisfeld [24], Bamberg, Law & Penttila [1] and De Wispelaere & Van Maldeghem [22]. The 1-ovoids are precisely those cocliques of the collinearity graph Γ of maximum size $|\mathcal{P}|/(s+1)$.

We first state some lemmas.

Lemma 3.10. [41, Lemma 4] *If $\mathcal{O}$ is an m-ovoid of $\mathcal{S}$, then its characteristic vector $\chi_{\mathcal{O}}$ can be written as $(m/(s+1))\chi_{\mathcal{P}} + Mw$ for some $(|\mathcal{P}| \times 1)$-matrix w.*

Lemma 3.11. [41, Lemma 5] *If $\mathcal{O}$ is an m-ovoid of $\mathcal{S}$, then*

$$|\Gamma_i(x) \cap \mathcal{O}| = k_i\Big(\frac{m}{s+1} + \Big(-\frac{1}{s}\Big)^i\Big(1 - \frac{m}{s+1}\Big)\Big)$$

for every point $x \in \mathcal{O}$ and every $i \in \{0, 1, \ldots, d\}$.

Theorem 3.12. *Suppose $\mathcal{O}$ is an m-ovoid of a finite regular near $2d$-gon $\mathcal{S}$, with $s \geq 2$ and $d \geq 3$. Let x and y be points at distance $i \geq 3$ from each other, and put $N_x := |\Gamma_1(x) \cap \Gamma_{i-1}(y) \cap \mathcal{O}|$ and $N_y := |\Gamma_{i-1}(x) \cap \Gamma_1(y) \cap \mathcal{O}|$.*

(a) *If $x, y \in \mathcal{O}$ and $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$, then $N_x = N_y$.*

(b) *If* $x \in \mathcal{O}$, $y \notin \mathcal{O}$ *and* $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$, *then* $N_x - N_y = -s(c_{i-1} - (-1)^i s^{i-2})$.

(c) *If* $x, y \notin \mathcal{O}$ *and* $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$, *then* $N_x = N_y$.

(d) *If* $x, y \in \mathcal{O}$ *and* $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$, *then*

$$N_x + N_y = (c_{i-1} + (-1)^i s^{i-2})\Big(2m\frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i} - 2s\Big).$$

(e) *If* $x \in \mathcal{O}$, $y \notin \mathcal{O}$ *and* $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$, *then*

$$N_x + N_y = (c_{i-1} + (-1)^i s^{i-2})\Big(2m\frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i} - s\Big).$$

(f) *If* $x, y \notin \mathcal{O}$ *and* $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$, *then*

$$N_x + N_y = (c_{i-1} + (-1)^i s^{i-2})\Big(2m\frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i}\Big).$$

Proof. Put $Z := \Gamma_1(x) \cap \Gamma_{i-1}(y)$ and $Z' := \Gamma_{i-1}(x) \cap \Gamma_1(y)$. We know from Lemma 3.10 that we can write $\chi_{\mathcal{O}} = \frac{m}{s+1}\chi_{\mathcal{P}} + Mw$ for some $(|\mathcal{P}| \times 1)$-matrix w. Hence when choosing v as in Theorem 3.2, we have in every case:

$$\Big(\chi_{\mathcal{O}} - \frac{m}{s+1}\chi_{\mathcal{P}}\Big)^T v = w^T(Mv) = 0.$$

Note that $(\chi_{\mathcal{P}})^T\chi_{\{x\}} = (\chi_{\mathcal{P}})^T\chi_{\{y\}} = 1$, $(\chi_{\mathcal{P}})^T\chi_Z = (\chi_{\mathcal{P}})^T\chi_{Z'} = c_i$ and $(\chi_{\mathcal{O}})^T\chi_Z = N_x$, $(\chi_{\mathcal{O}})^T\chi_{Z'} = N_y$. Also, $(\chi_{\mathcal{O}})^T\chi_{\{x\}} = 1$ if $x \in \mathcal{O}$, $(\chi_{\mathcal{O}})^T\chi_{\{x\}} = 0$ if $x \notin \mathcal{O}$, $(\chi_{\mathcal{O}})^T\chi_{\{y\}} = 1$ if $y \in \mathcal{O}$ and $(\chi_{\mathcal{O}})^T\chi_{\{y\}} = 0$ if $y \notin \mathcal{O}$. Working out the above expression, we now obtain the desired results. □

The following is a consequence of Theorem 3.12.

Corollary 3.13. *Suppose* $\mathcal{O}$ *is a* 1*-ovoid of a finite regular near* $2d$*-gon* $\mathcal{S}$, *with* $s \geq 2$ *and* $d \geq 3$, *and let* x, y *be two points of* $\mathcal{S}$ *at distance* $i \geq 3$ *from each other such that* $x \in \mathcal{O}$ *and* $y \notin \mathcal{O}$.

(a) *If* $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$, *then*

$$|\Gamma_1(y) \cap \Gamma_{i-1}(x) \cap \mathcal{O}| = s(c_{i-1} - (-1)^i s^{i-2}).$$

(b) *If* $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$, *then*

$$|\Gamma_1(y) \cap \Gamma_{i-1}(x) \cap \mathcal{O}| = (c_{i-1} + (-1)^i s^{i-2})\Big(2\frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i} - s\Big).$$

Proof. Since $\mathcal{O}$ is a 1-ovoid and $x \in \mathcal{O}$, we have $|\Gamma_1(x) \cap \Gamma_{i-1}(y) \cap \mathcal{O}| = 0$. The exact value of $|\Gamma_1(y) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|$ then follows from Theorem 3.12 (b)+(e). □

The following corollary of Theorem 3.12 is already known. Indeed, the dual polar space $DQ(2d, q)$, $d \geq 3$ and q odd, cannot have 1-ovoids since none of its quads have 1-ovoids (Payne & Thas [33, 3.4.1]). A result of Thomas [38, Theorem 3.2] implies that the dual polar space $DW(5, q)$ cannot have 1-ovoids either. An argument of Shult (as exposed in Pasini & Shpectorov [32, Proposition 2.8]) gives an alternative proof for the non-existence of 1-ovoids in $DQ(6, q) \cong DW(5, q)$, q even. Another proof for the non-existence of 1-ovoids in $DW(5, q)$, q odd, can be found in Cooperstein & Pasini [11] (see also De Bruyn & Pralle [17, Appendix]). Finally, observe that the dual polar spaces $DQ(2d, q)$ and $DW(2d - 1, q)$ $(d \geq 3)$ have full subgeometries (i.e. every line of the subgeometry is incident with the same set of points in both geometries) that are isomorphic to $DQ(6, q)$ or $DW(5, q)$. Hence the non-existence of 1-ovoids in these subgeometries implies the non-existence of 1-ovoids in the dual polar spaces themselves.

Corollary 3.14. *Let* $\mathcal{S}$ *be one of the dual polar spaces* $DQ(2d, q)$ *and* $DW(2d-1, q)$. *If* $d \geq 3$, *then* $\mathcal{S}$ *has no* 1*-ovoids.*

Proof. In both cases, the dual polar space is a regular near $2d$-gon with parameters $s = q$ and $c_i = (q^i - 1)/(q - 1)$, $i \in \{0, 1, \ldots, d\}$. Suppose $\mathcal{O}$ is a 1-ovoid of $\mathcal{S}$. It follows from Lemma 3.11 that there exist two elements x and y in $\mathcal{O}$ at distance 3. Theorem 3.12(d) with $i = 3$ now implies that $|\Gamma_1(x) \cap \Gamma_2(y) \cap \mathcal{O}| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}| = 2$. This is however impossible, because as $\mathcal{O}$ is a coclique of the collinearity graph, we must have $\Gamma_1(x) \cap \mathcal{O} = \Gamma_1(y) \cap \mathcal{O} = \emptyset$. □

The following is a special case of Theorem 3.12.

Corollary 3.15. *Suppose* $\mathcal{O}$ *is an* m*-ovoid of a generalized hexagon of order* (s, s^3), $s \geq 2$, *and let* x, y *be two points at distance* 3. *Then* $|\Gamma_1(x) \cap \Gamma_2(y) \cap \mathcal{O}| = |\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}|$ *if either* $x, y \in \mathcal{O}$ *or* $x, y \notin \mathcal{O}$ *and* $|\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}| - |\Gamma_1(x) \cap \Gamma_2(y) \cap \mathcal{O}| = s(s + 1)$ *if* $x \in \mathcal{O}$ *and* $y \notin \mathcal{O}$.

The only known generalized hexagons of order (s, s^3), $s \geq 2$, are the dual twisted triality hexagons $\mathrm{T}(q, q^3)$ where q is some prime power. It is not known whether $\mathrm{T}(q, q^3)$ can have m-ovoids with $0 < m < q+1$. As we shall see in Corollary 3.19, the dual twisted triality hexagon $\mathrm{T}(q, q^3)$ cannot have 1-ovoids.

Lemma 3.16. *Suppose $\mathcal{S}$ is a maximal regular near hexagon, $\mathcal{O}$ is a 1-ovoid of $\mathcal{S}$ and x, y are points of $\mathcal{S}$ such that $x \in \mathcal{O}$ and $y \notin \mathcal{O}$. Then $|\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}| = t_2+1$ if $\mathrm{d}(x, y) = 2$ and $|\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}| = s(t_2+s+1)$ if $\mathrm{d}(x, y) = 3$.*

Proof. Suppose $\mathrm{d}(x, y) = 2$. Then every neighbor of y at distance 2 from x must be on one of the $t_2 + 1$ lines through y at distance 1 from x. Each of these $t_2 + 1$ lines contains exactly one point of $\mathcal{O}$, and that point must be at distance 2 from x since $\mathcal{O}$ is a coclique of Γ. Hence, $|\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}| = t_2+1$.

The claim in the case $\mathrm{d}(x, y) = 3$ is a consequence of Corollary 3.13(a). □

It was shown in Vanhove [41, Theorem 4] that $(q+1)/2$-ovoids in the dual polar space $DH(2d-1, q^2)$ induce distance-regular graphs. We now prove a somewhat similar result for 1-ovoids in maximal regular near hexagons.

Theorem 3.17. *Suppose $\mathcal{S}$ is a maximal regular near hexagon and $\mathcal{O}$ is a 1-ovoid of $\mathcal{S}$. Then the distance-2-relation R_2 induces a strongly regular graph $\mathrm{srg}(v, k, \lambda, \mu)$ on $\mathcal{O}$ with parameters*

$$v = \frac{(s^2-s+1)(s^2+st_2+s+1)(s^4+s^3t_2-s^2t_2-s^2+t_2+1)}{t_2+1},$$
$$k = \frac{s(s^2-s+1)(s+t_2+1)(s^3+s^2t_2-st_2+t_2)}{t_2+1},$$
$$\lambda = \frac{(s^2-s+1)(s^4+2s^3t_2+2s^3+t_2^2s^2+2t_2s^2-st_2-s-t_2-1)}{t_2+1},$$
$$\mu = \frac{(s^2-s+1)(s+t_2+1)(s^3+s^2t_2-st_2-s+t_2+1)}{t_2+1},$$

eigenvalues $\theta_0 = k$, $\theta_1 = (s-1)(s^2-s+1)(s+t_2+1)^2/(t_2+1)$, $\theta_2 = -s^2+s-1$, and corresponding multiplicities $m_0 = 1$,

$$m_1 = \frac{s^2(s-1)(s^2+st_2+s+1)(s^3+s^2t_2-st_2+t_2)}{(s^2+st_2+s-t_2-1)(s+t_2)},$$
$$m_2 = \frac{s(s^2+st_2-t_2)(s^3+s^2t_2-st_2+t_2)(s+t_2+1)(s^4+s^3t_2-s^2t_2-s^2+t_2+1)}{(t_2+1)(s^2+s+st_2-t_2-1)(s+t_2)}.$$

Proof. The size v follows from $|\mathcal{O}| = (k_0 + k_1 + k_2 + k_3)/(s+1)$. Each element of $\mathcal{O}$ is at distance 2 from exactly $k_2/s = st(t+1)/(t_2+1)$ elements of $\mathcal{O}$ by Lemma 3.11. Now consider $x, x' \in \mathcal{O}$ at distance 2 or 3 from each other. We will count in two ways the number N of ordered pairs (p, y) with $\mathrm{d}(x, p) = 2$, $\mathrm{d}(p, y) = 1$, $\mathrm{d}(y, x') = 1$ and $p \in \mathcal{O} \setminus \{x'\}$. Note that if (p, y) is such an ordered pair, then p must be at distance 2 from x'. For a fixed $p \in \Gamma_2(x) \cap \Gamma_2(x') \cap \mathcal{O}$, there are $c_2 = t_2 + 1$ possibilities for y, and hence

$$N = |\Gamma_2(x) \cap \Gamma_2(x') \cap \mathcal{O}| \cdot (t_2 + 1).$$

Firstly, suppose $\mathrm{d}(x, x') = 2$. There are t_2+1 possibilities for y with $\mathrm{d}(x, y) = 1$, $(s-1)(t_2+1)$ for y with $\mathrm{d}(x, y) = 2$ and $s(t-t_2)$ for y with $\mathrm{d}(x, y) = 3$. In the first case, there are $t-1$ lines through y, not containing x or x', and hence $t-1$ possibilities for p as they all intersect $\mathcal{O}$ in one point. In the second case, there are $(t_2+1)-1$ possibilities for p (since we have to exclude x') by Lemma 3.16. Finally, in the third case, there are $s(t_2+s+1)-1$ possibilities for p (since we have to exclude x'), again by Lemma 3.16. Hence

$$N = (t_2+1)(t-1) + (s-1)(t_2+1)t_2 + s(t-t_2)(s(t_2+s+1)-1).$$

Next, suppose $\mathrm{d}(x, x') = 3$. There are $t+1$ possibilities for y with $\mathrm{d}(x, y) = 2$, and $(s-1)(t+1)$ for y with $\mathrm{d}(x, y) = 3$. In the first case, there are t_2+1 possibilities for p by Lemma 3.16. In the second case, there are $s(t_2+s+1)$ possibilities for p, again by Lemma 3.16. Hence

$$N = (t+1)(t_2+1) + (s-1)(t+1)s(t_2+s+1).$$

Using $t+1 = (s^2-s+1)(t_2+s+1)$ and solving for $|\Gamma_2(x) \cap \Gamma_2(x') \cap \mathcal{O}|$ yields the desired constants λ and μ. Note that $\mu > 0$ and $1 \le k < v-1$. The eigenvalues and their multiplicities can now be computed in a straightforward way (see for instance Brouwer, Cohen and Neumaier [5, Theorem 1.3.1]). □

Remark 3.18. The fact that a 1-ovoid $\mathcal{O}$ as in Theorem 3.17 yields a strongly regular graph in fact immediately follows from Brouwer, Godsil, Koolen & Martin [6, Theorem 2], since the so-called *degree* of $\mathcal{O}$ is 2 and the so-called *dual width* of $\mathcal{O}$ is 1.

With the aid of a computer we determined all parameters (s, t_2, t) with $2 \leq s \leq 10^7$, $0 \leq t_2 \leq s^2$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$ for which all the parameters v, k, λ, μ, m_1, m_2 defined in Theorem 3.17 are nonnegative integers, and t/t_2 is an integer if $t_2 > 0$ (as forced by Brouwer & Wilbrink [8, Theorem 6]). We found the following three possibilities.

- We have $(s, t_2, t) = (2, 1, 11)$ and $(v, k, \lambda, \mu) = (243, 132, 81, 60)$. In this case, $\mathcal{S} \cong \mathbb{E}_1$. By De Bruyn [13, Theorem 4.2], $\mathbb{E}_1$ has up to isomorphism a unique 1-ovoid. In the appendix, we show that the strongly regular graph associated with each 1-ovoid of $\mathbb{E}_1$ is isomorphic to the graph whose vertices are the codewords of the $(11, 5)$ ternary Golay code, with two codewords being adjacent whenever their difference has weight 6. This strongly regular graph, which was first constructed in [19, Example 2, p. 54], is known as the *Delsarte graph* in the literature.
- We have $(s, t_2, t) = (2, 2, 14)$ and $(v, k, \lambda, \mu) = (253, 140, 87, 65)$. In this case, $\mathcal{S} \cong \mathbb{E}_2$. All 1-ovoids of this regular near hexagon have been classified by Brouwer & Lambeck [7, p. 105] (see also De Bruyn [14, Section 6.6.2]). Every 1-ovoid of $\mathbb{E}_2$ consists of all 253 blocks of $S(5, 8, 24)$ through a distinguished point of $S(5, 8, 24)$. The strongly regular graph associated with each such 1-ovoid is easily seen to be isomorphic to the graph whose vertices are the blocks of the unique Steiner system $S(4, 7, 23)$, with two distinct blocks being adjacent whenever they intersect in precisely three points.
- We have $(s, t_2, t) = (4, 1, 77)$ and $(v, k, \lambda, \mu) = (47125, 12012, 3575, 2886)$. We do not know whether there exists a regular near hexagon or a strongly regular graph with these parameters. The collinearity graph of such a regular near hexagon would have classical parameters $(3, -4, -5/3, 24)$ and spectrum $(312)^1(22)^{89232}(-13)^{145464}(-78)^{928}$ (see for instance Brouwer, Cohen & Neumaier [5, §8.4] or De Bruyn [14, Section 3.3]). The strongly regular graph, if it exists, would have spectrum $(12012)^1(702)^{840}(-13)^{46284}$.

The 1-ovoids in generalized $2d$-gons are also known as *distance-2-ovoids*. In general, distance-2-ovoids in generalized hexagons of order (s, t) with

$s, t > 1$ are hard to find, but some constructions were given in De Wispelaere & Van Maldeghem [20, 21, 23]. De Wispelaere & Van Maldeghem [20, Subsection 7.2] proved that the dual twisted triality hexagons $\mathrm{T}(2, 8)$ and $\mathrm{T}(3, 27)$ cannot have 1-ovoids, and suggested that this might hold for all prime powers. The following corollary of Theorem 3.17 extends these latter results to all generalized hexagons of order (s, s^3), $s \geq 2$.

Corollary 3.19. *A generalized hexagon $\mathcal{S}$ of order (s, s^3), $s \geq 2$, cannot have* 1*-ovoids.*

Proof. Suppose $\mathcal{O}$ is a 1-ovoid of $\mathcal{S}$. We can apply Theorem 3.17 with $t_2 = 0$ to obtain a strongly regular graph with eigenvalue multiplicity $m_1 = (s^5 - s^4 + 2s^3 - 4s^2 + 6s - 10) + (16s - 10)/(s^2 + s - 1)$, which is impossible since this number is never an integer if $s \geq 2$. □

A generalized polygon of order (s', t') with $s' > 1$ finite and t' infinite is called *semi-finite*. The question whether semi-finite generalized polygons exist is one of the most important problems in the theory of generalized polygons (see Problem 5 of Van Maldeghem [42, Appendix E]). The following consequence of Corollary 3.19 shows the non-existence of certain semi-finite generalized polygons.

Corollary 3.20. *A (not necessarily finite) generalized hexagon of order (s, t') cannot have maximal generalized hexagons as full proper subgeometries.*

Proof. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized hexagon of order $(s, t) = (s, s^3)$, $s \in \mathbb{N} \setminus \{0, 1\}$, and suppose $\mathcal{S}$ is a full proper subgeometry of a generalized hexagon $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ of order (s, t'). Then $t' > t = s^3$, and hence t' must be infinite by the Haemers-Roos inequality.

Suppose z is a point of $\mathcal{S}'$ at distance 2 from $\mathcal{P}$. Every line of $\mathcal{S}$ contains a unique point nearest to z which necessarily lies at distance 2 from z. So, the set of points of $\mathcal{P}$ at distance 2 from z is a 1-ovoid of $\mathcal{S}$. But that is impossible by Corollary 3.19. So, every point of $\mathcal{S}'$ lies at distance at most 1 from $\mathcal{P}$.

Now, let x be a point of $\mathcal{S}'$ at distance 1 from $\mathcal{P}$. If x is collinear with two distinct points y_1 and y_2 of $\mathcal{S}$, then y_1 and y_2 lie at distance 2 from each other in $\mathcal{S}$ and hence their unique common neighbor x would be a point of $\mathcal{S}$, a contradiction. So, there is only one line L through x that meets $\mathcal{P}$ in a point.

Let X denote the set of points of $\mathcal{S}$ at distance 2 from x and let Y denote the set of all neighbors y of x that lie on a path of length two connecting x with one of the points of X. Since X is finite, also Y must be finite.

If y is a neighbor of x not contained in L, then $y \notin \mathcal{P}$ is collinear with a unique point in $\mathcal{P}$ which necessarily belongs to X, implying that $y \in Y$. So, x has only a finite number of neighbors. This is in contradiction with the fact that t' is infinite. □

Appendix: The strongly regular graphs associated with 1-ovoids of $\mathbb{E}_1$

We know from Theorem 3.17 that 1-ovoids of the regular near hexagon $\mathbb{E}_1$ yield strongly regular graphs srg$(243, 132, 81, 60)$. In this appendix, we will verify that all strongly regular graphs arising in this way are isomorphic to the Delsarte graph.

Let V be a 6-dimensional vector space over the field $\mathbb{F}_3$ of order 3 with basis $\mathcal{B} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}$ and let N be the following matrix over $\mathbb{F}_3$:

$$N := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 \end{bmatrix}.$$

The subspace C of $(\mathbb{F}_3)^{12}$ generated by the six rows of N is the so-called *extended ternary Golay code*. With respect to the basis $\mathcal{B}$, the 12 columns of N define a set $\mathcal{K}$ of 12 points of $\Sigma := \mathrm{PG}(V) \cong \mathrm{PG}(5,3)$. For every point x of Σ, let $i_{\mathcal{K}}(x)$ denote the smallest strictly positive integer k with the property that there exist k points $y_1, y_2, \ldots, y_k \in \mathcal{K}$ for which $x \in \langle y_1, y_2, \ldots, y_k \rangle$. We call $i_{\mathcal{K}}(x)$ the *$\mathcal{K}$-index of the point x*. The set $\mathcal{K}$ satisfies several nice properties (see e.g. Coxeter [12] and De Bruyn [14, Section 6.5]). Several of these properties are mentioned in the following proposition.

Proposition A.1. (*a*) *Every point of* Σ *has* $\mathcal{K}$*-index* 1, 2 *or* 3.

(*b*) *The stabilizer* G *of* $\mathcal{K}$ *in* $\mathrm{PGL}(V) \cong \mathrm{PGL}(6,3)$ *is isomorphic to the Mathieu group* M_{12} *and acts sharply* 5*-transitive on the set* $\mathcal{K}$.

(*c*) *Every* $i \in \{1, 2, \ldots, 5\}$ *distinct points of* $\mathcal{K}$ *generate a subspace of* Σ *with projective dimension* $i - 1$.

(*d*) *If* $x_1, x_2, \ldots, x_5$ *are five distinct points of* $\mathcal{K}$*, then the hyperplane* $\langle x_1, x_2, \ldots, x_5 \rangle$ *of* Σ *contains precisely* 6 *points of* $\mathcal{K}$*.*

(*e*) *Let* $\mathcal{D}$ *be the point-line geometry whose points are the elements of* $\mathcal{K}$ *and whose lines are all the sets of* 6 *points that arise as intersections of* $\mathcal{K}$ *with suitable hyperplanes of* Σ *(natural incidence). Then* $\mathcal{D}$ *is isomorphic to the unique Steiner system* $S(5, 6, 12)$*.*

By Proposition A.1(a)+(c), we know that there are 12 points with $\mathcal{K}$-index 1, 132 points with $\mathcal{K}$-index 2 and 220 points with $\mathcal{K}$-index 3.

Suppose that Σ is embedded as a hyperplane in the projective space $\mathrm{PG}(6, 3)$. Then let $\mathbb{E}_1$ be the point-line geometry whose points are the points of $\mathrm{PG}(6, 3)$ not contained in Σ, and whose lines are those lines of $\mathrm{PG}(6, 3)$ not contained in Σ and containing a unique point of $\mathcal{K}$ (natural incidence). By De Bruyn & De Clerck [16] (see also De Bruyn [14, Theorem 6.59]), $\mathbb{E}_1$ is a regular near hexagon with parameters $(s, t_2, t) = (2, 1, 11)$ (the unique one with these parameters). If x and y are two distinct points of $\mathbb{E}_1$ and the line $\langle x, y \rangle$ of $\mathrm{PG}(6, 3)$ through them intersects Π in a point z, then by De Bruyn & De Clerck [16, Lemma 4.2] the distance between x and y in the near hexagon $\mathbb{E}_1$ is equal to the $\mathcal{K}$-index of z.

Lemma A.2. *Every hyperplane of* Σ *intersects* $\mathcal{K}$ *in either* 6*,* 3 *or* 0 *points. Every three distinct points of* $\mathcal{K}$ *are contained in a unique hyperplane that intersects* $\mathcal{K}$ *in precisely three points. There are precisely* 132 *hyperplanes of* Σ *that contain* 6 *points of* $\mathcal{K}$*,* 220 *hyperplanes that contain precisely* 3 *points of* $\mathcal{K}$ *and* 12 *hyperplanes that are disjoint from* $\mathcal{K}$*.*

Proof. This easily follows from the fact that every $i \in \{1, 2, \ldots, 5\}$ distinct points of $\mathcal{K}$ are contained in precisely $\frac{3^{6-i}-1}{2}$ hyperplanes of Σ and $\binom{12-i}{5-i}/\binom{6-i}{5-i}$ blocks of the Steiner system $\mathcal{D}$ (see Proposition A.1). The fact that every hyperplane of Σ intersects $\mathcal{K}$ in either 6, 3 or 0 points is also a consequence of the fact that the code C only contains vectors of weight 0, 6, 9 or 12, see e.g. De Bruyn [14, Lemma 6.57]. □

We denote the hyperplane of Σ with equation $a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 = 0$ also by $[a_1, a_2, a_3, a_4, a_5, a_6]$. We denote by $\mathcal{K}^*$ the set of 12 hyperplanes of Σ that are disjoint from $\mathcal{K}$. After some straightforward calculations, we find that $\mathcal{K}^*$ consists of the following 12 hyperplanes:

$$\alpha_1 := [1, 1, 1, 1, 1, 1], \quad \alpha_2 := [1, 1, 1, 1, -1, -1], \quad \alpha_3 := [1, 1, 1, -1, -1, 1],$$

$\alpha_4 := [1, 1, -1, 1, -1, -1], \quad \alpha_5 := [1, -1, -1, -1, -1, -1], \quad \alpha_6 := [1, -1, 1, -1, -1, 1],$

$\alpha_7 := [1, 1, -1, -1, 1, -1], \quad \alpha_8 := [1, -1, 1, -1, 1, -1], \quad \alpha_9 := [1, 1, -1, -1, 1, 1],$

$\alpha_{10} := [1, -1, -1, 1, 1, 1], \quad \alpha_{11} := [1, -1, 1, 1, 1, -1], \quad \alpha_{12} := [1, -1, -1, 1, -1, 1].$

The set $\mathcal{K}^*$ is a set of points of the dual Σ^* of the projective space Σ. Now, let B be the following nonsingular symmetric matrix over $\mathbb{F}_3$:

$$B := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Let ζ denote the orthogonal polarity of the projective space Σ which maps each point $(X_1, X_2, \ldots, X_6)$ of Σ to the hyperplane $[a_1, a_2, \ldots, a_6]$ of Σ, where $[a_1, a_2, \ldots, a_6]^T = B \cdot [X_1, X_2, \ldots, X_6]^T$.

Proposition A.3. *The polarity ζ maps the 12 points of $\mathcal{K}$ to the 12 hyperplanes of $\mathcal{K}^*$, the 132 points of $\mathcal{K}$-index 2 to the 132 hyperplanes intersecting $\mathcal{K}$ in precisely 6 points and the 220 points of $\mathcal{K}$-index 3 to the 220 hyperplanes intersecting $\mathcal{K}$ in precisely 3 points.*

Proof. If C_i, $i \in \{1, 2, \ldots, 12\}$, denotes the i-th column of the matrix $B \cdot N$, then the row vector C_i^T describes the hyperplane α_i defined above. Hence ζ maps the 12 points of $\mathcal{K}$ to the 12 hyperplanes of $\mathcal{K}^*$. In order to prove the remaining claims of the lemma, it suffices to prove that ζ maps each of the 132 points of $\mathcal{K}$-index 2 to one of the 132 hyperplanes intersecting $\mathcal{K}$ in precisely 6 points.

Let q be a point of $\mathcal{K}$-index 2 and suppose $q \in \langle p_1, p_2 \rangle$ where p_1 and p_2 are two distinct points of $\mathcal{K}$. Put $\beta_1 := p_1^\zeta$, $\beta_2 := p_2^\zeta$ and $\gamma_1 := q^\zeta$. Then γ_1 is one of the two hyperplanes through $\beta_1 \cap \beta_2$ distinct from β_1 and β_2. Let γ_2 denote the other hyperplane. Since $\beta_1 \cap \mathcal{K} = \beta_2 \cap \mathcal{K} = \emptyset$ and $\beta_1 \cup \beta_2 \cup \gamma_1 \cup \gamma_2$ consists of all points of Σ, we necessarily have that $|\gamma_1 \cap \mathcal{K}| = |\gamma_2 \cap \mathcal{K}| = 6$ by Lemma A.2. □

Now, suppose Π is one of the 12 hyperplanes belonging to $\mathcal{K}^*$ and put $p := \Pi^\zeta$. The projection of $\mathcal{K} \setminus \{p\}$ from the point p on the hyperplane Π is a set $\mathcal{K}'$ of 11 points of Π. The Mathieu group M_{11}, which is isomorphic to the stabilizer of p in G, acts as a group of collineations of Π fixing $\mathcal{K}'$. The set $\mathcal{K}'$ is a *two-character set* of Π, that is a set of points of Π having two possible intersection sizes with the hyperplanes of Π. Indeed, Lemma A.2 implies that every hyperplane of Π intersects $\mathcal{K}'$ in either 2 or 5 points. The two-character set $\mathcal{K}'$ is isomorphic to the set described as example RT6 in Calderbank & Kantor [9, p. 112], which is up to isomorphism the unique two-character set of that size with those intersection numbers in $\mathrm{PG}(4,3)$. The rows of the (5×11)-matrix with the 11 coordinate vectors as columns generate the $(11,5)$ ternary Golay code with nonzero weights 6 and 9. The set of 66 hyperplanes of Π that intersect $\mathcal{K}'$ in precisely 5 points is a two-character set X^* of the dual Π^* of Π with intersection sizes 30 and 21. This two-character set is a so-called projective dual of $\mathcal{K}'$ and is precisely example RT6^d of Calderbank & Kantor [9, p. 112]. The two-character sets RT6 and RT6^d of [9] were already described in Delsarte [18].

Lemma A.4. *The set X of points of $\mathcal{K}$-index 2 contained in Π is a two-character set of Π which is isomorphic to the two-character set X^* of Π^* defined above.*

Proof. If x is a point of Π, then $x^{\zeta'} := \langle p, x\rangle^{\zeta} = p^{\zeta} \cap x^{\zeta}$ is a hyperplane of $\Pi = p^{\zeta}$. The map ζ' defines a polarity of Π. Now, x is a point of $\mathcal{K}$-index 2 if and only if the hyperplane x^{ζ} (which contains p) intersects $\mathcal{K}$ in precisely 6 points by Proposition A.3, i.e. if and only if $x^{\zeta'} \in X^*$. □

With every nontrivial two-character set X of the projective space $\mathrm{PG}(k-1,q)$, $k \geq 2$, there is associated a strongly regular graph Γ_X, see Calderbank & Kantor [9, Theorem 3.2]. To define Γ_X, embed $\mathrm{PG}(k-1,q)$ as a hyperplane in $\mathrm{PG}(k,q)$. Then the vertices of Γ_X are the points of $\mathrm{PG}(k,q)$ not contained in $\mathrm{PG}(k-1,q)$, and two distinct vertices x_1 and x_2 of Γ are adjacent whenever the unique line of $\mathrm{PG}(k,q)$ through them has a point in common with X.

So, we know that with the two-character set X^* of Π^* there is associated a strongly regular graph Γ_{X^*}. By Calderbank & Kantor [9, Theorem 5.7], the strongly regular graph Γ_{X^*} must be isomorphic to the Delsarte graph on 243 codewords as defined in Section 3.

By Theorem 3.17, we also know that with every 1-ovoid $\mathcal{O}$ of $\mathbb{E}_1$ there is associated a strongly regular graph $\Gamma_{\mathcal{O}}$. We can now prove the following.

Theorem A.5. *If $\mathcal{O}$ is a 1-ovoid of $\mathbb{E}_1$, then the strongly regular graph $\Gamma_{\mathcal{O}}$ is isomorphic to the strongly regular graph Γ_{X^*} associated with the two-*

character set X^* of Π^*. As a consequence, $\Gamma_{\mathcal{O}}$ is isomorphic to the Delsarte graph.

Proof. By De Bruyn [13, Theorem 4.2], there exists a hyperplane $\alpha \neq \Sigma$ in $PG(6,3)$ such that $\Pi := \alpha \cap \Sigma \in \mathcal{K}^*$ and $\mathcal{O} = \alpha \setminus \Pi$. Let X be the set of vertices of Π whose $\mathcal{K}$-index is equal to 2. By Lemma A.4, X is a two-character set of Π and $\Gamma_X \cong \Gamma_{X^*}$.

Now, the vertices of $\Gamma_{\mathcal{O}}$ are the points of $\alpha \setminus \Pi$ and two vertices are adjacent whenever they lie at distance 2 from each other, i.e. whenever the unique line of $PG(6,3)$ through them contains a point of X. It follows that $\Gamma_{\mathcal{O}} \cong \Gamma_X$. So, $\Gamma_{\mathcal{O}} \cong \Gamma_{X^*}$ as we needed to prove. We have seen in the preceeding paragraph that Γ_{X^*} is isomorphic to the Delsarte graph. □

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